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Partition problems in discrete geometry

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I, Pablo Soberon Bravo confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

''Just as every human undertaking pursues certain objectives, so also the mathematical research requires its problems. It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon.''

David Hilbert, 1900

Abstract

This Thesis deals with the following type of problems, which we denote *partition problems*,

Given a set X in $\mathbb{R}^d$, is there a way to partition X such that the convex hulls of all parts satisfy certain combinatorial properties?

We focus on the following two kinds of partition problems.

- **Tverberg type partitions.** In this setting, one of the properties we ask the sets to satisfy is that their convex hulls all intersect.
- **Ham sandwich type partitions.** In this setting, one of the properties we ask the sets to satisfy is that the interior of their convex hulls are pairwise disjoint.

The names for these types of partitions come from the quintessential theorem from each type, namely Tverberg's theorem and the ham sandwich theorem. We present a generalisation and a variation of each of these classic results.

The generalisation of the ham sandwich theorem extends the classic result to partitions into any arbitrary number of parts. This is presented in chapter 2. Then, in chapter 3, variations of the ham sandwich Theorem are studied when we search for partitions such that every hyperplane avoids an arbitrary number of sections. These results appear in two papers, [39, 35].

The generalisation of Tverberg's theorem consists of adding a condition of tolerance to the partition. Namely, that we may remove an arbitrary number of points and the partition still is Tverberg type. This is presented in chapter 4. Then, in chapter 5, "colourful" variations of Tverberg's Theorem are studied along their applications to some purely combinatorial problems. These results appear in two papers, [41, 40].

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I wanted to thank Armando and Pilar; they made the UK feel like a second home to me.

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A mi abuelo Guillermo,
Que es a quien más admiro, tanto como científico, hombre de familia y persona excelente; y es mi mejor ejemplo a seguir.

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Chapter 1

Introduction

This chapter contains an introduction to the necessary machinery to tackle the problems contained in the Thesis. It will also settle the notation for the forthcoming chapters. The expert reader may prefer to skip it.

1.1 Combinatorial geometry

Given n points $a_1, a_2, \ldots, a_n \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$, we say that y is a *convex combination* of the a_i if y can be written as a linear combination of the a_i using non-negative coefficients that sum to 1. We say that y is a *non-negative combination* of the a_i if y can be written as a linear combination of the a_i using non-negative coefficients.

Given a set $A \subset \mathbb{R}^d$, we define the *convex hull of* A, denoted by $\langle A \rangle$, as the set of all convex combinations of finite sets of points of A. We define the *convex cone of* A, denoted by $\langle A \rangle_{cone}$, as the set of all non-negative combinations of finite sets of points of A.

We say that A is *convex* if $A = \langle A \rangle$. It should be noted that the convex hull and the convex cone of any set are convex.

It is known that the intersection structure of families of convex sets is incredibly rich (see [15], for example), and a plethora of results deal with this subject. Thus, it is natural to ask the following kind of questions.

Given a set X in $\mathbb{R}^d$, is there a way to partition X such that the convex hulls of each part satisfy certain combinatorial properties?

We refer to this as a *partition problem.* We distinguis two kinds of partition problems. Namely,

- **Tverberg type partitions.** In this setting, one of the properties we ask the sets to satisfy is that their convex hulls all intersect.
- **Ham sandwich type partitions.** In this setting, one of the properties we ask the sets to satisfy is that the interiors of their convex hulls are pairwise disjoint.

The names for these types of partitions come from the quintessential theorem from each type, namely Tverberg's theorem and the ham sandwich theorem. The first example of a Tverberg type result is due to Radon. Namely,

Theorem 1.1.1 (Radon, 1921 [33])**.** *Given a set S of $d+2$ points in $\mathbb{R}^d$, there is a partition of S into two sets A_1 and A_2 such that*

$$\langle A_1 \rangle \cap \langle A_2 \rangle \neq \emptyset.$$

It should be noted that Radon stated this as a lemma in [33], where the focus of the paper was to prove Helly's theorem characterising intersecting families of convex sets in $\mathbb{R}^d$. This is why Radon's theorem is often referred to as Radon's lemma. Even though the proof of this theorem requires only basic linear algebra and may seem conspicuous at first sight, it has been the basis for a wide number of generalisations.

One natural way to generalise this theorem is asking to partition S into any number, say k, of parts. Tverberg's theorem gives the number of points necessary for this to happen. Namely,

Theorem 1.1.2 (Tverberg, 1966 [42])**.** *Given a set S of $(k-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of S into k sets $A_1, A_2, \ldots, A_k$ such that*

$$\bigcap_{i=1}^{k} \langle A_i \rangle \neq \emptyset.$$

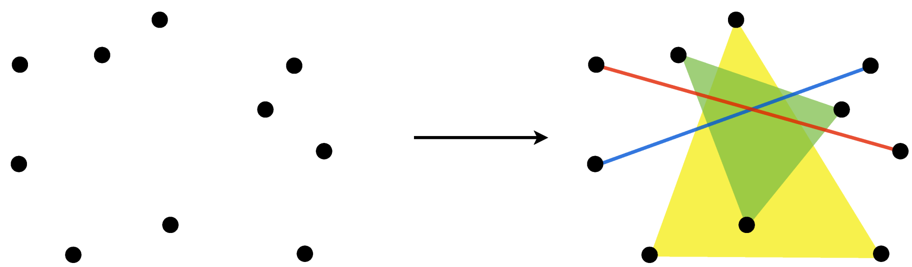

Figure 1.1: Example of Tverberg partition for 10 points in the plane, i.e. the case $d = 2, k = 4$

We call a partition of this type a *Tverberg partition*, or *Radon partition* if $k = 2$. It should be noted that the number $(k-1)(d+1)+1$ is optimal. The sudden scaling of difficulty from Radon's to Tverberg's theorem is made evident by the 45 year gap between them. Tverberg's proof of his own theorem involves moving points of an arbitrary configuration continuously and swapping points in the partition as problems appear. However, this proof method is difficult to generalise and can be heavily case-based. We refer to this as *Tverberg's method*. A simpler proof of Tverberg's theorem was found by Karanbir Sarkaria [38] using, in part, a clever linear-algebraic trick. We refer to it as *Sarkaria's method*. Since we will use it extensively in the chapters dealing with Tverberg type partitions, we present a detailed sketch of his proof in the next section.

1.2 Sarkaria's method

Given a set $X \subset \mathbb{R}^d$, we say that X *captures the origin* if $0 \in \langle X \rangle$. The main idea behind Sarkaria's proof of Tverberg's theorem is to make a transformation to represent the set of points in $\mathbb{R}^d$ in a space $\mathbb{R}^n$ of higher dimension, so that a Tverberg partition in $\mathbb{R}^d$ corresponds to a set in $\mathbb{R}^n$ capturing the origin. Then, we can use the following colourful generalisation of Carathéodory's theorem to finish.

Theorem 1.2.1 (Bárány, 1982 [5]). *Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{d+1}$ be families of points in $\mathbb{R}^d$. If each $\mathcal{F}_i$ captures the origin, we can find points $x_1 \in \mathcal{F}_1, x_2 \in \mathcal{F}_2, \dots, x_{d+1} \in \mathcal{F}_{d+1}$ such that the set $\{x_1, x_2, \dots, x_{d+1}\}$ captures the origin.*

Carathéodory's classic theorem is the case $\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{F}_{d+1}$. This is called a *colourful* version since the families $\mathcal{F}_1, \dots, \mathcal{F}_{d+1}$ are usually called *colour classes.*

In order to motivate the constructions that are going to follow, let us prove the case $k = 2$ of Tverberg's theorem. Given any $d+2$ points $a_1, a_2, \dots, a_{d+2}$ in $\mathbb{R}^d$, we first lift them to $\mathbb{R}^{d+1}$ by defining $b_i = (a_i, 1) \in \mathbb{R}^{d+1}$ for each i. Since we have $d+2$ points in $\mathbb{R}^{d+1}$, there is a non-trivial linear combination of them that gives 0. Namely, there are coefficients $\beta_1, \beta_2, \dots, \beta_{d+2}$, not all equal to zero such that $\sum_{i=1}^{d+2} \beta_i b_i = 0$. Since the last coordinate of each b_i is 1, the sum of these coefficients must be zero. Consider A the set of indices i such that β_i is non-negative and B the set of indices i such that β_i is negative. Then,

$$\sum_{i \in A} \beta_i a_i = \sum_{i \in B} (-\beta_i) a_i$$

$$\sum_{i \in A} \beta_i = \sum_{i \in B} (-\beta_i)$$

Using the second equation, we can make the first one into an equality of convex combinations, as we wanted.

If one wants to extend this argument, ideally we would like to have coefficients *of k types*, not only positive and negative. It turns out that this is possible via a tensor product, using a special set of k points in $\mathbb{R}^{k-1}$ to represent the types of coefficients. One should note that Sarkaria's original argument used number fields, and the simplified version presented below is due to Bárány and Onn [8].

Given $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots, y_m) \in \mathbb{R}^m$, we consider $x \otimes y$ the *tensor product* of x and y as a vector in $\mathbb{R}^{n\times m}$ with entries

$$(x \otimes y)_{(i,j)} = x_i y_j.$$

Note that $\otimes : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n\times m}$ is bilinear. We are now ready to prove Tverberg's theorem.

Sarkaria's proof. Let $n = (k-1)(d+1)$ and $a_1, a_2, \dots, a_{n+1}$ be $n+1$ points in $\mathbb{R}^d$. Consider $b_i = (a_i, 1) \in \mathbb{R}^{d+1}$ for all i. Let $u_1, u_2, \dots, u_k \in \mathbb{R}^{k-1}$ be the k vertices of a regular simplex centred at the origin. These k points will be used to parametrise the partition of the points in $\mathbb{R}^d$. Note that for coefficients $\gamma_1, \gamma_2, \dots, \gamma_k \in \mathbb{R}$, $\sum_i \gamma_i u_i = 0$ if and only if $\gamma_1 = \gamma_2 = \ldots = \gamma_k$. Consider the points $b_i \otimes u_j \in \mathbb{R}^n$.

$b_1 \otimes u_1$	$b_2 \otimes u_1$	$\cdots$	$b_{n+1} \otimes u_1$
$b_1 \otimes u_2$	$b_2 \otimes u_2$	$\cdots$	$b_{n+1} \otimes u_2$
$\vdots$	$\vdots$		$\vdots$
$b_1 \otimes u_k$	$b_2 \otimes u_k$	$\cdots$	$b_{n+1} \otimes u_k$

Figure 1.2: Points in Sarkaria's transformation.

Note that the $n+1$ families $\mathcal{F}_i = \{b_i \otimes u_j \ : \ 1 \le j \le k\}$ (the columns in the diagram above) each capture the origin in $\mathbb{R}^n$. Thus, there are indices

$j_1, j_2, \ldots, j_{n+1}$ such that the set $X = \{b_1 \otimes u_{j_1}, b_2 \otimes u_{j_2}, \ldots, b_{n+1} \otimes u_{j_{n+1}}\}$ captures the origin. It remains to prove that these points induce the Tverberg partition we seek. For this purpose, consider the sets

$$I_1 = \{i : j_i = 1\}, I_2 = \{i : j_i = 2\}, \ldots, I_k = \{i : j_i = k\}$$

Since X captures the origin, there are coefficients $\beta_1, \beta_2, \ldots, \beta_{n+1}$ of a convex combination such that

$$\sum_{i=1}^{n+1} \beta_i (b_i \otimes u_{j_i}) = 0.$$

Factoring each u_i we obtain

$$\left(\sum_{i \in I_1} \beta_i b_i\right) \otimes u_1 + \left(\sum_{i \in I_2} \beta_i b_i\right) \otimes u_2 + \ldots + \left(\sum_{i \in I_k} \beta_i b_i\right) \otimes u_k = 0.$$

Note that the need for equal coefficients in any linear combination of the u_i that gives 0 is carried through the tensor product, giving us

$$\sum_{i \in I_1} \beta_i b_i = \sum_{i \in I_2} \beta_i b_i = \ldots = \sum_{i \in I_k} \beta_i b_i.$$

Using the fact that the last coordinate of each b_i is 1 it is easy to see that, if $\alpha_i = k\beta_i$, then, for each j, the elements of the set $\Gamma_j = \{\alpha_i \ : \ i \in I_j\}$ are the coefficients of a convex combination, and

$$\sum_{i \in I_1} \alpha_i a_i = \sum_{i \in I_2} \alpha_i a_i = \ldots = \sum_{i \in I_k} \alpha_i a_i.$$

Thus the sets $A_j = \{a_i \ : \ i \in I_j\}$ for $1 \leq j \leq k$ form the Tverberg partition we seek. □

Note that for the case $k = 2$, the proof above reduces to the simple argument explained in the beginning of the section. The colourful Carathéodory theorem with sets of size $k > 2$ is playing the role of basic linear algebra in this setting.

If we follow blindly the same method without the lift $a_i \mapsto b_i$, we obtain the following corollary

Corollary 1.2.2. *Let $n = d(k-1)$ and $a_1, a_2, \ldots, a_{n+1}$ be $n+1$ points in $\mathbb{R}^d$. Then, there is a partition $I_1, I_2, \ldots, I_k$ of $\{1, 2, \ldots, n+1\}$ and non-negative coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$, not all zero, such that*

$$\sum_{i \in I_1} \alpha_i a_i = \sum_{i \in I_2} \alpha_i a_i = \ldots = \sum_{i \in I_k} \alpha_i a_i.$$

1.3 Ham-Sandwich type partitions

Contrary to Tverberg type partitions, given a set X in $\mathbb{R}^d$, finding ham-sandwich type partitions of X is trivial. This can be done, for example, by splitting the set using a hyperplane, and then successively partition each part with the same method.

This gives us a lot of freedom to choose the partition. Thus, in order to obtain more interesting results regarding ham-sandwich type partitions, stronger conditions need to be imposed on the resulting partition. For example, finding a ham-sandwich type partition of a set of red and green points in $\mathbb{R}^2$ such that every set has the same number of red and green points is interesting.

In this type of problems, the sets to be partitioned usually fall into one of two categories. That is, it is customary to talk about partitioning finite sets of points, or partitioning a finite measure in $\mathbb{R}^d$ with some kind of smoothness condition. This does not cause a lack of generality, as standard approximation arguments show that (almost always) these kinds of partition problems are closely related and can be derived from each other. Since the proof methods are topological, in order for the underlying functions to be continuous, the finite sets of points are required to be in general position and the measures are required to be absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$, among other properties.

In this thesis we deal with the case of partitioning measures in $\mathbb{R}^d$. Since additional conditions on the measures can vary from theorem to theorem, it is common to denote the set of desired measures as *nice measures* and specify the exact conditions needed for that theorem if the possibility of confusion arises. We will use the term *AAH measures* for the set of measures used in chapter 2 (since they satisfy properties mentioned in a paper by Aurenhammer, Aronov and Hoffman [3] and the term *YY measures* for the set of measures used in chapter 3 (since they satisfy properties mentioned in a paper by Yao and Yao [46].

To be precise, in ham sandwich partion problems we are given one or more nice measures in $\mathbb{R}^d$ and we wish to partition $\mathbb{R}^d$ into pairwise interior-disjoint convex sets $C_1, C_2, \ldots, C_k$ that satisfy certain properties. This can be seen, for example, in the classic ham-sandwich theorem. In this case, a nice measure μ refers to a probability measure in $\mathbb{R}^d$ such that $\mu(H) = 0$ for every hyperplane H.

Theorem 1.3.1 (Ham-sandwich theorem)**.** *Given d nice measures $\mu_1, \mu_2, \ldots, \mu_d$ in $\mathbb{R}^d$, there is a hyperplane H such that its two half-spaces H^+ and H^- satisfy*

$$\mu_i(H^+) = \mu_i(H^-) = \frac{1}{2}$$

for all i.

One should note that the number of measures cannot be increased, as $d+1$ measures concentrated each near a vertex of a non-degenerate simplex in $\mathbb{R}^d$ cannot be simultaneously split by half by a single hyperplane. There is a standard way to approach ham sandwich partition problems, explained in the next section.

1.4 Test map scheme

The test map scheme is a systematic way to approach ham-sandwich type partition problems. For an elaborate description of this method and a broad list of examples, we recommend [29]. The idea is to reduce this kind of problems to purely topological ones, which can then (hopefully) be solved using machinery in this field. The procedure is the following

- First, parametrise the space of partitions with a space X.
- Secondly, consider a space Y of (other) parameters of the partitions considered in X. The parameters in Y are usually related to the way the partition splits the measures. This induces naturally a continuous function $f : X \longrightarrow Y$.
- Ideally, f should satisfy a certain set of properties ψ. These properties are related to the symmetries of the problem and the parametrisations that were used. Then, given a property τ on the partition, proving that *there is always a partition in X satisfying τ* is reduced to proving that *for every function $f : X \longrightarrow Y$ satisfying ψ, there is a point $x_0 \in X$ such that $f(x_0) \in Y_0$*, where $Y_0 \subset Y$ depends on τ. Equivalently, it is standard to prove the non-existence of maps $f : X \longrightarrow Y \setminus Y_0$ that satisfy ψ.

One should note that the test map scheme can also be applied to Tverberg type partition problems, as explained in [13]. As stated above, the properties ψ that the function should satisfy are usually related to symmetries of the problem, and often translate to the function behaving nicely with group actions (as explained in the next section).

To show this method concretely, let us prove the ham sandwich Theorem (Theorem 1.3.1). For this, the topological tool we need is one of the equivalent formulations of the well-known Borsuk-Ulam theorem. Namely, define the *sphere of dimension* n as

$$\mathbf{S}^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}.$$

Then,

Theorem 1.4.1 (Borsuk-Ulam theorem)**.** *For every continuous function* $f : \mathbf{S}^n \longrightarrow \mathbb{R}^n$ *such that* $f(x) = -f(-x)$ *for all* $x \in \mathbf{S}^n$*, there is an* $x_0 \in \mathbf{S}^n$ *such that* $f(x_0) = 0$.

