A Function-Analytic Development of Field Theory

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I confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

This thesis presents a system of coupled differential equations as a simple model of quantum electrodynamics (QED). A key feature of the model is the Riemann-Silberstein (RS) representation of the photon. The RS representation leads to a natural configuration-space description for a system of multiple, non-interacting electrons and photons. Relativistic covariance is shown by extending the dynamics to a representation of the Poincaré group on the space of configuration-space amplitudes. Because the differential system forms a well-posed initial-value problem, this model features a natural concept of time evolution, and concretely parametrises the system even at intermediate times during scattering processes. If QED could be formulated in a framework of this type, both the analysis and rigorous formulation of quantum field theory may benefit from a useful new perspective. Towards this aim, I consider deformation of the free theory, preserving the initial-value nature while incorporating interactions between particles. The deformation takes the form of a coupling between states of different particle content. I present some simple criteria to show whether the deformation is compatible with relativity. For a specific choice of the deformation, I perform a perturbative expansion on this system. I demonstrate agreement between some of the leading terms and QED. Although further extensions are required, these appear to be compatible with the existing framework, and these results are an encouraging first step towards a complete configuration-space/differential representation of QED.

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Chapter 1

Introduction

Quantum field theory is a framework for describing the interactions and evolution of systems of many particles. The first successful quantum field theory, quantum electrodynamics (QED), describes electrons, positrons, photons and their interactions [1, 2, 3, 4]. QED is consistent with special (but not general) relativity. Field theory developed subsequently to include non-Abelian gauge theories [5, 6, 7, 8], and the weak [9, 10] and strong [11, 12, 13, 14] interactions. Each of these sectors of field theory is extraordinarily successful; the comparison between theory and experiment provides some of the most precise agreement in *any* field of science. For example, the experimental determination of the electron anomalous magnetic moment [15] agrees to eight significant figures with the theoretical prediction to order $(\alpha/\pi)^4$ expansion in field theory by [16]. A review of precision tests of QED is available in [17]. Reviews of experimental data supporting QCD are available in [18, 19, 20]. In this way, field theory provided the basis for modern particle physics (the 'Standard Model'). Other developments have a wide range of application in atomic physics, nuclear physics, condensed matter theory, and astrophysics.

With such flourishing practical application, mathematical physicists had a clear mandate to try to systematise the theoretical framework, and to elucidate the underlying hypotheses and structure of the theory. After all, tighter constraints on the theory might narrow down candidate extensions beyond the Standard Model. Yet despite many decades of concerted effort, opening many promising avenues of research and yielding many useful by-products, decisive progress towards a rigorous mathematical foundation for quantum field theory remains elusive. As the prominent mathematical physicists Streater and Wightman wrote in their influential treatise [21, p.1], "[Field

theory] never reached a stage where one could say with confidence that it was free from internal contradictions — nor the converse." F. Dyson, one of the pioneers of QED, echoed this sentiment, writing [22]

> ...[A]ll attempts to give [quantum field theory] a rigorous mathematical definition have so far failed. The opportunity has been open to mathematicians for 20 years, and is still open to them, to ... create the first rigorous theory of quantized relativistic particles with local interactions in a 4-dimensional space-time.

Throughout his career, Dyson remained a vocal spokesperson for the need for close collaboration between mathematicians and physicists. Dyson regarded the lack of a rigorous mathematical foundation for field theory as one of the outstanding 'missed opportunities' in mathematics, lamenting that 'the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce' [22]. Most recently, the prominent theorist E. Witten wrote [23]

> It is frustrating that, at the outset of the new century, the main framework used by physicists for describing the laws of nature is not accessible mathematically. The same point has been made for decades, since the start of axiomatic and constructive quantum field theory.

Thus, it is broadly agreed that the mathematical foundations of QFT remains an important, long-standing, as-yet unresolved problem. *Constructive quantum field theory* (CQFT) developed as a promising early program for the systematic development of the original formalism. Wightman [24] proposed a minimal set of mathematical hypotheses (the *Wightman axioms*) and sought the rigorous development of quantum field theories from these. The first example, constructed by Glimm and Jaffe [25, 26], was a consistent, non-trivial Hamiltonian theory of the (1+1)-dimensional ϕ^4 field theory. These results were later extended to (1+2)-dimensional ϕ^4 theory [27]. (Reviews of these results are available in Jaffe [28] and Streater and Wightman [21, pp. 179–185].) Despite the success of these simpler theories, extension of these results to a theory on physical (1+3)-dimensional spacetime faces some challenging obstacles. Jaffe and Witten [29] wrote

> In contrast to the existence of quantum fields with a ϕ^4 nonlinearity in dimensions 2 and 3, the question of extending these results to four dimensions is problematic... Analysis of the borderline dimension 4 (between existence and non-existence) is more subtle; if one makes some reasonable (but not entirely proved) assumptions, one also can conclude triviality for the quartic coupling in four dimensions.

In particular, extension to (1+3)-dimensions would need to incorporate the phenomenon of charge renormalization, a significant additional obstacle not present in the theories of lower dimension [21, p.191]. A review of constructive field theory is available in [28]. Faced with these obstacles, subsequent efforts sought to abstract away from the underlying Hilbert space, focusing instead on the 'C^*-algebra of observables' [30, 31]. Much progress has been made in this field of *algebraic quantum field theory*, but the framework is yet to place decisive constraints on field theory over and above the content of the standard methods and techniques of (non-axiomatic) field theory.

Meanwhile, phenomenological techniques in field theory continued to develop, providing ever more precise agreement with myriad experimental data. In this context, it is not surprising that attention shifted away from efforts to systematise quantum field theory. Some theorists felt that QFT was too modest in its conceptual aims. According to Streater and Wightman [21, p.1],

> In the beginning... it was not expected that [quantum field theory] would provide a consistent description of Nature. After all, it was only a quantized version of the classical theory of Maxwell and Lorentz, a theory which was well known to be afflicted with diseases arising from the infinite electromagnetic intertia of point particles. Many physicists were of the opinion that any project to make the theory's mathematical foundation more rigorous was probably ill-advised.

Perhaps for this reason, many UK researchers in fundamental particle physics developed an enthusiasm for more radical, deeper structural changes to fundamental particle physics. These efforts, of which superstring theory [32] is a prominent example, have yielded many intriguing results and many useful by-products, but are yet to place decisive constraints on the structure of field theory or of the Standard Model.

Without compelling evidence in favour of these more radical changes to fundamental particle physics, the present is a great opportunity to revisit the early efforts of systematising QFT, while taking stock of new developments in related fields of physics. The Riemann-Silberstein (RS) representation of the photon (see §2.4) is one such new development: an old idea with new significance, gaining recent attention in the field of experimental quantum optics. In this thesis, I will consider properties of the RS photon that suggest a *differential* representation of relativistic quantum field theory. This representation comprises a system of first-order, linear, partial-differential equations. I will show that this representation forms a well-posed initial-value problem, and that the space of initial data provides a natural realisation of the space of states in field theory. We will perform a perturbative solution of the system, evaluating the leading contribution to scattering. It will be shown that some contributions are equal to certain terms of the perturbative series for scattering as specified in QED. It is not known whether this equality extends to the full perturbative series, but this preliminary study provides some encouraging early results. If equality could be shown, the differential representation of field theory may provide a useful new perspective from which to seek a rigorous foundation for field theory.

J. Schwinger, a pioneer of QED, remarked on the lack of progress in QFT during 1930–1940, 'most physicists were concerned not with careful analysis of the existing theory, but with changing it' [33]. I hope that this thesis may contribute towards a careful analysis of existing theory, and following Schwinger's advice, it would be good to contribute some results at the formal end of UK particle physics theory that are not string theory.

The remainder of this chapter will present a brief review of QED and an outline of the thesis.

1.1 Quantum electrodynamics

The QED Lagrangian is

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - eA\!\!\!/ - m\right)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{1.1.1}$$

The Dirac bispinor ψ is the lepton field and A_μ the electromagnetic field.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.1.2}$$

is the electromagnetic field strength. Here m is the electron mass and e is the coupling constant, usually quoted as the *fine structure constant* $\alpha = e^2/4\pi \simeq 1/137.036$. Expression (1.1.1) uses the 'slashed' notation

$$\not{k} = \gamma^\mu k_\mu \tag{1.1.3}$$

for 4-vectors k_μ, where γ^μ are the Dirac gamma-matrices. The 'Dirac adjoint' is defined

$$\bar{\psi} = \psi^\dagger \gamma^0, \tag{1.1.4}$$

where † denotes the Hermitian conjugate.

In QED, scattering is described by the S-matrix, and given as a power series in the coupling e. The coefficients at each order are specified mathematical expressions, given by the *Feynman rules* and *Feynman diagrams*. The formal relationship between the Lagrangian (1.1.1) and the Feynman rules for QED are given by the *Lehmann-Symanzik-Zimmermann reduction formulae* [34]. This topic is treated thoroughly in standard texts on introductory field theory (e.g. [35, 36]). As a simple example, for the scattering process $e\gamma \to e\gamma$, the S-matrix takes the form

$$S = \mathrm{Id} + (2\pi)^4 \delta^4(p + q - p' - q') \sum iM(p, q, p', q') \tag{1.1.5}$$

where p_μ, q_μ are the 4-momenta of the incoming photon and electron, and p'_μ, q'_μ are the 4-momenta of the outgoing photon and electron. Here Id is the identity operator (corresponding to no scattering). The terms M in the sum (the *matrix elements*) describe non-trivial scattering; there is one matrix element for each Feynman diagram. (This sum is sometimes denoted the T-matrix.) The delta-function constrains scattering to processes for which

$$\begin{aligned} \mathbf{p} + \mathbf{q} &= \mathbf{p}' + \mathbf{q}' \\ |\mathbf{p}| + E(\mathbf{q}) &= |\mathbf{p}'| + E(\mathbf{q}') \end{aligned} \tag{1.1.6}$$

in other words, processes consistent with conservation of energy-momentum. (Here we define the energy of a massive particle $E(\mathbf{k}) = +\sqrt{|\mathbf{k}|^2 + m^2}$.)

The leading non-trivial contributions to scattering arise at $O(e^2)$, and some examples are given in the sections below.

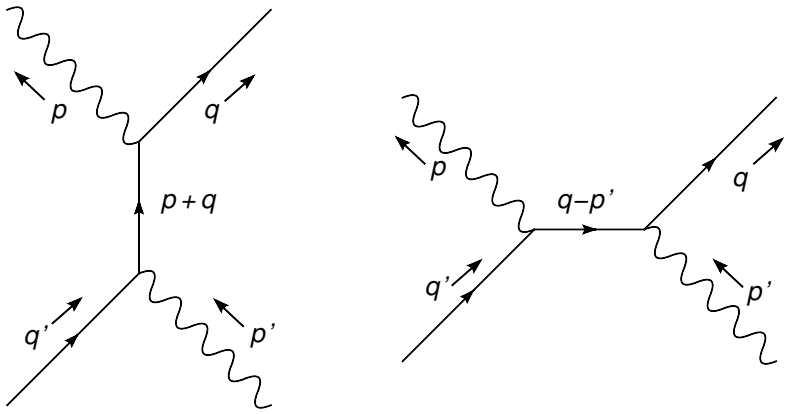

Figure 1.1: Leading-order contributions to Compton scattering. L-R: s-channel. t-channel. Time increases from bottom to top.

1.2 Example: Compton scattering

The leading-order contributions to electron-photon scattering (Compton scattering) are shown in Fig 1.1. According to the Feynman rules, the matrix element for the s-channel diagram (Fig 1.1a) is

$$iM = \bar{u}(q)\,(-ie\gamma^\mu)\,\varepsilon^*_\mu(p)\frac{i(\not{p}+\not{q}+m)}{(p+q)^2-m^2+i\epsilon}\,(-ie\gamma^\nu)\,\varepsilon'_\nu(p')u'(q'). \tag{1.2.1}$$

Here, p' and q' denote the incoming photon and electron momenta respectively, and ε' and u' their respective 4-vector and bispinor amplitudes. The unprimed variables denote the corresponding parameters of the outgoing particles. The presence of $\epsilon > 0$ shifts poles of M off the real axis. The limit $\epsilon \downarrow 0$ is implicit. Likewise, the matrix element for the t-channel diagram (Fig 1.1b) is

$$iM = \bar{u}(q)\,(-ie\gamma^\mu)\,\varepsilon^*_\nu(p)\frac{i(\not{q}-\not{p}'+m)}{(q-p')^2-m^2+i\epsilon}\,(-ie\gamma^\nu)\,\varepsilon'_\mu(p')u'(q'). \tag{1.2.2}$$

Positron scattering $e^+\gamma \to e^+\gamma$ is also treated by QED. The diagrams are identical to Fig. 1.1 except the fermion lines are labelled with arrows in the opposite direction. The Feynman rules specify the corresponding matrix elements. For example, the matrix element for s-channel scattering is

$$iM = \bar{v}'(q')\,(-ie\gamma^\mu)\,\varepsilon'_\mu(p')\frac{i(-\not{p}-\not{q}+m)}{(p+q)^2-m^2+i\epsilon}\,(-ie\gamma^\nu)\,\varepsilon^*_\nu(p)v(q). \tag{1.2.3}$$

Here we have followed a common convention of labelling the positron amplitudes as v' (incoming) and v (outgoing).

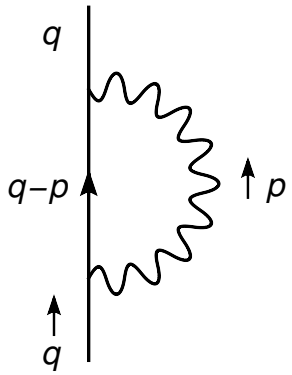

Figure 1.2: A contribution to the electron self energy.

1.3 Example: Electron self-energy

Consider the loop diagram (Fig 1.2), which contributes to the electron self-energy in QED. According to the Feynman rules, the matrix element is

$$iM = \int \frac{d^4p}{(2\pi)^4} \bar{u}(q)\,(-ie\gamma^\mu)\,\frac{i(\not{q}-\not{p}+m)}{(q-p)^2-m^2+i\epsilon}\,(-ie\gamma^\nu)\,u'(q)\frac{-ig_{\mu\nu}}{p^2+i\epsilon} \tag{1.3.1}$$

in the momentum representation. Because the integral in (1.3.1) diverges in the region of large momenta, the loop diagram in Figure 1.2 is called a (ultraviolet-)divergent diagram. According to *renormalization theory* developed by [4, 37, 38, 39], divergent diagrams are still a useful starting point for scattering calculations. Let us briefly review this procedure.

The divergent integral in (1.3.1) can be written

$$\tilde{M} \propto (-ie)^2 \int \frac{d^4p}{(2\pi)^4}\gamma^\mu \tilde{S}^F(q-p)\gamma^\nu \tilde{D}^F(p)g_{\mu\nu}. \tag{1.3.2}$$

where we have introduced the Feynman propagators

$$\begin{aligned} \tilde{D}^F(k) &= \frac{-1}{k^2+i\epsilon} \\ \tilde{S}^F(k) &= \frac{\not{k}+m}{k^2-m^2+i\epsilon}. \end{aligned} \tag{1.3.3}$$

(Conventions follow Itzykson and Zuber [36].) By a simple calculation from this definition, the position space representations $D^F(x)$ and $S^F(x)$ are Green's functions for the wave equation and free Dirac equation, respectively:

$$\Box^2 D^F(x) = \delta^4(x) \tag{1.3.4}$$

$$(i\not{\partial} - m)S^F(x) = \delta^4(x) \tag{1.3.5}$$

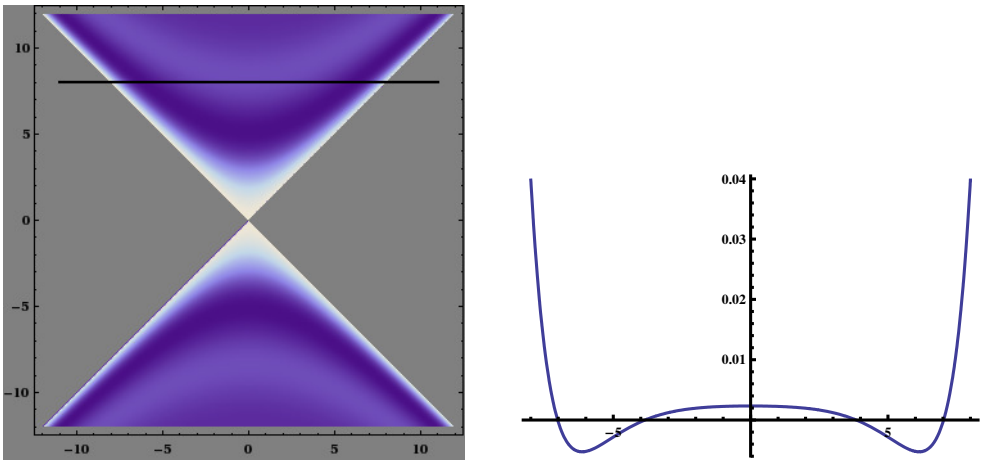

Figure 1.3: *a*) Re$(D^F(x))$ on the spacetime (t, r)-plane (units of m^{-1}). *b*) Re$(D^F(x))$ as a function of r at $t = 8m^{-1}$ (black line in *a*).

Here $\Box^2 = \partial_t^2 - \nabla^2$ is the wave operator; the $+,-,-,-$ convention for the spacetime metric will be used throughout this text. D^F and G^F are closely related to the Klein-Gordon propagator

$$\tilde{G}^F(k) = \frac{-1}{k^2 - m^2 + i\epsilon} \tag{1.3.6}$$

In particular, D^F is the massless limit of G^F, and S^F is related to G^F by

$$S^F(x) = -(i\partial\!\!\!/ + m)G^F(x) \tag{1.3.7}$$

The position space representation $G^F(x)$ is obtained from $\tilde{G}^F(k)$ by evaluating the inverse Fourier transform of (1.3.6),

$$G^F(x) = \frac{1}{4\pi}\delta(\tau^2) + \begin{cases} \dfrac{m}{8\pi\tau}H_1(m\tau) & x \text{ timelike} \\ \\ \dfrac{im}{4\pi^2\tau}K_1(m\tau) & x \text{ spacelike} \end{cases} . \tag{1.3.8}$$

Here $\tau = \sqrt{|t^2 - r^2|}$ and H_1 and K_1 are respectively the first Hankel and MacDonald functions [40]. Full details of the calculation may be found in [41]. It is evident that G^F is singular on the lightcone $(\tau = 0)$: the delta-function in the first term is singular at $\tau = 0$, and both K_1 and Im(H_1) diverge as $\tau \to 0$. Figure 1.3 shows the real part of G^F. The singular nature of G^F carries over to S^F and D^F; each is singular on the lightcone $\tau = 0$.

The singular behaviour of $G^F(x)$ may be rigorously described within the theory of Schwartz distributions [42, 41]. In this theory, G^F is characterised by convergence of the overlap integral

$$\int d^3\mathbf{x}\, G^F(t,\mathbf{x})\psi(\mathbf{x}) \tag{1.3.9}$$

for ψ belonging a suitable class of *test functions*. A common specification for the test functions ψ is

- smoothness: ψ is differentiable to all orders at all $\mathbf{x} \in \mathbb{R}^3$, and
- rapid decrease at infinity: ψ and its partial derivatives of all orders tend to zero as $|\mathbf{x}| \to \infty$ more rapidly than any inverse power of $\mathbf{x}$.

ψ is said to belong to the *Schwartz space* $\mathcal{S}$, and it can be shown that (1.3.9) defines a continuous linear map $\mathcal{S} \to \mathbb{C}$. G^F is said to be a *distribution* over $\mathcal{S}$, or to belong to the *dual space* of $\mathcal{S}$. It can be shown that S^F is also well defined as a distribution over $\mathcal{S}$ [41]. The expression for S^F corresponding to (1.3.9) has an important physical interpretation:

$$\int d^3\mathbf{x}'\; S^F(t,\mathbf{x}-\mathbf{x}')\psi(\mathbf{x}') \tag{1.3.10}$$

solves the initial-value problem for the Dirac equation, where ψ are initial data. Consequently, convergence of the overlap integral (1.3.9) is necessary by *physical* considerations: it ensures (1.3.10) can describe the evolution from arbitrary initial conditions.

Let us return to the divergent integral (1.3.2). If D^F and S^F were ordinary (integrable) functions, the convolution integral could be rewritten as

$$M(x) \propto (-ie)^2\gamma^\mu S^F(x)\gamma_\mu D^F(x). \tag{1.3.11}$$

using the Fourier convolution theorem. However, while the singular nature of either D^F or S^F is precisely constrained in terms of operators on $\mathcal{S}$, (1.3.11) attempts to multiply them together. With the singularities in the region $\tau = 0$ coinciding, the product simply cannot be defined in a consistent way. The divergence in the loop integral (1.3.1) can be understood as a direct consequence of the badly defined product (1.3.11) of singular functions [38, 41] (see also [43]).

In renormalization theory, the problem is solved by replacing the singular functions D^F, S^F by *regularized* counterparts D^F_r, S^F_r. For finite values of the regularization parameter $r > 0$, these are smooth, integrable functions, so the product (1.3.11) and the convolution (1.3.2) are well-defined. The specific form depends on the regularization scheme. Renormalization theory specifies *counterterm subtraction*, after which the remainder has a limit in the dual space of $\mathcal{S}$ as $r \to 0$ [38, 39]. In this way, renormalization shows term-by-term existence of the perturbative series (as distributions over $\mathcal{S}$). However, it should be emphasised that convergence of the series as a whole is a separate (unresolved) problem [44, 45, 46]. The presence of the Landau pole in QED is a possible indicator for divergence of the series.

1.4 The concept of *state* in field theory

In quantum field theory, the term *state* refers to a superposition of free-particle plane waves whose behaviour approximates the system in the asymptotic past ($t \to -\infty$ or 'in'- states) or future ($t \to \infty$ or 'out'- states) [47, 48]. According to Itzykson and Zuber ([36] §5-1)

> ... in-states must represent exactly the individual characteristics of isolated particles ... Long before the collision, well-separated wave packets evolve independently and freely. Long after the collision, free wave packets separate, representing the outgoing states.

This usage of 'state' is well-suited to scattering processes occurring in colliders and particle physics experiments. In these experiments, collision times are extremely short, and on macroscopic scales the incoming/outgoing particles are well separated. On macroscopic scales, the process resembles — to an accuracy well beyond experimental precision — the transition from an 'incoming' free-particle state (the *in-state*) to an 'outgoing' free-particle state (the *out-state*).

Field theory is often the means by which we compute the out-state of the system for a specified in-state. This is written

$$|\text{out}\rangle = S^\dagger |\text{in}\rangle \tag{1.4.1}$$

as a definition of the S-matrix [47, 48] (see also [36] §4-10). The space of free-particle states has a well-understood structure (the *Fock space*) and S is known to be linear

(the principle of superposition) and unitary (the principle of conserved probabilities). The perturbative expansion of S in terms of matrix elements (the *Dyson series*) and the accompanying Feynman rules give rise to the spectacularly accurate experimental predictions of field theory.

In this way field theory provides an extremely accurate description of scattering between in- and out-states. Yet the theory is (at best) vague with regards to an obvious further question:

Key Question 1. Is it possible to describe the system *during* the scattering? If so, what data furnish this unique description?

Indeed, Itzykson and Zuber wrote in their influential text 'it is out of the question to follow in detail the time evolution during the elementary scattering events' ([36] §5-1). Likewise, Weinberg in his text was careful to restrict attention to

> the paradigmatic experiment ... [involving] transitions between the initial and final states of distant and effectively non-interacting particles

([33] Ch3, p.107). Most texts (and research papers) skirt or skip the issue entirely — there is, after all, plenty to get on with to cover scattering and computation of the S-matrix! Yet meaningful answers to these questions would undoubtedly provide a richer and more complete conceptual framework for the theory. The term 'state' could be profitably extended to mean 'data which uniquely describe a system', and the scope of the theory would be decisively broadened to describe systems at *any* point of their evolution.

Furthermore, additional structure in the theory would hopefully provide additional criteria with which to constrain or exclude candidate extensions to the standard model (beyond-the-standard-model (BSM) extensions). A historical example is renormalisation, whereby improved understanding of divergences in known theories (Yang-Mills or non-abelian gauge theories) provided new criteria (renormalisability, or the removal or divergences and unitarity on physical state space) to test whether new field theories could be regarded as physically plausible.

Likewise, we hope that an improved understanding of *Key Question 1* might also place restrictions on what theories are plausible. This thesis will develop a framework

towards the solution of *Key Question 1*, and provide evidence that this framework is the correct choice. Consistency with known field theories (and their manifestly correct predictions) are demanded, and will be demonstrated by exhibiting transitions between in- and out-states as the limiting cases $t \to \pm\infty$. This thesis is not concerned with *changing* field theory; it is concerned with broadening the scope of field theory.

1.5 The initial-value problem

The principal theories of fundamental physics, except quantum field theory, are alike in one regard: they admit formulation as an *initial-value problem* (also known as a 'Cauchy problem'). In other words, each theory is built on the concept of *state* as 'data which uniquely describes a system', and *dynamics* specifying the evolution of states in time. Equivalently, dynamics are a family of transformations on a specified space of states, parametrised by the time t. One could argue that this is the most meaningful *definition* of 'time': it is the parameter which labels the system's trajectory through state space as it evolves.

Let us consider an example. In the classical mechanics of a (spinless) point particle, the system is fully specified at any time t_0 by the particle's position $\mathbf{x}$ and momentum $\mathbf{p}$. The data $(\mathbf{x}, \mathbf{p})$ are called the *state* of the system, and the set of all states $(\mathbb{R}^3 \times \mathbb{R}^3)$ is called the *state space*. The dynamics are governed by Hamilton's equations; together with initial data $(\mathbf{x}(t_0), \mathbf{p}(t_0))$ these form an initial-value problem, and completely constrain the past and future history of the system (at least for a finite interval $(t_0 - \epsilon, t_0 + \epsilon)$ in time). These results, stemming from the classical theory of ordinary differential equations, are a guarantee of both *existence* and *uniqueness* of solutions corresponding to arbitrary initial data. Conceptually, the system is fully specified by its state at a given instant, and the state is parametrised in terms of quantities which are (in principle) known or measurable at that instant.

Likewise, in the quantum theory of a non-relativistic, spinless scalar particle, a system may be described at any time t by a complex, square-integrable amplitude (wavefunction) $\psi_t : \mathbb{R}^3 \to \mathbb{C}$. Conversely, any square-integrable function can be regarded as the initial data for some system. Thus *state* means any square-integrable function and *state space* means the space of all such functions. The evolution of the system is given

by the Schrödinger equation

$$i\partial_t \psi_t(\mathbf{x}) = \left(\frac{-1}{2m}\nabla^2 + V(\mathbf{x})\right)\psi_t(\mathbf{x}). \tag{1.5.1}$$

According to the theory of linear partial differential equations, (1.5.1) provides a well-defined Cauchy problem, so the initial data $\psi_0(x) = \psi_t(x)|_{t=0}$ uniquely constrains the system at all past or future times. For example, for a free particle, we have $V \equiv 0$ and (1.5.1) may be written in momentum space

$$\frac{\partial}{\partial t}\tilde{\psi}_t(\mathbf{k}) = -i\frac{k^2}{2m}\tilde{\psi}_t(\mathbf{k}) \tag{1.5.2}$$

(where $\tilde{}$ represents the three-dimensional Fourier transform in $\mathbf{x}$). (1.5.2) integrates to

$$\tilde{\psi}_t(\mathbf{k}) = \exp\left(-i\frac{k^2 t}{2m}\right)\tilde{\psi}_0(\mathbf{k}). \tag{1.5.3}$$

Taking the Fourier transform, in real space this may be written

$$\psi_t(\mathbf{x}) = \int d^3\mathbf{x}'\; G_t(\mathbf{x}-\mathbf{x}')\psi_0(\mathbf{x}). \tag{1.5.4}$$

where

$$\begin{aligned} G_t(\mathbf{x}) &= \exp\left(-i\frac{k^2 t}{2m} - i\mathbf{k}\cdot\mathbf{x}\right) \\ &= \left(\frac{m}{2\pi t}\right)^{3/2}\exp\left(\frac{-x^2 m}{2t}\right) \end{aligned} \tag{1.5.5}$$

is known as the Green's function for the Schrödinger equation for a free particle. For any given initial data ψ_0, (1.5.4) uniquely fixes the state $\psi_t(\mathbf{x})$ at any past or future time t. Thus the Schrödinger equation for a free particle poses an initial value problem.

The initial value problem remains well-posed in the general case where the particle propagates in an external potential $V(\mathbf{x}) \neq 0$, although the solution generally cannot be expressed in closed form [49, 50, 51]. (Some features of field theory begin to appear, for with a sufficiently well-behaved scattering centre, the limits as $t \to \pm\infty$ are in- and out-states, in the sense of being superpositions of free-particle plane waves [52, 53, 54, 55]. This line of research will not be discussed further in this paper, in favour of pursuing results based on relativistically covariant theories.)

For further examples, consider the Dirac equation for an electron or positron propagating in a classical electromagnetic background potential A_μ,

$$(i\partial\!\!\!/ - m)\psi(x) = eA\!\!\!/(x)\psi(x). \tag{1.5.6}$$

This may also be written in Schrödinger form

$$i\partial_t \psi_t(\mathbf{x}) = \left(i\alpha^j \partial_j + \beta m\right) \psi_t(\mathbf{x}) + \gamma^0 e \not{A}(t, \mathbf{x}) \psi_t(\mathbf{x}) \tag{1.5.7}$$

and describes an initial-value problem with $\psi|_{t=0}$ comprising the initial data. The Maxwell equations are also a well-posed initial-value problem for the classical electromagnetic field: given a specified background current j, specification of $\mathbf{E}$ and $\mathbf{B}$ at time $t = 0$ (in any chosen frame) suffices to compute the complete past and future of the system. Formulation of general relativity as an initial-value problem was developed by Choquet-Bruhat [56] and Arnowitt, Deser, and Misner [57]; classical[1] Yang-Mills by Segal [58], classical Yang-Mills-Higgs by Eardley and Moncrief [59, 60, 61] and classical $N = 1$ supergravity by Bao et al [62]. These results do not consider 'second-quantised' systems (systems containing multiple, interacting particles). Lawrie [63] has recently considered time evolution in cosmology.

In contrast, there is presently no indication that quantum electrodynamics or any other known field theory can be formulated in terms of dynamics of Schrödinger type. As discussed in the preceding section, field theory is a theory of transitions between in- and out-states: it is not known how to describe a system in terms of a trajectory of states W_t for all $t \in \mathbb{R}$, such that the limit $t \to \pm\infty$ recovers in- and out-states.

Thus, each of these principal theories of theoretical physics has been formulated as initial-value problems, yet quantum field theory has not. We are motivated to ask

Key Question 2. Is it possible to write field theory as an initial-value problem? If so, what is the appropriate state space and dynamics?

It is clear from the properties of the initial-value problem that a meaningful 'yes' to *Key Question 2* will also resolve *Key Question 1* introduced previously. *Key Question 2* will be the primary focus of this thesis.

Relativistic covariance will be an important consideration for investigating the prospects of field theory as an initial-value problem. Dynamics of Schrödinger type appear to break relativistic covariance by implying the existence of a privileged inertial frame — namely, the inertial frame S in which the states W_t are labeled by the time t defined in S. It is not *a priori* true that the representation by states in S would be consistent with the representation by states in a different inertial frame S'. Whether or not this

[1] i.e., not quantised

is achievable depends on the choice of Hamiltonian; of the examples mentioned above some are relativistic (Dirac, Maxwell, Yang-Mills), and some are not (Schrödinger). Chapter 2 will develop criteria to help identify whether a given Hamiltonian is compatible with relativity. We will examine the Riemann-Silberstein (RS) representation of the photon, and use it as a simple example to illustrate these criteria.

In Chapter 3, we will develop further the characterisation of Dirac and RS states as configuration space amplitudes – that is, as *functions* of specified type. Chapter 4 will extend this framework to include multi-particle systems and we will show that the properties of the RS photon are particularly well-suited to satisfying the requirements of special relativity. Finally, Chapter 5 will explore deformation of the free theory to a non-trivial theory describing interacting particles. We expect that this deformation should be constrained by the criteria for relativistic covariance (Chapter 2). We will perform a perturbative analysis of this system and compare some of the leading terms to QED. Chapter 6 will present the conclusions and outlook for this study.

Chapter 2

Relativity and the initial-value problem

2.1 Representations of the Poincaré group/algebra

The Dirac and Schrödinger equations are both well-posed initial-value problems with dynamics of Schrödinger type,

$$\partial_t \psi = -iH\psi. \tag{2.1.1}$$

A quick comparison between the free theories (Table 2.1) indicates some close structural similarities between the two theories. Here $L^2(\mathbb{R}^3, \mathbb{C})$ denotes the space of square-integrable complex amplitudes on $\mathbb{R}^3$; $L^2(\mathbb{R}^3, \mathbb{C}^4)$ denotes the space of square-integrable Dirac (4-component) amplitudes. Both spaces possess a positive-definite, complete inner product (they are a *Hilbert space*). In each theory, the Hamiltonian is self-adjoint with respect to the inner product, so (by the Stone theorem [64]) the system evolves as a one-parameter family of unitary transformations on the state space Θ^1. Thus the initial data uniquely constrains the past and future history of the system; the system is a well-posed initial value problem.

Yet despite the structural similarities between the two theories, the Dirac system is relativistic, while the Schrödinger system is not. This result is not difficult to prove, for instance by solving for the explicit solutions of (2.1.1) (plane waves) and testing for invariance of the space of solutions under Poincaré transformation. Unfortunately, this approach will not work for some Hamiltonians that we will investigate in later chapters of this thesis: Hamiltonians that are not explicitly solvable, or for which solutions

		Schrödinger	Dirac
State space	Θ^1	$L^2(\mathbb{R}^3, \mathbb{C})$	$L^2(\mathbb{R}^3, \mathbb{C}^4)$
Inner product	$\langle \psi, \chi \rangle$	$\int d^3x\ \psi^*(x)\chi(x)$	$\int d^3x\ \psi^\dagger(x)\chi(x)$
Hamiltonian	H	$-\frac{1}{2m}\nabla^2$	$-i\alpha_k\partial_k + \beta m$

Table 2.1: Comparison of state space and dynamics of the free Schrödinger and Dirac theories.

are only known perturbatively. What tests can be performed in this case to indicate whether the Hamiltonian is consistent with relativity? In this section we will revisit some first principles of relativity theory to construct criteria for a given Hamiltonian to be relativistically covariant.

Special relativity is concerned with the Poincaré group of transformations between inertial frames. Let us recall some of its elementary properties (see e.g., [33], §2.4 or [65]). The Poincaré group is a 10-dimensional real Lie group whose generators will be denoted

$$\{\underbrace{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3}_{\text{translation}}, \underbrace{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3}_{\text{space rotation}}, \underbrace{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3}_{\text{Lorentz boost}}\}. \tag{2.1.2}$$

The *Poincaré algebra* refers to the 10-dimensional real vector space spanned by the generators (2.1.2). Like any Lie algebra, the Poincaré algebra is closed under the commutator bracket; the $\binom{10}{2} = 45$ brackets may be summarised as

$[\cdot,\cdot]$	$\mathcal{P}_0$	$\mathcal{P}_j$	$\mathcal{J}_j$	$\mathcal{K}_j$
$\mathcal{P}_0$	0	0	0	$\mathcal{P}_j$
$\mathcal{P}_i$	0	0	$\epsilon_{ijk}\mathcal{P}_k$	$\delta_{ij}\mathcal{P}_0$
$\mathcal{J}_i$	0	$\epsilon_{ijk}\mathcal{P}_k$	$\epsilon_{ijk}\mathcal{J}_k$	$\epsilon_{ijk}\mathcal{K}_k$
$\mathcal{K}_i$	$-\mathcal{P}_j$	$-\delta_{ij}\mathcal{P}_0$	$\epsilon_{ijk}\mathcal{K}_k$	$-\epsilon_{ijk}\mathcal{J}_k$.

(2.1.3)

Here ϵ_{ijk} is the totally antisymmetric quantity with $\epsilon_{123} = +1$. Finally, a *representation* of the Poincaré group refers to a group homomorphism of the Poincaré group into the automorphism group of some space Θ^1 (usually a state space). Heuristically, a representation of the Poincaré group is a group of transformations on Θ^1 that respect the group structure of the Poincaré group.

As emphasised by Foldy [66] (see also [67, §2.3]), relativistic theories must necessarily be built from representations of the Poincaré group. In a relativistic theory, there is no privileged inertial frame, so the state space Θ^1 must be common to all inertial frames. Suppose a system is observed in two inertial frames, S and S'. Since by hypothesis the system is uniquely described by some state, there must exist a one-to-one correspondence between all states W in S and W' in S'. As Weinberg wrote, 'different observerses see *equivalent* state vectors, but not the *same* state vector' ([33, §3.1]). If L denotes the Poincaré transformation between S and S', let

$$\rho_L : \Theta^1 \to \Theta^1 \tag{2.1.4}$$

denote the one-to-one correspondence between states of S and S'. The one-to-one correspondence holds in each direction, so ρ_L must be invertible. It can now be shown that ρ is the claimed representation. For suppose the system is viewed from *three* different inertial frames S, S', S''. Let L denote the transformation from $S \to S'$, and L' the transformation from $S' \to S''$,

$$S \xrightarrow{L} S' \xrightarrow{L'} S''. \tag{2.1.5}$$

Suppose the system is observed in S in the state W. In S', the system will be observed in the state $\rho_L(W)$, and in S'' in the state $\rho_{(L' \circ L)}(W)$. (Here $\circ$ denotes group composition in the Poincaré group). Yet transformation of the former to the latter corresponds to the Poincaré transformation L', so we must have

$$\rho_{(L \circ L')}(W) = \rho_{L'}\left(\rho_L(W)\right). \tag{2.1.6}$$

Since this relation holds for any state $W \in \Theta^1$, and all Poincaré transformations L, L', we conclude

$$\rho_{(L' \circ L)} = \rho_{L'}\rho_L \tag{2.1.7}$$

for all L, L'. In this way, the state space Θ^1 of a relativistic theory must admit a representation ρ of the Poincaré group.

