

The Evolution of Mixed Strategies in Population Games.

Ratul Lahkar* and Robert M. Seymour†

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Abstract

We study the evolution of mixed strategies in population games. At any time, the distribution of mixed strategies over agents in a population is described by a density function. A pair of players is chosen randomly in each round of the game. After each round, players update their mixed strategies using certain reinforcement driven rules. The distribution over mixed strategies thus changes. In a continuous-time limit, this change is described by non-linear continuity equations. The updating rules we use generate the replicator continuity equations, and we provide the asymptotic solution for these equations for general 2 player asymmetric and symmetric normal form games. We use these results to study in greater detail mixed strategy evolution in 2×2 symmetric and asymmetric games. A key finding is that, when agents carry mixed strategies, distributional considerations in general cannot be subsumed under a classical approach represented by the deterministic replicator dynamics.

1 Introduction

In this paper, we study the evolution of mixed strategies in population games. Evolutionary game theory has largely focused on the the evolution of pure actions. It is assumed that there exist populations of agents, with each population standing for a particular player role in the game. Each agent is primed to play a pure action at a particular time which is retained until a revision opportunity becomes available. The variables of interest are the proportions of agents playing a particular action in each population. The change in these proportions is tracked using systems of ordinary differential equations called evolutionary (or learning) dynamics.

This sole focus on pure actions introduces a sharp dichotomy between evolutionary game theory and conventional game theory. Mixed strategies are central to the technical foundations of game theory. Moreover, without mixed strategies, we impose a severe restriction both on the level of rationality at which individual agents operate and their behavioural flexibility. Sceptics can reasonably raise doubts about the ability of a theory to explain social behaviour if it imposes such severe behavioural restrictions on individual agents. Instead, it might be argued, the theory is more

*Institute for Financial Management and Research, 24, Kothari Road, Nungambakkam, Chennai, 600 034, India. email: r.lahkar@ifmr.ac.in. Much of the work on this paper was done while the author was a postdoctoral research fellow in ELSE and the Department of Mathematics, University College London. The author would like to thank ELSE and the mathematics department for their generous financial support during that period.

†Department of Mathematics and ELSE centre, University College London, Gower Street, London, WC1E 6BT, UK. email: rms@math.ucl.ac.uk.

suited to the study of biological evolution in which animals are genetically programmed to play individual pure strategies.

Further, the neglect of mixed strategies has created a divide between the otherwise closely related fields of evolutionary game theory and the theory of learning by individual agents. In learning theory, individual agents are construed as exhibiting *behavioural dispositions*, which, in the context of a particular role in a game, can be represented as mixed strategies. Players update these strategies according to some protocol based on their past experience. Individual mixed strategies then evolve according to some ordinary differential equations that are technically similar to the ODEs of evolutionary theory, although the state variables of the two types of equations have different interpretations.¹ In terms of its behavioural foundations, however, learning theory is closer to conventional game theory than to evolutionary game theory.

In this paper, we consider individual agents in population games who employ mixed strategies, and study how the share of agents using a particular mixed strategy changes over time when agents revise their mixed strategies in the light of experience. This is in the spirit of conventional evolutionary game theory, but without the restriction to agents primed only with pure strategies. The techniques we use are based on methods associated with what are called ‘continuity equations’, a type of first-order partial differential equation (PDE) derived from physics.² At a particular time, we envisage the existence of a density function over the set of mixed strategies that an agent from a particular population can employ. Intuitively, the density function describes the probability mass of agents in the population using any mixed strategy. Over time, as agents revise and update their strategies, the density functions for each population changes in continuous time. The continuity equations we obtain are partial differential equations that track these changes. However, our equations differ from classical versions encountered in physics in that they contain non-linearities.

We study two models of population games, asymmetric and symmetric. In the asymmetric case, there are two populations of agents with one player from each population randomly matched in each round to play a two-player asymmetric normal form game. Initially we model time as a discrete variable. Players play the game by choosing pure actions using the mixed strategies which they bring to the table. After the game, the players update these mixed strategies in light of their experience using some learning protocol such as reinforcement learning.³ This updating changes the distribution of mixed strategies over agents in their respective populations, and hence changes the probability density functions over these mixed strategies, leading to a new probability density function over the set of mixed strategies. If we let the time interval between each round of play

¹As an example is the logit dynamic that is used in both learning theory (Fudenberg and Levine, 1998) and in evolutionary game theory (Hofbauer and Sandholm, 2005). In learning theory, the state variable of the dynamic is the history of play by opponents while in evolutionary game theory, the state variable is the current population state. In both fields, the dynamic is generated by agents playing a perturbed best response to the relevant state variable. The nature of the perturbation and the functional form of the dynamic is identical in both interpretations.

²In physics, continuity equations are used in the study of conserved quantities, such as bulk fluids. The continuity equation is a linear partial differential equation that describes the rate of change in the mass of fluid in any part of the medium through which the fluid is flowing. See, for example, Margenau and Murphy (1962).

³We note the difference between our situation and the situation in conventional evolutionary game theory and learning theory. In conventional evolutionary game theory, players update their pure strategies when they receive a new revision opportunity. In learning theory, players do update mixed strategies but in each round of the game, they are matched with the same opponent. In our model, since there is random matching in each round, opponents would be almost sure to vary in each round.

go to zero, we obtain a coupled pair of continuity equations, one for each population, with each population density function being functionally dependent on the mean of the density function of the other population.

We also examine symmetric games in which players from the same population are matched in pairs. In this case, we obtain a single (non-linear) continuity equation describing the change in the density function which characterizes the distribution of mixed strategies over agents in the population.

The general form of the continuity equations we derive depends on unspecified updating protocols, by means of which individual agents update their mixed strategies after playing the game. We proceed to consider one particular form of this general scenario—yielding the *replicator continuity equations*. We provide microfoundations for these equations using two alternative strategy updating rules. Both are based on the idea of reinforcement under which the currently used action becomes more probable in the next round if its payoff is high. In the theory of learning, the expected change in mixed strategy of an agent under both these rules is given by the classical replicator dynamic, which leads to the name we adopt for the specific form of the continuity equations generated by these rules.

We solve the replicator continuity equations using standard methods for solving continuity equations based on Liouville’s formula.⁴ The solution thus obtained is presented in Proposition 6.2. To characterize this solution explicitly requires us to derive an associated characteristic ODE system whose solutions describe trajectories of certain aggregate quantities associated to the population means. We call this ODE system the *Distributional Replicator dynamics*.⁵

We then use the distributional replicator dynamics to analyze the continuity equations for the simplest normal form games: generic 2×2 symmetric games and 2×2 asymmetric games. For 2×2 symmetric games, the key conclusion is that the evolution of the mean population strategy under the replicator continuity equation exhibits identical asymptotic behaviour to that of the pure strategy distribution under the classical replicator dynamic, provided the initial point in the latter case is identical to the initial mean strategy in the former case, though the time lines in the two cases may differ. However, this conclusion does not hold for 2×2 asymmetric games where we provide a counter-example in which the two dynamics converge to different pure equilibria. Since, even allowing for the use of mixed strategies, it is only the mean population strategy that can be the observed social state, this example illustrates our main finding: that expanding the behavioural flexibility of agents to allow use of mixed strategies in evolutionary contexts has real consequences, in that it can lead to radically different conclusions about the observed social state.

The literature on both evolutionary game theory and learning theory has grown to impressive proportions.⁶ There have, however, been relatively few attempts to explore the link between the

⁴This formula expresses the time evolution of the probability density function as a function of the initial probability density and the deterministic trajectories of the underlying ‘characteristic’ ODE system, which describes the motion of individual agents in the population – see, sections 2 and 5 below. The classical Liouville formula describes the change in volume along flow lines of an underlying dynamical system – see, for example, Hartman (1964). Related versions are discussed in Weibull (1995) and Hofbauer and Sigmund (1998).

⁵The solution obtained through Liouville’s formula is a function of the aggregate payoff obtained up to the present time by each strategy. The distributional replicator dynamics provide the solution trajectories of the aggregate payoff.

⁶See Weibull (1995), Hofbauer and Sigmund (1988, 1998), and Sandholm (2007) for book level studies of evolutionary game theory. Young (2005) is an excellent summary of the theoretical advances in the learning literature.

two fields and these have focused largely on the expected changes in players’ mixed strategies under particular learning algorithms. For example, Börgers and Sarin (1997) and Hopkins (2002) show that the expected motion under reinforcement learning in a learning environment is given by the replicator dynamic.⁷ This paper, by applying learning algorithms to the evolution of mixed strategies in population games, strengthens the link between these fields. In terms of methodology, the paper to which our work is most closely related is Ramsza and Seymour (2007), which takes a continuity equation approach to study the evolution of pure-strategy distributions in a population of agents using a version of the fictitious play algorithm. Instead of mixed strategies, their continuity equation tracks the evolution of a probability density over fictitious play updating weights.⁸ Their equation is also much simpler than those considered in this paper, with a solution that can be derived explicitly.⁹ The present paper establishes continuity equations on a rigorous foundation as a more general tool in the arsenal of evolutionary game theory.

