Cylindric Algebra

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Chapter 1

Completions and Complete Representations

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1.1 Introduction

The title of this chapter indicates a rather technical topic, but it can also be thought of as a foundational issue in logic. The question to be considered is this: to what extent can we use an abstract mathematical language to express and reason about relations? Going back at least as far as Augustus de Morgan [dM60], a relation can be *defined* explicitly, as a set of tuples of some fixed length. This allows us to focus on the mathematical aspects of relations and ignore other more problematic features that might arise from other approaches, such as a linguistic analysis of the use of relations in natural language. In order to treat relations algebraically, we consider them abstractly, identify certain relational operations (e.g., the operation of taking the *converse* of a binary relation) and write down some equational axioms which are sound for the chosen kind of relations (e.g., a binary relation is equal to the converse of its converse). Ideally, our set Γ of equations will be *equationally complete,* so that any equation valid over fields of relations of a certain rank equipped with the chosen set-theoretically definable operators will be entailed by Γ.

The finite set of equational axioms for boolean algebra is very successful in this respect, for handling *unary relations.* The chosen operators are union and complementation together with constants 0 denoting the empty set and 1 for the unit of the boolean algebra. Other operators, like intersection, can be defined within this signature. The finite set of axioms defining a boolean algebra is complete and every boolean algebra is isomorphic to a genuine field of sets.

Similarly, for binary relations, we treat the boolean operators plus some additional operators and try to write down some equational axioms that are equationally complete for binary relations. Different choices are possible for a set of operators for binary relations — for relation algebras we use the boolean operators together with the unary operator of taking the converse, the binary operator of composition and a constant for the identity. Axiomatising binary relations with the relation algebra operators turns out to be more difficult than was the case for unary relations, and we know that any complete set of axioms is necessarily infinite [Mon64], but recursively enumerable, complete, equational axiomatisations are known [Lyn56, HH02a]. For relations of higher finite rank, different choices of algebras can be considered — cylindric algebra, polyadic algebra, diagonal-free algebra — but for ranks at least three, the situation is largely similar to the relation algebra case. All of the operators of these algebras are additive in each argument, and normal, meaning that their value is 0 whenever any argument is 0. All algebras mentioned above are *boolean algebras with operators* (BAOs).

Let $\mathcal{F}$ be one of the following: (i) the class of fields of sets equipped with the boolean operators, (ii) the class of fields of binary relations with the relation algebra operators, (iii) the class of fields of n-ary relations (for some n) with the cylindric algebra operators, (iv) the class of fields of n-ary relations with the polyadic operators, (v) the class of fields of n-ary relations with the diagonal free operators. Let Γ be a set of equations of the appropriate signature equationally complete over $\mathcal{F}$. Since the closure of $\mathcal{F}$ under isomorphism is known to be a variety, every model of Γ is isomorphic to a member of $\mathcal{F}$. A *representation* is an isomorphism from an algebra to a field of relations, and its *base* is the underlying set of objects that the relations relate.

But the correspondence between algebras and fields of relations may not be quite as close as we had hoped. By completeness, any equation valid in $\mathcal{F}$ is entailed by Γ, and since $\mathcal{F}$ is a variety, Γ entails all first-order sentences valid over $\mathcal{F}$. But there might be other true properties of $\mathcal{F}$, not expressible by equations or even first-order sentences, that do not follow from Γ. At least some second-order properties do follow from Γ in these cases. Since each k-ary operator f of each algebra in $\mathcal{F}$ is *conjugated,* it follows that f is *completely additive* in each argument [JT51], meaning that if an arbitrary non-empty set S of elements of some $\mathcal{A} \in \mathcal{F}$ has a supremum $\sup(S)$, and if $i < k$ and $a_0, \ldots, a_{i-1}, a_{i+1}, \ldots a_{k-1} \in \mathcal{A}$, then $b = \sup\{f(a_0, \ldots, a_{i-1}, s, a_{i+1}, \ldots, a_{k-1}) : s \in S\}$ exists and

$$f(a_0, a_1, \ldots, a_{i-1}, \sup(S), a_{i+1}, \ldots, a_{k-1}) = b.$$

But there are other second-order properties of $\mathcal{F}$ that might not be properly captured in our algebraic framework. The first problem is that a model $\mathcal{A}$ of Γ might be *incomplete* — there could be a set S of elements of $\mathcal{A}$ that has no supremum in $\mathcal{A}$. With a field of concrete relations, we can always extend the field to include a supremum of any set of relations, simply by taking the set-theoretic union of each set of relations, and generating a field of relations. A construction of Monk [Mon70] gives us, for any completely additive BAO $\mathcal{A}$, a complete extension $\mathsf{Com}(\mathcal{A})$ in which $\mathcal{A}$ is dense, and which respects all existing suprema in $\mathcal{A}$. Such an extension is unique up to isomorphism, and is called the *completion* of $\mathcal{A}$. However, a potential problem is that $\mathcal{A}$ could be a model of Γ, so $\mathcal{A}$ is isomorphic to a field of relations, but $\mathsf{Com}(\mathcal{A})$ could fail some of the axioms in Γ and have no representation. For binary and higher order relations, this problem is real, as we will see.

The second problem is that even if h is a representation of $\mathcal{A}$, so that h is an isomorphism from the algebra to a field of relations, there are certain operators definable in second order logic that might not be preserved by h. We say that h is a *complete representation* of $\mathcal{A}$ if

$$h(\sup(S)) = \bigcup_{s \in S} h(s)$$

for any subset S of $\mathcal{A}$ where the supremum $\sup(S)$ exists in $\mathcal{A}$. By the De Morgan Laws, a complete representation also preserves arbitrary infima wherever they are defined. Every representation of a finite algebra is of course complete. A saturation argument shows that all infinite algebras, even boolean ones, have incomplete representations. So the main question is when an algebra has some complete representation. Complete representability is connected to the omitting types theorem for the corresponding logic: see chapter **??**• for more on this. We will devise an infinite game to characterise when an algebra has a complete representation, and we will use this game to analyse the class of completely representable algebras.

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1.2 Boolean Algebra

We start with the easiest case: algebras of unary relations. We can define an ordering in a boolean algebra by $x \leq y \iff x + y = y$. An *atom* of a boolean algebra is a $\leq$-minimal non-zero element and the algebra is *atomic* if every non-zero element of the algebra is above some atom. For a boolean algebra $\mathcal{B}$ write $At(\mathcal{B})$ for the set of all atoms of $\mathcal{B}$. All non-trivial finite boolean algebras are atomic but there are boolean algebras with no atoms at all. (For example, let X be an infinite set and define the equivalence relation over the subsets of X by $S \sim T$ iff the symmetric difference of S and T is finite. The boolean operators have a well-defined action on the equivalence classes yielding a boolean algebra with no atoms.)

A representation h of a boolean algebra $\mathcal{B}$ is called *atomic* if for all $x \in h(1)$ there is an atom $b \in At(\mathcal{B})$ with $x \in h(b)$.

THEOREM 1.2.1 [HH97] *Let h be a representation of the boolean algebra $\mathcal{B}$. The following are equivalent.*

1. *h is a complete representation.*
2. *h is an atomic representation.*

Proof. If h is an atomic representation then for all $b \in \mathcal{B}$, $h(b) = \bigcup\{h(a) : a \in At(\mathcal{B}),\ a \leq b\}$. Let S be a set of elements of $\mathcal{B}$ with a supremum $\sup(S) \in \mathcal{B}$. An atom a is below $\sup(S)$ iff there is $s \in S$ with $a \leq s$. So

$$\begin{aligned} x \in h(\sup(S)) &\iff x \in \bigcup\{h(a) : a \in At(\mathcal{B}), \exists s \in S, a \leq s\} \\ &\iff \exists s \in S\ \exists a \in At(\mathcal{B}),\ (a \leq s, x \in h(a)) \\ &\iff \exists s \in S\ (x \in h(s)) \\ &\iff x \in \bigcup\{h(s) : s \in S\} \end{aligned}$$

so h is a complete representation.

Conversely, suppose h is complete. Let $x \in h(1)$. The set $\gamma = \{b \in \mathcal{B} : x \in h(b)\}$ is an ultrafilter of $\mathcal{B}$. 0 is a lower bound of γ. 0 cannot be the greatest lower bound of γ, else

$$x \in \bigcap\{h(b) : b \in \gamma\} \setminus h(\inf(\gamma))$$

contradicting the assumed completeness of h. So there must be a non-zero lower bound of γ, say a. Since $a \not\leq -a$, $-a \notin \gamma$, so $a \in \gamma$. Since if $b + c \in \gamma$ then $b \in \gamma$ or $c \in \gamma$, it follows that a is an atom. Hence $x \in h(a)$ for some atom a, and h is an atomic representation. □

COROLLARY 1.2.2 [HH97] *The class of completely representable boolean algebras is the same as the class of atomic boolean algebras.*

Other potential problems that we mentioned earlier do not arise for boolean algebras. Since every boolean algebra is representable it follows trivially that the completion of a boolean algebra is always representable.

1.3 Completely Representable Relation Algebras

The main focus of this article is about completions and complete representations of n-dimensional cylindric algebras for finite $n \geq 3$. Historically the main results were all established first for relation algebra, and we outline these results here, without including any proofs. In the following sections we will go through the corresponding material for cylindric algebras in more detail.

In 1950 Roger Lyndon published a set **LC** of axioms (now called the *Lyndon conditions*) and proved that a finite relation algebra satisfies the conditions iff it is representable [Lyn50]. His proof can be extended to work for arbitrary relation algebras. He also claimed to prove that his conditions were valid over complete, representable atomic relation algebras, but in fact his proof only works for finite relation algebras. His main result was to construct a finite relation algebra which failed some of his conditions and was therefore not representable. This showed that Tarski's set of equations for relation algebra [CT51] was not complete. He then defined two infinite atomic relation algebras $\mathcal{M}, \mathcal{M}'$, and showed that (i) $\mathcal{M}$ was representable, (ii) $\mathcal{M}'$ failed one of the first Lyndon conditions, and (iii) every finitely generated subalgebra of either relation algebra was isomorphic to a finitely generated subalgebra of the other. He concluded from (ii) that $\mathcal{M}'$ was not representable and from (iii) that there could be no equational axiomatisation of the class of representable relation algebras (**RRA**). But in 1955 Tarski proved that **RRA** was closed under homomorphic image, subalgebra and direct product and was therefore an equational variety [Tar55]. The situation appeared contradictory.

