SPACE TILING ZONOTOPES

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Abstract. A d-dimensional zonotope Z in E^d which is the vector sum of n line segments is linearly equivalent to the image of a regular n-cube under some orthogonal projection. The zonotope $\bar{Z}$ in E^{n-d} which is the image of the same cube under projection on to the orthogonal complementary subspace is said to be associated with Z. In this paper is proved a conjecture of G. C. Shephard, which asserts that, if Z tiles E^d by translation, with adjacent zonotopes meeting facet against facet, then $\bar{Z}$ tiles E^{n-d} in the same manner. A number of conditions, conjectured by Shephard and H. S. M. Coxeter to be equivalent to the tiling property, are also proved.

§1. *Introduction.* In a recent paper, Shephard [1974b] has considered a number of properties of zonotopes, or vector sums of line segments. One main theme of his paper, which carries on his and the present author's earlier work (McMullen [1971], Shephard [1974a]), is the investigation of the relationship between associated zonotopes. This term will be defined more rigorously below, but, roughly speaking, two zonotopes are associated if they are the images of a regular cube under orthogonal projection on to orthogonal complementary subspaces.

The basic property he considered was that of tiling space; that is, covering space by translates of the zonotope, in such a way that the intersection of any two translates is empty or a common face of each. He showed that, provided the zonotope is at most four-dimensional, if it tiles space, then so does the associated zonotope. He also showed, under the same restrictions, that various conditions on zonotopes are equivalent to the tiling property; one of these is due, originally, to Coxeter [1962].

Shephard conjectured that his relationships held without the restriction on the dimension; it is this conjecture that we shall establish here. We shall often, in fact, prove results which are slightly stronger than Shephard's. For example, we shall show that, if a zonotope tiles space, then every zonotope which is combinatorially isomorphic to it must actually be equivalent to it (that is, linearly equivalent, except possibly for changes in the lengths of its component line segments), and so it tiles space also.

Throughout we shall adopt Shephard's notation (with minor changes); however, to make the paper self contained, we shall repeat the statements of his conditions.

§2. *Statement of conditions and theorems.* By a *zonotope* we mean a vector (or Minkowski) sum of line segments, say $Z = S_1 + \dots + S_n$. The line segments S_i are called the *components* of Z; there is no essential loss of generality in assuming them to be of the form $S_i = \operatorname{conv}\{x_i, -x_i\}$ $(i = 1, \dots, n)$, so that the segments, and hence Z itself, are centrally symmetric about the origin o. We shall assume that $\operatorname{aff} Z = E^d$, and, to avoid certain trivial complications, that Z is not a prism, and that no $x_i = o$. (If these latter conditions are violated, some of the statements below may need modification, but these modifications will not affect the truth of the theorems.)

However, we shall allow two components to be parallel, so that the particular representation of a zonotope as a sum of segments is not irrelevant.

Each face of Z is of the form

$$F = \varepsilon_1 x_{1\sigma} + \dots + \varepsilon_k x_{k\sigma} + S_{(k+1)\sigma} + \dots + S_{n\sigma},$$

for some $\varepsilon_i = \pm 1$ and some permutation σ of $\{1, \dots, n\}$. The components S_i are just those which lie in the hyperplane through o parallel to one which supports Z in F, and the points $\varepsilon_i x_{i\sigma}$ all lie on one side of this hyperplane. (For proofs of these and subsequent combinatorial properties of zonotopes, see McMullen [1971].) F is itself a zonotope, whose centre is $\varepsilon_1 x_{1\sigma} + \dots \varepsilon_k x_{k\sigma}$. In particular, let the facets ($(d-1)$-faces) of Z be denoted $\pm F_1, \dots, \pm F_r$, and the centre of F_j by

$$c_j = \sum_{i=1}^{n} \varepsilon_{ij} x_i,$$

where $\varepsilon_{ij} \in \{-1, 0, 1\}$. We shall write

$$e_i = (\varepsilon_{i1}, \dots, \varepsilon_{ir}) \qquad (i = 1, \dots, n),$$

$$\tilde{e}_j = (\varepsilon_{1j}, \dots, \varepsilon_{nj}) \qquad (j = 1, \dots, r).$$

Later, u_j shall denote an outer normal vector to the facet F_j.