The remainder of this paper is organized as follows. In section 2, we present an elementary discussion of continuity equations and their use adapted to a simple learning context. In section 3, we derive the general, non-linear continuity equations for 2 player asymmetric and symmetric games. Section 4 presents two updating rules that generate continuity equations based on the classical Replicator dynamics. In section 5, we introduce Liouville’s formula in a general context. We use this formula to solve a generalized form of the Replicator continuity equation in section 6. In section 7, we introduce the Distributional Replicator dynamics, a system of ODEs whose solutions determine solutions of the Replicator continuity equations. Sections 8 and 9 contain the analysis of 2×2 symmetric and asymmetric games respectively. Section 10 contains a discussion of the paper and concludes. Certain proofs and additional technical material are presented in the appendix.

2 Continuity Equations: General Discussion

Continuity equations are used widely in physics to study various mechanisms, collectively known as transport phenomenon, most simply of bulk materials such as fluids, and arise from an assumption that mass is conserved over time (‘what goes in must come out’). To gain an understanding of these equations and to provide intuition, we outline here the elementary ‘physics proof’ of the continuity equation (e.g. Margenau and Murphy, 1962) adapted to a simple learning theory context.

Consider a population of agents who interact with a fixed environment E . At any time, the environment can be in one of a number of states, $1, 2, \dots, m$. The state of the environment is determined by a stationary probability distribution over these states, which is not accessible to

⁷Some other papers that have explored the analogy between learning and biological evolution are Binmore and Samuelson (1997), Cabrales (2000), and Schlag (1998).

⁸As well as the first-order, continuity equation approximation, these authors also consider a higher order, diffusion approximation to fictitious play.

⁹The actual learning process referred to in Ramsza and Seymour (2007) is, however, more complex than the reinforcement type learning processes we use. In fictitious play, agents call on more substantial cognitive resources to keep track of their opponents’ past moves, and then to construct a best reply. In this paper, we consider agents who are boundedly rational in the sense that they respond only to their own current payoff information. Nevertheless, at the level of learning dynamics, this can lead to more complex trajectories, such as those described by the classical replicator dynamics.

the agents, who experience only the ‘state of the world’ at any given time. In response to this environment, agents can take one of a number of actions, $1, 2, \dots, n$. These actions do not influence the state of the environment, but provide the agents with feedback in the form of payoffs which depend on both the agent’s action and the state of the world. An agent chooses her action only on the basis of an individual mixed strategy which characterizes her likely behaviour. Adaptation to the environment over time consists of adjustments to this mixed strategy in the light of experience, according to some learning rule.

Mixed strategies are points in a simplex $\Delta \subset \mathbb{R}^n$, and we assume that the population of agents is characterized by a probability density function $p(x)$ over Δ , with $p(x)dV(x)$ denoting the probability that an agent’s mixed strategy lies in the infinitesimal volume element $dV(x)$ at x . We can think of $p(x)dV(x)$ as a ‘probability mass’, analogous to the fluid mass considered in a physics interpretation.

Consider a connected open subset $U \subset \text{int}\Delta$, with closed, smooth boundary ∂U . The total probability mass of agents using mixed strategies in U at time t is:

$$\int_U p(x, t)dV(x). \quad (1)$$

Hence, the rate of increase (which may be negative) of probability mass in U is given by

$$\frac{d}{dt} \int_U p dV = \int_U \frac{\partial p}{\partial t} dV. \quad (2)$$

Now consider the flow of probability mass into and out of U across the boundary ∂U . We assume that the individual learning process in the population is represented (in a continuous time limit) by a deterministic dynamics of the form $\dot{x} = L_E(x)$, where $L_E : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a smooth vector field. Thus, $L_E(x)\delta t$ represents the expected change in mixed strategy x in the small time interval δt in response to an agent’s interaction with the environment E . The flow of probability mass out of U is given by the vector field $(L_E \cdot u)p$ on ∂U , where u is the outward pointing unit normal to ∂U . Thus, the aggregate flow of probability mass *into* U across its boundary is

$$- \int_{\partial U} (L_E \cdot u)p dA, \quad (3)$$

where dA is the induced element of area on ∂U , and the negative sign indicates that the flow is into U (i.e. $-u$ is the inward pointing unit normal to ∂U).

The divergence theorem allows us to express (3) as an integral over U . The ‘compressibility’ of the probability mass flow at a point $x \in U$ is measured by the divergence of the flow $\nabla \cdot (L_E p)$.¹⁰ If the divergence at x is positive, then the flow expands from a small volume around x , whereas if the divergence is negative, the flow contracts. The divergence theorem allows us to express the net probability mass inflow in terms of the summed effects of all these infinitesimal expansions and

¹⁰Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a vector field. Then, the divergence of f is given by the scalar-valued function $\nabla \cdot f(x) = \sum_{i=1}^n \frac{\partial f_i(x)}{\partial x_i}$.

contractions, and states that (3) is equal to

$$-\int_U \nabla \cdot (L_E p) dV, \quad (4)$$

We assume that, on the relevant time scale, agents do not either enter or leave the population. Thus, the total probability mass of agents is conserved, and hence probability mass inside U can change only via transport across the boundary ∂U . It follows that, since (1) is the rate of accumulation of probability mass inside U , for mass balance this must be equal to (3), and hence to (4). That is, we obtain

$$\int_U \left(\frac{\partial p}{\partial t} + \nabla \cdot (L_E p) \right) dV = 0. \quad (5)$$

Since (5) holds for any arbitrary U inside Δ , the integrand in (5) must be equal to zero at every point in the interior of Δ . We therefore obtain the equation

$$\frac{\partial p}{\partial t} + \nabla \cdot (L_E p) = 0, \quad (6)$$

which governs the evolution of the probability density function $p(x)$, given the underlying deterministic learning dynamics represented by the vector field $L_E(x)$. This equation has the form of a classical continuity equation. It is a linear, first-order PDE in the density p . Given an initial density $p_0(x)$ at time $t = 0$, it may be solved to obtain the time evolution of the density, $p(x, t)$ for all $t \geq 0$.

This simple continuity equation arises because of the assumption that the environment E is represented by a fixed stationary process. Agents then behave under the learning dynamic like fluid particles, moving passively in parallel with each other, responding only to the environment. However, in the more complex scenarios we shall consider below, agents interact not only with a fixed environment, but with other agents, who are also learning. This latter fact leads to non-linearity in the resulting continuity equations.

3 The General Continuity Equation for Population Games

We derive the continuity equations in the setting of population games. First, we consider the case in which two players, each chosen from a separate population, are randomly matched to play an asymmetric game.¹¹ Next, we look separately at the case where two players chosen from the same population are randomly matched to play a symmetric game.

3.1 Two-population Asymmetric Games

Consider a society consisting of the set of populations $\mathcal{P} = \{1, 2\}$. We assume both populations are of fixed probability mass 1. Let S_l be the strategy set and n_l be the number of strategies of

¹¹We confine ourselves to two-player asymmetric games merely for notational convenience. All the ideas involved can be easily extended to multipopulation asymmetric games at the cost of more cumbersome notation.

population $l \in \mathcal{P}$. We denote by Δ_l the simplex corresponding to population l . Thus,

$$\Delta_l = \left\{ x \in \mathbb{R}_+^{n_l} : \sum_{i \in S_l} x_i = 1 \right\}. \quad (7)$$

A mixed strategy used by a player in population l belongs to Δ_l . We will use x and y to denote a typical mixed strategy of a player in populations 1 and 2 respectively. Then $\Delta = \Delta_1 \times \Delta_2$ is the set of mixed strategies of pairs of players, one from each population.

Let $p : \Delta_1 \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the time track of the probability density function over the space of mixed strategies for population 1. Let $A \subseteq \Delta_1$ be a measurable set. Then,

$$p(A, t) = \int_A p(x, t) dV(x) \quad (8)$$

is the proportion of agents in population 1 playing mixed strategies in A at time $t \geq 0$, where $dV(x)$ denotes the volume element at $x \in \Delta_1$. The mean strategy in population 1 at time t , $\langle x \rangle_t \in \Delta_1$, is given by

$$\langle x \rangle_t = \int_{\Delta_1} xp(x, t) dV(x). \quad (9)$$

Similarly, we define $q(y, t)$ as the probability density function over the space of mixed strategies for population 2. The mean strategy $\langle y \rangle_t \in \Delta_2$ for population 2 is defined analogously to (9).

Two players, one from each population, are randomly matched to play an asymmetric normal form game. We assume that players from population 1 play the role of the row player while those from population 2 are column players. If the row player plays action $i \in S_1$ and the column player plays action $j \in S_2$, the payoff to the row player is u_{ij} and to the column player is v_{ji} . The expected payoff to $i \in S_1$ against mixed strategy $y \in \Delta_2$ is $(Uy)_i$, where U is the $n_1 \times n_2$ payoff matrix (u_{ij}) . Similarly, the payoff to $j \in S_2$ against mixed strategy $x \in \Delta_1$ is $(Vx)_j$, where V is the $n_2 \times n_1$ payoff matrix (v_{ji}) . Thus

$$(Uy)_i = \sum_{j \in S_2} u_{ij}y_j, \quad (10)$$

$$(Vx)_j = \sum_{i \in S_1} v_{ji}x_i. \quad (11)$$

Our objective is to track the evolution of the two density function $p(x, t)$ and $q(y, t)$ over time. We derive the continuity equation for this purpose as follows.