We can slightly strengthen the above results on other physical grounds. We may assume that ρ_L is linear, so that quantum superpositions of states are preserved under a change of frame. Furthermore, we will typically consider Θ^1 to carry a geometry so

that the notion of a limit is defined, and we may assume that ρ_L is continuous with respect to this geometry, so that limits are preserved under a change of frame. The latter assumption is a simple and concise guarantee for the validity of perturbative calculations: the convergence of a perturbative calculation in one frame should be reflected in any other frame. (This requirement seems desirable even though QFT perturbative series are not convergent in general.) Thus we suppose that the one-to-one correspondence ρ_L between state spaces is in fact continuous, linear, and invertible: a so-called *toplinear isomorphism* of state spaces.

The existence of a representation ρ is the direct test to check whether a given theory is relativistic. If ρ cannot be constructed unambiguously, then some frame must be chosen as a privileged inertial frame in which the description by states is valid. Doing so would violate the principle of relativity: physics should be insensitive to the choice of inertial frame. Conversely, a well-defined representation implies the one-to-one correspondence between the state spaces of *all* inertial frames, so the description by states is unambiguous and consistent across all inertial frames, and the theory is compatible with relativity.

As an example, a simple representation of the Poincaré group is provided by the space of *all amplitudes on spacetime*. This is evidently a relativistic concept, because an amplitude viewed in some other inertial frame is (of course) another amplitude. This representation is a well-known result whose generators take the form (see, e.g., [36, §2-1-3])

$$\begin{aligned} \mathcal{P}_0 &= \frac{\partial}{\partial t} \\ \mathcal{P}_j &= \partial_j \\ \mathcal{J}_j &= -\epsilon_{jkl} x_k \partial_l + J_j \\ \mathcal{K}_j &= x_j \frac{\partial}{\partial t} + t\partial_j + K_j \end{aligned} \tag{2.1.8}$$

where $\{J_j, K_j\}$ are any matrix representation of the Lorentz group. For a scalar theory, J_j, K_j are identically zero; for a theory of Dirac amplitudes,

$$J_j = -\frac{1}{2}\begin{pmatrix} i\sigma_j & 0 \\ 0 & i\sigma_j \end{pmatrix}, \qquad K_j = \frac{1}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix} \tag{2.1.9}$$

(block-diagonal form with σ_j denoting the Pauli matrices) provide the standard (chiral) representation of the Lorentz group. It is a straightforward exercise to check that

the generators (2.1.8) satisfy the bracket relations (2.1.3). According to group theory, the isomorphism of Lie algebras is sufficient to guarantee the desired group homomorphism.

This example illustrates a general tip when looking for a representation of the Poincaré group: usually it is simpler and more convenient to check for the algebra than the group. For example, an explicit expression for ρ_L may be unavailable or not easy to work with; it may for instance only be available as an infinite perturbative series. In these cases it is impossible or impractical to prove group homomorphism (2.1.7). Group theory simplies the problem to identifying a set of 10 operators on Θ^1 which satisfy the same bracket relations as the Poincaré algebra (2.1.3).

2.2 Poincaré algebra in a configuration space theory

As discussed in §1.4–1.5, this thesis aims to formulate field theory both as an initial-value problem and as a *configuration space representation*. The initial-value nature of the problem complicates the question of relativistic covariance. Let us illustrate this with some familiar examples. For a system consisting of a single particle, the configuration space representation means that the system may be represented by an amplitude W on space ($\mathbb{R}^3$). The non-relativistic Schrödinger and relativistic Dirac theories mentioned at the start of the chapter (Table 2.1) are familiar examples. In these cases, W is simply the Schrödinger or Dirac amplitude $\psi(t, \cdot)$ at a specified time t in a specified inertial frame. The evolution in time of a system from an initial state W_0 to another state W_t is an example of the one-to-one correspondence between states observed in different inertial frames: in this case, between inertial frames which are comoving, and whose origins coincide in space and are displaced in time by an interval t. Because the system evolves according to a well-posed initial-value problem, W_t is fixed uniquely by W_0, and vice versa. So the one-to-one correspondence between states of these inertial frames is assured. (As mentioned earlier, the Stone theorem indicates that the correspondence is unitary, so it is linear, invertible and continuous as required.) In an initial-value theory, state space Θ^1 is not the space of 'all amplitudes on spacetime', but rather the space of initial-value data. The space translations $\mathcal{P}_j$ and $\mathcal{J}_j$ act naturally on this domain and remain unchanged from (2.1.8). However, the equation of motion fixes the time derivative of all amplitudes in terms of a specified

Hamiltonian,

$$\frac{\partial}{\partial t} = -iH, \tag{2.2.1}$$

thereby modifying the generators (2.1.8) to

$$\begin{aligned} \mathcal{P}_0 &= -iH \\ \mathcal{P}_j &= \partial_j \\ \mathcal{J}_j &= -\epsilon_{jkl} x_k \partial_l + J_j \\ \mathcal{K}_j &= -ix_j H + t\partial_j + K_j. \end{aligned} \tag{2.2.2}$$

Though the change may appear cosmetic, the Lie bracket structure (2.1.3) required of the new generators does constrain the Hamiltonian. For example, relations such as $[\frac{\partial}{\partial t}, \partial_j] = 0$, which used to be trivial, have now become $[H, \partial_j] = 0$, a non-trivial constraint on H. The change has come about because evolution in time corresponds to a representation on Θ^1 of the subgroup of time translations. In a relativistic theory, this subgroup of time translations is contained in the larger group of all Poincaré transformations, and the generators (2.2.2) reflect this by satisfying the Poincaré brackets. Conversely, if the Hamiltonian is not relativistic, then time evolution is not consistently defined in different inertial frames and there is no larger Poincaré group extending the group of time translations. A Hamiltonian which is not relativistic will cause the generators (2.2.2) to fail at least some of the bracket relations. This test will therefore distinguish whether the Hamiltonian of a given configuration-space theory is relativistic.

To perform the test on a given theory, the Hamiltonian H (Table 2.1) is substituted into the generators (2.2.2), and the brackets evaluated and compared to the requirements of Poincaré algebra (2.1.3). There next two lemmas will help reduce the labour enormously. The first allows us to set $t = 0$ in the expression for $\mathcal{K}_j$.

Lemma 2.2.1. *The generators* (2.2.2) *form a Poincaré representation for all* $t \in \mathbb{R}$ *if and only if they form a Poincaré representation at* $t = 0$.

Proof. Suppose the generators (2.2.2) form a Poincaré representation with $t = 0$. We recover arbitrary $t \in \mathbb{R}$ by the replacement $\mathcal{K}_j \to \mathcal{K}_j + t\partial_j$. We can check that this change does not break the requirements of the Poincaré algebra (2.1.3). For $t \in \mathbb{R}$,

$$[\mathcal{K}_j + t\partial_j, \mathcal{P}_0] = [\mathcal{K}_j, \mathcal{P}_0] + t[\mathcal{P}_j, \mathcal{P}_0] = -\mathcal{P}_j \tag{2.2.3}$$

and

$$[\mathcal{K}_j + t\partial_j, \mathcal{P}_k] = [\mathcal{K}_j, \mathcal{P}_k] + t[\mathcal{P}_j, \mathcal{P}_k] = -\delta_{jk}\mathcal{P}_0, \tag{2.2.4}$$

as required. Likewise,

$$\begin{aligned}[\mathcal{K}_j + t\partial_j, \mathcal{J}_k] &= [\mathcal{K}_j, \mathcal{J}_k] + t[\mathcal{P}_j, \mathcal{J}_k] \\ &= \epsilon_{jkl}\mathcal{K}_l + t\epsilon_{jkl}\mathcal{P}_l \\ &= \epsilon_{jkl}\left(\mathcal{K}_l + t\partial_l\right)\end{aligned} \tag{2.2.5}$$

as required. Finally

$$\begin{aligned}[\mathcal{K}_j + t\partial_j, \mathcal{K}_j + t\partial_j] &= [\mathcal{K}_j, \mathcal{K}_k] + t[\mathcal{P}_j, \mathcal{K}_k] + t[\mathcal{K}_j, \mathcal{P}_k] + t^2[\mathcal{P}_j, \mathcal{P}_k] \\ &= [\mathcal{K}_j, \mathcal{K}_k] + t\delta_{jk}\mathcal{P}_0 - t\delta_{jk}\mathcal{P}_0 + 0 \\ &= -\epsilon_{jkl}\mathcal{J}_l.\end{aligned} \tag{2.2.6}$$

Thus (2.2.2) provide a Poincaré representation for all t. □

The next lemma shows that it is not necessary to evaluate all 45 brackets: a particular subset suffices. Though this is a simple result, it does not seem to have appeared in the literature before.

Lemma 2.2.2. *Sufficient (and necessary) conditions on H for the generators* (2.2.2) *to provide a representation of the Poincaré algebra are:*

$$[H, \mathcal{P}_k] = [H, \mathcal{J}_k] = 0 \tag{2.2.7}$$

and

$$-[H, x_k]H - i[H, K_k] = \partial_k. \tag{2.2.8}$$

Proof. According to lemma 2.2.1, it suffices to show that the generators (2.2.2) form a Poincaré representation with $t = 0$. (2.2.7) is equivalent to the brackets

$$[\mathcal{P}_0, \mathcal{P}_k] = [\mathcal{P}_0, \mathcal{J}_k] = 0 \tag{2.2.9}$$

of the Poincaré algebra. Likewise, by substituting the generators (2.2.2) into

$$[\mathcal{P}_0, \mathcal{K}_k] = \mathcal{P}_k, \tag{2.2.10}$$

we obtain

$$[-iH, -ix_kH + K_k] = \partial_k, \tag{2.2.11}$$

equivalent to (2.2.8) after expanding. We will now show that the correct brackets for $[\mathcal{P}_j, \mathcal{K}_k]$, $[\mathcal{J}_j, \mathcal{K}_k]$ and $[\mathcal{K}_k, \mathcal{K}_k]$ follow from (2.2.7) and (2.2.8). Firstly,

$$\begin{aligned}[\mathcal{P}_j, \mathcal{K}_k] &= [\partial_j, x_k\mathcal{P}_0 + K_k] \\ &= [\partial_j, x_k]\mathcal{P}_0 \\ &= \delta_{jk}\mathcal{P}_0\end{aligned} \tag{2.2.12}$$

as required. Remember the J_k and K_k are constant matrices, and consequently commute with x_j and ∂_j. Secondly,

$$\begin{aligned}[\mathcal{J}_j, \mathcal{K}_k] &= [\mathcal{J}_j, -ix_kH + K_k] \\ &= -i\mathcal{J}_jx_kH + ix_kH\mathcal{J}_j + [J_j, K_k] \\ &= -i[\mathcal{J}_j, x_k]H + ix_k[H, \mathcal{J}_j] + [J_j, K_k].\end{aligned} \tag{2.2.13}$$

We can simplify

$$\begin{aligned}[\mathcal{J}_j, x_k] &= [-\epsilon_{jpq}x_p\partial_q + J_j, x_k] \\ &= \epsilon_{jkl}x_l.\end{aligned} \tag{2.2.14}$$

Therefore

$$\begin{aligned}[\mathcal{J}_j, \mathcal{K}_k] &= -i\epsilon_{jkl}x_lH + \epsilon_{jkl}K_l \\ &= \epsilon_{jkl}\mathcal{K}_l\end{aligned} \tag{2.2.15}$$

as required. Finally, to evaluate $[\mathcal{K}_j, \mathcal{K}_k]$, expand $[\mathcal{K}_j, \mathcal{K}_k]$ as

$$\begin{aligned}[\mathcal{K}_j, \mathcal{K}_k] &= [-ix_jH + K_j, -ix_kH + K_k] \\ &= -[x_jH, x_kH] - ix_j[H, K_k] + ix_k[H, K_j] + [K_j, K_k].\end{aligned} \tag{2.2.16}$$

We can expand

$$[x_jH, x_kH] = x_j[H, x_k]H - x_k[H, x_j]H \tag{2.2.17}$$

because x_j and x_k commute. Therefore

$$\begin{aligned}[\mathcal{K}_j, \mathcal{K}_k] &= -x_j[H, x_k]H + x_k[H, x_j]H - ix_j[H, K_k] + ix_k[H, K_j] + [K_j, K_k] \\ &= x_j\big(-[H, x_k]H - i[H, K_k]\big) - x_k\big(-[H, x_j]H - ix_k[H, K_j]\big) + [K_j, K_k].\end{aligned} \tag{2.2.18}$$

Each of the expressions in parentheses are precisely the left-hand side of (2.2.8). Thus

$$\begin{aligned}[\mathcal{K}_j, \mathcal{K}_k] &= x_j\partial_k - x_k\partial_j - \epsilon_{jkl}J_l \\ &= -\epsilon_{jkl}\left(-\epsilon_{lpq}x_p\partial_q + J_l\right) \\ &= -\epsilon_{jkl}\mathcal{J}_l,\end{aligned} \tag{2.2.19}$$

as required. The remaining bracket relations of the Poincaré algebra (2.1.3) do not depend on H, so the Poincaré representation is complete. □

Incidentally, once the Poincaré algebra representation is known it is possible to reconstruct a group representation[1] by integrating the equation

$$\partial_\sigma W = \mathcal{R}W \tag{2.2.20}$$

where $\sigma \in \mathbb{R}$ and $\mathcal{R}$ is the appropriate real linear combination of the ten generators. For example, when $\mathcal{R} = \mathcal{P}_0$, (2.2.20) is the equation of motion, and the solutions of (2.2.20) describe the transformation of states W over a finite interval σ of time. Likewise, transforming to a boosted inertial frame S' can be achieved by integrating the appropriate combination of boosts

$$\partial_\eta W_\eta = n^j\mathcal{K}_j W_\eta \tag{2.2.21}$$

to describe the change of inertial frame from S to S'. Here η is the rapidity of the boost, and n^j is the unit vector in the direction of the boost[2]. Checking for a representation of the Poincaré algebra is not only a useful test for relativistic covariance. It is also a starting point for explicitly constructing the states observed in different inertial frames.

We remark that our choice of generators (2.2.2) differs from the early work of [65] or later variations proposed by [68, 69, 70]. Our choice follows from straightforward substitution $\partial_t = -iH$ into the canonical representation (2.1.8) and appears to be the simplest and most natural choice.

We now provide some examples of relativistic and non-relativistic configuration space theories.

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[1] In this case, a *projective* or *spinor* group representation. See [67] for further details.

[2] The term *rapidity* has several different uses in high-energy physics. We refer to the parameter η in which a boost in the x-direction takes the form

$$\begin{aligned}t' &= \cosh(\eta)t - \sinh(\eta)x \\ x' &= \cosh(\eta)x - \sinh(\eta)t\end{aligned} \tag{2.2.22}$$

2.3 Examples

The test for relativistic covariance is now a straightforward check of the conditions (2.2.7) and (2.2.8). Let us apply this test to the non-relativistic Schrödinger and relativistic Dirac theories, and show that the former fails while the latter passes. These examples show that the 'Poincaré algebra test' does indeed provide a useful discriminant between non-relativistic and relativistic Hamiltonians.

First, consider the Dirac Hamiltonian $H = -i\alpha_j\partial_j + \beta m$. The first condition (2.2.7) may be checked by direct computation and is simply equivalent to the translation- and rotation-invariance of H. For the other condition (2.2.8), we first evaluate

$$\begin{aligned}[H, x_k] &= [-i\alpha_j\partial_j + \beta m, x_k] \\ &= -i\alpha_j[\partial_j, x_k] \\ &= -i\alpha_k\end{aligned} \tag{2.3.1}$$

and

$$\begin{aligned}[H, K_k] &= [-i\alpha_j\partial_j + \beta m, K_k] \\ &= \tfrac{1}{2}i[\alpha_j, \alpha_k]\partial_j - \tfrac{1}{2}[\beta, \alpha_k]m\end{aligned} \tag{2.3.2}$$

since for Dirac amplitudes, $K_k = -\frac{1}{2}\alpha_k$ according to (2.1.9). We also recall well-known identities $\{\alpha_j, \alpha_k\} = 2\delta_{jk}$ and $\{\beta_j, \alpha_k\} = 0$ satisfied by the Dirac matrices. The condition (2.2.8) becomes

$$\begin{aligned}-[H, x_k]H - i[H, K_k] &= i\alpha_k\left(-i\alpha_j\partial_j + \beta m\right) - i\left(\tfrac{1}{2}i[\alpha_j, \alpha_k]\partial_j - \tfrac{1}{2}[\beta, \alpha_k]m\right) \\ &= \alpha_k\alpha_j\partial_j + \tfrac{1}{2}[\alpha_j, \alpha_k]\partial_j + i\alpha_k\beta m + \tfrac{1}{2}i[\beta, \alpha_k]m \\ &= \tfrac{1}{2}\{\alpha_j, \alpha_k\}\partial_j + \tfrac{1}{2}i\{\beta, \alpha_k\}m \\ &= \partial_k\end{aligned} \tag{2.3.3}$$

as required. Thus the Dirac Hamiltonian satisfies both conditions (2.2.7) and (2.2.8), and the test also correctly indicates that the Dirac theory is relativistic.

In contrast to the Dirac theory, consider the Schrödinger Hamiltonian $H = -\frac{1}{2m}\nabla^2 = -\frac{1}{2m}\partial_j\partial_j$. The Schrödinger Hamiltonian is also translation- and rotation-invariant and therefore satisfies the first condition (2.2.7). However for the other con-

dition (2.2.8), we have

$$\begin{aligned}[H, x_k] &= -\frac{1}{2m}[\partial_j\partial_j, x_k] \\ &= -\frac{1}{m}\partial_k\end{aligned} \tag{2.3.4}$$

and $K_k \equiv 0$ in this scalar theory. Therefore

$$-[H, x_k]H - i[H, K_k] = -\frac{1}{2m^2}\partial_k\nabla^2 \neq \partial_k \tag{2.3.5}$$

and the Schrödinger Hamiltonian fails (2.2.8). Thus the test correctly indicates that the Schrödinger theory is not relativistic.

2.4 Riemann-Silberstein representation of the photon

Subsequent chapters of this thesis will draw upon a configuration-space Hamiltonian theory for the photon known as the *Riemann-Silberstein* (RS) representation. Let us briefly review the features of this theory and, as a final example of the Poincaré algebra test, prove relativistic covariance of the theory.

The RS representation is attributed variously to [71], and [72, 73]. It has received recent attention in relation to localisation properties of photons and the concept of a photon wave-function [74, 75, 76, 77, 78, 79]. In the simplest version of these models, the single-photon state space Θ^1 is contained in the space of square-integrable amplitudes W with 3 complex components,

$$\begin{aligned}\Theta^1 &\subset L^2(\mathbb{R}^3, \mathbb{C}^3), \\ \langle V, W\rangle &= \int d^3x\; V^\dagger(x)W(x) \qquad V, W \in \Theta^1.\end{aligned} \tag{2.4.1}$$

The amplitudes W transform under a spin-1 representation of the Lorentz group with J_j, K_j given by

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.4.2}$$

and $K_j = -iJ_j$. Equivalently, $(J_j)_{kl} = -\epsilon_{jkl}$ and $(K_j)_{kl} = i\epsilon_{jkl}$ [3]. Dynamics are

[3] Definition (2.4.2) follows a convention where J_j and K_j differ by a factor of i from some physics texts. See the note in Appendix §7.1.

governed by the equation of motion

$$\begin{aligned}\frac{\partial}{\partial t}W_t &= -iHW_t \\ &= -i\nabla \times W_t \\ &= K_j\partial_j W_t \qquad (W_t \in \Theta^1)\end{aligned} \tag{2.4.3}$$

with the additional condition

$$\nabla \cdot W_t = 0. \tag{2.4.4}$$

This additional condition characterises Θ^1 as a proper (linear) subspace of $L^2(\mathbb{R}^3, \mathbb{C}^3)$. The real and imaginary parts of (2.4.3) and (2.4.4) correspond to the four Maxwell equations in free space when

$$W_t(x) = \frac{\mathbf{E}(t,x)}{\sqrt{2\epsilon_0}} + i\frac{\mathbf{B}(t,x)}{\sqrt{2\mu_0}} \tag{2.4.5}$$

Thus, in this model the amplitude (or 'wave function') representing a photon state is constructed from the field strengths, and not from the potential. The magnitude-squared describes the classical energy density [76],

$$|W_t(x)|^2 = \frac{|\mathbf{E}(t,x)|^2}{2\epsilon_0} + \frac{|\mathbf{B}(t,x)|^2}{2\mu_0} \tag{2.4.6}$$

and not a probability density as in Schrodinger wave mechanics. A square-integrable amplitude corresponds not to a state of finite probability, but to a state of finite total energy.[4]

The RS representation describes an amplitude $W \in \Theta^1$ which is

- transverse: the condition (2.4.4) implies that each Fourier component $\hat{W}(k)$ satisfies $k \cdot \hat{W}(k) = 0$;
- transforming in a spin-1 representation of the Lorentz group: the generators (2.4.2) satisfy $-(J_1^2 + J_2^2 + J_3^2) = s(s+1)$ with $s = 1$; and
- massless: in the sense that the dynamics (2.4.3)–(2.4.4) constrain the Fourier modes $e^{\pm i(Et - \mathbf{p}\cdot\mathbf{x})}$ of W to satisfy $E^2 - p^2 = 0$. Thus, W is a wave with both group and phase velocity equal to the speed of light.

[4] A word on notation: (2.4.5) and following chapters will not necessarily follow the convention of 'boldface for vectors'. Later chapters will deal extensively with p-fold tensor products of such amplitudes, and there is no clear advantage to a distinguishing typeface for vectors (tensors of rank 1) among tensors of all other ranks.

I propose that these are precisely the key physical characteristics of the free photon, and in the following chapters we will use the RS photon in the foundations of a simple model of QED.

The RS photon satisfies a first order equation of motion (2.4.3), and provides a natural choice for the generator of time translations,

$$\mathcal{P}_0 = \partial_t = -i\nabla \times . \tag{2.4.7}$$

The existence of a generator of time translations is a key feature of the RS theory. Shortly, we will show that this generator of time translations extends to a Poincaré representation, in accord with the requirements of relativistic covariance laid out in §2.2.

Adoption of the RS photon is a clear departure from usual formulations of QED, where photons are parametrised by the electromagnetic potential A_μ. The need for this change arises because parametrisation by A_μ implies dynamics governed by the wave equation. Since the wave equation involves the second derivative in time, there is no natural choice for a generator of time translations. There is no Poincaré representation, and relativistic covariance is not explicit. The existence of a Poincaré representation, present in the RS representation but lacking in the A_μ representation, will also be crucial to the construction of multi-particle configuration-space theories discussed in Chapter 4.

Furthermore, we will see in Chapter 3 that the RS photon has a natural choice of state space, well-suited to the free dynamics but not specific to the free dynamics. We will therefore be free to choose different (interacting) dynamics in Chapter 5. Such transparency is lacking in the A_μ representation. Indeed, most standard texts (e.g. [33]) simply define the photon state space as the set of all linear superpositions of free-particle states indexed by momentum and spin. The reader is then left to study interacting dynamics on a state space defined in terms of free dynamics. This is not a fatal problem, but it lacks transparency. In this way, the RS parametrisation possesses some advantageous features in comparison to the A_μ representation.

As promised, let us establish relativistic covariance of the Riemann-Silberstein theory by applying the Poincaré algebra test. (Relativistic covariance is shown or assumed in [74, 79]. We include this proof for completeness and as an additional example

of application of the Poincaré algebra test of §2.2.) This claim has two parts. Firstly, we claim that the space of states Θ^1 is a relativistically invariant subspace of $L^2(\mathbb{R}^3, \mathbb{C}^3)$; that if a photon state satisfies the vanishing-divergence condition (2.4.4) at some time in some inertial frame, it will have vanishing divergence at all times in all inertial frames. In other words, we substitute the RS Hamiltonian $H = \nabla\times$ into the Poincaré generators (2.2.2), and claim that each generator maps Θ^1 into itself. As the second part of the claim, we substitute the RS Hamiltonian $H = \nabla\times$ into the Poincaré generators (2.2.2), and claim that they form Poincaré representation on Θ^1.

For the first part of the claim, we have $\mathcal{P}_0 = -i\nabla\times$, so $\nabla\cdot(\mathcal{P}_0 W) = 0$ for all W. Hence $\mathcal{P}_0 W \in \Theta^1$ and $\mathcal{P}_0$ maps Θ^1 into itself. Likewise, substituting $H = \nabla\times$ into the expression for $\mathcal{K}_j$ in (2.2.2), we have

$$\begin{aligned}\mathcal{K}_j W &= (x_j K_k \partial_k + t\partial_j + K_j)\, W \\ &= -i\nabla\times(x_j W) + t\partial_j W.\end{aligned} \tag{2.4.8}$$

The second line follows from expression (2.4.3) for K_j. (2.4.8) demonstrates that

$$\nabla\cdot(\mathcal{K}_j W) = 0 \tag{2.4.9}$$

whenever $\nabla\cdot W = 0$. Hence $\mathcal{K}_j$ also maps Θ^1 into itself. Finally, since vanishing divergence is a translation- and rotation- invariant property, Θ^1 is also an invariant subspace for $\mathcal{P}_j$ and $\mathcal{J}_j$. So all of the generators (2.2.2) are operators on Θ^1 into itself.

For the second part of the claim, we apply the criteria of Lemma 2.2.2 to show that the generators form a Poincaré representation. We will first develop the following lemma to help characterise the photon state space Θ^1. Θ^1 has been identified in (2.4.4) as the space of amplitudes with vanishing divergence. The lemma merely rewrites this condition in terms of the K_j matrices.

Lemma 2.4.1. $\nabla\cdot W = 0$ *if and only if*

$$(\partial_k - K_j K_k \partial_j)\, W = 0 \qquad \textit{for each } k = 1, 2, 3. \tag{2.4.10}$$

Proof. The lemma can be verified by explicitly writing out the products $K_j K_k$, or from the following argument. Let $W = (W_1(x), W_2(x), W_3(x))$ be a vector-valued function

and let $c = (c_1, c_2, c_3)$ be an arbitrary constant vector. It is straightforward to verify from the explicit form of K_j (2.4.2) that

$$c \times W = ic_k K_k W. \tag{2.4.11}$$

We also have $\nabla \times \; = iK_j\partial_j$. Hence

$$\nabla \times (c \times W) = -c_k K_j K_k \partial_j W. \tag{2.4.12}$$

From elementary vector calculus, we have the identity

$$\nabla \times (c \times W) = c(\nabla \cdot W) - (c \cdot \nabla)W - W(\nabla \cdot c) + (W \cdot \nabla)c \tag{2.4.13}$$

with the last two terms vanishing since c is constant. Thus

$$\begin{aligned} c(\nabla \cdot W) &= (c \cdot \nabla)W + \nabla \times (c \times W) \\ &= (c_k\partial_k - c_k K_j K_k \partial_j)\, W \\ &= c_k\, (\partial_k - K_j K_k \partial_j)\, W. \end{aligned} \tag{2.4.14}$$

If $\nabla \cdot W = 0$, then the left-hand side of (2.4.14) vanishes for all c. This can only be possible if expression (2.4.10) is true, as claimed by the forward ('only if') implication of the lemma. The reverse ('if') implication is also true, by a similar argument. □

Returning to the Poincaré algebra test, the RS Hamiltonian $H = \nabla \times \; = -iK_j\partial_j$ is clearly translation- and rotation-invariant, and thereby satisfies the first condition (2.2.7). For the other condition (2.2.8),

$$\begin{aligned} [H, x_k] &= iK_j[\partial_j, x_k] \\ &= iK_k \end{aligned} \tag{2.4.15}$$

so

$$\begin{aligned} -[H, x_k]H - i[H, K_k] &= -iK_k(iK_j\partial_j) - i(i[K_j, K_k]\partial_j) \\ &= K_k K_j \partial_j + [K_j, K_k]\partial_j \\ &= K_j K_k \partial_j. \end{aligned} \tag{2.4.16}$$

Using Lemma 2.4.1, we obtain

$$-[H, x_k]H - i[H, K_k] = \partial_j \tag{2.4.17}$$

on Θ^1. In this way, the RS Hamiltonian satisfies both conditions of the Poincaré test (Lemma 2.2.2), and the generators (2.2.2) form a representation of the Poincaré algebra on Θ^1. So the RS representation of the photon is a relativistic theory.

Restriction to the space Θ^1 of amplitudes with vanishing divergence was a crucial part of demonstrating the Poincaré representation. Without the condition of vanishing divergence, the Poincaré bracket relations are not satisfied. Thus in the RS representation, photon states with non-zero divergence do not admit a Poincaré representation, are not relativistically covariant, cannot be physical, and must be excluded from the theory. This conclusion agrees with the condition of vanishing divergence (2.4.4) demanded by Maxwell's equations.

Without doubt, there are more concise ways to show relativistic covariance of the RS theory (it is, after all, structurally equivalent to Maxwell's equations). I wished to illustrate how the Poincaré algebra test can be used to prove relativistic covariance of a configuration space theory if only the Hamiltonian is known, and without solving for closed-form solutions to look directly for relativistic covariance. The technique will be handy in later chapters, with far less trivial Hamiltonians possessing no closed-form solutions.

Note that we have not considered scalar particles, whose equation of motion is the Klein-Gordon equation

$$\frac{\partial^2}{\partial t^2}\psi - \nabla^2\psi + m^2\psi = 0, \tag{2.4.18}$$

or (for massless scalar particles) the wave equation. Interacting scalar theories are widely regarded as the simplest available non-trivial field theories, probably because the lack of spin degrees-of-freedom in such theories simplifies calculations vastly. Nearly all popular texts begin with ψ^4, as do many new lines of research (such as the early constructive field theories mentioned in the Introduction). I will not consider scalar theories in this thesis. The Dirac and RS theories are naturally posed as initial-value problems, and their time evolution is naturally embedded within a Poincaré representation. Scalar theories fit less well with this framework; with the second-order equation of motion, there is no natural choice for a generator of time translations. It is possible to write (2.4.18) as a first-order system with both ψ and $\partial_t\psi$ at $t = 0$ required as initial data (see, e.g. [80]). If the Poincaré representation can be constructed on top

of this, it is a moot point, because by this point the theory is certainly no simpler than the Dirac or RS systems. Moreover, I will argue that the Dirac and RS theories are the optimal entry points to quantum electrodynamics, the prototype field theory with the strongest experimental evidence.

Representations of the Poincaré group on the the single-particle state space pop up perennially in the literature — [70, 69, 68] develop variations on the original results of Foldy [65] without providing a really decisive argument about which is most relevant. The requirements of relativity combined with the initial-value problem on configuration space appear to present a new angle on this old question.

In the next chapter, we will look more closely at the characterisation of Dirac and RS states as functions of specified type on $\mathbb{R}^3$.

Chapter 3

Geometry of state space

The previous chapter showed how relativity — in the guise of Poincaré representations — placed a decisive, structural constraint on dynamics permitted in a configuration-space theory. In this chapter, we will show how these constraints strongly favour a particular choice of *geometry* on the underlying state space. Here, the term 'geometry' (in the sense of metric spaces) means a prescription of limits and convergence in the state space Λ. These are prescribed using a *norm* $\|\cdot\|$ on state space with the properties

- $\|W\| \geq 0$ for all states $W \in \Lambda$, and $\|W\| = 0$ if and only if $W = 0$,
- $\|cW\| = |c|\|W\|$ for all $W \in \Lambda, c \in \mathbb{C}$, and
- $\|V + W\| \leq \|V\| + \|W\|$ for all $V, W \in \Lambda$.

Given a norm on state space, a sequence (W_n) is *convergent* if there exists $W \in \Lambda$ such that $\|W_n - W\| \to 0$ as $n \to \infty$; W is called the limit of the sequence (the above hypotheses guarantee the limit is unique). The norm is a means of quantifying the error in approximating one state by another; a convergent sequence is precisely a sequence whose terms approximate the limit in such a way that the error tends to zero.

As mentioned in §2.2 and §2.4, both the (Dirac) lepton state space Λ^1 and the RS photon state space Θ^1 possess natural L^2-norms of the above form. This is one very favourable aspect of the RS photon. In comparison, the parametrisation by the potential A_μ is most naturally equipped with the geometry of an *indefinite* metric (the Gupta-Bleuler metric [81, 82]). This geometry does not possess the powerful notions of convergence of the L^2 Hilbert spaces. As a further point in its favour, the RS photon state space has a gauge-invariant geometry, since the amplitude W is gauge-invariant.

In contrast, it seems difficult to achieve this under parametrisation by A_μ, since A_μ itself is gauge-dependent. These two fundamental considerations in favour of the RS parametrisation of the photon are almost always overlooked in standard developments of QED.

In this way, both the lepton state space Λ^1 and the RS photon state spaces Θ^1 are provided with a concrete notion of convergence. On physical grounds, this convergence should be a relativistic invariant. States 'approximately equal' in one inertial frame should be approximately equal in any other frame, or rather, a perturbative method of computing some state in one frame should be just as valid in any other frame. In other words, we should demand that all Poincaré transformations U between inertial frames be continuous (bounded),

$$\|UW\| \leq C\|W\| \qquad \text{for all } W \in \Lambda \tag{3.0.1}$$

for some $C > 0$. Note that (3.0.1) is sufficient to ensure that if $W_n \to W$ as $n \to \infty$, then $UW_n \to UW$ as $n \to \infty$. So limits are preserved under Poincaré transformations. Continuity is a non-trivial condition on the possible choices of state space and norm, and will be demonstrated later (§3.3–3.4) for the specific choices of lepton and photon state space proposed in Chapter 2.

Additionally, we may require that a smooth transition between inertial frames should be observed as a smooth transition between states: if $U(\sigma)$ is the subgroup of transformations generated by any element $\mathcal{R}$ of the Poincaré algebra, then we require $U(\sigma)W \to W$ as $\sigma \to 0$ for all $W \in \Lambda$. Heuristically, a state W in a given inertial frame should be well approximated by the state $U(\sigma)W$ measured in 'nearby' inertial frames. We say that we require the Poincaré group representation to be a *strongly continuous group* of transformations on Λ. This places a further constraint on the possible choices of norm $\|\cdot\|$ on state space.

This chapter will revisit the two relativistic theories introduced in the previous chapter — the Dirac lepton and the RS photon. It is well-known that the Dirac lepton occupies a state space constructed from the L^2 function space. We will begin with a brief review of some common function spaces appearing in quantum theory and scattering theory, and briefly review Lebesgue integration and measure theory. We will recall that the L^2 geometry makes the Poincaré representation into a 'strongly continuous

group of bounded transformations' and is therefore consistent with both the continuity requirements mentioned so far. We will then argue for a similar geometry for the RS photon state space and also demonstrate both boundedness and strong continuity of the representation. In this way, the L^2-geometry seems to be the optimal choice for each state space.

Although the argument for this state space geometry is based on physical considerations, the underlying motivation is to provide additional structure to the theory. This additional structure will be justified largely by consideration of free field theories. The key focus of both this chapter, and the preceding one, is to identify features of the free theory that might admit deformation of the theory to include interactions. We hypothesise that geometry of state space is one such feature, and that both the free theory and the interacting theory are Poincaré representations on the same state space and geometry (but with different dynamics). The arguments of this chapter, based on the free theory, motivate a particular choice of geometry for that state space. It is hoped that having settled on this geometry, the continuity requirements outlined above may constrain how interactions might be introduced to the theory — they might decisively constrain the Hamiltonian, in a similar fashion to the constraints of the Poincaré test proposed in §2.2.

The question of geometry of state space has received little attention from the physics community; many common texts do not mention it, or e.g. refer only to 'technical assumptions that allow us to take limits' [33, §2.1]. Arguably, the question is irrelevant for experimentalists who deal only with finitely many terms of the perturbative expansion. Yet these would be in the minority, for the *resummation* of infinitely many terms is a central technique in particle physics phenomenology (see, e.g. [83]). The validity (convergence) of the resummation is a question which can only be posed with respect to a specific choice of geometry for state space. So the question is definitely relevant. Possibly, the lack of attention arises because the perturbative series that form the centrepiece of field theory do not seem to converge [44]. The mood is captured by the noted mathematical physicist B. Simon, who wrote [45]:

> It has been known for some time that the perturbation series in [certain] field theories diverge... [and] it seems unlikely that perturbation theory converges in any Lagrangian field theory.

Yet electrodynamics in particular provides a perturbative series that matches reality spectacularly well. It is hard to argue, on the one hand, that experimental agreement to one part in 10^8 can be obtained by a truncation the QED perturbative series, but on the other hand that the same series diverges catastrophically. Perhaps for this reason, Simon also wrote [*ibid.*], 'it has certainly not been proven that fantastic cancellations do not occur'. It seems overly pessimistic to completely ignore the question of geometry of state space, quite apart from the stated reasons of constraining the dynamics of a field theory.

3.1 $C_c(\mathbb{R}^n)$ and the Schwartz space: test functions.

Under the standard interpretation of a (lepton) wave-function f, $|f_t(\mathbf{x})|^2$ represents the probability density in $\mathbf{x}$ of finding the particle. It is therefore necessary that the L^2-*norm*

$$\|f\| = \left(\int d^3\mathbf{x} \, |f_t(\mathbf{x})|^2 \right)^{1/2} \tag{3.1.1}$$

be both well-defined and finite. (f can then be normalised so that (3.1.1) integrates to unity.) One simple way to capture this requirement is to designate state space as

$$C_c(\mathbb{R}^n) = \{ f : \mathbb{R}^3 \to \mathbb{C}, \text{continuous, compact support} \}, \tag{3.1.2}$$

treating the integral in (3.1.1) as a Riemann integral. Since f is continuous on a compact domain, it is bounded, so the integral (3.1.1) is finite. Here (3.1.1) defines a *norm*, as defined above, and $C_c(\mathbb{R}^n)$ is called a *normed vector space*. On integrability requirements only, $C_c(\mathbb{R}^n)$ is a reasonable choice of state space.