As a general point of notation, a capital letter shall denote the matrix whose rows are the corresponding minuscule letters. Thus,

$$X = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

We shall also use $X = (x_1, \dots, x_n)$ (and so on) to denote the ordered set of these vectors; however, no confusion should be caused by this, as the particular meaning will be clear from the context. As usual, X^T is the transpose of the matrix X.

We shall be concerned in this paper with a number of conditions on the zonotope Z or on the set of vectors X; we shall label these conditions by roman numerals. The first of these is the most basic.

I *Z tiles E^d*. By this we mean that there exists a set Λ of vectors of E^d, such that the translated zonotopes $Z + t$ $(t \in \Lambda)$ cover E^d, and any intersection $(Z + t_1) \cap (Z + t_2)$ is empty or a common face of each translate. In particular, this implies that Λ is a lattice (discrete additive subgroup of E^d), and that $2c_j \in \Lambda$ $(j = 1, \dots, r)$. In fact, we shall later show that $\Lambda = 2\langle C \rangle$, the lattice generated by $2c_1, \dots, 2c_r$.

The next condition is due to Coxeter [1962], who proved that it is a necessary condition for Z to tile E^d. We can derive from Z in two ways zonotopes which are the sum of fewer segments. First, we write

$$Z \sim S_i = S_1 + \dots + S_{i-1} + S_{i+1} + \dots + S_n,$$

which we say is obtained from Z by *contracting* S_i. In general, the sum of a subset of $S_1, \dots, S_n$ is a *contraction* of Z. Second, let ϕ_i be a linear map with kernel lin S_i, and let $S'_k = S_k \phi_i$. Then we write

$$Z/S_i = S'_1 + \dots + S'_{i-1} + S'_{i+1} + \dots + S'_n,$$

which we say is obtained from Z by *natural projection* in direction S_i. More generally, if L is a *spanned subspace* of E^d, by which we mean a linear subspace spanned by the segments S_i (or points $x_i \in X$) which it contains, and Z' is the image of Z under a linear map with kernel L, we say that Z' is the image of Z under a *natural projection*. In talking about Z', we ignore the segments of zero length arising from the components of Z lying in L.

II *Every spanned $(d-2)$-space of E^d is contained in* 2 *or* 3 *spanned hyperplanes.* Equivalently, every 2-dimensional image of Z under a natural projection is a parallelogram or a hexagon.

The next two conditions relate to the matrix E, whose rows are $e_1, \ldots, e_n$, and whose columns are $\tilde{e}_1, \ldots, \tilde{e}_r$.

III $\operatorname{rank} E = d$.

IV *Let* $T_i = \operatorname{conv}\{e_i, -e_i\}$ $(i = 1, \ldots, n)$, *and* $Y = T_1 + \ldots + T_n \subseteq E^r$. *Then Y is a zonotope combinatorially isomorphic* (*or equivalent, see below*) *to Z.*

We say that an ordered set $X' = (x'_1, \ldots, x'_n)$ is *equivalent* to $X = (x_1, \ldots, x_n)$, if there is a linear mapping ϕ, one to one on $\operatorname{lin} X$, and scalars $\lambda_1, \ldots, \lambda_n > 0$, such that $x'_i = \lambda_i x_i \phi$ $(i = 1, \ldots, n)$. This induces an *equivalence* of the corresponding zonotopes. Clearly, equivalent zonotopes are combinatorially isomorphic, but the converse is, in general, false. It is also clear that, if Z tiles E^d, then so does every zonotope equivalent to Z. Our next two conditions are concerned with the set X which determines Z.