Suppose the two chosen players use the mixed strategy profile $(x, y) \in \Delta$. The probability that they play the action profile $(i, j) \in S = S_1 \times S_2$ is given by

$$\pi_{ij}(x, y) = x_i y_j. \quad (12)$$

Of course, $\sum_{i,j} \pi_{ij}(x, y) = 1$ for all (x, y) ¹². After a play of the game, a player updates his mixed

¹²In what follows in this section we do not assume that $\pi_{ij}(x, y)$ is necessarily given by the uncorrelated expression (12). More generally, we could assume that $\pi_{ij}(x, y) = \pi_i^1(x, y)\pi_j^2(x, y)$, where $\pi_r^l(x, y)$ is the probability that player- l chooses action r , given that the players use mixed strategies x and y . This dependency on an opponent's strategy

strategy in the following manner. Given that the action profile (i, j) has been played, the row player updates his strategy $x \in \Delta_1$ to x' given by an updating rule of the form:

$$x' = x + \tau f_{ij}(x, y), \quad (13)$$

where τ is a small time parameter representing the length of a round in which the game is played. Similarly, the column player updates her strategy $y \in \Delta_2$ to y' given by an updating rule of the form:

$$y' = y + \tau g_{ij}(x, y). \quad (14)$$

Thus, f_{ij} and g_{ij} are functions,¹³ $f_{ij} : \Delta \rightarrow \mathbb{R}_0^{n_1}$ and $g_{ij} : \Delta \rightarrow \mathbb{R}_0^{n_2}$, where $\mathbb{R}_0^n = \{z \in \mathbb{R}^n : \sum_i z_i = 0\}$. We call these the *forward state change* functions: they specify how the players' states change going forward in time and therefore are rules to update the mixed strategies x and y respectively.

The associated *backward state change* functions specify where current states came from, going backward in time. Thus the backward state changes are functions $b_{ij} : \Delta \rightarrow \mathbb{R}_0^{n_1}$ and $c_{ij} : \Delta \rightarrow \mathbb{R}_0^{n_2}$ which satisfy:

$$(x, y) = (u + \tau f_{ij}(u, v), v + \tau g_{ij}(u, v)) \iff (u, v) = (x - \tau b_{ij}(x, y), y - \tau c_{ij}(x, y)). \quad (15)$$

Between times t and $t + \tau$, the two density functions make the transition from $p(x, t)$ and $q(y, t)$ to $p(x, t + \tau)$ and $q(y, t + \tau)$ respectively. The relationships between the density functions at the two time periods are given by

$$p(x, t + \tau)dV(x) = \sum_{i,j \in S} \int_{z \in \Delta_2} [\pi_{ij}(\cdot, z)p(\cdot, t)dV(\cdot)](x - \tau b_{ij}(x, z))q(z, t)dV(z), \quad (16)$$

$$q(y, t + \tau)dV(y) = \sum_{i,j \in S} \int_{w \in \Delta_1} [\pi_{ij}(z, \cdot)q(\cdot, t)dV(\cdot)](y - \tau c_{ij}(w, y))p(w, t)dV(w). \quad (17)$$

In order to derive the continuity equations, we multiply (16) and (17) by smooth 'test functions' $\phi(x)$ and $\psi(y)$ respectively, and then integrate. We therefore obtain

$$\langle \phi \rangle_{t+\tau} = \sum_{i,j \in S} \int_{y \in \Delta_2} \int_{x \in \Delta_1} \phi(x) [\pi_{ij}(\cdot, y)p(\cdot, t)dV(\cdot)](x - \tau b_{ij}(x, y))q(y, t)dV(y) \quad (18)$$

$$\langle \psi \rangle_{t+\tau} = \sum_{i,j \in S} \int_{x \in \Delta_1} \int_{y \in \Delta_2} \psi(y) [\pi_{ij}(x, \cdot)q(\cdot, t)dV(\cdot)](y - c_{ij}(x, y))p(x, t)dV(x). \quad (19)$$

We now focus on (18) to obtain the continuity equation for population 1. The equation for population 2 may be derived analogously. Making the change of notation $x - \tau b_{ij}(x, y) \rightarrow x$ (for each could arise, for example, if the players exchange pre-play signals that convey some information about the opponent's state.

¹³In this general discussion, we retain y as an argument of f_{ij} to account for the possibility that the row player may have some pre-play information about the mixed strategy that will be used by his column player opponent - cf footnote 12. The two forward state change vectors we cite in the next section are examples in which players do not possess any such information about an opponent's strategy. In these cases, f_{ij} is a function only of the player's own mixed strategy x . A similar point applies to g_{ij} .

where we have written

$$\mathcal{F}(x | q_t) = \langle \hat{f}(x, \cdot) \rangle_t = \int_{\Delta_2} \hat{f}(x, y) q(y, t) dV(y) = \sum_{i, j \in S} \int_{\Delta_2} f_{ij}(x, y) \pi_{ij}(x, y) q(y, t) dV(y). \quad (23)$$

We therefore obtain

$$\int_{\Delta_1} \phi(x) \left\{ \frac{\partial p(x, t)}{\partial t} + \nabla \cdot [\mathcal{F}(x | q_t) p(x, t)] \right\} dV(x) = 0.$$

Since this holds for all differentiable test functions $\phi(x)$ which vanish on $\partial\Delta_1$, we obtain the differential form of the continuity equation:

$$\frac{\partial p_t(x)}{\partial t} + \nabla \cdot [\mathcal{F}(x | q_t) p_t(x)] = 0, \quad x \in \text{int } \Delta_1, t > 0, \quad (24)$$

where we have now written $p_t(x)$ for $p(x, t)$.

A similar derivation gives the continuity equation for $q(y, t)$ in the form analogous to (22):

$$\int_{y \in \Delta_2} \psi(y) \frac{\partial q(y, t)}{\partial t} dV(y) = \int_{x \in \Delta_1} \int_{y \in \Delta_2} \nabla \psi(y) \cdot \hat{g}(x, y) p(x, t) q(y, t) dV(x) dV(y), \quad (25)$$

where $\hat{g}(x, y)$ is defined as in (21). We then obtain the form analogous to (24):

$$\frac{\partial q_t(y)}{\partial t} + \nabla \cdot [\mathcal{G}(y | p_t) q_t(y)] = 0, \quad y \in \text{int } \Delta_2, t > 0, \quad (26)$$

where

$$\mathcal{G}(y | p_t) = \langle \hat{g}(\cdot, y) \rangle_t = \int_{\Delta_1} \hat{g}(x, y) p(x, t) dV(x) = \sum_{i, j \in S} \int_{\Delta_1} g_{ij}(x, y) \pi_{ij}(x, y) p(x, t) dV(x). \quad (27)$$

Note that, given q_t , the form (24) is linear in p_t , and given p_t , the form (26) is linear in q_t . However, taken together, this pair of equations is a coupled non-linear system.

Equations (24) and (26) form the system of partial differential equations that describes the evolution of the density functions $p_t(x)$ and $q_t(y)$. Intuitively, as in the discussion in section 2, $\mathcal{F}(x | q_t)$ represents the adaptation ‘velocity’ of mixed strategy x .¹⁵ That is, $\mathcal{F}(x | q_t)\tau$ is the expected change in mixed strategy x in the small time interval τ in response to a play of the game. Since the mass of x is represented by $p_t(x)$, $\mathcal{F}(x | q_t)p_t(x)$ gives the probability mass flow of x . The divergence of $\mathcal{F}(x | q_t)p_t(x)$ therefore gives the rate at which the probability mass in a small neighbourhood of x is expanding or contracting. Since $\frac{\partial p_t(x)}{\partial t}$ is precisely the rate of change of the probability mass of x , we are led to the continuity equation (24).

¹⁵In the next section, we provide two strategy updating rules in which this velocity is given by the replicator dynamic.

y) and using (15), we obtain

$$\langle \phi \rangle_{t+\tau} = \sum_{i,j \in S} \int_{x \in \Delta_1} \int_{y \in \Delta_2} \phi(x + \tau f_{ij}(x, y)) \pi_{ij}(x, y) p(x, t) q(y, t) dV(x) dV(y).$$

Now Taylor expand the $\phi(\cdot)$ term up to terms of order τ :

$$\langle \phi \rangle_{t+\tau} = \sum_{i,j \in S} \int_{x \in \Delta_1} \int_{y \in \Delta_2} \{\phi(x) + \tau \nabla \phi(x) \cdot f_{ij}(x, y)\} \pi_{ij}(x, y) p(x, t) dV(x) q(y, t) dV(y).$$

Noting that $\sum_{i,j} \pi_{ij}(x, y) = 1$, and $\int_{\Delta_2} q(y, t) dV(y) = 1$, this can be written in the form:

$$\begin{aligned} \int_{x \in \Delta_1} \phi(x) \frac{1}{\tau} \{p(x, t + \tau) - p(x, t)\} dV(x) \\ &= \sum_{i,j \in S} \int_{x \in \Delta_1} \int_{y \in \Delta_2} \nabla \phi(x) \cdot f_{ij}(x, y) \pi_{ij}(x, y) p(x, t) q(y, t) dV(x) dV(y) \\ &= \int_{x \in \Delta_1} \int_{y \in \Delta_2} \nabla \phi(x) \cdot \hat{f}(x, y) p(x, t) q(y, t) dV(x) dV(y), \end{aligned} \quad (20)$$

where we have written

$$\hat{f}(x, y) = \sum_{i,j \in S} f_{ij}(x, y) \pi_{ij}(x, y). \quad (21)$$

This is the expected forward change of state vector for player 1, given that the players' pre-play states are (x, y) .