While continuity on a compact domain is sufficient to satisfy the integrability requirement, further physical considerations appear to favour even tighter restrictions on the precise form of the states. For both the Dirac and RS theories, the Poincaré representations discussed in the previous chapter were generated by linear first-order partial-differential operators (cf. (2.2.20) and (2.2.2)). To directly apply these, a simple option is to further restrict state space to *differentiable* functions f. We will also be interested in the momentum operator

$$f \mapsto -i \frac{\partial f}{\partial \mathbf{x}} \tag{3.1.3}$$

and the position operator

$$f \mapsto \mathbf{x} f. \tag{3.1.4}$$

The *Schwartz space* $S(\mathbb{R}^n)$ is defined as the space of functions for which all finite polynomials of both (3.1.3) and (3.1.4) are well-defined: functions f for which

$$p\left(\mathbf{x}, -i\frac{\partial}{\partial \mathbf{x}}\right) f(\mathbf{x}) \tag{3.1.5}$$

is bounded on $\mathbb{R}^n$ for all polynomials p. (3.1.5) implies all Schwartz-class functions are differentiable to all orders, and 'rapidly-decaying' at infinity (decaying faster than any power of $|\mathbf{x}|$). Examples of Schwartz-class functions are

$$p(\mathbf{x}) e^{-|\mathbf{x}|^2} \tag{3.1.6}$$

with p any polynomial, or more generally any smooth function f with compact support. A classical theorem [42, 84] states that the Fourier transform

$$\begin{aligned} \tilde{f}(k) &= \int d^3x \; f(x) \; e^{-ikx} \\ f(x) &= \int \frac{d^3k}{(2\pi)^3} \; \tilde{f}(k) \; e^{ikx}. \end{aligned} \tag{3.1.7}$$

maps the Schwartz space into itself, and the mapping is isometric with respect to the norm (3.1.1). This is one reason why the Schwartz space features prominently in many mathematical analyses of quantum theory (e.g. [38, 41]) or scattering theory (e.g. [80]).

Taking state space as either $C_c(\mathbb{R}^n)$ or the Schwartz space $S(\mathbb{R}^n)$ are both reasonable choices. Either specification could be used as a precise characterisation of the notion of 'state'. This thesis, however, will pursue the rigorous construction of field theory based on an extension of both of these spaces. The extension is called the L^2-completion, and it adds significant capability to the standard computational problem of quantum theory mentioned above: perturbative calculations. The extra capability arises because the norm (3.1.1), extended to the L^2-completion, is *complete*, and the Cauchy criterion for convergence of a sequence may be applied. The Cauchy criterion states if (f_n) is a sequence such that

$$\|f_n - f_m\| \to 0 \tag{3.1.8}$$

as $n, m \to \infty$, then a limit f is guaranteed to exist and (f_n) converges to f. In other words, the limit f can be shown to exist even if it has not been explicitly evaluated.

This would not hold in $C_c(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$, and is easy to demonstrate: Figure 3.1 shows a sequence of continuous, compactly-supported functions f_n on $\mathbb{R}$ defined by

$$f_n(x) = \begin{cases} (1-x^2)^{1/n}, & x \in [-1,1] \\ 0, & |x| > 1 \end{cases} . \tag{3.1.9}$$

The sequence f_n is Cauchy convergent under the L^2-norm, but the limit (the box function) has a step discontinuity at $x = \pm 1$ and is therefore not continuous. In this way, the sequence has a limit in $L^2(\mathbb{R})$, but not in $C_c(\mathbb{R})$. Similar examples can be constructed for $S(\mathbb{R}^n)$. We observe that in $C_c(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$, the limit of a Cauchy sequence does not necessarily possess the same characteristics as the elements of the sequence. This is a significant shortcoming of either of these choices of state space.

Because completeness is a condition of the strict mathematical definition of a *Hilbert space*, the L^2-completion reconciles the somewhat informally-defined 'Hilbert space of states' in physics with its counterpart in pure mathematics. Regarding practical outcomes, the limit of a Cauchy sequence exists and is unique, and it belongs to the state space. The limit satisfies the same hypotheses as the sequence of states which converge to it. I speculate that this might provide a crucial advantage in field theory, if a convergent perturbative series of a non-trivial field theory could not be summed in closed form. Completeness is also used to prove that L^2 is *self-dual*: L^2 is isomorphic to the space of bounded linear functionals on L^2. Completeness therefore naturally provides the assumed isomorphism of Dirac 'bras' and 'kets' in widespread physics usage. This does not hold in $C_c(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$. Finally, the L^2-spaces are the setting for some celebrated results of pure mathematics (e.g. the Carleson theorem on pointwise convergence of the Fourier transform on $L^2(\mathbb{R})$ [85, 86, 87]). It is speculative, but these may have some interesting consequences in physics.

We will review briefly the characteristics of a typical L^2 function in the next section.

3.2 L^2: the protypical Hilbert space

$L^2(X, \mathbb{C})$ is the space of square-integrable complex amplitudes on a space X. It is a central topic in measure theory and the theory of integration. Let us recall briefly some features of the theory in the present context, where the base space $X = \mathbb{R}^3$. For more

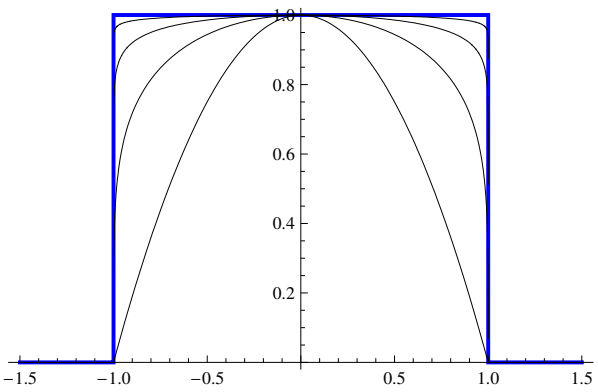


Figure 3.1: A Cauchy sequence of continuous functions may have a discontinuous limit. Shown: f_n defined in (3.1.9) for $n = 1, 5, 20, 100$. The limit (bold line) is the box function.

detail, see e.g. [88, 89].

A primitive concept in constructing a theory of integration of functions on X is *measure*: the association of a positive real number $\mu(E)$ with certain subsets $E \subset X$. Measure is intended to correspond to the intuitive concept of 'volume' and is postulated to satisfy the following conditions:

- A sufficiently large class of 'physically relevant' subsets $E \subset X$ should have a well-defined measure/volume (i.e., $\mu(E)$ is well-defined). These are called *measurable* sets. It is assumed

  i. the empty set is measurable, and

  ii. if $E \subset X$ is measurable, then its complement $X \backslash E$ is also measurable, and

  iii. if $E_1, E_2, E_3, \ldots$ is a countable sequence of measurable sets, then their union $\cup_{k=1}^{\infty} E_k$ is also measurable.

  It is consistent with the above conditions to suppose μ is defined on *all* subsets of X. However, in many cases this assumption unavoidably leads to a trivial (zero) measure and trivial theory of integration. The standard measure on $\mathbb{R}^n$, the Lebesgue measure, adopts a next-best approach and is defined on the 'Borel σ-algebra' of sets — a large class of sets which includes (among others) all rectangular volumes (with or without boundary), all open sets, all closed sets, all finite sets, and all countable intersections and unions thereof. In this way, mea-

sure or volume is defined for the subsets of $\mathbb{R}^3$ encountered in the context of a plausible physical theory.

- Intuitively, volume is additive and additional hypotheses are placed on the measure μ to reflect this. It is assumed

  iv. the empty set has zero measure, and

  v. if $E_1, E_2, E_3, \ldots$ is a countable sequence of mutually disjoint measurable sets, then

  $$\mu(\bigcup_{k=1}^{\infty} E_k) = \sum_{k=1}^{\infty} \mu(E_k). \tag{3.2.1}$$

  In particular, if A, B are disjoint measurable sets, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

In the Lebesgue measure on $\mathbb{R}^n$, the primitive open set is the box without boundary

$$E = I_1 \times I_2 \times \ldots \times I_n \tag{3.2.2}$$

where for $k = 1, \ldots, n$, $I_k = (a_k, b_k)$ are open intervals in $\mathbb{R}$. The measure of E is defined, as expected, as the volume of the box

$$\mu(E) = \prod_{k=1}^{n} (b_k - a_k). \tag{3.2.3}$$

and the class of measurable subsets of $\mathbb{R}^n$ is the smallest class of subsets of $\mathbb{R}^n$ containing all boxes of the form (3.2.2) and satisfying (i-iii). It can be shown that the measure (3.2.3) extends consistently to all measurable sets (the Borel σ-algebra in $\mathbb{R}^n$).

The link with integration stems from the interpretation of $\mu(E)$ as the integral of the *characteristic function*

$$\chi_E(x) = \begin{cases} 0, & x \notin E \\ 1, & x \in E. \end{cases} \tag{3.2.4}$$

Thus,

$$\int \chi_E \, d\mu = \mu(E). \tag{3.2.5}$$

A finite linear sum of characteristic functions is called a *step map* (Figure 3.2a). The definition (3.2.5) extends by linearity to all step maps. The integral in this case is a sum

of finitely many (finite) terms. The space of step maps is an inner product space with inner product

$$\langle s, s' \rangle = \int s^*(x) s'(x) \, d\mu(x) \tag{3.2.6}$$

and norm

$$\|s\|^2 = \langle s, s \rangle. \tag{3.2.7}$$

The key to integration of a more general function f on X is realising f as the pointwise limit of a sequence of step maps $s_n, n = 1, 2, \ldots$ (Figure 3.2b). It can be shown that if (s_n) are Cauchy-convergent with respect to the norm (3.2.7), then the limit

$$\lim_{n \to \infty} \int s_n(x) \, d\mu(x) \tag{3.2.8}$$

exists. In this case, f is said to be *integrable* and $\int f(x) \, d\mu(x)$ is defined to be the limit (3.2.8). It can be shown that the limit (3.2.8) is not sensitive to the particular Cauchy-convergent sequence of step functions chosen. In this way, the Lebesgue integral of f is defined. It can be shown that the Lebesgue integral is an extension of the well-known Riemann integral, in the sense that if a function f on $\mathbb{R}^n$ is Riemann integrable, then f is Lebesgue integrable and the integrals are equal. On the other hand, it can be shown that any Cauchy-convergent sequence of step maps (s_n) converges in norm to some square-integrable function f, with f unique up to redefinition on a set of zero measure. L^2 is the space of such functions; it is a complete inner-product space with inner product

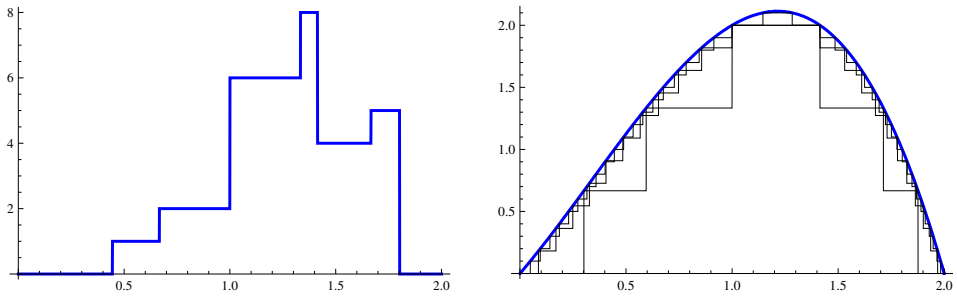
$$\langle f, g \rangle = \int d^3\mathbf{x}^n \; f^\dagger(x) g(x) \tag{3.2.9}$$

and norm

$$\|f\|^2 = \langle f, f \rangle. \tag{3.2.10}$$

Thus L^2 is a *Hilbert space* by construction.

It is this last feature that sets L^2 apart as a starting point for field theory. On one hand, $L^2(\mathbb{R}^n)$ is sufficiently general to include all 'physically relevant' probability amplitudes on $\mathbb{R}^n$, including $C_c(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ and all continuous square-integrable functions. On the other hand, $L^2(\mathbb{R}^n)$ is also sufficiently inclusive to include its limit points:



(a) (b)

Figure 3.2: (a) A step map takes only finitely many values. Its integral is a sum of finitely many terms. (b) Any continuous function is the pointwise limit of step maps.

it is a complete metric space; every Cauchy sequence converges in L^2. It is not possible to construct a sequence that 'converges to a point outside L^2'. Finally, it can be shown that $S(\mathbb{R}^n)$ and $C_c(\mathbb{R}^n)$ are dense in $L^2(\mathbb{R}^n)$, and the Fourier transform (3.1.7) thereby extends by continuity from $S(\mathbb{R}^n)$ to the whole of L^2.

3.3 Lepton state space

The relationship between functional analysis and wave mechanics in the lepton sector has a well-developed literature, e.g. [67, 90, 91, 92]. As indicated in the previous chapter, the state space for a single particle is $\Lambda^1 \equiv L^2(\mathbb{R}^3, \mathbb{C}^4)$, that is, four-component Dirac spinors in three space dimensions with inner product

$$\langle f, g \rangle = \int d^3\mathbf{x}\; f^\dagger(\mathbf{x}) g(\mathbf{x}) \tag{3.3.1}$$

[67, §1.3]. (That is, each of the four components of f are L^2 functions.)

This choice of state space differs from many of the formal developments of field theory. Renormalizability, for instance, is usually discussed in the setting of $S(\mathbb{R}^{4n})$ (the Schwartz space over copies of *spacetime* rather than space) [38, 41]. This permits a rigorous definition of the operator products encountered in field theory and proof of counterterm subtraction order by order in perturbation theory, but there is no sense of time evolution, nor of an initial value problem.

$\Lambda^1 = L^2(\mathbb{R}^3, \mathbb{C}^4)$ is well-suited to the standard physical interpretation of a state $f \in \Lambda^1$ as a probability amplitude, since by construction $|f(x)|^2 = f^\dagger(x) f(x)$ is square-integrable. Conversely, it is very easy to characterise a typical state $f \in \Lambda^1$:

it is square-integrable in the sense of Lebesgue. No assumptions are made on differentiability/smoothness or continuity of f. We now quote the following theorem, stating that all transformations between inertial frames are continuous with respect to the L^2-norm on Λ^1 (the 'continuity requirement' mentioned at the start of the chapter).

Theorem 3.3.1. *The Poincaré representation* (2.2.2) *with the Dirac Hamiltonian* $H = \alpha_k\partial_k + \beta m$ *generates a group of unitary transformations on* Λ.

Proof. See [67], Thm 2.19. The result is proved by demonstrating that each generator is anti-Hermitian with respect to the inner product (3.3.1). As an example, the anti-Hermitian character of $\mathcal{K}_j$ will be demonstrated. $\mathcal{K}_j$ is defined as part of the Poincaré representation (2.2.2). Integration by parts is used to evaluate the Hermitian adjoint of the term $-x_j\alpha_k\partial_k$ contributing to $\mathcal{K}_j$:

$$\begin{aligned}\langle f, -x_j\alpha_k\partial_k g\rangle &= -\int d^3x\ f^\dagger(x)x_j\alpha_k\partial_k g(x)\\ &= \int d^3x\ \partial_k(x_j f^\dagger(x))\alpha_k g(x)\\ &= \int d^3x\ \left(x_j\partial_k f^\dagger(x) + \delta_{jk}f^\dagger(x)\right)\alpha_k g(x)\\ &= \langle (x_j\alpha_k\partial_k + \alpha_j)f, g\rangle.\end{aligned} \tag{3.3.2}$$

Since

$$\alpha_j = -2K_j \tag{3.3.3}$$

and K_j is Hermitian, we have

$$\langle f, (-x_j\alpha_k\partial_k + K_k)g\rangle = -\langle(-x_j\alpha_k\partial_k + K_k)f, g\rangle. \tag{3.3.4}$$

Hence the sum $(-x_j\alpha_k\partial_k + K_j)$ is anti-Hermitian, even though the two individual terms are not. The remaining $-ix_j\beta m$ term in $\mathcal{K}_j$ is also anti-Hermitian, so $\mathcal{K}_j$ itself is anti-Hermitian. The anti-Hermitian character of the other generators may be shown similarly. Since all ten generators of the Poincaré group are anti-Hermitian, so is any linear combination and it follows that finite transformations are unitary [1]. Hence the free Dirac theory admits a unitary representation of the Poincaré group. □

[1] This thesis uses a convention for generators that differs from many physics texts by a factor i. See note in the Appendix §7.1. Hence, unitary transformations are generated by *anti*-Hermitian (rather than Hermitian) operators. This is also evident in expression (2.2.20) relating the generators to the finite transformations.

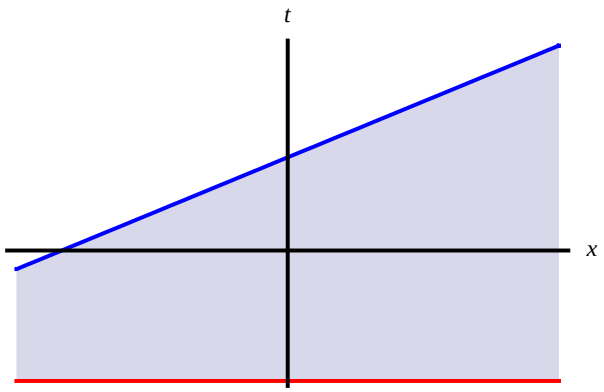


Figure 3.3: The difference between the flux integral (3.3.5) evaluated in different inertial frames is equivalent to a spacetime surface integral.

Remark. This result stems from the fact that

$$\|\psi\|^2 = \int d^3x \; \psi^\dagger(x)\psi(x) \tag{3.3.5}$$

is the flux integral of the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi$ over a hypersurface of constant t. The difference between the flux integrals (3.3.5) evaluated in different inertial frames is equivalent to the surface integral over the boundary of the region shown in Figure 3.3. By the divergence theorem (Gauss' theorem), the surface integral vanishes. So $\|\psi\|$ is an invariant of change of frame (including translation in time). That $\|\psi\|$ is invariant under change of frame is equivalent to saying that the change of frame acts unitarily on Λ^1. Note that a unitary transformation is necessarily bounded/continuous; the definition of boundedness (3.0.1) is satisfied with $C = 1$.

3.4 Photon state space

The Riemann-Silberstein (RS) representation of the photon was introduced in §2.4. It was shown how the space of RS amplitudes admits a Poincaré representation and a relativistic initial-value problem. In this section, we will show that the RS representation is also well-suited to an L^2 state space. It was mentioned earlier that the L^2 norm of the RS representation corresponds to the energy of a given solution to Maxwell's equations. Hence a bound on the L^2 norm under change of inertial frame is equivalent to a bound on the energy of an EM field under change of inertial frame. We can formalise this in

Theorem 3.4.1. *The Poincaré representation* (2.2.2) *with the RS Hamiltonian* $H = iK_j\partial_j$ *generates a group of bounded/continuous transformations on* Θ^1.

Proof. It is a straightforward calculation to show that $\mathcal{P}_0, \mathcal{P}_j$ and $\mathcal{J}_j$ are anti-Hermitian and therefore generate unitary transformations. In particular, the time evolution is unitary. However the same is not true of the boost operators $\mathcal{K}_j$. These are not anti-Hermitian, so finite boosts are not unitary transformations on Θ^1. In other words, the RS Hamiltonian does not give rise to a *unitary* representation of the Poincaré group.

However, it is possible to show that boosts are continuous (bounded) transformations on Θ^1. Let W_η denote the state W_0 boosted by rapidity η in the direction of the unit-vector $\mathbf{n} = (n_1, n_2, n_3)$. Then

$$\begin{aligned}\frac{d}{d\eta}\|W_\eta\|^2 &= \frac{d}{d\eta}\int d^3x\ W_\eta^\dagger(x)W_\eta(x)\\ &= \int d^3x\ \left(\frac{d}{d\eta}W_\eta(x)\right)^\dagger W_\eta(x) + W_\eta^\dagger(x)\frac{d}{d\eta}W_\eta(x)\\ &= \int d^3x\ \left[(n_jx_jK_k\partial_k + n_jK_j)\,W_\eta(x)\right]^\dagger W_\eta(x)\\ &\qquad + W_\eta^\dagger(x)\,(n_jx_jK_k\partial_k + n_jK_j)\,W_\eta(x)\end{aligned} \tag{3.4.1}$$

The final line has used expression (2.2.21) to substitute for the boost generator $\mathcal{K}_j$, with the RS boost generator $\mathcal{K}_j$ given in §2.4. We recall $K_j^\dagger = K_j$. Hence

$$\frac{d}{d\eta}\|W_\eta\|^2 = \int d^3x\ n_jx_j\partial_kW^\dagger K_kW + n_jx_jW^\dagger K_j\partial_kW + 2n_jW^\dagger K_jW \tag{3.4.2}$$

(explicit x, η-dependence has been suppressed for brevity). Integrating the first term by parts,

$$\begin{aligned}\frac{d}{d\eta}\|W_\eta\|^2 &= \int d^3x\ \big(-n_jW^\dagger K_jW - n_jx_jW^\dagger K_k\partial_kW\\ &\qquad\qquad + n_jx_jW^\dagger K_k\partial_kW + 2n_jW^\dagger K_jW\big)\\ &= \int d^3x\ n_jW^\dagger K_jW\end{aligned} \tag{3.4.3}$$

Consistent with the left-hand side being the derivative of a real quantity, the right-hand side is real as a consequence of K_j being Hermitian. We may use the estimate

$$\left|n_jW^\dagger K_jW\right| \leq |\mathbf{n}||W|^2 \tag{3.4.4}$$

on the integrand. (This stems from the inequality $|\mathbf{v}^* \times \mathbf{v}| \leq |\mathbf{v}|^2$ for any complex 3-vector $\mathbf{v}$, and the Cauchy-Schwarz inequality $|\mathbf{n}\cdot\mathbf{v}| \leq |\mathbf{n}||\mathbf{v}|$). Moreover $|\mathbf{n}| = 1$ by

definition. Hence

$$\begin{aligned}\left|\frac{d}{d\eta}\|W_\eta\|^2\right| &\leq \int d^3x\ |W_\eta(x)|^2 \\ &= \|W_\eta\|^2.\end{aligned} \tag{3.4.5}$$

By a straightforward estimate using the mean value theorem we have

$$\|W_\eta\|^2 \leq e^{|\eta|}\|W_0\|^2 \tag{3.4.6}$$

for all η. So the boost transformation $W_0 \mapsto W_\eta$ is bounded/continuous. An arbitrary Poincaré transformation can be decomposed into a combination of translations, rotations and boosts. Such a combination of continuous of maps is continuous. □

Since the L^2 norm is not invariant under change of inertial frame, the Poincaré representation for the RS photon is not a unitary representation. A probabilistic interpretation of the RS photon is therefore impossible, because a normalisation $\|W\| = 1$ is not conserved under change of frame. This breakdown of unitarity agrees with early studies of [93] and [24] concluding on the impossibility of a probability interpretation for the photon (and attributed to the lack of a rest frame for a massless particle) [2]. See also [74] and references therein for a recent review. However, the weaker condition of boundedness (continuity) still holds and this is sufficient to ensure the physical requirement that limits are preserved under change of inertial frame.

The low research profile of the RS photon may stem from this lack of a probabilistic interpretation. I do not regard the lack of a probabilistic interpretation as a fatal flaw. On the contrary, I am pleased to take advantage of the same function-analytic geometric features enjoyed by the lepton state space Λ^1.

Incidentally, we remark that the (quadratic) L^2 norm on the RS photon state space is consistent with interesting recent experimental limits on the validity of Born's rule (ruling out 'multi-order interference' of photons) [95].

3.5 Note about Fourier transforms

Note that the L^2 norms do not adhere to the widespread convention of defining the Fourier transform in terms of the Lorentz-invariant measure $d^3k/(2\pi)^3 2E_k$, so it is not

[2] [94] have suggested a different, *non-local* inner product for which a unitary representation may exist.

manifestly Lorentz covariant. On the other hand, the Fourier isometry (3.1.7) guarantees that any $\hat{h}$ in L^2 corresponds to a state h in L^2, a fact that will be put to good effect in the following; the same cannot be said of $\hat{h}/(2\pi)^3 2E_k$. Additionally, use will be made of the Parseval relation, which takes the simple form

$$\langle g, h\rangle = \langle \hat{g}, \hat{h}\rangle \tag{3.5.1}$$

under the L^2 norm.

3.6 Energy Projections

In standard formulations of QED, the positive-energy (electron) and negative-energy (positron) subspaces of Λ^1 correspond to different particle species, and are described in the theory by distinct Fock spaces. Because of this decomposition, the interpretation of configuration space amplitudes is largely lost. The model developed in this thesis attempts to keep Λ^1 intact and thereby preserve the configuration space representation.

However, it is worth looking in detail at the decomposition of Λ^1 into positive- and negative-energy subspaces. The decomposition takes the form of a divergent convolution integral, and I will introduce two methods of regularisation of these integrals. I have included these results because they might prove useful in the regularisation of other divergent integrals present in the theory of Chapter 5. I will also discuss the feature of 'exponential localisation' demonstrated by these results.

The Hilbert transform of a function $f : \mathbb{R} \to \mathbb{C}$ is defined [96, 97, 98]

$$(\mathcal{H}f)(x) = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty}\frac{dy}{y}f(x-y) \tag{3.6.1}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\mathbb{R}\setminus(-\epsilon,\epsilon)}\frac{dy}{y}f(x-y) \tag{3.6.2}$$

provided the right-hand side is well-defined. The basic idea of the transform is to convolve f with the kernel $1/\pi y$. However, even if f is smooth, the divergence at $y = 0$ is generally too strong to simply use a 'standard' (Lebesgue) integral. Instead, the principal-value integral (PV) prescribes 'regularising' the integral by removing a neighbourhood $(-\epsilon, \epsilon)$ around the divergence, and then taking the limit $\epsilon \to 0$. For $f \in L^2(\mathbb{R})$, it can be shown the limit exists for almost all $x \in \mathbb{R}$ and the function $x \mapsto \mathcal{H}f(x)$ also belongs to $L^2(\mathbb{R})$. Hence, Fourier transforms of both f and $\mathcal{H}f$ are

well-defined. It is in this context that a remarkable property of the Hilbert transform becomes apparent: denoting the Fourier transform by a tilde, it can be shown that

$$\widetilde{\mathcal{H}f}(k) = \begin{cases} \tilde{f}(k) & k > 0 \\ -\tilde{f}(k) & k < 0 \end{cases}. \tag{3.6.3}$$

In other words, $\mathcal{H}$ introduces a factor of -1 into the negative-frequency modes of f. This property of the Hilbert transform provides an intriguing hook into quantum theory, where a key property of a Dirac amplitude is its unique decomposition into positive-energy (electron) and negative-energy (positron) components. Indeed, a quantum-theoretic counterpart to the Hilbert transform would provide this decomposition by defining projection operators

$$Q_+ f = \frac{1}{2}(f + \mathcal{H}f) \tag{3.6.4}$$

$$Q_- f = \frac{1}{2}(f - \mathcal{H}f). \tag{3.6.5}$$

which project onto the positive- and negative-energy subspaces respectively. The construction of a Hilbert transform for Dirac amplitudes, and its space representation, will be presented below.

The Dirac Hamiltonian is

$$H = -i\boldsymbol{\alpha} \cdot \boldsymbol{\nabla} + \beta m, \tag{3.6.6}$$

where $\boldsymbol{\alpha} \equiv (\alpha^1, \alpha^2, \alpha^3)$ and β are standard notation for certain constant 4-by-4 matrices (see, e.g. [67, 36]). The Fourier representation of H is

$$\tilde{f}(\mathbf{k}) \mapsto (\boldsymbol{\alpha} \cdot \mathbf{k} + \beta m)\tilde{f}(\mathbf{k}), \tag{3.6.7}$$

that is, pointwise multiplication by the 4-by-4 matrix $\tilde{H}(\mathbf{k}) \equiv (\boldsymbol{\alpha} \cdot \mathbf{k} + \beta\, m)$. At each $\mathbf{k} \in \mathbb{R}^3$, $\tilde{H}(\mathbf{k})$ is Hermitian and has eigenvalues $\pm E \equiv \pm\sqrt{k^2 + m^2}$.

Definition 3.6.1. A state $f \in \Lambda^1$ is *positive-energy* if, for almost all $\mathbf{k} \in \mathbb{R}^3$, $\tilde{f}(\mathbf{k})$ lies in the $+E$ eigenspace of $\tilde{H}(\mathbf{k})$.

The subspace of positive-energy states will be denoted Λ_+. *Negative-energy* states are defined analogously; the subspace of negative-energy states will be denoted Λ_-. Because $\tilde{H}(k)$ is Hermitian at each $\mathbf{k}$, any state $f \in \Lambda^1$ possesses a unique decomposition $f = f_+ + f_-$, where $f_+ \in \Lambda_+$ is positive-energy and f_- is negative-energy.

Definition 3.6.2. Let $f \in \Lambda^1$. The *Hilbert transform for Dirac amplitudes* $\mathcal{H}f$ is defined to introduce a factor of -1 into the negative-energy component of f. In other words,

$$\mathcal{H}f = \begin{cases} f & f \in \Lambda_+ \\ -f & f \in \Lambda_- \end{cases} \tag{3.6.8}$$

and extending by linearity to $\Lambda^1 = \Lambda_+ \oplus \Lambda_-$.

The main result of this section is a space representation for $\mathcal{H}$, in the following theorem. As described in (3.6.5), such an expression provides a space representation for projecting an arbitrary Dirac amplitude into its electron and positron components. The existence of this expression provides additional evidence that $\Lambda^1 \equiv L^2(\mathbb{R}^3, \mathbb{C}^4)$ with inner product (3.3.1) is an appropriate realisation of the state space for a Dirac particle.

The statement of the theorem needs a three-dimensional equivalent of the principal-value integral encountered in (3.6.1).

Definition 3.6.3. Suppose $h : \mathbb{R}^3 \to \mathbb{C}$. The 3-dimensional principal-value integral is

$$\text{PV} \int h(\mathbf{x})\, d^3\mathbf{x} = \lim_{\epsilon \to 0} \int_{|\mathbf{x}| \geq \epsilon} h(\mathbf{x})\, d^3\mathbf{x} \tag{3.6.9}$$

provided the right-hand side is well-defined and the limit exists.

As with the 1D case, if $h \in L^1$ the principal-value integral coincides with the standard integral. The prescription is designed to deal with certain cases where the integrand h is badly-behaved in the vicinity of 0 (e.g. strongly divergent and not L^1-integrable).

Theorem 3.6.4. *Let $f \in \Lambda^1$. Then the Hilbert transform for Dirac amplitudes has the two equivalent representations*

$$\mathcal{H}f(\mathbf{x}) = \lim_{\epsilon \to 0} \int d^3\mathbf{y} \left(\boldsymbol{\alpha} \cdot \hat{\mathbf{y}} \frac{im^2}{2\pi^2 y^{1-\epsilon}} K_{2-\epsilon}(my) + \beta \frac{m^2}{2\pi^2 y} K_1(my) \right) f(\mathbf{x} - \mathbf{y}) \tag{3.6.10}$$

and

$$\mathcal{H}f(\mathbf{x}) = \text{PV} \int d^3\mathbf{y} \left(\boldsymbol{\alpha} \cdot \hat{\mathbf{y}} \frac{im^2}{2\pi^2 y} K_2(my) + \beta \frac{m^2}{2\pi^2 y} K_1(my) \right) f(\mathbf{x} - \mathbf{y}) \tag{3.6.11}$$

where $y = |\mathbf{y}|$, $\hat{\mathbf{y}} = \mathbf{y}/y$, and K_p is the modified Bessel function of the second kind of order p.

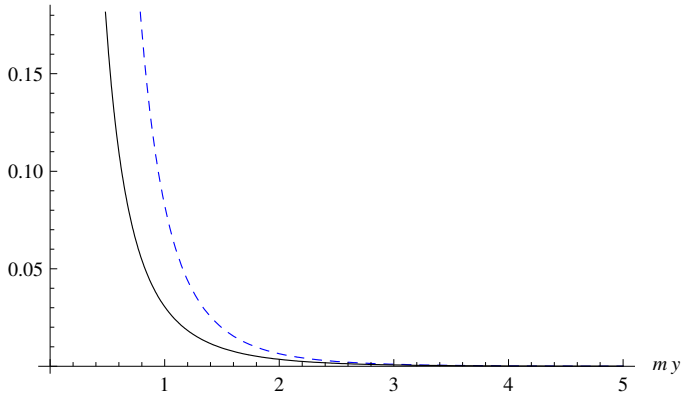


Figure 3.4: Two terms contributing to the convolution integrals, as a function of distance y (units of $1/m$). Black/solid: The integrable term $K_1(y)/2\pi^2 y$. Blue/dashed: The singular term $K_2(y)/2\pi^2 y$.

Before proceeding to a proof of the main theorem, let us remark briefly on the form of the Hilbert transform for Dirac particles. As with the 1D transform (3.6.1), the transform resembles a convolution integral where the kernel is singular at $\mathbf{y} = 0$ (Figure 3.4). The short-distance behaviour of the Bessel functions is [40]

$$K_p(u) \simeq \frac{\Gamma(p)}{2}\left(\frac{2}{u}\right)^p \tag{3.6.12}$$

for $u \ll 1$. Therefore, the term involving K_1 in (3.6.10) and (3.6.11) behaves like $1/y^2$ for $y \ll 1/m$ and is absolutely integrable (L^1-integrable). The convolution of an L^2 function with an L^1 kernel is well-defined almost everywhere (see, e.g. [88] or [89]); no regularisation prescription is required in this case. It is the steeper $1/y^3$ singularity arising from the term involving K_2 which requires special treatment: either the principal-value prescription of (3.6.11), or 'Bessel regularisation' of the divergence as in (3.6.10).

At large distances, the Bessel functions are exponentially suppressed, behaving like $e^{-my}/\sqrt{y}$ for $y \gg 1/m$. From a physical perspective, the exponential suppression means that the energy projections $P_\pm$ do not 'smear' the amplitude over a range significantly larger than $1/m$. For example, if f is a compactly-supported amplitude, then the projection $P_+ f$ is suppressed like $\exp(-my)$ times a negative power of y at large distance. In this case, the exponential fall-off demonstrates 'exponential localisation' as defined by [76]. In that paper, the author improved existing localization

estimates on photons [99] by constructing a class of massless photon amplitudes which approach, but do not attain, exponential fall-off. The results presented in Theorem 3.6.4 provide an explicit example of exponential fall-off, in the complementary context of massive spin-half particles. This represents a slight improvement on the published results above. Localising massless or massive particles more sharply than exponential fall-off is impossible [76].

The 'Bessel regularisation' (3.6.10) and 'principal-value regularisation' (3.6.11) expressions will be dealt with individually. The proof for Bessel regularisation is simpler, can be evaluated in closed form, and admits a direct proof involving no initial regularity assumptions on f.

3.7 Bessel Regularised Convolution

Consistent with (3.6.8), the Fourier representation of $\mathcal{H}$ is

$$\tilde{f}(\mathbf{k}) \mapsto \frac{\tilde{H}(\mathbf{k})}{E}\tilde{f}(\mathbf{k}) \equiv \left(\frac{\boldsymbol{\alpha}\cdot\mathbf{k}}{E} + \frac{\beta m}{E}\right)\tilde{f}(\mathbf{k}), \tag{3.7.1}$$

that is, pointwise multiplication by the 4-by-4 matrix $\tilde{H}/E$. The Fourier convolution theorem suggests looking for an inverse Fourier transform of $\tilde{H}/E$. An inverse Fourier transform of $\tilde{H}/E$ in L^1 would naturally give rise to a convolution integral, and indeed this line of reasoning is valid for the term $\beta m/E$ contributing to $\tilde{H}/E$. The following lemma shows that $\beta m/E$ possesses an inverse Fourier transform in L^1 and gives rise to the absolutely integrable term (involving K_1) in (3.6.11) and (3.6.10).

Lemma 3.7.1. *The function* $\mathbf{k} \mapsto 1/E$ *has an inverse Fourier transform in* $L^1(\mathbb{R}^3)$ *given by*

$$G(\mathbf{y}) = \frac{m}{2\pi^2 y}K_1(my) \tag{3.7.2}$$

This expression is an example of the 'Bessel potential' treated in great detail by [100, 97]. It also appears in the projection operator for scalar particles ([33, §5.2]). The calculation will carried through again, in order to fix notation, to introduce methods needed in subsequent lemmas, and to make the exposition as self-contained as possible.

$G \in L^1(\mathbb{R}^3)$ and G depends only on the radial coordinate, i.e. $G(\mathbf{y}) = G(y)$. The

angular integrals in the Fourier transform can be completed,

$$\tilde{G}(\mathbf{k}) = \int d^3\mathbf{y}\; G(\mathbf{y}) e^{-i\mathbf{k}\cdot\mathbf{y}} \tag{3.7.3}$$

$$= \frac{4\pi}{k} \int_0^\infty G(y)\; y \sin(ky)\; dy, \tag{3.7.4}$$

where $k = |\mathbf{k}|$. Substituting the integral representation [101]

$$K_1(u) = u \int_1^\infty e^{-us} \left(s^2 - 1\right)^{1/2} ds, \tag{3.7.5}$$

(3.7.4) becomes the double integral

$$\tilde{G}(\mathbf{k}) = \frac{2m^2}{\pi k} \int_0^\infty \int_1^\infty e^{-mys} \left(s^2 - 1\right)^{1/2} y \sin(ky)\; ds\; dy \tag{3.7.6}$$

By the Fubini theorem, the order of integration can be reversed. The y-integral can be completed by elementary methods,

$$\int_0^\infty y e^{-mys} \sin(ky)\; dy = \frac{2kms}{(k^2 + m^2 s^2)^2}, \tag{3.7.7}$$

leaving

$$\tilde{G}(\mathbf{k}) = \frac{4m^3}{\pi} \int_1^\infty \frac{s(s^2-1)^{1/2}}{(k^2 + m^2 s^2)^2} ds. \tag{3.7.8}$$

The integral in (3.7.8) can be completed by the successive substitutions $s^2 - 1 = v^2$ and $mv = \sqrt{k^2 + m^2} \tan\theta$,

$$\tilde{G}(\mathbf{k}) = \frac{4}{\pi} \int_0^{\pi/2} \frac{1}{E} \sin^2\theta\; d\theta = \frac{1}{E}, \tag{3.7.9}$$

and Lemma 3.7.1 follows. Incidentally, as G is non-negative, (3.7.9) shows that the L^1-norm of G is $1/m$.