Proof of Theorem 1.3.1. Given a half-space H^+ in $\mathbb{R}^d$, it is of the form $\{x \in \mathbb{R}^d : x \cdot c \geq t\}$ for some non-zero vector $c \in \mathbb{R}^d$ and some $t \in \mathbb{R}$. This is equivalent to $\{x \in \mathbb{R}^d : (x, 1) \cdot (c, -t) \geq 0\}$. Moreover, we can normalise $(c, -t)$ so that it lies on $\mathbf{S}^d$. By associating $H^+ \leftrightarrow v = \frac{(c,-t)}{||(c,-t)||}$, the set of half-spaces in $\mathbb{R}^d$ is parametrised as $\mathbf{S}^d$ minus the north and south poles. We can include these two points, which would correspond to all of $\mathbb{R}^d$ and the empty set, respectively (see figure 1.3). These can be thought as the half-spaces determined by the hyperplane at infinity.

Note that two antipodal points in $\mathbf{S}^d$ correspond to two complementary half-spaces. With this parametrisation in mind, it makes sense to evaluate a measure in $\mathbb{R}^d$ for points $x \in \mathbf{S}^d$, as $\mu(x)$. Consider the function

$$\begin{array}{rcl} f : \mathbf{S}^d & \longrightarrow & \mathbb{R}^d \\ x & \mapsto & \left(\mu_1(x) - \frac{1}{2}, \mu_2(x) - \frac{1}{2}, \ldots, \mu_d(x) - \frac{1}{2}\right) \end{array}$$

Note that the condition on the measures imply that f is continuous. Moreover, since $\mu_i(x) + \mu_i(-x) = 1$ for all x, $f(x) = -f(-x)$. Thus, there is an x_0 such that $f(x_0) = 0$. One should note as well that x_0 cannot be the south pole nor the north pole, so it does represent a half-space of $\mathbb{R}^d$. The pair of antipodal points $\{x_0, -x_0\}$ represents the hyperplane we seek. □

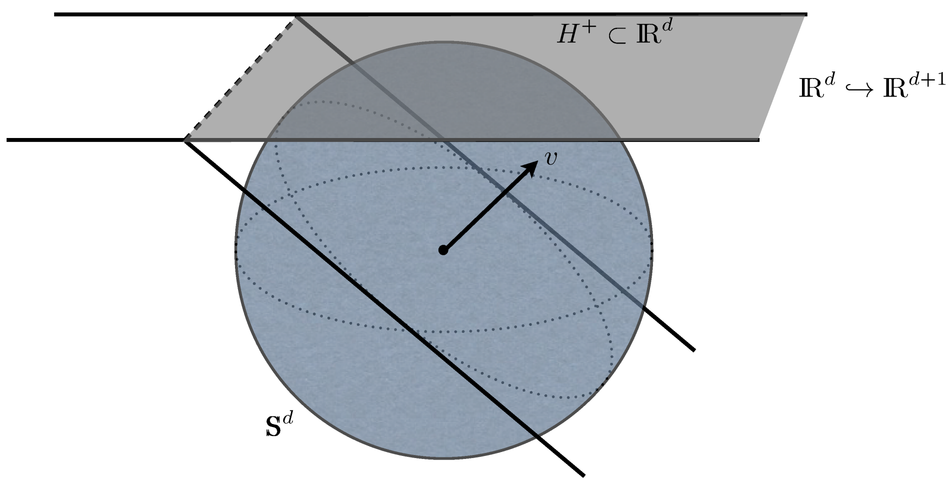

Figure 1.3: Given a half-space $H^+ \subset \mathbb{R}^d$, the figure shows how it can be represented by a vector $v = \frac{(c,-t)}{||(c,-t)||} \in \mathbf{S}^d$. If $\mathbb{R}^d$ is embedded in $\mathbb{R}^{d+1}$ as the set of vectors whose last coordinate is 1, consider H to be the hyperplane that results from extending the hyperplane in $\mathbb{R}^d$ supporting H^+ through the origin in $\mathbb{R}^{d+1}$. Then $v \in \mathbf{S}^d$ is the vector orthogonal to H pointing towards the side of H that contains H^+. The operation can be reversed, and given $v \in \mathbf{S}^d$, one can obtain H^+.

1.5 Borsuk-Ulam type theorems

In the test map scheme, problems are usually reduced to topological statements similar to the Borsuk-Ulam theorem. In this section we mention a nice generalisation of the Borsuk-Ulam theorem by Dold, which we will use to prove Theorem 2.1.2. For the proof of Theorem 3.0.4, we will prove a different Borsuk-Ulam type theorem, adapted to that specific case.

Given a topological space X and a group G, we say that G *acts on* X or that *there is an action of* G *in* X if there is a continuous function

$$\begin{array}{rcl} G \times X & \longrightarrow & X \\ (g,x) & \mapsto & gx \end{array}$$

such that for every $g, h \in G$, $x \in X$, $g(hx) = (gh)x$ and $ex = x$ if e is the neutral element in G. If the topology of G is not specified, we assume it has the discrete topology. In this thesis we will only use spaces with actions of the discrete cyclic group with k elements, denoted by $\mathbb{Z}_k$.

We say that the action of G in X is *free* or that X *is a G-space* if, for all $x \in X$, the equation $gx = x$ implies $g = e$. For example, if we consider $\mathbb{Z}_2 = \{-1, 1\}$ with multiplication, we have that $\mathbf{S}^n$ is a free $\mathbb{Z}_2$-space with the natural action

$$\begin{aligned} (1)x &\mapsto x \\ (-1)x &\mapsto -x \end{aligned}$$

Given two spaces X, Y with an action of G, we say that a function $f : X \longrightarrow Y$ is *equivariant* if $f(gx) = gf(x)$ for all $x \in X$, $g \in G$.

Given a topological space X, we say that X is n-connected if every continuous function $f : \mathbf{S}^n \to X$ can be extended to a continuous function $f^* : \mathbb{B}_{n+1} \to X$, where $\mathbb{B}_{n+1}$ is the ball of dimension $n+1$ (with $\mathbf{S}^n$ as boundary); namely

$$\mathbb{B}_{n+1} = \{x \in \mathbb{R}^{n+1} : ||x|| \leq 1\}.$$

Note that 0-connectedness simply means connectedness. One can think of n-connectedness as a higher dimensional version of connectedness. For example, $\mathbf{S}^n$ is a space that is $(n-1)$-connected but is not n-connected. Then Dold's theorem says

Theorem 1.5.1 (Dold, 1983 [14]). *Let G be a finite group, $|G| > 1$, X be an n-connected G-space and Y be a (paracompact) G-space of dimension at most n. Then there is no equivariant function $f : X \longrightarrow Y$.*

Note that the case $X = \mathbf{S}^{n+1}$, $Y = \mathbf{S}^n$ with their natural $\mathbb{Z}_2$-action is one of the many equivalent forms of the Borsuk-Ulam theorem. Namely, that there is no antipodal mapping from $\mathbf{S}^{n+1}$ to $\mathbf{S}^n$ (see [29], for example). The advantage of Dold's theorem is that it is not specific to two spaces, and gives us the non-existence of equivariant functions by checking two simple properties of X and Y. One could intuitively think that the reason for Dold's theorem to work is that X is *too thick* to be arranged properly into Y (by properly we mean respecting the group action).

The need for free actions in this kind of theorems is necessary for the topological machinery to work, and often translates to apparently artificial conditions on ham-sandwich type theorems (or topological versions of Tverberg type theorems) such as some parameters being prime numbers or prime powers. When such conditions appear, the question of whether they are actually necessary is, more often than not, unresolved.

Chapter 2

Ham sandwich type partitions

2.1 Balanced convex partitions

The ham sandwich theorem (Theorem 1.3.1) shows that given any d nice measures in $\mathbb{R}^d$, there is always a hyperplane that splits them by half simultaneously. The main goal of the following sections is to prove an extension of the ham sandwich theorem where we want to split the measures into more than 2 sections. The contents of this chapter can be found in [39].

In $\mathbb{R}^2$, it was shown that given two measures μ_1 and μ_2, if one wants to divide them simultaneously into more than two parts, it is possible to do so with convex sections. This was done by Sakai [36] for measures and by Bespamyatnikh, Kirkpatrick and Snoeyink [9] for finite families of points. Namely,

Theorem 2.1.1 (Bespamyatnikh *et al.* 2000 [9], Sakai 2002 [36])**.** *Given a positive integer k and two nice probability measures μ_1, μ_2 of $\mathbb{R}^2$, there is a partition of $\mathbb{R}^2$ into k pairwise interior-disjoint convex sets $C_1, C_2, \ldots, C_k$ such that*

$$\mu_i(C_j) = \frac{1}{k} \text{ for all } i, j.$$

Sakai's conditions on the measures were that they had to be absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^2$ and that there had to be a bounded set B such that the measures vanished outside of B.

Imre Bárány conjectured that this should hold for d measures in $\mathbb{R}^d$. In this section we give an affirmative answer to his conjecture. Namely, we prove the following theorem.

Theorem 2.1.2 (Soberón , 2012 [39]). *Let k and d be positive integers. Let $\mu_1, \mu_2, \ldots, \mu_d$ be AAH measures in $\mathbb{R}^d$ such that $\mu_i(\mathbb{R}^d) = k$ for all i. Then, there is a convex partition of $\mathbb{R}^d$ into sets $C_1, C_2, \ldots, C_k$ such that $\mu_i(C_j) = 1$ for all i, j.*

The conditions we need on the measures are the same as the ones required in section 2.2. That is, we say that a measure μ in $\mathbb{R}^d$ is an AAH measure if it is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$, there is a bounded convex set K such that μ vanishes outside of K and μ has positive value on every open set in K. Note that if A is a convex set of positive measure on μ, then the restriction $\mu|_A$ is also a nice measure.

This result was proven independently by R.N. Karasev [25] using stronger topological methods. We prove this theorem using the test map scheme and Dold's generalisation of the Borsuk-Ulam theorem, presented in section 1.5. In order to parametrise the convex partitions of $\mathbb{R}^d$, power diagrams are used (defined in the next section). However, a simple approach of this kind gives a base space which lacks the necessary connectedness to use Dold's theorem. This is because the construction of the power diagrams forces the sites to be different points in $\mathbb{R}^d$. In order to fix this problem, we use power diagrams to parametrise the set of convex partitions of $\mathbb{R}^d$ into at most k parts, instead of only the convex partitions with exactly k parts. This is done in section 2.3, when we allow the points in the configuration space to coincide.

2.2 Power diagrams

Power diagrams are a family of partitions of $\mathbb{R}^d$ into pairwise interior-disjoint convex sets which admit a nice parametrisation. They generalise Voronoi diagrams, and we will use them in the construction of the configuration space for the test map scheme to prove Theorem 2.1.2.

Given k different points $x_1, x_2, \ldots, x_k$ in $\mathbb{R}^d$ (which we refer to as sites), the *Voronoi diagram* of $S = (x_1, x_2, \ldots, x_k)$ is a partition of $\mathbb{R}^d$ into k convex sets $C_1, C_2, \ldots, C_k$ defined by

$$C_i = \{x \in \mathbb{R}^d \ : \ d(x, x_i) \leq d(x, x_j) \text{ for all } 1 \leq j \leq k\}$$

That is, the points of C_i are those that are closer to x_i than to any other site.

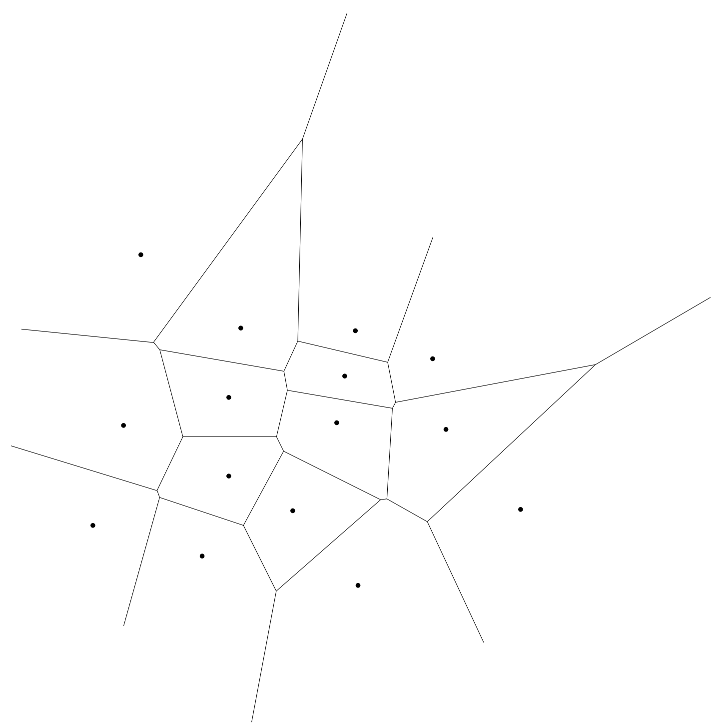

Figure 2.1: Voronoi diagram of a set of 15 points in $\mathbb{R}^2$

This definition can be extended if we allow the sections C_i to grow or shrink according to some functions. This is parametrised by a weight vector $w = (w_1, w_2, \dots, w_k) \in \mathbb{R}^k$. Using these numbers, we can define the *power functions* $h_i(x) = d(x, x_i)^2 - w_i$ for all i. Then, the *power diagram* $C(S, w)$ is a partition of $\mathbb{R}^d$ into k sets $C_1, C_2, \dots, C_k$ defined by

$$C_i = \{x \in \mathbb{R}^d \ : \ h_i(x) \leq h_j(x) \text{ for all } 1 \leq j \leq k\}.$$

Note that if $w = (0, 0, \dots, 0)$, then $C(S, w)$ is the Voronoi diagram of S. It may happen that some C_i are empty. Each C_i is the intersection of the half-spaces

$$H_{i,j} = \{x \in \mathbb{R}^d \ : \ h_i(x) \leq h_j(x)\} = \{x \in \mathbb{R}^d \ : \ d(x, x_i)^2 - d(x, x_j)^2 \leq w_i - w_j\}.$$

This implies that each C_i is convex. Moreover, the hyperplane $H_{i,j}$ is orthogonal to the line $x_i - x_j$ and its position depends entirely on $w_i - w_j$. Thus, if $v_0 = (1, 1, \dots, 1) \in \mathbb{R}^d$, then $C(S, w) = C(S, w + \alpha v_0)$ for all $\alpha \in \mathbb{R}$.

Power diagrams are simple enough to be determined by a set of different sites and a weight vector, but rich enough to split measures into sets of pre-described sizes. Namely,

Theorem 2.2.1 (Aronov, Aurenhammer and Hoffman, 1998 [3])**.** *Let μ be an AAH measure in $\mathbb{R}^d$, S a set of k different sites and $c = (c_1, c_2, \dots, c_k) \in \mathbb{R}^k$ such that $c_i \geq 0$ for all i and $\sum_i c_i = \mu(\mathbb{R}^d)$. Then there is a weight vector $w \in \mathbb{R}^k$ such that for the power diagram $C(S, w)$ we have that $\mu(C_i) = c_i$ for all i.*

In this theorem, the conditions we need on the measure μ are that it is absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^d$ and there is a bounded convex set K such that μ vanishes outside of K and has positive value on every open set in K. In the original paper, the conditions actually were that K was a hypercube, but the proof follows the same way in any convex set.

A vector c as above is called a *capacity vector*. Note that since translations of w by scalar multiplications of v_0 do not change the partition, we may suppose that the dot product $w \cdot c$ is 0. With this type of conditions, the theorem above can be improved.

Proposition 2.2.2 (Aronov and Hubard, 2010 [2]). *The weight vector found in 2.2.1 is unique up to translations of scalar multiplications of v_0. Moreover, if a condition such as $c \cdot w = 0$ has been imposed, then $w = w(S)$ is a continuous function of S.*

The non-vanishing condition of the measure is essential for the uniqueness. This proposition is what makes the functions used in section 2.1 to be well defined and continuous.

Given a measure μ in $\mathbb{R}^d$, a capacity vector $c = (c_1, c_2, \ldots, c_k)$ and a power diagram $C(S, w)$ where the dimension of w and c coincide, we say that *the parts of $C(S, w)$ agree with c* if and only if $\mu(C_i) = c_i$ for all $1 \leq i \leq k$.

2.3 Configuration space for merging power diagrams

We are interested in analysing the behaviour of power diagrams if we allow the sites to coincide. Namely, consider the problem of having a measure in $\mathbb{R}^d$ and a capacity vector $c = (c_1, c_2, \ldots, c_k)$. If we are given k different sites, we know by Theorem 2.2.1 that we can choose weights to partition μ with parts of sizes according to c. Namely, we can think as the sites representing the sections of a partition, and the weight vector simply represents how to go from one to the other. However, if some sites coincide, then even the definition of power diagrams gives nothing. In this case, we will consider this repeated point to represent a large section with capacity equal to the sum of the capacities associated to it. The purpose of this section is to define properly these partitions and to show some of their properties that will allow us to prove Theorem 2.1.2.

The following lemma is necessary to show that the functions used in the main proof are continuous. Let μ be an AAH probability measure in $\mathbb{R}^d$ and let $c = (c_1, c_2, \ldots, c_k)$ be a capacity vector with positive entries. Given an ordered set S of k different sites in $\mathbb{R}^d$, denote by $w(S) = (w_1, w_2, \ldots, w_k)$ the weight vector such that the parts of the partition $C(S, w(S))$ agree with the capacity vector and $w(S) \cdot c = 0$. Using this notation we have the following

Lemma 2.3.1. *Let S be an ordered set of k points in $\mathbb{R}^d$ that move continuously during the time interval $[0,1]$. Suppose that no pair of points of S coincide during the time interval $[0,1)$; and that there are two points $x_1, x_2 \in S$ such that $x_1 = x_2$ at time 1. Then there is a real number w' such that w_1 and w_2 tend to w as the time t tends to 1.*

Another way to state this is that if two points have the same limit, their weights also have the same limit. We will also show that this limit is related to another power diagram. Note that even though we are supposing that w_1 and w_2 coincide at time 1, this does not exclude the possibility that other pairs of points of S have the same limit as the time t approaches 1.