In fact, by Tarski's result, both algebras were representable, but the fact that $\mathcal{M}'$ failed a Lyndon condition did not prove it to be unrepresentable, but only that it had no complete representation. The mistake in Lyndon's paper turned out to be a very fruitful one, mainly because it led him to publish a second paper with the first correct axiomatisation of **RRA** [Lyn56], but also because it led to a thorough investigation of the relationship between representability, the Lyndon conditions, complete representability etc. It was shown in [Hir95] that the class of completely representable relation algebras is non-elementary. In [HH97] this was extended to $\mathbf{RCA}_n$ for all $n \geq 3$. [Hod97] showed that **RRA** (and $\mathbf{RCA}_n$ for finite $n \geq 3$) is not closed under completions.

1.4 Complete representations of Cylindric Algebras and Games

We now consider complete representations of n-dimensional cylindric algebras, for $3 \leq n < \omega$. It is clear that determining whether a cylindric algebra has a complete representation or not can be tricky. (Indeed we will see that the class of completely representable cylindric algebras of dimension n is not even elementary.) We saw in theorem 1.2.1 that a representation of a boolean algebra is complete if and only if it is atomic. This theorem generalises to algebras of higher order relations, since their representations are, *inter alia,* boolean representations. It follows that only atomic algebras can have complete representations, although Lyndon's relation algebra $\mathcal{M}'$ shows that not every representable atomic algebra need have a complete representation, and similarly (as it turns out), not every atomic representable cylindric algebra need have a complete representation.

We will introduce a two-player game that tests complete representability of an atomic cylindric algebra, but we have some preliminaries concerning networks first.

The dimension n (where $3 \leq n < \omega$) remains fixed until section 1.9. $\mathbf{RCA}_n$ denotes the class of representable n-dimensional cylindric algebras and $\mathbf{CCA}_n$ denotes the class of completely representable n-dimensional cylindric algebras. In the following, we often suppress references to n, so it is implicit that all cylindric algebras are n-dimensional. To avoid unnecessary checking, it will often be convenient to consider a slightly wider class of algebras: by a *cylindric-type algebra* we will mean a completely additive BAO of the signature of n-dimensional cylindric algebras. Note that every n-dimensional cylindric algebra is such an algebra (because it is conjugated: see §1.1), and every representable cylindric-type algebra is a cylindric algebra. A cylindric-type algebra $\mathcal{A}$ is said to be *atomic* if its boolean reduct is atomic, and in that case we let $At(\mathcal{A})$ denote the set of atoms of its boolean reduct.

We consider functions from n to A, where A is a set. (The set of all functions from a set X to a set Y is as usual denoted by ${}^XY$.) We identify the function $x \in {}^nA$ with the sequence $(x(0), x(1), \ldots, x(n-1))$, and we sometimes write x as $\bar{x} = (x_0, \ldots, x_{n-1})$. Given $x, y \in {}^nA$ and $i < n$, we write $x \equiv_i y$ if for all $j < n$, if $j \neq i$ then $x(j) = y(j)$. For $i < n$ and $a \in A$, we write $x[i/a]$ for the function that is identical to x except $x[i/a]$ maps i to a.

Definitions 1.4.1 and 1.4.2 below appeared first as [HH97, Definition 27].

DEFINITION 1.4.1 (Network) Let $\mathcal{A}$ be an atomic cylindric-type algebra. An *$\mathcal{A}$-pre-network* $N = (N_1, N_2)$ consists of a set of nodes N_1 and a 'labelling' function $N_2 : {}^nN_1 \to At(\mathcal{A})$. N is said to be a *network* if it satisfies, for all $x, y \in {}^nN_1$ and $i, j < n$,

- $N_2(x) \leq \mathsf{d}_{ij} \iff x(i) = x(j)$,
- if $x \equiv_i y$ then $N_2(x) \leq \mathsf{c}_i N_2(y)$.

Write $(M_1, M_2) \subseteq (N_1, N_2)$ if $(M_1, M_2), (N_1, N_2)$ are networks, $M_1 \subseteq N_1$ and $M_2 = N_2\restriction_{M_1}$. For a limit ordinal λ and a sequence of networks $(N_1^0, N_2^0) \subseteq (N_1^1, N_2^1) \subseteq \cdots \subseteq (N_1^\mu, N_2^\mu) \subseteq \cdots$ $(\mu < \lambda)$, define the *limit* of the sequence to be the network $(N_1, N_2) = \bigcup_{\mu<\lambda}(N_1^\mu, N_2^\mu)$ with nodes $N_1 = \bigcup_{\mu<\lambda} N_1^\mu$ and labelling $N_2 = \bigcup_{\mu<\lambda} N_2^\mu$: that is, $N_2(m, n) = N_2^\mu(m, n)$ for any $\mu < \lambda$ such that $m, n \in N_1^\mu$.

The elements of ${}^nN_1$ are called n-dimensional hyperedges (or simply hyperedges) of the network. We will frequently drop the suffices and let N denote the network (N_1, N_2), the set of nodes N_1 and the labelling function N_2, distinguishing cases by context.

A complete representation of an atomic cylindric-type algebra $\mathcal{A}$ can be identified with a set $\{N_a : a \in At(\mathcal{A})\}$ of $\mathcal{A}$-networks such that

$$\begin{array}{l}\text{for each } a \in At(\mathcal{A}) \text{ there is } x \in {}^nN_a \text{ with } N_a(x) = a, \text{ and} \\ \text{whenever } x \in {}^nN_a,\ b \in At(\mathcal{A}),\ i < n, \text{ and } N_a(x) \leq \mathsf{c}_i b, \\ \text{there is } y \in {}^nN_a \text{ with } x \equiv_i y \text{ and } N_a(y) = b.\end{array} \tag{1.1}$$

By dint of theorem 1.2.1, such a set of networks can easily be constructed from a complete representation. Conversely, by renaming the nodes of the networks, we can arrange that the nodes of N_a and N_b are disjoint, when a and b are distinct atoms. An atomic (hence complete) representation h of $\mathcal{A}$ whose base is the union of the sets of nodes of the N_a, for $a \in At(\mathcal{A})$, is defined by

$$h(b) = \{x : \exists a \in At(\mathcal{A}), x \in {}^nN_a,\ N_a(x) \leq b\},$$

for each element b of $\mathcal{A}$.

DEFINITION 1.4.2 (Atomic Game) Let $\mathcal{A}$ be an atomic cylindric-type algebra and let $\kappa > 0$ be a cardinal. The two player game $G^\kappa(\mathcal{A})$ is defined as follows. A *play* of the game is a sequence $N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t \subseteq \cdots$ of $\mathcal{A}$-networks ($t < \kappa$). In round 0, $\forall$ picks an atom $a \in \mathcal{A}$ and $\exists$ plays a network N_0. If there is no $x \in {}^n(N_0)$ such that $N_0(x) = a$ then $\forall$ wins the play.

For a limit ordinal $\lambda < \kappa$ let $N_\lambda = \bigcup_{t<\lambda} N_t$. $\forall$ does not win in the round of a limit ordinal.

For successor ordinals, suppose the play has proceeded $N_0 \subseteq \cdots \subseteq N_t$ for some t with $t + 1 < \kappa$. In the $(t + 1)$th round, $\forall$ picks $i < n$, $x \in {}^nN_t$, and an atom $a \in At(\mathcal{A})$ such that $N_t(x) \leq \mathsf{c}_i a$. Such a move by $\forall$ is denoted (i, x, a). $\exists$ responds with a network $N_{t+1} \supseteq N_t$. If there is no node $l \in N_{t+1}$ such that $N_{t+1}(x[i/l]) = a$ then $\forall$ wins.

The *limit* of the play is defined to be $\bigcup_{t<\kappa} N_t$. If $\forall$ does not win in any round then $\exists$ wins the play.

The next theorem generalises [HH97, Theorem 28] to the uncountable case.

THEOREM 1.4.3 *Let $\mathcal{A}$ be an atomic cylindric-type algebra with κ atoms. The following are equivalent.*

- *$\mathcal{A}$ is completely representable.*
- *$\exists$ has a winning strategy in $G^{\kappa+\omega}(\mathcal{A})$.*

Proof. If $\mathcal{A}$ has a complete representation then by theorem 1.2.1 it has an atomic representation and $\exists$'s winning strategy is to maintain an embedding of the current network in a play of the game into the base of the atomic representation. Conversely, if $\exists$ has a winning strategy in $G^{\kappa+\omega}(\mathcal{A})$, then for each $a \in At(\mathcal{A})$ consider a play of the game in which $\exists$ plays networks with fewer than $\kappa + \omega$ nodes, and $\forall$ picks the atom a initially and picks all possible $i < n$, all hyperedges and all legitimate atoms eventually. Let the limit of the play be N_a. Then $\{N_a : a \in At(\mathcal{A})\}$ satisfies (1.1). $\square$

For finite m, we can define a first-order sentence ρ_m such that for any atomic cylindric-type algebra $\mathcal{A}$, $\exists$ has a winning strategy in $G^m(\mathcal{A})$ iff $\mathcal{A} \models \rho_m$. These formulas ρ_m correspond, roughly, to the Lyndon conditions that we mentioned in the section on Relation Algebra. By König's lemma, a *finite* n-dimensional cylindric algebra $\mathcal{A}$ satisfies $\{\rho_m : m < \omega\}$ iff $\exists$ has a winning strategy in $G^\omega(\mathcal{A})$, iff $\mathcal{A}$ is representable. This can fail for infinite algebras, but still there is a generalisation to arbitrary algebras (corollary 1.4.5 below).

THEOREM 1.4.4 *If $\mathcal{A}$ is an atomic cylindric-type algebra and $\exists$ has a winning strategy in $G^m(\mathcal{A})$ (all $m < \omega$), then $\exists$ has a winning strategy in the game $G^\omega(\prod_U \mathcal{A})$ on the ultrapower $\prod_U \mathcal{A}$, for any non-principal ultrafilter U over ω.*

Proof. (See [HH97, Theorem 28(2)] for details.) Let X be a finite set and suppose N^i is an $\mathcal{A}$-pre-network with nodes X, for all $i < \omega$. The *ultraproduct* of $(N^i : i < \omega)$ is defined to be the $\prod_U \mathcal{A}$-pre-network N with nodes X and labelling defined by $N(\bar{x}) = [(N^i(\bar{x}) : i < \omega)] \in \prod_U \mathcal{A}$. Łoś's theorem can be used to prove that this is a network iff $\{i < \omega : N^i \text{ is a network}\} \in U$.