In contrast to the $\beta m/E$ term, the $\boldsymbol{\alpha}\cdot\mathbf{k}/E$ term in (3.7.1) does not vanish as $|k| \to \infty$. By the Riemann-Lebesgue lemma, it does not possess an inverse Fourier transform in L^1. The following lemma introduces a *regularisation* of $\mathbf{k}/E$ which does possess an inverse Fourier transform in L^1. Because the regularisation is implemented through the index on the Bessel function, I will refer to this regularisation as 'Bessel regularisation'.

Lemma 3.7.2. *Let $\epsilon > 0$. The function*

$$\mathbf{F}_\epsilon(\mathbf{y}) = \hat{\mathbf{y}} \frac{i m^{2+\epsilon}}{2\pi^2 y^{1-\epsilon}} K_{2-\epsilon}(my) \tag{3.7.10}$$

belongs to L^1, and the Fourier transform $\tilde{\mathbf{F}}_\epsilon$ tends pointwise to $\mathbf{k}/E$ as $\epsilon \to 0$. Furthermore, $|\tilde{\mathbf{F}}_\epsilon|$ is uniformly bounded as $\epsilon \to 0$.

According to (3.6.12), $F_\epsilon(y) \sim y^{-3+2\epsilon}$ for $y \ll 1/m$ and $\mathbf{F}_\epsilon \in L^1$. Moreover, $\mathbf{F}_\epsilon(\mathbf{y})$ is radially directed and depends only on the radial coordinate y. In a similar calculation to (3.7.4), the angular integrals can be completed to give

$$\tilde{\mathbf{F}}_\epsilon(\mathbf{k}) = \frac{4\pi i}{k^2}\hat{\mathbf{k}} \int_0^\infty F_\epsilon(y)\,(ky\cos(ky) - \sin(ky))\ dy. \tag{3.7.11}$$

where $F_\epsilon(y) = |\mathbf{F}_\epsilon(\mathbf{y})|$. In this way, the problem reduces to evaluation of the scalar integral

$$I_\epsilon(k) \equiv \int_0^\infty F_\epsilon(y)\,(ky\cos(ky) - \sin(ky))\ dy. \tag{3.7.12}$$

As in the preceding lemma, we appeal to the integral representation

$$K_p(u) = \frac{\sqrt{\pi}}{2^p\Gamma(p+1/2)}u^p \int_1^\infty e^{-us}\left(s^2-1\right)^{p-1/2} ds \qquad (p > -\tfrac{1}{2}). \tag{3.7.13}$$

with $u = my$ and $p = 2-\epsilon$. The $y^{1-\epsilon}$ factor in the denominator of (3.7.10) cancels with a corresponding $y^{2-\epsilon}$ factor arising in (3.7.13). This convenient simplification will facilitate evaluation of I_ϵ in closed form by avoiding tricky fractional powers of y under the integral. This is the decisive advantage to the form $K_{2-\epsilon}(y)/y^{1-\epsilon}$ in (3.7.10), compared to apparently simpler alternatives such as $K_{2-\epsilon}(y)/y$.

In this way,

$$I_\epsilon(k) = \frac{im^4}{2^{3-\epsilon}\pi^{3/2}\Gamma(\frac{5}{2}-\epsilon)} \int_0^\infty \int_1^\infty ye^{-mys}\left(s^2-1\right)^{3/2-\epsilon} (ky\cos(ky) - \sin(ky))\ ds\ dy. \tag{3.7.14}$$

The integral is absolutely convergent for each $\epsilon > 0$ and the order of integration may be interchanged. The integral over y may be completed by elementary methods,

$$\int_0^\infty ye^{-mys}\,(ky\cos(ky) - \sin(ky))\ dy = -\frac{8k^3ms}{(k^2+m^2s^2)^3}, \tag{3.7.15}$$

and substituting into (3.7.14),

$$I_\epsilon(k) = \frac{-im^5k^3}{2^{-\epsilon}\pi^{3/2}\Gamma(\frac{5}{2}-\epsilon)} \int_1^\infty \frac{s(s^2-1)^{3/2-\epsilon}}{(k^2+m^2s^2)^3}\ ds. \tag{3.7.16}$$

The same substitutions used for (3.7.8) can be used to reduce (3.7.16) to

$$I_\epsilon(k) = \frac{-ik^3m^{2\epsilon}}{2^{-\epsilon}\pi^{3/2}\Gamma(\frac{5}{2}-\epsilon)E^{1+2\epsilon}} \int_0^{\pi/2} \sin^{4-2\epsilon}\theta\cos^{2\epsilon}\theta\ d\theta. \tag{3.7.17}$$

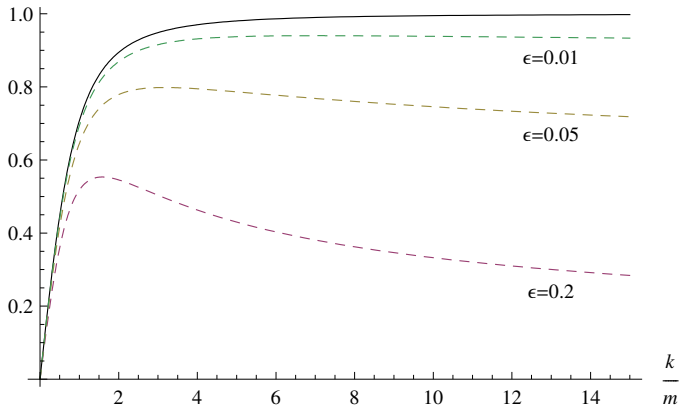


Figure 3.5: $\tilde{\mathbf{F}}_\epsilon(\mathbf{k})$ tends pointwise to $\mathbf{k}/E$ as $\epsilon \to 0$. The plot shows $|\tilde{\mathbf{F}}_\epsilon|$ for $\epsilon = 0.2, 0.05, 0.01$ (colour/dashed) tending pointwise to k/E (black/solid).

The trigonometric integral in (3.7.17) may be evaluated as $\frac{1}{4}(\frac{3}{2}-\epsilon)(\frac{1}{2}-\epsilon)\pi\sec(\pi\epsilon)$ [102]. Using this result, and the reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)} \tag{3.7.18}$$

and elementary properties of the gamma function, (3.7.17) may be simplified to

$$I_\epsilon(k) = \frac{-ik^3 m^{2\epsilon}}{2^{2-\epsilon}\pi^{3/2}E^{1+2\epsilon}}\Gamma(\tfrac{1}{2}+\epsilon). \tag{3.7.19}$$

This expression, proven for each $\epsilon > 0$, is sufficient for the present section. However, the result is also valid for $\epsilon = 0$. Near $y = 0$, the factor $(ky\cos(ky) - \sin(ky)) \simeq k^3y^3/3$ suppresses the pole, such that the integral is L^1-integrable for each $\epsilon \geq 0$ and each $k > 0$. Dominated convergence applies to the limit $\epsilon \to 0$. The case $\epsilon = 0$ will be needed in §3.8.

Substituting (3.7.19) into (3.7.11),

$$\tilde{\mathbf{F}}_\epsilon(\mathbf{k}) = \frac{\mathbf{k}}{E}\left(\frac{m}{E}\right)^{2\epsilon}\frac{2^\epsilon}{\sqrt{\pi}}\Gamma(\tfrac{1}{2}+\epsilon). \tag{3.7.20}$$

This surprisingly simple expression for the Fourier transform of $\mathbf{F}_\epsilon$ exhibits the desired limit $\mathbf{k}/E$ with a 'correction factor' tending pointwise to 1 as $\epsilon \to 0$ (Figure 3.5). For $\epsilon \lesssim 0.3$, convergence is monotone increasing and uniformly bounded above by 1. Lemma (3.7.2) follows.

Let

$$\mathcal{H}_\epsilon f(\mathbf{x}) = \int d^3\mathbf{y}\,(\boldsymbol{\alpha}\cdot\mathbf{F}_\epsilon(\mathbf{y}) + \beta G(\mathbf{y}))\,f(\mathbf{x}-\mathbf{y}). \tag{3.7.21}$$

$\mathbf{F}_\epsilon$ and G belong to L^1, while f belongs to L^2. It can be shown that such a convolution of L^1 and L^2 functions is well-defined and itself belongs to L^2 (Young's theorem: see, e.g., [88]). Indeed, $\mathcal{H}_\epsilon$ defines an endomorphism on L^2 with Fourier representation

$$\tilde{f}(\mathbf{k}) \mapsto (\boldsymbol{\alpha} \cdot \tilde{\mathbf{F}}_\epsilon(\mathbf{k}) + \beta \tilde{G}(\mathbf{k}))\tilde{f}(\mathbf{k}). \tag{3.7.22}$$

By lemmas 3.7.1 and 3.7.2, and comparing with (3.7.1), the Fourier transform $\widetilde{\mathcal{H}_\epsilon f}$ tends pointwise to $\widetilde{\mathcal{H}f}$. $|\tilde{\mathbf{F}}_\epsilon|$ is uniformly bounded as $\epsilon \to 0$, so by dominated convergence

$$\widetilde{\mathcal{H}_\epsilon f} \to \widetilde{\mathcal{H}f} \tag{3.7.23}$$

in L^2. As the Fourier transform is an L^2-isometry,

$$\mathcal{H}f = \lim_{\epsilon \to 0} \mathcal{H}_\epsilon f = \lim_{\epsilon \to 0} \int d^3\mathbf{y}\, (\boldsymbol{\alpha} \cdot \mathbf{F}_\epsilon(\mathbf{y}) + \beta G(\mathbf{y}))\, f(\mathbf{x} - \mathbf{y}) \tag{3.7.24}$$

with $\mathbf{F}_\epsilon$ and G given by (3.7.10) and (3.7.2), respectively. This establishes the first claim of the main theorem.

3.8 Principal-Value Regularised Convolution

Lemma (3.7.2) illustrated the role of the regularisation $K_{2-\epsilon}(my)/y^{1-\epsilon}$ in controlling the singularity at $y \to 0$. The principal-value regularisation (3.6.11) performs the same function in a different way. Let $\delta > 0$ and denote $\mathbf{F} \equiv \mathbf{F}_\epsilon|_{\epsilon=0}$. $\mathbf{F}$ does not belong to L^1, but the restriction of $\mathbf{F}$ to $|\mathbf{y}| > \delta$ does, and it therefore has a well-defined Fourier transform. Unfortunately, in this case the Fourier transform does not have a simple closed-form expression for non-zero δ.

The preceding section contains most of the computations necessary for demonstration of the second claim of the main theorem. To begin with, the claim will be demonstrated with the domain of $\mathcal{H}$ restricted to the Schwartz space $\mathcal{S} \subset \Lambda^1$ (i.e., 4-component spinors where each component is Schwartz-class).

Lemma 3.8.1. *Let $f \in \mathcal{S}$. Then*

$$\mathrm{PV} \int d^3\mathbf{y}\; \boldsymbol{\alpha} \cdot \mathbf{F}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \boldsymbol{\alpha} \cdot \frac{\mathbf{k}}{E} \tilde{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.8.1}$$

For brevity of notation, let

$$J(\mathbf{x}) = \mathrm{PV} \int d^3\mathbf{y}\; \boldsymbol{\alpha} \cdot \mathbf{F}(\mathbf{y}) f(\mathbf{x} - \mathbf{y}). \tag{3.8.2}$$

Then

$$J(\mathbf{x}) = \lim_{\delta\to 0}\int_{|\mathbf{y}|>\delta} d^3\mathbf{y}\ \boldsymbol{\alpha}\cdot\mathbf{F}(\mathbf{y})f(\mathbf{x}-\mathbf{y}) \tag{3.8.3}$$

$$= \lim_{\delta\to 0}\int_{|\mathbf{y}|>\delta} d^3\mathbf{y}\ \boldsymbol{\alpha}\cdot\mathbf{F}(\mathbf{y})\int\frac{d^3\mathbf{k}}{(2\pi)^3}\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} \tag{3.8.4}$$

$$= \lim_{\delta\to 0}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\ \boldsymbol{\alpha}\cdot\left(\int_{|\mathbf{y}|>\delta} d^3\mathbf{y}\ \mathbf{F}(\mathbf{y})e^{-i\mathbf{k}\cdot\mathbf{y}}\right)\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.8.5}$$

In (3.8.4), the restriction of $\mathbf{F}$ to $|\mathbf{y}| > \delta$ is bounded and behaves like $e^{-my}/\sqrt{y}$ at large y. As a member of the Schwartz class, $\tilde{f}$ is continuous and decays faster than any polynomial at large y. Hence the integrand belongs to L^1 and the order of integration may be reversed (3.8.5). The expression in parentheses may be evaluated using (3.7.11),

$$\int_{|\mathbf{y}|>\delta} d^3\mathbf{y}\ \mathbf{F}(\mathbf{y})e^{-i\mathbf{k}\cdot\mathbf{y}} = \frac{4\pi i}{k^2}\hat{\mathbf{k}}\int_\delta^\infty dy\ F(y)(ky\cos(ky)-\sin(ky)) \tag{3.8.6}$$

In order to apply dominated convergence as $\delta \to 0$ to (3.8.5), it is sufficient to show (3.8.6) is bounded in magnitude by some polynomial in k which does not depend on δ (because $\tilde{f}$ falls off faster than any polynomial, by the hypothesis $f \in \mathcal{S}$). Using $|ky\cos(ky) - \sin(ky)| \le k^3y^3/3$,

$$\begin{aligned}\left|\int_{|\mathbf{y}|>\delta} d^3\mathbf{y}\ \mathbf{F}(\mathbf{y})e^{-i\mathbf{k}\cdot\mathbf{y}}\right| &\le \frac{4\pi k}{3}\int_\delta^\infty dy\ y^3F(y)\\ &= O(k)\end{aligned} \tag{3.8.7}$$

because the integral on the right-hand side of (3.8.7) is finite and does not depend on k. (3.8.7) is the desired polynomial bound. Consequently, (3.8.5) becomes

$$J(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\ \frac{4\pi i}{k^2}\boldsymbol{\alpha}\cdot\hat{\mathbf{k}}\left(\int_0^\infty dy\ F(y)(ky\cos(ky)-\sin(ky))\right)\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{3.8.8}$$

The integral in parentheses is (3.7.12) with $\epsilon = 0$. So

$$J(\mathbf{x}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\ \boldsymbol{\alpha}\cdot\frac{\mathbf{k}}{E}\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.8.9}$$

and lemma 3.8.1 follows directly. Together with lemma 3.7.1,

$$\text{PV}\int d^3\mathbf{y}\ (\boldsymbol{\alpha}\cdot\mathbf{F}(\mathbf{y}) + \beta G(\mathbf{y}))f(\mathbf{x}-\mathbf{y}) = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\ \left(\boldsymbol{\alpha}\cdot\frac{\mathbf{k}}{E} + \beta\frac{m}{E}\right)\tilde{f}(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}} \tag{3.8.10}$$

$$= \mathcal{H}f(\mathbf{x}) \tag{3.8.11}$$

as desired. $\mathcal{H}$ provides an isometric embedding of $\mathcal{S}$ in Λ^1, and admits a unique extension to all Λ^1. □

The preceding two sections have studied in detail the separation of Λ^1 into its electron- and positron subspaces. In the next section, we will find that the projections $Q_\pm$ are also relevant to the free evolution of the system.

3.9 Lepton propagator

For smooth functions, we have a differential equation (the Dirac equation) describing their evolution. In this section, we give an explicit characterisation of the time evolution of free fields.

The time evolution in the Dirac theory, $H = -i\alpha_j\partial_j + \beta m$,

$$\begin{aligned}\frac{\partial}{\partial t}W_t &= -iHW_t \\ &= (-\alpha_k\partial_k - i\beta m)\, W_t, \qquad (W_t \in \Lambda^1)\end{aligned} \tag{3.9.1}$$

has an explicit solution in Fourier space,

$$\begin{aligned}\hat{W}_t(k) &= \exp(-itH(k))\hat{W}_0(k) \\ &\equiv \hat{S}_t(k)\hat{W}_0(k).\end{aligned} \tag{3.9.2}$$

Here the hats denote the Fourier transform on $\mathbb{R}^3$ and $H(k) = \alpha_j k_j + \beta m$. Thus evolution in time is described in Fourier space by pointwise multiplication by the 4-by-4 matrix $\hat{S}_t(k) = \exp(-itH(k))$. $\hat{S}_t(k)$ is uniformly bounded in k, so evolution in time is bounded from $\Lambda^1 \to \Lambda^1$. Towards a space representation, suppose that $\hat{S}_t(k)$ was an L^1-integrable function. One could define the space representation $S(x)$ as the Fourier transform

$$S_t(x) = \int d^3x\ \exp(-iH(k)t)\, e^{ikx}, \tag{3.9.3}$$

and since multiplication in Fourier space is equivalent to convolution in real space, evolution in real space would be simply characterised as

$$W_t(x) = (S_t * W_0)(x) = \int d^3x'\ S_t(x - x')W_0(x'). \tag{3.9.4}$$

According to Young's theorem (see e.g. [88]), the convolution (3.9.4) would be well-defined for all $W_0 \in L^2$ (indeed, it is an isometry). Physically, S_t represents

the evolution of the field from point-like initial conditions, since at $t = 0$ we have $\exp(-iHt) \equiv 1$ and $S_0 * W_0 = W_0$ identically. In other words, $S_0(x) = \delta^3(x)$. For this reason, S is known as the *retarded propagator*.

Unfortunately, $\hat{S}_t(k)$ is not an element of L^1, so the hypotheses of the above analysis are not met, and the integral in (3.9.3) is divergent. The above analysis can be completed rigorously within the theory of *distributions* over the Schwartz space; in this way S_t can be characterised in terms of its action on fields in the Schwartz space. The Schwartz space is dense in Λ^1, so the convolution (3.9.4) extends by continuity to all of Λ^1. However, these conclusions are obvious in the Fourier representation, with no need for the additional machinery of the theory of distributions. In the remainder of this thesis, we will avoid the extra complexity of distributions and stick to the L^2 spaces Λ^1 and Θ^1.

The bounded linear operator

$$\hat{S}_t : \Lambda^1 \to \Lambda^1, \tag{3.9.5}$$

will be referred to as the *lepton propagator*. $\hat{S}_t$ can be explicitly evaluated using

$$\begin{aligned} (H(k))^2 &= \alpha^j \alpha^k k_j k_k + \alpha^j \beta k_j m + \beta \alpha_j k_j m + \beta^2 m^2 \\ &= |k|^2 + m^2 \\ &= E(k)^2 \end{aligned} \tag{3.9.6}$$

where the second line follows from elementary properties of the Dirac α^j and β matrices. In the third line, $E(k) = \sqrt{|k|^2 + m^2} > 0$ is the energy. Using (3.9.6), the matrix exponential in (3.9.2) can be written

$$\begin{aligned} \hat{S}_t(k) &= 1 - itH(k) + \frac{1}{2!}(-itH(k))^2 + \frac{1}{3!}(-itH(k))^3 + \dots \\ &= \left(1 - \frac{1}{2!}(Et)^2 + \dots\right) + \frac{iH(k)}{E}\left(-Et + \frac{1}{3!}(Et)^3 \dots\right) \\ &= \cos(Et) - \frac{iH(k)}{E}\sin(Et) \\ &= \frac{H(k)+E}{2E}e^{-iEt} + \frac{-H(k)+E}{2E}e^{iEt}. \end{aligned} \tag{3.9.7}$$

This result is readily interpreted, with the two contributing terms describing dynamical evolution on each of the two eigenspaces of H. In particular, (3.9.7) can be rewritten

$$\hat{S}_t(k) = Q_+(k)e^{-iE(k)t} + Q_-(k)e^{iE(k)t} \tag{3.9.8}$$

where

$$\begin{aligned} Q_+(k) &= \frac{H(k)+E(k)}{2E(k)} \\ Q_-(k) &= \frac{-H(k)+E(k)}{2E(k)} \end{aligned} \tag{3.9.9}$$

$Q_\pm$ are orthogonal projection operators which project a vector $\hat{W}(k)$ onto its components lying in the $\pm E$ eigenspaces of H respectively,

$$H(k)Q_\pm(k)\hat{W}(k) = \pm E(k)Q_\pm(k)\hat{W}(k). \tag{3.9.10}$$

The form (3.9.8) reflects the fact that on each eigenspace, the matrix $H(k)$ in the exponential may be substituted for the corresponding eigenvalue. Some straightforward calculations using (3.9.9) verify the claim of orthogonal projection operators, namely

$$\begin{aligned} Q_\pm^2 &= Q_\pm \\ Q_+Q_- = Q_-Q_+ &= 0 \\ Q_+ + Q_- &= 1. \end{aligned} \tag{3.9.11}$$

From (3.9.9), $Q_\pm$ are Hermitian,

$$Q_\pm = Q_\pm^\dagger \tag{3.9.12}$$

and using (3.9.8)

$$\begin{aligned} \hat{S}_t^\dagger \hat{S}_t &= \left(Q_+e^{iEt} + Q_-e^{-iEt}\right)\left(Q_+e^{-iEt} + Q_-e^{iEt}\right) \\ &= Q_+^2 + Q_-^2 + Q_+Q_-e^{2iEt} + Q_-Q_+e^{-2iEt} \\ &= Q_+ + Q_- \\ &= 1 \end{aligned} \tag{3.9.13}$$

Similarly, $\hat{S}_t\hat{S}_t^\dagger = 1$. Thus $\hat{S}_t$ is a unitary operator on Λ^1 for all $t \in \mathbb{R}$. By Fourier isometry, a space representation S_t is well-defined and unitary on Λ^1.

3.10 Photon Propagator

The free dynamics for the RS photon (2.4.3) also have an explicit solution in Fourier space,

$$\begin{aligned} \hat{W}_t(k) &= \exp(-itH(k))\hat{W}_0(k) \\ &\equiv \hat{D}_t(k)\hat{W}_0(k), \end{aligned} \tag{3.10.1}$$

where the RS Hamiltonian was given in §2.4 as $H(k) = K_j k_j$. (K_j were defined in (2.4.2).) The linear operator

$$\hat{D}_t : \Theta^1 \to \Theta^1 \tag{3.10.2}$$

will be referred to as the *photon propagator*. $\hat{D}_t$ can be explicitly evaluated using the identities

$$(k_j K_j)^2 = |k|^2 - k \otimes k \tag{3.10.3}$$

$$(k_j K_j)^3 = |k|^2 \, k_j K_j. \tag{3.10.4}$$

These identities can be verified by direct calculation using the representation of K_j given in the appendix, and can be understood from a variety of different standpoints – they are, for instance, equivalent to the identities

$$k \times (k \times W) = -|k|^2 W + k(k \cdot W) \tag{3.10.5}$$

$$k \times (k \times (k \times W)) = -|k|^2 k \times W \tag{3.10.6}$$

of vector calculus. The latter could also be derived from the Cayley-Hamilton theorem, which states that any matrix M is a solution to its own characteristic equation $p(\lambda) = 0$, where

$$p(\lambda) = \det(\lambda I - M). \tag{3.10.7}$$

In this case, putting $M = k_j K_j$, direct computation gives

$$p(\lambda) = \lambda^3 - |k|^2 \lambda \tag{3.10.8}$$

and the Cayley-Hamilton theorem states that $p(k_j K_j) = 0$; (3.10.4) follows. Using (3.10.4), the matrix exponential in (3.10.1) can be written

$$\begin{aligned}
\hat{D}_t(k) &= \left(1 - \frac{t^2}{2!}(k_j K_j)^2 + \frac{t^4}{4!}(k_j K_j)^4 - \frac{t^6}{6!}(k_j K_j)^6 \ldots\right) \\
&\quad + i\left(t k_j K_j - \frac{t^3}{3!}(k_j K_j)^3 + \frac{t^5}{5!}(k_j K_j)^5 + \ldots\right) \\
&= 1 + \left(|k|^2 - k \otimes k\right)\left(-\frac{t^2}{2!} + \frac{t^4}{4!}|k|^2 - \frac{t^6}{6!}|k|^4 \ldots\right) \\
&\quad + i k_j K_j \left(t - \frac{t^3}{3!}|k|^2 + \frac{t^5}{5!}|k|^4 + \ldots\right) \\
&= 1 + \frac{|k|^2 - k \otimes k}{|k|^2}\left(\cos(|k|t) - 1\right) + \frac{i k_j K_j}{|k|}\sin(|k|t)
\end{aligned} \tag{3.10.9}$$

(Rodrigues' rotation formula). $\hat{D}_t$ may be resolved into modes,

$$\hat{D}_t(k) = \frac{1}{2}\left(1 - \frac{k_j K_j}{|k|} - \frac{k \otimes k}{|k|^2}\right) e^{-i|k|t} + \frac{1}{2}\left(1 + \frac{k_j K_j}{|k|} - \frac{k \otimes k}{|k|^2}\right) e^{i|k|t} + \frac{k \otimes k}{|k|^2}. \tag{3.10.10}$$

This result is readily interpreted, with each of the three contributing terms describing the evolution on one of the three eigenspaces of H. In particular, (3.10.10) can be rewritten

$$\hat{D}_t(k) = P_+(k)e^{-i|k|t} + P_-(k)e^{i|k|t} + P_0(k) \tag{3.10.11}$$

where

$$\begin{aligned} P_0(k) &= \frac{k \otimes k}{|k|^2} \\ P_+(k) &= \frac{1}{2}\left(1 + \frac{k_j K_j}{|k|} - \frac{k \otimes k}{|k|^2}\right) \\ P_-(k) &= \frac{1}{2}\left(1 - \frac{k_j K_j}{|k|} - \frac{k \otimes k}{|k|^2}\right). \end{aligned} \tag{3.10.12}$$

P_0 and $P_\pm$ are orthogonal projection operators, projecting an arbitrary vector $\hat{W}(k)$ onto its components lying in the 0 and $\pm|k|$ eigenspaces of H respectively:

$$\begin{aligned} H(k)P_0(k)\hat{W}(k) &= 0 \\ H(k)P_\pm(k)\hat{W}(k) &= \pm|k|P_\pm(k)\hat{W}(k). \end{aligned} \tag{3.10.13}$$

The relation (3.10.11) indicates that on each eigenspace, the action of H in the exponential may be substituted for its associated eigenvalue. Some straightforward calculations verify the claim for orthogonal projection operators, namely

$$\begin{aligned} P_0^2 &= P_0 \\ P_\pm^2 &= P_\pm \\ P_0P_\pm = P_\pm P_0 = P_+P_- = P_-P_+ &= 0 \\ P_0 + P_+ + P_- &= 1.) \end{aligned} \tag{3.10.14}$$

From the first two relations, the projection P_+W of any state $W \in \Theta^1$ lies in the eigenspace of P_+ (likewise P_- and P_0). From (3.10.12), $P_\pm$ and P_0 are Hermitian,

$$P_\pm = P_\pm^\dagger \text{ and } P_0 = P_0^\dagger. \tag{3.10.15}$$

A short calculation using (3.10.11) establishes

$$\begin{aligned}\hat{D}_t^\dagger \hat{D}_t &= \left(P_+(k)e^{-i|k|t} + P_-(k)e^{i|k|t} + P_0(k)\right)\left(P_+(k)e^{+i|k|t} + P_-(k)e^{-i|k|t} + P_0(k)\right) \\ &= 1\end{aligned} \tag{3.10.16}$$

after expanding and using the projection properties (3.10.14). (This is a similar calculation to (3.9.13).) Similarly, $\hat{D}_t\hat{D}_t^\dagger = 1$. So $\hat{D}_t$ is a unitary operator on Θ^1 for all $t \in \mathbb{R}$. By Fourier isometry, a space representation D_t is well-defined and unitary on Θ^1.

Incidentally, the eigenspaces of $P_\pm$ have a useful physical interpretation. If W lies in the eigenspace of $P_\pm$, then

$$P_0 W = P_0 P_\pm W = 0, \tag{3.10.17}$$

so from the definitions (3.10.10) we have

$$\begin{aligned}\hat{W}(k) &= P_\pm(k)\hat{W}(k) \\ &= \frac{1}{2}\left(1 \pm \frac{k_j K_j}{|k|}\right)\hat{W}(k).\end{aligned} \tag{3.10.18}$$

Rearranging and using $K_j = -iJ_j$,

$$\pm\hat{W}(k) = i\frac{k_j}{|k|}J_j\hat{W}(k). \tag{3.10.19}$$

The right-hand side is recognisable as the (spin-one) *helicity* operator. In this way, the $P_\pm$ operators project on to the ± 1 helicity subspaces of the RS photon. Note that (3.10.19) agrees with the standard result that the spin-one helicity operator has ± 1 eigenspaces. A photon in the $+1$ eigenspace is called *right-handed*; a photon in the -1 eigenspace is called *left-handed*.

In the next chapter, we will review 'second-quantisation', or generalisation of the preceding ideas to include systems with multiple particles. We will see that the process can be performed without affecting any of the results so far.

Chapter 4

Particle statistics

4.1 Second Quantisation

Second quantisation is the process of generalising the parametrisation and dynamics of single-particle systems to multi-particle systems. In the preceding chapters, it was argued that the single-particle state space for both leptons and photons is best represented by an L^2 function space on $\mathbb{R}^3$. In this chapter, we will generalise these ideas to describe systems of multiple photons and leptons.

Second quantisation is extremely natural in a configuration-space theory, and the essential ideas appeared shortly after the development of the earliest configuration-space theory, the non-relativistic Schrödinger electron. Within several years of Schrödinger's work, Fock developed the essentials of second quantisation in a general configuration space theory [103]. The Dirac theory also rapidly led to a configuration space theory [104, 105]. These methods led to the Hartree-Fock methods employed to great success in various problems in quantum chemistry.

However, after the development of QFT, configuration-space methods never gained much attention in the field theory community. The obstacle appears to be that the A_μ (EM potential) representation of the photon is not naturally a configuration-space theory. Lacking a generator of time translation, the single-particle dynamics do not simply extend to the second-quantised state space. This will be shown in §4.6, and we will also show that the RS photon avoids this problem. There also appears to be a wide (mis)perception that the methods of configuration-space theory are somehow intrinsically incompatible with relativity and, consequently, limited in scope to non-relativistic problems. (A recent example in the literature is [106] who specifically

study Fock space as a construction relevant to non-relativistic QED only.) §4.6 will show that the second-quantised theory of free Dirac leptons and RS photons is fully relativistic, and Chapter 5 will demonstrate the possibility of modification to include interactions.

4.2 The tensor algebra

The framework of the *tensor algebra* elegantly captures the extra degrees of freedom associated with multi-particle systems (see, e.g. [107, 108]). To illustrate the core of the idea, suppose two particles have amplitudes $f(x), g(x)$ respectively (spin indices are suppressed). The following ideas apply equally to the lepton state space Λ^1 and the RS photon state space Θ^1, and in the following I will denote either of these spaces by Ω. The tensor product $f \otimes g$ is, by definition, the function

$$(f \otimes g)(x_1, x_2) = f(x_1)g(x_2) \tag{4.2.1}$$

defined on the set of *pairs* of space coordinates. The amplitude-square of $f \otimes g$ may be interpreted as the probability density of finding one particle x_1, conditional on finding the other particle near x_2. Since

$$|(f \otimes g)(x_1, x_2)|^2 = |f(x_1)|^2|g(x_2)|^2, \tag{4.2.2}$$

the locations of the two particles are uncorrelated (x_1, x_2 are independent random variables).

A state of the form (4.2.1) is called *unentangled*. Quantum-mechanical systems generally admit *superpositions* of states: it is assumed that for any two states f, f', the system can also exist in the mixed (superposed) state $(\alpha f + \beta f')$ for arbitrary $\alpha, \beta \in \mathbb{C}$. Consequently, state space is assumed to be a vector space over $\mathbb{C}$. The set of unentangled states is not a vector space, for a sum such as $(f \otimes g + f' \otimes g')$ can not generally be written in the form $f'' \otimes g''$. Such states are called *entangled*, and the locations of the two particles are dependent random variables.

The smallest complete normed space which includes all unentangled states is the space of 'all' functions of the form $h(x_1, x_2)$. This space is called the tensor product of Ω with itself, and is denoted $\Omega \otimes \Omega$. By a general result of measure theory, $h \in \Omega \otimes \Omega$ if and only if h is square-integrable on $\mathbb{R}^3 \times \mathbb{R}^3$. This is yet another helpful feature of

L^2 function spaces, and hence of the RS representation. Thus, we may characterise a general two-particle system by a square-integrable amplitude on $\mathbb{R}^3 \times \mathbb{R}^3$. The question of defining dynamics on this state space will be addressed in §4.6.

The preceding discussion generalises readily to p-particle systems and particles of arbitrary spin, but has not addressed the issue of indistinguishability of particles (Bose and Fermi statistics). Two particles (of the same species) in a two-particle system are indistinguishable, and this requires that the probability distribution $|f(x, x')|^2$ be symmetric under interchange of x, x'. An obvious way to achieve this is to require either symmetry or antisymmetry of the amplitude f itself; these two cases, called Bose and Fermi statistics respectively, are treated below. We will now briefly review these ideas for p-particle systems. The conventions follow [109].

We will consider first the lepton state space with Fermi statistics. The RS photon state space, with Bose statistics, is defined analogously. Let $\otimes^p\Omega$ denote the space of p-particle amplitudes. A typical p-particle configuration space amplitude $f \in \otimes^p\Omega$ is a square integrable function

$$f(x_1, x_2, \ldots, x_p) \tag{4.2.3}$$

defined on $\mathbb{R}^{3p}$, the set of p-tuples of space coordinates. (f carries p spinor indices which have been suppressed.) I will refer to such amplitudes as *configuration space amplitudes*. According to Fermi (Bose) statistics, we require the amplitude for a p-particle state to be antisymmetric (symmetric) under permutation of the underlying single-particle states. Formally, a permutation is an invertible map σ on the set $\{1, \ldots, p\}$ to itself. There are $p!$ such permutations and the set of all permutations shall be denoted Π_p. The canonical action of a permutation $\sigma \in \Pi_p$ on the space of tensors of rank p is defined

$$\sigma f(x_1, \ldots x_p) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(1)}). \tag{4.2.4}$$

It may help to consider the action of (4.2.4) on a typical product $f = g_1 \otimes \ldots \otimes g_p$.

$$\begin{aligned} \sigma(g_1 \otimes \ldots \otimes g_p)(x_1, \ldots, x_p) &= (g_1 \otimes \ldots \otimes g_p)(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(1)}) \\ &= g_1(x_{\sigma^{-1}(1)}) \ldots g_p(x_{\sigma^{-1}(p)}) \\ &= g_{\sigma(1)}(x_1) \ldots g_{\sigma(p)}(x_p) \\ &= (g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(p)})(x_1, \ldots, x_p). \end{aligned} \tag{4.2.5}$$

Thus

$$\sigma(g_1 \otimes \ldots \otimes g_p) = g_{\sigma(1)} \otimes \ldots \otimes g_{\sigma(p)}. \tag{4.2.6}$$

In other words, on an unentangled state, the permutation σ simply rearranges the order the single-particle states appear in the tensor product. Since unentangled states span the entire p-particle state space, (4.2.6) is an alternative to (4.2.4) characterising the representation of Π_p on $\otimes^p\Omega$.

Having carefully defined permutations on the p-particle state space, it is straightforward to characterise particle statistics:

- A p-particle state f is symmetric if

$$f = \sigma f \qquad \forall \sigma \in \Pi_p. \tag{4.2.7}$$

  The system is said to satisfy *Bose statistics*.

- A p-particle state f is antisymmetric if

$$f = \mathrm{sgn}(\sigma)\, \sigma f \qquad \forall \sigma \in \Pi_p, \tag{4.2.8}$$

  where $\mathrm{sgn}(\sigma) = \pm 1$ according to whether σ is odd or even. The system is said to satisfy *Fermi statistics*.

The photon satisfies Bose statistics; the lepton satisfies Fermi statistics.

4.3 Symmetric and Exterior Algebras

The tensor product $\otimes$ is unsuited to systems constrained to either the symmetric or antisymmetric state spaces, because if f, g are symmetric (antisymmetric), in general $f \otimes g$ is not symmetric (antisymmetric). As a more suitable alternative, a tensor $g \in \otimes^p\Omega$ possesses unique symmetric and antisymmetric projections

$$\mathrm{Sym}\, g = \frac{1}{p!} \sum_{\sigma \in \Pi_p} \sigma g \tag{4.3.1}$$

$$\mathrm{Alt}\, g = \frac{1}{p!} \sum_{\sigma \in \Pi_p} \mathrm{sgn}(\sigma)\, \sigma g. \tag{4.3.2}$$

$\mathrm{Sym}\, g$ is symmetric in the sense of (4.2.7), and $\mathrm{Alt}\, g$ is antisymmetric (or 'alternating') in the sense of (4.2.8). Sym and Alt are called projections because

$$\begin{aligned} \mathrm{Sym}\,\mathrm{Sym}\, g &= \frac{1}{p!}\sum_{\sigma\in\Pi_p} \sigma\ \ \mathrm{Sym}\, g \\ &= \frac{1}{p!}\sum_{\sigma\in\Pi_p} \mathrm{Sym}\, g \\ &= \mathrm{Sym}\, g, \end{aligned} \tag{4.3.3}$$

i.e., a symmetric tensor is the symmetric projection of itself. Similarly, $\mathrm{Alt}\,\mathrm{Alt} = \mathrm{Alt}$.

The projections give natural candidates for symmetric and antisymmetric tensor products, namely

$$f \odot g = \mathrm{Sym}(f \otimes g) \tag{4.3.4}$$

$$f \wedge g = \mathrm{Alt}(f \otimes g). \tag{4.3.5}$$

$\wedge$ is also known as the 'alternating' or 'wedge' product. It is a straightforward exercise to show $\odot$ and $\wedge$ are associative. (Thus, the boson state space Θ will carry the product $\odot$, while the fermion state space Λ will carry the product $\wedge$. I hope the notation will be easy to remember!)