Proof. Suppose that w_1 and w_2 have limits w_1' and w_2' as t tends to 1, respectively. We first show that these limits coincide. If this is not true, suppose that $w_1' > w_2'$. As they are the limits of w_1 and w_2, we may suppose that $w_1 > w_2$ without loss of generality. Denote by y the point where the hyperplane $H_{1,2} = \{x \ : \ d(x,x_1)^2 - d(x,x_2)^2 = w_1 - w_2\}$ intersects the line through x_1 and x_2; and define $u := d(y, x_2)$, $v := d(x_1, x_2)$. Since $w_1 > w_2$, $d(y, x_1) = u + v$. Thus $w_1 - w_2 = (u+v)^2 - u^2 = v(2u + v)$. Since $w_1 - w_2$ has a positive limit and $v \longrightarrow 0$, we have that $u \longrightarrow \infty$. If p is the limit of x_1 and x_2, this means that the distance $d(p, C_2)$ tends to infinity, and so does the distance $d(0, C_2)$. However, this is impossible since $\mu(C_2) = c_2 > 0$ for all $t \in [0, 1)$.

Now it only remains to show that these limits exist. Using the same argument and $w \cdot c = 0$, we can deduce that the weight vector is bounded. Thus, we only need to show that if any sequence of values of w converges as the time tends to 1, it does so to the same limit. Let $w^{(1)}, w^{(2)}, w^{(3)}, \dots$ be such a sequence, with limit w'. We replace S, c, w' by $\tilde{S}, \tilde{c}, \tilde{w}$ in the following way

- $\tilde{S}$ is obtained by replacing all the points in S converging to q by a single copy of q, for all $q \in \mathbb{R}^d$;
- $\tilde{c}$ is obtained by replacing the capacities of all the points in S converging to q by a single copy of their sum, for all $q \in \mathbb{R}^d$;
- $\tilde{w}$ is obtained by replacing all the weights in w' corresponding to to points in S converging to q by a single copy of that number (we already showed all these weights must have the same limit), for all $q \in \mathbb{R}^d$.

It is clear that the parts of the power diagram $C(\tilde{S}, \tilde{w})$ have measures that agree with $\tilde{c}$ and $\tilde{w} \cdot \tilde{c} = 0$. Thus there is only one possible value for $\tilde{w}$, which implies that there is only one possible value for w', as we wanted. □

The proof of this lemma not only shows that weights have the same limit if the corresponding sites do, but that if we think that when sites come together their sections merge, then the weights vectors behave as expected. To state this formally, we need to define a function f that represents the weight vectors even when sites coincide. We will say that this function represents a *merging power diagram*. Consider $Y \equiv \mathbb{R}^{kd}$ to be the set of vectors $(y_1, y_2, \dots, y_k)$ such that $y_i \in \mathbb{R}^d$ and μ an AAH measure in $\mathbb{R}^d$.

Given a capacity vector $(c_1, c_2, \dots, c_k)$ such that all the c_i are positive, let us define a function $f : Y \longrightarrow \mathbb{R}^k$ in the following way. For $y \in Y$, let $S(y) = (s_1, s_2, \dots, s_t)$ be the t-tuple of different points in y. For $1 \leq i \leq r$, define

$$A_i = \{j : y_j = s_i\}$$

to index the elements of y that are equal to each element in $S(y)$. Define $c(y) = (\gamma_1, \gamma_2, \ldots, \gamma_t)$, where

$$\gamma_i = \sum_{j \in A_i} c_j.$$

Denote by $Sf(y) = (w_1, w_2, \ldots, w_t)$ the weight vector such that the measure of the parts of $C(S(y), Sf(y))$ agrees with $c(y)$ and $Sf(y) \cdot c(y) = 0$. Finally define $f(y) = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where $\alpha_j = w_i$ if and only if $j \in A_i$.

There is another way to describe this function. First, note that $S : \left(\mathbb{R}^d\right)^k \to \left(\mathbb{R}^d\right)^t$ is induced by some function $\tau : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, t\}$. Namely, if $(y_1, y_2, \ldots, y_k) \mapsto (s_1, s_2, \ldots, s_t)$, we have that $y_i = s_j$ if and only if $\tau(i) = j$. Then, $f(y)$ is constructed so that it follows this same relation with $Sf(y)$. This is perhaps easier to represent with a commutative diagram, as the one below.

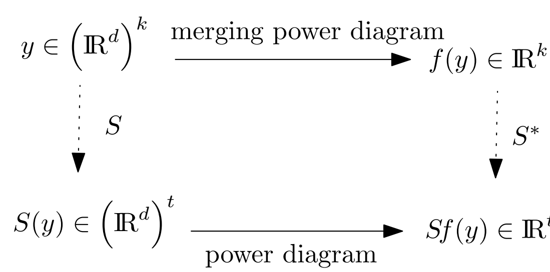

Figure 2.2: S and S^* are induced in the same way by τ, and f is constructed so that the diagram commutes. Note as well that t depends on y.

From the proof of Lemma 2.3.1, we obtain the following corollary,

Corollary 2.3.2. *Let μ be a AAH measure in $\mathbb{R}^d$, and c be a capacity vector. If f is the function representing the merging power diagram for μ and c, then f is continuous and, for all $y \in Y$, $f(y) \cdot c = 0$. Note that the dimension of $f(y)$ is, by construction, the same as that of c.*

2.4 Proof of Theorem 2.1.2

Using Corollary 2.3.2, we are now ready to prove the following lemma, which is the core of the proof of Theorem 2.1.2.

Lemma 2.4.1 (Lemma 1 in [39]). *Let p, d be positive integers such that p is prime. Let $\mu_1, \mu_2, \ldots, \mu_d$ be AAH measures in $\mathbb{R}^d$ such that $\mu_i(\mathbb{R}^d) = p$ for all i. Then, there is an integer $2 \leq r \leq p$ and a partition of $\mathbb{R}^d$ in r convex parts $C_1, C_2, \ldots, C_r$ such that $\mu_i(C_j) = \mu_{i'}(C_j)$ for all i, i', j and all these measures are positive integers.*

Proof. Let X be the set of all ordered p-tuples of vectors in $\mathbb{R}^d$ such that not all the vectors are equal. Let $c_0 \in \mathbb{R}^p$ be the vector with all entries equal to 1. For $1 \leq i \leq d$, let g_i be the function associated with μ_i and c_0 as in Corollary 2.3.2 (with $k = p$) and f_i be the restriction of g_i to X. Note that since $f(x) \cdot c_0 = 0$, we have that $f_i : X \longrightarrow \mathbb{R}^{p-1} \hookrightarrow \mathbb{R}^p$, where we are considering

$$\mathbb{R}^{p-1} = \{(z_1, z_2, \ldots, z_p) \ : \ z_1 + z_2 + \ldots + z_p = 0\}.$$

By Corollary 2.3.2, all the functions f_i are continuous. Moreover, if we consider the actions of $\mathbb{Z}_p$ in $\mathbb{R}^{p-1}$ and X where $\sigma(z_1, z_2, \ldots, z_p) = (z_2, z_3, \ldots, z_p, z_1)$ and $\sigma(x_1, x_2, \ldots, x_p) = (x_2, x_3, \ldots, x_p, x_1)$ for a generator $\sigma \in \mathbb{Z}_p$, we have that each f_i is equivariant.

Consider the function f defined as

$$\begin{aligned} f : X &\longrightarrow \left(\mathbb{R}^{p-1}\right)^{d-1} \\ f &= (f_1 - f_2, f_1 - f_3, \ldots, f_1 - f_d). \end{aligned}$$

We show that there is an $x \in X$ such that $f(x) = 0$. If there is no such

x, we can reduce the dimension of the target space by defining

$$\begin{aligned} g : X &\longrightarrow \mathbf{S}^{(p-1)(d-1)-1} \\ x &\mapsto \frac{f(x)}{||f(x)||}. \end{aligned}$$

Note that both f and g are equivariant. Moreover, since p is prime, the actions of $\mathbb{Z}_p$ are free on X and $\mathbf{S}^{(p-1)(d-1)-1}$. However, X is $\mathbb{R}^{pd}$ with a hole of dimension d, so its connectedness is reduced by $d+1$. Thus, it is at least $[(pd-1)-(d+1)]$-connected. Since $(pd-1)-(d+1) \geq (p-1)(d-1)-1$, this contradicts Dold's theorem.

Thus, there must be an x such that $f(x) = 0$. In this case the weight vector associated to x by the definition of each f_i must be the same, so the partition induced by it is the one we wanted. □

Lemma 2.4.2. *Given positive integers a and b, if the conclusion for Theorem 2.1.2 holds for $k = a$ and $k = b$, then it does so for $k = ab$.*

Proof. Suppose we have d AAH measure $\mu_1, \mu_2, \dots, \mu_d$ in $\mathbb{R}^d$ such that $\mu_i(\mathbb{R}^d) = ab$ for all $1 \leq i \leq d$. Since Theorem 2.1.2 is true for $k = b$ we can find a partition of $\mathbb{R}^d$ into pairwise interior-disjoint convex sets $C_1, C_2, \dots, C_b$ such that $\mu_i(C_j) = a$ for all $1 \leq i \leq d$, $1 \leq j \leq b$. For $1 \leq j \leq b$, $\mu_i|_{C_j}$ is an AAH measure in $\mathbb{R}^d$. Thus we can use the fact that Theorem 2.1.2 holds for $k = a$ to obtain a partition of $\mathbb{R}^d$ into pairwise interior-disjoint convex sets $C_{j,1}, C_{j,2}, \dots, C_{j,a}$ such that $\mu_i|_{C_j}(C_{j,h}) = 1$ for $1 \leq h \leq a$. Consider the sets $D_{j,h} = C_j \cap C_{j,h}$. These sets satisfy that $\mu_i(D_{j,h}) = \mu_i|_{C_j}(C_{j,h}) = 1$ and they form a partition of $\mathbb{R}^d$ into ab pairwise interior-disjoint convex sets. Thus Theorem 2.1.2 holds for $k = ab$. □

Using these two lemmas we can now prove Theorem 2.1.2.

Proof of Theorem 2.1.2. We use strong induction on k. If $k = 1$, then $C_1 = \mathbb{R}^d$ is the partition we want. Suppose now that $k > 1$ and the statement holds for all $1 \leq k' < k$. There are two cases

- If k is not prime, then $k = ab$ for some positive integers a, b such that $k > a$ and $k > b$. Applying Lemma 2.4.2 we are done.
- If k is prime, using Lemma 2.4.1 there is an integer $2 \leq r \leq k$ and a partition of $\mathbb{R}^d$ into pairwise interior-disjoint convex sets $C_1, C_2, \ldots, C_r$ such that $\mu_i(C_j)$ depends only on j and is a positive integer for all j. We can apply the same argument of the proof of Lemma 2.4.2 to subdivide each C_j and obtain the desired partition.

□

Using standard approximation arguments, we can make one of the measures a Dirac measure centred at the origin. This gives the following corollary:

Corollary 2.4.3 (Theorem 2.1.2 for spheres, [39]). *Given a positive integer k and $\mu_1, \mu_2, \ldots, \mu_d$ AAH measures in $\mathbf{S}^d$ such that $\mu(\mathbf{S}^d) = k$ for all $1 \leq i \leq d$, there is a convex cone partition $C_1, C_2, \ldots, C_k$ of $\mathbb{R}^{d+1}$ with apices at the origin such that $\mu_j(C_i) = 1$ for all i, j*

The equivalence of Theorem 2.1.2 and its version for spheres was proven by Imre Bárány.

One should note that the last inequality of Lemma 2.4.1 translates to $p \geq 2$, so there is some degree of freedom in the proof. A similar thing happens in Sakai's proof of the version in $\mathbb{R}^2$, where he uses similar partitions into either 2 or 3 pieces. When the partitions have to be in 3 pieces, he can impose further conditions on the shape of the parts. However, it is unclear if these additional degrees of freedom can be used to obtain a meaningful improvement of the main result.

2.5 An application to sets of fixed measure

It should be noted that the number of measures in the ham sandwich theorem and Theorem 2.1.2 cannot be increased. However, if instead of asking for a partition of $\mathbb{R}^d$ where the sets have equal measure we look for one convex set of equal pre-described size in each measure, positive results can be obtained. This was shown by Arseniy Akopyan and Roman N. Karasev in the following theorem

Theorem 2.5.1 (Akopyan and Karasev, 2012 [26]). *Let $x = \frac{1}{n}$, where n is a positive integer and $\mu_1, \mu_2, \ldots, \mu_{d+1}$ be $d+1$ AAH probability measures in $\mathbb{R}^d$. Then there is a convex set $K \subset \mathbb{R}^d$ such that $\mu_i(K) = x$ for all i.*

Here we present a proof of this result using Theorem 2.1.2. It should be noted that Akopyan and Karasev also showed that if x is not of the form $\frac{1}{n}$, then this result does not hold. The proof below is joint work with Edgardo Roldán-Pensado.

Proof. Let $f : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}$ be a strictly convex function. We can lift $\mathbb{R}^d$ to $\mathbb{R}^{d+1}$ by mapping $x \mapsto (x, f(x))$, denote by p this lifting. Note that given a hyperplane H in $\mathbb{R}^{d+1}$ and H^+ one of its half-spaces, then $p^{-1}(H^+)$ is convex if H^+ contains infinite rays in the direction $(0, 0, \ldots, 0, -1)$. Thus, if we have a power diagram in $\mathbb{R}^{d+1}$ with sites $x_1, x_2, \ldots, x_k$ and parts $C_1, C_2, \ldots, C_k$, if x_j is the site with the smallest $(d+1)$-th coordinate, then $p^{-1}(C_j)$ is convex.

Using p, we can also lift the measures $\mu_1, \mu_2, \ldots, \mu_{d+1}$ to $\mathbb{R}^{d+1}$. We may now apply Theorem 2.1.2 to obtain a partition of $\mathbb{R}^d$ into n sets $C_1, C_2, \ldots, C_n$ of equal measure in each μ_i. Moreover, this partition comes from the iteration of power diagrams. Since in each power diagram at least one section projects back to $\mathbb{R}^d$ to a convex set, we know that there is a j_0 such that C_{j_0} must project back to a convex set. $p^{-1}(C_{j_0})$ is the set we were looking for. □

The case with d measures is also interesting, as the condition $x = \frac{1}{n}$ for some positive integer n seems unnecessary. Namely, we have the following problem by Jorge Urrutia and Ruy Fábila-Monroy.

Problem 2.5.2 (Fábila-Monroy, Urrutia [17]). *Given d nice measures in $\mathbb{R}^d$, and a real number $x \in (0, 1/2]$, show that there is a convex set K in $\mathbb{R}^d$ such that $\mu_i(K) = x$ for all i.*

This is solved completely only for $d = 2$ by a result by Blagojević and Dimitrijević Blagojević [10].

Chapter 3

Partitions avoiding hyperplanes

The contents of this chapter are part of a joint work with Edgardo Roldán-Pensado, and can be found in [35]. Searching for a ham-sandwich partition of a measure in $\mathbb{R}^d$ where no hyperplane intersects all the parts is an interesting problem by itself. However, this problem has also a motivation from computational geometry.

This is in the setting of geometric queries. In a geometric range query problem, we are given a family $\mathcal{C}$ of n sets in $\mathbb{R}^d$. Then, we are given a point $p \in \mathbb{R}^d$, and we want to know how many sets of $\mathcal{C}$ contain p. Moreover, if we suppose that checking whether $p \in C$ or $p \notin C$ takes unit time for all sets $C \subset \mathbb{R}^d$, then we are interested in knowing how fast we can obtain the answer.

Of course, in this setting we can check every set in $\mathcal{C}$ individually and obtain an answer in time n. However, if the family $\mathcal{C}$ is fixed and a large number of points p are going to be sampled, a faster way to obtain the answer is desirable. If we analyse $\mathcal{C}$, it may be possible to obtain a way to solve the query using less than n operations. This is called *pre-processing* $\mathcal{C}$. This is only possible if $\mathcal{C}$ has a nice intersection structure.

Yao and Yao showed that such a solution could be found if $\mathcal{C}$ is a family of half-spaces [46]. The number of operations needed is $O(n^{\alpha(d)})$, where

$\alpha(d) = \frac{\log_2(2^d-1)}{d} < 1$. One should note that many geometric queries can be reduced to this case, so this gave fast solutions to many problems in computational geometry.

To solve this problem, Yao and Yao worked on the dual version, in which $\mathcal{C}$ is a set of points, P is a half-space, and we want to know how many points of $\mathcal{C}$ are contained in P. In order to answer this question efficiently, they used iterations of partitions of $\mathcal{C}$ where no hyperplane intersected all sections. The following theorem was the core of their construction.

Theorem 3.0.3 (Yao and Yao, 1985 [46])**.** *Given a YY measure μ in $\mathbb{R}^d$, there is a partition of $\mathbb{R}^d$ into 2^d pairwise interior-disjoint convex sets of equal μ-measure such that every hyperplane in $\mathbb{R}^d$ avoids the interior of at least one section.*

In this section by a *YY measure* μ in $\mathbb{R}^d$ we mean a finite measure absolutely continuous to the Lebesgue measure such that there is a closed ball B around 0 with $\mu(B) = \mu(\mathbb{R}^d)$.

We are interested in extending this theorem. Namely, we are interested in finding partitions where each hyperplane avoids the interior of more than one section.

Let $N_d(k)$ be the smallest positive integer such that for every YY measure μ in $\mathbb{R}^d$ there is a partition of $\mathbb{R}^d$ into $N_d(k)$ pairwise interior-disjoint convex sets of equal μ-measure such that every hyperplane in $\mathbb{R}^d$ avoids the interior of at least k sections. Theorem 3.0.3 can be restated as $N_d(1) \leq 2^d$.

In the next sections we will study the behaviour of $N_d(k)$, showing the following two main results.

Theorem 3.0.4. $N_d(2) \leq 3 \cdot 2^{d-1}$

Theorem 3.0.5. $N_d(1) \geq 2^{(d-2)/2}$ *for all d.*

The question whether $N_d(1)$ was polynomial was asked by Boris Bukh, and the theorem above gives a negative answer. Even if the convex sets are allowed to overlap, as long as the sum of their measures is at most 1, an exponential number of parts is needed. This is shown in Theorem 3.0.7 below.