Consider a play $N_0 \subseteq N_1 \subseteq \ldots \subseteq N_m \subseteq \ldots$ of $G^\omega(\prod_U \mathcal{A})$. For each $m < \omega$, $\exists$ maintains a sequence of $\mathcal{A}$-pre-networks $(N_m^j : j < \omega)$, each with the same nodes as N_m, such that N_m is the ultraproduct of the N_m^j's. Inductively, she also arranges that there is a set $X_m \in U$ such that for all $j \in X_m$, $j \geq m$ and the sequence $N_0^j \subseteq N_1^j \subseteq \ldots \subseteq N_m^j$ is the initial segment of a play of $G^j(\mathcal{A})$ in which $\exists$ uses her winning strategy. Let $X_0 = \omega$. In round m, suppose $\forall$ plays $i < n$, $\bar{x}$ and an atom $[(a_j : j < \omega)]$ of $\prod_U \mathcal{A}$. By Łoś's theorem, $L = \{j < \omega : j > m, (i, \bar{x}, a_j) \text{ is a legal } \forall\text{-move}\} \in U$, so $X_m \cap L \in U$. Fix a new node x_m. For each $j \in X_m \cap L$, $\exists$ uses her winning strategy to determine a network $M_{m+1}^j \supseteq N_m^j$. We can assume that M_{m+1}^j extends N_m^j by at most the single node x_m. If $\{j \in X_m \cap L : M_{m+1}^j \text{ has same nodes as } N_m^j\} \in U$ then $\exists$ plays $N_{m+1} = N_m$ in the main game. Otherwise, $Y = \{j \in X_m \cap L : M_{m+1}^j \text{ extends } N_m^j \text{ by a single node } x_m\} \in U$, and she lets N_{m+1} extend N_m by the single new node x_m. For all $j \in Y$ she lets $N_{m+1}^j = M_{m+1}^j$ and for $j \notin Y$ she lets N_{m+1}^j be an arbitrary pre-network with the same nodes as N_{m+1}. By Łoś's theorem again, this maintains the induction hypothesis and defines a valid move for $\exists$ in round m. Since she can do this in all rounds she will win the play. $\square$

COROLLARY 1.4.5 *Let $\mathcal{A}$ be an atomic cylindric-type algebra. Then $\exists$ has a winning strategy in $G^m(\mathcal{A})$ for all finite m, iff $\mathcal{A}$ is elementarily equivalent to a completely representable cylindric algebra.*

Proof. If $\mathcal{B}$ is a completely representable cylindric algebra, then by theorem 1.4.3, $\exists$ has a winning strategy in $G^m(\mathcal{B})$ for all finite m, and hence $\mathcal{B} \models \{\rho_m : m < \omega\}$. If $\mathcal{A}$ is elementarily equivalent to $\mathcal{B}$ then $\mathcal{A} \models \{\rho_m : m < \omega\}$ as well, so $\exists$ has a winning strategy in $G^m(\mathcal{A})$ for all finite m.

Conversely, if $\exists$ has a winning strategy in $G^m(\mathcal{A})$ for all finite m, then by theorem 1.4.4, $\exists$ has a winning strategy in $G^\omega(\prod_U \mathcal{A})$ where $\prod_U \mathcal{A}$ is a non-principal ultrapower of $\mathcal{A}$. It can be checked using elementary chains that $\prod_U \mathcal{A}$ has a countable elementary subalgebra $\mathcal{B}$ where $\exists$ still has a winning strategy in $G^\omega(\mathcal{B})$. By theorem 1.4.3, $\mathcal{B}$ is completely representable, and plainly, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent. $\square$

1.5 Atom structures

Duality has been important in the theory of BAOs since [JT51], and much earlier for boolean and other algebras. In the rest of the chapter we will consider representations from the dual perspective of atom structures.

The action of the non-boolean operators in a completely additive atomic BAO is determined by their behaviour over the atoms, and this in turn is encoded by the *atom structure* of the algebra.

DEFINITION 1.5.1 (Atom Structure) *Let $\mathcal{A} = (A, 0, 1, +, -, \Omega_i : i \in I)$ be an atomic boolean algebra with operators $\Omega_i : i \in I$. Let the rank of Ω_i be $\rho(i)$. The atom structure* $\mathsf{At}(\mathcal{A})$ *of $\mathcal{A}$ is a relational structure*

$$(At(\mathcal{A}), R_{\Omega_i} : i \in I)$$

where $At(\mathcal{A})$ is the set of atoms of $\mathcal{A}$ as before, and R_{Ω_i} is a $(\rho(i)+1)$-ary relation over $At(\mathcal{A})$ defined by

$$R_{\Omega_i}(a_0, \ldots, a_{\rho(i)}) \iff \Omega_i(a_1, \ldots, a_{\rho(i)}) \geq a_0.$$

Similar 'dual' structures arise in other ways, too. For any not necessarily atomic BAO $\mathcal{A}$ as above, its *ultrafilter frame* is the structure

$$\mathcal{A}_+ = (\mathrm{Uf}(\mathcal{A}), R_{\Omega_i} : i \in I),$$

where $\mathrm{Uf}(\mathcal{A})$ is the set of all ultrafilters of (the boolean reduct of) $\mathcal{A}$, and for $\mu_0, \ldots, \mu_{\rho(i)} \in \mathrm{Uf}(\mathcal{A})$, we put $R_{\Omega_i}(\mu_0, \ldots, \mu_{\rho(i)})$ iff $\{\Omega_i(a_1, \ldots, a_{\rho(i)}) : a_j \in \mu_j \text{ for } 0 < j \leq \rho(i)\} \subseteq \mu_0$.

DEFINITION 1.5.2 (Complex algebra) *Conversely, if we are given an arbitrary structure $\mathcal{S} = (S, r_i : i \in I)$ where r_i is a $(\rho(i)+1)$-ary relation over S, we can define its* complex algebra

$$\mathfrak{Cm}(\mathcal{S}) = (\wp(S), \emptyset, S, \cup, \setminus, \Omega_i : i \in I),$$

where $\wp(S)$ *is the power set of* S*, and* Ω_i *is the* $\rho(i)$*-ary operator defined by*

$$\Omega_i(X_1,\ldots,X_{\rho(i)}) \\ = \{s \in S : \exists s_1 \in X_1 \ldots \exists s_{\rho(i)} \in X_{\rho(i)},\ r_i(s, s_1, \ldots, s_{\rho(i)})\},$$

for each $X_1, \ldots, X_{\rho(i)} \in \wp(S)$*.*

It is easy to check that, up to isomorphism, $\mathsf{At}(\mathfrak{Cm}(\mathcal{S})) \cong \mathcal{S}$ always (we identify the two), and $\mathcal{A} \subseteq \mathfrak{Cm}(\mathsf{At}(\mathcal{A})) \cong \mathsf{Com}(\mathcal{A})$ for any completely additive atomic BAO $\mathcal{A}$. If $\mathcal{A}$ is finite then of course $\mathcal{A} \cong \mathfrak{Cm}(\mathsf{At}(\mathcal{A}))$.

Atom structures of cylindric-type algebras have the form $(S, R_{\mathsf{c}_i}, R_{\mathsf{d}_{ij}} : i, j < n)$, where the R_{c_i} and $R_{\mathsf{d}_{ij}}$ are binary and unary relations on S, respectively. We call such objects *cylindric-type atom structures.* One can construct from the standard axiomatisation of cylindric algebras [HMT71, definition 1.1.1] a *Sahlqvist correspondent:* a first-order sentence true in all atom structures of atomic cylindric algebras, and such that the complex algebra of any atom structure in which it is true is a cylindric algebra. We call any model of this sentence a *cylindric algebra atom structure.*

It turns out that if $\mathcal{A}$ is any cylindric algebra, $\mathcal{A}_+$ is a cylindric algebra atom structure. Its complex algebra $\mathfrak{Cm}(\mathcal{A}_+)$ is often written $\mathcal{A}^\sigma$, and is called the *canonical extension* of $\mathcal{A}$ [JT51]. $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{A}^\sigma$ and the isomorphism is $a \mapsto \{f \in \mathcal{A}_+ : a \in f\}$. This $\mathcal{A}^\sigma$ is a different kind of complete extension of $\mathcal{A}$ to the Monk completion $\mathsf{Com}(\mathcal{A})$ that we mentioned in the introduction. Whereas suprema and infima are preserved from $\mathcal{A}$ to $\mathsf{Com}(\mathcal{A})$, this is not the case for $\mathcal{A}^\sigma$ if $\mathcal{A}$ is infinite. On the other hand, $\mathcal{A}^\sigma$ is always complete and atomic, while $\mathsf{Com}(\mathcal{A})$ will be atomic iff $\mathcal{A}$ is. Monk proved that $\mathbf{RCA}_n$ is *canonical* (closed under taking canonical extensions): see [HMT71, Theorem 2.7.23]. In fact,

THEOREM 1.5.3 *An* n*-dimensional cylindric algebra* $\mathcal{A}$ *is representable iff* $\mathcal{A}^\sigma$ *has a* complete *representation.*

(See [HH02a, theorem 3.36] for the analogous result for relation algebras.)

1.6 Representability and atom structures

Given an atomic cylindric-type algebra $\mathcal{A}$, the games $G^\kappa(\mathcal{A})$ are effectively played on the atom structure $\mathsf{At}(\mathcal{A})$, so by theorem 1.4.3, *whether* $\mathcal{A}$ *has a complete representation or not depends only on its atom structure.* It follows that if a cylindric algebra $\mathcal{A}$ has a complete representation then any cylindric algebra with the same atom structure as $\mathcal{A}$ is completely representable, and in particular the completion $\mathsf{Com}(\mathcal{A})$ of $\mathcal{A}$ is also completely representable. At any rate, the atom structures of completely representable cylindric algebras form an important class, which we would like to characterise, perhaps by first-order sentences.

But whether the plain representability of $\mathcal{A}$ is determined by $\mathsf{At}(\mathcal{A})$ is not so clear. On the one hand, $\mathcal{A}$ is determined by its boolean structure and by $\mathsf{At}(\mathcal{A})$, and since boolean algebras are easy to represent, one might surmise that impediments to representing $\mathcal{A}$ reside in its atom structure. On the other hand, the boolean and atom structure of $\mathcal{A}$ may interact, perhaps allowing two atomic cylindric algebras with the same atom structure, one being representable, the other not. This happens iff there is a representable atomic cylindric algebra whose completion is not representable. It would lead to two different kinds of 'representability' for a cylindric algebra atom structure, depending on whether *some* or *all* atomic cylindric algebras with that atom structure are representable. This turns out to be the case: it is possible to construct a weakly but not strongly representable cylindric algebra atom structure [Hod97], as we will see below.

In this section, we examine these issues (see [HH02b] for the corresponding definitions and results for relation algebra).