4.4 Geometry of multiparticle state space

We introduce a geometry to the multi-particle state spaces as follows. Chapter 3 argued for specific choices for the single-particle space Ω for leptons and photons: in both cases an L^2 space. Both of these are Hilbert spaces, equipped with a complete inner product $\langle\cdot,\cdot\rangle$. The inner product structure on $\otimes^p\Omega$ is inherited from Ω according to

$$\langle f_1 \otimes \ldots \otimes f_p, g_1 \otimes \ldots \otimes g_p\rangle = p!\ \langle f_1, g_1\rangle \ldots \langle f_p, g_p\rangle \tag{4.4.1}$$

where the factor $p!$ is included for convenience; together with the definition of $\wedge$, it implies

$$\langle f_1 \wedge \ldots \wedge f_p, g_1 \wedge \ldots \wedge g_p\rangle = \det \begin{pmatrix} \langle f_1, g_1\rangle & \ldots & \langle f_1, g_p\rangle \\ \vdots & \ddots & \vdots \\ \langle f_p, g_1\rangle & \ldots & \langle f_p, g_p\rangle \end{pmatrix}. \tag{4.4.2}$$

The inner product (4.4.1), defined for unentangled states, extends by linearity to all of $\otimes^p\Omega$. The geometry on $\otimes^p\Omega$ is defined as usual by the norm

$$\|f\|^2 = \langle f, f\rangle, \qquad f \in \otimes^p\Omega, \tag{4.4.3}$$

and $\otimes^p\Omega$ is complete with respect to this norm. The subspaces of p-particle symmetric states, denoted Θ^p, and p-particle antisymmetric states, denoted Λ^p, inherit this inner product.

In general, a quantum system will not have definite particle number, but will be in a superposition of states of different particle number, in the sense that

$$f = f_{(0)} + f_{(1)} + f_{(2)} + \ldots + f_{(p)} + \ldots \tag{4.4.4}$$

where each of the components $f_{(p)}$ are amplitudes on p-particle configuration space, i.e. $f_{(p)} \in \Theta^p$ (respectively Λ^p) for each p. A p-particle state is a square-integrable amplitude

$$f_{(p)} \equiv f_{(p)}^{r_1 \ldots r_p}(x_1, \ldots, x_p) \tag{4.4.5}$$

where r_j are the appropriate spin indices. We hypothesise that $\|f_{(p)}\|^2$ gives the total probability the system will be in the p-particle state. Since a system of specified particle number is definitely *not* in a state of any other particle number, the components $f_{(p)}, f_{(q)}$ are defined to be orthogonal for $p \neq q$. To complete the picture, $f_{(0)}$ is defined to be a complex number whose magnitude square gives the probability the system is in a state with *no* particles (the vacuum). Thus, the total state space available to a system consisting of indeterminate number of bosons (respectively fermions) is

$$\Theta = \Theta^0 \oplus \Theta^1 \oplus \ldots \oplus \Theta^p \oplus \ldots, \tag{4.4.6}$$

$$\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \ldots \oplus \Lambda^p \oplus \ldots, \tag{4.4.7}$$

where the vacuum is the one-dimensional space $\Theta^0 = \Lambda^0 = \mathbb{C}$. This thesis is concerned with systems of both fermions and bosons, for which the total state space is $\Theta \otimes \Lambda$.

4.5 Commutation relations

Creation operators are a convenient way to describe how the state of a p-particle system changes if a single additional particle is included. By definition, the creation operator $a_g^\dagger$ for fermions acts on a state $f \in \Lambda^p$ by adding a single particle in the one-particle state $g \in \Lambda^1$, that is,

$$\begin{aligned} &a_g^\dagger : \Lambda^p \to \Lambda^{p+1}, \qquad g \in \Lambda^1 \\ &a_g^\dagger f = f \wedge g \end{aligned} \tag{4.5.1}$$

Similarly, for bosons,

$$b_g^\dagger : \Theta^p \to \Theta^{p+1}, \qquad g \in \Theta^1$$
$$b_g^\dagger f = f \odot g. \tag{4.5.2}$$

By virtue of the identities $f \odot g \odot h = f \odot h \odot g$ and $f \wedge g \wedge h = -f \wedge h \wedge g$ when g and h are single-particle states, the relations

$$[b_g^\dagger, b_h^\dagger] = b_g^\dagger b_h^\dagger - b_h^\dagger b_g^\dagger = 0, \qquad g, h \in \Theta^1 \tag{4.5.3}$$
$$\{a_g^\dagger, a_h^\dagger\} = a_g^\dagger a_h^\dagger + a_h^\dagger a_g^\dagger = 0, \qquad g, h \in \Lambda^1 \tag{4.5.4}$$

are valid operator identities on Θ and Λ respectively. These are the first of the ubiquitous (anti-)commutation relations in field theory.

Brief calculations show that, although $a_g^\dagger$ and $b_g^\dagger$ are unbounded on Λ and Θ respectively, they are bounded on Λ^p and Θ^p respectively (for fixed p). We may therefore define the *annihilation operators* b_g and a_g as the Hermitian adjoints of $b_g^\dagger$ and $a_g^\dagger$, with respect to the inner product $\langle \cdot, \cdot \rangle$ defined in §4.4. In other words, b_g and a_g are the unique operators for which

$$\langle f, a_g h \rangle = \langle a_g^\dagger f, h \rangle \qquad \forall f \in \Lambda^p, h \in \Lambda^{p+1} \tag{4.5.5}$$
$$\langle f, b_g h \rangle = \langle b_g^\dagger f, h \rangle \qquad \forall f \in \Theta^p, h \in \Theta^{p+1}. \tag{4.5.6}$$

Taking Hermitian adjoints of (4.5.3) and (4.5.4) yields the second (anti-)commutation relations

$$[b_g, b_h] = 0 \qquad g, h \in \Theta^1 \tag{4.5.7}$$
$$\{a_g, a_h\} = 0 \qquad g, h \in \Lambda^1 \tag{4.5.8}$$

Though a_g and b_g are uniquely defined, it is nevertheless useful to obtain some explicit expressions describing the action of the annihilation operators on separable states. Let

$f = f_1 \wedge \ldots \wedge f_{p-1}$ and $h = h_1 \wedge \ldots \wedge h_p$. Then

$$\begin{aligned}\langle f, a_g h\rangle &= \langle a_g^\dagger f, h\rangle \\ &= \langle f_1 \wedge \ldots \wedge f_{p-1} \wedge g, h_1 \wedge \ldots \wedge h_p\rangle \\ &= \det \begin{pmatrix} \langle f_1, h_1\rangle & \ldots & \langle f_1, h_p\rangle \\ \vdots & \ddots & \\ \langle f_{p-1}, h_1\rangle & \ldots & \langle f_{p-1}, h_p\rangle \\ \langle g, h_1\rangle & \ldots & \langle g, h_p\rangle \end{pmatrix}.\end{aligned} \tag{4.5.9}$$

Expanding the determinant by cofactors along the bottom line, starting from the bottom right corner,

$$\begin{aligned}\langle f, a_g h\rangle = &\langle f_1 \wedge \ldots \wedge f_{p-1}, h_1 \wedge \ldots \wedge h_{p-2} \wedge h_{p-1}\rangle\langle g, h_p\rangle \\ &- \langle f_1 \wedge \ldots \wedge f_{p-1}, h_1 \wedge \ldots \wedge h_{p-2} \wedge h_p\rangle\langle g, h_{p-1}\rangle \\ &+ \ldots - \ldots \\ &+ (-1)^{p-1}\langle f_1 \wedge \ldots \wedge f_{p-1}, h_2 \wedge \ldots \wedge h_{p-1} \wedge h_p\rangle\langle g, h_1\rangle.\end{aligned} \tag{4.5.10}$$

This equation, of the form $\langle f, a_g h\rangle = \langle f, \cdot\rangle$, uniquely fixes $a_g h$ because the separable states f span the entire space Λ^p. In this way,

$$\begin{aligned}a_g(h_1 \wedge \ldots \wedge h_p) = &\, h_1 \wedge \ldots \wedge h_{p-2} \wedge h_{p-1}\ \langle g, h_p\rangle \\ &- h_1 \wedge \ldots \wedge h_{p-2} \wedge h_p\ \langle g, h_{p-1}\rangle \\ &+ \ldots - \ldots \\ &+ (-1)^{p-1} h_2 \wedge \ldots \wedge h_{p-1} \wedge h_p\ \langle g, h_1\rangle\end{aligned} \tag{4.5.11}$$

A similar calculation for the bosonic case is longer to write down (as it can not be paraphrased in terms of determinants) but it gives a similar answer:

$$\begin{aligned}b_g(h_1 \odot \ldots \odot h_p) = &\, h_1 \odot \ldots \odot h_{p-2} \odot h_{p-1}\ \langle g, h_p\rangle \\ &+ h_1 \odot \ldots \odot h_{p-2} \odot h_p\ \langle g, h_{p-1}\rangle \\ &+ \ldots \\ &+ h_2 \odot \ldots \odot h_{p-1} \odot h_p\ \langle g, h_1\rangle.\end{aligned} \tag{4.5.12}$$

These explicit expressions (4.5.11) and (4.5.12) give the action of annihilation operators on separable states. The results extend by linearity to the whole p-particle space Λ^p or Θ^p, so (4.5.11) and (4.5.12) could be regarded as defininitions for a_g and b_g.

The explicit expressions permit direct calculation of the (anti-)commutators $\{a_g^\dagger, a_h\}$ and $[b_g^\dagger, b_h]$. For example, starting from (4.5.12),

$$\begin{aligned} b_g b_f^\dagger(h_1 \odot \ldots \odot h_p) &= b_g(h_1 \odot \ldots \odot h_p \odot f) \\ &= h_1 \odot \ldots \odot h_p \, \langle g, f\rangle \\ &\quad + h_1 \odot \ldots \odot h_{p-2} \odot h_{p-1} \odot f \, \langle g, h_p\rangle \\ &\quad + h_1 \odot \ldots \odot h_{p-2} \odot h_p \odot f \, \langle g, h_{p-1}\rangle \\ &\quad + \ldots \\ &\quad + h_2 \odot \ldots \odot h_{p-1} \odot h_p \odot f \, \langle g, h_1\rangle \\ &= h_1 \odot \ldots \odot h_p \, \langle g, f\rangle + b_f^\dagger b_g(h_1 \odot \ldots \odot h_p). \end{aligned} \tag{4.5.13}$$

Therefore $b_g b_f^\dagger - b_f^\dagger b_g = \langle g, f\rangle$ on separable states. Extending by linearity to Θ^p,

$$[b_g, b_f^\dagger] = \langle g, f\rangle \qquad\qquad g, f \in \Theta^1. \tag{4.5.14}$$

Similarly, for fermions,

$$\{a_g, a_f^\dagger\} = \langle g, f\rangle \qquad\qquad g, f \in \Lambda^1. \tag{4.5.15}$$

(4.5.14) and (4.5.15) are the third of the famous (anti-)commutation relations. The usual forms

$$[b_x, b_{x'}^\dagger] = \delta^3(x - x') \tag{4.5.16}$$

$$\{a_x, a_{x'}^\dagger\} = \delta^3(x - x') \tag{4.5.17}$$

may be established by taking the limits $f \to \delta^3(x')$ and $g \to \delta^3(x)$. This limit was *not* taken at the start of this chapter, as calculation of the relations relied critically on the existence of a complete inner product on state space. The delta-functions are not contained in any inner-product space, and (4.5.16) and (4.5.17) should be regarded as derived, rather than fundamental, relations.

It is worth emphasising that the (anti-)commutation relations are purely a consequence of the structure of the (anti-)symmetric tensor algebra, and are not sensitive to the specific form of the single-particle state space Ω (provided Ω is a Hilbert space). Consequently, we are free to use the RS photon as the single-photon state space, rather

than (say) the representation by the electromagnetic potential A_μ: the commutation relations remain valid. Similarly, the choice of dynamics defined on the state space has some freedom; in Chapter 5 we will modify the free Hamiltonian to include interactions. With the state space unchanged, the (anti-)commutation relations will remain valid as required.

This section constructed the bosonic/fermionic Fock spaces over a given single-particle state space (Λ^1 or Θ^1). According to (4.4.4), a state is a superposition of p-particle states for all p, and according to (4.4.5), each p-particle state is a square-integrable amplitude on $(\mathbb{R}^3)^p$. I will refer to this as a *configuration space amplitude*.

4.6 Relativity and dynamics under second-quantisation

The previous sections have constructed a second-quantised theory of Dirac leptons and RS photons in a configuration space representation. The following lemma shows that this theory admits formulation as an initial-value problem, and is relativistic in the sense of Chapter 2.

Lemma 4.6.1. *A representation ρ of the Poincaré group on Θ^1, Λ^1 extends to the Fock spaces Θ, Λ and $\Theta \otimes \Lambda$.*

Proof: Standard result; see e.g. [109]. In essence, one defines a representation on Θ^{p+1} by

$$\rho_L(f \otimes g) = (\rho_L f) \otimes (\rho_L g), \qquad f \in \Theta^1, g \in \Theta^p \tag{4.6.1}$$

for all Poincaré transformations L, extends by linearity and induction on p. Evidently, ρ is a representation in the sense of §2.1. □

The differential of (4.6.1) is

$$\mathcal{R}(f \otimes g) = (\mathcal{R}f) \otimes g + f \otimes (\mathcal{R}g) \tag{4.6.2}$$

where $\mathcal{R}$ is any generator (element of the Poincaré algebra). In particular, putting $\mathcal{R}$ equal to the generator of time translations $\mathcal{P}_0 = -iH$, the second-quantised state space naturally inherits the dynamics of the underlying single-particle state spaces:

$$\begin{aligned} \partial_t(f \otimes g) &= -iH(f \otimes g) \\ &= (-iHf) \otimes g + f \otimes (-iHg) \end{aligned} \tag{4.6.3}$$

and extending by linearity.

As noted at the start of the chapter, it is possible begin with scalars (the Klein-Gordon equation) and/or the electromagnetic potential (the wave equation) and proceed to a second-quantised configuration space representation. An example is the early study of Wightman and Schweber [110]. However, lacking a generator of time translations $\mathcal{P}_0$, it is then extremely awkward (if not impossible) to construct evolution equations like (4.6.3) and, at the same time, retain relativistic covariance. To the author's knowledge, no previous research in field theory has used the RS photon to avoid these problems.

To illustrate (4.6.3), let us consider the time evolution of a state $W \in \Theta^p \otimes \Lambda^q$ for some specific small values of p, q. (Summation is implied over repeated indices.)

- For a single-lepton state $W_t \in \Lambda^1$ (i.e. $p = 0, q = 1$), the free equation of motion is the free Dirac equation,

$$\frac{\partial}{\partial t} W_t^{s_1}(y_1) = \left(\alpha^{r s_1 s_1'} \frac{\partial}{\partial y^r} + im\beta^{s_1 s_1'} \right) W_t^{s_1'}(y_1). \tag{4.6.4}$$

  Summation over $s_1' = 1, 2, 3, 4$ and $r = 1, 2, 3$ is implied, and $s_1 = 1, 2, 3, 4$ is a free index on each side of the equation. Here $\alpha^{r s_1 s_1'}$ and $\beta^{s_1 s_1'}$ are the Dirac matrices α^r and β with spin indices explicitly written in.

- For a single-photon state $W_t \in \Theta^1$ (i.e. $p = 1, q = 0$),

$$\frac{\partial}{\partial t} W_t^{r_1}(x_1) = -K^{r r_1 r_1'} \frac{\partial}{\partial x_1^r} W_t^{r_1'}(x_1). \tag{4.6.5}$$

  Summation over $r = 1, 2, 3$ and $r_1' = 1, 2, 3$ is implied, and $r_1 = 1, 2, 3$ is a free index. The matrices K^r were defined in (2.4.2).

- For a 2-lepton state $W_t \in \Lambda^2$ (i.e. $p = 0, q = 2$),

$$\begin{aligned} \frac{\partial}{\partial t} W_t^{s_1 s_2}(y_1, y_2) = &\left(\alpha^{r s_1 s_1'} \frac{\partial}{\partial y_1^r} + im\beta^{s_1 s_1'} \right) W_t^{s_1' s_2}(y_1, y_2) \\ &+ \left(\alpha^{r s_2 s_2'} \frac{\partial}{\partial y_2^r} + im\beta^{s_2 s_2'} \right) W_t^{s_1 s_2'}(y_1, y_2). \end{aligned} \tag{4.6.6}$$

- For a state with one lepton and one photon $W_t \in \Theta^1 \otimes \Lambda^1$ (i.e. $p = q = 1$),

$$\begin{aligned} \frac{\partial}{\partial t} W_t^{r_1 s_1}(x_1, y_1) = &-K^{r r_1 r_1'} \frac{\partial}{\partial x_1^r} W_t^{r_1' s_1}(x_1, y_1) \\ &+ \left(\alpha^{r s_1 s_1'} \frac{\partial}{\partial y_1^r} + im\beta^{s_1 s_1'} \right) W_t^{r_1 s_1'}(x_1, y_1). \end{aligned} \tag{4.6.7}$$

The generalisation to larger p, q should be obvious. As presaged by (4.6.1), the solution to the multiparticle equation of motion (4.6.3) is

$$\left(e^{-iHt}f_0\right) \otimes \left(e^{-iHt}g_0\right). \tag{4.6.8}$$

Consequently, this representation is a non-interacting or free theory; the single-particle solutions do not mix in time or under change of frame. There are no transitions between states of different particle content (the representation (4.6.1) acts *homogeneously* on the subspaces $\Theta^p \otimes \Lambda^q$).

In the next chapter, we will consider a simple deformation of the foregoing free theory to incorporate some non-trivial scattering. The deformation will modify $\mathcal{P}_0$ and $\mathcal{K}_j$ relative to their action in the free theories. We will consider a linear, self-adjoint deformation of H that is *not* homogeneous on the subspaces of different particle content. Doing so will couple the free-particle equations (4.6.4)–(4.6.7) to each other. With a Hamiltonian of this type, the theory will be able to describe processes which change the particle content of the system. For consider initial conditions

$$W_0(x'_1, \ldots, x'_m) \tag{4.6.9}$$

describing an m-particle state at time t_0. (Spin indices and species/statistics of the particles have been ignored, but do not affect the argument.) Let us denote the evolved state

$$SW_0 = e^{-iH(t-t_0)}W_0 \tag{4.6.10}$$

on the understanding that the S-matrix of Chapter 1 is the limit $t \to \infty, t_0 \to -\infty$. It follows that if H is not homogeneous on the m-particle subspaces, then the evolved state will include components of different content, that is

$$(SW_0)(x_1, \ldots, x_n) \tag{4.6.11}$$

for $n \neq m$, i.e. different particle content to the initial conditions (4.6.9). Note that with H remaining self-adjoint, evolution in time will remain unitary ($\|W\|$ constant in time).

While the theory will modify H, and therefore $\mathcal{P}_0$, there is a strong argument for leaving $\mathcal{P}_j$ and $\mathcal{J}_j$ unchanged. As suggested by the motivating ideas of §4.1, the configuration space representation is intended to give a close physical interpretation to the

amplitudes W defined on copies of space ($\mathbb{R}^3$), and this interpretation does not appear to be limited to the free theory. Thus with $\mathcal{P}_j$ and $\mathcal{J}_j$ having a natural action on $\mathbb{R}^3$ (and, by extension, on the full second-quantised configuration space), we suppose their action is unchanged in the interacting theory. In consequence, the interacting theory must satisfy the hypotheses of the Poincaré test of §2.2 in order to fulfil relativistic covariance. The generators will be given by (2.2.2) as usual. From those expressions, it follows that $\mathcal{K}_j$ will change accordingly if H is modified.

4.7 Example: Translation invariance of H in a second-quantised theory

To give an example of one useful feature of the second-quantised configuration space representation, let us partially derive the relation (1.1.5) describing energy-momentum conservation during a scattering process. We will give a (somewhat heuristic) derivation that this relation follows from the multiparticle representation of the Poincaré group (4.6.1) in a configuration space theory. Firstly, according to the Poincaré test, $\mathcal{P}_j$ must commute with H (the deformed Hamiltonian must be translation-invariant). Acting with $\mathcal{P}_j$ on the definition (4.6.10),

$$\mathcal{P}_j S W_0 = S \mathcal{P}_j W_0 \qquad (j = 1, 2, 3). \tag{4.7.1}$$

Suppose the initial data W_0 are an m-particle state, and consider the n-particle component (4.6.11) of the evolved state. In particular, we leave open the possibility that $n \neq m$. We will be considering self-adjoint deformations of H, so S will still be unitary on Θ, and its projection to Θ^n will be bounded (by unity). A general form for this bounded operator from Θ^m to Θ^n is

$$(SW_0)(x_1, \ldots, x_n) = \int d^3x'_1 \ldots d^3x'_m \; S(x_1, \ldots, x_n, x'_1, \ldots, x'_m) W_0(x'_1, \ldots, x'_m). \tag{4.7.2}$$

(Whether this is the most general form, and the general properties and regularity of the kernel $S(x_1, \ldots, x_n, x'_1, \ldots, x'_m)$, is left for future work. However, the Hamiltonian introduced in the next chapter is of this form.) Eq.(4.6.2) provides the action of the space-

translation generators $\mathcal{P}_j$ on Θ^m and Θ^n. Using a vector notation $\boldsymbol{\nabla} = (\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3)$,

$$\begin{aligned}(\boldsymbol{\nabla}_{x_1} + \ldots + \boldsymbol{\nabla}_{x_n})\,(SW_0)(x_1, \ldots, x_n) \\ = \int d^3x'_1 \ldots d^3x'_m\; S(x_1, \ldots, x_n, x'_1, \ldots, x'_m) \\ \left(\boldsymbol{\nabla}_{x'_1} + \ldots + \boldsymbol{\nabla}_{x'_m}\right) W_0(x'_1, \ldots, x'_m)\end{aligned} \tag{4.7.3}$$

Taking Fourier transforms,

$$\begin{aligned}(p_1 + \ldots + p_n)(\widehat{SW_0})(p_1, \ldots, p_n) \\ = \int d^3p'_1 \ldots d^3p'_m\; \hat{S}(p_1, \ldots, p_n, p'_1, \ldots, p'_m) \\ (p'_1 + \ldots + p'_m)\hat{W}_0(p'_1, \ldots, p'_m).\end{aligned} \tag{4.7.4}$$

Hence

$$\int d^3p'_1 \ldots d^3p'_m\; \hat{S}(p_1, \ldots, p_n, p'_1, \ldots, p'_m) \left(\sum_{k=1}^{n} p_k - \sum_{l=1}^{m} p'_l\right) \hat{W}_0(p'_1, \ldots, p'_m) = 0 \tag{4.7.5}$$

The vanishing of expression (4.7.5) holds true for arbitrary initial conditions W_0. It follows that the support of $\hat{S}$ must be contained in the region where

$$\left(\sum_{k=1}^{n} p_k - \sum_{l=1}^{m} p'_l\right) = 0. \tag{4.7.6}$$

Hence we may write

$$\hat{S}(p_1, \ldots, p_n, p'_1, \ldots, p'_m) = \delta^3\left(\sum_{k=1}^{n} p_k - \sum_{l=1}^{m} p'_l\right) \cdot \hat{S}'(p_1, \ldots, p_n, p'_1, \ldots, p'_m) \tag{4.7.7}$$

where $\hat{S}'$ is some other function. Let us now compare this expression to the desired form of the S-matrix, (1.1.5). (1.1.5) is the special case of (4.7.7) for 2-to-2 scattering ($n = m = 2$), and it includes an additional delta-function factor

$$\delta\left(\sum_{k=1}^{n} E(p_k) - \sum_{l=1}^{m} E(p'_l)\right) = 0, \tag{4.7.8}$$

implying conservation of energy during scattering. (Here $E(p)$ is the energy corresponding to the momentum p.) However, (4.7.7) does correctly identify the other three delta-function factors implying conservation of momentum during scattering. (Note

that the identity operator Id in (1.1.5) is also of the form (4.7.7): its support is contained in the region (4.7.6).) It is a neat consequence of the requirements of relativity in a configuration space theory. To the author's knowledge, the missing energy-conserving delta-function (4.7.8) cannot be derived from such general principles and requires a specific choice of Hamiltonian. The free Hamiltonian is one such choice, largely because it contains such trivial dynamics. We will study an interacting Hamiltonian in the next chapter, and show that our choice provides this missing delta-function to the S-matrix.

Transitions to and from the vacuum can be treated as a special case. For vacuum initial conditions $W_0 \in \mathbb{C}$, $m = 0$ and the constraint (4.7.5) implies $\hat{S}(p_1, \ldots, p_n) \equiv 0$ for all $n > 0$. So there is no possibility of transition from vacuum to non-vacuum state in this theory. As another special case, the vacuum component of SW_0 is a bounded linear functional from Θ^m to $\mathbb{C}$. Here we may use self-duality of the L^2 spaces, from which it follows that the kernel $\hat{S}(p'_1, \ldots, p'_m)$ is measurable and integrable. According to the constraint (4.7.5), this kernel vanishes except on the set $\{p'_l : \sum p'_l = 0\}$, a set of measure zero. Hence the kernel vanishes almost everywhere and the vacuum component of SW_0 is identically zero for all non-vacuum initial conditions W_0. So there is no possibility of transition from non-vacuum to vacuum state in this theory. This compares well with QED, in which transitions to and from the vacuum are also disallowed.

None of the results in the preceding section relied on the specific structure of the underlying single-particle space (Λ^1 or Θ^1). The bosonic/fermionic Fock space, equivalent to the symmetric/anti-symmetric tensor algebra, can be constructed on *any* vector space Ω and will always provide a representation of the canonical commutation/anti-commutation relations. Moreover, if relativity is implemented in the theory as a Poincaré representation on the single-particle space, the representation will always extend to the second-quantised theory. Under these conditions, one cannot use second-quantisation as an argument for or against specific choices of the single-particle space.

The tensor representations (4.6.1) and (4.6.2) provide a strong additional argument for building field theory from theories formulated as initial-value problems, namely the Dirac lepton and the RS photon. The crucial property of these theories is that they have first-order equations of motion which extend naturally to the multiparticle state spaces

via (4.6.3). A second-order equation of motion such as the wave equation or Klein-Gordon equation would not generalise in this simple way. Standard formulations of field theory generally settle for a representation of the canonical commutation relations (CCRs) on an abstract Hilbert space, performing an ad-hoc replacement of classical ('c-number') amplitudes with operators satisfying the CCRs [36]. Some popular texts (e.g. Peskin and Schroeder [35], Weinberg [33]) justify this by drawing the analogy with the simple harmonic oscillator (SHO), with each mode of the photon corresponding to an independent SHO. These arguments and procedures are ubiquitous in field theory, but I feel that the concepts are vague, while the end-result lacks the concreteness of the configuration-space theory advocated in this thesis. Perhaps in a similar vein, the prominent theorist P. Cvitanović (responsible for one of the early higher-order loop calculations in QED [111, 112]) wrote [113]:

> Usually somebody tells you that the quantum mechanics is obtained from the classical mechanics by replacing Poisson brackets by commutators... this gives me no intuition about quantum mechanics.

In this chapter, I have tried to emphasise the simplicity of second-quantisation of the configuration space theory considered in this thesis. The next chapter will consider a simple deformation of the multiparticle dynamics to incorporate non-trivial scattering into the theory.

Chapter 5

Interactions

Chapter 2 introduced Poincaré representations describing free dynamics of the Dirac lepton and the RS photon. Chapter 3 introduced state spaces for individual particles of each species, and Chapter 4 described free dynamics for systems of arbitrary numbers of leptons and photons. This chapter will present results for the primary aim of the thesis, the deformation of the free theory with a view to recovering interacting dynamics as described by quantum electrodynamics. I will introduce into the equations of motion a coupling between states of different particle content, in view of the known QED transitions between states of different particle number.

I will consider a simplified model introducing a linear coupling between the subspace of states containing one electron and one photon ($p = q = 1$) to the subspace of states containing one electron only ($p = 0, q = 1$). It is then possible to perform a perturbative expansion on the resulting differential equations around a free-particle solution $U_t \in \Theta^1 \otimes \Lambda^1$ describing the passage of one photon and one electron through some region of overlap. At the first order of perturbation, the coupling between $\Theta^1 \otimes \Lambda^1$ and $\Theta^0 \otimes \Lambda^1$ describes the excitation of a state containing one electron only (and no photon). Because the coupling conserves energy-momentum, the one-electron state will be off-shell and cannot excite the propagating modes described by (4.6.4), and in fact this component describes a transient excitation localised around the interaction region. However, back-reaction of the off-shell excitation on the free-particle solution is non-trivial, and enters at the second order of perturbation as $e\gamma \to e\gamma$ elastic scattering. I will evaluate this scattering and show that it agrees with the contribution from one of the leading-order Feynman diagrams for Compton scattering. Thus, the simplified model presented here reproduces a non-trivial part of tree-level scattering as described

by QED. I will also consider a (divergent) contribution to the propagation of a single electron state, also appearing at the second order of perturbation. I will interpret this as a contribution to the electron self-energy as described by QED. These encouraging results suggest that the model may extend to a successful differential formulation of QED. Further extensions to this simple model are required, but these appear to be compatible with the framework presented in this thesis.

5.1 Perturbative expansion

We are interested in the solution of an equation of motion of the form

$$\partial_t W_t = -iHW_t - igRW_t \tag{5.1.1}$$

where H is the free Hamiltonian (solvable in closed form), $g > 0$ is a coupling constant and R is some linear interaction term[1]. In this thesis, H and R are assumed independent of time. (5.1.1) is a *linear* deformation of the free dynamics, in the sense that if V_t and W_t are solutions, then for any $b, c \in \mathbb{C}$ the *superposition* $bV_t + cW_t$ is also a solution. Thus the linear interaction term ensures that superpositions of quantum states are preserved in time. The linear representation of translation in time anticipates the extension to a representation of the full Poincaré group by linear operators – so superpositions of quantum states would also be preserved under change of inertial frame.

The perturbative solution of (5.1.1) is covered in standard texts (e.g. [36, 35, 33]). For completeness, let us briefly review the method. The free dynamics are assumed solvable in closed form, so $e^{\mp iH(t-t_0)}$ is known. We can commence a perturbative analysis of (5.1.1) by introducing $e^{iH(t-t_0)}$ as an integrating factor,

$$\begin{aligned} \partial_t \left(e^{iH(t-t_0)} W_t\right) &= -ige^{iH(t-t_0)} RW_t \\ &= -ig \left(e^{iH(t-t_0)} Re^{-iH(t-t_0)}\right) \left(e^{iH(t-t_0)} W_t\right) \\ &= -igR(t) \left(e^{iH(t-t_0)} W_t\right). \end{aligned} \tag{5.1.2}$$

Here we have introduced

$$R(t) = \left(e^{iH(t-t_0)} Re^{-iH(t-t_0)}\right), \tag{5.1.3}$$

[1] H', H_1 and H_{int} are also common notation for the interaction term. Unfortunately, these clash with the need for both sub- and super-scripts of H elsewhere in this thesis.

the *interaction picture* representation of R. A perturbative solution of (5.1.2) for arbitrary initial conditions W_{t_0} is given by

$$\begin{aligned} e^{iH(t-t_0)}W_t = W_{t_0} &+ (-ig)\int_{t_0}^{t} dt_1\ R(t_1)W_{t_0} + \\ &+ (-ig)^2 \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2\ R(t_2)R(t_1)W_{t_0} + \dots \\ &+ (-ig)^n \int_{t_0}^{t} dt_1 \int_{t_1}^{t} dt_2 \cdots \int_{t_{n-1}}^{t} dt_n\ R(t_n)\dots R(t_2)R(t_1)W_{t_0} + \dots \end{aligned} \tag{5.1.4}$$

(5.1.4) can be verified by direct substitution into (5.1.2): the time derivative of each term in the series evaluates to $-igR(t)$ acting on the previous term. Incidentally, note that the initial conditions W_{t_0} constrain the entire past and future history of the system, so the system (5.1.1) is a well-defined initial-value problem.

Although the multiple integrals of (5.1.4) are taken over the domain $t_0 \le t_1 \le t_2 \le \dots \le t_n \le t$, it is conventional to rewrite the expression so that the integrals are taken over the rectangular domain $t_1, t_2, \dots, t_n \in [t_0, t]$ in order to ease the passage to the limit $t_0 \to -\infty$ and $t \to \infty$. This can be achieved by introducing the time-ordering operation T,

$$T\left(R(t_1)R(t_2)\right) = \begin{cases} R(t_1)R(t_2) & \text{if } t_1 > t_2 \\ R(t_2)R(t_1) & \text{if } t_1 < t_2 \end{cases} \tag{5.1.5}$$

and likewise for higher products. Then (5.1.4) can be rewritten term-by-term as

$$\begin{aligned} e^{iH(t-t_0)}W_t = W_{t_0} &+ (-ig)\int_{t_0}^{t} dt_1\ R(t_1)W_{t_0} + \\ &+ \frac{(-ig)^2}{2!} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\ T\left(R(t_1)R(t_2)\right) W_{t_0} + \dots \\ &+ \frac{(-ig)^n}{n!} \int_{t_0}^{t} dt_1 \cdots \int_{t_0}^{t} dt_n\ T\left(R(t_1)\dots R(t_n)\right) W_{t_0} + \dots \\ &\equiv T\exp\left(-ig\int_{t_0}^{t} dt'\ R(t')\right) W_{t_0}. \end{aligned} \tag{5.1.6}$$

The series (5.1.6) is (by definition) called the *time-ordered exponential*. The leading term in (5.1.4), at zeroth order in g, is simply the free-particle solution

$$W_t = e^{-iH(t-t_0)}W_{t_0} + O(g). \tag{5.1.7}$$

This is not scattering — indeed, by the term *scattering* we refer to the departure or deviation of the exact solution W_t from this free-particle solution. In other words, we will be interested in the asymptotic limit

$$\lim_{t\to\infty,\, t_0\to-\infty} \left(e^{iH(t-t_0)}W_t - W_{t_0}\right). \tag{5.1.8}$$

describing the map between in-states (W_{t_0} as $t_0 \to -\infty$) and out-states (W_t as $t \to \infty$). These are the higher order (in g) terms of (5.1.4). The first of these,

$$-ige^{-iH(t-t_0)} \int_{t_0}^{t} dt_1\ R(t_1)W_{t_0}, \tag{5.1.9}$$

can be heuristically understood as follows. From (5.1.3), the action of $R(t_1)$ on a state W_{t_0} is equivalent to the combined effect of three sequential operations:

- $e^{-iH(t_1-t_0)}$ propagating the state forward in time, from t_0 to t_1, according to the free dynamics H;
- the interaction operator R acting instantaneously on the propagated state, at time t_1; and
- $e^{iH(t_1-t_0)}$ propagating the state backward in time, from t_1 to t_0, again according to the free dynamics H.

According to (5.1.9), the contribution to scattering is the integrated effect of $R(t_1)$ for all t_1 between t_0 and t.

This heuristic interpretation leads to the *diagrammatic interpretation* of the field theory perturbative series [3, 4]. The diagrams in Figure 5.1 denote the $O(g)$-contribution to scattering (5.1.9) in some theory of leptons (straight lines) and photons (wavy lines). The diagrams assume that the interaction term R couples $e\gamma$ to e states, and vice versa. The diagrams depict the three sequential parts of $R(t)$: the free evolution between t_0 and t_1, the instantaneous action of R at t_1, and the free evolution thereafter. According to (5.1.9), the contribution to scattering is the cumulative effect of all such contributions for t_1 integrated between t_0 and t.

In practice, the first-order contributions to scattering in QED invariably vanish. This is one manifestation of the conservation of energy/momentum in real scattering

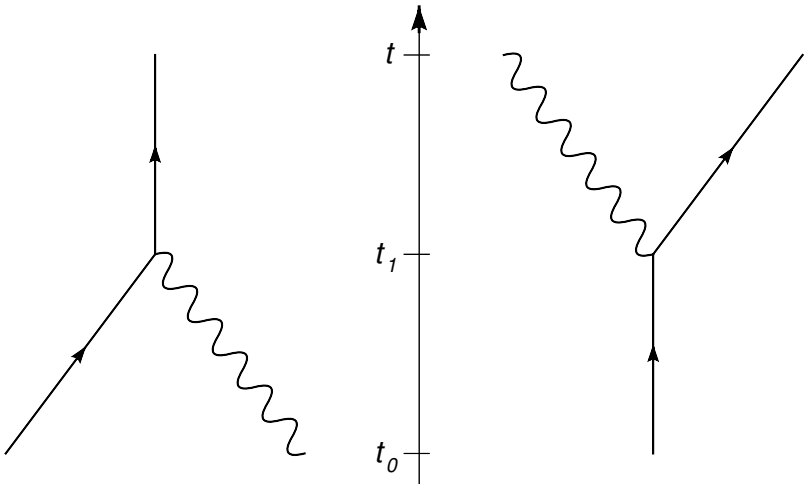


Figure 5.1: $O(g)$ contributions to the perturbative series (5.1.4).

events. To observe non-zero scattering, it is necessary to consider the second-order term in (5.1.4),

$$\frac{(-ig)^2}{2!}e^{-iH(t-t_0)}\int_{t_0}^{t}dt_1\int_{t_0}^{t}dt_2\;T\left(R(t_2)R(t_1)\right)W_{t_0}. \tag{5.1.10}$$

The cumulative effect of $R(t_2)$ and $R(t_1)$ can be interpreted using a very similar heuristic: free dynamics on the intervals $[t_0,t_1),(t_1,t_2),(t_2,t]$ with the interaction operator acting instantaneously at t_1 and t_2. Because of the time-ordering operator T, the contribution to scattering is the integrated effect of such processes for all t_1,t_2 with $t_0<t_1<t_2<t$. In the simple theory with a coupling between $e\gamma$ states and e states, one of the second-order corrections is represented by the diagram in Figure 5.2. The elements of the diagram have the same interpretation as the $O(g)$ contributions, with the lines representing free evolution and the vertices corresponding to the instantaneous action of R at t_1 and t_2. In this way, diagrams each represent a specific contribution to the perturbative series (5.1.4). Each vertex of the diagram corresponds to one of the variables of integration t_j in (5.1.6), so a diagram with n vertices represents some term in the expansion of the $O(g^n)$ term in the series.