Taking the limit as $\tau \rightarrow 0$ in (20) therefore gives:

$$\int_{x \in \Delta_1} \phi(x) \frac{\partial p(x, t)}{\partial t} dV(x) = \int_{x \in \Delta_1} \int_{y \in \Delta_2} \nabla \phi(x) \cdot \hat{f}(x, y) p(x, t) q(y, t) dV(x) dV(y). \quad (22)$$

Finally, assume that $\phi(x) = 0$ for $x \in \partial \Delta_1$, and integrate by parts on the right-hand side to obtain¹⁴:

$$\begin{aligned} \int_{x \in \Delta_1} \phi(x) \frac{\partial p(x, t)}{\partial t} dV(x) &= - \int_{x \in \Delta_1} \int_{y \in \Delta_2} \phi(x) \nabla \cdot [\hat{f}(x, y) p(x, t)] dV(x) q(y, t) dV(y) \\ &= - \int_{\Delta_1} \phi(x) \nabla \cdot [\mathcal{F}(x | q_t) p(x, t)] dV(x), \end{aligned}$$

¹⁴The formal argument has the following form. For X a vector field on the domain Δ , we use the identity $\nabla \cdot [\phi X] = \phi \nabla \cdot X + \nabla \phi \cdot X$ to obtain

$$\int_{\Delta} \nabla \phi \cdot X dV = \int_{\Delta} \nabla \cdot [\phi X] dV - \int_{\Delta} \phi \nabla \cdot X dV$$

Now use the divergence theorem together with the assumption that $\phi = 0$ on $\partial \Delta$ to obtain:

$$\int_{\Delta} \nabla \cdot [\phi X] dV = \int_{\partial \Delta} (\phi \cdot X) dA = 0.$$

[cf. equation (3).]

3.2 Symmetric Games

We now consider a symmetric game with players chosen from a single population.¹⁶ We denote by $S = \{1, 2, \dots, n\}$ the set of actions in the game. The set of mixed strategies is the n -simplex $\Delta = \{x \in \mathbb{R}_+^n : \sum_{i \in S} x_i = 1\}$. The derivation of the continuity equation for a one population symmetric game now proceeds analogously to the asymmetric case. The only difference is that instead of two probability densities, we need only track the evolution of a single density. The microfoundations of the continuity equation are the same as in the asymmetric case. Players are matched in pairs at each time interval to play the game, although in this case, both players in a pair are from the same population. The event that player 1 uses mixed strategy x and player 2 uses mixed strategy y occurs with probability $p(x, t)p(y, t)dV(x)dV(y)$, where $p(x, t)$ is the probability density of players using x in the population at time t . Thus, given that the chosen players use strategies $x, y \in \Delta$, the probability, $\pi_{ij}(x, y)$, that they play the pair of pure strategies $i, j \in S$ is given by (12). As in the asymmetric case, players update their mixed strategies using a rule of the form (13).

The updating equation corresponding to (16) is:

$$p(x, t + \tau)dV(x) = \sum_{i, j \in S} \int_{y \in \Delta} [\pi_{ij}(\cdot)p(\cdot, t)dV(\cdot)] (x - \tau b_{ij}(x, y)) p(y, t)dV(y). \quad (28)$$

The difference between (16) and (28) is that the $q(y, t)$ term in (16) is replaced by $p(y, t)$. The derivation now proceeds as in the asymmetric case to obtain the symmetric form corresponding to (22):

$$\int_{x \in \Delta} \phi(x) \frac{\partial p(x, t)}{\partial t} dV(x) = \int_{x \in \Delta} \int_{y \in \Delta} \nabla \phi(x) \cdot \hat{f}(x, y) p(x, t) p(y, t) dV(x) dV(y), \quad (29)$$

with $\phi(x)$ a smooth test function, and $\hat{f}(x, y)$ given by (21). From this, we derive the symmetric analogue of (24):

$$\frac{\partial p_t(x)}{\partial t} + \nabla \cdot [\mathcal{F}(x | p_t) p_t(x)] = 0, \quad x \in \text{int } \Delta, t > 0, \quad (30)$$

where now

$$\mathcal{F}(x | p_t) = \langle \hat{f}(x, \cdot) \rangle_t = \int_{\Delta} \hat{f}(x, y) p(y, t) dV(y) = \sum_{i, j \in S} \int_{\Delta} f_{ij}(x, y) \pi_{ij}(x, y) p(y, t) dV(y). \quad (31)$$

Note that the form (30) is non-linear in p_t .

4 Replicator Continuity Equations

Equations (24) and (26) give the general form of the continuity equations for 2-population, asymmetric games. In this section, we derive a particular form of the continuity equations—the *replicator continuity equations*. We first introduce two alternative forward state change rules $f_{ij}(x)$ and $g_{ij}(y)$.

¹⁶To describe the symmetric case, we adopt the notation of population 1 of the asymmetric case but drop the population subscript.

Both these rules are based on the idea of reinforcement. We then show that these updating rules lead to the replicator continuity equations. These updating rules therefore provide the microfoundations to the replicator continuity dynamic.

Reinforcement models have been widely studied in the learning literature. A group of players, one in each role in the game, employ mixed strategies in each round of a game. Reinforcement models are based on the idea that if the action currently employed obtains a high payoff, then the probability assigned to it increases in the next round of play. Reinforcement models are therefore extremely naive models of learning. Agents mechanically respond to stimuli from their environment without seeking to create any model of the situation or strategically evaluate how they are doing. Hence, they do not seek to exploit the pattern of opponents' past play and predict the future behaviour of their opponents.¹⁷ In this sense, agents are boundedly rational.

The two forward state change rules we consider are described below. We consider a player in a 2-player game who employs strategy $x \in \Delta$, uses action i and encounters an opponent who uses action j in the current round. The player then updates her strategy to x' according to an updating rule $f_{ij}(x)$, as in (13). For brevity, we present only the rules for population 1. For population 2, the updated strategy y' and the updating vector $g_{ij}(y)$ take analogous forms, as in (14).

In enumerating the two rules, we need to assume that all payoffs are positive for Rule 1 and negative for Rule 2 in order to ensure that all probabilities x'_r are less than 1.¹⁸ Since it is always possible to rescale payoffs to make them all positive or negative without affecting incentives, we do not consider this a severe restriction.

1. This rule is from Börgers and Sarin (1997) and is a special case of a general class of reinforcement rules introduced in Börgers and Sarin (2000). Under this rule, the mixed strategy x' and the forward state change vector take the form

$$x'_r = \delta_{ir} u_{ij} \tau + (1 - u_{ij} \tau) x_r, \quad (32)$$

$$f_{ij,r}(x) = (\delta_{ir} - x_r) u_{ij}. \quad (33)$$

For τ small enough, a sufficient condition for (32) to represent an updating rule is $u_{ij} > 0$, for all $i, j \in S$.

The general class of rules in Börgers and Sarin (2000) is based on the idea of *aspiration*. To explain this rule, let us momentarily set $\tau = 1$. Suppose that at round t of play, a player aspires to a payoff of a_t . The probability of playing a strategy $r \neq i$ is then $x'_r = x_r + (a_t - u_{ij}) x_r$.¹⁹ Hence, if $u_{ij} > a_t$, then action i gets reinforced. By setting a_t identically equal to zero, we obtain (32). Note that in this case, the current action i is always reinforced.

¹⁷Börgers and Sarin (1997) provide some justification of why agents respond to very limited information in these models—only their own payoffs. They argue that the acquisition or processing of new information may be too costly relative to benefits. Hence, they say, reinforcement models may be more plausible if agents' behaviour is habitual rather than the result of careful reflection.

¹⁸For large τ , we would also need to assume that the payoffs are less than 1 (more than -1) for Rule 1 (Rule 2) to ensure that the probabilities are positive. Since we are primarily concerned with the case where τ is arbitrarily small, we dispense with this restriction.

¹⁹For the moment, we are ignoring the requirement of imposing restrictions on a_t and u_{ij} such that the probability x'_r actually makes sense

2. We now consider a revision rule which applies when all payoffs u_{ij} are negative. The updated strategy and the state change rule we consider is

$$x'_r = x_r + \tau u_{rj} x_r, \quad \text{for } r \neq i, \quad (34)$$

$$f_{ij,r}(x) = u_{rj} x_r, \quad \text{for } r \neq i, \quad (35)$$

with the residual probability being allotted to i . For small τ , it is sufficient to assume that u_{rj} is negative for x' to be a probability distribution.