DEFINITION 1.6.1 *Let $\mathcal{S}$ be an n-dimensional cylindric algebra atom structure.*

1. *$\mathcal{S}$ is completely representable if some (equivalently, every) atomic n-dimensional cylindric algebra $\mathcal{A}$ with $\mathsf{At}(\mathcal{A}) = \mathcal{S}$ has a complete representation. $\mathbf{CRAS}_n$ denotes the class of completely representable (n-dimensional) cylindric algebra atom structures.*

2. *$\mathcal{S} \in \mathbf{LCAS}_n$ if $\exists$ has a winning strategy in $G^m(\mathfrak{Cm}\,\mathcal{S})$ for all $m < \omega$ — i.e., $\mathcal{A} \models \{\rho_n : n < \omega\}$, for some (equivalently, all) $\mathcal{A}$ where $\mathsf{At}(\mathcal{A}) = \mathcal{S}$.*

3. *$\mathcal{S}$ is* strongly representable *if every atomic cylindric algebra $\mathcal{A}$ with $\mathsf{At}(\mathcal{A}) = \mathcal{S}$ is representable. We write $\mathbf{SRAS}_n$ for the class of strongly representable (n-dimensional) cylindric algebra atom structures.*

4. *$\mathcal{S}$ is* weakly representable *if there is a representable, atomic cylindric algebra $\mathcal{A}$ with $\mathsf{At}(\mathcal{A}) = \mathcal{S}$. We let $\mathbf{WRAS}_n$ denote the class of weakly representable (n-dimensional) cylindric algebra atom structures.*

Note that for any n-dimensional cylindric algebra $\mathcal{A}$ and atom structure $\mathcal{S}$, if $\mathsf{At}(\mathcal{A}) = \mathcal{S}$ then $\mathcal{A}$ embeds into $\mathfrak{Cm}\,\mathcal{S}$, and hence $\mathcal{S}$ is strongly representable iff $\mathfrak{Cm}\,\mathcal{S}$ is representable.

We want to investigate these classes, and the relationships between them. It is easily seen that

$$\mathbf{CRAS}_n \subseteq \mathbf{LCAS}_n \subseteq \mathbf{SRAS}_n \subseteq \mathbf{WRAS}_n. \tag{1.2}$$

The last inclusion is trivial, and the first is immediate from the proof of theorem 1.4.3: $\exists$ may use a complete representation of a cylindric algebra to guide her to victory in any atomic game played on the algebra. For the middle inclusion, let $\mathcal{S}$ be an atom structure in $\mathbf{LCAS}_n$. To show $\mathcal{S} \in \mathbf{SRAS}_n$ we must show that an arbitrary atomic cylindric algebra $\mathcal{A}$ with $\mathsf{At}(\mathcal{A}) = \mathcal{S}$ is representable. By corollary 1.4.5, $\mathcal{A}$ is elementarily equivalent to some (completely) representable algebra, and since $\mathbf{RCA}_n$ is an elementary class, $\mathcal{A}$ is representable too. This shows that $\mathbf{LCAS}_n \subseteq \mathbf{SRAS}_n$. (Also, by corollary 1.4.5, $\mathbf{LCAS}_n$ is the elementary closure of $\mathbf{CRAS}_n$.)

We now ask *which of the inclusions in (1.2) are strict,* and *which of the classes are elementary.* $\mathbf{LCAS}_n$ is elementary and it is defined by $\{\rho_m : m < \omega\}$. The fact that $\mathbf{WRAS}_n$ is elementary is a special case of a more general result: given any variety V of completely additive BAOs, Venema showed in [Ven97a] that the class $\mathsf{At}\,\mathsf{V}$ of atom structures of atomic algebras in V is elementary. The idea of the proof is as follows. Any atom structure $\mathcal{S}$ of a completely additive atomic BAO is also the atom structure of the subalgebra, say $\mathcal{A}$, generated by the atoms. Then $\mathcal{S} \in \mathsf{At}\,\mathsf{V}$ iff $\mathcal{A} \in \mathsf{V}$, and this holds iff each equation ε defining V is valid in $\mathcal{A}$. But each element of $\mathcal{A}$ is the value (in $\mathcal{A}$) of some term $t(\bar{x})$ of the signature of $\mathcal{A}$, whose variables $\bar{x}$ are instantiated by atoms. So the statement that ε is valid in $\mathcal{A}$ is equivalent to the truth in α of an infinite set T_ε of first-order sentences in the signature of α, obtained by replacing the $\bar{x}$ by arbitrary terms, rewriting all function symbols into first-order formulas over α (using complete additivity), and then taking the universal ($\forall$) closure. The union of the T_ε, taken over all equations ε defining V, is then a set of first-order axioms defining $\mathsf{At}\,\mathsf{V}$.

This leaves the classes $\mathbf{CRAS}_n$ and $\mathbf{SRAS}_n$. It turns out that they are not elementary [HH97, HH09]. (Hence, all inclusions in (1.2) are strict.)

1.7 Monk and rainbow algebras

How are these non-elementary results proved? The games introduced earlier are potentially powerful tools for problems like this, since they can be used to determine when an atom structure lies in one of the classes. But to take advantage of them, we need a source of examples of atom structures whose game-theoretic properties we can control.

We will give two types of example, both obtained from graphs. Aspects of our constructions can be traced back to [Mon69, Hir95, Hod97, HH02b, HH09]. A *graph* is a structure $\Gamma = (V, E)$ where V is a non-empty set of 'nodes' or 'vertices', and $E \subseteq V \times V$ is a symmetric binary 'edge' relation on V. Note that our graphs can have 'loops': a *reflexive node* is a node $x \in V$ such that $(x, x) \in E$. A set $X \subseteq V$ is a *clique* if $(x, y) \in E$ for all distinct $x, y \in X$, and *independent* if $(X \times X) \cap E = \emptyset$. The *chromatic number* $\chi(\Gamma)$ of Γ is the least natural number k such that V is the union of k independent sets, and ∞ if no such k exists. For economy's sake, we often identify (notationally) Γ with V. In the same way, we identify (notationally) a model-theoretic structure M with its domain, the cardinality of which we write as $|M|$. We will write $M \subseteq N$ to mean that M is a substructure of N.

Proofs in this section are only sketched, owing to lack of space. More details can be found in the references. Recall that n is fixed ($3 \leq n < \omega$).

1.7.1 Strong homomorphisms

Before we proceed, a little more duality will be helpful.

DEFINITION 1.7.1 Let $\mathcal{S} = (S, R_{\mathsf{c}_i}, R_{\mathsf{d}_{ij}} : i, j < n)$ and $\mathcal{S}' = (S', R'_{\mathsf{c}_i}, R'_{\mathsf{d}_{ij}} : i, j < n)$ be cylindric-type atom structures. A map $\theta : S \to S'$ is said to be a *strong homomorphism from* $\mathcal{S}$ *to* $\mathcal{S}'$ if for each $x, y \in S$ and $i, j < n$ we have

1. $(x, y) \in R_{\mathsf{c}_i} \iff (\theta(x), \theta(y)) \in R'_{\mathsf{c}_i}$,
2. $x \in R_{\mathsf{d}_{ij}} \iff \theta(x) \in R'_{\mathsf{d}_{ij}}$.

LEMMA 1.7.2 *Let $\mathcal{S}, \mathcal{S}'$ be cylindric-type atom structures and let $\theta : \mathcal{S} \to \mathcal{S}'$ be a strong homomorphism.*

1. *Let N_1 be a set and let $N_2 : {}^nN_1 \to S$. Then $N = (N_1, N_2)$ is a $\mathfrak{Cm}\,\mathcal{S}$-network iff $\theta(N) = (N_1, \theta \circ N_2)$ is a $\mathfrak{Cm}\,\mathcal{S}'$-network.*
2. *If θ is surjective and the cylindric-type algebra $\mathfrak{Cm}\,\mathcal{S}$ is a completely representable cylindric algebra then $\mathfrak{Cm}\,\mathcal{S}'$ is a completely representable cylindric algebra.*[1]

Proof. The first part is a consequence of definitions 1.4.1 and 1.7.1. For the second part, θ induces a map $\theta^{-1} : \mathfrak{Cm}\,\mathcal{S}' \to \mathfrak{Cm}\,\mathcal{S}$ by $\theta^{-1}(X) = \{s \in S : \theta(s) \in X\}$, for $X \subseteq S'$. This can be checked to be an algebra embedding that preserves all meets and joins. Hence, $\mathfrak{Cm}\,\mathcal{S}'$ is a cylindric algebra, and if h is a complete representation of $\mathfrak{Cm}\,\mathcal{S}$ then $h \circ \theta^{-1}$ is a complete representation of $\mathfrak{Cm}\,\mathcal{S}'$. □

1.7.2 Algebras from classes of structures

Both of our examples will be based on an underlying class of structures.

DEFINITION 1.7.3 Let L be a first-order signature consisting of relation symbols of arity $< n$, and let $\mathcal{K}$ be a non-empty class of L-structures with the property that an L-structure M is in $\mathcal{K}$ iff every substructure of M with at most n elements is in $\mathcal{K}$.

[1] A slightly weaker property suffices for this than θ being a strong homomorphism, namely, being a 'bounded morphism', but we will need the stronger version in lemma 1.7.4.

1. Let X be a set, $M, N \in \mathcal{K}$, $f \in {}^X M$, and $g \in {}^X N$. We write $f \sim g$ if $\{(f(x), g(x)) : x \in X\}$ is a well defined partial isomorphism from M to N.

2. Let $\mathcal{F} = \mathcal{F}(\mathcal{K}) = \bigcup\{{}^n M : M \in \mathcal{K}\}$. For each $f \in \mathcal{F}$, we fix some $M_f \in \mathcal{K}$ with $f \in {}^n(M_f)$. The class relation $\sim$ induces an equivalence relation on $\mathcal{F}$, and we write the equivalence class of $f \in \mathcal{F}$ as $[f]$. For $f \in \mathcal{F}$, if we write $\ker f = \{(x, y) \in n \times n : f(x) = f(y)\}$, we may identify $[f]$ with the L-structure induced on the set $n/\ker f$ of $(\ker f)$-equivalence classes by pulling back from M_f in the obvious way. Therefore, we may treat each equivalence class $[f]$, and $\mathcal{F}$ itself, as a set. Then, $\mathcal{F}/{\sim}$ denotes the set of $\sim$-equivalence classes.

3. We now define a structure $\rho(\mathcal{K}) = (\mathcal{F}/{\sim}, R_{\mathsf{c}_i}, R_{\mathsf{d}_{ij}} : i, j < n)$, which will be the atom structure of the cylindric-type algebra $\mathfrak{Cm}\,\rho(\mathcal{K})$, as follows:

   - $R_{\mathsf{c}_i} = \big\{([f], [g]) : f, g \in \mathcal{F},\ f \restriction (n \setminus \{i\}) \sim g \restriction (n \setminus \{i\})\big\}$,
   - $R_{\mathsf{d}_{ij}} = \{[f] : f \in \mathcal{F},\ f(i) = f(j)\}$,

   where $i, j < n$.

4. As usual, we will identify any $[f] \in \mathcal{F}/{\sim}$ with the singleton $\{[f]\} \in \mathfrak{Cm}\,\rho(\mathcal{K})$.