These examples were chosen to resemble a field theory with particle content similar to QED, but the perturbative expansion and the diagrammatic method are quite general, and they apply to any Hamiltonian theory which can be written in the $H+gR$ form (5.1.1). We will demonstrate this in the next section with an interacting field theory stripped down to its barest essentials. The example will illustrate some of the

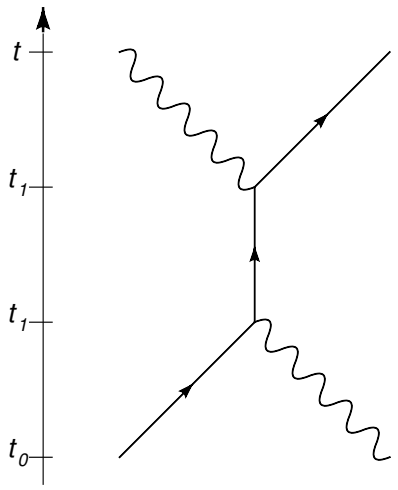


Figure 5.2: An $O(g^2)$ contribution to the perturbative series (5.1.4).

inherent limitations and shortcomings of the perturbative expansion, and will be a useful precursor to the more involved analysis of a QED-like theory later in the chapter.

5.2 Example: displaced SHO as a field theory

The simplest possible example of a field theory is a system for which a single particle has only one available state. In other words, the single-particle state space Θ^1 is equal to $\mathbb{C}$. Let e be any state; it forms a basis for Θ^1. We will consider the second-quantised system Θ with Bose statistics[2]. According to Chapter 4, a basis for the state space for the second-quantised system is

$$1, e, e \otimes e, \ldots, \underbrace{e \otimes \ldots \otimes e}_{n \text{ copies}}, \ldots \tag{5.2.1}$$

denoting states in which the system has $0, 1, 2, \ldots, n, \ldots$ particles respectively. We adopt a concise notation borrowed from the quantum theory of the simple harmonic oscillator, writing (5.2.1) as

$$|0\rangle, |1\rangle, |2\rangle, \ldots, |n\rangle, \ldots \tag{5.2.2}$$

where $|n\rangle$ is the state in which the system has n particles. According to the conventions of Chapter 4, $|n\rangle$ are orthogonal but not orthonormal, with

$$\langle n|n\rangle = n! \,, \qquad\qquad n = 0, 1, 2, \ldots \tag{5.2.3}$$

[2]under Fermi statistics, the second-quantisation of a one-dimensional state space would be almost trivial since $\Theta^1 \wedge \Theta^1 = \text{Span}(e \wedge e) = \{0\}$

(This convention helps avoid obscure factors of $\sqrt{n!}$ appearing in various results presented in this following.) Thus the typical state $W \in \Theta$ is of the form

$$W = \sum_{n=0}^{\infty} W_n |n\rangle \tag{5.2.4}$$

such that

$$\|W\|^2 = \sum_{n=0}^{\infty} n! |W_n|^2 < \infty. \tag{5.2.5}$$

Θ is a Hilbert space with inner product

$$\langle V, W \rangle = \sum_{n=0}^{\infty} n!\ V_n^* W_n. \tag{5.2.6}$$

In this system there is only one creation operator $a_e^\dagger \equiv a^\dagger$, defined following Chapter 4 as

$$a^\dagger W = e \odot W. \tag{5.2.7}$$

It is a brief exercise to show

$$\begin{aligned} a^\dagger |n\rangle &= |n+1\rangle \\ a|n\rangle &= n|n-1\rangle. \end{aligned} \tag{5.2.8}$$

The canonical commutation relations (CCRs) immediately follow,

$$[a^\dagger, a^\dagger] = [a, a] = 0 \tag{5.2.9}$$

$$[a, a^\dagger] = 1. \tag{5.2.10}$$

Defined in this way, the second-quantised state space Θ contains sufficient structure to introduce some simple dynamics, together with a small deformation representing an interaction. We will use this example to illustrate perturbative expansion about a free solution.

For the free dynamics, we simply suppose $|n\rangle$ describes a state with n non-interacting particles of mass $m > 0$ at rest, and by appropriate choice of units we may put $m = 1$. Thus we may write down a Hamiltonian operator,

$$H|n\rangle = n|n\rangle \tag{5.2.11}$$

H is unbounded (since $\|H\| \geq N$ for all $N \in \mathbb{N}$) but self-adjoint. It is a brief exercise from (5.2.8) to show $H = a^\dagger a$. The free equation of motion is

$$\partial_t W_t = -iHW_t. \tag{5.2.12}$$

H is diagonal with respect to the basis (5.2.2), so the spectrum of H is

$$\sigma(H) = \mathbb{N} \cup \{0\} \tag{5.2.13}$$

and the free dynamics can be easily solved in closed form. For initial conditions $W_{t_0} = |n\rangle$, the solution to (5.2.12) is

$$W_t = \exp(-in(t - t_0))|n\rangle, \tag{5.2.14}$$

and may be extended to arbitrary initial conditions by linearity. Once we introduce an interaction, we will perform the perturbative expansion about these free dynamics.

We will consider a simple interaction term R modifying the free Hamiltonian H (cf. equation (5.1.1)). The distinguishing feature of an interacting field theory is the coupling of states of different particle number. A simple choice is

$$R = (a^\dagger + a), \tag{5.2.15}$$

and this will provide our illustrative example for this section. R is not diagonal with respect to the basis of free particle states (5.2.2) and mixes (or *couples*) the free particle states over time. We will pursue two contrasting analyses. First, we will perform a direct analysis on the interacting system to determine the nature and extent of its spectrum. We will then contrast these results to the somewhat more ambiguous results arising from the standard perturbative/diagrammatic techniques of field theory. The perturbative expansion shares some recognisable features with real field theories.

Towards a direct analysis of the equation of motion (5.1.1), the operator $H + gR$ acts on basis states $|n\rangle$ as

$$(H + gR)|n\rangle = n|n\rangle + g|n + 1\rangle + gn|n - 1\rangle. \tag{5.2.16}$$

Hence, $H + gR$ can be written explicitly as a tridiagonal matrix with respect to the

basis (5.2.2),

$$\begin{bmatrix} 0 & g & & & \\ g & 1 & 2g & & \\ & g & 2 & 3g & \\ & & g & 3 & 4g \\ & & & g & \ddots \end{bmatrix} \tag{5.2.17}$$

(where all other entries are 0). The mixing of free particle states is evident from the off-diagonal terms for $g > 0$. Eigenvalues of (5.2.17) are not obvious at first glance but can be readily obtained from the relation

$$\begin{aligned} H + gR &= a^\dagger a + g(a^\dagger + a) \\ &= (a^\dagger + g)(a + g) - g^2 \end{aligned} \tag{5.2.18}$$

with $(a^\dagger + g)$ and $(a + g)$ satisfying the CCRs. (5.2.18) shows that the 'interaction' we have introduced is almost trivial. Indeed the 'interacting' system (5.2.18) does not differ essentially from the free system, apart from a shift of $-g^2$ in the spectrum. The spectrum is simply

$$\sigma(H + gR) = (\{0\} \cup \mathbb{N}) - g^2 \tag{5.2.19}$$

The ground state W_g is easily constructed by solving

$$(a + g)W_g = 0, \tag{5.2.20}$$

an equation well-known in quantum optics as defining the *coherent state* [114, 115, 116]. The (normalised) solution can be written as

$$W_g = e^{-\frac{1}{2}g^2} \sum_{p=0}^{\infty} \frac{(-g)^p}{p!} |p\rangle. \tag{5.2.21}$$

We observe that 'free' solution $|0\rangle$ is not a solution of the interacting dynamics. The non-zero coupling g has introduced contributions of the higher states $|1\rangle, |2\rangle, \ldots$ to the ground state $|0\rangle$ of the free theory, and shifted the energy by $-g^2$. A complete basis of eigenstates of $H + gR$ may be produced by iterating the ladder operator $(a^\dagger + g)$ on W_g, namely

$$W_g, (a^\dagger + g)W_g, \ldots, (a^\dagger + g)^n W_g, \ldots \tag{5.2.22}$$

Thus the 'interacting' theory with $g > 0$ may be regarded as solved in closed form. The closed-form solution was a special feature of such a simple system and in general will not be available. Let us apply the perturbative techniques of the previous section and draw the comparison.

The interaction picture representation of R was defined in (5.1.3) and may be evaluated as

$$\begin{aligned} R(t)|n\rangle &= e^{iH(t-t_0)}\left(a^\dagger + a\right)e^{-iH(t-t_0)}|n\rangle \\ &= e^{iH(t-t_0)}\left(a^\dagger + a\right)e^{-in(t-t_0)}|n\rangle \\ &= e^{iH(t-t_0)}e^{-in(t-t_0)}\left(|n+1\rangle + n|n-1\rangle\right) \\ &= e^{i(t-t_0)}|n+1\rangle + e^{-i(t-t_0)}n|n-1\rangle \end{aligned} \tag{5.2.23}$$

Therefore

$$R(t) = e^{i(t-t_0)}a^\dagger + e^{-i(t-t_0)}a. \tag{5.2.24}$$

The leading-order contribution to the scattering is given by (5.1.9). The integral can be evaluated, giving

$$-ig\int_{t_0}^{t} dt_1\, R(t_1) = -ig\left(-ie^{i(t-t_0)} + i\right)a^\dagger + -ig\left(ie^{-i(t-t_0)} - i\right)a \tag{5.2.25}$$

and suggesting that the interaction has perturbed the free eigenstates $|n\rangle$ by introducing $O(g)$ contributions of the adjacent eigenstates $|n \pm 1\rangle$. The standard approach of quantum field theory is to take the limit $t \to \infty, t_0 \to -\infty$ — but the limit clearly does not exist for the oscillatory terms in (5.2.25). If we simply ignored these terms, the $O(g)$ correction to the ground state would be

$$\begin{aligned} W_g &= -ig(ia^\dagger - ia)|0\rangle + O(g^2) \\ &= g|1\rangle + O(g^2) \end{aligned} \tag{5.2.26}$$

which compares favourably with the exact expression (5.2.21) for the perturbed ground state. Likewise, the $O(g)$ correction to the first excited state would be

$$\begin{aligned} W &= -ig(ia^\dagger - ia)|1\rangle + O(g^2) \\ &= g|2\rangle - g|0\rangle + O(g^2). \end{aligned} \tag{5.2.27}$$

and contains components of both $|0\rangle$ and $|2\rangle$. The $g|2\rangle$ term in (5.2.27) corresponds to a diagram similar to the left-hand diagram of Fig 5.1, coupling a single-particle state

to a two-particle state. The $g|0\rangle$ term in (5.2.27) couples a single-particle state to the vacuum and corresponds to a diagram with one ingoing leg terminating at a vertex. It would be an interesting further study to look in depth at the intricacies of the full perturbative series of this system; perhaps the troublesome oscillatory terms cancel with other terms elsewhere in the series. However, we leave the further details of this simple system as an open problem and turn our attention to a more realistic field theory.

The preceding analysis illustrated how the perturbative method may be used to solve interacting Hamiltonian systems of $H + gR$ form, and how the terms of the resulting series may be conveniently denoted by diagrams. However, this example also illustrates that if a direct analysis of the underlying system is available, it may offer far more insight into the actual features of the theory.

The time-ordered perturbative method is standard material in an introductory field theory course and is covered by many standard texts [36, 35, 33]). However, none of these texts actually specify what H and the R should be for a field theory such as QED, or what space they act on. In §5.4, I will consider a theory of $H + gR$ form as a simple model for QED. I will first briefly review another elementary example of an initial-value problem that can be solved perturbatively [107, 117, 92], such that the terms in the perturbative expansion also have a simple representation by diagrams.

5.3 Example: Dirac particle in an external potential

The propagation of a spin-half particle ψ in an electromagnetic field A_μ has been considered by [107, 117, 92]. The equation of motion is the Dirac equation,

$$(i\partial\!\!\!/ - m)\psi(x) = eA\!\!\!/(x)\psi(x) \tag{5.3.1}$$

for $x = (t, x)$. Fourier transformed, (5.3.1) reads

$$i\partial_t\hat{\psi}_t(k) = \left(\alpha^j k_j + \beta m\right)\hat{\psi}_t(k) + \gamma^0 e\mathcal{F}\left(A\!\!\!/_t\psi_t\right)(k) \tag{5.3.2}$$

where $\mathcal{F}$ denotes the Fourier transform. $\mathcal{F}(A\!\!\!/_t\psi_t)$ is well-defined only if $A\!\!\!/$ is bounded measurable ($A_\mu \in L^\infty$) for all t; the following will assume this condition is met. By direct computation, (5.3.2) is equivalent to the integral equation

$$\hat{\psi}_t(k) = \hat{\phi}_t(k) + e\int dt'\ \hat{S}_{t-t'}(k)\ \mathcal{F}\left(A\!\!\!/_{t'}\psi_{t'}\right)(k) \tag{5.3.3}$$

where $\hat{\phi}_t$ is any free space solution. Taking the Fourier transform to return to the space representation, (5.3.3) may be written as the convolution integral

$$\psi_t(x) = \phi_t(x) + e \int dt' d^3x' \; S_{t-t'}(x - x') \; \not{A}_{t'}(x')\psi_{t'}(x'). \tag{5.3.4}$$

We may write (5.3.3) or (5.3.4) concisely as

$$\psi = \phi + K\psi \tag{5.3.5}$$

with the K defined by (5.3.3). Further conditions on the potential A_μ are required to ensure K is well-defined and bounded on Λ^1; a simple choice is $A_\mu \neq 0$ only for some finite interval of time, $a \leq t \leq b$. Then the integrand in (5.3.3) is non-zero only for $a \leq t \leq b$, and the integral necessarily converges. This restriction on A_μ, together with $\|S_t\| = 1$, provides an estimate on the operator-norm of K,

$$\|K\| \leq e(b-a) \sup_{t\in[a,b]} \|\not{A}_t\|_\infty. \tag{5.3.6}$$

Equation (5.3.5), a Fredholm equation, has a unique solution if and only if $\|K\| < 1$. According to (5.3.6), this is generally true when each of e, A_μ and $(b-a)$ are small. These correspond to small coupling, weak field, and short duration respectively. Under these conditions, the solution to (5.3.5) is the absolutely convergent perturbative series

$$\psi = (1 + K + K^2 + K^3 + \ldots)\phi. \tag{5.3.7}$$

Assuming condition (5.3.6) is met, convergence of the series (5.3.7) is trivial in the Hilbert space Λ^1, by completeness. In contrast, convergence within the Schwartz space is not guaranteed. This is an example of how completeness of the state space can simplify the conceptual structure of the theory.

The perturbative series can be interpreted diagrammatically as in Figure 5.3. For example, the second-order term $K^2\phi$ corresponds to the diagram with two photon legs. We can explicitly evaluate this term by substituting twice for K from (5.3.4), obtaining

$$\begin{aligned}(K^2\phi)_t(x) &= e \int_{t_0}^{t} dt_2 \; d^3x_2 \; S_{t-t_2}(x - x_2) \; \not{A}_{t_2}(x_2)(K\phi)_{t_2}(x_2) \\ &= e^2 \int_{t_0}^{t} dt_2 \; d^3x_2 \; \int_{t_0}^{t_2} dt_1 \; d^3x_1 \; S_{t-t_2}(x - x_2) \; \not{A}_{t_2}(x_2)\times \\ &\qquad \times S_{t_2-t_1}(x_2 - x_1) \; \not{A}_{t_1}(x_1)\phi_{t_1}(x_1).\end{aligned} \tag{5.3.8}$$

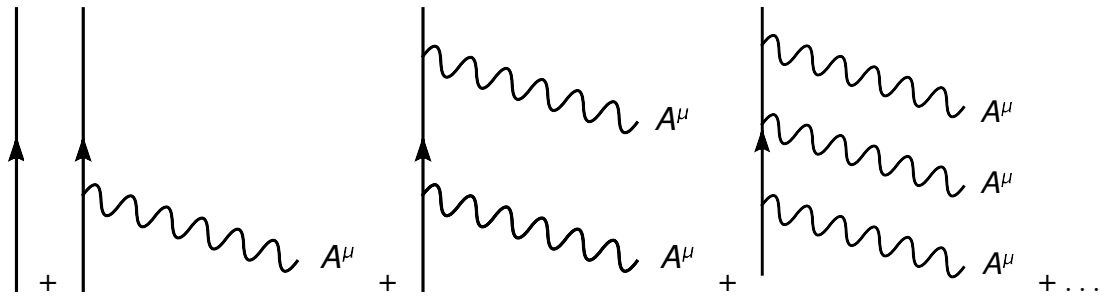


Figure 5.3: Perturbative series for a lepton propagating in an external potential A_μ.

From this expression, we can see that the diagrams of Figure 5.3 are consistent with the diagrammatic notation introduced in §5.1. The vertices represent one of the variables of integration t_1 or t_2, the vertices represent the instantaneous action of the interaction at t_1 or t_2, and legs of the diagram represent free propagation outside from those times. The expression as a whole is the integrated effect of all such processes with $t_0 < t_1 < t_2 < t$.

Equipped with the perturbative expansion and the diagrammatic method, I will now apply these to a simple deformation of the free, second-quantised theory constructed in Chapters 2–4.

5.4 A simple model for electrodynamics

Chapter 4 introduced the configuration space representation for systems of arbitrary number of free photons and free leptons. In this representation, the system is uniquely described by a state W equal to the orthogonal sum of components $W_{(p,q)}$ representing states of definite particle content,

$$W = \sum_{p,q=0,1,2,\ldots} W_{(p,q)}, \tag{5.4.1}$$

Here each $W_{(p,q)}$ is a configuration space amplitude representing the component containing p photons and q leptons; it is a function on $p+q$ copies of space with $p+q$ spin indices,

$$W_{(p,q)}^{r_1\ldots r_p s_1\ldots s_q}(x_1,\ldots,x_p,y_1,\ldots,y_q). \tag{5.4.2}$$

Here $r_1,\ldots,r_p \in \{1,2,3\}$ are spin indices for photons in the Riemann-Silberstein representation, and $s_1,\ldots,s_q \in \{1,2,3,4\}$ are Dirac spin indices for leptons. x_j and

y_j are space coordinates, i.e. $x_j, y_k \in \mathbb{R}^3$. According to Chapter 2, the variation of W in time or under a change of inertial frame is described by a representation of the Poincaré group on the space of states.

We will consider a model where the system is described by a state W_t whose two components are an e state (belonging to $\Theta^0 \otimes \Lambda^1$) and an $e\gamma$ state (belonging to $\Theta^1 \otimes \Lambda^1$). Thus

$$W = \begin{bmatrix} W^s_{(0,1)}(y) \\ W^{rs}_{(1,1)}(x,y) \end{bmatrix} \tag{5.4.3}$$

where time dependence (the subscript t) has been made implicit. Chapter 4 constructed a free, relativistic theory on such a system, with Hamiltonian

$$H = \begin{bmatrix} H_{(0,1)} & 0 \\ 0 & H_{(1,1)}. \end{bmatrix} \tag{5.4.4}$$

Here $H_{(p,q)}$ are the appropriate free Hamiltonians given in (4.6.4) and (4.6.7). Note that $e^{\mp iH(t-t_0)}$ is the time evolution of the free theory, and is known in closed form: (4.6.8) gave the tensor representation in terms of the single-particle propagators, and the single-particle propagators were evaluated in Chapter 3. Explicitly,

$$e^{-iHt} = \begin{bmatrix} S_t & 0 \\ 0 & D_t \otimes S_t \end{bmatrix} \tag{5.4.5}$$

where

$$(S_t W)^s(y) = \int d^3y' \; S_t^{ss'}(y - y') W^{s'}(y') \tag{5.4.6}$$

$$((D_t \otimes S_t)W)^{rs}(x,y) = \int d^3x' d^3y' \; D_t^{rr'}(x - x') S_t^{ss'}(y - y') W^{r's'}(x', y'), \tag{5.4.7}$$

with S_t and D_t given in (3.9.8) and (3.10.11). In line with the discussion of §4.7, we will consider a self-adjoint deformation of the free Hamiltonian, which we write in the form

$$H \mapsto H + gR \tag{5.4.8}$$

where R is a linear operator representing the deformation of H to an interacting theory. $g > 0$ is a dimensionless coupling constant. Within this framework, we will choose R to couple $(\Theta^0 \otimes \Lambda^1)$ to $(\Theta^1 \otimes \Lambda^1)$. With R having non-zero terms off the diagonal

(compare with the free Hamiltonian (5.4.4)), the dynamics will couple the components of different particle content. The dynamics are of the $H + gR$ form (5.1.1), and we will use the time-ordered exponential discussed in §5.1 to compute the scattering. Finally, we will compare the results to quantum electrodynamics.

We will now make a specific choice for the interaction operator R, and define R on $(\Theta^0 \otimes \Lambda^1) \oplus (\Theta^1 \otimes \Lambda^1)$ by

$$\begin{bmatrix} (RW)^s_{(0,1)}(y) \\ (RW)^{rs}_{(1,1)}(x,y) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{m}\int d^3x\ \delta^3(x-y)\alpha^{r'ss'} \\ \delta^3(x-y)\alpha^{rss'} & 0 \end{bmatrix} \begin{bmatrix} W^{s'}_{(0,1)}(y) \\ W^{r's'}_{(1,1)}(x,y) \end{bmatrix}, \tag{5.4.9}$$

where m is the electron mass. The factor $1/m$ has been inserted to ensure consistency of dimensions. According to the equation of motion (5.1.1), this choice of R is equivalent to the coupled system of partial differential equations

$$\frac{\partial}{\partial t} W^s_{(0,1)}(y) = -iHW^s_{(0,1)}(y) + \frac{-ig}{m}\alpha^{rss'} W^{rs'}_{(1,1)}(y,y) \tag{5.4.10}$$

$$\frac{\partial}{\partial t} W^{rs}_{(1,1)}(x,y) = -iHW^{rs}_{(1,1)}(x,y) - ig\ \delta^3(x-y)\alpha^{rss'} W^{s'}_{(0,1)}(y) \tag{5.4.11}$$

This form exhibits coupling between the components $W_{(0,1)}$ and $W_{(1,1)}$ of different particle content. (5.4.10) is the free Dirac equation for a lepton state, with a source term depending on $W_{(1,1)}(y,y)$. If the source term is non-zero, the equation implies the excitation of modes of $W_{(0,1)}$ not present in the initial conditions, and this is how scattering arises within the theory. In particular, as a configuration space amplitude for the $e\gamma$ state, $W_{(1,1)}$ is non-zero at (y,y) if and only if there is non-zero probability of both lepton and photon coinciding at y. In other words, scattering occurs where the lepton and photon can be found in overlapping regions of space.

Similarly, (5.4.11) is the free-particle equation of motion for an $e\gamma$ state, with a source term depending on $W_{(0,1)}$. This coupling excites modes of $W_{(1,1)}$ wherever $W_{(0,1)}$ is non-zero, including the incoming and outgoing states. The free-particle e solution describing a single propagating lepton will now include an $O(g)$ correction in the $e\gamma$ subspace. This correction is qualitatively similar to the $O(g)$ contribution to the ground state of the displaced SHO (5.2.26).

A particularly interesting result arises at $O(g^2)$ in perturbation theory about a free-particle $e\gamma$ state. In the region where the particles overlap (the collision region), a

transient e state is excited at $O(g)$. The back-reaction of this transient on the $e\gamma$ state will be observed as scattering. Thus, scattering will arise as a combination of the above two effects. Let us now explicitly calculate the scattering using the perturbative expansion (5.1.4), and find the contributions at $O(g)$ and $O(g^2)$.

Firstly, $H + gR$ is translation-invariant and therefore satisfies one of the criteria for the Poincaré test of §2.2. Additionally, from §4.7, the translation-invariance implies that scattering will conserve momentum (the S-matrix will include the required δ^3 function).

From here on I will adopt a different notation to aid readability: let us drop the subscript (p, q) on the components $W_{(p,q)}$, because the particle content of any component can be inferred from the number/type of spin indices (r, s etc.) and space arguments (x, y etc.). (Compare with (5.4.2).) Also we will write

$$\begin{bmatrix} W^s_{(0,1)}(y) \\ W^{rs}_{(1,1)}(x,y) \end{bmatrix} = \begin{bmatrix} \langle s,y| \\ \langle r,x;s,y| \end{bmatrix} W. \tag{5.4.12}$$

This change is inspired by the concept and notation of the Dirac 'bra-state' $\langle s, y|$. The bra-state is generally defined as a distribution (generalised function)

$$\langle s,y|^{s'}(y') = \delta(y-y')\delta^{ss'} \tag{5.4.13}$$

(see, e.g. [41]). Here $\delta^{ss'}$ is the usual Kronecker symbol. This thesis does not discuss the time evolution or other properties of such a 'state'; in fact the delta-function (5.4.13) is not square-integrable, and therefore not a state as defined in this thesis: $\langle s, y|$ does not belong to either Θ^1 or Λ^1. We will regard (5.4.12) as strictly a change of notation only; a relation such as

$$\langle s,y|W = W^s_{(0,1)}(y) \tag{5.4.14}$$

should be regarded merely as two equivalent types of notation for a square-integrable function $W_{(0,1)}$ on $\mathbb{R}^3$. The bra-notation seems to be slightly easier to read. Likewise, a relation in Fourier space

$$\langle s,q|W = \hat{W}^s_{(0,1)}(q) \tag{5.4.15}$$

should also be regarded merely as equivalent notation for a square-integrable function $\hat{W}_{(0,1)}$; there is no intention to introduce a 'momentum eigenstate' $\langle s, q|$ as an indepen-

dent object. In thesis, the term *state* is reserved for the square-integrable amplitudes (and tensor products and superpositions thereof) introduced in earlier chapters.

According to the perturbation theory of §5.1, the interacting system (5.4.10)–(5.4.11) admits a perturbative solution via the time-ordered exponential (5.1.6). Scattering arises from the higher-order terms in the perturbative series, and as a first step towards evaluating these higher-order perturbative terms, let us consider evaluation of the interaction-picture operator $R(t)$ defined in (5.1.3). It is simpler to switch to Fourier space at this point. Taking the Fourier transform of (5.4.9) in all space variables yields

$$\begin{bmatrix} \langle s,q| \\ \langle r,p;s,q| \end{bmatrix} RW = \begin{bmatrix} 0 & \mathbf{II} \\ \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \langle s',q'| \\ \langle r',p';s',q'| \end{bmatrix} W \tag{5.4.16}$$

where

$$\mathbf{I} = \int d^3q' \alpha^{rss'} \delta^3(q'-p-q) \tag{5.4.17}$$

and

$$\mathbf{II} = \frac{1}{m} \int \frac{d^3p' d^3q'}{(2\pi)^3}\ \alpha^{r'ss'} \delta^3(p'+q'-q). \tag{5.4.18}$$

Likewise, the propagator (5.4.5) reads

$$\begin{bmatrix} \langle s,q| \\ \langle r,p;s,q| \end{bmatrix} e^{-iH(t-t_0)} W = \begin{bmatrix} \hat{S}^{ss'}_{t-t_0}(q) & 0 \\ 0 & \hat{D}^{rr'}_{t-t_0}(p)\hat{S}^{ss'}_{t-t_0}(q) \end{bmatrix} \begin{bmatrix} \langle s',q| \\ \langle r',p;s',q| \end{bmatrix} W. \tag{5.4.19}$$

We can use (5.4.16) and (5.4.19) to evaluate $R(t)$ as the product (5.1.3). The product is fairly straightforward to evaluate but some care is needed to keep track of all the spin indices. The result is

$$\begin{bmatrix} \langle s,q| \\ \langle r,p;s,q| \end{bmatrix} R(t_1) W = \begin{bmatrix} 0 & \mathbf{II} \\ \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \langle s_0,q'| \\ \langle r_0,p';s_0,q'| \end{bmatrix} W_{t_0}, \tag{5.4.20}$$

where

$$\mathbf{I} = \int d^3q'\ \hat{D}^{rr_1}_{t_0-t_1}(p) \hat{S}^{ss_1}_{t_0-t_1}(q) \alpha^{r_1s_1s'_1} \hat{S}^{s'_1s_0}_{t_1-t_0}(q') \delta^3(p+q-q') \tag{5.4.21}$$

and

$$\mathbf{II} = \frac{1}{m} \int \frac{d^3p' d^3q'}{(2\pi)^3}\ \hat{S}^{ss_1}_{t_0-t_1}(q) \alpha^{r'_1s_1s'_1} \hat{D}^{r'_1r_0}_{t_1-t_0}(p') \hat{S}^{s'_1s_0}_{t_1-t_0}(q') \delta^3(p'+q'-q). \tag{5.4.22}$$

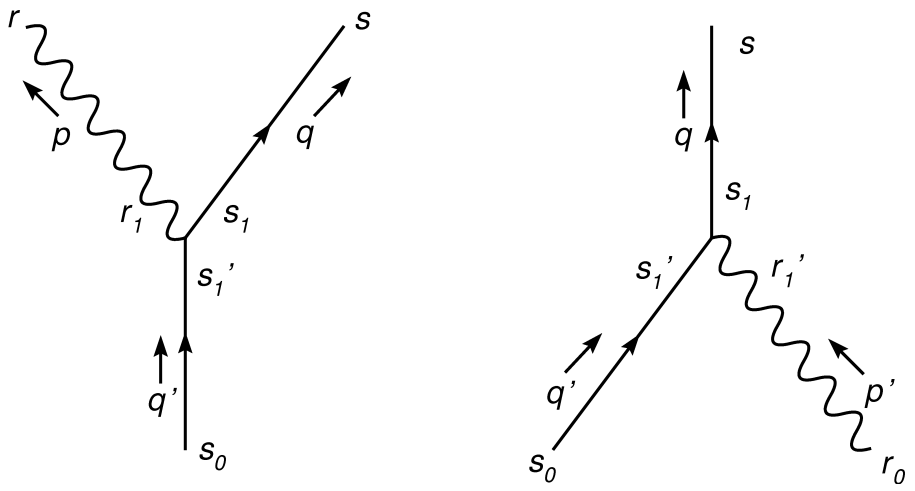


Figure 5.4: L-R: diagrams for expressions **I** (5.4.21) and **II** (5.4.22).

We will abbreviate these expressions by the diagrams shown in Figure 5.4. In the same way that Feynman diagrams are the abbreviated, or shorthand, representation of the mathematical expressions for matrix elements (such as (1.2.1)), the diagrams in Figure 5.4 are also intended as a shorthand for mathematical expressions similar to **I** and **II**. Each leg of the diagram represents a free propagator: wavy lines for the free photon propagator $\hat{D}$ and straight lines for the free lepton propagator $\hat{S}$. The arguments and spin indices of each propagator are marked on the diagram. Vertices represent contraction with the Dirac α matrices.

For the sake of completeness, the space representation of (5.4.20) is obtained by taking Fourier transforms in all variables, and may be written down as

$$\begin{bmatrix} \langle s,y| \\ \langle r,x;s,y| \end{bmatrix} R(t_1)W_{t_0} = \begin{bmatrix} 0 & \mathbf{II} \\ \mathbf{I} & 0 \end{bmatrix} \begin{bmatrix} \langle s_0,y_0| \\ \langle r_0,x_0;s_0,y_0| \end{bmatrix} W_{t_0}, \tag{5.4.23}$$

where

$$\mathbf{I} = \int d^3y_1 d^3y_0 \; D^{rr_1}_{t_0-t_1}(x-y_1) S^{ss_1}_{t_0-t_1}(y-y_1) \alpha^{r_1 s_1 s_1'} S^{s_1' s_0}_{t_1-t_0}(y_1-y_0) \tag{5.4.24}$$

and

$$\mathbf{II} = \frac{1}{m} \int d^3y_1 d^3x_0 d^3y_0 \; S^{ss_1}_{t_0-t_1}(y-y_1) \alpha^{r_1' s_1 s_1'} D^{r_1' r_0}_{t_1-t_0}(y_1-x_0) S^{s_1' s_0}_{t_1-t_0}(y_1-y_0). \tag{5.4.25}$$

We can now compute some terms in the series.

5.5 Vanishing contribution at $O(g)$

The $O(g)$ contribution to the solution of the interacting system is given in (5.1.6) as

$$(-ig)\int_{t_0}^{t} dt_1\ R(t_1)W_{t_0} \tag{5.5.1}$$

We can evaluate this term and show that it vanishes in the asymptotic limit ($t_0 \to -\infty, t \to \infty$). This term describes a non-propagating, transient excitation in $\Theta^0 \otimes \Lambda^1$, localised both in space and time to the region of overlap of the colliding electron and photon wavepackets.

We will treat the e and $e\gamma$ components separately. $R(t_1)W_{t_0}$ is given in (5.4.20), and the lower ($e\gamma$) component reads

$$\langle r,p;s,q|R(t_1)W = \int d^3q'\ \hat{D}^{rr_1}_{t_0-t_1}(p)\hat{S}^{ss_1}_{t_0-t_1}(q)\alpha^{r_1s_1s'_1}\hat{S}^{s'_1s_0}_{t_1-t_0}(q')\delta^3(p+q-q')\langle s',q'|W \tag{5.5.2}$$

Consequently, the first-order contribution to scattering (5.5.1) is

$$\begin{aligned}\langle r,p;s,q|(-ig)\int_{t_0}^{t} dt_1\ R(t_1)W \\ = (-ig)\int d^3q' \int_{t_0}^{t} dt_1\ \hat{D}^{rr_1}_{t_0-t_1}(p)\hat{S}^{ss_1}_{t_0-t_1}(q)\alpha^{r_1s_1s'_1}\hat{S}^{s'_1s_0}_{t_1-t_0}(q')\delta^3(p+q-q')\langle s',q'|W.\end{aligned} \tag{5.5.3}$$

This expression describes the excitation of an $e\gamma$ component of W from initial conditions containing e only. Obviously, such a scattering event is disallowed in QED, by relativistic kinematics, and the S-matrix element for any such process must vanish. So a proof that this expression vanishes is consistent with QED.

We will substitute the closed-form expressions (3.10.11) and (3.9.8) for the propagators $\hat{D}$ and $\hat{S}$, conveniently written in abbreviated form

$$\hat{D}_t(k) = \sum_{\eta=-1,0,1} P_\eta(k)e^{-i\eta|k|t} \tag{5.5.4}$$

$$\hat{S}_t(k) = \sum_{\xi=-1,1} Q_\xi(k)e^{-i\xi E(k)t}. \tag{5.5.5}$$

(5.5.3) becomes

$$\begin{aligned}\langle r,p;s,q|(-ig)\int_{t_0}^{t}dt_1\ R(t_1)W\\ =(-ig)\int d^3q'\int_{t_0}^{t}dt_1\sum_{\substack{\eta=0,\pm1\\ \xi,\xi'=\pm1}}P_\eta^{rr_1}(p)Q_\xi^{ss_1}(q)\alpha^{r_1s_1s_1'}Q_{\xi'}^{s_1's_0}(q')\delta^3(p+q-q')\\ \exp\Big(-i\eta|p|(t_0-t_1)-i\xi E(q)(t_0-t_1)-i\xi'E(q')(t_1-t_0)\Big)\langle s',q'|W.\end{aligned} \tag{5.5.6}$$

The integral over t_1 can be done in the limit $t_0\to-\infty, t\to\infty$,

$$\begin{aligned}\int_{t_0}^{t}dt_1\ \exp\Big(-i\eta|p|(t_0-t_1)-i\xi E(q)(t_0-t_1)-i\xi'E(q')(t_1-t_0)\Big)\\ =2\pi\delta(\eta|p|+\xi E(q)-\xi'E(q')).\end{aligned} \tag{5.5.7}$$

Hence the $O(g)$-contribution to scattering is

$$\begin{aligned}\langle r,p;s,q|(-ig)\int_{t_0}^{t}dt_1\ R(t_1)W\\ =(-ig)\int d^3q'\sum_{\substack{\eta=0,\pm1\\ \xi,\xi'=\pm1}}P_\eta^{rr_1}(p)Q_\xi^{ss_1}(q)\alpha^{r_1s_1s_1'}Q_{\xi'}^{s_1's_0}(q')\\ 2\pi\delta^3(p+q-q')\delta(\eta|p|+\xi E(q)-\xi'E(q'))\langle s',q'|W.\end{aligned} \tag{5.5.8}$$

This expression is a sum of 12 terms corresponding to the summations over $\eta=0,\pm1$ and $\xi,\xi'=\pm1$. We will now show that all 12 terms vanish individually.

Firstly, consider the 8 terms corresponding to $\eta=\pm1$. The four delta-functions in (5.5.8) are equivalent to the constraints

$$\begin{aligned}p+q&=q'\\ \eta|p|+\xi E(q)&=\xi'E(q')\end{aligned} \tag{5.5.9}$$

on the domain of integration. We will show that these constraints have no solution for all $p,q\in\mathbb{R}^3$, so the domain of integration over q' in (5.5.8) is empty. Recall $E(q')=\sqrt{|q'|^2+m^2}$ by definition. Hence

$$\begin{aligned}m^2&=E(q')^2-|q'|^2\\ &=(\eta|p|+\xi E(q))^2-|p+q|^2\end{aligned} \tag{5.5.10}$$

where the second line uses (5.5.9). Expanding,

$$\begin{aligned}m^2&=|p|^2+E(q)^2+2\eta\xi|p|E(q)-|p|^2-|q|^2-2p\cdot q\\ 0&=2\eta\xi|p|E(q)-2p\cdot q.\end{aligned} \tag{5.5.11}$$

The last line violates the Schwartz identity ($|p \cdot q| \leq |p||q|$), so there is no solution. In this way, the constraints (5.5.8) are not met for any $p, q, q' \in \mathbb{R}^3$, so at least one of the four delta-functions in (5.5.8) is zero over the entire domain of integration. So terms with $\eta = \pm 1$ are zero identically.

Incidentally, there is a close relation between the constraints (5.5.9) and the kinematic constraints of 2-to-1 relativistic scattering process. The above proof that the constraints have no solution is similar to the proof that a $e\gamma \to e$ scattering process cannot conserve energy-momentum.