Revision rule (34) has a similar interpretation to (32). We interpret the negative payoffs as costs that the consumer incurs. Suppose a_t is the maximum (non-negative) cost that the consumer is willing to incur in period t . The probability of playing $r \neq i$ in the next round is given by²⁰ $x'_r = x_r + (u_{rj} - a_t)x_r$. Action i is therefore reinforced if $u_{rj} < a_t$, for all $r \neq i$. By setting a_t identically equal to zero, we obtain (34) and ensure that the current action i is always reinforced when all payoffs are negative.

In the present context, we may use the form (12) to write (23) and (27) as

$$\mathcal{F}(x | q) = \sum_{i,j \in S} x_i f_{ij}(x) \langle y_j \rangle, \quad (36)$$

$$\mathcal{G}(y | p) = \sum_{i,j \in S} \langle x_i \rangle g_{ij}(y) y_j, \quad (37)$$

where $\langle x_i \rangle$ is the expected value of x_i with respect to $p(x)$, and $\langle y_j \rangle$ is the expected value of y_j with respect to $q(y)$.

Recalling the notation of (10) and (11), we introduce the following operators

$$R_i^1(x) y = x_i \{ (Uy)_i - x \cdot Uy \}, \quad (38)$$

$$R_j^2(y) x = y_j \{ (Vx)_j - y \cdot Vx \}. \quad (39)$$

Clearly, the vector field generated by the bimatrix replicator dynamic on $\Delta = \Delta_1 \times \Delta_2$ at (x, y) is identical to the vector field generated by the two operators in (38) and (39). Hence, we call the $n_1 \times n_2$ matrix operator $R^1(x)$ and the $n_2 \times n_1$ matrix operator $R^2(y)$, the *Replicator operators* for the two populations.

We now establish that the two updating rules described above generate the replicator operators for the two populations.

Lemma 4.1 *For each of the updating protocols enumerated earlier in this section, $\mathcal{F}(x | q) = R^1(x) \langle y \rangle$ and $\mathcal{G}(y | p) = R^2(y) \langle x \rangle$.*

Proof. We prove the result only for Rule 1 for population 1. The proof for Rule 2 is similar.

²⁰We once again momentarily set $\tau = 1$ and ignore any restriction we need to put on a_t for x' to be a probability distribution. We also temporarily drop the assumption that the u_{rj} are negative.

We show that for $f_{ij}(x)$ given by (33), $\mathcal{F}_r(x | q) = R_r^1(x)\langle y \rangle$, for $r \in S_1$. From (36) we have

$$\begin{aligned}
\mathcal{F}_r(x | q) &= \sum_{i,j \in S} x_i f_{ij,r}(x)\langle y_j \rangle \\
&= \sum_{i,j \in S} x_i \langle y_j \rangle (\delta_{ir} - x_r) u_{ij} \\
&= x_r \left(\sum_{j \in S_2} u_{rj} \langle y_j \rangle - \sum_{i,j \in S} x_i u_{ij} \langle y_j \rangle \right) \\
&= x_r \{ (U\langle y \rangle)_r - x \cdot U\langle y \rangle \} \\
&= R_r^1(x)\langle y \rangle.
\end{aligned}$$

The proof for population 2 and $\mathcal{G}(y | p)$ is similar. ■

The following proposition is now immediate.

Proposition 4.2 *Under the forward state change rules (33) and (35), the continuity equations (24) and (26) are given by*

$$\frac{\partial p_t(x)}{\partial t} + \nabla \cdot [p_t(x) R^1(x) \langle y \rangle_t] = 0, \quad (40)$$

$$\frac{\partial q_t(y)}{\partial t} + \nabla \cdot [q_t(y) R^2(y) \langle x \rangle_t] = 0, \quad (41)$$

where $\langle x \rangle_t = \int_{\Delta_1} xp(x, t) dV(x)$ and $\langle y \rangle_t = \int_{\Delta_2} yq(y, t) dV(y)$.

We call (40) and (41) the *Replicator continuity equations*.

In a similar way, we obtain the Replicator continuity equation for a single population, symmetric game. Let $f_{ij}(x)$ be the mixed strategy rule in a symmetric game where $f_{ij}(x)$ can take the form in (33) or (35). We write $R = R^1$, as in (38), for the Replicator operator in the symmetric case.

Corollary 4.3 *Let $p(x, t)$ be the density function over mixed strategies in a symmetric game. Then, under each of the updating protocols (33)-(35), the continuity equation (30) is given by*

$$\frac{\partial p_t(x)}{\partial t} + \nabla \cdot [p_t(x) R(x) \langle x \rangle_t] = 0, \quad (42)$$

where $\langle x \rangle_t = \int_{\Delta} xp(x, t) dV(x)$.

5 Solution of the general continuity equation: Liouville's Formula

Our approach to solving the non-linear continuity equations we have constructed is to begin by solving a different, but related problem. Thus, instead of confronting the non-linearities directly, we first consider a linear continuity equation, but one defined by an explicitly time-dependent vector field. We will later show how a solution of the non-linear continuity equations of interest can be constructed from explicit solutions of linear continuity equations of this type.

5.1 Liouville's formula

Let $X = X(x, t) \in \mathbb{R}^n$ be a (possibly time-dependent) smooth vector field defined for x in a state space $\Omega \subset \mathbb{R}^n$ (a connected, open domain with piecewise smooth boundary), and suppose that $p(x, t)$ is a probability density on Ω satisfying the linear continuity equation

$$\frac{\partial p(x, t)}{\partial t} + \nabla \cdot [pX](x, t) = 0, \quad \text{with } p(x, 0) = p_0(x), \quad (43)$$

where $p_0(x)$ is a specified initial density. The solution to this initial-value problem is well-known, and given by *Liouville's formula*, which may be described as follows.

We first introduce some notation to describe the solution trajectories to the (non-autonomous) differential equation defined by X ,

$$\frac{dx}{dt} = X(x, t). \quad (44)$$

Let $x_{t_0, t}(x), t \in \mathbb{R}$, denote the solution trajectory to (44) that passes through the point x at time t_0 . Thus, the trajectory that passes through x at time t starts at the point $x_{-t}(x) = x_{t, 0}(x)$ when $t = 0$ ²¹. After time $s \geq 0$, this trajectory has reached the point $x_{t, s}(x) = x_{0, s}(x_{t, 0}(x))$. In particular, $x_{t, t}(x) = x_{0, t}(x_{t, 0}(x)) = x$, and by definition $x_{t, 0}(x_{0, t}(x)) = x$.

We can now write down the solution to the initial value problem (43):

$$p(x, t) = p_0(x_{t, 0}(x)) \exp \left\{ - \int_0^t [\nabla \cdot X](x_{t, s}(x), s) ds \right\}. \quad (45)$$

This is Liouville's formula. A proof is given in Appendix A.1.

5.2 Expected values

Liouville's formula for the density $p(x, t)$ allows us to calculate expected values of associated variables in terms of the initial density $p_0(x)$ and solutions of the characteristic system (44). Thus, for a function $\phi \in L_2(\Omega)$, define its expected value with respect to the probability density $p(x, t)$ satisfying (43) by:

$$\langle \phi \rangle_t = \int_{\Omega} \phi(x) p(x, t) dV(x). \quad (46)$$

Then we have:

Proposition 5.1 *The expected value $\langle \phi \rangle_t$ may be expressed in the form:*

$$\langle \phi \rangle_t = \int_{\Omega} \phi(x_{0, t}(x)) p_0(x) dV(x). \quad (47)$$

A proof is given in Appendix A.2.

As an example of the use of (47), the following Corollary shows that the trajectories of the underlying characteristic dynamics (44) may be recovered as solutions of the continuity equation (43) for initial conditions which are mass points.

²¹Note that the situation for a non-autonomous vector field is more complicated than for the more familiar autonomous case. This is because the explicit time dependence of $X(x, t)$ imposes an absolute, rather than a relative, time-scale on the dynamics. In particular, the initial time $t = 0$ is exogenously determined.

Corollary 5.2 *Suppose $p_0(x) = \delta(x - x_0)$ for some $x_0 \in \Omega$. Then $p(x, t) = \delta(x - x_{0,t}(x_0))$ for all $t \geq 0$, where $x_{0,t}(x_0)$ is the solution trajectory of the characteristic equations $\dot{x} = X(x, t)$ with initial condition x_0 .*

Proof. If $p_0(x) = \delta(x - x_0)$ in (43), it follows from (47) that $\langle \phi \rangle_t = \phi(x_{0,t}(x_0))$. Since this is true for any continuous function $\phi(x)$, it follows from (46) that $p(x, t) = \delta(x - x_{0,t}(x_0))$. ■

6 Solution of the Replicator Continuity Equation

In this section, we use Liouville’s formula (45) to lay the foundations for a solution to the pair of coupled continuity equations (40) and (41). To do this, we first “freeze” population 2 in the following sense. Suppose that mixed strategies are distributed over agents in population 2 by a fixed, time dependent probability density $q(y, t)$ that is independent of any process in population 1. This density determines a mean history, $y(t) = \langle y \rangle_t \in \Delta_2$, which determines the evolution of the density function $p(x, t)$ for population 1 via the continuity equation (40). In effect, this is a generalization of the scenario considered in section 2, in the sense that it replaces the population 2 of responsive agents by a non-stationary environment with which agents in population 1 interact, and whose behaviour is determined by the fixed, but now non-stationary process $q(y, t)$. To reclaim a version of the situation described in section 2, we need only assume that $q(y, t) = q(y)$ is stationary.