Fix $L, \mathcal{K}$ as in definition 1.7.3, and write $\mathcal{A}$ for $\mathfrak{Cm}\,\rho(\mathcal{K})$. Notions to do with $\mathcal{A}$-networks and complete representations of $\mathcal{A}$ have analogues in terms of structures in $\mathcal{K}$. This can be seen as follows. We leave the reader to check the (quite standard) details.

There is a one-one correspondence between $\mathcal{A}$-networks and structures in $\mathcal{K}$. In one direction, we may view any $M \in \mathcal{K}$ as an $\mathcal{A}$-network Net$M$ via Net$M(\bar{a}) = [\bar{a}]$, for each $\bar{a} \in {}^n M$. Conversely, let N be an $\mathcal{A}$-network. We define an L-structure Str$N$ on the same domain as N. For each k-ary $R \in L$ and $a_0, \ldots, a_{k-1} \in N$, we define Str$N \models R(a_0, \ldots, a_{k-1})$ iff

$$N(a_0, \ldots, a_{k-1}, \underbrace{a_0, \ldots, a_0}_{n-k \text{ times}}) = [f] \text{ and } M_f \models R(f(0), \ldots, f(k-1)). \quad (1.3)$$

This is independent of the choice of f in (1.3). Using the networkhood of N, it can be checked that for every $\bar{a} = (a_0, \ldots, a_{n-1}) \in {}^n N$, if $N(\bar{a}) = [f]$ then the partial map $\{(a_i, f(i)) : i < n\} : \mathrm{Str}N \to M_f$ is a partial isomorphism. This is a useful property to bear in mind. Among other things, it implies that (i) every substructure of Str$N$ with at most n elements is in $\mathcal{K}$, and hence Str$N \in \mathcal{K}$ too, and (ii) $\bar{a} : n \to \mathrm{Str}N$ and $\bar{a} \sim f$, so $\mathrm{Net}(\mathrm{Str}N)(\bar{a}) = [\bar{a}] = [f] = N(\bar{a})$, and hence $\mathrm{Net}(\mathrm{Str}N) = N$. Similarly, for $M \in \mathcal{K}$ we have $\mathrm{Str}(\mathrm{Net}M) = M$.

Taking account of (1.1), this correspondence allows us to view a complete representation of $\mathcal{A}$ as a set[2] $\{N_{[f]} : [f] \in \rho(\mathcal{K})\} \subseteq \mathcal{K}$ of structures such that $f : n \to N_{[f]}$ for each $[f]$, and:

$$\begin{array}{l}\text{whenever } F \subseteq N_{[f]},\ F \subseteq A \in \mathcal{K}, \text{ and } |A| \leq n,\\ \text{the inclusion map } \iota : F \to N_{[f]}\\ \text{extends to an embedding } \iota' : A \to N_{[f]}.\end{array} \tag{1.4}$$

The correspondence also allows us to construe the game $G^\kappa(\mathcal{A})$ of Definition 1.4.2 as a game played to build a chain of structures $M_t \in \mathcal{K}$ $(t < \kappa)$ as follows. In the initial round, $\forall$ picks a structure $M_0 \in \mathcal{K}$ with $|M_0| \leq n$. In successor rounds $t + 1 < \kappa$, supposing that $M_t \in \mathcal{K}$ is the structure at the start of the round, $\forall$ picks a substructure $F \subseteq M_t$ and $A \in \mathcal{K}$ with $|A| \leq n$ and $F \subseteq A$. $\exists$ must respond by finding $M_{t+1} \in \mathcal{K}$ with $M_t \subseteq M_{t+1}$ and $\forall$ wins unless the identity map on F extends to an embedding of A into M_{t+1}. At limit rounds $\lambda < \kappa$ we take unions and define $M_\lambda = \bigcup_{t<\lambda} M_t$.

1.7.3 Algebras over graphs

We now present our first specific example of this construction. The algebras we construct are related to ones in [HH09] and have some affinity to algebras devised by Monk [Mon69]. We will use them to study $\mathbf{SRAS}_n$. Let Γ be a graph. We write $\Gamma \times n$ for the graph consisting of n pairwise disjoint copies of Γ, and with an edge added between every two nodes lying in different copies. Formally, if $\Gamma = (V, E)$,

$$\Gamma \times n = \big(V \times n, \{((x, i), (y, j)) : i, j < n,\ (x, y) \in E \vee i \neq j\}\big).$$

We regard $\Gamma \times n$ as a signature by regarding each node of it as an $(n-1)$-ary relation symbol. Let $\mathcal{I}(\Gamma)$ be the class of $(\Gamma \times n)$-structures M satisfying:

M1. all relations in M are irreflexive and symmetric: whenever $p \in \Gamma \times n$, $a_0, \ldots, a_{n-2} \in M$, and $M \models p(a_0, \ldots, a_{n-2})$, then $a_0, \ldots, a_{n-2}$ are pairwise distinct and $M \models p(a_{\pi(0)}, \ldots, a_{\pi(n-2)})$ for each permutation π of $(n-1)$,

M2. whenever $a_0, \ldots, a_{n-2} \in M$ are pairwise distinct, $M \models p(a_0, \ldots, a_{n-2})$ for some unique $p \in \Gamma \times n$,

M3. whenever $a_0, \ldots, a_{n-1} \in M$, $p_0, \ldots, p_{n-1} \in \Gamma \times n$, and $M \models \bigwedge_{i<n} p_i(a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{n-1})$, there are $i < j < n$ such that (p_i, p_j) is an edge of $\Gamma \times n$.[3]

[2] In the case where $\mathcal{A}$ is simple, this set may be taken to be a singleton. Cf. [HMT85, corollary 3.1.81].

[3] The conclusion is a stronger condition than '$\{p_0, \ldots, p_{n-1}\}$ is not independent' in the case where Γ has reflexive nodes.

Note that $\mathcal{I}(\Gamma)$ satisfies the conditions in definition 1.7.3. We will write $\mathcal{M}(\Gamma)$ for the algebra $\mathfrak{Cm}\,\rho(\mathcal{I}(\Gamma))$. Our aim is to prove (proposition 1.7.8) that if Γ is infinite then $\mathcal{M}(\Gamma) \in \mathbf{RCA}_n$ iff $\chi(\Gamma) = \infty$.

LEMMA 1.7.4 *Let Γ be a graph that contains a reflexive node.. Let $\mathcal{S} = (S, R_{\mathsf{c}_i}, R_{\mathsf{d}_{ij}} : i, j < n)$ be a cylindric-type atom structure, and suppose that $\theta : \mathcal{S} \to \rho(\mathcal{I}(\Gamma))$ is a surjective strong homomorphism. Then $\mathfrak{Cm}\,\mathcal{S}$ is a completely representable cylindric algebra.*

Proof. By theorem 1.4.3, it is enough to show that $\exists$ has a winning strategy in the game $G^{|S|+\omega}(\mathfrak{Cm}\,\mathcal{S})$. Let N be a $\mathfrak{Cm}\,\mathcal{S}$-network. By lemma 1.7.2(1), $\theta(N)$ is an $\mathcal{M}(\Gamma)$-network. By the equivalence between networks and structures in $\mathcal{I}(\Gamma)$, $\theta(N)$ can be identified in a well defined way with a structure $N{\downarrow} \in \mathcal{I}(\Gamma)$ with the same domain as N and with the following property: for each $x_0, \ldots, x_{n-1} \in N$ with $\theta(N(x_0, \ldots, x_{n-1})) = [f]$, say, each $i < n$, and each $p \in \Gamma \times n$, we have $N{\downarrow} \models p(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})$ iff $M_f \models p(f(0), \ldots, f(i-1), f(i+1), \ldots, f(n-1))$. This identification will provide $\exists$ with a winning strategy in the game.

The initial round and rounds indexed by limit ordinals pose no problems for her. In some successor round, suppose that the current $\mathfrak{Cm}\,\mathcal{S}$-network is N, say. Let $\forall$ choose $x \in {}^nN$, $i < n$, and $a \in \mathcal{S}$ with $a \leq \mathsf{c}_i N(x)$. If $N(x[i/x_j]) = a$ for some $j < n$ then $\exists$ simply responds to $\forall$'s move with the current network N.

Assume from now on that no such j exists. It follows that $a \leq -\mathsf{d}_{ij}$ for each $j \in n \setminus \{i\}$. Let $\theta(a) = [f]$, say. So $f(i) \neq f(j)$ for all $j \neq i$. Let $z \notin N$ be a new node, and write

$$\begin{array}{rcl} y & = & x[i/z] \in {}^n(N \cup \{z\}), \\ Y & = & \{y_0, \ldots, y_{n-1}\}. \end{array} \tag{1.5}$$

Then $f(j) = f(k)$ iff $y_j = y_k$ for each $j, k < n$.

We now extend $N{\downarrow}$ to a structure $M \in \mathcal{I}(\Gamma)$ defined as follows. Its domain is the domain of $N{\downarrow}$ together with z. We specify that

S1. $N{\downarrow} \subseteq M$, and $f \sim y$.

To complete the specification, we first define elements $q_j \in \Gamma \times n$ ($j \in n \setminus \{i\}$) as follows. If $y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1}$ (as in (1.5)) are pairwise distinct, we let $q_j \in \Gamma \times n$ be the unique element satisfying $M \models q_j(y_0, \ldots, y_{j-1}, y_{j+1}, \ldots, y_{n-1})$ according to S1. If they are not all distinct, we choose $q_j \in \Gamma \times n$ arbitrarily. Next, recalling that $\Gamma \times n$ consists of n pairwise disjoint copies of Γ, choose one of these copies that does not contain any of $q_0, \ldots, q_{i-1}, q_{i+1}, \ldots, q_{n-1}$. Let d be a reflexive node in this copy. We now specify that

S2. $M \models d(t_0, \ldots, t_{n-2})$ whenever $t_0, \ldots, t_{n-2} \in M$ are distinct and $z \in \{t_0, \ldots, t_{n-2}\} \not\subseteq Y$, where Y is as in (1.5). It can be checked that this specifies a well defined $(\Gamma \times n)$-structure M. We check that $M \in \mathcal{I}(\Gamma)$. Properties M1 and M2 are easy to verify. We pass to M3. Let $t_0, \ldots, t_{n-1} \in M$ be distinct, let $p_0, \ldots, p_{n-1} \in \Gamma \times n$, and suppose that $M \models p_j(t_0, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n-1})$ for each $j < n$. We need to show that (p_j, p_k) is an edge of $\Gamma \times n$, for some $j < k < n$. Using S1, it can be seen that this holds if $t_0, \ldots, t_{n-1} \in N{\downarrow}$, since $N{\downarrow} \in \mathcal{I}(\Gamma)$, and it holds if $\{t_0, \ldots, t_{n-1}\} = Y$, because $M_f \in \mathcal{I}(\Gamma)$. So assume that $z \in \{t_0, \ldots, t_{n-1}\} \not\subseteq Y$. Clearly, there are at least $n-2$ indices $j < n$ with

$$z \in \{t_0, \ldots, t_{j-1}, t_{j+1}, \ldots, t_{n-1}\} \not\subseteq Y.$$

By S2, there are at least $n-2$ indices $j < n$ with $p_j = d$. Since $n \geq 3$, there is at least one such j. There are now two cases.