V *There is some set X' equivalent to X, such that $\pm X'$ lies in each spanned hyperplane, and two hyperplanes parallel to it.*

VI *There is some set X' equivalent to X, such that whenever*

$$\{\varepsilon_1 x'_{1\sigma}, \ldots, \varepsilon_k x'_{k\sigma}\} \subseteq \pm X'$$

is the set of vertices of a simplex with o in its relative interior, then

$$\varepsilon_1 x'_{1\sigma} + \ldots + \varepsilon_k x'_{k\sigma} = o.$$

The *Voronoi polytope* of a lattice in E^d is the set of points of E^d which are no further from o than from any other lattice point. A Voronoi polytope need not be a zonotope; for example, the regular 24-cell in E^4 is the Voronoi polytope of the centres of the densest lattice packing of spheres. If it is, then the centre c_j is a normal vector to its facet F_j, in which case we call the zonotope *regular*.

VII *Some zonotope equivalent to Z is the Voronoi polytope of a lattice.*

We can now state our first theorem.

THEOREM 1. *The conditions* I–VII *on a zonotope Z are equivalent.*

Before we can formulate the next result, we must formally define an associated zonotope. We first do this geometrically, enlarging on the remarks in the introduction. Every d-zonotope Z with n components is linearly equivalent to the image under orthogonal projection of some regular n-cube. (In this context, we shall regard E^d as embedded as a subspace of E^n.) Any zonotope $\bar{Z}$ which is linearly equivalent to the image of the same n-cube under orthogonal projection on to the orthogonal complementary subspace E^{n-d} is said to be *associated* with Z. (In McMullen [1971], where this concept was introduced, the term *derived* was used; we now prefer the term used here, which is due to Shephard [1974a].)

Algebraically, the condition is as follows. If $X = (x_1, \ldots, x_n)$ linearly spans E^d, let $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_n)$ be any set of vectors in E^{n-d} such that the $n \times n$ matrix

$$[X \mid \bar{X}] = \left[\begin{array}{c|c} x_1 & \bar{x}_1 \\ \vdots & \vdots \\ x_n & \bar{x}_n \end{array}\right]$$

is non-singular, and is such that each of its last $n - d$ columns is orthogonal to each of its first d. Then $\bar{X}$ is called a *linear transform* (or *representation*) of X. If X corresponds to the zonotope Z, and we define $\bar{S}_i = \operatorname{conv}\{\bar{x}_i, -\bar{x}_i\}$ $(i = 1, \ldots, n)$ and $\bar{Z} = \bar{S}_1 + \ldots + \bar{S}_n$, then $\bar{Z}$ is *associated* with Z. We may observe that the relationship (as defined here, or geometrically as above) between Z and $\bar{Z}$ (or between X and $\bar{X}$) is symmetrical. It should be noted that each *linear dependence* of X, that is, a vector $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 x_1 + \ldots + \alpha_n x_n = o$, is such that $\alpha_i = \langle a, \bar{x}_i \rangle$ $(i = 1, \ldots, n)$ for some $a \in E^{n-d}$; conversely, every $a \in E^{n-d}$ gives rise to a linear dependence of X in this way. Thus the last $n - d$ columns of the matrix $[X \mid \bar{X}]$ form a basis for the space $L(X)$ of linear dependences of X, and conversely, any basis of $L(X)$ gives rise to a linear transform of X. In particular, the linear transform $\bar{X}$ of X is determined by X up to linear equivalence.

We shall label with I*, ..., VII* the conditions I, ..., VII as applied to the associated zonotope $\bar{Z}$. Our second result is then:

THEOREM 2. *The conditions* I *and* I* *are equivalent*; *that is, the zonotope* Z *tiles* E^d, *if, and only if, its associated zonotope* $\bar{Z}$ *tiles* E^{n-d}.

We have two further results of rather less importance. We call a zonotope Z *zonally stable*, if every zonotope combinatorially isomorphic to Z is equivalent to Z. The third result is, in fact, a corollary of the proof of the first two theorems.

THEOREM 3. *If* Z *tiles* E^d, *then* Z *is zonally stable.*

Our final result is fairly intriguing. We have already introduced the notation $\langle C \rangle$ for the set of all integer combinations of the centres $c_1, \ldots, c_r$ of the facets of Z; $\langle \bar{C} \rangle$ is derived similarly from $\bar{Z}$. If Z tiles E^d, then $\langle C \rangle$ is a lattice.