Besides geometric queries, theorem of partitions avoiding hyperplanes can be applied to the following problem by Imre Bárány.

Problem 3.0.6 (The (α, β) problem)**.** *Given a fixed positive integer d, determine all pairs $(\alpha, \beta) \in \mathbb{R}_+^2$ such that for any finite set X of points in $\mathbb{R}^d$ and any finite set Y of hyperplanes in $\mathbb{R}^d$, there are subsets $A \subset X$, $B \subset Y$ such that*

- $|A| \geq \alpha |X|$,
- $|B| \geq \beta |Y|$ *and*
- *no hyperplane in B lies between any pair of points in A.*

Even though Theorem 3.0.5 can be proved without making reference to problem 3.0.6, the bounds obtained for this problem imply the lower bound for the Yao-Yao theorem. Moreover, working with the (α, β) problem makes the arguments needed more natural.

We can extend the notion of $N_d(k)$ and avoid the need for the convex sets to form a partition of $\mathbb{R}^d$. Namely, consider $M_d(k, \alpha)$ the smallest positive integer such that for every YY measure in $\mathbb{R}^d$ there is a family A of $M_d(k, \alpha)$ convex sets, each of measure at least α, such that every hyperplane avoids the interior of at least k elements of A. With this in mind, we immediately have

$$N_d(k) \geq M_d\left(k, \frac{1}{N_d(k)}\right).$$

Lower bounds for $M_d(k, \alpha)$ can be obtained in terms of the areas of spherical caps. For this, consider $\mathbf{S}^d$ with its usual probability measure. Denote by $h_d(t)$ the measure of a spherical cap in $\mathbf{S}^d$ with central angle t. Then,

Theorem 3.0.7. *Let μ be a YY measure in $\mathbb{R}^d$, α a positive real number and $\mathcal{A}$ a family of convex sets in $\mathbb{R}^d$ such that the following properties hold*

- *For all $A \in \mathcal{A}$, we have $\mu(A) \geq \alpha$,*
- *Every hyperplane in $\mathbb{R}^d$ avoids the interior of at least one set in $\mathcal{A}$*
- *The sum of the measure of all sets in A is at most 1.*

Then $\alpha = \Omega(2^{-d/2})$. More precisely,

$$\frac{1}{\alpha} \geq \frac{1}{2}\left[h_d\left(\frac{\pi}{4}\right)\right]^{-1} \geq \frac{1}{2} \cdot 2^{d/2}.$$

This implies Theorem 3.0.5, setting $\alpha = \frac{1}{N_d(1)}$. The conditions of the theorem above can be restated as $1 \geq \alpha \cdot M_d(1, \alpha)$.

3.1 Behaviour of $N_d(k)$

The core of this section is the following theorem.

Theorem 3.1.1 ([46]). *Given a positive integer d, the following holds*

$$\lim_{k\to\infty} \frac{N_d(k)}{k} = 1$$

In order to prove this, we need the following two lemmas

Lemma 3.1.2. $N_d(a+b) \leq N_d(a) + N_d(b)$

Proof. Given a YY measure μ, consider a hyperplane H that divides it in proportions $N_d(a) : N_d(b)$. We can find a partition of one side into $N_d(a)$ convex sets of equal μ-measure such that every hyperplane avoids at least a of them. We can find a partition of the other side into $N_d(b)$ convex sets of equal μ-measure such that every hyperplane avoids at least b of them. This gives a partition of $\mathbb{R}^d$ into $N_d(a) + N_d(b)$ convex sets with the desired properties. □

Lemma 3.1.3. $N_d(a)N_d(b) \geq N_d(aN_d(b) + bN_d(a) - ab)$

Proof. Given μ a YY measure in $\mathbb{R}^d$, partition $\mathbb{R}^d$ into $N_d(a)$ convex sets of equal μ-measure such that every hyperplane avoids at least a of them. Then, partition each section into $N_d(b)$ convex sets of equal μ-measure such that every hyperplane avoids at least b of them. This gives a partition of $\mathbb{R}^d$ into $N_d(a)N_d(b)$ convex sets of equal μ-measure such that every hyperplane intersects at most $(N_d(a) - a)(N_d(b) - b)$ of them, showing the desired inequality. $\square$

Now we are ready to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. Note that $N_d(k) \geq k + d$, as through every d points in $\mathbb{R}^d$ there is a hyperplane. Thus, $\liminf_{k\to\infty} \frac{N_d(k)}{k} \geq 1$.

If we start iterating Yao-Yao partitions in Lemma 3.1.3, we obtain that, for each n,

$$2^{dn} \geq N_d(2^{dn} - (2^d - 1)^n).$$

Using that $\lim_{n\to\infty} (2^d - 1)^n/(2^d)^n = 0$, we obtain that the sequence $k_n = 2^{dn} - (2^d - 1)^n$ satisfies $\limsup_{n\to\infty} \frac{N_d(k_n)}{k_n} \leq 1$. Using Fekete's lemma for subadditive sequences, we obtain the desired result. $\square$

In lower dimensions, the behaviour of $N_d(k)$ and $M_d(k, \alpha)$ is easier to describe. For example, in the plane we have the following result

Lemma 3.1.4. *Let $p \leq q$ be non-negative integers, then $M_2(q - p, \frac{p}{2q}) \leq 2q$.*

Proof. We will construct a family of $2q$ convex sets of $\mathbb{R}^2$ such that the boundaries are contained in the union of q lines $\ell_1, \ell_2, \ldots, \ell_q$ and every point of $\mathbb{R}^d$ is covered p times. If this is achieved, note that whenever a given line intersects an ℓ_i, it enters a new region. Thus, the line starts in points contained in p regions, it cannot intersect more than $p+q$ sets in the partition, giving the desired result. First, we fix a parameter $t \geq 0$, and we construct the lines ℓ_i inductively. In the construction they will be oriented halving lines. Thus, each ℓ_i will have a right side, which we denote ℓ_i^+, and a left side ℓ_i^- and $\mu(\ell_i^+) = \mu(\ell_i^-)$. We choose ℓ_1 be an oriented halving line (i.e. a line that splits $\mathbb{R}^2$ into two parts of equal μ-measure). Once that ℓ_i has been constructed, let ℓ_{i+1} be the oriented halving line such that the regions

$$A_i = \ell_{i+1}^+ \cap \ell_i^-$$
$$A_{q+i} = \ell_{i+1}^- \cap \ell_i^+$$

have μ-measure $\frac{p}{2q} + t$, for $i = 1, \ldots, q$ (see Fig. 3.1(a)). If $t = 0$ then the sum of the measures of these regions up to $i = q$ is p, but the regions may overlap and not cover almost every point of $\mathbb{R}^2$ at least p times. Let t be the smallest real number such that almost every point of $\mathbb{R}^2$ is covered at least p times. For this choice of t, the non-oriented lines determined by ℓ_1 and ℓ_{q+1} are equal. Thus, we have a construction induced by q lines as we wanted. ☐

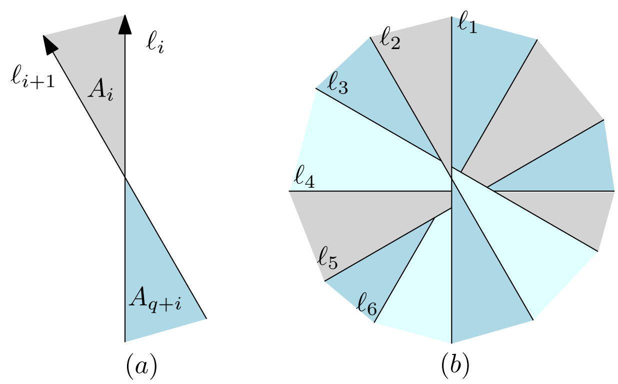

Figure 3.1: Regions for Lemma 3.1.4 and Lemma 3.1.5.

In order to obtain bounds for $N_2(k)$ with this method, one has to take additional considerations in the construction. This is done in the following way

Lemma 3.1.5. $N_2(k) \leq 2k + 2$.

Proof. This is similar to the proof above with $p = 1$ and $q = k + 1$, but this time we define the lines ℓ_i such that the regions

$$A_i = \ell_{i+1}^{+} \cap \ell_i^{-} \setminus \bigcup_{j<i} A_j$$
$$A_{q+i} = \ell_{i+1}^{-} \cap \ell_i^{+} \setminus \bigcup_{j<i} A_{q+j}$$

have μ-measure $\frac{1}{2k+2}$. To see that it is possible to find such ℓ_i, consider μ_i the measure μ restricted to $\mathbb{R}^2 \setminus \bigcup_{j<i} (A_j \cup A_{q+j})$. Note that ℓ_{i-1} is a halving line of μ_i, so we need l_i to be a halving line of μ_i such that $\mu_i(A_i) = \frac{1}{2k+2}$. This implies that $\mu_i(A_{q+i}) = \frac{1}{2k+2}$. Since μ and μ_i coincide in A_i and A_{q+i}, we obtain the desired line.

We end up with something like Fig. 3.1(b). This is a partition as, once again, ℓ_1 and ℓ_{q+1} are equal as non-oriented lines. Since $\mathbb{R}^2 \setminus \bigcup_{j<i} A_j$ consists of two convex components of equal μ-measure for all i, every A_i is convex. The same argument as above shows that every line avoids at least k regions.

□

3.2 The (α, β) problem

In some sense, the (α, β) problem deals with how well behaved are the sets of points in $\mathbb{R}^d$ with respect to hyperplanes. The way we stated this problem in the previous section, it seems not to be self-dual. That is, there is no reason for the pair (β, α) to work if (α, β) does. This is why it is convenient to use the following reformulation of it

Problem 3.0.6 ((α, β) problem, second version)**.** *Find all pairs $(\alpha, \beta) \in \mathbb{R}_+^2$ such that for any two YY probability measures μ_1, μ_2 in $\mathbf{S}^d$, there are subsets $A, B \subset \mathbf{S}^d$ such that $\mu_1(A) \geq \alpha, \mu_2(B) \geq \beta$ and either*

- *$a \cdot b \geq 0$ for all $a \in A, b \in B$ or*
- *$a \cdot b \leq 0$ for all $a \in A, b \in B$.*

Proof of equivalence between the two versions. Consider $\mathcal{C}'_d$ the set of valid pairs for the first version of the (α, β) problem and $\mathcal{C}_d$ the set of points for the second version. We show that $(\alpha, \beta) \in \mathcal{C}'_d$ if and only if $(\frac{\alpha}{2}, \frac{\beta}{2}) \in \mathcal{C}_d$. Consider the embedding $\mathbb{R}^d \hookrightarrow \mathbb{R}^{d+1}$ that maps $x \mapsto (x, 1)$. Then, projecting from the origin, we can assign to every $a \in \mathbb{R}^d$ a pair of points in $\mathbf{S}^d$

$$a \mapsto \left\{ \frac{(a,1)}{||(a,1)||}, -\frac{(a,1)}{||(a,1)||} \right\}.$$

To every hyperplane H^+ in $\mathbb{R}^d$ defined by $H = \{x \in \mathbb{R}^d : x \cdot x_0 = \lambda\}$ we can assign two antipodal points in $\mathbf{S}^d$ in the following way

$$H \mapsto \left\{ \frac{(x_0, -\lambda)}{||(x_0, -\lambda)||}, \frac{-(x_0, -\lambda)}{||(x_0, -\lambda)||} \right\}.$$

Note that $(x_0, -\lambda)$ is never the 0 vector. With his in mind, given a set X of points in $\mathbb{R}^d$ and a set Y of hyperplanes in $\mathbb{R}^d$, we can assign to X the finite measure μ_1 in $\mathbf{S}^d$ induced by the pairs of points assigned to X and to Y the finite measure induced by the pairs of points assigned to the hyperplanes in Y. We may normalise both measures so that $\mu_1(\mathbf{S}^d) = \mu_2(\mathbf{S}^d) = 1$. Note that good sets with these measures for the second problem correspond to good sets for the first problem, but each a factor of 2 involved. Note that every pair of YY centrally symmetric measures μ_1, μ_2 can be approximated by finite ones. Also, the pair (α, β) works in the second version of the problem for μ_1 and μ_2 if and only if it works for the centrally symmetric measures μ'_1, μ'_2, where

$$\mu'_i(X) = \frac{\mu_i(X) + \mu_i(-X)}{2}.$$

With this association we obtain that if $(\alpha, \beta) \in \mathcal{C}'_d$, then $(\frac{\alpha}{2}, \frac{\beta}{2}) \in \mathcal{C}_d$. The other inclusion can be proved the same way, since finite measures can also be approximated by YY measures. □

From now on we continue to use $\mathcal{C}_d$ to denote the set of pairs (α, β) that work in the second version of problem 3.0.6. Consider as well M^d the usual probability measure in $\mathbf{S}^d$ and $h_d(t)$ the M^d-measure of a spherical cap in $\mathbf{S}^d$ with central angle t. That is, the set of points at spherical distance at most t from a certain fixed point $p \in \mathbf{S}^d$ (the center of the cap). For example, we consider a half-sphere as a spherical cap with central angle $\frac{\pi}{2}$, and not π.

With this in mind, we can prove the following result.

Proposition 3.2.1. *The set $\mathcal{C}_d$ lies below or on the curve*

$$\left\{\left(h_d(t), h_d\left(\frac{\pi}{2} - t\right)\right) : 0 \leq t \leq \frac{\pi}{2}\right\}.$$

In order to prove this, for a set $A \subset \mathbf{S}^d$ consider the sets $A^{\perp}$ and A_{ϵ} defined as

$$\begin{aligned} A^{\perp} &= \{x \in \mathbf{S}^d : x \cdot a = 0 \text{ for some } a \in A\} \\ A_{\epsilon} &= \{x \in \mathbf{S}^d : \arccos(x \cdot a) < \epsilon \text{ for some } a \in A\} \end{aligned}$$

That is, A_{ϵ} is the set of points that are at distance at most ϵ in the $\mathbf{S}^d$ metric. Using this notation, we have the following result

Theorem 3.2.2 (T. Figiel, J. Lindenstrauss, and V. Milman [18])**.** *Given $A \subset \mathbf{S}^d$ such that $M^d(A) = h_d(t)$ for some t, then for all $\epsilon > 0$, we have that*

$$M^d(A_{\epsilon}) \geq h_d(t + \epsilon)$$

Using this we can prove 3.2.1.

Proof of Proposition 3.2.1. Consider μ_1 and μ_2 to be M^d. If $\epsilon = \frac{\pi}{2}$ and A is connected, then $\mathbf{S}^d \backslash A^{\epsilon}$ is one of the two connected components of $\mathbf{S}^d \backslash A^{\perp}$. This implies that if $A, B \subset \mathbf{S}^d$ are such that $a \cdot b \geq 0$ for all $a \in A, b \in B$ and $M^d(A) = h_d(t)$, then

$$M^d(B) \leq 1 - M^d(A_{\epsilon}) = 1 - h_d\left(\frac{\pi}{2} + t\right) = h_d\left(\frac{\pi}{2} - t\right).$$

Since M^d is symmetric, the case $a \cdot b \leq 0$ is analogous. □

The Yao Yao partition type problems are closely related to the (α, β) problem, as the first version of the problem suggests. This is made clear using the following lemma

Lemma 3.2.3. *Let $0 \leq \rho \leq 1$. Suppose that for any YY measure μ in $\mathbf{S}^d$ we can find a family F of subsets of $\mathbf{S}^d$ and a probability measure μ_F on F such that the following properties hold*

- *$\mu(A) \geq \alpha$ for all $A \in F$, and*
- *for every $b \in \mathbf{S}^d$, the set $F_b = \{A \in F : A \cap \{b\}^\perp \neq \emptyset\}$ is μ_F-measurable and $\mu_F(F_b) \leq \rho$.*

Then $(\alpha, \frac{1-\rho}{2}) \in \mathcal{C}_d$.

Proof. Let μ_1 and μ_2 be YY measures. Given α and ρ, construct F as above for $\mu = \mu_1$. Then, by Fubini's theorem,

$$\begin{aligned}\int_F \mu_2(A^\perp) d\mu_F &= \int_F \int_{\mathbf{S}^d} \chi(A^\perp) d\mu_2 d\mu_F \\ &= \int_{\mathbf{S}^d} \int_F \chi(F_b) d\mu_F d\mu_2 \\ &= \int_{\mathbf{S}^d} \mu_F(F_b) d\mu_2 \leq \rho\end{aligned}$$

Thus, there must be at least one $A_0 \in F$ such that $\mu_2(A_0^\perp) \leq \rho$. Thus one of there must be a B_0 such that the sign of $a \cdot b$ is constant for all $a \in A_0$, $b \in B_0$ and $\mu(B_0) \geq \frac{1-\rho}{2}$, as we wanted. □

Using this, we can prove the following theorem

Theorem 3.2.4. *If $\alpha > 0$, then*

$$\left(\frac{\alpha}{2}, \frac{k}{2M_d(k, \alpha)}\right) \in \mathcal{C}_d.$$

Proof. Let μ be a YY measure in $\mathbf{S}^d$. Consider $\mathbb{R}^d$ embedded in $\mathbb{R}^{d+1}$ as a hyperplane not containing the origin. Then, using a radial projection, we obtain a YY measure μ' in $\mathbb{R}^d$, so we may find $M_d(k,\alpha)$ convex sets of μ'-measure at least α such that every hyperplane in $\mathbb{R}^d$ avoids at least k of them. If we pull them back to $\mathbf{S}^d$ via the radial projection, we obtain a family F of $2M_d(k,\alpha)$ sets of measure at least $\frac{\alpha}{2}$ each. Also, note that every great circle intersects at most $2M_d(k,\alpha) - 2k$ of them. The great circle that is parallel to the hyperplane representing $\mathbb{R}^d$ causes no problems. By choosing μ_F to be the uniform probability measure on them and applying Lemma 3.2.3, we are done. □

If we set $\alpha = \frac{1}{N_d(k)}$, we obtain the following corollary

Corollary 3.2.5.

$$\left(\frac{1}{2N_d(k)}, \frac{k}{2N_d(k)}\right) \in \mathcal{C}_d.$$

We are now ready to prove Theorem 3.0.7

Proof of Theorem 3.0.7. Suppose that $1 \geq \alpha \cdot M_d(1,\alpha)$. Then, by Theorem 3.2.4 we have that $\left(\frac{\alpha}{2}, \frac{\alpha}{2}\right) \in \mathcal{C}_d$. However, by proposition 3.2.1 we have that this must be at most $\left(h_d\left(\frac{\pi}{4}\right), h_d\left(\frac{\pi}{4}\right)\right)$. Showing that $h_d\left(\frac{\pi}{4}\right) \leq 2^{-d/2}$ is a standard calculation, see the proof of Lemma 2.2 in [4] for details. □

Applying the results obtained for $N_d(k)$, we can show the following.