1. If there are $j < k < n$ with $p_j = p_k = d$, then as d is a reflexive node, (p_j, p_k) is an edge of $\Gamma \times n$ as required.

2. If there is a unique $j < n$ with $p_j = d$, we plainly must have $n = 3$. Let $k, l \in 3 \setminus \{j\}$ satisfy $z = t_k$ and $t_l \notin Y$. Then $M \models p_l(t_j, t_k)$ by the above, and $t_j, t_k \in Y$. So $p_l \in \{q_j : j \in n \setminus \{i\}\}$. But d lies in a copy of Γ that does not contain p_l. As there are edges of $\Gamma \times n$ connecting all nodes in distinct copies of Γ, (p_j, p_l) and (p_l, p_j) are edges of $\Gamma \times n$ as required.

We now extend N to a network $N' \supseteq N$ whose set of nodes is the domain of M, with $N'(y) = a$, and with $N'{\downarrow} = M$, in any way at all; lemma 1.7.2(1) guarantees that the result will be a $\mathfrak{Cm}\,\mathcal{S}$-network. $\exists$ responds to $\forall$ with this network N', and thus has the capability to win the game. $\square$

DEFINITION 1.7.5 Let Γ be a graph. We write $\mathfrak{Ue}\,\Gamma$, the *ultrafilter extension* of Γ, for the graph whose nodes are the ultrafilters on Γ (i.e., ultrafilters of the boolean algebra of subsets of Γ), and such that (μ, ν) is an edge of $\mathfrak{Ue}\,\Gamma$ iff for every $X \in \mu$, $Y \in \nu$, there are $p \in X$, $q \in Y$ such that (p, q) is an edge of Γ.

LEMMA 1.7.6 *Let Γ be any graph.*

1. *$\mathfrak{Ue}(\Gamma \times n) \cong (\mathfrak{Ue}\,\Gamma) \times n$ (we will identify the two).*

2. *$\chi(\mathfrak{Ue}\,\Gamma) = \chi(\Gamma)$.*

3. *$\chi(\Gamma) = \infty$ iff $\mathfrak{Ue}\,\Gamma$ has a reflexive node.*

Proof. Write Δ for $\mathfrak{Ue}\,\Gamma$ in the proof. The first part is easy and we leave it to the reader. Let $k < \omega$. If $\Gamma = \bigcup_{i<k} I_i$ for independent $I_i \subseteq \Gamma$, then for each i, $I_i^\Delta = \{\nu \in \Delta : I_i \in \nu\}$ is an independent subset of Δ, and $\bigcup_{i<k} I_i^\Delta = \Delta$. Similarly, suppose $\Delta = \bigcup_{j<k} J_j$ for independent $J_j \subseteq \Delta$. For

$p \in \Gamma$ let $\langle p \rangle \in \Delta$ be the principal ultrafilter generated by $\{p\}$. Then for each j, the set $J_j^\Gamma = \{p \in \Gamma : \langle p \rangle \in J_j\}$ is an independent subset of Γ, and $\Gamma = \bigcup_{j<k} J_j^\Gamma$.

For the last part, if there is finite k with $\Gamma = \bigcup_{i<k} I_i$ for independent $I_i \subseteq \Gamma$, then each ultrafilter on Γ contains some I_i and so cannot be reflexive. Conversely, if $\chi(\Gamma) = \infty$ then the set of independent subsets of Γ generates a proper ideal of subsets of Γ. Any ultrafilter on Γ disjoint from this ideal contains no independent sets and is therefore a reflexive node of Δ. $\square$

Recall that $\mathcal{A}_+$ (for a BAO $\mathcal{A}$) was defined in §1.5.

LEMMA 1.7.7 *For any graph Γ, there is a surjective strong homomorphism $\theta : \mathcal{M}(\Gamma)_+ \to \rho(\mathcal{I}(\mathfrak{Ue}\,\Gamma))$.*

Proof. First, for any $x_0, \ldots, x_{n-2} < n$ and $X \subseteq \Gamma \times n$, define the following element of $\mathcal{M}(\Gamma)$:

$$X^{(x_0,\ldots,x_{n-2})} = \big\{[f] \in \rho(\mathcal{I}(\Gamma)) : \exists p \in X\big[M_f \models p(f(x_0), \ldots, f(x_{n-2}))\big]\big\}.$$

Now let μ be an ultrafilter of $\mathcal{M}(\Gamma)$. Define an equivalence relation $\sim$ on n by $i \sim j \iff \mathsf{d}_{ij} \in \mu$. Let $g : n \to n/{\sim}$ be given by $g(i) = i/{\sim}$. Define a $(\mathfrak{Ue}\,\Gamma \times n)$-structure M_μ with domain $n/{\sim}$ as follows. For each $\nu \in \mathfrak{Ue}\,\Gamma \times n$ and $x_0, \ldots, x_{n-2} < n$, we let

$$M_\mu \models \nu(g(x_0), \ldots, g(x_{n-2})) \iff X^{(x_0,\ldots,x_{n-2})} \in \mu \text{ for each } X \in \nu. \quad (1.6)$$

It is straightforward (though lengthy) to check that this is well defined and that $M_\mu \in \mathcal{I}(\mathfrak{Ue}\,\Gamma)$. So $g : n \to M_\mu$ and hence $g \in \mathcal{F}(\mathcal{I}(\mathfrak{Ue}\,\Gamma))$ (see definition 1.7.3(2)). Then we define $\theta(\mu) = [g] \in \rho(\mathcal{I}(\mathfrak{Ue}\,\Gamma))$.

Using M1 and M2, it can now be checked that θ is a strong homomorphism. We show it is surjective. Let $[g] \in \rho(\mathcal{I}(\mathfrak{Ue}\,\Gamma))$ be given, so $g : n \to M_g \in \mathcal{I}(\mathfrak{Ue}\,\Gamma)$. Let $D_g = \{\mathsf{d}_{ij} : i, j < n, g(i) = g(j)\} \cup \{-\mathsf{d}_{ij} : i, j < n, g(i) \neq g(j)\} \subseteq \mathcal{M}(\Gamma)$. There are three cases.

Case 1: g is one-one. By M2, for each $i < n$ there is a unique $\nu_i \in \mathfrak{Ue}\,\Gamma$ with $M_g \models \nu_i(g(0), \ldots, g(i-1), g(i+1), \ldots, g(n-1))$, and by M3, there are $i < j < n$ such that (ν_i, ν_j) is an edge of $\mathfrak{Ue}\,\Gamma$. We show that

$$\mu_0 = D_g \cup \{X^{(0,\ldots,l-1,l+1,\ldots,n-1)} : l < n, X \in \nu_l\} \subseteq \mathcal{M}(\Gamma) \quad (1.7)$$

has the finite intersection property. As the ν_l are ultrafilters, it suffices to check that whenever $X_l \in \nu_l$ $(l < n)$, we have

$$G = D_g \cap \bigcap_{l<n} X_l^{(0,\ldots,l-1,l+1,\ldots,n-1)} \neq \emptyset.$$

We may choose $p_l \in X_l$ (each l) such that (p_i, p_j) is an edge of $\Gamma \times n$. Then we can define a $(\Gamma \times n)$-structure $M \in \mathcal{I}(\Gamma)$ with domain n by specifying that $M \models p_l(0, \ldots, l-1, l+1, \ldots, n-1)$ for each l. Because (p_i, p_j) is an edge, M3 is satisfied. If $f : n \to M$ is the identity map on n, then $[f] \in G$, which is therefore non-empty as required. So μ_0 extends to an ultrafilter μ of $\mathcal{M}(\Gamma)$. By (1.6) and (1.7) we have $\theta(\mu) = [g]$.

Case 2: there are unique $i < j < n$ **with** $g(i) = g(j)$**.** Using M2, let $\nu \in \mathfrak{Ue}\,\Gamma$ be such that $M_g \models \nu(g(0), \ldots, g(i-1), g(i+1), \ldots, g(n-1))$. Using the irreflexivity condition in M1, it can be verified that

$$D_g \cup \{X^{(0,\ldots,i-1,i+1,\ldots,n-1)} : X \in \nu\}$$

has the finite intersection property and so extends to a (unique) ultrafilter μ of $\mathcal{M}(\Gamma)$, and $\theta(\mu) = [g]$.

Case 3: otherwise. By irreflexivity (M1), all structures in $\mathcal{I}(\Gamma)$ with fewer than $n-1$ elements are isomorphic, so $\bigwedge D_g$ is an atom of $\mathcal{M}(\Gamma)$. Let μ be the unique ultrafilter of $\mathcal{M}(\Gamma)$ containing D_g. Then $\theta(\mu) = [g]$. □

Combining these lemmas and with a little more work, we reach our goal:

PROPOSITION 1.7.8 *For any infinite graph* Γ*, we have* $\mathcal{M}(\Gamma) \in \mathbf{RCA}_n$ *iff* $\chi(\Gamma) = \infty$.

Proof. Suppose that $\mathcal{M}(\Gamma)$ is representable. By theorem 1.5.3, $\mathcal{M}(\Gamma)^\sigma$ is completely representable. By lemmas 1.7.7 and 1.7.2(2), so is $\mathcal{M}(\mathfrak{Ue}\,\Gamma)$, and hence, choosing any $[f] \in \rho(\mathcal{I}(\mathfrak{Ue}\,\Gamma))$, there is $M = M_{[f]} \in \mathcal{I}(\mathfrak{Ue}\,\Gamma)$ satisfying (1.4). Since Γ is infinite, it can be checked that M is also infinite.

Suppose for contradiction that $\chi(\Gamma) < \infty$. By lemma 1.7.6, $\chi(\mathfrak{Ue}\,\Gamma) < \infty$, and it is clear that $\chi(\mathfrak{Ue}\,\Gamma \times n) = \chi(\mathfrak{Ue}\,\Gamma) \cdot n < \infty$ as well. So there are $k < \omega$ and independent sets $I_0, \ldots, I_{k-1} \subseteq \mathfrak{Ue}\,\Gamma \times n$ with $\mathfrak{Ue}\,\Gamma \times n = \bigcup_{i<k} I_i$. Choose pairwise distinct $x_0, x_1, \ldots \in M$. By M2, for each $i_0 < \cdots < i_{n-2} < \omega$ there is $c(i_0, \ldots, i_{n-2}) < k$ such that $M \models p(x_{i_0}, \ldots, x_{i_{n-2}})$ for some $p \in I_{c(i_0,\ldots,i_{n-2})}$. By Ramsey's theorem, we may assume that c has constant value c_0, say. Then for each $i < n$ there is some $p_i \in I_{c_0}$ with $M \models p_i(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n-1})$. Since $\{p_i : i < n\} \subseteq I_{c_0}$, an independent set, this contradicts M3.