The outcome of this “freezing” process is that we can consider the population 1 continuity equation (40) as decoupled from (41). In the next section, we shall recover this coupling by considering a simultaneous “freezing” procedure for both populations.

We have defined the Replicator operators $R^1(x) : \Delta_2 \rightarrow \mathbb{R}^{n_1}$ in (38). Suppose given a specified history $y(t) \in \Delta_2$, as described above. We associate a *pseudo Replicator dynamics* to this trajectory, whose solutions specify the time-development of row-player responses to this history. This takes the form of the explicitly time-dependent dynamical system

$$\dot{x}_i = R_i^1(x)y(t) = x_i(e_i^1 - x) \cdot Uy(t), \quad (48)$$

where $e_i^1 \in \mathbb{R}^{n_1}$ is the i -th standard basis vector. This is an explicitly time-dependent dynamical system, which we consider as the characteristic ODE (44) for a general continuity equation (43). To solve this continuity equation, we begin by solving the characteristic system (48). We can then find the solution of any associated initial value problem of the form (43) by means of Liouville’s formula (45).

6.1 Solution of the pseudo Replicator dynamics

Write $c(t) = Uy(t) \in \mathbb{R}^{n_1}$, a time-dependent vector-payoff stream to row players. Then the pseudo-Replicator equations (48) can be written as:

$$\frac{dx_i}{dt} = x_i(e_i^1 - x) \cdot c(t). \quad (49)$$

Write

$$C(t) = \int_0^t c(s) ds. \quad (50)$$

Then we can express the solutions of (49) as follows.

Proposition 6.1 *The solution trajectory of the pseudo-Replicator dynamics (49) passing through $x \in \Delta_1$ at time $t = t_0$ is:*

$$x_{t_0,t}(x)_i = \frac{x_i e^{C_i(t,t_0)}}{x \cdot e^{C(t,t_0)}}, \quad (51)$$

where $C(t, t_0) = C(t) - C(t_0)$. In particular:

$$x_{0,t}(x)_i = \frac{x_i e^{C_i(t)}}{x \cdot e^{C(t)}}, \quad \text{and} \quad x_{t,0}(x)_i = \frac{x_i e^{-C_i(t)}}{x \cdot e^{-C(t)}}. \quad (52)$$

Proof. With $x_{t_0,t}(x)$ given by (51), we have:

$$\begin{aligned} \frac{d}{dt} [x_{t_0,t}(x)_i] &= \frac{x_i c_i(t) e^{C_i(t,t_0)}}{x \cdot e^{C(t,t_0)}} - \frac{x_i e^{C_i(t,t_0)}}{(x \cdot e^{C(t,t_0)})^2} \sum_{j=1}^n x_j c_j(t) e^{C_j(t,t_0)} \\ &= x_{t_0,t}(x)_i c_i(t) - x_{t_0,t}(x)_i \sum_{j=1}^n c_j(t) \frac{x_j e^{C_j(t,t_0)}}{x \cdot e^{C(t,t_0)}} \\ &= x_{t_0,t}(x)_i \left\{ c_i(t) - \sum_{j=1}^n c_j(t) x_{t_0,t}(x)_j \right\} \\ &= x_{t_0,t}(x)_i \{ e_i - x_{t_0,t}(x) \} \cdot c(t), \end{aligned}$$

which shows that $x_{t_0,t}(x)$ is a solution of (49). Since $C(t_0, t_0) = 0$, it follows from the definition (51) that $x_{t_0,t_0}(x) = x$, as required. ■

6.2 Solution of the Replicator continuity equation

We now use Liouville's formula (45), together with Proposition 6.1, to compute the solution to the Replicator continuity equation associated with a pseudo-Replicator vector field of the form (49). This is given in the following proposition, proved in Appendix A.3.

Proposition 6.2 *The solution of the initial value problem (43) associated to the characteristic vector field (49) is:*

$$p(x, t) = p_0 \left(\frac{x e^{-C(t)}}{x \cdot e^{-C(t)}} \right) \left(\frac{1}{x \cdot e^{-C(t)}} \right)^{n_1} \exp \{ -e^1 \cdot C(t) \}, \quad (53)$$

where $C(t) \in \mathbb{R}^{n_1}$ is given by (50).

We may also obtain the expected value of a function $\phi \in L_2(\Delta_1)$ from (47) and (52):

$$\langle \phi \rangle_t = \int_{\Delta_1} \phi \left(\frac{\xi e^{C(t)}}{\xi \cdot e^{C(t)}} \right) p_0(\xi) dV(\xi). \quad (54)$$

Corollary 6.3 *Suppose there exists an i such that $[C_i(t) - C_j(t)] \rightarrow \infty$ as $t \rightarrow \infty$ for all $j \neq i$, and that the i -th face, $\partial\Delta_1^{(i)} = \{x \in \Delta_1 : x_i = 0\}$, has measure zero with respect to $p_0(x)$. Then $p(x, t) \rightarrow \delta(x - e_i^1)$ as $t \rightarrow \infty$.*

Proof. For $\xi \in \Delta_1 \setminus \partial\Delta_1^{(i)}$, we have $\xi_i > 0$. Thus:

$$\frac{\xi_i e^{C_i(t)}}{\xi \cdot e^{C(t)}} = \frac{\xi_i e^{C_i(t)}}{\xi_i e^{C_i(t)} + \sum_{j \neq i} \xi_j e^{C_j(t)}} = \frac{\xi_i}{\xi_i + \sum_{j \neq i} \xi_j e^{-[C_i(t) - C_j(t)]}} \rightarrow \frac{\xi_i}{\xi_i} = 1 \quad \text{as } t \rightarrow \infty.$$

For $k \neq i$, we have:

$$\frac{\xi_k e^{C_k(t)}}{\xi \cdot e^{C(t)}} = \frac{\xi_k e^{C_k(t)}}{\xi_i e^{C_i(t)} + \sum_{j \neq i} \xi_j e^{C_j(t)}} = \frac{\xi_k e^{-[C_i(t) - C_k(t)]}}{\xi_i + \sum_{j \neq i} \xi_j e^{-[C_i(t) - C_j(t)]}} \rightarrow \frac{0}{\xi_i} = 0 \quad \text{as } t \rightarrow \infty.$$

Hence,

$$\frac{\xi e^{C(t)}}{\xi \cdot e^{C(t)}} \rightarrow e_i^1 \quad \text{as } t \rightarrow \infty, \text{ for any } \xi \in \Delta_1 \setminus \partial\Delta_1^{(i)}.$$

Since $\partial\Delta_1^{(i)}$ has zero probability mass with respect to p_0 , we have:

$$\langle \phi \rangle_t = \int_{\Delta_1 \setminus \partial\Delta_1^{(i)}} \phi \left(\frac{\xi e^{C(t)}}{\xi \cdot e^{C(t)}} \right) p_0(\xi) dV(\xi) \rightarrow \phi(e_i^1) \int_{\Delta_1} p_0(\xi) dV(\xi) = \phi(e_i^1) \quad \text{as } t \rightarrow \infty,$$

for any continuous function $\phi(x)$. It follows that $p(x, t) \rightarrow \delta(x - e_i^1)$ as $t \rightarrow \infty$. ■

Corollary 6.4 *Suppose the $n_1 \times n_2$ payoff matrix U has a strictly dominant strategy i for the row player, and that $\partial\Delta_1^{(i)}$ has zero probability mass with respect to $p_0(x)$. Then the distributional dynamics for the row player associated with any mixed strategy time path $y : \mathbb{R} \rightarrow \Delta_2$ for the column player, satisfies $p(x, t) \rightarrow \delta(x - e_i^1)$ as $t \rightarrow \infty$.*

Proof. If i is a strictly dominant strategy, then $u_{ir} > u_{jr}$ for all $j \neq i$ and all column-player strategies r . Thus, for any path of mixed strategies $y(t)$ used by the column player, we have $c_i(t) = [Uy(t)]_i = \sum_r u_{ir} y_r(t) > \sum_r u_{jr} y_r(t) = c_j(t)$ for all $j \neq i$. Let $u_* = \min_{j \neq i, r} \{u_{ir} - u_{jr}\}$. Then $u_* > 0$ and $c_i(t) - c_j(t) > u_*$ for all $j \neq i$. Thus, from (50), $[C_i(t) - C_j(t)] > u_* t \rightarrow \infty$ as $t \rightarrow \infty$. The result therefore follows from Corollary 6.3 . ■

7 Distributional Replicator Dynamics

In this section we show how, in the asymmetric case, a solution to the pair of coupled continuity equations (40) and (41), or, in the symmetric case, to the corresponding single continuity equation

(42), can be obtained from the “frozen” solution (53) for population 1, and an analogous frozen solution for population 2. The coupling of these solutions is then tracked by solutions of an associated ODE system, which we term the *Distributional Replicator Dynamics*.