Corollary 3.2.6. *For any two non-negative integers k_1 and k_2, not both equal to* 0*, we have*

$$\frac{1}{2}\left(\left(\frac{1}{2^d}\right)^{k_1}\left(\frac{1}{3\cdot 2^{d-1}}\right)^{k_2}, 1 - \left(1-\frac{1}{2^d}\right)^{k_1}\left(1-\frac{1}{3\cdot 2^{d-2}}\right)^{k_2}\right) \in \mathcal{C}_d.$$

This is done by iterating Lemma 3.1.3 to the results on $N_d(1)$ and $N_d(2)$ and applying Corollary 3.2.5. In particular, we obtain that $\left(\frac{1}{2^{d+1}}, \frac{1}{2^{d+1}}\right) \in \mathcal{C}_d$ and $\left(\frac{1}{3\cdot 2^d}, \frac{1}{3\cdot 2^{d-1}}\right) \in \mathcal{C}_d$. The fact that $\left(\frac{1}{2^{d+1}}, \frac{1}{2^{d+1}}\right) \in \mathcal{C}_d$ was obtained earlier in [1] using a similar method. In Fig. 3.2 there are plots of these points together with the bound obtained in Proposition 3.2.1 in dimensions 2 and 3.

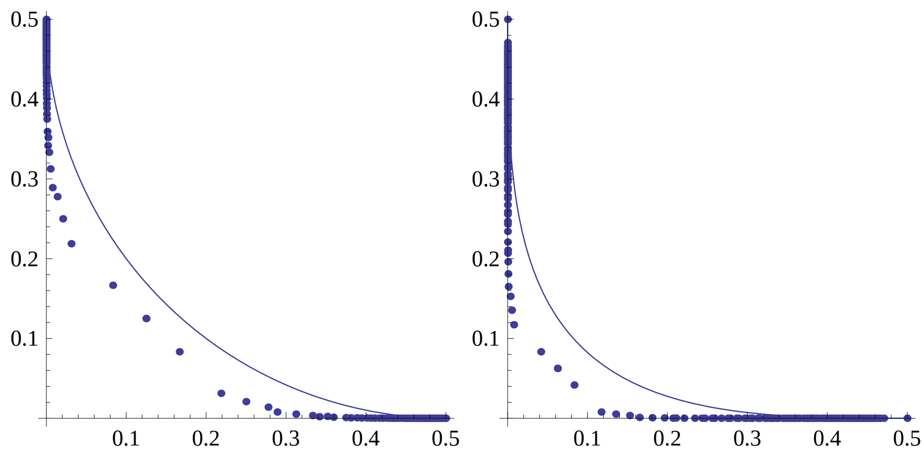

Figure 3.2: Bounds for $\mathcal{C}_2$ and $\mathcal{C}_3$.

Corollary 3.2.7. *There are pairs* $(\alpha, \beta) \in \mathcal{C}_d$ *arbitrarily close to* $(0, \frac{1}{2})$*.*

The drawback of this corollary is that the sequences of pairs we can obtain that converge to $(0, \frac{1}{2})$ come from using iterations of Yao-Yao's theorem and/or Theorem 3.0.4. These sequences are extremely close to the x-axis if d is large. We can obtain better sequences of pairs that converge to $(0, \frac{1}{2})$ if we impose additional conditions to one of the measures.

Let $\mathcal{C}_d(\Delta)$ be the set of pairs (α, β) such that, for any two YY measures μ_1, μ_2 in $\mathbf{S}^d$ such that μ_1 is the integral of a Lipschitz function f with $\text{Lip}(f) \leq \Delta$, we can find two sets $A, B \subset \mathbf{S}^d$ with $\mu_1(A) \geq \alpha$, $\mu_2(B) \geq \beta$ such that for all $a \in A, b \in B$, the sign of $a \cdot b$ is the same.

Theorem 3.2.8. *For all* $0 < \lambda \leq 1$*, and* $0 < r < \frac{1-\lambda}{\Delta}$*,*

$$\left(\lambda h_d(r), h_{d-1}\left[\frac{\pi}{2} - \left(\frac{\sin(r)}{\sin(\frac{1-\lambda}{\Delta} - r)}\right)\right]\right)$$

Note that if r is close to 0, then these pairs are similar to $\left(\lambda h_d(r), h_{d-1}\left(\frac{\pi}{2} - cr\right)\right)$ for a fixed constant c. This is similar to proposition 3.2.1, but changing the dimension of the second term.

The idea of the proof is to find a smaller $\mathbf{S}^{d-1}$ where f is large, and then construct sets that can be used in Lemma 3.2.3. Since f is large, bounding their μ_1-measure is easy, and the only problem is choosing the family of sets so that every hyperplane avoids many of them. If instead of the construction used in the proof we find sets that are close to a hypercube, then we obtain pairs that behave like $(\frac{c_1}{m^{d-1}}, \frac{1}{2} - \frac{c_2 d}{m})$. These are worse than the ones presented in the theorem above, but are easier to understand.

Proof of Theorem 3.2.8. Since $\mu_1(\mathbf{S}^d) = 1$, there must be a point $x_0 \in \mathbf{S}^d$ such that $f(x_0) \geq 1$. Consider $R = \min(\frac{1-\lambda}{\Delta}, \frac{\pi}{2})$. Since $\text{Lip}(f) \leq \Delta$, it follow that $f(x) \geq \lambda$ for all $x \in \mathbf{S}^d$ at distance at most R from x_0. Consider $r \leq \frac{R}{4}$. Given two points $x, y \in \mathbf{S}^d$, denote by $\text{dist}_{\mathbf{S}^d}(x, y)$ their distance in the sphere. Namely, the angle they sustain at the origin. For each $x \in \mathbf{S}^d$, define

$$S(x, r) = \{y \in \mathbf{S}^d : \text{dist}_{\mathbf{S}^d}(x, y) \leq r\}.$$

We construct the family F as follows (see figure 3.3 below)

$$F = \{S(x, r) : \text{dist}(x, x_0) = R - r\}.$$

We can note that

- Each set in F has measure at least $\lambda h_d(r)$
- $S(x, r)$ is the intersection of $\mathbf{S}^d$ with a ball with centre x and radius $\sin(r)$.
- The locus of the centers of the balls in F is $S(x_0, R-r)$; the intersection of $\mathbf{S}^d$ with a ball with centre x_0 and radius $\sin(R - r)$.

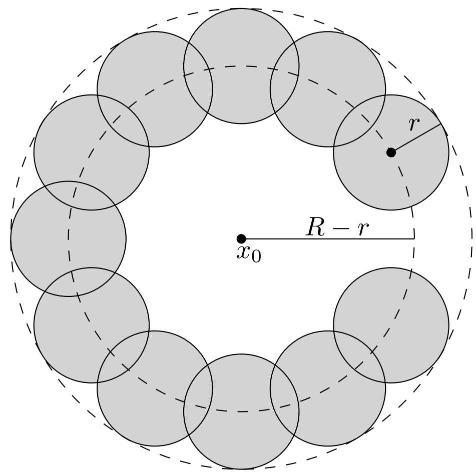

Figure 3.3: Construction of sets for the proof of Theorem 3.2.8.

Consider μ_F the usual probability measure on $S(x_0, R-r)$. Then, every hyperplane intersects a subset of F of size at most

$$1 - 2h_{d-1}\left(\frac{\pi}{2} - 2\arcsin\left(\frac{\sin(r)}{\sin(R-r)}\right)\right).$$

Then, the same argument for Lemma 3.2.3 finishes the proof. □

3.3 The Yao-Yao partition theorem

The next 3 sections deal with the proof of Theorem 3.0.4, via the test map scheme. We first give a detailed sketch of the original proof by Yao and Yao of their partition theorem, since it will be important in our constructions. Then, in section 3.4 we prove the topological result that will be used in the main proof. Finally, in section 3.5, we give the geometric construction that reduces the problem to topology.

Thus, we now show Yao and Yao's proof of the following theorem.

Theorem 3.0.3 (Yao and Yao, 1985 [46]). *Given a YY measure μ in $\mathbb{R}^d$, there is a partition of $\mathbb{R}^d$ into 2^d pairwise interior-disjoint convex sets of equal*

μ-measure such that no hyperplane intersects the interior of all of them.

Sketch of proof of Theorem 3.0.3. Consider $O(d)$ the space of all orthonormal (ordered) bases $u_1, u_2, \dots, u_d$ of $\mathbb{R}^d$. Note that if we consider the u_i as rows of a $d \times d$ matrix, $O(d)$ is identified with the set of matrices U such that $U^T U = I$. Consider $SO(d) \subset O(d)$ those bases that correspond to matrices of determinant 1.

Given a basis $u_1, u_2, \dots, u_d$ an orthonormal basis of $\mathbb{R}^d$ and a hyperplane H orthogonal to u_1, we can define its two half-spaces as

$$\begin{aligned} H^+ &= \{x + tu_1 : x \in H, t > 0\} \\ H^- &= \{x + tu_1 : x \in H, t < 0\} \end{aligned}$$

Given a normal vector v not orthogonal to u_1, it induces a projection

$$\begin{aligned} p_v : \mathbb{R}^d &\rightarrow H \\ x + tv &\mapsto x \text{ for all } x \in H, t \in \mathbb{R} \end{aligned}$$

Using p_v, we can project the measure μ restricted to H^+ and H^- to H to obtain two YY measures, μ_v^+ and μ_v^- respectively, in H.

We define a centre $c \in \mathbb{R}^d$ which depends on μ and $u_1, u_2, \dots, u_d$ inductively on d in the following way

- If $d = 1$, then c is the midpoint of the closed interval of points that divides $\mathbb{R}^1$ into two part of equal μ-measure

- If $d > 1$, consider H the hyperplane orthogonal to u_1 that divides $\mathbb{R}^d$ into two parts of equal μ-measure (if there is more than one option, choose the central hyperplane of this set). If there is a normal vector v (not orthogonal to u_1) such that μ_v^+ and μ_v^- have the same center c using the basis $(u_2, u_3, \dots, u_d)$ in H, we define c as the center for μ and $u_1, u_2, \dots, u_d$.

Yao and Yao showed that in the second step, this vector v always existed. For this they used that if c^+ and c^- are the centres of μ_v^+, μ_v^- respectively, then the function $v \mapsto c^+ - c^-$ can be extended continuously to $\mathbf{S}^{d-1}$ and is antipodal. Thus, using the Borsuk-Ulam theorem, they could prove the existence of v. Moreover, using the centre c, one can define inductively on the dimension the partition we seek in the following way

- For $d = 1$, the partition is given by c.
- For $d > 1$, consider H, v and c as in the construction of the centre c. Then, we have two partitions, $\mathcal{P}^+$ and $\mathcal{P}^-$, of H into 2^{d-1} parts each induced by μ_v^+ and μ_v^-, respectively. We pull $\mathcal{P}^+$ back to H^+ using p_v and we pull $\mathcal{P}^-$ back to H^- using p_v. This gives the partition we seek.

Note that, by induction, all the parts are convex cones with c as apex. It is then showed that if we consider l the line with direction v through c, if a hyperplane Y intersect l in H^+ it avoids the interior of at least one of the sections of the partition in H^- and if it intersects l in H^- it avoids the interior of at least one of the sections in H^+, proving the theorem.

One should note that c, v are unique (v up to a multiplication of -1, but we can fix $v \cdot u_1 > 0$) and vary continuously as the basis $u_1, u_2, \ldots, u_d$ does. $\square$

3.4 Borsuk-Ulam type theorems via homotopy

In order to prove Borsuk-Ulam type theorems, one usually has to use tools from topology. The information regarding the existence of equivariant continuous maps between two spaces is completely determined by the existence of sections of some fibre bundles, which in turn can be determined by their characteristic classes. Which characteristic classes actually carry this information and how to compute them are part of what is known as obstruction theory.

However, some results of this kind can be proved without relying on the higher-end topological techniques. For example, Dold's generalisation of the Borsuk-Ulam theorem (presented in section 1.5) can be shown using only the topological degree of a map.

The method we use to prove the topological result of this section is based on a geometric proof of the Borsuk-Ulam theorem by Bárány, which is very intuitive. This proof is completely contained in section 2.2 of [29]. The method we follow is also used and expanded in [32].

We first explain the proof method and then how the spaces and group actions we are using fit this scheme.

Consider two spaces X and Y of dimension n, each with an action of a finite group G. Moreover, suppose the action of G in X is free. Let y_0 be a special point of Y, which is fixed by the action of G in Y. We want to show that for every equivariant map $f : X \to Y$ there is a point $x_0 \in X$ such that $f(x_0) = y_0$.

We will prove this by contradiction, supposing that f sends no points of X to y_0. The key point of the proof is to find a special function f_0 such that there is an odd number of G-orbits of points of X sent to y_0 by f_0. Let I be the interval $[0, 1]$ and suppose that there is an equivariant homotopy $F : X \times I \to Y$ between f_0 and f.

Consider the preimage $F^{-1}(y_0)$. If F is generic enough, the co-dimension of this space in $X \times I$ should be the same as the co-dimension of $\{y_0\}$ in Y. That is, $F^{-1}(y_0)$ should be a set of paths and cycles. Note that the paths need to have their endpoints at the extreme copies of X, namely $X \times \{0\}$ and $X \times \{1\}$. Since the action of G in X is free, this implies that the parity of the number of G-orbits of points of X that are sent to y_0 should be the same in f, and f_0. This is the contradiction we wanted, as it is 0 in f and odd in f_0 (see figure 3.4).

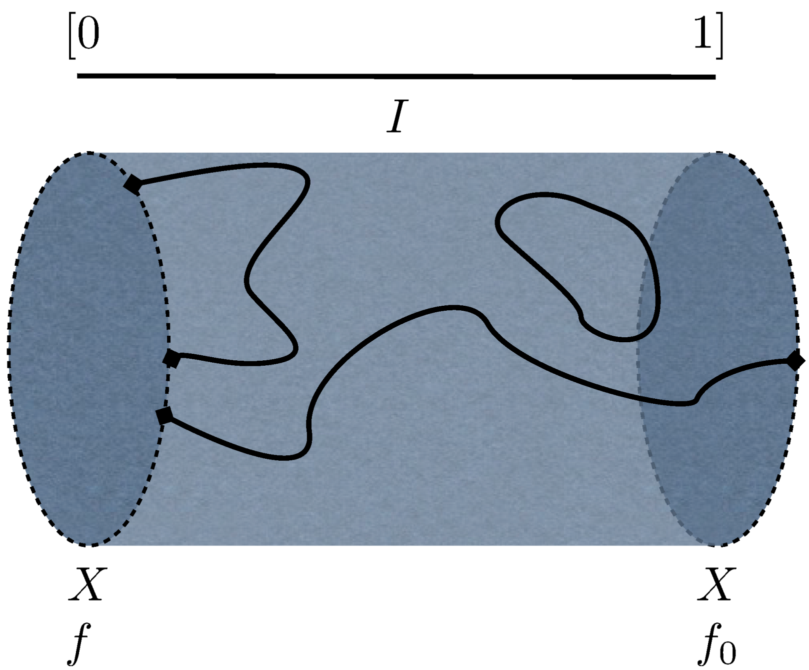

Figure 3.4: The figure above shows why the equivariant homotopy preserves the parity of the number of orbits of preimages of y_0 in f and f_0. This is based on the figure of page 31 of [29]. Note that every such path or cycle is repeated according to the group action.

If Y is a vector space and the action of G is nice enough (namely, multiplying by any element of G is a linear function), then one natural candidate for F is simply

$$\begin{aligned} F : X \times I &\to Y \\ (x,t) &\mapsto tf_0(x) + (1-t)f(x) \end{aligned}$$

This function may fail to be generic enough for our purposes. However, finding an generic equivariant function $H : X \times I \to Y$ that approximates F may yield the contradiction we seek. The existence of H depends on further properties of X, Y and f_0, but we will mention what is needed in the proof at the end of the section.

Let us then construct the spaces and group actions we will need. For our construction, we will parametrise a family of partitions with $O(d)$, the space of orthonormal basis of $\mathbb{R}^d$. Note that there is a natural action of $(\mathbb{Z}_2)^d$ in $O(d)$. Given $u \in O(d)$ and $g \in (\mathbb{Z}_2)^d$, gu is the result of changing the sign of some elements of u, according the coordinates of g.

The target space (what we usually denote as Y) will be of the form $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}$ for some d_i, so it is convenient to settle some operations on these spaces. Given $d_1, d_2, \ldots, d_n$ positive integers, consider $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}$. For $i = 1, 2, \ldots, n$, we define the function

$$g_i : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n} \to \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \times \ldots \times \mathbb{R}^{d_n}$$

as the result of changing the sign of x_i. We will use this notation regardless of the values $d_1, d_2, \ldots, d_n$, unless there is need to specify. We denote by $x_i^{(j)}$ the j-th coordinate of $x_i \in \mathbb{R}^{d_i}$.

Given $v = (v_1, v_2, \ldots, v_{d-1}) \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$, we define $v^{(j)} \in \mathbb{R}^{d-j}$ as $(v_1^{(j)}, \ldots, v_{d-j}^{(j)})$. Thus, we can consider $v^T \in \mathbb{R}^{d-1} \times \cdots \times \mathbb{R}^1$ as $v^T = (v^{(1)}, \ldots, v^{(d-1)})$.