Conversely, suppose that $\chi(\Gamma) = \infty$. By lemma 1.7.6, $\mathfrak{Ue}\,\Gamma$ contains a reflexive node. By lemmas 1.7.7 and 1.7.4, $\mathcal{M}(\Gamma)^\sigma$ is completely representable. By theorem 1.5.3, $\mathcal{M}(\Gamma)$ is representable. □

1.7.4 'Rainbow algebras' over graphs

Our second specific instance of the construction of section 1.7.2 are so-called 'rainbow algebras'. They are similar to algebras constructed in [HH97] and will be used to study $\mathbf{CRAS}_n$.

DEFINITION 1.7.9 Let Γ be a graph.

1. Let $L = L(\Gamma)$ be the signature

$$\Gamma \cup \{\mathsf{g}_0^j : j < \omega\} \cup \{\mathsf{g}_i : 1 \leq i \leq n-2\} \cup \{\mathsf{w}_i : i \leq n-2\} \\ \cup \{\mathsf{y}_S : S \subseteq \omega, |S| < \omega\}.$$

   Each y_S is an $(n-1)$-ary relation symbol, regarded as yellow. All the others are binary relation symbols. We regard the g_0^j and g_i as green and the w_i as white. We define the following formulas:

   - $G(x, y) = \bigvee_{j<\omega} \mathsf{g}_0^j(x, y) \vee \bigvee_{1\leq i\leq n-2} \mathsf{g}_i(x, y)$ (an $L_{\omega_1\omega}$-formula),
   - $\chi^j(x_0, \ldots, x_{n-2}, y) = \mathsf{g}_0^j(x_0, y) \wedge \bigwedge_{1\leq i\leq n-2} \mathsf{g}_i(x_i, y)$, for $j < \omega$.

2. We let $\mathcal{K} = \mathcal{K}(\Gamma)$ be the class of L-structures M such that:

   R1. all relations in M are irreflexive: if $R \in L$ is k-ary, $a_0, \ldots, a_{k-1} \in M$, and $M \models R(a_0, \ldots, a_{k-1})$, then $a_0, \ldots, a_{k-1}$ are pairwise distinct,

   R2. all non-yellow binary relations are symmetric,

   R3. exactly one non-yellow binary relation holds on each pair of distinct elements of M,

   R4. M has no green triangles: $M \models \neg\exists xyz(G(x, y) \wedge G(y, z) \wedge G(x, z))$,

   R5. M has no green-green-white triangles with equal lower indices: $M \models \neg\exists xyz(\mathsf{g}_0^j(x, y) \wedge \mathsf{g}_0^k(y, z) \wedge \mathsf{w}_0(x, z))$ for each $j, k < \omega$, and $M \models \neg\exists xyz(\mathsf{g}_i(x, y) \wedge \mathsf{g}_i(y, z) \wedge \mathsf{w}_i(x, z))$ for $1 \leq i \leq n-2$,

   R6. $M \models \neg\exists xyz(p(x, y) \wedge q(y, z) \wedge r(x, z))$ whenever $p, q, r \in \Gamma$ and $\{(p, q), (q, r), (p, r)\} \not\subseteq E$,

   R7. $M \models \neg\exists x_0 \ldots x_{n-2}(\mathsf{y}(x_0, \ldots, x_{n-2}) \wedge \bigvee_{i<j\leq n-2} G(x_i, x_j))$ for each yellow $\mathsf{y} \in L$,

   R8. if $S \subseteq \omega$ is finite and $j \in \omega \setminus S$ then $M \models \neg\exists x_0 \ldots x_{n-2} y(\mathsf{y}_S(x_0, \ldots, x_{n-2}) \wedge \chi^j(x_0, \ldots, x_{n-2}, y))$.

   Note that $\mathcal{K}$ satisfies the conditions in definition 1.7.3.

3. We write $\mathcal{R}(\Gamma)$ for the 'rainbow' algebra $\mathfrak{Cm}\, \rho(\mathcal{K}(\Gamma))$.

1.7.5 Complete representability of the algebras $\mathcal{R}(\Gamma)$

PROPOSITION 1.7.10 *Let Γ be any graph. If $\mathcal{R}(\Gamma)$ is completely representable then Γ has a reflexive node or an infinite clique.*

Proof. Assume that $\mathcal{R}(\Gamma)$ has a complete representation, viewed as a structure $M \in \mathcal{K} = \mathcal{K}(\Gamma)$ satisfying (1.4) above. By (1.4), we can find elements $a_0, \ldots, a_{n-2} \in M$ such that $M \models \neg G(a_i, a_j)$ for each $i, j < n-1$, and $M \models \neg \mathsf{y}(a_0, \ldots, a_{n-2})$ for each yellow $\mathsf{y} \in L$. By (1.4) again, for each $j < \omega$ there is $b_j \in M$ such that $M \models \chi^j(a_0, \ldots, a_{n-2}, b_j)$. Since $M \in \mathcal{K}$, M satisfies conditions R3–R5 of definition 1.7.9. It follows that for each $j < k < \omega$ there is $p_{jk} \in \Gamma$ with $M \models p_{jk}(b_i, b_j)$. Considering triangles (b_0, b_j, b_k) and using the definition of $\mathcal{K}(\Gamma)$, we see that (p_{0j}, p_{0k}) is an edge of Γ for all $0 < j < k < \omega$. So $\{p_{0j} : 1 \leq j < \omega\}$ is either an infinite clique in Γ or contains a reflexive node. □

PROPOSITION 1.7.11 *If Γ is a countable graph containing a reflexive node or an infinite clique, then $\mathcal{R}(\Gamma)$ is completely representable.*

Proof. Assume that $C \subseteq \Gamma$ is an infinite clique or a singleton consisting of a reflexive node of Γ. By theorem 1.4.3, it suffices to show how $\exists$ can win $G^\omega(\mathcal{R}(\Gamma))$, construed as above as a game on structures in $\mathcal{K} = \mathcal{K}(\Gamma)$. Let $M \in \mathcal{K}$ be the structure at the start of some round t $(1 \leq t < \omega)$. Suppose inductively that M is finite. In round t, suppose that $\forall$ chooses $F \subseteq M$ with $|F| < n$, and an extension $A \in \mathcal{K}$ of F with $|A| \leq n$. We can assume without loss of generality that $|A \setminus F| = 1$ and that $A \setminus F = \{d\}$, say, where $d \notin M$. $\exists$ must extend M to some $M^\sharp \in \mathcal{K}$ in such a way that the inclusion map $\iota : F \to M$ extends to an embedding $\iota^\sharp : A \to M^\sharp$.

If there is already such an $\iota^\sharp : A \to M$, then $\exists$ lets $M^\sharp = M$. So assume not. $\exists$ defines an extension $M^\sharp$ of M with domain $M \cup \{d\}$ as follows. Let $M^\flat$ be the union of M and A over F.[4]

- For each $a_0, \ldots, a_{n-2} \in M^\flat$ such that $d \in \{a_0, \ldots, a_{n-2}\} \not\subseteq A$ and $M^\flat \models \neg G(a_i, a_j)$ for each $i < j \leq n-2$, $\exists$ defines

$$M^\sharp \models \mathsf{y}_S(a_0, \ldots, a_{n-2}),$$
$$\text{where} \quad S = \{j < \omega : M^\flat \models \exists x \chi^j(a_0, \ldots, a_{n-2}, x)\}.$$

Then, for each $b \in M \setminus F$, $\exists$ chooses a binary relation symbol x_b and lets $M^\sharp \models x_b(b, d) \wedge x_b(d, b)$. In each case she chooses $x_b \in \{\mathsf{w}_i : i \leq n-2\} \cup C$. She chooses these elements in turn, as follows.

[4]That is, we assume that $M \cap A = F$; for each k-ary $R \in L$ and k-tuple $\bar{a}$ of elements of $M \cup A$, we define $M^\flat \models R(\bar{a})$ iff the elements of $\bar{a}$ lie in M and $M \models R(\bar{a})$ or the elements of $\bar{a}$ lie in A and $A \models R(\bar{a})$.

- If there are no $a_0 \in F$ and $j, j' < \omega$ such that $M^\flat \models \mathsf{g}_0^j(a_0, d) \wedge \mathsf{g}_0^{j'}(a_0, b)$, then $\exists$ defines $M^\sharp \models \mathsf{w}_0(b, d) \wedge \mathsf{w}_0(d, b)$.

- Otherwise, if there is $1 \leq i \leq n-2$ such that for no $a_i \in F$ do we have $M^\flat \models \mathsf{g}_i(d, a_i) \wedge \mathsf{g}_i(a_i, b)$, then $\exists$ chooses the least such i and defines $M^\sharp \models \mathsf{w}_i(b, d) \wedge \mathsf{w}_i(d, b)$.

- Otherwise, there must be $\bar{a} \in {}^{n-1}F$ and $j, j' < \omega$ with $M^\flat \models \chi^j(\bar{a}, b) \wedge \chi^{j'}(\bar{a}, d)$. If C consists of a single reflexive node ρ, say, she lets $M^\sharp \models \rho(b, d) \wedge \rho(d, b)$. If C is an infinite clique then she picks some $x \in C$ that has not been used as a label so far (either in a previous round or for some other 'b' in the current round) and lets $M^\sharp \models x(b, d) \wedge x(d, b)$.

Note that $\exists$ never defines any green relations, so

$$M^\sharp \models \neg G(d, b) \text{ for every } b \in M \setminus F. \tag{1.8}$$

This strategy can be checked to be winning for $\exists$. We have no space for a full proof, but the chief point to check is that $M^\sharp \in \mathcal{K}$, and in particular that $M^\sharp$ satisfies condition R6 of definition 1.7.9. This boils down to checking that whenever $b, c \in M \setminus F$, $\exists$ defines $M^\sharp \models p(b, d) \wedge q(c, d)$ for $p, q \in C$ as per her strategy, and also $M \models r(b, c)$ for some $r \in \Gamma$, then $(p, q), (p, r), (q, r)$ are edges of Γ. Certainly (p, q) is an edge, since $p, q \in C$ are chosen successively by $\exists$ as already outlined. So it is sufficient to show that $r \in C$.