THEOREM 4. *If* Z *tiles* E^d, *with* X *itself* (*and not merely some set equivalent to* X) *satisfying the condition of* V, *then for integers* $v_1, \ldots, v_n$, $\sum_{i=1}^{n} v_i x_i \in \langle C \rangle$, *if, and only if,* $\sum_{i=1}^{n} v_i \bar{x}_i \in \langle \bar{C} \rangle$.

§3. *Proofs of the theorems.* We shall set out our proof of Theorems 1 and 2 in the form of a series of lemmas. For the reader's convenience, we give a scheme of this proof.

$$\begin{tikzcd}
\mathrm{I} \arrow[r, Rightarrow] & \mathrm{II} \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & \mathrm{III} \arrow[rr, Rightarrow] & & \mathrm{I} \\
& & & \mathrm{IV} \arrow[ul, Rightarrow] & \\
& \mathrm{VI}^* \arrow[r, Leftrightarrow] & \mathrm{V} \arrow[uu, Rightarrow] \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] \arrow[d, Rightarrow] & & \mathrm{VII} \arrow[uu, Rightarrow] \\
& & \mathrm{VI} & &
\end{tikzcd}$$

Our first lemma has already been stated informally in the previous section; it and the second lemma are due to Shephard [1974b], where the proofs can be found.

LEMMA 1. *If the zonotope Z tiles E^d, then so does every zonotope equivalent to Z.*

LEMMA 2. *If $Z = S_1 + \ldots + S_n$ tiles E^d, then $Z \sim S_i$ tiles E^d and Z/S_i tiles E^{d-1}.*

LEMMA 3 (Coxeter [1962]). I $\Rightarrow$ II.

For, the lemma is trivial if $d = 2$. For $d \geqslant 3$, just take the natural projection of Z in the direction of the given spanned $(d-2)$-space.

For convenience, we now adopt a certain convention. We suppose (after suitable relabelling, if necessary) $\{x_1, \ldots, x_d\}$ to be linearly independent vectors of X. For each $j = 1, \ldots, d$, the subset of $d-1$ vectors omitting x_j spans a hyperplane H_j parallel to the affine hulls of a pair of facets $\pm F_j$. Interchanging $\pm$ if necessary, we see that the centre of F_j is of the form

$$c_j = x_j + \sum_{i=d+1}^{n} \varepsilon_{ij} x_i,$$

for some $\varepsilon_{ij} \in \{-1, 0, 1\}$. Clearly, both $\{e_1, \ldots, e_d\}$ and $\{\tilde{e}_1, \ldots, \tilde{e}_d\}$ are linearly independent; $\{c_1, \ldots, c_d\}$ is also linearly independent, as will become more clear from the next lemma.

LEMMA 4. II $\Rightarrow$ III.

Since, clearly, rank $E \geqslant d$, we must show that each other column vector $\tilde{e}_k$ of E is linearly dependent on $\tilde{e}_1, \ldots, \tilde{e}_d$. Consider the corresponding centre c_k, and its facet F_k. Pick an arbitrary linearly independent set $\{y_1, \ldots, y_d\} \subseteq \pm X$, $d-1$ of whose vectors are parallel to F_k. We can find a sequence of linearly independent subsets of $\pm X$, going from $\{x_1, \ldots, x_d\}$ to $\{y_1, \ldots, y_d\}$, of which successive sets differ by one vector. In considering the relationship between the centres of the facets corresponding to the subsets of the sequence, it is clear that we need only consider the change in passing to an adjacent subset. There is no loss in generality in going from $\{x_1, \ldots, x_d\}$ to $\{x_1, \ldots, x_{d-1}, x_{d+1}\}$ (say). Let the new centres derived from the latter set be $c'_1, \ldots, c'_{d-1}, c'_{d+1}$, with the corresponding columns of E being $\tilde{e}'_1, \ldots, \tilde{e}'_{d-1}, \tilde{e}'_{d+1}$. By changing signs and reordering, if necessary, we can suppose