An easier way to visualise this last construction is to consider a $(d-1) \times (d-1)$ matrix V induced by v in the following way. In the k-th row write the coordinates of v_k followed by $k-1$ signs "$\times$",

$$V = \begin{matrix} v_1 \\ v_2 \\ \vdots \\ v_{d-2} \\ v_{d-1} \end{matrix} \begin{pmatrix} v_1^{(1)} & v_1^{(2)} & \cdots & v_1^{(d-2)} & v_1^{(d-1)} \\ v_2^{(1)} & v_2^{(2)} & \cdots & v_2^{(d-2)} & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{d-2}^{(1)} & v_{d-2}^{(2)} & \cdots & \times & \times \\ v_{d-1}^{(1)} & \times & \cdots & \times & \times \end{pmatrix}.$$

Then v^T is the set of vectors induced in the same way by the transpose V^T of V, namely

$$V^T = \begin{matrix} v^{(1)} \\ v^{(2)} \\ \vdots \\ v^{(d-2)} \\ v^{(d-1)} \end{matrix} \begin{pmatrix} v_1^{(1)} & v_2^{(1)} & \cdots & v_{d-2}^{(1)} & v_{d-1}^{(1)} \\ v_1^{(2)} & v_2^{(2)} & \cdots & v_{d-2}^{(2)} & \times \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_1^{(d-2)} & v_2^{(d-2)} & \cdots & \times & \times \\ v_1^{(d-1)} & \times & \cdots & \times & \times \end{pmatrix}.$$

Note that $g_i(v^T)^T$ is the result in changing the i-th coordinate of every vector in v.

The main result of this section is the following

Lemma 3.4.1. *Given $f : O(d) \to \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ such that for all $u \in O(d)$*

- $f(g_1(u)) = g_2(f(u))$,
- $f(g_2(u)) = g_{d-1}(f(u)^T)^T$ *and*
- $f(g_{i+2}(u)) = g_i(f(u)^T)^T$ *for* $i = 1, 2, \ldots, d-2$.

there is a $u_0 \in O(d)$ such that $f(u_0) = 0$.

Proof. Suppose that the lemma does not hold, and let f be a function with the properties above a no zeros. Note that if we consider $g_1, g_2, \ldots, g_d$ as generators of the group $\mathbb{Z}_2^d$, it defines a natural action of this group on $O(d)$. We may define an action of $\mathbb{Z}_2^d$ on $\mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ so that f is equivariant. Note that in this case, the elements $g_1 \circ g_d,\ g_2 \circ g_d, \ldots,\ g_{d-1} \circ g_d$ define a free action of $\mathbb{Z}_2^{d-1}$ in $SO(d)$. We can define analogously an action of $\mathbb{Z}_2^{d-1}$ in $\mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ so that $f_1 = f|_{SO(d)}$ is equivariant. Note that the dimension of $SO(d)$ and $\mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ is the same.

With this action in mind, we will construct an equivariant function f_0 between these two spaces that has exactly one $(\mathbb{Z}_2)^{d-1}$-orbit of zeros.

Namely, let f_0 be the function defined as

$$\begin{aligned} f_0 : SO(d) &\to \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1 \\ w &\mapsto (v_1, v_2, \ldots, v_{d-1}) \end{aligned}$$

where

$$\begin{aligned} v_1 &= \left(w_3^{(1)}, \ldots, w_d^{(1)}, w_2^{(1)}\right), \\ v_2 &= w_1^{(1)} \left(w_3^{(2)}, \ldots, w_d^{(2)}\right), \\ v_{i+2} &= \left(w_3^{(i+2)}, \ldots, w_{d-i}^{(i+2)}\right) \text{ for } i = 1, 2, \ldots, d-2 \end{aligned}$$

Note that f_0 is equivariant under the group actions defined above. Moreover, if $f(w)$ is 0, then w_i has to be either the i-th element of the canonical basis or its negative if $i = 1, 2$, and w_i has to be the $(d+3-i)$-th element of the canonical basis if $3 \leq i \leq d$. Thus f_0 has exactly 2^{d-1} zeros in $SO(d)$. Moreover, 0 is a regular value of f_0.

Let F be the equivariant homotopy between f_0 and f_1, defined as

$$\begin{aligned} F : SO(d) \times I &\to \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1 \\ u &\mapsto t f_1(u) + (1-t) f_0(u) \end{aligned}$$

If F is not generic enough, it may be perturbed slightly to a function H that satisfies the following,

- H is an equivariant homotopy between two (equivariant) functions f_1' and f_0' on the same spaces as f_1 and f_0,
- zero is a regular value of H,
- f_1' has no zeros,

- f_0' has exactly one orbit of zeros (it is essential that zero is a regular value of f_0 to be able to guarantee this).

Then, using H we may obtain a contradiction as described by the method at the beginning of the section, which completes the proof. □

Approximating F by generic functions as above is a standard technique in differential geometry.

3.5 Proof of Theorem 3.0.4

We are now ready to prove Theorem 3.0.4.

Proof of Theorem 3.0.4. Let μ be a YY measure on $\mathbb{R}^d$. Let $u = (u_1, u_2, \ldots, u_d)$ be an orthonormal basis of $\mathbb{R}^d$. Given a hyperplane H orthogonal to u_1, we can define denote its open half-spaces by

$$\begin{aligned} H^+ &= \{h + tu_1 : h \in H, t > 0\} \\ H^- &= \{h + tu_1 : h \in H, t < 0\} \end{aligned}$$

let H_1, H_2 be hyperplanes orthogonal to u_1, so that they divide $\mathbb{R}^d$ into three regions, $A = H_1^+, B = H_1^- \cap H_2^+$ and $C = H_2^-$, of equal μ-measure. Consider the measures $\mu_1 = \mu|_{A\cup B}$ and $\mu_2 = \mu|_{B\cup C}$. Note that H_1 splits μ_1 by half and H_2 splits μ_2 by half.

Thus, if we follow the proof of Theorem 3.0.3 explained in section 3.3, there is a Yao-Yao center $O_1 \in H_1$ and a projection vector v_1 in $\mathbf{S}^{d-1}$, not orthogonal to u_1, that induce a Yao-Yao partition $\mathcal{P}_1$ for μ_1. Moreover, if we impose the condition $u_1 \cdot v_1 < 0$, then O_1 and v_1 are unique. The same happens for μ_2 and H_2, where we can find $O_2 \in H_2$ and $v_2 \in \mathbf{S}^{d-1}$ with $u_1 \cdot v_2 > 0$ that induce a Yao-Yao partition $\mathcal{P}_2$ for μ_2 (See figure 3.5).

If the vectors v_1, v_2 and $O_1 - O_2$ are parallel, then we can construct the partition we seek. For this, consider $\mathcal{P}$ the partition of $\mathbb{R}^d$ that consists of the following sections

- the parts of $\mathcal{P}_1$ contained in A,
- the intersections of B with the parts of $\mathcal{P}_1$ contained in $B \cup C$ and
- the parts of $\mathcal{P}_2$ contained in C.

Consider l the line through O_1 and O_2 and H an arbitrary hyperplane. Since l is parallel to v_1 and v_2, depending on whether H intersects l in A, B, or C, then it avoids at least one section of $\mathcal{P}$ in B and C, A and C or A and B, respectively. This follows from the fact that $\mathcal{P}_1$ and $\mathcal{P}_2$ were Yao-Yao partitions. The only case that needs a closer analysis is to see that if H intersects l in C, it does avoid a part of $\mathcal{P}$ in B. However, since the partition in B is symmetric with respect to the midpoint of O_1O_2, this case is equivalent to the one when H intersect l in A, which is covered by the fact that $\mathcal{P}_1$ is a Yao-Yao partition. If H does not intersect l, then it avoids one section of $\mathcal{P}$ in each of A, B and C. Thus, it suffices to find an orthogonal basis u so that $O_1 - O_2, v_1$ and v_2 are parallel.

Denote by J_1 the flat of codimension 2, orthogonal to u_1 and u_2 through O_1. Note that $J_1 \subset H_1$. The half-hyperplane $\{j + tv_1 : j \in J_1, t > 0\}$ splits B into two sets of equal μ_1-measure. The flat $J_2 \subset H_2$, parallel to J_1 through O_2, satisfies that the half-hyperplane $\{j + tv_2 : j \in J_2, t > 0\}$ splits B into two sets of equal μ_2-measure. Note that since μ_1 and μ_2 are the same measure in B, these two sets intersect in a $(d-2)$-flat J.

Consider $r_1 : \mathbb{R}^d \to H_1$ the projection such that $r_1(O_2) = O_1$, and consider $r_2 : H_1 \to \mathbb{R}^{d-1}$ the projection such that $r_2(O_1) = 0$. Their composition $r = r_2 \circ r_1 : \mathbb{R}^d \to \mathbb{R}^{d-1}$ satisfies $r(O_1) = r(O_2) = 0$. We may consider $(u_2, u_3, \ldots, u_d)$ the basis for $\mathbb{R}^{d-1}$. Since J is orthogonal to u_2, so is $r(J)$. Thus, there is a $\lambda \in \mathbb{R}$ such that $r(J) = \{x \in \mathbb{R}^{d-1} : x \cdot u_2 = \lambda\}$. Let $x \in \mathbb{R}^{d-2}$ and $y \in \mathbb{R}^{d-2}$ be the last $d-2$ coordinates of $r(v_1)$ and $r(v_2)$ respectively.

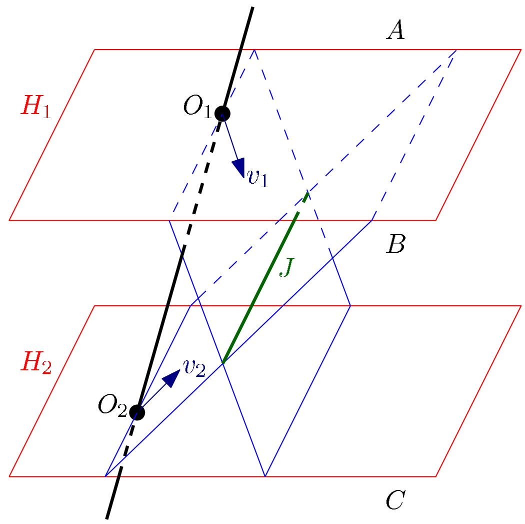

Figure 3.5: Construction of the partition (figure from [35])

Consider $h(u) = (x, y, \lambda) \in \mathbb{R}^{d-2} \times \mathbb{R}^{d-2} \times \mathbb{R}$. If $h(u) = 0$, then v_1, v_2 and $O_1 - O_2$ are parallel, as we wanted.

Note that the map h satisfies the following

- $h(g_1(u)) = (y, x, \lambda)$, as changing the sign of u_1 exchanges the roles of H_1, H_2.
- $h(g_2(u)) = (x, y, -\lambda)$, by the definition of λ, and
- $h(g_{i+2}(u) = (g_i(x), g_i(y), \lambda)$ for $i = 1, 2, \dots, d-2$, as x, y were formed by the last $d-2$ coordinates of the projections of v_1, v_2 using the basis $(u_2, u_3, \dots, u_d)$.

Consider the function f defined as

$$\begin{aligned} f : O(d) &\to \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1 \\ u &\mapsto ((x+y, \lambda), x-y, 0, \dots, 0) \end{aligned}$$

Note that finding a zero of f is equivalent to finding a zero of h. Given

$v \in \mathbb{R}^{d-1} \times \mathbb{R}^{d-2} \times \cdots \times \mathbb{R}^1$ recall the definition of v^T at the beginning of the section. With this notation, the conditions of h translate to

- $f(g_1(u)) = g_2(f(u))$,
- $f(g_2(u)) = g_{d-1}(f(u)^T)^T$ and
- $f(g_{i+2}(u)) = g_i(f(u)^T)^T$ for $i = 1, 2, \ldots, d-2$.

Using Lemma 3.4.1, we are done.

□

Chapter 4

Tverberg type partitions

The next two chapters will deal with Tverberg type partitions. The core of the proofs are based on a deeper analysis of Sarkaria's proof of Tverberg's theorem, explained in section 1.2.

Namely, we will rely on the following two simple observations about this proof (using the notation of section 1.2).

- An action of $\mathbb{Z}_k$ can be defined on the set $\{b_i \otimes u_j \ : \ 1 \le i \le n+1, 1 \le j \le k\}$. Moreover, that action sends sets capturing the origin to sets capturing the origin, which the colourful Carathéodory theorem ignores.

- If additional structure is added to the partition, the lift $a_i \mapsto b_i$ can be avoided.

The contents of this chapter are contained in [41].

4.1 Partitions with tolerance

In this chapter we will prove a version of Tverberg's theorem with tolerance. We say that a property $\mathcal{P}$ is true in a set X with tolerance r if $\mathcal{P}$ is true in X even if we remove any r points of X. For example, *captures the origin*, or *the convex hull of the red points of X and the convex hull of the blue points of X intersect* are examples of properties that may be true with a certain tolerance. The first theorem with tolerance is due to David Larman. While working on a problem of McMullen about sending arbitrary sets of points to convex position via projective transformations, he showed the following equivalent result.

Theorem 4.1.1 (Larman, 1972 [27])**.** *Let $d \geq 1$ be an integer. Given a set A of $2d+3$ points in $\mathbb{R}^d$, there is a partition of A into two sets A_1 and A_2 such that for any $x \in A$,*

$$\langle A_1 \backslash \{x\} \rangle \cap \langle A_2 \backslash \{x\} \rangle \neq \emptyset.$$

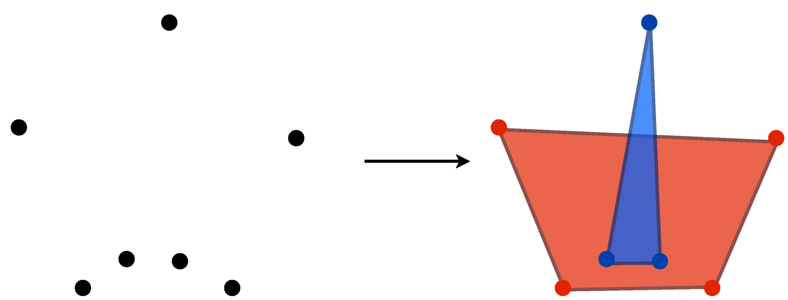

Figure 4.1: An example of a Radon partition with tolerance 1 in the plane. Note that with any 6 points in the figure such partitions do not exist.

This is what we call a Radon theorem with tolerance 1. One should note that the theorem is trivial with $2d+4$ points, as one can split the set into two sets of $d+2$ points each and find a Radon partition on each side (A, B) and (A', B'). The partition $(A \cup A', B \cup B')$ clearly satisfies Larman's condition, as removing one point can only break one of the two original Radon partitions. However, with $2d+3$ we are gaining meaningful information about the intersection of the convex hulls of sets of points.

The number $2d+3$ is known to be optimal for $d \leq 4$ [20]. Currently the best lower bound in general is $3 + \lceil \frac{5d}{3} \rceil$ by Ramirez-Alfonsín [34]. The condition $d \geq 1$ is a strange necessity. In the case $d = 0$, $\mathbb{R}^d$ is a single point. Given 3 copies of that point, in any partition of them into two parts there is one part with at most one point. Removing that point shows that the conclusion of the theorem is false.

This result was generalised by Natalia García-Colín as part of her PhD thesis at UCL for any tolerance (under David Larman's supervision). Namely

Theorem 4.1.2 (García-Colín, 2007 [21]). *Let $d \geq 1$ be an integer. Given A set A of $(r+1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them into two sets A_1 and A_2 such that for any set $C \subset A$ of r points,*

$$\langle A_1 \backslash C \rangle \cap \langle A_2 \backslash C \rangle \neq \emptyset$$

This is what we would call a Radon with tolerance r. Note that the case $r = 0$ is Radon's theorem and the case $r = 1$ is Larman's result. Again, for any value of $r \geq 1$, the condition $d \geq 1$ is necessary. For $r > 1$, García-Colín showed as well that in any dimension $2d + r + 3$ points may be necessary for the theorem to hold [21]. She conjectured a version of her result for Tverberg partitions. This was answered affirmatively in a joint paper with Ricardo Strausz [41] and is presented in the next section.

One should note that García-Colín's proof of her conjecture is similar to Tverberg's proof of his own theorem. Namely, she proved that a special set of $(r+1)(d+1)+1$ points had a Radon partition with tolerance r, and then moved the points continuously. If the partition stopped working, she showed that at that moment one could swap points in the partition to avoid any problems. This way one could reach any configuration of points and have a Radon partition with tolerance r. The proof of the generalisation of her conjecture for Tverberg partitions is based on Sarkaria's proof of Tverberg's theorem, presented in section 1.2.

One should note that there are version of other results in combinatorial geometry with tolerance. This includes classic results like Helly's theorem and Carathéodory's theorem (see [31] and the references therein).

4.2 Tverberg with tolerance

Given a set X, we denote by $\binom{X}{r}$ the family of subsets of X of size r. Using this notation,

Theorem 4.2.1 (Soberón, Strausz 2012 [41]). *Let r, k, d be non-negative integers with $d \geq 1$. Given a set A of $(r+1)(k-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of S into k sets $A_1, A_2, \ldots, A_k$ such that for any $C \in \binom{S}{r}$,*

$$\bigcap_{i=1}^{k} \langle A_i \backslash C \rangle \neq \emptyset$$

In order to prove Theorem 4.2.1, we shall prove a version of the colourful Carathéodory theorem with tolerance. Groups action are essential for this to work. Note that the result would be trivial with $(r+1)\left[(k-1)(d+1)+1\right]$ points.

Let $A' \subset \mathbb{R}^d$ be a set of points and G a group such that there is a group action of G in A'. Given $A \subset A'$ we say that the group action of G is *compatible* with A if the following two conditions are met

- If $B \subset A'$ captures the origin, then gB captures the origin for any $g \in G$
- Given a point $a \in A$, then Ga captures the origin.

These conditions allow us to extend the colourful Carathéodory theorem to a version with tolerance, stated below. It should be noted that the original proof of the theorem by Bárány is metric, and the proof we present here follows the same line of thought.