By $\exists$'s strategy, this will certainly be the case if $\exists$ defined $M \models r(b, c)$ herself in an earlier round of the game. We will show that she did. $\exists$ is currently defining $M^\sharp \models p(b, d) \wedge q(c, d)$, so according to her strategy there must be $\bar{a}, \bar{a}' \in {}^{n-1}F$ and $j, k, l, l' < \omega$ with $M^\flat \models \chi^j(\bar{a}, b) \wedge \chi^l(\bar{a}, d) \wedge \chi^k(\bar{a}', c) \wedge \chi^{l'}(\bar{a}', d)$. As $|F| \leq n-1$, by R3 we have $\bar{a} = \bar{a}'$ and $l = l'$. So

$$M^\flat \models \chi^j(\bar{a}, b) \wedge \chi^k(\bar{a}, c) \wedge \chi^l(\bar{a}, d). \tag{1.9}$$

Now M has been built by the game: its elements were added one at a time in earlier rounds. Let $\bar{a} = (a_0, \ldots, a_{n-2})$. Clearly, $a_0, \ldots, a_{n-2}, b, c$ are pairwise distinct. Consider the round in which the final element among them, say d', was added. In his move in that round, suppose that $\forall$ chose $F' \subseteq M$ with $|F'| < n$.

Suppose for contradiction that $d' = a_i$ for some $i \leq n-2$. By (1.9), $M \models G(a_i, b) \wedge G(a_i, c)$, and by (1.8) applied to the earlier round we must have $b, c \in F'$. As $M \in \mathcal{K}$, by (1.9) and R4 we have $M^\flat \models \neg G(a_i, a_j)$ for $i < j \leq n-2$. As $|F'| < n$, there is $i' \leq n-2$ with $i' \neq i$ and $a_{i'} \notin F'$. Referring to the strategy showed that $\exists$ defined $M \models \mathsf{y}_S(\bar{a})$, where S was the set of all $m < \omega$ such that $\exists x \chi^m(a_0, \ldots, a_{n-2}, x)$ was true in the structure existing at the start of that round. This structure is a substructure of M, so $S \subseteq \{m < \omega : M \models \exists x \chi^m(a_0, \ldots, a_{n-2}, x)\}$. Now we return our attention to the current round. Since $A \in \mathcal{K}$ and $A \models \mathsf{y}_S(\bar{a}) \wedge \chi^l(\bar{a}, d)$ (see (1.9)), by R8 we must have $l \in S$, so there must be some $d' \in M$ with $M \models \chi^l(\bar{a}, d')$. It follows by condition R7 of the definition of $\mathcal{K}$ that $\iota^\sharp = \iota \cup \{(d, d')\}$ embeds A into M, contradicting our assumption that there is no such embedding.

So $d' \in \{b, c\}$. Suppose that $d' = b$ (the case where $d' = c$ is similar). For each $i \le n-2$, $M \models G(b, a_i)$, and by (1.8) applied to the earlier round, $a_i \in F'$. Since $|F'| < n$, we have $c \notin F'$, and by (1.9), $\exists$'s strategy would have defined $M \models r(b, c)$ for $r \in C$, as required. $\square$

1.8 Consequences

It is now easy to derive several corollaries. We will use a few common graph constructions. The *disjoint union* of graphs $\Gamma_i = (V_i, E_i)$ $(i \in I)$ is the graph

$$\bigoplus_{i \in I} \Gamma_i = \Big(\bigcup \{ V_i \times \{i\} : i \in I \}, \big\{ \big((x, i), (y, i) \big) : i \in I, (x, y) \in E_i \big\} \Big). \quad (1.10)$$

For a cardinal $\kappa > 0$, we write K_κ for the complete graph $(\kappa, \{(i, j) : i, j < \kappa,\ i \neq j\})$. For finite $n > 0$, we have $\chi(K_n) = n$. Also, $\chi(\bigoplus_{i \in I} \Gamma_i) = \max\{\chi(\Gamma_i) : i \in I\}$ if this exists, and ∞ otherwise.

COROLLARY 1.8.1 [HH97] $\mathbf{CRAS}_n$ *is not an elementary class.*

Proof. Write $\Gamma = \bigoplus_{1 \le n < \omega} K_n$. We know from proposition 1.7.10 that $\mathcal{R}(\Gamma)$ is not completely representable. Therefore, its atom structure $\rho(\mathcal{K}(\Gamma))$ is not in $\mathbf{CRAS}_n$.

However, since Γ has arbitrarily large finite cliques, there is a countable graph Δ that is elementarily equivalent to Γ and has an infinite clique. By proposition 1.7.11, $\mathcal{R}(\Delta)$ is completely representable, so $\rho(\mathcal{K}(\Delta)) \in \mathbf{CRAS}_n$.

It can be checked that $\rho(\mathcal{K}(\Gamma))$ is elementarily equivalent to $\rho(\mathcal{K}(\Delta))$. This shows that $\mathbf{CRAS}_n$ is not closed under elementary equivalence and so cannot be elementary. $\square$

In fact, $\mathbf{CRAS}_n$ is pseudo-elementary, and so closed under ultraproducts [CK90, exercise 4.1.17, corollary 6.1.16]. Hence [CK90, theorems 4.1.12 and 6.1.15], it is not closed under ultraroots.

In contrast, $\mathbf{SRAS}_n$ is closed under ultraroots [Gol89, 3.8.1(1)], but not ultraproducts, and hence is not elementary:

COROLLARY 1.8.2 [HH09] $\mathbf{SRAS}_n$ *is not an an elementary class.*

Proof. We use a celebrated theorem of Erdős [Erd59] stating that for each finite n, there exists a finite graph Γ_n with chromatic number at least n and with no cycles of length at most n. (For our purposes, a *cycle of* *length* n in a graph is a sequence $v_1, \ldots, v_n$ of distinct nodes such that $(v_1, v_2), \ldots, (v_{n-1}, v_n)$, and (v_n, v_1) are edges.) Let $\Delta_n = \bigoplus_{n<m<\omega} \Gamma_m$. Then $\chi(\Delta_n) = \infty$, and Δ_n is countably infinite and has no cycles of length at most n. Therefore, by proposition 1.7.8, $\mathcal{M}(\Delta_n)$ is representable, and so $\rho(\mathcal{I}(\Delta_n)) \in \mathbf{SRAS}_n$.

Now let Δ be a non-principal ultraproduct of the Δ_n. It follows from Łoś's theorem that Δ has no cycles of any finite length. So by a well known result from graph theory (cf. [Die97, proposition 1.6.1]), $\chi(\Delta) \leq 2$. By proposition 1.7.8 again, $\mathcal{M}(\Delta)$ is not representable, so $\rho(\mathcal{I}(\Delta)) \notin \mathbf{SRAS}_n$.

But it is easily seen that $\rho(\mathcal{I}(\Delta))$ is isomorphic to an ultraproduct of the $\rho(\mathcal{I}(\Delta_n))$. As elementary classes are closed under ultraproducts, it follows that $\mathbf{SRAS}_n$ is non-elementary. □

COROLLARY 1.8.3 [Hod97] $\mathbf{RCA}_n$ *is not closed under completions.*

Proof. In the notation of the preceding proof, let $\mathcal{A}$ be a non-principal ultraproduct of the $\mathcal{M}(\Delta_n)$. For each n we know $\mathcal{M}(\Delta_n) \in \mathbf{RCA}_n$, so as this class is elementary, by Łoś's theorem we have $\mathcal{A} \in \mathbf{RCA}_n$ as well. But $\mathcal{A}$ is atomic with atom structure $\rho(\mathcal{I}(\Delta))$, so its completion is $\mathcal{M}(\Delta)$, which is not representable. □

It follows that $\mathbf{RCA}_n$ is not Sahlqvist-axiomatisable [Ven97b]. As $\mathsf{At}(\mathcal{A}) \in \mathbf{WRAS}_n \setminus \mathbf{SRAS}_n$, or as only one of them is elementary, we see that these classes are indeed distinct.

We conclude that:

THEOREM 1.8.4 *For finite* $n \geq 3$*, we have* $\mathbf{CRAS}_n \subset \underline{\mathbf{LCAS}_n} \subset \mathbf{SRAS}_n \subset \underline{\mathbf{WRAS}_n}$*, the elementary classes being underlined.*

Related results for relation algebras are proved in [Hir95, HH02b, HV05].

1.9 Cylindric Algebras of Low or High Dimension

We end by considering $\mathbf{RCA}_n$ for $n \leq 2$ and the infinite dimensional case. For $n \leq 2$ there are analogues of Corollary 1.2.2 for these classes:

PROPOSITION 1.9.1 *For* $n \leq 2$*, an* n*-dimensional cylindric algebra is completely representable iff it is representable and atomic.*

Proof. '⇒' is immediate from Corollary 1.2.2, since a cylindric representation is *inter alia* a boolean representation. We sketch the proof of '⇐'. The case $n = 0$ follows from Corollary 1.2.2, as 0-dimensional cylindric algebras are just boolean algebras. Let $\mathcal{A} \in \mathbf{RCA}_1$ be atomic. Consider the equivalence relation on $\mathsf{At}\,\mathcal{A}$ defined by $x \sim y \iff \mathsf{c}_0 x = \mathsf{c}_0 y$. Let E be the set of $\sim$-equivalence classes, and for $e \in E$ write $\mathcal{C}(e)$ for the full

1-dimensional cylindric set algebra with base e. Then $f : \mathcal{A} \to \prod_{e\in E} \mathcal{C}(e)$ given by $f(a) = \langle a \cap e : e \in E\rangle$ is an embedding preserving all meets and joins that exist in $\mathcal{A}$.

Let $\mathcal{A} \in \mathbf{RCA}_2$ be atomic. As $\mathbf{RCA}_2$ is conjugated and defined by (algebraic versions of) Sahlqvist equations given in [Ven95, Definition 2.2]. it is closed under completions [GV99]. So the equations are valid over the frame (atom structure) $\mathsf{At}\,\mathcal{A}$. By [Ven95, Theorem 2.4], $\mathsf{At}\,\mathcal{A}$ is a bounded morphic image of a disjoint union of square frames $\mathcal{F}_i$ $(i \in I)$. Each $\mathfrak{Cm}\,\mathcal{F}_i$ is a full 2-dimensional cylindric set algebra. By duality, the inverse of the bounded morphism is an embedding from $\mathcal{A}$ into $\prod_{i\in I} \mathfrak{Cm}\,\mathcal{F}_i$ that can be checked to preserve all meets and joins existing in $\mathcal{A}$. □

By [GV99], for $n \leq 2$, since $\mathbf{RCA}_n$ is a conjugated variety defined by Sahlqvist equations, it is closed under completions.

For $n \geq \omega$, a simple cardinality argument will show that the class of completely representable n-dimensional cylindric algebras is not elementary [HH97, corollary 26]. Other results established in this chapter for the finite dimensional case have not yet been considered for $n \geq \omega$.

PROBLEM 1.9.2 *Which parts of theorem 1.8.4 remain true when n is infinite?*

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