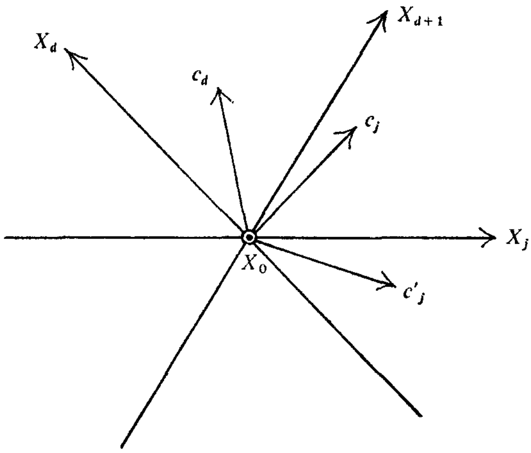

Fig. 1

that $x_{d+1} \in \text{rel int pos}\{x_{k+1}, \ldots, x_d\}$, with $k < d - 1$, since otherwise the result is trivial. For $j \leqslant k$,

$$\text{lin}\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d\} = \text{lin}\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d-1}, x_{d+1}\},$$

so that $c'_j = c_j$ and $\tilde{e}'_j = \tilde{e}_j$. Clearly also $c'_{d+1} = c_d$, so that $\tilde{e}'_{d+1} = \tilde{e}_d$. Finally (compare Fig. 1), II implies that, for $k+1 \leqslant j \leqslant d-1$, $c'_j = c_j - c_d$ and $\tilde{e}'_j = \tilde{e}_j - \tilde{e}_d$. For, the vectors in X fall into four classes: X_0 consisting of those in $\text{lin}\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{d-1}\}$, and, for $i = j, d, d+1$, X_i consisting of those in $X \setminus X_0$ lying in the hyperplane $H_i = \text{lin}(X_0 \cup \{x_i\})$. Changing signs in X if necessary, we can suppose that X_i lies on the same side of the other hyperplanes as x_i. Then, with an obvious notation, $c_j = \sum (X_j \cup X_{d+1})$, $c'_j = \sum (X_j \cup (-X_d))$, $c_d = c'_{d+1} = \sum (X_d \cup X_{d+1})$, and the lemma follows.

We may remark that the converse of Lemma 4 is very easy to prove.

LEMMA 5. III *implies that* $\langle C \rangle = \langle c_1, \ldots, c_d \rangle$ *is a lattice.*

For, we observe that, for $1 \leqslant i, j \leqslant d$, $\varepsilon_{ij} = \delta_{ij}$ ($= 0$ or 1 as $i =$ or $\neq j$). Thus, for $j = d+1, \ldots, r$, $\tilde{e}_j = \sum_{i=1}^{d} \varepsilon_{ij} \tilde{e}_i$, since $\text{rank}\, E = d$, and $\tilde{e}_1, \ldots, \tilde{e}_d$ are linearly independent. We conclude that $c_j = \sum_{i=1}^{d} \varepsilon_{ij} c_i$ for $j = d+1, \ldots, r$, and the assertion of the lemma is now obvious.

We now quote a result of Shephard [1974b], which is comparatively easy to check. Temporarily, we write $E(Z)$ for E, to emphasize its dependence on Z.

LEMMA 6. $\text{rank}\, E(Z \sim S_i) \leqslant \text{rank}\, E(Z)$; $\text{rank}\, E(Z/S_i) \leqslant \text{rank}\, E(Z) - 1$.

In particular, if $\text{rank}\, E(Z) = d$, then $\text{rank}\, E(Z \sim S_i) = d$ and

$$\text{rank}\, E(Z/S_i) = d - 1.$$

LEMMA 7. III $\Rightarrow$ I.

We assume the result of Lemma 5, so that $\langle C \rangle$ is a lattice. Since the tiling property is clear for a single segment, we can make the inductive assumption that the lemma holds for zonotopes which are the sum of $n-1$ segments. It is clear that the translates $Z + 2c$ $(c \in \langle C \rangle)$ cover E^d, since Z is completely surrounded by the translates $Z \pm 2c_j$ $(j = 1, \ldots, r)$, so we have only to show that no two of these translates overlap. Since the hyperplane $H = \operatorname{lin}\{c_2, \ldots, c_d\}$ is not parallel to S_1, we can take the natural projection of Z in direction S_1 to be on to H. Then, by Lemma 6 and the inductive assumption, Z/S_1 tiles H, with lattice of centres $2\langle c_2, \ldots, c_d \rangle$. We deduce that the zonotopes $Z + 2c$ $(c \in \langle c_2, \ldots, c_d \rangle)$ do not overlap, and so form a layer, $\mathscr{L}_0$ say, about H.