Secondly, consider the 4 remaining terms corresponding to $\eta = 0$. The four delta-functions are equivalent to the constraints

$$\begin{aligned} p + q &= q' \\ \xi E(q) &= \xi' E(q'). \end{aligned} \tag{5.5.12}$$

The two terms with $\xi \neq \xi'$ cannot satisfy the latter constraint, since $E(q) > 0$ by definition. So the domain of integration is empty and these two terms vanish. For the remaining two terms, $\xi = \xi'$ and $E(q) = E(q')$. In contrast to the previous cases, the delta-function constraints (5.5.12) in this case do admit solutions, so the domain of integration (over q') for these terms is non-empty. However, we will now show that the integrand

$$P_0^{rr_1}(p) Q_\xi^{ss_1}(q) \alpha^{r_1 s_1 s_1'} Q_{\xi'}^{s_1' s_0}(q') \tag{5.5.13}$$

vanishes on this domain, so the integral as a whole also vanishes. The projection P_0 was defined in (3.10.12) as

$$P_0^{rr'} = \left(\frac{p \otimes p}{|p|^2} \right)^{rr'} = \frac{p^r p^{r'}}{|p|^2}, \tag{5.5.14}$$

so the product (5.5.13) is proportional to the matrix product

$$Q_\xi(q)(\alpha \cdot p) Q_{\xi'}(q') \tag{5.5.15}$$

(matrix/spin indices have been suppressed). From the constraints (5.5.12) and using $E(q) = E(q')$,

$$\begin{aligned} \alpha \cdot p &= \alpha \cdot q' - \alpha \cdot q \\ &= 2E(q) \left(\frac{\alpha \cdot q' + \beta m - E(q')}{2E(q')} - \frac{\alpha \cdot q + \beta m - E(q)}{2E(q)} \right) \\ &= 2E(q)(-Q_-(q') + Q_-(q)) \end{aligned} \tag{5.5.16}$$

(Note that the above relation holds only when $E(q) = E(q')$.) This equality suffices to show that the $\xi = \xi' = +1$ term vanishes, because

$$\begin{aligned} Q_+(q)(\alpha\cdot p)Q_+(q') &= 2E(q)Q_+(q)\Big(-Q_-(q')+Q_-(q)\Big)Q_+(q') \\ &= 2E(q)\Big(-Q_+(q)Q_-(q')Q_+(q')+Q_+(q)Q_-(q)Q_+(q')\Big) \\ &= 0 \end{aligned} \tag{5.5.17}$$

by the identities $Q_+Q_- = Q_-Q_+ = 0$ discussed in §3.9. Thus the integrand of the $\xi = \xi' = +1$ term vanishes over its domain of integration and is therefore equal to zero. Finally, the last remaining term with $\xi = \xi' = -1$ is dealt with in a similar way; an identity similar to (5.5.16) is

$$\begin{aligned} \alpha\cdot p &= \alpha\cdot q' - \alpha\cdot q \\ &= 2E(q)\left(\frac{\alpha\cdot q'+\beta m+E(q')}{2E(q')}-\frac{\alpha\cdot q+\beta m+E(q)}{2E(q)}\right) \\ &= 2E(q)(Q_+(q')-Q_+(q)) \end{aligned} \tag{5.5.18}$$

and the product (5.5.15) vanishes according to

$$\begin{aligned} Q_-(q)(\alpha\cdot p)Q_-(q') &= 2E(q)Q_-(q)\Big(Q_+(q')-Q_+(q)\Big)Q_-(q') \\ &= 2E(q)\Big(Q_-(q)Q_+(q')Q_-(q')-Q_-(q)Q_+(q)Q_-(q')\Big) \\ &= 0. \end{aligned} \tag{5.5.19}$$

This completes the proof that all 12 terms of the first-order contribution to scattering (5.5.8) vanish individually. This completes the proof that the $e \to e\gamma$ component (5.5.2) of scattering vanishes at first order.

The preceding argument for the vanishing of expression (5.5.2) was the $(e\gamma)$-component of the full first-order contribution (5.5.1). The (e)-component also vanishes, by an almost identical analysis to the above. (This could also be inferred from unitarity — evolution is unitary, so in a perturbative expansion the $O(g)$ coefficient must be self-adjoint. According to (5.5.1), this coefficient is

$$(-ig)\int_{t_0}^{t} d_{t_1}R(t_1). \tag{5.5.20}$$

Thus if either of the off-diagonal terms vanish, then both do. It is because of self-adjointness that the two analyses are almost identical.)

As mentioned earlier, the vanishing of all $O(g)$ contributions agrees with QED. It is interesting that to arrive at this result, neither kinematics (the delta-function constraints) nor the amplitude part (product of Ps and Qs) were individually sufficient to prove the vanishing of all terms; both were needed.

5.6 Contributions at $O(g^2)$

In the previous section, the $O(g)$-contribution to scattering was shown to vanish identically. Let us now produce a more recognisable result by evaluating the leading non-zero contribution to $e\gamma \to e\gamma$ scattering (Compton scattering). This arises at $O(g^2)$ in the expansion (5.1.4)

$$\lim_{t\to\infty,\, t_0\to-\infty} \frac{g^2}{2!}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2\; T\left(R(t_1)R(t_2)\right) W_{t_0} \tag{5.6.1}$$

with initial conditions W_{t_0} in the $\Theta^1 \otimes \Lambda^1$ subspace. Firstly, let us evaluate the product $R(t_2)R(t_1)W_{t_0}$ by using the expression (5.4.20) for $R(t)$,

$$\begin{bmatrix} \langle s,q| \\ \langle r,p;s,q| \end{bmatrix} R(t_2)R(t_1)W_{t_0} = \frac{1}{m}\begin{bmatrix} \mathbf{III} & 0 \\ 0 & \mathbf{IV} \end{bmatrix}\begin{bmatrix} \langle s_0,q| \\ \langle r_0,p';s_0,q'| \end{bmatrix} W_{t_0}, \tag{5.6.2}$$

where

$$\mathbf{III} = \int \frac{d^3p'}{(2\pi)^3}\hat{S}^{ss_2}_{t_0-t_2}(q)\alpha^{r'_2s_2s'_2}\hat{D}^{r'_2r_1}_{t_2-t_1}(p')\hat{S}^{s'_2s_1}_{t_2-t_1}(q-p')\alpha^{r_1s_1s'_1}\hat{S}^{s'_1s_0}_{t_1-t_0}(q) \tag{5.6.3}$$

and

$$\begin{aligned}\mathbf{IV} = \int \frac{d^3p'\; d^3q'}{(2\pi)^3}&\hat{D}^{rr_2}_{t_0-t_2}(p)\hat{S}^{ss_2}_{t_0-t_2}(q)\alpha^{r_2s_2s'_2}\hat{S}^{s'_2s_1}_{t_2-t_1}(p+q)\\ &\alpha^{r'_1s_1s'_1}\hat{D}^{r'_1r_0}_{t_1-t_0}(p')\hat{S}^{s'_1s_0}_{t_1-t_0}(q')\delta^3(p+q-p'-q')\end{aligned} \tag{5.6.4}$$

Using the same rules as (5.4.21)–(5.4.22), **III** and **IV** may be represented by the diagrams shown in Figure 5.5.

For completeness, here is the space representation of the product $R(t_2)R(t_1)W_{t_0}$.

$$\begin{bmatrix} \langle s,y| \\ \langle r,x;s,y| \end{bmatrix} R(t_2)R(t_1)W_{t_0} = \begin{bmatrix} \mathbf{III} & 0 \\ 0 & \mathbf{IV} \end{bmatrix}\begin{bmatrix} \langle s_0,y_0| \\ \langle r_0,x_0;s_0,y_0| \end{bmatrix} W_{t_0}, \tag{5.6.5}$$

where

$$\begin{aligned}\mathbf{III} = -\frac{1}{m}\int d^3y_2 d^3y_1 d^3y_0\; &S^{ss_2}_{t_0-t_2}(y-y_2)\alpha^{r'_2s_2s'_2}D^{r'_2r_1}_{t_2-t_1}(y_2-y_1)\\ &S^{s'_2s_1}_{t_2-t_1}(y_2-y_1)\alpha^{r_1s_1s'_1}S^{s'_1s_0}_{t_1-t_0}(y_1-y_0)\end{aligned} \tag{5.6.6}$$

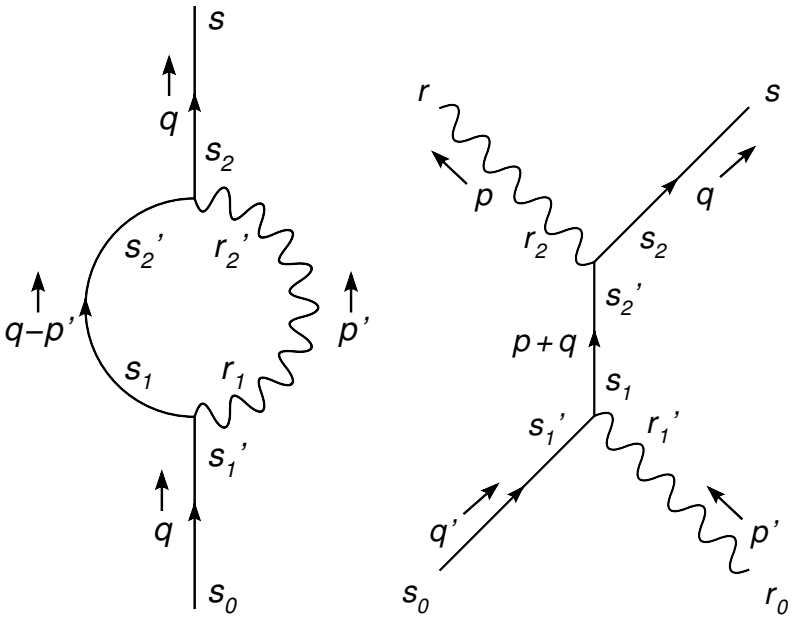


Figure 5.5: L-R: Diagrams for expressions **III** (5.6.3) and **IV** (5.6.4).

and

$$\mathbf{IV} = -\frac{1}{m}\int d^3y_2 d^3y_1 d^3x_0 d^3y_0\; D^{rr_2}_{t_0-t_2}(x-y_2) S^{ss_2}_{t_0-t_2}(y-y_2)\alpha^{r_2 s_2 s_2'} \\ S^{s_2' s_1}_{t_2-t_1}(y_2-y_1)\alpha^{r_1' s_1 s_1'} D^{r_1' r_0}_{t_1-t_0}(y_1-x_0) S^{s_1' s_0}_{t_1-t_0}(y_1-y_0). \qquad (5.6.7)$$

III is undefined without regularizing of the divergences. This can be seen roughly in the momentum representation (5.6.3), because the integrand is $O(1)$ and the integral is taken over the (infinite) volume $p' \in \mathbb{R}^3$. Thus **III** is divergent in the region of large momenta. It can be seen more clearly in the space representation (5.6.6), where the singular functions $D_{t_2-t_1}(y_2-y_1)$ and $S_{t_2-t_1}(y_2-y_1)$ are multiplied together. As mentioned in §1.3, the singularities overlap on the region $|y_2-y_1| = |t_2-t_1|$ and it is impossible to define the product. We will assume that a suitable regularization process has been introduced to soften the singularities sufficiently.

We will evaluate the second-order contribution (5.6.1) by integrating (5.6.2) over the (t_1, t_2) plane, suitably accounting for the time-ordering operation $T(R(t_1)R(t_2))$. Performing the integral will preserve a key property of (5.6.2): it is diagonal and contains no terms describing the transition between an e state and and an $e\gamma$ state, nor the transition between an $e\gamma$ state and and an e state. This agrees with QED. In contrast to the $O(g)$ calculation of the previous section, at $O(g^2)$ no calculations are needed

to show vanishing of these forbidden processes. **III** modifies dynamics of the single-lepton state $W^r(y)$, and we will interpret this as a contribution to the electron self-energy. We will return to this later. We first evaluate **IV**.

5.7 Compton Scattering

IV describes the $O(g^2)$ contribution to $e\gamma \to e\gamma$ scattering and it is here that we will first focus our attention. We will evaluate the second-order term of the time-ordered perturbative expansion (5.1.6),

$$\langle r, p; s, q| \, \frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; T\left(R(t_2)R(t_1)\right) W_{t_0}. \tag{5.7.1}$$

This expression can be written in the form

$$\begin{aligned} &\langle r, p; s, q| \, \frac{1}{2} \int_{-t_0}^{t} dt_1 \int_{-t_0}^{t} dt_2 \; T\left(R(t_2)R(t_1)\right) W_{t_0} \\ &\qquad = \int \frac{d^3p' \; d^3q'}{(2\pi)^3} \mathcal{T}(r, p, r_0, p'; s, q, s_0, q') \langle r_0, p'; s_0, q'| W_{t_0}. \end{aligned} \tag{5.7.2}$$

(because expression **IV** in (5.6.4) is of this form). We will evaluate the kernel $\mathcal{T}$. Appearing in the integral, $\mathcal{T}$ quantifies how much different modes p', q' in the initial data contribute to a specified mode p, q of the evolved state. For example, non-trivial scattering occurs if non-zero contributions arise from regions of integration where $p' \neq p$ and $q' \neq q$. Following §4.7, $\mathcal{T}$ will contain the momentum-conserving delta-functions $\delta^3(p+q-p'-q')$. In the following calculations, we will evaluate $\mathcal{T}$ and show that in the limit $t_0 \to -\infty, t \to \infty$ it also includes the necessary energy-conserving delta-function $\delta(|p| + E(q) - |p'| - E(q'))$ required by QED. The second-order contribution (5.7.1) is therefore of the required 'matrix-element' form (1.1.5) and gives a definite expression for the matrix element $\mathcal{M}$. We will show that this result agrees precisely with QED. So for this particular process, the Hamiltonian theory developed in this thesis agrees with QED at leading order. The agreement of the theories is much stronger than simply being able to draw similar-looking diagrams, such as Figure 5.5 resembling Figure 1.1 in the Introduction. We will show equality of mathematical expressions.

§4.7 argued on general grounds that $\mathcal{T}$ must include a delta-function $\delta^3(p + q - p' - q')$. This is also directly apparent from (5.6.4). The delta-function constrains the momentum integrals to the region

$$p + q = p' + q'. \tag{5.7.3}$$

In other words, contributions to the scattered outgoing state can only arise from incoming states with the same total momentum. Thus conservation of momentum is ensured. This is a (small) neat result of the theory developed in this thesis — momentum conservation is correctly predicted within the theory and does not need to be put in 'by hand'.

Everything is in place to evaluate $\mathcal{T}$; the techniques are similar to but more involved than the $O(g)$ calculation of the previous section. Firstly, substitute the free propagators (5.5.5) into the expression **IV** of (5.6.4). Each $\hat{D}$ is a sum of three terms; each $\hat{S}$ is a sum of two terms. With two factors of $\hat{D}$ and three of $\hat{S}$, the expression is a sum of $(3 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 72)$ terms in total. (Compare to the 12 of the previous section!) The propagators will contribute factors of the projection operators P_η, Q_ξ as

$$P_\eta^{rr_2}(p) Q_\xi^{ss_2}(q) \alpha^{r_2 s_2 s_2' s_1'} Q_\zeta^{s_2' s_1}(p+q) \alpha^{r_1' s_1 s_1'} P_{\eta'}^{r_1' r_0}(p') Q_{\xi'}^{s_1' s_0}(q'), \tag{5.7.4}$$

in that order. Here $\eta, \eta' = 0, \pm 1$ and $\xi, \xi', \zeta = \pm 1$. Meanwhile, let us turn attention to the time-ordered integral over t_1, t_2. With time-ordering explicitly written out, the integral is

$$\begin{aligned} &\frac{1}{2} \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ \left(R(t_1) R(t_2) \theta(t_1 - t_2) + R(t_2) R(t_1) \theta(t_2 - t_1) \right) W_{t_0} \\ &\qquad = \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ R(t_2) R(t_1) \theta(t_2 - t_1) W_{t_0}. \end{aligned} \tag{5.7.5}$$

In the second line, the two summands were collected together after relabelling $t_1 \leftrightarrow t_2$. Recall θ is the (Heaviside) unit step function. The (t_1, t_2)-dependence of the integrand arises solely from the exponential terms in the five factors of $\hat{S}$ and $\hat{D}$, contributing

$$e^{-i\eta|p|(t_0 - t_2)} e^{-i\xi E(q)(t_0 - t_2)} e^{-i\zeta E(p+q)(t_2 - t_1)} e^{-i\eta'|p'|(t_1 - t_0)} e^{-i\xi' E(q')(t_1 - t_0)} \tag{5.7.6}$$

respectively. We may factor out t_0-dependence

$$e^{-i(\eta|p| + \xi E(q) - \eta'|p'| - \xi' E(q'))t_0}, \tag{5.7.7}$$

and we will later show this to be identically equal to unity over the domain of integration. The (t_1, t_2)-integral becomes

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \ e^{i\eta|p|t_2 + i\xi E(q) t_2 - i\zeta E(p+q)(t_2 - t_1) - i\eta'|p'|t_1 - i\xi' E(q') t_1} \theta(t_2 - t_1) \tag{5.7.8}$$

The integrand of (5.7.8) is bounded in magnitude by unity, so for all $p, q, p', q' \in \mathbb{R}^3$, the integral is bounded by the area of integration, $\frac{1}{2}(t - t_0)^2$. For finite limits $t_0, t \in \mathbb{R}$, it is possible to evaluate the integral exactly, but the result is not very illuminating. It is much more informative to take the limit $t_0 \to -\infty$, $t \to \infty$. In this limit, we may change variables to $s_1 = t_1 + t_2$ and $s_2 = t_2 - t_1$, and rewrite (5.7.8) as

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} ds_1 \ \exp\left(i(\eta|p| + \xi E(q) - \eta'|p'| - \xi' E(q'))\frac{s_1}{2}\right) \times \\ \times \int_{-\infty}^{\infty} ds_2 \ \theta(s_2) \exp\left(i\left(\frac{\eta|p|}{2} + \frac{\xi E(q)}{2} - \zeta E(p+q) + \frac{\eta'|p'|}{2} + \frac{\xi' E(q')}{2}\right)s_2\right)\end{aligned} \tag{5.7.9}$$

(The leading factor of $1/2$ arises from the Jacobian determinant $|\frac{\partial(t_1,t_2)}{\partial(s_1,s_2)}| = 1/2$.) The integral over s_1 can be done, yielding a delta-function,

$$4\pi \ \delta(\eta|p| + \xi E(q) - \eta'|p'| - \xi' E(q')). \tag{5.7.10}$$

This delta-function constrains the region of integration to momenta p', q' such that

$$\eta|p| + \xi E(q) = \eta'|p'| + \xi' E(q'). \tag{5.7.11}$$

So we are justified to set (5.7.7) equal to unity, as claimed earlier. The remaining integral (over s_2) may be evaluated using the Fourier representation of θ

$$\int_{-\infty}^{\infty} ds \ \theta(s) e^{iks} = \frac{i}{k + i\epsilon} \qquad\qquad (\epsilon \to 0^+). \tag{5.7.12}$$

We obtain

$$\begin{aligned}\int_{-\infty}^{\infty} ds_2 \ \theta(s_2) \exp\left(i(\eta|p| + \xi E(q) - \zeta E(p+q))s_2\right) \\ = \frac{i}{\eta|p| + \xi E(q) - \zeta E(p+q) + i\epsilon}.\end{aligned} \tag{5.7.13}$$

Combining these two results, the (t_1, t_2)-integral (5.7.8) is evaluated as

$$2\pi i \frac{\delta(\eta|p| + \xi E(q) - \eta'|p'| - \xi' E(q'))}{\eta|p| + \xi E(q) - \zeta E(p+q) + i\epsilon}. \tag{5.7.14}$$

We can now write down a complete expression for the scattering kernel $\mathcal{T}$,

$$\begin{aligned}\mathcal{T} = \frac{2\pi i}{m} P_{\eta}^{rr_2}(p) Q_{\xi}^{ss_2}(q) \alpha^{r_2 s_2 s_2'} Q_{\zeta}^{s_2' s_1}(p+q) \alpha^{r_1' s_1 s_1'} P_{\eta'}^{r_1' r_0}(p') Q_{\xi'}^{s_1' s_0}(q') \times \\ \times \delta^3(p + q - p' - q') \frac{\delta(\eta|p| + \xi E(q) - \eta'|p'| - \xi' E(q'))}{\eta|p| + \xi E(q) - \zeta E(p+q) + i\epsilon}\end{aligned} \tag{5.7.15}$$

with a sum taken across the 72 possible combinations of $\eta, \eta' = -1, 0, 1$ and $\xi, \xi', \zeta = \pm 1$. The sum over ζ can be performed. Dependence on ζ arises only from two factors: Q_ζ and the denominator of (5.7.14). Let us temporarily abbreviate these by X. Performing the sum,

$$\begin{aligned} X &= \sum_{\zeta=\pm 1} Q_\zeta(p+q) \frac{1}{\eta|p| + \xi E(q) - \zeta E(p+q) + i\epsilon} \\ &= \sum_{\zeta=\pm 1} \frac{\zeta H(p+q) + E(p+q)}{2E(p+q)} \frac{1}{\eta|p| + \xi E(q) - \zeta E(p+q) + i\epsilon} \\ &= \frac{H(p+q) + \eta|p| + \xi E(q)}{(\eta|p| + \xi E(q))^2 - E(p+q)^2 + i\epsilon}, \end{aligned} \tag{5.7.16}$$

where the last expression follows from putting the two terms of the sum over a common denominator and simplifying. We may substitute the definitions $H(p+q) = \alpha^j(p_j + q_j) + \beta m$ and $E(p+q)^2 = |p+q|^2 + m^2$ to obtain

$$X = \frac{\alpha^j(p_j + q_j) + \beta m + (\eta|p| + \xi E(q))}{(\eta|p| + \xi E(q))^2 - |p+q|^2 - m^2 + i\epsilon} \tag{5.7.17}$$

and use $\alpha^j = -\gamma^j\gamma^0$ and $\beta = \gamma^0$ to extract a common factor of γ^0 in the numerator,

$$X = \frac{\gamma^0(\eta|p| + \xi E(q)) - \gamma^j(p_j + q_j) + m}{(\eta|p| + \xi E(q))^2 - |p+q|^2 - m^2 + i\epsilon}\gamma^0. \tag{5.7.18}$$

(5.7.18) has a familiar form: it is (up to a surplus factor γ^0) the Feynman propagator for a lepton internal line,

$$X = \frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - m^2 + i\epsilon}\gamma^0, \tag{5.7.19}$$

whose four-momentum $k_\mu = (k_0, k_1, k_2, k_3)$ satisfies

$$\begin{aligned} k_0 &= \eta|p| + \xi E(q) \\ k_j &= -(p_j + q_j). \end{aligned} \tag{5.7.20}$$

Consequently, expression (5.7.15) for $\mathcal{T}$ can be simplified to

$$\begin{aligned} \mathcal{T} = \frac{2\pi i}{m} \sum_{\eta,\xi,\eta',\xi'} &\delta^3(p+q-p'-q')\delta(\eta|p| + \xi E(q) - \eta'|p'| - \xi' E(q')) \times \\ &\times P_\eta^{rr_2}(p) Q_\xi(q) \gamma^0 \gamma^{r_2} \frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - m^2 + i\epsilon} \gamma^{r'_0} P_{\eta'}^{r'_0 r_0}(p') Q_{\xi'}(q') \end{aligned} \tag{5.7.21}$$

where spinor indices have been made implicit. (We have also substituted for factors of $\alpha^j = \gamma^0\gamma^j$.) The sum is taken over the 36 combinations of $\eta, \eta' = 0, \pm 1$ and $\xi, \xi' = \pm 1$.

Of these 36 terms, we will show that many are precisely the expressions required by QED for this process.

For example, the term corresponding to $\eta = \eta' = \xi = \xi' = 1$ is one of the terms that agree with QED. This term contains delta-functions

$$\delta^3(p + q - p' - q')\delta(|p| + E(q) - |p'| - E(q')) \tag{5.7.22}$$

describing conservation of energy-momentum. Note that the presence of these delta-functions agrees with the form of the S-matrix (1.1.5) required by QED. We will show full agreement with QED by demonstrating that the coefficients of these delta-functions in (5.7.21) agree precisely with the s-channel matrix element $i\mathcal{M}$ introduced in (1.2.1).

To show this agreement, we recall that $i\mathcal{M}$ describes a typical scattering event where the system evolves from separable initial conditions $W_{t_0} = \varepsilon' \otimes u'$, where $\varepsilon' \in \Theta^1$ is a RS photon state and $u' \in \Lambda^1$ is a lepton state. This evolved state is specified in terms of the overlap with some other separable state $\varepsilon \otimes u$ for $\varepsilon \in \Theta^1, u \in \Lambda^1$. In other words, we identify

$$\delta^3(p + q - p' - q')\delta(|p| + E(q) - |p'| - E(q'))(2\pi)^4 i\mathcal{M} = (-ig)^2\langle \varepsilon \otimes u, \mathcal{T}(\varepsilon' \otimes u')\rangle. \tag{5.7.23}$$

Considering now the term $\eta = \eta' = \xi = \xi' = +1$, we may substitute for $\mathcal{T}$ using (5.7.21),

$$(2\pi)^4 i\mathcal{M} = (-ig)^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}(\hat{\varepsilon}^*)^r(p)\hat{u}^\dagger(q) \cdot \frac{2\pi i}{m} m \int \frac{d^3p'\, d^3q'}{(2\pi)^3} \times \\ \times P_+^{rr_2}(p)Q_+(q)\gamma^0\gamma^{r_2}\frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - m^2 + i\epsilon}\gamma^{r'_0}P_+^{r'_0 r_0}(p')Q_+(q')(\hat{\varepsilon}')^{r_0}(p')\hat{u}'(q'). \tag{5.7.24}$$

The trailing and leading factors P_+, Q_+ project both domain and range of $i\mathcal{M}$ onto the positive-energy subspaces of both Θ^1 and Λ^1. For the leptons, this is exactly what the Feynman diagram for $e^-\gamma \to e^-\gamma$ scattering requires: by definition, the electron lies in the positive-energy subspace of Λ^1. In other words, the relation (5.7.24) is defined by QED to be valid only for u, u' such that $Q_+u = u$ and $Q_+u' = u'$. Furthermore, note the presence of the 'surplus' factor γ^0; this is precisely as required by the definition of the Dirac adjoint $\bar{u} = u^\dagger\gamma^0$ appearing in $i\mathcal{M}$. We deal with the P_+ factors similarly to the Q_+: they are equivalent to assuming $\varepsilon, \varepsilon'$ lie in the positive-energy subspace of Θ^1,

or in other words that $P_+\varepsilon' = \varepsilon'$ and $P_+\varepsilon = \varepsilon$. We therefore obtain

$$(2\pi)^4 i\mathcal{M} = -i(2\pi)^4(-ig)^2 \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3p'}{(2\pi)^3}\frac{d^3q'}{(2\pi)^3}\times \\ \times (\hat{\varepsilon}^*)^r(p)\hat{\bar{u}}(q)\gamma^r\frac{\gamma^\mu k_\mu + m}{k^\mu k_\mu - m^2 + i\epsilon}\gamma^{r_0}(\hat{\varepsilon}')^{r_0}(p')\hat{u}'(q') \tag{5.7.25}$$

One remaining difference from the QED expression (1.2.1) is that the amplitudes $\varepsilon, \varepsilon'$ (5.7.25) are RS 3-component amplitudes, and summation ranges over $r, r_0 = 1, 2, 3$. The corresponding variables in (1.2.1) are 4-component electromagnetic potential amplitudes $\mu, \nu = 0, 1, 2, 3$. However, it is a well-known result of QED that the time-like and longitudinal components of the 4-component photon are unphysical, and cancel in their contribution to any physical scattering process. The non-zero contribution to scattering comes only from the tranverse (spacelike) part of the amplitude. This is also true of the amplitudes in (5.7.25), since using the condition $\varepsilon = P_+\varepsilon$,

$$P_0\varepsilon = P_0P_+\varepsilon = 0 \tag{5.7.26}$$

and ε is therefore transverse. (The same argument holds for ε'.) According to equation (3.10.19) regarding helicity states of the RS photon, $\varepsilon, \varepsilon'$ correspond to $+1$ helicity-photons; these are equal to the space-components of a $+1$-helicity photon in QED. Thus while the RS photons have a different interpretation to the standard EM potential photons of QED, the amplitudes in either theory contribute similarly to the matrix element. Finally, QED matrix elements relate to the plane wave limit with $\varepsilon, \varepsilon', u, u'$ tending to delta-functions in momentum space, effectively removing the integrals in (5.7.25). (Suitable normalising factors of $(2\pi)^{3/2}$ are to be included). Finally, we have obtained

$$i\mathcal{M} = (-ig)^2(\varepsilon^*)^r(p)\bar{u}(q)\gamma^r\frac{-i(\gamma^\mu k_\mu + m)}{k^\mu k_\mu - m^2 + i\epsilon}\gamma^{r_0}(\varepsilon')^{r_0}(p')u'(q') \tag{5.7.27}$$

where

$$k_0 = |p| + E(q) \\ k_j = -(p_j + q_j) \tag{5.7.28}$$

is the correct 4-momentum for the internal electron line as required by the Feynman rules for QED[3]. Finally, we identify the coupling g with the electron charge e of QED. This completes the derivation of the QED matrix element (1.2.1) for s-channel Compton scattering of an electron and a $+1$-helicity photon.

[3] The odd-looking minus sign in front of k_j derives from a widespread convention in high energy

5.8 Other terms contributing to scattering

The preceding analysis treated only one of the 36 terms contributing to $\mathcal{T}$ in (5.7.21): the term with $\eta, \eta', \xi, \xi' = 1$. Now let us consider the term with $\eta, \eta', \xi, \xi' = -1$. The overall minus-sign makes no change to the delta-functions (5.7.22) describing conservation of energy-momentum. The matrix element $i\mathcal{M}$ is modified from (5.7.24) by replacing

$$\begin{aligned} k_0 &= -|p| - E(q) \\ k_j &= -(p_j + q_j) \end{aligned} \tag{5.8.1}$$

and by replacing the trailing and leading factors P_+, Q_+ with P_-, Q_-. The latter change describes the projection of domain and range of $i\mathcal{M}$ onto the negative-energy subspace of Λ^1. Consequently, we interpret this term as the scattering of positrons by photons, $e^+\gamma \to e^+\gamma$. The analysis is similar to the preceding case; the matrix element is identical to (5.7.25) with $Q_-u = u$, $Q_-u' = u'$ and k_μ given by (5.8.1). We again seek to match the results of QED by taking the plane wave limit, so let $\epsilon, \epsilon', u, u'$ tend to delta-functions in momentum-space and effectively remove the integrals in (5.7.25). The resulting expression for $i\mathcal{M}$ is identical to (5.7.27) with k_μ now given by (5.8.1), $\epsilon, \epsilon' \in P_-\Theta^1$ and $u, u' \in Q_-\Lambda^1$.

To connect this expression to the textbook result (1.2.3) for this process, we have to account for another widespread convention in high energy physics of changing the sign on all momenta for amplitudes in the negative-energy subspace of Λ^1. This convention derives from labelling electron/positron plane waves as

$$u \exp(-ik_\mu x^\mu), \qquad v \exp(+ik_\mu x^\mu), \tag{5.8.2}$$

respectively — note the differing signs on the exponent. Under this convention, if an amplitude $u(k)$ lies in the negative-energy subspace of Λ^1, it is relabelled as $v(-k)$. (For further details see, e.g., the popular text of Peskin and Schroeder [35], §3.3.) We

physics of defining $E = i\partial_t$ but $\mathbf{p} = -i\boldsymbol{\nabla}_j$, following Dirac's influential work [105]. So $i\partial_\mu = (E, -\mathbf{p})$. The standard formulae of Chapter 1 are quoted from a popular modern text following this convention, Peskin and Schroeder [35, pp. xix–xx]. Under this convention, (5.7.28) can also be written as the less odd-looking $k^0 = |p| + E(q), k^j = p_j + q_j$.

Personally, I would prefer $p_\mu = -i\partial_\mu = (E, \mathbf{p})$, but the textbook expressions in Chapter 1 would have to be adjusted accordingly. This would just confuse everybody.

can account for this convention by substituting $v(-q)$ for $u(q)$ and $v'(-q')$ for $u(q')$ in the matrix element (5.7.27). However, in order to get the correct answer, a similar transformation must be applied to the left-handed RS photons: we will also need to substitute $\epsilon^*(-p)$ for $\epsilon(p)$ and $(\epsilon')^*(-p')$ for $\epsilon'(p')$. The physical interpretation of this relabelling, which affects only the left-handed modes of the RS photon, is unclear. However, it seems to be as reasonable as the conventional relabelling the positron amplitudes. Note that there is no contradiction with the preceding analysis of the $\eta = \eta' = \xi = \xi' = +1$ case, which correctly described scattering of right-handed RS photons without any relabelling.

When these substitutions are performed, we obtain

$$i\mathcal{M} = (-ig)^2 \varepsilon^r(p)\bar{v}(q)\gamma^r \frac{-i(\gamma^\mu k_\mu + m)}{k^\mu k_\mu - m^2 + i\epsilon}\gamma^{r_0}(\varepsilon'^*)^{r_0}(p')v'(q') \tag{5.8.3}$$

where we now have $(k_\mu = -|p| - E(q), p_j + q_j)$. This is not exactly the matrix element $i\mathcal{M}$ predicted by QED and quoted in (1.2.3); but a straightforward calculation shows that it is exactly its complex conjugate. This is only a small discrepancy, and it is possible that the discrepancy may eventually be explained away purely as convention. If the same discrepancy could be shown to occur with all other processes interfering with this one, then it would be unobservable because the contribution to scattering cross-section is proportional to $|\sum \mathcal{M}|^2$. Future calculations may be able to confirm this conjecture if they describe processes which interfere with the s-channel processes described above. (The t-channel scattering of (1.2.2) is an example.) In QED, there is no possibility of interference between the two processes considered so far; $e^-\gamma \to e^-\gamma$ and $e^+\gamma \to e^+\gamma$ lie in orthogonal subspaces. This completes the derivation of the QED matrix element (1.2.3) for s-channel Compton scattering of a positron and a -1-helicity photon, obtaining a correct answer 'up to complex conjugacy'.

We return now to the 36 terms appearing in the expression for $\mathcal{T}$ in (5.7.21). Table 5.1 classifies the 36 terms into separate cases *i–vii*, which we will deal with case by case. Cases *i* and *ii* have already been discussed, and we have found these to agree with QED. We will now use methods similar to the $O(g)$ calculation in §5.5 to show that many of the other terms vanish identically.

Consider the two terms in Table 5.1 labelled *iii*. The first of these, with $\eta = \xi = 1$

	η	ξ	η'	ξ'	*No.*	Remarks
i	1	1	1	1	1	$e^-\gamma \to e^-\gamma$
ii	-1	-1	-1	-1	1	$e^+\gamma \to e^+\gamma$
iii	1	1	-1	-1	1	Vanish by $4 \to 0$ kinematics
	-1	-1	1	1	1	
iv	$\{+,+,+,-\}$ any order				8	Vanish by $3 \to 1$ kinematics
	$\{-,-,-,+\}$ any order					
v	1	-1	1	-1	4	May cancel
	1	-1	-1	1		
	-1	1	1	-1		
	-1	1	-1	1		
vi	any	any	0	any	12	Vanish by initial conditions
vii	0	any	± 1	any	8	Expect to vanish
				Total:	36	

Table 5.1: The 36 terms appearing in the $O(g^2)$ expansion. *No.* indicates the number of terms of this type.

and $\eta' = \xi' = -1$ contains the delta-function

$$\delta(|p| + E(q) + |p'| + E(q')). \tag{5.8.4}$$

The argument is a sum of positive terms only (recall $E(p) = +\sqrt{|p|^2 + m^2}$), of which two are positive definite, so this constraint has no solution. So this term vanishes identically. By a similar argument, the other term in case *iii* also vanishes. Table 5.1 labels these as 'vanishing by $4 \to 0$ kinematics' because the delta-function (5.8.4) is equivalent to energy conservation in the impossible process $ee\gamma\gamma \to$ (vacuum).

Let us move on to the 8 terms in Table 5.1 labelled *iv*. The term corresponding to $\eta = \eta' = \xi = 1$ and $\xi' = -1$ contains delta-functions whose arguments are equivalent to the the conditions

$$\begin{aligned} p + q &= p' + q' \\ |p| + E(q) &= |p'| - E(q'). \end{aligned} \tag{5.8.5}$$

Since $E(q') = E(-q')$, (5.8.5) can be rewritten

$$\begin{aligned} p + q + (-q') &= p' \\ |p| + E(q) + E(-q') &= |p'|. \end{aligned} \tag{5.8.6}$$

These constraints are equivalent to the kinematic constraints of the impossible process $\gamma ee \leftrightarrow \gamma$: a straightforward proof similar to (5.5.10)–(5.5.11) suffices to prove that no choice of p, q, p', q' can satisfy all four constraints. As with the preceding case, the domain of integration over p' and q' is empty and this term vanishes. This type of straightforward 'kinematic' argument also applies to the other 7 terms in case *iv*. So far, so good: the two non-zero terms *i* and *ii* and the 10 vanishing terms in cases *iii* and *iv* agree with QED. We will discuss terms *v*–*vii* in the next sections.

5.9 Need for coupling to higher subspaces

Let us now consider cases *v*. For example, the term with $\eta = \eta' = 1, \xi = \xi' = -1$, the internal momentum is $k_\mu = (|p| - E(q), -p_j - q_j)$, and the delta-functions take the form

$$\delta^3(p + q - p' - q')\delta(|p| - E(q) - |p'| + E(q')). \tag{5.9.1}$$

The constraints are equivalent to

$$\begin{aligned}(-p) + q' &= (-p') + q \\ |-p| + E(q') &= |-p'| + E(q)\end{aligned} \tag{5.9.2}$$

This is precisely the kinematics and form of the QED t-channel diagram (1.2.2) for an *incoming* photon/electron momenta $-p, q'$ and *outgoing* photon/electron momenta $-p', q$. So there is a discrepancy: in the framework of this thesis, the initial state is most definitely parametrised by p', q' and the final state by p, q. The other terms of case *v* are similar.