7.1 Asymmetric Games

Consider solutions to equations (40) and (41). From (49) and (50), these equations can be construed in the first instance as independent (“frozen”) continuity equations associated with the time-dependent vectors (one for each population):

$$c(t) = \frac{dC(t)}{dt} = U\langle y \rangle_t \in \mathbb{R}^{n_1}, \quad (55)$$

$$d(t) = \frac{dD(t)}{dt} = V\langle x \rangle_t \in \mathbb{R}^{n_2}. \quad (56)$$

Thus, (54) gives:

$$\langle x \rangle_t = \int_{\Delta_1} \left(\frac{\xi e^{C(t)}}{\xi \cdot e^{C(t)}} \right) p_0(\xi) dV(\xi), \quad (57)$$

$$\langle y \rangle_t = \int_{\Delta_2} \left(\frac{\zeta e^{D(t)}}{\zeta \cdot e^{D(t)}} \right) q_0(\zeta) dV(\zeta). \quad (58)$$

We therefore obtain the system of $n_1 + n_2$ differential equations in the variables C_1, \dots, C_{n_1} and D_1, \dots, D_{n_2} :

$$\frac{dC_i}{dt} = \sum_{k=1}^m u_{ik} \int_{\Delta_2} \left(\frac{\zeta_k e^{D_k}}{\zeta \cdot e^D} \right) q_0(\zeta) dV(\zeta), \quad C_i(0) = 0, \quad 1 \leq i \leq n_1, \quad (59)$$

$$\frac{dD_j}{dt} = \sum_{l=1}^n v_{jl} \int_{\Delta_1} \left(\frac{\xi_l e^{C_l}}{\xi \cdot e^C} \right) p_0(\xi) dV(\xi), \quad D_j(0) = 0, \quad 1 \leq j \leq n_2. \quad (60)$$

We call these equations the asymmetric *Distributional Replicator Dynamics* associated with the pair of initial densities $p_0(x)$ and $q_0(y)$. The solutions of these equations with the given initial conditions define trajectories $C(t)$ and $D(t)$, in terms of which the continuity dynamics can be completely specified as in (53) and (54), with analogous formulae for population 2.

Note that at most $n_1 - 1$ of the C_i 's and at most $n_2 - 1$ of the D_j 's are independent.²² For example, setting $A_i = C_i - C_{n_1}$ and $B_j = D_j - D_{n_2}$, equations (59) and (60) can be reduced to:

$$\frac{dA_i}{dt} = \sum_{k=1}^m (u_{ik} - u_{n_1 k}) \int_{\Omega_2} \left(\frac{\zeta_k e^{B_k}}{\zeta \cdot e^B} \right) q_0(\zeta) dV(\zeta), \quad A_i(0) = 0, \quad 1 \leq i \leq n_1 - 1, \quad (61)$$

$$\frac{dB_j}{dt} = \sum_{l=1}^n (v_{jl} - v_{n_2 l}) \int_{\Omega_1} \left(\frac{\xi_l e^{A_l}}{\xi \cdot e^A} \right) p_0(\xi) dV(\xi), \quad B_j(0) = 0, \quad 1 \leq j \leq n_2 - 1, \quad (62)$$

where Ω_1 and Ω_2 are the projections of Δ_1 and Δ_2 onto \mathbb{R}^{n_1-1} and \mathbb{R}^{n_2-1} , respectively, given by $x_{n_1} = 1 - \sum_{i=1}^{n_1-1} x_i$ and $y_{n_2} = \sum_{j=1}^{n_2-1} y_j$ (see definition (94) of Appendix A.3). Of course,

²²Because of the constraints $\sum_j \langle y_j \rangle_t = \sum_i \langle x_i \rangle_t = 1$.

$$A_{n_1} = B_{n_2} = 0.$$

7.2 Symmetric Games

We consider the continuity equation (42) associated with a 2-player, n -strategy symmetric game having $n \times n$ payoff matrix U . In terms of the theory of section 6, this is the continuity equation associated to the time-dependent mixed strategy profile $y : \mathbb{R} \rightarrow \Delta$ given by $y(t) = \langle x \rangle_t$. That is, $c(t) = U \langle x \rangle_t$. Thus, from (50) we have

$$c(t) = \frac{dC(t)}{dt} = U \langle x \rangle_t, \quad (63)$$

and by (54),

$$\langle x_j \rangle_t = \int_{\Delta} \left(\frac{\xi_j e^{C_j(t)}}{\xi \cdot e^{C(t)}} \right) p_0(\xi) dV(\xi).$$

We therefore obtain the system of n differential equations in the variables C_1, \dots, C_n :

$$\frac{dC_i}{dt} = \sum_{j=1}^n u_{ij} \int_{\Delta} \left(\frac{\xi_j e^{C_j}}{\xi \cdot e^C} \right) p_0(\xi) dV(\xi), \quad C_i(0) = 0, \quad 1 \leq i \leq n. \quad (64)$$

Following section 7.1, we call equations (64) the symmetric *Distributional Replicator Dynamics* associated with the initial density $p_0(x)$. The solutions of these equations with the given initial conditions define trajectories $C(t)$, in terms of which the continuity dynamics can be completely specified as in (53) and (54).

Again, at most $n - 1$ of equations (64) are independent. For example, setting $A_i = C_i - C_n$, equations (64) can be reduced to

$$\frac{dA_i}{dt} = \sum_{j=1}^n (u_{ij} - u_{nj}) \int_{\Omega} \left(\frac{\xi_j e^{A_j}}{\xi \cdot e^A} \right) p_0(\xi) dV(\xi), \quad A_i(0) = 0, \quad 1 \leq i \leq n - 1, \quad (65)$$

where $\Omega \subset \mathbb{R}^{n-1}$ is the projection of Δ onto \mathbb{R}^{n-1} obtained by setting $x_n = 1 - \sum_{i=1}^{n-1} x_i$. Of course $A_n = 0$. Note that the formulae (53) and (54) can be expressed in terms of the A_i 's.

8 Application: 2×2 symmetric games

8.1 The Replicator Dynamic

We use the ideas introduced in Section 7 to study the dynamics of mixed strategies in the simplest form of games, namely 2×2 symmetric games. We consider the game with payoff matrix

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}. \quad (66)$$

Players play a mixed strategy $(x_1, x_2) \in \Delta$. Write $x = x_1$ and $1 - x = x_2$. We are interested in the case in which there are three symmetric Nash equilibria, two pure strategies at $x = 0$ and $x = 1$,

and a mixed strategy at

$$\bar{x} = \frac{u_{22} - u_{12}}{(u_{22} - u_{12}) + (u_{11} - u_{21})}. \quad (67)$$

This is an allowable mixed strategy provided $0 < \bar{x} < 1$; i.e. provided the payoff differences $(u_{22} - u_{12})$ and $(u_{11} - u_{21})$ are non-zero and have the same sign.

Set

$$\lambda_U = (u_{22} - u_{12}) + (u_{11} - u_{21}). \quad (68)$$

The Replicator dynamics associated with the game (66) reduce to the 1-dimensional system defined for $x \in [0, 1]$

$$\dot{x} = \lambda_U x(1-x)(x - \bar{x}). \quad (69)$$

If $\lambda_U > 0$, $x = 0$ and $x = 1$ are locally asymptotically stable, and \bar{x} is unstable. This is the case of most interest since there is then an equilibrium selection problem. Thus, if the initial condition $x_0 \in (0, \bar{x})$, then $x = 0$ is attracting under the Replicator dynamic (69), and if $x_0 \in (\bar{x}, 1)$, then $x = 1$ is attracting.

Our objective is to compare evolution under the standard replicator dynamic (69) with evolution under the replicator continuity equation (42). In the replicator dynamic, the state space is Δ and the variable of interest is the proportion of players playing each strategy. We can compare the evolution of this variable under the replicator dynamic with the evolution of the mean of mixed strategies under the replicator continuity dynamic. In this section, we shall show that for the simple case of 2×2 games, there is a very close relationship between the two types of evolution. Thus, suppose that from an initial point $x_0 \in (0, 1)$, the replicator dynamic converges to a particular Nash equilibrium x^* . Then from an initial density function $p_0(x)$ with mean $\langle x \rangle_0 = x_0$, the replicator continuity equation converges to a point-mass probability measure concentrated at x^* .