The zonotopes $Z + 2c$ $(c \in \langle C \rangle)$ now fall into parallel layers $\mathscr{L}_k$, where k is determined by $c = kc_1 + \ldots$. Since each zonotope $Z + 2c$ contains a translate of S_1, we easily see that we need now only show that adjacent layers do not overlap. This, in turn, will clearly follow, if $\operatorname{int}(Z_0 \cap (Z + 2c_1)) = \varnothing$ for each $Z_0 \in \mathscr{L}_0$. We now consider the contraction of Z in direction S_1. If $Z' = Z \sim S_1$ and $c'_1 = c_1 - x_1$, then, again by Lemma 6 and the inductive hypothesis, $2\langle c'_1, c_2, \ldots, c_d \rangle$ is (possibly a sublattice of) the lattice of centres of the tiling of E^d by Z'. (We enlarge on this comment below.) If $\mathscr{L}'_0$ is the corresponding contraction of the layer $\mathscr{L}_0$, then $\operatorname{int}(Z'_0 \cap (Z' + 2c'_1)) = \varnothing$ for $Z'_0 \in \mathscr{L}'_0$. But we obtain Z_0 from Z'_0 by adding S_1, and $Z + 2c_1$ from $Z' + 2c'_1$ by adding S_1 and translating by $2x_1$. It follows that $\operatorname{int}(Z_0 \cap (Z + 2c_1)) = \varnothing$, as we wished to show. This proves the lemma.

We remark that an apparent difficulty may arise in contracting Z to $Z' = Z \sim S_1$, since we may contract the facet F_j, with centre c_j, to a $(d-2)$-face F, so that c_j is no longer a facet centre of Z'. If b_1 and b_2 are the centres of the facets of Z' which contain F, the corresponding facets of Z have centres $b_1 \pm x_1$ and $b_2 \mp x_1$. Since $\operatorname{rank} E = d$, by choosing a basis of E^d from X with $d-2$ of its vectors in the linear subspace parallel to $\operatorname{aff} F$, and the remaining two parallel to these facets, we see that $c_j = (b_1 \pm x_1) + (b_2 \mp x_1) = b_1 + b_2$ is in the lattice generated by the facet centres of Z' (see Fig. 2). (Compare here the remark after Lemma 4.) We note incidentally

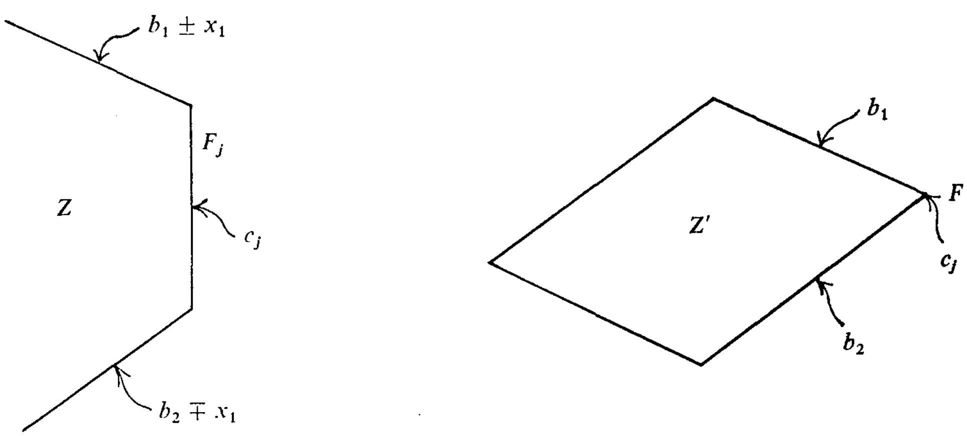

Fig. 2

that the proof of the lemma shows that $2\langle c'_1, c_2, \ldots, c_d\rangle$ must be the whole lattice of centres for the tiling of E^d by Z'.