Lemma 4.2.2 (Colourful Carathéodory with tolerance)**.** *Let $m \geq 1$ and $r \geq 0$ be integers, $A \subset \mathbb{R}^m$ a set of $n = (r+1)m + 1$ points $a_1, a_2, \dots a_m$ and G a group with $|G| \leq m$. If there is a set A' such that $A \subset A' \subset \mathbb{R}^d$ and an action of G in A' which is compatible with A, then for each a_i there is a $g_i \in G$ such that the set $\{g_1 a_1, g_2 a_2, \dots, g_n a_n\}$ captures the origin with tolerance r.*

Proof. We proceed by induction on r. For $r = 0$, this lemma is a direct consequence of the colourful Carathéodory theorem, taking $Ga_1, Ga_2, \dots, Ga_n$ as the colour classes.

Suppose the lemma is true for $r-1$ but not for r, and we look for a contradiction. Let $\{h_1, h_2, \dots, h_t\}$ be the elements of G with $t \leq m$. Given any vector $\alpha = (g_1, g_2, \dots, g_n) \in G^n$, let $\alpha \cdot A = \{g_1 a_1, g_2 a_2, \dots, g_n a_n\}$. Since we are supposing that the lemma is false, for any α there is a subset $C \subset \alpha \cdot A$ of r points such that $(\alpha \cdot A) \backslash C$ does not capture the origin. For each α, let

$$P(\alpha) = \max_{|C|=r} \operatorname{dist}\left(\langle (\alpha \cdot A) \backslash C \rangle, 0\right).$$

Observe that $P(\alpha) > 0$ for all α.

Let α_0 be a vector in G^n such that $P(\alpha_0)$ is minimal, and let $C_0 \subset \alpha_0 \cdot A$ be a set of r points such that realises this distance, namely $\operatorname{dist}\left(\langle (\alpha_0 \cdot A) \backslash C_0 \rangle, 0\right) = P(\alpha_0)$. If p_0 is the point of $\langle (\alpha_0 \cdot A) \backslash C_0 \rangle$ closest to the origin, p_0 must be in a face of $\langle (\alpha_0 \cdot A) \backslash C_0 \rangle$. Note that this face is contained in a flat of co-dimension at least 1. Thus, there is a set $X \subset (\alpha_0 \cdot A) \backslash C_0$ of at most m points such that p_0 is in the relative interior of $\langle X \rangle$. Let $B = (\alpha_0 \cdot A) \backslash X$ and H be a hyperplane that contains X and leaves the origin in one of its open half-spaces H^-.

By induction, since the action of G is compatible with B and B has at least $mr + 1$ points, there is a vector β of $G^{|B|}$ such that $\beta \cdot B$ captures the origin with tolerance $r - 1$. Since Gx captures the origin for all $x \in B$, for each $b \in B$ there must be a $g \in G$ such that $gb \in H^-$. Consider the sets

$(h_1\beta)\cdot B$, $(h_2\beta)\cdot B$, …, $(h_t\beta)\cdot B$. Among them, there must be at least $mr+1$ points in H^-. Since $t \leq m$, we can use the pigeonhole principle to find a $g \in G$ such that $(g\beta)\cdot B$ contains at least $\lceil \frac{mr+1}{t} \rceil \geq r+1$ points in H^-.

Let α_1 be the vector in G^n that results in changing in α_0 the elements corresponding to B for those of $(g\beta)\cdot B$. We claim that $P(\alpha_1) < P(\alpha_0)$.

For this we have to show that for any subset $C \subset \alpha_1 \cdot A$ of r points, we have that

$$\operatorname{dist}(\langle \alpha_1 \cdot A \backslash C \rangle, 0) < P(\alpha_0).$$

There are two cases we need to analyse.

- If among these points there are at most $r-1$ points of $(g\beta)\cdot B$, then $(\alpha_1 \cdot A)\backslash C$ captures the origin, so the distance above is 0.
- If $C \subset (g\beta)\cdot B$, then there is a point $x \in H^- \cap (g\beta)\cdot B$ that is not in C. It follows that $\langle X \cup x\rangle$ is closer to the origin than $\langle X \rangle$, so the distance above is smaller than $P(\alpha_0)$.

Thus $P(\alpha_1) < P(\alpha_0)$, contradicting the minimality of $P(\alpha_0)$. □

If we ignore the condition on the group actions, the result no longer holds. In fact, a direct version of the colourful Carathéodory theorem with tolerance only holds with the trivial number of colour classes. Namely,

Claim 4.2.3. *Let $n = (r+1)(d+1)$. Given n sets $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ in $\mathbb{R}^d$ such that each captures the origin, it is possible to find elements $x_1 \in \mathcal{F}_1, \ldots, x_n \in \mathcal{F}_n$ such that the set $\{x_1, \ldots, x_n\}$ captures the origin with tolerance r. Moreover, the value of n is optimal.*

Proof. The case $n = (r+1)(d+1)$ is a direct consequence of the colourful Carathéodory theorem. Let us construct a counter-example for $n < (r+1)(d+1)$.

Let S be a non-degenerate simplex in $\mathbb{R}^d$ that captures the origin. Consider $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ all equal to the set of vertices of S. In any colourful choice of points of the $\mathcal{F}_i$, there is one vertex of S that was chosen at most r times. Removing all the copies of this vertex leaves us with a set that does not capture the origin. □

Proof of Theorem 4.2.1. Let $m = (k-1)(d+1)$, and $a_1, a_2, \ldots, a_{(r+1)m+1}$ be $(r+1)m+1$ points in $\mathbb{R}^d$. Consider the points $b_i = (a_i, 1) \in \mathbb{R}^{d+1}$ for all i. Let $u_1, u_2, \ldots, u_k \in \mathbb{R}^{k-1}$ be the k vertices of a regular simplex centred at the origin. Consider the sets $A' = \{b_i \otimes u_j : \text{ for all } i, j\}$ and $A = \{b_i \otimes u_1 : \text{ for all } i\}$ in $\mathbb{R}^m$.

Let σ be a generator of the group $\mathbb{Z}_k$. Note that there is an action of $\mathbb{Z}_k$ in A' given by $\sigma(b_i \otimes u_j) = b_i \otimes u_{j+1}$. Moreover, this action is compatible with A. Since $d \geq 1$, we have that $k \leq m$. Thus, we can apply Lemma 4.2.2 and obtain indices $j_1, j_2, \ldots, j_{(r+1)n+1}$ such that the set $\{b_1 \otimes u_{j_1}, b_2 \otimes u_{j_2}, \ldots, b_{(r+1)n+1} \otimes u_{j_{(r+1)n+1}}\}$ captures the origin with tolerance r. The last arguments are analogous to those in the proof of Tverberg's theorem in section 1.2. □

If we avoid using the inclusion $\mathbb{R}^d \hookrightarrow \mathbb{R}^{d+1}$ we obtain a (slightly more general) version of this theorem for convex cones. Namely

Corollary 4.2.4. *Let r, k, d be non-negative integers with $d \geq 2$ and $n = (r+1)(k-1)d$. Given a set A of $n+1$ points $a_1, a_2, \ldots, a_{n+1}$ in $\mathbb{R}^d$, there is a partition $I_1, I_2, \ldots, I_k$ of $\{1, 2, \ldots, n+1\}$ into k sets such that for any $C \in \binom{S}{r}$, there are non-negative coefficients $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$, not all zero, such that*

$$\sum_{i \in I_1} \alpha_i a_i = \sum_{i \in I_2} \alpha_i a_i = \ldots = \sum_{i \in I_k} \alpha_i a_i.$$

The following lower bound comes from a simple example.

Claim 4.2.5. *Given r, d, k non-negative integers, there is a set A of $k(\lfloor \frac{d}{2} \rfloor + r + 1) - 1$ points in $\mathbb{R}^d$ such that for every partition of A into k sets $A_1, A_2, \ldots, A_k$, there is a subset $C \in \binom{A}{r}$ such that*

$$\cap_{i=1}^k \langle A_i \backslash C \rangle = \emptyset.$$

Proof. Let A be a set of $k(\lfloor \frac{d}{2} \rfloor + r + 1) - 1$ points in the moment curve $\gamma = \{(t, t^2, \ldots, t^d) \ : \ t \in \mathbb{R}\}$. It is known that every finite subset of γ is the set vertices of a $\lfloor \frac{d}{2} \rfloor$-neighbourly polytope. That is, they are a set of points such that any $\lfloor \frac{d}{2} \rfloor$ of them can be separated from the rest using a hyperplane. If A_i is the smallest section of the partition, it must have at most $\lfloor \frac{d}{2} \rfloor + r$ points. If we remove any r of them, the points left in A_i can be separated from the rest of A by a hyperplane, so the intersection of the convex hulls of each part is empty. □

If $k = 2$, this is equal to $d + 2r$ if d is odd and $d + 2r + 1$ if d is even. This improves García-Colín's bound of $2d + r + 3$ if $r \geq d + 3$. We conjecture that Theorem 4.2.1 is optimal. Namely,

Conjecture 4.2.6. *Given r, k, d non-negative integers with $d \geq 1$, there is a set A of $(r+1)(k-1)(d+1)$ points in $\mathbb{R}^d$ such that for any partition $A_1, A_2, \ldots, A_k$ of A into k parts, there is a set $C \in \binom{A}{r}$ such that*

$$\bigcap_{i=1}^{k} \langle A_i \backslash C \rangle = \emptyset.$$

Chapter 5

Colourful Tverberg Partitions

Another way to generalise Radon's theorem is via colourful partitions. In this setting additional structure is given to the original set of points, and we ask that the resulting Radon or Tverberg partition satisfies certain properties regarding this structure. Namely, we are dealing with the following problem

Conjecture 5.0.7 (Colourful Tverberg, [7]). *Given $F_1, F_2, \ldots, F_{d+1}$ sets of k points each of $\mathbb{R}^d$, we can partition their union into k sets $A_1, A_2, \ldots, A_k$ such that $|A_i \cap F_j| = 1$ for each i, j and*

$$\bigcap_{i=1}^{k} \langle A_i \rangle \neq \emptyset.$$

We refer to $F_1, F_2, \ldots, F_{d+1}$ as the colour classes. We call a family of pairwise-disjoint subsets $A_1, A_2, \ldots, A_k$ of their union satisfying $|A_i \cap F_j| = 1$ a colourful k-partition. Even in the case when the F_j have more than k points each we will denote this a k-partition, even though it is not a partition per se.

The contents of this chapter deal with variations of the conjecture above, and have been accepted in [40]. Conjecture 5.0.7 was made by Bárány and Larman [7]. The case $d = 2, k = 3$ was solved previously by Bárány, Füredi

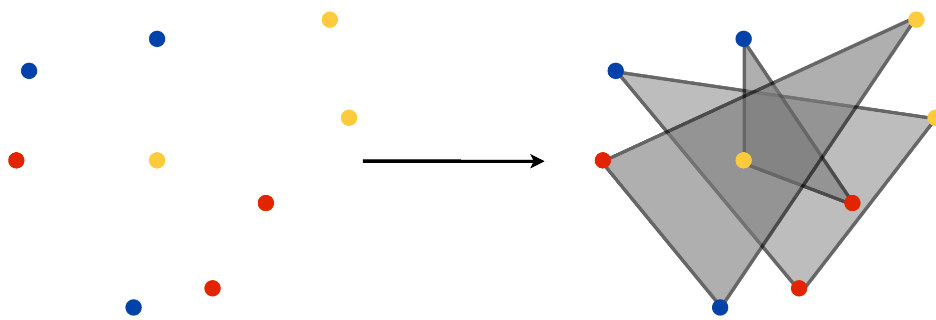

Figure 5.1: A colourful Tverberg partition for the case $d = 2$, $k = 3$.

and Lovász [6]. In [7] the authors proved the case $d = 2$ and any k and presented Lovász proof of the case $k = 2$ and any d (also known as the coloured Radon theorem). The proof of the case $d = 2$, any k is much like Tverberg's proof of his own theorem or García-Colín's proof of the Radon theorem with tolerance. Again, points are moved continuously and it is shown that the points may be swapped in the partition to avoid the condition from breaking. The colourful Tverberg theorem was proved for the case when $k+1$ is prime by Blagojević, Matschke and Ziegler with topological methods [12], [11].

Historically, when the conjecture was made, it was asked if the was a number $t = t(k, d)$ such that for any $d + 1$ families $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_{d+1}$ of t points each in $\mathbb{R}^d$, one could find a coulourful k-partition $A_1, A_2, \ldots, A_k$ such that the convex hulls of the parts intersect. The colourful Tverberg theorem is equivalent to showing that $t(k, d) = k$. The first general bound on t was given by Vrećica and Živaljević. They showed that if k was prime, then $t(k, d) \leq 2k - 1$ [47]. This implies the bound $t(k, d) \leq 4k - 3$ for all values of k. One should note that the result by Blagojević, Matschke and Ziegler mentioned above improves this bound to $t(k, d) \leq 2k - 2$ for all values of k.

With the exception of the proof for $d = 2$ by Bárány and Larman, all advances have been using topological methods. There is a proof of the Blagojević, Matschke, Ziegler result that does not use topology [30]. However, it follows the scheme of the simplified topological proofs of this result, in which only the computation of the degree of a map is needed [12, 44].

In the next sections we show two proofs of the coloured Radon theorem, and then a version of the coloured Tverberg theorem where convex combinations have equal coefficients. The proofs in both sections are non-topological.

5.1 Two proofs of colourful Radon

We present here two non-topological proof of the colourful Radon theorem. We refer to Lovász's original proof as the first proof, and to the proofs below as second and third. Lovász's proof consists of lifting of the pairs of points to $\mathbb{R}^{d+1}$ and then applying the Borsuk-Ulam theorem on an octahedron.

Second proof of colourful Radon. Denote the elements of each pair $F_i = \{x_i, y_i\}$ arbitrarily. Then the $d+1$ vectors $x_i - y_i$ are linearly dependent. This means that there is a linear combination $\sum_{i=1}^{d+1} \alpha_i(x_i - y_i) = 0$ such that not all the coefficients are 0. We may relabel the points so that no α_i is negative. Using a scalar multiplication we may also assume that they have sum 1. Thus,

$$\sum_{i=1}^{d+1} \alpha_i x_i = \sum_{i=1}^{d+1} \alpha_i y_i.$$

This convex combination gives the result. □

This proof avoids the topological arguments and gives an algorithmic way to find the colourful Radon partition in polynomial time, as it reduces the problem of finding a colourful Radon partition to finding a linear dependence. It not only shows that the partition exists, but that we may use the same coefficients in A_1 and A_2 to find a point of intersection of their convex hulls.

This fact can also be deduced from Lovász's topological proof, as the images of antipodal points in the construction used in his proof have this property. This will be exploited in the following section, where a colourful Tverberg theorem with equal coefficients is proved.

For the third proof we use the Gale transform. As described verbatim in [40],

The Gale transform of a set of n points $a_1, a_2, \ldots, a_n$ in $\mathbb{R}^d$ that are not all contained in a hyperplane is a set of n points $b_1, b_2, \ldots, b_n$ in $\mathbb{R}^{n-d-1}$ such that the following two conditions hold

- $\sum_i b_i = 0$ and
- for every two disjoint subsets $X, Y \subset [n]$, the convex hull of the sets $\{a_i : i \in X\}$ and $\{a_i : i \in Y\}$ intersect if and only if there is a hyperplane H through the origin in $\mathbb{R}^{n-d-1}$ that leaves $\{b_i : i \in X\}$ in one (closed) side, $\{b_i : i \in Y\}$ in the other (closed) side and goes through every other b_i.

Third proof of colourful Radon. Let $F_1, F_2, \ldots, F_{d+1}$ be the sets of pairs and denote by F their union. Without loss of generality we can suppose that they are in general position. If we consider J the Gale transform of F, we have that J is a set of $d+1$ pairs of points in $\mathbb{R}^{d+1}$. Denote by $J_1, J_2, \ldots, J_{d+1}$ these pairs. We need to find a hyperplane H that splits each J_i and contains the origin. Consider m_i the midpoint of each J_i. Since the points in J sum 0, the sum of all the m_i is 0 as well. Thus the hyperplane that goes through $m_1, m_2, \ldots, m_{d+1}$ contains the origin and splits each pair. $\square$

Note that the equal coefficients can also be deduced using this proof. This is because every linear function $f : \mathbb{R}^{d+1} \to \mathbb{R}$ that has H as kernel gives antipodal values to the elements of each J_i (since it goes through m_i). Due to the Gale transform properties, this translates to equal coefficients in the convex combinations that give the same point. The assumption of general position does not bring problems for this conclusion.

5.2 Colourful Tverberg with equal coefficients

Given $d+1$ families $F_1, F_2, \ldots, F_n$ of points in $\mathbb{R}^d$, each with at least k points, and $A_1, A_2, \ldots, A_k$ a colourful partition of them we can denote the elements of each A_i by $A_i = \{x_j^i \ : \ x_j^i \in F_j\}$. If we can find coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ of a convex combination such that

$$\sum_{j=1}^{n} \alpha_j x_j^1 = \sum_{j=1}^{n} \alpha_j x_j^2 = \ldots = \sum_{j=1}^{n} \alpha_j x_j^n,$$

we say that the convex hulls of $A_1, A_2, \ldots, A_k$ *intersect with equal coefficients*.

Asking for intersection with equal coefficients in conjecture 5.0.7 is too much, even if the colour classes are allowed to have more than k points. However, if the number of colour classes is increased, then such a theorem can be proved. The theorem below shows the minimum number of colour classes needed for this.

Theorem 5.2.1 (Colourful Tverberg with equal coefficients, [40])**.** *Consider $F_1, F_2, \ldots, F_n$ families of t points of $\mathbb{R}^d$ each, with $t \geq k$. If $n = (k-1)d+1$ and $t = k$, there is a colourful k-partition $A_1, A_2, \ldots, A_k$ of them such that their convex hulls intersect with equal coefficients. If $n \leq (k-1)d$, there may not be such colourful partitions, regardless of the value of t.*

It should be noted that the theorem above, even though it is a variation of the colourful Tverberg theorem (which is actually a conjecture), is actually a generalisation of Tverberg's theorem (Theorem 1.1.2). This is made clear in the next proof.