I will speculate that the discrepancy may arise as an artifact of the very simple choice for the interaction R (5.4.9). This interaction effectively truncates the Fock space $\Theta \otimes \Lambda$ to the two subspaces $\Theta^0 \otimes \Lambda^1$ (e-states) and $\Theta^1 \otimes \Lambda^1$ ($e\gamma$-states). (States of different particle content are included in the theory, but are uncoupled, and evolve according to the free theory.) In particular the theory lacks a coupling of the $e\gamma$ initial state to the $e\gamma\gamma$ subspace. A modification to R that would achieve this would be of the form

$$\begin{bmatrix} (RW)^{s}_{(0,1)}(y) \\ (RW)^{rs}_{(1,1)}(x,y) \\ (RW)^{r_1 r_2 s}_{(2,1)}(x_1,x_2,y) \end{bmatrix} = \begin{bmatrix} 0 & * & 0 \\ * & 0 & \sharp \\ 0 & \sharp & 0 \end{bmatrix} \begin{bmatrix} W^{s'}_{(0,1)}(y) \\ W^{r's'}_{(1,1)}(x,y) \\ W^{r'_1 r'_2 s'}_{(1,1)}(x_1,x_2,y) \end{bmatrix}, \tag{5.9.3}$$

where the off-diagonal entries $*$ are the same as in the expression for R (5.4.9) used in this chapter, and the entries $\sharp$ are some suitable new expressions. These would not depend on g, nor introduce any new couplings; I speculate that these off-diagonal terms might take a similar form to the off-diagonal terms in the Hamiltonian of the displaced SHO (5.2.17). The $\sharp$ entries would imply new contributions to Compton scattering at $O(g^2)$, mediated by a transient $e\gamma\gamma$ state between the initial and final $e\gamma$ states. These contributions would likely interfere with the t-channel contributions to Compton scattering and might easily cancel with the unwanted terms in case *v*. Some preliminary work assuming a simple conjecture on the form of the $\sharp$ entries in (5.9.3) indicate that the second-order expansion would involve even more terms than the 36 considered in this case. This would be a sizeable computation which is unfortunately beyond the scope of this thesis. Dealing with the large number terms in the expansion may require a new, more efficient methodology.

Incidentally, the conjecture on the form of R as (5.9.3) is suggested as the 3-by-3 truncation of an infinite, tri-diagonal matrix similar to the Hamiltonian (5.2.17) of the displaced SHO discussed in §5.2.

Likewise, this thesis has not considered a coupling between states of different lepton content. Such a coupling must exist in order to account for observed e^+e^- pair-annihilation and pair-creation processes. The likely mechanism for this coupling would be coupling of the $e\gamma$ initial state (from the $\Theta^1 \otimes \Lambda^1$ subspace) to a transient eee state (from the $\Theta^0 \otimes \Lambda^3$ subspace), and vice versa. Such a coupling would also contribute to Compton scattering at $O(g^2)$ and would likely interfere with the t-channel diagram.

5.10 Contributions from the zero-mode RS photon

Cases *vi* and *vii* describe scattering of the zero-mode (longitudinal) RS photons: 20 terms for which $\eta = 0$ or $\eta' = 0$ (or both). According to previous sections, such modes of the RS photon are non-physical, and for the theory to agree with QED, all such terms should vanish. The terms with $\eta' = 0$ (case *v*) couple a zero-mode RS photon in the initial state to a possibly physical photon in the final state. These terms can be safely ignored by assuming that the initial conditions W_{t_0} do not include these non-physical modes: $P_0(p)W_{t_0}(p) = 0$ for all $p \in \mathbb{R}^3$.

However, the terms with $\eta = 0$ and $\eta' = \pm 1$ (case *vii*) should be dealt with more carefully. These describe the coupling of a (physical) transverse RS photon in the initial state to a zero-mode RS photon in the final state. I hoped to use similar methods to the $O(g)$ calculation of §5.5, where all contributions to scattering of zero-modes modes were shown to vanish individually. Unfortunately, it is not the case, but the expressions simplify significantly. For completeness I will describe the computation and simplification of these terms. It is possible that including couplings to higher subspaces would lead to cancellation of these unwanted terms, similar to the previous section. As mentioned earlier, this would be a sizeable computation, and is left for future research.

The simplification of the terms with $\eta = 0$ is as follows. Though $\mathcal{T}$ is given in (5.7.21), it turns out that it is simpler to proceed from the earlier expression (5.7.15). Using this expression and substituting for $\eta = 0$, the delta-functions in $\mathcal{T}$ read

$$\delta^3(p + q - p' - q')\delta(\xi E(q) - \eta'|p'| - \xi' E(q')) \tag{5.10.1}$$

and their coefficient is

$$\frac{P_0^{rr_2}(p)Q_\xi^{ss_2}(q)\alpha^{r_2s_2s_1'}Q_\zeta^{s_1's_1}(p+q)\alpha^{r_0's_1s_0'}P_{\eta'}^{r_0'r_0}(p')Q_{\xi'}^{s_0's_0}(q')}{\xi E(q)-\zeta E(p+q)+i\epsilon}. \tag{5.10.2}$$

(Some constant multipliers have been neglected.) Using the same reasoning as (5.5.15) in the previous section, the first few factors of the numerator may be written as the matrix product

$$\frac{p^r}{|p|^2}Q_\xi(q)(\alpha\cdot p)Q_\zeta(p+q)\dots \tag{5.10.3}$$

We will substitute for $\alpha\cdot p$,

$$\begin{aligned}\alpha\cdot p &= \alpha\cdot(p+q)-\alpha\cdot q\\ &= E(p+q)(Q_+(p+q)-Q_-(p+q))-E(q)(Q_+(q)-Q_-(q))\end{aligned} \tag{5.10.4}$$

where the second line can be verified by comparing with the definition (3.10.12) of $Q_\pm$. Substitution of this expression into the product (5.10.3) is a straightforward calculation, yielding

$$Q_\xi(q)(\alpha\cdot p)Q_\zeta(p+q)=(-\xi E(q)+\zeta E(p+q))Q_\xi(q)Q_\zeta(p+q) \tag{5.10.5}$$

Note that the factor in parentheses neatly cancels with the denominator of (5.10.2). In this way, we simplify (5.10.2) as

$$\frac{p^r}{|p|^2}Q_\xi(q)Q_\zeta(p+q)\alpha^{r'}Q_{\xi'}(q')P_{\eta'}^{r'r_0}(p') \tag{5.10.6}$$

There is no other ζ-dependence in the expression, so the sum over $\zeta=\pm1$ can be performed using $Q_+(p+q)+Q_-(p+q)=1$. The result is

$$\mathcal{T}=\frac{2\pi i}{m}\delta^3(p+q-p'-q')\delta(\xi E(q)-\eta'|p'|-\xi' E(q'))\frac{p^r}{|p|^2}Q_\xi(q)\alpha^{r'}Q_{\xi'}(q')P_{\eta'}^{r'r_0}(p') \tag{5.10.7}$$

Though this expression has simplified significantly as a result of the substitution $\eta=0$, it does not appear to vanish altogether. (Keep in mind that we only require vanishing on the domain defined by the delta-function constraints.) For $p=0$, the expression reduces to the $O(g)$-expression (5.5.8) and therefore vanishes in the same way. However, for many values of q,p',q' there is a non-zero $p\in\mathbb{R}^3$ such that all four delta-functions are satisfied. Meanwhile, the coefficients do not appear to vanish. So there appear to

be non-zero contributions from such terms, and we reiterate that these terms describe scattering from an $e\gamma$ initial state with a physical, transverse RS photon to an $e\gamma$ final state with a non-physical, longitudinal RS photon. This is in contradiction with QED. It is possible that including couplings to other subspaces would lead to cancellation of these unwanted terms, similar to the argument put forth in the previous section. Because of the simplifications, the hoped-for cancellation seems more plausible; note that (5.10.7) contains no propagator-like terms and is largely a tensor product of projection operators. It is possible that a more inclusive coupling would generate further terms of similar form.

We remark that these contributions to scattering also violate basic kinematics, with the delta-functions in (5.10.7) not depending at all on the energy $|p|$ of the outgoing, longitudinal RS photon. I wish to emphasise once again that relativistic kinematics is not an *assumption* or hypothesis of the framework developed in this thesis; it is rather a derived *feature* of the theory. In this context it is less puzzling that (5.10.7) violates relativistic kinematics, and rather more remarkable that cases *i* and *ii* successfully contain the correct kinematics.

5.11 Electron Self-Energy

Let us also consider the other contribution to scattering at $O(g^2)$, the expression **III** in (5.6.3). This contribution to scattering was denoted by the diagram in Figure 5.5. As mentioned in §5.6, this term describes a contribution to the evolution of pure e states and we will seek to interpret it as the electron self-energy diagram (Figure 1.2) of QED. We saw in the previous section that **IV** describes an operator mapping $e\gamma$ states to $e\gamma$ states. It is a convolution integral over the momenta p', q', so its value at p, q contains contributions from the initial state at *all* values of p', q' where the integrand is non-zero. In contrast, **III** does not describe a convolution over the momentum of the initial state; its value at q contains a contribution from the inital state at the same value of q only. This simplification arose because the delta-functions appearing in $R(t)$ in (5.4.20) combined precisely to reduce the domain of integration to a single point. This result is equivalent to the discussion in §4.7, arguing on general grounds that $\mathcal{T}$ must include a delta-function $\delta^3(q' - q)$.

Consequently, in contrast to the scattering kernel $\mathcal{T}$ (5.7.2) for Compton scattering,

we are interested simply in a *multiplier* $\mathcal{T}^{ss_0}(q)$ defined by

$$\langle s,q|\,\frac{1}{2}\int_{-t_0}^{t}dt_1\int_{-t_0}^{t}dt_2\;T\left(R(t_2)R(t_1)\right)W_{t_0}=\mathcal{T}^{ss_0}(q)\langle s_0,q|W_{t_0}. \tag{5.11.1}$$

They have slightly different interpretations, but in the following calculations we will show some resemblance between the multiplier $\mathcal{T}$ defined by (5.11.1), and the QED matrix element $i\mathcal{M}$.

We obtain an expression for $\mathcal{T}$ by substituting for the time-ordered integral and expression **III** in (5.6.3). As mentioned in §5.6, **III** is divergent in the region of large momenta, so $\mathcal{T}$ will diverge in the same way. This also agrees qualitatively with the divergent loop integral (1.3.1) of QED. In the following sections, we will try to perform a more detailed comparison to see whether this expression agrees quantitatively with QED. Unfortunately, this expression is divergent/infinite in each theory and it is difficult to draw a direct comparison. Ideally the problem could be approached by explicitly regularizing all divergent quantities, for example using a UV cutoff, or perhaps by using one of the regularisation techniques considered in §3.7–3.8. It may be possible to draw a direct comparison as the cutoff is removed. However, the contribution to scattering by this loop integral is effectively unobservable; it is a correction to the freely-propagating e state only. I feel it would be far more informative to consider a loop integral which does contribute an observable correction to scattering; an example is the vertex correction (Figure 5.6). The vertex correction is observable and contributes a non-zero correction to the magnetic moment of the electron (and other phenomena). I anticipate that the vertex correction term will arise naturally within the 3-by-3 truncation proposed in (5.9.3). For this reason, a careful analysis of regularisation and renormalisation awaits development of that extension to the model.

Nevertheless, I will present some heuristic calculations to manipulate the divergent expressions for $\mathcal{T}$ into a form which bears a much closer resemblance to the QED matrix element (1.3.1).

Let us write down an expression for $\mathcal{T}$ using the definition (5.11.1), substituting for the time-ordered integral and expression **III** in (5.6.3). We obtain

$$\begin{aligned}\mathcal{T}^{ss_0}(q)=(-ig)^2\int_{-t_0}^{t}dt_1\int_{-t_0}^{t}dt_2\;\theta(t_2-t_1)\int\frac{d^3p'}{(2\pi)^3}\times\\ \times\,\hat{S}^{ss_2}_{t_0-t_2}(q)\alpha^{r'_2s_2s'_2}\hat{D}^{r'_2r_1}_{t_2-t_1}(p')\hat{S}^{s'_2s_1}_{t_2-t_1}(q-p')\alpha^{r_1s_1s'_1}\hat{S}^{s'_1s_0}_{t_1-t_0}(q).\end{aligned} \tag{5.11.2}$$

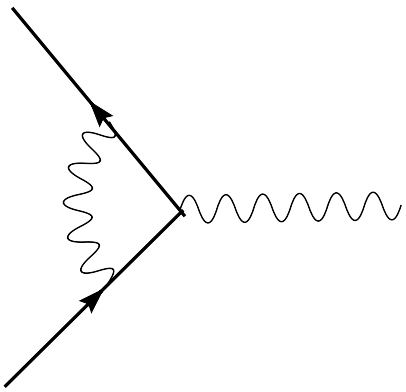


Figure 5.6: The vertex correction, a divergent loop integral.

The methods of calculation are similar to the Compton scattering of the previous section. We substitute the free propagators (5.5.5) and recall that $\hat{D}$ is a sum of three terms, and each $\hat{S}$ is a sum of two terms. Consequently, this expression will be a sum of $(2 \cdot 3 \cdot 2 \cdot 2 = 24)$ terms in total. Substituting into (5.11.2), the propagators will contribute factors of the projection operators P_η, Q_ξ as

$$Q_\xi^{ss_2}(q)\alpha^{r_2's_2s_2'}P_\eta^{r_2'r_1}(p')Q_\zeta^{s_2's_1}(q-p')\alpha^{r_1s_1s_1'}Q_{\xi'}^{s_1's_0}(q), \tag{5.11.3}$$

in that order. Here $\eta = 0, \pm 1$ and $\xi, \xi', \zeta = \pm 1$. The (t_1, t_2)-dependence of the integrand arises solely from the exponential terms in the four factors of $\hat{S}$ and $\hat{D}$, contributing

$$e^{-i\xi E(q)(t_0-t_2)}e^{-i\eta|p'|(t_2-t_1)}e^{-i\zeta E(q-p')(t_2-t_1)}e^{-i\xi' E(q)(t_1-t_0)} \tag{5.11.4}$$

respectively. We may factor out t_0-dependence

$$e^{-i(\xi-\xi')E(q)t_0}, \tag{5.11.5}$$

and we will later show this to be identically equal to unity over the domain of integration. The (t_1, t_2)-integral becomes

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \; e^{i\xi E(q)t_2 - i\zeta E(p+q)(t_2-t_1) - i\eta'|p'|t_1 - i\xi' E(q)t_1}\theta(t_2 - t_1) \tag{5.11.6}$$

We remark that the integrand of (5.11.6) is bounded in magnitude by unity, so for all $p', q, q' \in \mathbb{R}^3$, the integral is bounded by the area of integration, $\frac{1}{2}(t-t_0)^2$. The integral can be done for finite t, t_0, but the answer is not very illuminating. As in the case of

Compton scattering, we are most interested in the limit $t_0 \to -\infty, t \to \infty$. In this limit, we may change variables to $s_1 = t_1 + t_2$ and $s_2 = t_2 - t_1$, and rewrite (5.11.6) as

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{\infty} ds_1 \ \exp\left(i(\xi-\xi')E(q)\frac{s_1}{2}\right) \times \\ \times \int_{-\infty}^{\infty} ds_2 \ \theta(s_2) \exp\left(i\frac{\xi+\xi'}{2}E(q)s_2 - i\eta|p'|s_2 + i\zeta E(q-p')s_2\right)\end{aligned} \tag{5.11.7}$$

(The leading factor of $1/2$ arises from the Jacobian determinant $|\frac{\partial(t_1,t_2)}{\partial(s_1,s_2)}| = 1/2$.) Consider first the terms for which ξ, ξ' have opposite sign. By the identity

$$\int_{-\infty}^{\infty} e^{iks} ds = 2\pi\delta(k), \tag{5.11.8}$$

the s_1-integral is proportional to $\delta(\pm E(q))$. Since $E(q) > m > 0$, this delta-function vanishes for all q. So when ξ, ξ' are of opposite sign, the integral vanishes identically. This is agrees with QED, for otherwise the term would contribute a non-zero positronic correction to the electron, and vice versa.

Consider now the other case of interest, when ξ, ξ' are of like sign. In this case, the s_1 integral becomes the integral of unity over $(-\infty, \infty)$ and diverges. Roughly speaking, this is the correct behaviour if we identify this term with the divergent loop integral (1.3.1). Since both are infinite, it is difficult to make an objective comparison between these two contributions. However, I propose we denote this divergent/infinite quantity by $\delta(0)$ and push on regardless, for the remaining quantities do begin to resemble some recognisable expressions. The s_2-integral can be evaluated using the Fourier representation of the Heaviside function θ (5.7.12) as

$$\begin{aligned}\int_{-\infty}^{\infty} ds_2 \ \theta(s_2) \exp\left(i\xi E(q)s_2 - i\eta|p'|s_2 + i\zeta E(q-p')s_2\right) \\ = \frac{i}{\xi E(q) - \eta|p'| - \zeta E(q-p') + i\epsilon}\end{aligned} \tag{5.11.9}$$

Putting all the pieces of (5.11.2) together,

$$\begin{aligned}\mathcal{T}^{ss_0}(q) = (-ig)^2 \int \frac{d^3p'}{(2\pi)^3} Q_\xi(q)\alpha^{r'} P_\eta^{r'r}(p') Q_\zeta(q-p')\alpha^r Q_\xi(q) \times \\ \times \delta(0)\frac{i}{\xi E(q) - \eta|p'| - \zeta E(q-p') + i\epsilon}\end{aligned} \tag{5.11.10}$$

(Dirac spinor indices have been suppressed). We can relabel the dummy variable p' to p. The ζ-dependence is similar to that of the quantity X computed in (5.7.16), and by

a similar calculation,

$$\mathcal{T}^{ss_0}(q) = \delta(0) \int \frac{d^3p}{(2\pi)^3} Q_\xi(q) P_\eta^{r'r}(p) \gamma^0 \gamma^{r'} \frac{i(\gamma^\mu k_\mu + m)}{k_\mu k^\mu - m^2 + i\epsilon} \gamma^r Q_\xi(q) \tag{5.11.11}$$

where $k = (\xi E(q) - \eta|p|, -q_j + p_j)$. Note that the Feynman propagator has appeared, and the resemblance to the QED matrix element (1.3.1) has begun to appear. As in §5.7, the trailing and leading factors of Q_ξ project both domain and range of $\mathcal{T}$ onto the positive-energy (or negative-energy) subspaces of Λ^1. Similar to (5.7.23), we construct the matrix element $\mathcal{M}$ as the contraction

$$i\mathcal{M} = \langle u, \mathcal{T} u' \rangle \tag{5.11.12}$$

with $Q_\xi u = u$ and $Q_\xi u' = u'$. Substituting,

$$i\mathcal{M} = \delta(0) \sum_{\eta=-,0,+} \int \frac{d^3p}{(2\pi)^3} \bar{u}(q) P_\eta^{r'r}(p) \gamma^{r'} \frac{i(\gamma^\mu k_\mu + m)}{k_\mu k^\mu - m^2 + i\epsilon} \gamma^r u(q). \tag{5.11.13}$$

An obvious remaining difference is the 3-dimensional momentum integral in (5.11.13) compared to the 4-dimensional momentum integral in the QED expression, (1.3.1). However, there are a variety of ways that an additional dummy variable of integration (say p_0) could be introduced. For instance, the residue theorem states

$$\frac{-f(-|p|)}{2|p|} = -i \int_{-\infty}^{\infty} \frac{dp_0}{2\pi} \frac{f(p_0)}{p_\mu p^\mu + i\epsilon} \tag{5.11.14}$$

where f is regular and decays appropriately on the upper complex half-plane. The expression (5.11.13) for $i\mathcal{M}$ is of the same form as the right-hand side of (5.11.14) and the substitution can be made. Note that (5.11.14) brings in a factor of $(p_\mu p^\mu + i\epsilon)^{-1}$ (the Feynman propagator for the photon) in agreement with QED. The result is

$$i\mathcal{M} = 2i\delta(0) \sum_{\eta=-,0,+} \int \frac{d^4p}{(2\pi)^4} \bar{u}(q) \gamma^{r'} \frac{i(\gamma^\mu k_\mu + m)}{k_\mu k^\mu - m^2 + i\epsilon} \gamma^r u(q) \frac{|p| P_\eta^{r'r}(p)}{p_\mu p^\mu + i\epsilon} \tag{5.11.15}$$

where now $k = (\xi E(q) + \eta p_0, -q_j + p_j)$. In this way, the integral is taken over all four spacetime components of the loop momentum p. (5.11.15) can now be compared to the QED expression (1.3.1). There are some intriguing similarities: the necessary factors of the lepton and photon propagators $\tilde{S}^F(q-p)$ and $\tilde{D}^F(p)$ are present in the term $\xi = +1, \eta = -1$. The couplings γ^μ are present. Yet there are also some obvious differences: the presence of the photon projection P_η and an additional factor of $|p|$. I wish to again emphasise that the comparison of divergent quantities must necessarily be considered in

the context of renormalization theory: these quantities are not-well defined without the procedure of regularisation and counterterm subtraction. I am optimistic that analogues of these procedures could be constructed in the differential formalism of this thesis. It is possible that the discrepancies might be resolved in such a framework.

5.12 Remarks

In this chapter, we began with the free-particle dynamics of systems of multiple leptons and photons, described by uncoupled systems of first-order partial differential equations. I proposed a coupling of the e and $e\gamma$ subspaces and performed a perturbative expansion about the free-particle $e\gamma$ solution. Owing to the initial-value nature of the system, this theory gives a concrete parametrisation of the system *during* the scattering, and carries a natural realisation of time evolution. This model therefore provides an answer to *Key Questions 1–2* posed in §1.4–1.5.

However, this answer to *Key Questions 1–2* is only useful if it agrees with QED, since this is known to work so well. Scattering was computed as the deviation from the free-particle $e\gamma$ solution in the limits $t_0 \to -\infty, t \to \infty$. On a particular subspace (cases *i*,*ii*), the scattering was shown to be precisely equal to the s-channel QED diagram for Compton scattering. On other subspaces, it was suggested that a coupling to higher subspaces as in (5.9.3) may resolve the discrepancies. A divergent contribution to the electron self-energy was also found, qualitatively in agreement with QED. However, a more precise comparison of divergences would be most informative in a physically observable loop diagram, such as the vertex correction. This analysis awaits the study of the 3-by-3 extension of R (5.9.3) including a coupling to the $e\gamma\gamma$ subspace.

Case *i* illustrated that both positive- and negative-energy subspaces were needed in the perturbation theory: both signs of $\zeta = \pm 1$ were needed in (5.7.16), corresponding to the internal lepton line in s-channel scattering. This illustrates one important feature arising from the representation of lepton states by functions (namely $\Lambda^1 \equiv L^2(\mathbb{R}^3, \mathbb{C}^4)$). Both positive- and negative-energy subspaces sit naturally in this representation, and both were needed. In contrast, in most presentations of QED the lepton state space is generally divided into electron and positron spaces $\Lambda^1 \equiv Q_+\Lambda^1 \oplus Q_-\Lambda^1$. (As mentioned in case *ii*, §5.8, the separation is further compounded by reversing the momenta k in $Q_-\Lambda^1$.) In the differential system presented in this chapter, both subspaces are

handled together and contribute on the same footing. I feel this is one particularly appealing feature of this framework.

Relativistic covariance of the coupled system was not proved, but the Poincaré test of Chapter 2 should provide the necessary tools to do so — once a fully correct Hamiltonian is identified. As the analysis of the cases *iii–vii* showed, agreement with QED most likely requires modification of the coupling R to include coupling to higher subspaces. This would be required anyway, in order to describe interactions in systems with more particles than just e or $e\gamma$ as considered in this simple model. While such extensions were not considered in this thesis, it seems plausible that a judicious choice of Hamiltonian could also satisfy the Poincaré test. Conversely, the Poincaré test could be used to narrow down candidates for the extended coupling R.

In the next (final) chapter, we will briefly recap the results of this thesis and consider prospects for future work.

Chapter 6

Conclusion and outlook

6.1 Prospects for a differential representation of QED

This thesis presented a coupled system of differential equations as a simple model of QED (Chapter 5). One key feature is the incorporation of the Riemann-Silberstein (RS) representation of the photon (§2.4). The RS representation is based on the electromagnetic field strength and describes a massless particle, with transverse amplitude, transforming in the spin-1 representation of the Lorentz group, and therefore appears to possess all of the key physical characteristics of the photon. However, the RS representation offers significant conceptual advantages over the usual parametrisation of the photon. These include a natural state space with a complete L^2-norm, and relativistic dynamics given by a well-posed initial-value problem.

Because of the initial-value nature of the system, this theory gives a concrete parametrisation of the system during the scattering, and carries a natural realisation of time evolution. This framework therefore contains natural answers to *Key Questions 1–2* posed in §1.4–1.5. The perturbative expansion of this simple model was shown to bear close similarities to the perturbative series of QED, with precise equality on certain subspaces. These encouraging results suggest that a suitable extension of the coupling R may yield a successful differential formulation of QED. Relativistic covariance of the coupled system was not proved, but it seems plausible that the 'Poincaré test' proposed in §2.2 should provide the necessary tools to do so.

As mentioned at the end of §5.9, a mechanism describing pair production/annihilation $e^+e^- \leftrightarrow \gamma$ was not discussed in this thesis, but it seems likely that this process could be included in the framework presented. This would take the form of a

coupling between Θ^1 and the $P_+ \otimes P_-$ subspace of Λ^2. Similar to the e–$e\gamma$ coupling of §5.4, this linear interaction term would couple the free-particle equations of motion on subspaces of different particle content. I anticipate that scattering would again arise at $O(g^2)$, and the results could be compared to e^+e^--scattering of QED.

As mentioned in the concluding remarks of the previous chapter, agreement with QED will likely require modification of the coupling R to include coupling to higher subspaces. I expect that the coupling to the full photon Fock space would take a tridiagonal form similar to the Hamiltonian of the displaced SHO (5.2.17) in §5.2. The expressions for R considered in (5.4.9) and (5.9.3) would be the 2-by-2 and 3-by-3 truncations of an infinite tridiagonal array of couplings. This system would provide an interesting future study. In particular, this system might admit a direct analysis similar to the displaced SHO (cf. (5.2.20)). We speculate that the ground state solutions in this theory might be qualitatively similar to the coherent states (5.2.21). This possibility is particularly appealing, because coherent states are the correct quantum description for the electromagnetic field in the macroscopic limit (a point emphasised by R. Glauber [114, 115], one of the pioneers of quantum optics).

6.2 Non-Abelian field theories and SUSY

QED is only the simplest experimentally proven field theory. An immediate question is whether the framework presented in this thesis could be extended to the other experimentally proven field theories, the non-Abelian gauge theories: electroweak theory and quantum chromodynamics (QCD). The conceptual aims of *Key Questions 1–2* (§1.4–1.5) are equally relevant in these cases, and I speculate that the methods of this thesis might generalise to include these. The non-Abelian analogue of the RS photon would derive from a field-strength representation of the gauge field, just as the RS photon derives from the field strength $\mathbf{E}$ and $\mathbf{B}$ of the electromagnetic field. This question clearly awaits a complete analysis of the simpler case of QED, but at least no obvious obstacles are apparent.

As stated in the introduction, one of the principle motivations of this research was to seek constraints on possible extensions of the standard model. As a speculative example, consider one of the leading contenders for a beyond-the-standard-model theory, supersymmetry (some influential early papers are [118, 119]). The key hypothesis

is a new symmetry of nature relating each species of boson to a fermion 'superpartner', and vice versa. The symmetry transformation mapping a particle's state space to its superpartner's state space is known as a *supergauge transformation*. In the literature, the supergauge transformation is generally only exhibited as an isomorphism of single-particle states. The theoretical framework presented in this thesis disfavours supersymmetry, or at least major structural changes would be required to incorporate the supergauge symmetry. The reason is that the configuration space representations of fermions and bosons are sufficiently different that a symmetry of single-particle state spaces probably does not imply a symmetry of the second-quantised state spaces. In other words, there is no canonical isomorphism between $\mathrm{Sym}(V)$ and $\mathrm{Alt}(V)$, the symmetric and anti-symmetric tensor algebras of a space V. (For example, this is obvious in the case of a finite-dimensional space V, where $\mathrm{Sym}(V)$ is infinite-dimensional while $\mathrm{Alt}(V)$ is finite-dimensional.) A full study would be an interesting area for future research. At the time of writing, recent LHC results have eliminated some previously favoured regions of the supersymmetry parameter space [120], and no direct evidence of supersymmetry has been observed.

Chapter 7

Appendix

7.1 The Poincaré group

The *Poincaré group* describes the set of symmetries of Minkowski spacetime. The generators are

$$\{\underbrace{\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3}_{\text{translation}}, \underbrace{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3}_{\text{space rotation}}, \underbrace{\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3}_{\text{Lorentz boost}}\} \tag{7.1.1}$$

with Lie bracket [65]

$[\cdot,\cdot]$	$\mathcal{P}_0$	$\mathcal{P}_j$	$\mathcal{J}_j$	$\mathcal{K}_j$
$\mathcal{P}_0$	0	0	0	$\mathcal{P}_j$
$\mathcal{P}_i$	0	0	$\epsilon_{ijk}\mathcal{P}_k$	$\delta_{ij}\mathcal{P}_0$
$\mathcal{J}_i$	0	$\epsilon_{ijk}\mathcal{P}_k$	$\epsilon_{ijk}\mathcal{J}_k$	$\epsilon_{ijk}\mathcal{K}_k$
$\mathcal{K}_i$	$-\mathcal{P}_j$	$-\delta_{ij}\mathcal{P}_0$	$\epsilon_{ijk}\mathcal{K}_k$	$-\epsilon_{ijk}\mathcal{J}_k$,

(7.1.2)

where ϵ_{ijk} is the totally antisymmetric quantity with $\epsilon_{123} = 1$. This thesis follows the convention of mathematical texts on Lie group theory (e.g. [121]) and defines generators differing by a factor i from many physics texts. This convention emphasises the real (as opposed to complex) group properties of the Poincaré group. For example, the bracket relation of the rotation generators

$$[\mathcal{J}_i, \mathcal{J}_j] = \epsilon_{ijk}\mathcal{J}_k \tag{7.1.3}$$

is closed with *real* coefficients ϵ_{ijk}, thereby indicating that the rotations form a real subgroup. This fact is obscured if a factor i is introduced into the definition of $\mathcal{J}_j$.

7.2 The Lorentz group

The Lorentz group describes the set of linear isometries of Minkowski space. The generators are

$$\{\underbrace{J_1, J_2, J_3}_{\text{space rotation}}, \underbrace{K_1, K_2, K_3}_{\text{Lorentz boost}}\} \tag{7.2.1}$$

with Lie bracket

$$\begin{aligned} [J_i, J_j] &= \epsilon_{ijk} J_k \\ [K_i, K_j] &= -\epsilon_{ijk} J_k \\ [J_i, K_j] &= \epsilon_{ijk} K_k. \end{aligned} \tag{7.2.2}$$

7.3 Pauli and Dirac matrices

The Pauli matrices σ^μ are Hermitian 2-by-2 matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.3.1}$$

The Dirac α_j and β matrices are Hermitian 4-by-4 matrices

$$\alpha_j = \begin{pmatrix} -\sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.3.2}$$

Properties:

$$\{\alpha^j, \alpha^k\} = \alpha^j \alpha^k + \alpha^k \alpha^j = 2\delta^{jk} \tag{7.3.3}$$

$$\alpha^j \beta = -\beta \alpha^j \tag{7.3.4}$$

$$\beta^2 = 1 \tag{7.3.5}$$

The Dirac γ^μ matrices (in the chiral representation) are defined by

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}. \tag{7.3.6}$$

Equivalently, $\beta = \gamma^0$ and $\alpha^j = \gamma^0 \gamma^j$. Properties:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \tag{7.3.7}$$

7.4 Spin-half representation

The *spin-half representation of the Lorentz group* is

$$J_1 = \frac{-i}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad J_2 = \frac{-i}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad J_3 = \frac{-i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.4.1}$$

and $K_j = \pm iJ_j$. Concisely, $J_j = \frac{-i}{2}\sigma^j$ and $K_j = \pm\frac{1}{2}\sigma^j$. The spin-half representation generates $SL(2,\mathbb{C})$, the group of complex 2-by-2 matrices with unit determinant.

7.5 Spin-one representation

There are various *spin-one representations of the Lorentz group*; this thesis uses

$$J_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \qquad J_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{7.5.1}$$

and $K_j = \pm iJ_j$. Concisely, $(J_j)_{kl} = -\epsilon_{jkl}$ and $(K_j)_{kl} = \mp i\epsilon_{jkl}$. The spin-one representation generates $SO^*(3)$, the group of complex 3-by-3 matrices R for which $R^{-1} = R^T$.

7.6 Left- and Right-handed representations

Left and *right* representations arise because the transformation

$$\{J_j, K_j\} \to \{-J_j^\dagger, -K_j^\dagger\} \tag{7.6.1}$$

gives a distinct representation of the Lorentz algebra (7.2.2), accounting for the $\pm$ in the definitions of K_j above. (7.6.1) maps a finite transformation R to $(R^\dagger)^{-1}$.

7.7 Leptons

The amplitude for a massive free Dirac particle is a pair

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{7.7.1}$$

with ψ_L and ψ_R belonging to the left- and right- handed spin-half representations of the Lorentz group. Thus under a transformation $R \in SL(2,\mathbb{C})$,

$$\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \to \begin{pmatrix} R\psi_L \\ (R^\dagger)^{-1}\psi_R \end{pmatrix}. \tag{7.7.2}$$

The equation of motion for free particles (the Dirac equation) is

$$(i\gamma^\mu \partial_\mu - m)\,\psi = 0. \tag{7.7.3}$$

(7.7.3) can be written in the form

$$\frac{\partial \psi}{\partial t} = -iH\psi \tag{7.7.4}$$

where

$$H = -i\alpha^j \partial_j + \beta m \tag{7.7.5}$$

(These conventions follow Itzykson and Zuber [36]).

7.8 Riemann-Silberstein photon

Photons are described by an $SO^*(3)$-covariant amplitude

$$W \to RW, \qquad R \in SO^*(3) \tag{7.8.1}$$

The equation of motion in free space (Maxwell's equations) is

$$\frac{\partial}{\partial t} W_t = -iHW_t \tag{7.8.2}$$

$$= -i\nabla \times W_t \tag{7.8.3}$$

$$= -K_j \partial_j W_t \tag{7.8.4}$$

7.9 Fourier transforms

A review of properties of the Fourier transform on $\mathbb{R}^n$ follows; this will also serve to fix notation. Conventions follow Lang [88].

The Fourier transform for $f \in L^2 \cap L^1$ is defined by the integral transform

$$\tilde{f}(k) = \int d^3x\; f(x) e^{-ikx}. \tag{7.9.1}$$

The Fourier transform is the unique extension of (7.9.1) to all L^2. The Fourier transform is an isometry of L^2 onto itself, in the sense that

$$\langle f, g\rangle = \int d^3x\; f^*(x) g(x) = \int \frac{d^3k}{(2\pi)^3} \tilde{f}^*(k)\tilde{g}(k) = \langle \tilde{f}, \tilde{g}\rangle \tag{7.9.2}$$

(the *Parseval relation*). As an isometric automorphism, the Fourier transform possesses a well-defined inverse $\tilde{f} \mapsto f$; this inverse also has an integral representation,

$$f(x) = \int \frac{d^3k}{(2\pi)^3} \tilde{f}(k) e^{ikx} \tag{7.9.3}$$

for $\tilde{f} \in L^2 \cap L^1$.

Alternatively, the integral transform (7.9.1) is well-defined for all $f \in L^1$; this transform on L^1 is also called the Fourier transform. In this case, $f \mapsto \tilde{f}$ is a map from L^1 into L^∞ with the bound

$$\|\tilde{f}\|_\infty \leq \|f\|_1. \tag{7.9.4}$$

L^∞ and L^1 are Banach algebras, where the product on L^∞ is pointwise multiplication and the product on L^1 is the *convolution*

$$(f * g)(x) = \int d^3x' \; f(x') g(x - x'). \tag{7.9.5}$$

The Fourier transform is a Banach-algebra endomorphism, in the sense that it 'preserves products':

$$\widetilde{f * g}(k) = \tilde{f}(k)\tilde{g}(k). \tag{7.9.6}$$

(7.9.6) is known as the *Fourier convolution theorem*.

The Fourier transform on L^1 is not surjective; there exist L^∞ functions which are not the Fourier transform of any L^1 function. This is not really surprising, for an integral transform like (7.9.3) applied to an L^∞ function would probably diverge, and neither L^1 nor L^2 are dense in L^∞. Examples of L^∞ functions which do *not* possess an inverse Fourier transform in L^1 are important in the energy projections discussed later in this section.

If $G \in L^1(\mathbb{R}^3)$ and G depends only on the radial coordinate r, i.e. $G(\mathbf{x}) = G(r)$, then the angular integrals in (7.9.1) can be done. As a result, $\tilde{G}$ depends only on $k = |\mathbf{k}|$ and

$$\tilde{G}(\mathbf{k}) = \frac{4\pi}{k} \int_0^\infty G(r) r \sin(kr) \; dr. \tag{7.9.7}$$

Furthermore, let $\hat{\mathbf{r}}$ denote the unit vector in the radial direction. The vector-valued function $G(r)\hat{\mathbf{r}}$ is also $L^1(\mathbb{R}^3)$ and

$$\tilde{G\hat{\mathbf{r}}}(\mathbf{k}) = \frac{4\pi i}{k^2} \hat{\mathbf{k}} \int_0^\infty G(r) \left(kr\cos(kr) - \sin(kr)\right) \; dr \tag{7.9.8}$$

7.10 Causal functions

The retarded Green's function is

$$D^{\text{ret}}(x) = \frac{1}{2\pi}\left(\delta(\tau^2) - \Theta(\tau^2)\frac{m}{2\tau}J_1(m\tau)\right) \tag{7.10.1}$$

for x forward timelike (with $\tau = \sqrt{t^2 - r^2}$), and 0 otherwise.

The Pauli-Jordan commutation function D is real and antisymmetric under time reversal.

$$D(x) = \frac{1}{2\pi}\text{sgn}(x^0)\left(\delta(\tau^2) - \Theta(\tau^2)\frac{m}{2\tau}J_1(m\tau)\right) \tag{7.10.2}$$

$D(x)$ may be decomposed into the retarded and advanced Green's functions:

$$D^{\text{ret}}(x) = \Theta(x^0)D(x)$$
$$D^{\text{adv}}(x) = \Theta(-x^0)D(-x)$$

We reserve the term 'Green's function' for describing a distribution which solves the initial-value problem. Any such distribution must vanish for spacelike x. Further details are available in [122, pp. 353–354] and [123, pp. 87–93]. A detailed treatment of Bessel functions is available in [40].

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