We have now established the equivalence of I, II and III, which we shall henceforth assume without comment. We shall now show that the remaining conditions are implied by II, and imply III, except that VII clearly implies I directly.

LEMMA 8. II $\Rightarrow$ VI*.

By our conventions, $\{x_1, \ldots, x_d\}$ is a basis of E^d, and hence, by a standard result in the theory of linear transforms (it follows easily from the definition), $\{\bar{x}_{d+1}, \ldots, \bar{x}_n\}$ is a basis of E^{n-d}. Let $u_1, \ldots, u_d$ be normal vectors to the facets $F_1, \ldots, F_d$, chosen so that $\langle x_i, u_j\rangle = \delta_{ij}$ $(i, j = 1, \ldots, d)$, and let $\lambda_{ij} = \langle x_i, u_j\rangle$ $(i = d+1, \ldots, n;$ $j = 1, \ldots, d)$. Thus $\varepsilon_{ij} = \operatorname{sgn} \lambda_{ij}$ $(= -1, 0$ or 1 as λ_{ij} is negative, zero or positive). For $j = 1, \ldots, d$, we have the relations corresponding to c_j and $\tilde{e}_j$: $\bar{x}_j + \sum_{i=d+1}^{n} \lambda_{ij} \bar{x}_i = o$ (see the discussion before the statement of Theorem 2). The condition II implies that the ratios $\lambda_{i_1 j_1}/\lambda_{i_2 j_1}$ and $\lambda_{i_1 j_2}/\lambda_{i_2 j_2}$ are equal whenever all four λ_{ij} are non-zero (compare Fig. 1 again). Hence, if we define $\bar{x}'_i = \bar{x}_i$ for $i = d+1, \ldots, n$, and $\bar{x}'_1, \ldots, \bar{x}'_d$ by $\bar{x}'_j + \sum_{i=d+1}^{n} \varepsilon_{ij} \bar{x}_i = o$, then the mapping

$$(\bar{x}_1, \ldots, \bar{x}_n) \mapsto (\bar{x}'_1, \ldots, \bar{x}'_n)$$

is an equivalence. (The reader may find it helpful here to think in terms of cross-ratios and projectivities.) To check the condition VI*, suppose we have a relation $\sum_{i=1}^{n} \mu'_i \bar{x}'_i = o$, giving the origin as a relatively interior point of a simplex with vertices from $\pm X'$. We then have a corresponding relation $\sum_{i=1}^{n} \mu_i \bar{x}_i = o$ (since $\bar{X}$ and $\bar{X}'$ are equivalent), and so there is a vector $v \in E^d$, such that $\mu_i = \langle x_i, v\rangle$ $(i = 1, \ldots, n)$, and those x_i for which $\mu_i = 0$ span a hyperplane (see McMullen [1971]). That is, v is a normal vector to some facet F_j, and $\varepsilon_{ij} = \operatorname{sgn} \mu_i$ $(i = 1, \ldots, n)$. Since $\tilde{e}_j = \sum_{i=1}^{d} \varepsilon_{ij} \tilde{e}_i$ is a linear combination of $\tilde{e}_1, \ldots, \tilde{e}_d$ by III, it follows from the definition of $\bar{x}'_i$ that $\sum_{i=1}^{n} \varepsilon_{ij} \bar{x}'_i = o$. This completes the proof of the lemma.

LEMMA 9. VI* $\Leftrightarrow$ V.

This is, in fact, an elementary exercise in the theory of linear transforms. For, if two sets of points are equivalent, then so are their linear transforms. So, if (after performing a suitable equivalence) $\bar{X}$ satisfies VI*, then X has the property that, corresponding to each spanned hyperplane H_j and pair of facets $\pm F_j$ of Z, is a normal vector u_j such that $\langle x_i, u_j\rangle = \varepsilon_{ij} \in \{-1, 0, 1\}$. (Compare the proof of Lemma 8.) That is, X satisfies V. The argument is completely reversible, so we have proved the lemma.