Proof that Theorem 5.2.1 implies Theorem 1.1.2. Suppose that Theorem 5.2.1 is true. Let $n = (k-1)(d+1)+1$, and $A = \{a_1, a_2, \ldots, a_n\}$ be a set of n

points in $\mathbb{R}^d$. Consider the points $b_i = (a_i, 1) \in \mathbb{R}^{d+1}$, for all i. Consider $F_i = \{b_i, 0, 0, \dots, 0\} \subset \mathbb{R}^{d+1}$. That is, the multiset of points made of b_i and $k-1$ copies of the 0 vector in $\mathbb{R}^{d+1}$. We can apply Theorem 5.2.1 to the families $F_1, F_2, \dots, F_n$ to obtain a colourful partition $B_1, B_2, \dots, B_k$ where the convex hulls of the parts intersect with equal coefficient. Note that a colourful partition of $F_1, F_2, \dots, F_n$ induces a partitions of A into k sets, $A_1, A_2, \dots, A_k$. Moreover, since the 0 vectors are unimportant, we may assign the coefficient used for a certain F_i to the respective a_i. Since the last coordinate of the b_i is 1, this implies that the sum of the coefficients assigned to $A_1, A_2, \dots, A_k$ is equal. Thus, we have obtain a Tverberg partition. □

In order to prove Theorem 5.2.1, we first need to represent the colourful partitions of the F_j as vectors of injective functions. This can be done in the following way.

Given $F_1, F_2, \dots, F_n$ families of t points each in $\mathbb{R}^d$, we can denote their elements by

$$F_j = \{z_j^1, z_j^2, \dots, z_j^t\}.$$

By $\Sigma_{k,t}$ we refer to the set of injective functions from $[k] = \{1, 2, \dots, k\}$ to $[t]$. We can assign a vector $(\sigma_1, \sigma_2, \dots, \sigma_n)$ in $(\Sigma_{k,t})^n$ to a colourful k-partition $(A_1, A_2, \dots, A_k)$ if we consider

$$\sigma_j(i) = m \text{ if and only if } x_j^i = z_j^m.$$

This is equivalent to

$$A_i = \{z_j^{\sigma_j(i)} \ : 1 \le j \le n\}$$

The vector $(\sigma_1, \sigma_2, \dots, \sigma_n)$ is called the *function representation* of the partition $(A_1, A_2, \dots, A_k)$.

One way to visualise this is the following. We first write the elements of each F_j k times in k rows. From the first row we can choose the elements of A_1, from the second row we can choose the elements of A_2, and so on. Then we can see the function representation of $(A_1, A_2, \dots, A_k)$, as in the following diagram of [40].

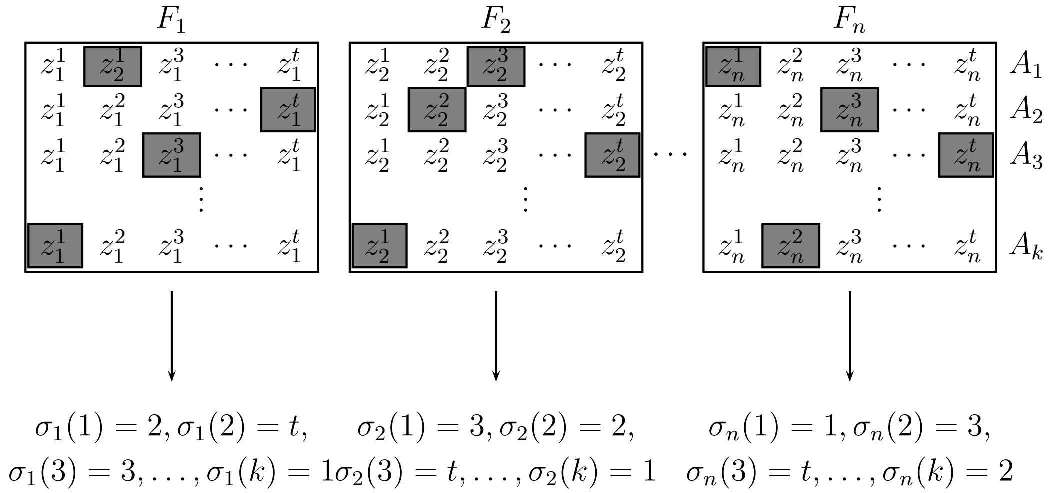

Figure 5.2: Representation of colourful partitions as vectors of injective functions.

Let $u_1, u_2, \dots, u_k$ be the vertices of a regular simplex in $\mathbb{R}^{k-1}$ centred at the origin. We use them to represent the distribution of the partition following the Sarkaria method.

Definition 5.2.2. *Given $\sigma \in \Sigma_{k,t}$ and $F_j = \{z_j^1, z_j^2, \dots, z_j^t\}$ a family of t points of $\mathbb{R}^d$, we define $F_j(\sigma) \in \mathbb{R}^{(k-1)d}$ as*

$$F_j(\sigma) = \sum_{i=1}^{k} u_i \otimes z_j^{\sigma(i)}.$$

Note that this construction gives us points in $\mathbb{R}^{d(k-1)}$, unlike the usual Sarkaria argument that gives points in $\mathbb{R}^{(d+1)(k-1)}$. It is the additional structure of the partition that allows us to work in a lower-dimensional space. The following lemma relates the previous definition to the intersection with equal coefficients

Lemma 5.2.3 (Lemma 6 in [40]). *Let $F_1, F_2, \ldots, F_n$ be sets of t points each in $\mathbb{R}^d$ and $(A_1, A_2, \ldots, A_k)$ be a colourful k-partition of them. Then for coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$ we have that $\sum_{j=1}^n \alpha_j x_j^i$ is the same point for all i if and only if $\sum_{j=1}^n \alpha_j F_j(\sigma_j) = 0$, where $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is the function representation of $(A_1, A_2, \ldots, A_k)$.*

Proof. Note that it is enough to show this for $d = 1$, as the argument can be repeated for each coordinate in $\mathbb{R}^d$. Thus $u_i \otimes z_j^m$ becomes simply $z_j^m u_i$. If we expand $\sum_{j=1}^n \alpha_j F_j(\sigma_j)$ and factor u_i, we obtain that its coefficient is $\sum_{j=1}^n \alpha_j z_j^{\sigma_j(i)}$. Since a linear combination of $u_1, u_2, \ldots, u_k$ is 0 if and only if all the coefficients are equal, we obtain the desired result. $\square$

We are now ready to prove theorem 5.2.1.

Proof of theorem 5.2.1. Suppose $n = (k-1)d+1$ and $t = k$. Each of the sets $F_j(\Sigma_{k,k})$ captures the origin. We thus have n sets, each of which captures the origin in $\mathbb{R}^{n-1}$, so by the colourful version of Carathéodory's theorem, we can find n permutations $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that the set $\{F_1(\sigma_1), F_2(\sigma_2), \ldots, F_n(\sigma_n)\}$ captures the origin. By Lemma 5.2.3 this is the permutation representation of the colourful partition we seek. Consider the case $n \leq (k-1)d$ and suppose σ_1 is given. Note that by an appropriate choice of F_1, $F_1(\sigma_1)$ can be any point of $\mathbb{R}^(k-1)d$. Suppose we are given $F_2, F_3, \ldots, F_n$ and want to know what sets F_1 would make the conclusion of the theorem true. Note that for any colourful partition $(A_1, A_2, \ldots, A_k)$ there is a permutation we can apply to each colour class such that in its function representation σ_1 becomes the identity. If we can find $\sigma_2, \sigma_3, \ldots, \sigma_n$ in $\Sigma_{k,t}$ such that $\{F_1(\sigma_1), F_2(\sigma_2), \ldots, F_n(\sigma_n)\}$ capture the origin, then $F_1(\sigma_1)$ must be in the subspace of dimension at most $n-1$ generated by $F_2(\sigma_2), F_3(\sigma_3), \ldots, F_n(\sigma_n)$. Since a finite number of subspaces of positive co-dimension cannot cover $\mathbb{R}^{d(k-1)}$, there are choices of F_1 that do not satisfy the theorem, as we wanted. $\square$

The last argument also shows that if $n \leq (k-1)d$ the points of each F_j are randomly distributed according to a (possibly different) measure where hyperplanes have measure 0, then the probability that there is a colourful k-partition $(A_1, A_2, \ldots, A_k)$ of them such that the convex hulls intersect with equal coefficients is 0.

5.3 Number of partitions and tolerance

One interesting aspect of Tverberg's theorem is that even though we know that for $(k-1)(d+1)+1$ points there are Tverberg partitions, little is known about how many of them we can find. It is conjecture that for any set of $(k-1)(d+1)+1$ points there are always at least $(k-1)!^d$ Tverberg partitions, but this is still an open problem. This conjecture is usually referred to as the Dutch cheese conjecture or Sierksma's conjecture. Lower bounds have been obtained with topological methods if k is a prime power [22], [45], which are around $\sqrt{(k-1)!^d}$. Apart from the trivial $d = 1$ or $k = 2$ cases, the only completely solved case for this conjecture is $(d, k) = (2, 3)$ [23], with a positive answer.

A deeper analysis of the proof of Theorem 5.2.1 shows that there are many such coloured partitions.

Proposition 5.3.1. *Let $n = (k-1)d + 1$ and $F_1, F_2, \ldots, F_n$ be families of k points each in $\mathbb{R}^d$. Then there are at least $(k-1)!^{d(k-1)}$ colourful k-partitions of them $(A_1, A_2, \ldots, A_k)$ such that their convex hulls intersect with equal coefficients.*

Proof. In the proof of Theorem 5.2.1, the fact that each $F_j(\Sigma_{k,k})$ captured the origin was necessary for the proof. However, we can find small subsets of $F_j(\Sigma_{k,k})$ that also capture the origin. For example, given β a cycle of length k, the set $F_j(\beta, \beta^2, \ldots, \beta^k)$ also captures the origin. If we fix the order of F_1 and assign a cyclic order to each F_j, $j > 1$ (with a cyclic order we mean an order up to iterated applications of β), we can use these sets to find a good colourful partition. However, for each way to assign cyclic orders we obtain a different partition, so there are at least $(k-1)!^{d(k-1)}$ partitions. □

We do not know if this number of partitions is optimal. In order for the result above to imply Sierksma's conjecture (using that Theorem 5.2.1 implies Tverberg's theorem), we would need to show there are at least $(k-1)!^{kd-1}$ good partitions. This is what we conjecture to be optimal. However, we have nothing other than personal intuition to support this fact.

Note that there is an action of $\Sigma_{k,k}$ in $F_j(\Sigma_{k,k})$ given by $\sigma F_j(\tau) \mapsto F_j(\sigma\tau)$. If we let β be a cycle of length k, the subgroup $\{\beta, \beta^2, \dots, \beta^k\}$ is isomorphic to $\mathbb{Z}_k$, and thus gives us an action of $\mathbb{Z}_k$ in $F_j(\Sigma_{k,k})$. This group action allows us to prove a version of Theorem 5.2.1 with tolerance. Namely,

Theorem 5.3.2. *Let $d \geq 2$, $n = (r+1)(k-1)d + 1$ and $F_1, F_2, \dots, F_n$ be families of k points each in $\mathbb{R}^d$. Then we can find a colourful k-partition $(A_1, A_2, \dots, A_k)$ of them such that for any set C of r colour classes, the convex hulls of $A_1 \backslash C, A_2 \backslash C, \dots, A_k \backslash C$ intersect with equal coefficients. Moreover, there are at least $(k-1)!^{(r+1)(k-1)d}$ such colourful k-partitions.*

Proof. Note that the sets $F_1(\Sigma_{k,k}), F_2(\Sigma_{k,k}), \dots, F_n(\Sigma_{k,k})$ with their action of $\mathbb{Z}_k$ satisfy the conditions of Lemma 4.2.2. Combining this with Lemma 5.2.3 we obtain the result. For the number of partitions, an argument analogous to the proof of proposition 5.3.1 gives the result. □

5.4 Variations of colourful Radon

We follow the idea in the third proof of colourful Radon (presented in section 5.1) in order to obtain variations of this theorem. Namely,

Theorem 5.4.1. *Given a set A of $k + d + 2$ points in $\mathbb{R}^d$ each of which has one of k possible colours, we can find two subsets A_1, A_2 such that they both have the same number of points of each colour and their convex hulls intersect.*

Note that this theorem does not imply the colourful Radon theorem. When $k = d+2$ or $k = d+1$ we can obtain results which are slightly weaker than colourful Radon. This is because in this setting the precise structure of each colour class is ignored.

Proof. We may suppose without loss of generality that A is in general position. Consider $B \subset \mathbb{R}^{k+1}$ the Gale transform of A. Consider each set of points as a measure, and a Dirac measure concentrated in the origin. Since we have $k+1$ measures in $\mathbb{R}^{k+1}$, we can use the ham sandwich theorem for finite sets of points to find a hyperplane that splits them by half simultaneously. This hyperplane goes through the origin and thus induces the partition we need. $\square$

Corollary 5.4.2. *Given $d+3$ points in $\mathbb{R}^d$, there are two disjoint subset A and B of the same size such that their convex hulls intersect.*

This corollary is already known, the earliest references we could find to it are [19] and [24]. We mention it since it would be interesting to find an analogous statement for Tverberg partitions. This has been asked earlier by Eckhoff [16]. Namely, finding the smallest $n = n(d,k)$ such that for every set of n points in $\mathbb{R}^d$ we can find k pairwise disjoint subsets of the same size whose convex hulls intersect. A direct application of Carathéodory's Theorem gives an upper bound, so we obtain $(k-1)(d+1) < n \leq k(d+1)$. A result by Sarkaria (Theorem 1.3 in [37]) implies that *if k is prime,* $n(d,k) \leq (k-1)(d+2)+1$. This improves the trivial upper bound if k is prime and $k \leq d+1$. Eckhoff conjectured $n(d,k) = (k-1)(d+1)+2$.

5.5 Families of sets with equal unions

Given a set X of n elements and a family $\mathcal{F}$ of non-empty subsets of X, if $\mathcal{F}$ is large enough, then many subfamilies of $\mathcal{F}$ have the same union. A theorem of Lindström makes this clear.

Theorem 5.5.1 (Lindström, 1972 [28])**.** *Let X be a set of n element and $\mathcal{F}$ a family of non-empty subsets of X. If $|\mathcal{F}| > n(k-1)$, then we can find k*

disjoint subsets $I_1, I_2, \ldots, I_k$ *of* $\mathcal{F}$ *such that*

$$\bigcup I_1 = \bigcup I_2 = \ldots = \bigcup I_k.$$

This result was also proved by Tverberg using his theorem on intersection of convex hulls [43]. Here we prove variations of this theorem using the Tverberg-type results contained in the last two chapters. First, let us show a version of Lindström's theorem with tolerance.

Theorem 5.5.2 (Lindström with tolerance)**.** *Let* X *be a set of* $n \geq 2$ *elements and* $\mathcal{F}$ *a family of non-empty subsets of* X*. If* $|\mathcal{F}| > n(k-1)(r+1)$*, then we can find* k *disjoint subfamilies* $I_1, I_2, \ldots, I_k$ *of* $\mathcal{F}$ *such that for any* $C \in \binom{\mathcal{F}}{r}$ *there are families* $U_i \subset I_i \backslash C$ *for* $i = 1, 2, \ldots, k$ *that satisfy*

$$\bigcup U_1 = \bigcup U_2 = \ldots = \bigcup U_k.$$

Proof. Assign to each $F \in \mathcal{F}$ its incidence vector $v_F \in \mathbb{R}^n$. Denote by $v(\mathcal{F})$ the image of $\mathcal{F}$ under this function. We can normalise the vector v_F to obtain $u_F = \frac{1}{|F|} v_F$ (note that the non-emptiness of F is essential for this step). The family $u(\mathcal{F})$ is a family of at least $n(k-1)(r+1)+1$ points in a flat H of dimension $n-1$. Then, we can apply Theorem 4.2.1 to obtain a partition $I_1, I_2, \ldots, I_k$ of $\mathcal{F}$. Given $C \in \binom{\mathcal{F}}{r}$, we know that there is a $v \in H$ that is contained in the convex hull of each $u(I_i \backslash C)$. Note that $v \neq 0$, as H does not contain the origin. For each i we know that there is a linear combination of the elements of $u(I_i \backslash C)$ using only non-negative coefficients that gives v.

Let U_i be the subfamily of those sets in $I_i \backslash C$ whose incidence vector used positive coefficients. Clearly, the union of the sets in U_i is the set of non-zero entries of v. □

Note that reducing the dimension to H may not be necessary, as Corollary 4.2.5 gives the result directly. However, this reduction is necessary when we apply the colourful versions of Tverberg's theorem instead of Theorem 4.2.1.

Since the proof of the following results is analogous to the one above, we omit them and only mention which version of Tverberg's theorem gives the result.

Using Theorem 5.2.1, we obtain the following,

Proposition 5.5.3 (Colourful Lindström). *Let X be a set of of n elements, $m > (k-1)(n-1)$ and $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ be families of k subsets of X each. Then, we can find a non-empty subset I of $[m]$ such that there is a colourful partition of the families $\mathcal{F}_i$ with $i \in I$ into families $I_1, I_2, \dots, I_k$ such that*

$$\bigcup I_1 = \bigcup I_2 = \ldots = \bigcup I_k.$$

Note that if we used the classic colourful Tverberg we would not obtain this. The condition of equal coefficients allows us to say that if a family I_i has a set of some $\mathcal{F}_j$, then all the families $I_1, I_2, \dots, I_k$ have each a set of $\mathcal{F}_j$. Of course, this can be generalised to a theorem with tolerance using Theorem 5.3.2.

Theorem 5.5.4 (Colourful Lindström with tolerance). *Let X be a set of $n \geq 2$ elements, $m > (r+1)(k-1)(n-1)$ and $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_m$ be families of k subsets of X each. Then, there is a colourful partition of the families $\mathcal{F}_i$ with into families $I_1, I_2, \dots, I_k$ such that, for all $C \in \binom{[m]}{r}$, there is a $U \subset [m] \backslash C$ such that the families $U_i = I_i \cap (\cup_{j \in U} \mathcal{F}_j)$ satisfy*

$$\bigcup U_1 = \bigcup U_2 = \ldots = \bigcup U_k.$$

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