LEMMA 10. V $\Leftrightarrow$ VI.

For, V and VI are both equivalent to the assertion that (after a suitable equivalence has been performed) if we choose any basis of E^d from X, then every other vector of X is a linear combination of these basis vectors, with coefficients 0 or ± 1.

LEMMA 11. V $\Rightarrow$ III, IV *and* VII.

Again, let us suppose that X itself satisfies V. Then, with the normal vectors u_j chosen as in the proof of Lemma 9, we have $E = XU^T$, and so

$$d \leqslant \operatorname{rank} E \leqslant \min\{\operatorname{rank} X, \operatorname{rank} U\} = d.$$

This proves III, and also shows that E is linearly equivalent to X, which proves IV. Moreover, by performing some linear equivalence, we can suppose that the columns of X form an orthonormal set (again, we refer the reader back to the definition of a linear transform). Thus, $X^T X = I_{n-d}$, and hence $C = E^T X = UX^T X = U$. In other words, the centres of the facets are themselves the normal vectors to the facets. Since I holds, it follows that Z is a Voronoi polytope, and we have proved VII. This establishes the lemma.

We now note in conclusion that the assertion VII $\Rightarrow$ I is trivial, as is the assertion IV $\Rightarrow$ III. Thus we see that we have completed the proofs of Theorems 1 and 2.

In fact, we have proved Theorem 3 in passing. For, we have shown that the initial set X is always equivalent to the set E (under the assumption that Z tiles space). But E depends only upon the combinatorial type of Z; the assertion of Theorem 3 follows at once.

It remains for us to prove Theorem 4. For this, first suppose $\mu_1, \ldots, \mu_n$ to be integers such that $\sum_{i=1}^{n} \mu_i \bar{x}_i = o$. Subtracting the relation

$$\sum_{j=1}^{d} \mu_j \left(\bar{x}_j + \sum_{i=d+1}^{n} \varepsilon_{ij} \bar{x}_i\right) = o$$

(compare the proof of Lemma 8), we conclude (since $\bar{x}_{d+1}, \ldots, \bar{x}_n$ are linearly independent) that, for $i = d+1, \ldots, n$, $\mu_i - \sum_{j=1}^{d} \varepsilon_{ij} \mu_j = 0$. Recalling the definition of c_j, we see that $\sum_{i=1}^{n} \mu_i x_i = \sum_{j=1}^{d} \mu_j c_j \in \langle C \rangle$.

Now suppose $\nu_1, \ldots, \nu_n$ to be integers such that $\sum_{i=1}^{n} \nu_i x_i \in \langle C \rangle$. Then there are integers $\lambda_1, \ldots, \lambda_d$ such that $\sum_{i=1}^{n} \nu_i x_i = \sum_{j=1}^{d} \lambda_j c_j$. Thus

$$\sum_{i=1}^{n} \left(\nu_i - \sum_{j=1}^{d} \varepsilon_{ij} \lambda_j\right) x_i = o,$$

and it follows, by the first part of the proof, and the symmetry between X and $\bar{X}$, that

$$\sum_{i=1}^{n} \nu_i \bar{x}_i = \sum_{i=1}^{n} \left(\nu_i - \sum_{j=1}^{d} \varepsilon_{ij} \lambda_j\right) \bar{x}_i \in \langle \bar{C} \rangle.$$

This completes the proof of Theorem 4.

Theorem 4 has a suggestive geometrical interpretation. Let W be a regular n-cube in E^n, whose images under orthogonal projection on to orthogonal complementary subspaces of E^n are Z and $\bar{Z}$. Then Theorem 4 states that the same cubes in the tiling of E^n by W (in the usual way), which are projected onto translates of Z in its tiling of E^d, are also projected onto translates of $\bar{Z}$ in its tiling of E^{n-d}. This hints at a possibly more direct proof of Theorem 2, although we should emphasize that Z and $\bar{Z}$ are specially chosen to satisfy the condition of V. However, we have been unable to find such a proof.

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