8.2 The Replicator Continuity Equation

We use the reduced form (65) of the Distributional Replicator dynamics to analyze the evolution of mixed strategies in a 2×2 symmetric games. Since there are only two strategies, there is only one such independent variable, which we write as $A(t)$. The Distributional Replicator dynamic associated with an initial probability dynamic $p_0(x)$ is therefore

$$\begin{aligned} \dot{A} &= (u_{11} - u_{21}) \int_0^1 \left(\frac{\xi e^A}{1 - \xi + \xi e^A} \right) p_0(\xi) d\xi + (u_{12} - u_{22}) \int_0^1 \left(\frac{1 - \xi}{1 - \xi + \xi e^A} \right) p_0(\xi) d\xi \\ &= -(u_{22} - u_{12}) + \{(u_{22} - u_{12}) + (u_{11} - u_{21})\} \int_0^1 \left(\frac{\xi e^A}{1 - \xi + \xi e^A} \right) p_0(\xi) d\xi, \end{aligned}$$

which can be written as

$$\dot{A} = \lambda_U \left\{ -\bar{x} + \int_0^1 \left(\frac{\xi e^A}{1 - \xi + \xi e^A} \right) p_0(\xi) d\xi \right\}, \quad A(0) = 0. \quad (70)$$

For a variable $k \geq 0$, and a probability density $p(x)$ on $[0, 1]$, define a function

$$F(k | p) = \int_0^1 \frac{z}{z + k(1-z)} p(z) dz. \quad (71)$$

Then we can write (70) as

$$\dot{A} = \lambda_U \{ -\bar{x} + F(e^{-A} | p_0) \}, \quad A(0) = 0. \quad (72)$$

The function $F(k | p)$ is monotonically decreasing in k with $F(0 | p) = 1$ and, provided $p(x)$ has no mass point at $x = 1$, $F(k | p) \rightarrow 0$ as $k \rightarrow \infty$. In addition, $F(1 | p) = \langle x \rangle$, the mean of x with respect to p . Hence, $F(e^{-A} | p_0)$ is monotonically increasing in A with $F(e^{-A} | p_0) \rightarrow 0$ as $A \rightarrow -\infty$ and $F(e^{-A} | p_0) \rightarrow 1$ as $A \rightarrow \infty$.

The following lemma relates the asymptotic behaviour of $A(t)$ to the initial density function.

Lemma 8.1 *Let $\langle x \rangle_0$ be the mean with respect to the initial density function $p_0(x)$.*

1. *If $\langle x \rangle_0 < \bar{x}$, then $A(t)$ is monotonically decreasing in t , and $A(t) \rightarrow -\infty$ as $t \rightarrow \infty$.*
2. *If $\langle x \rangle_0 > \bar{x}$, then $A(t)$ is monotonically increasing in t , and $A(t) \rightarrow \infty$ as $t \rightarrow \infty$.*

Proof. From (72), we have $\dot{A}(0) = \lambda_U (-\bar{x} + \langle x \rangle_0)$. Since $\lambda_U > 0$, $\dot{A}(0) > 0$ if $\langle x \rangle_0 > \bar{x}$ and $\dot{A}(0) < 0$ if $\langle x \rangle_0 < \bar{x}$. Moreover, the monotonicity properties of $F(e^{-A} | p_0)$ imply that the initial conditions are self-reinforcing as t increases. Hence, if $\langle x \rangle_0 > \bar{x}$, then $\dot{A}(t) > 0$, if $\langle x \rangle_0 < \bar{x}$, then $\dot{A}(t) < 0$, for all $t > 0$. ■

We now use Corollary 6.3 and Lemma 8.1 to derive the following proposition.

Proposition 8.2 *Consider a 2×2 symmetric game. Let $\langle x \rangle_t$ be the mean with respect to the density $p(x, t)$. If $\langle x \rangle_0 > \bar{x}$, then $p(x, t) \rightarrow \delta(x - 1)$, and hence $\langle x \rangle_t \rightarrow 1$, and if $\langle x \rangle_0 < \bar{x}$, then $p(x, t) \rightarrow \delta(x)$, and hence $\langle x \rangle_t \rightarrow 0$ as $t \rightarrow \infty$.*

Proposition 8.2 implies in the type of 2×2 symmetric games we are considering, there is no difference at the observational level between evolution of pure strategies under the replicator dynamic and the evolution of mixed strategies under the replicator continuity equation. Even if agents are updating mixed strategies and the relevant evolutionary dynamic is the replicator continuity equation, an outside observer will only be able to see the proportion of agents using each particular pure strategy. This is just the mean of the mixed strategies currently in use in the population. Since the convergence of the mean over time replicates the dynamic of the social state under the replicator dynamic (possibly with a time lag), it is impossible to distinguish over the long run whether agents actually play only pure strategies or they employ mixed strategies. An example of trajectories obtained from these two dynamics is shown in Fig 1.

This is, however, not a general result, and does not hold for $n \times n$ symmetric games with $n > 2$. We shall not show this, but instead, in the next section we present an example of a 2×2 *asymmetric* game in which the long run social state differs radically under the replicator dynamics and the replicator continuity dynamics.

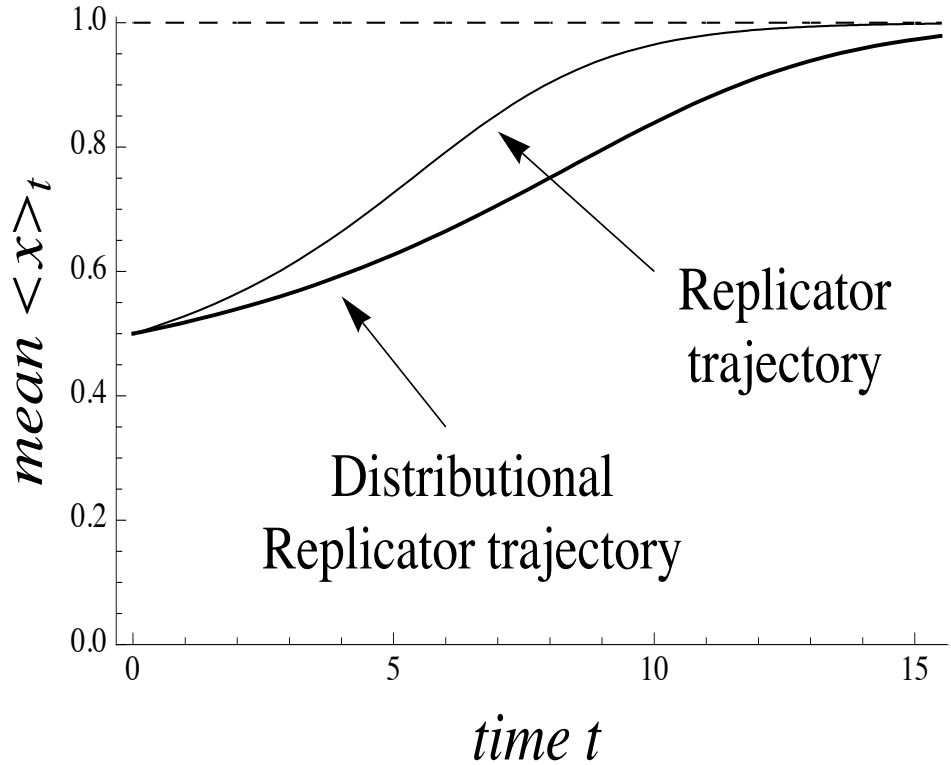


Figure 1: Trajectories of the classical replicator dynamic (69) (thin curve) starting from initial condition x_0 , and of the mean $\langle x \rangle_t$ under the dynamic (72) (thick curve), starting from an initial condition with mean, $\langle x \rangle_0 = x_0$. These trajectories converge to the Nash equilibrium $x = 1$, but with different time lines. In this example, the Distributional Replicator dynamics has uniform initial density $p_0(x) = 1$. Other parameters are $\bar{x} = 0.4$ and $\lambda_U = 1$.

9 Application: 2×2 Asymmetric Games

9.1 The Replicator Dynamics

As in the last section, we use the distributional replicator dynamics for asymmetric games to analyze the dynamics of the densities over mixed strategies for 2×2 asymmetric games. The payoff matrices to the row and column players are

$$U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}, \quad V = \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix}. \quad (73)$$

Since each player has only two strategies, we denote a mixed strategy for population 1 as $(x, 1-x) \in \Delta_1$, and a mixed strategy for population 2 by $(y, 1-y) \in \Delta_2$. The standard Replicator dynamics is then

$$\dot{x} = \lambda_U x(1-x)(y - \bar{y}), \quad (74)$$

$$\dot{y} = \lambda_V y(1-y)(x - \bar{x}), \quad (75)$$

where

$$\lambda_U = (u_{11} - u_{21}) + (u_{22} - u_{12}), \quad \bar{y} = \frac{u_{22} - u_{12}}{(u_{11} - u_{21}) + (u_{22} - u_{12})}, \quad (76)$$

$$\lambda_V = (v_{11} - v_{21}) + (v_{22} - v_{12}), \quad \bar{x} = \frac{v_{22} - v_{12}}{(v_{11} - v_{21}) + (v_{22} - v_{12})}. \quad (77)$$

The dynamics (74), (75) have equilibria at $(x, y) = (0, 0), (0, 1), (1, 1), (1, 0)$ and (\bar{x}, \bar{y}) . The latter lies in the interior the state space $0 \leq x, y \leq 1$ provided the payoff differences $(u_{11} - u_{21})$ and $(u_{22} - u_{12})$ are non-zero and have the same sign, and similarly for $(v_{11} - v_{21})$ and $(v_{22} - v_{12})$. In particular, if these signs are all positive, then λ_U and λ_V are both positive, and in this case $(0, 0)$ and $(1, 1)$ are locally asymptotically stable Nash Equilibria, with all other equilibria unstable. There is therefore an equilibrium selection problem in this case. Which of the two stable Nash equilibria is the asymptotic outcome of a Replicator dynamic trajectory depends on the initial condition.

9.2 The Distributional Replicator Dynamics

The reduced Distributional Replicator dynamics (61) and (62) is 2-dimensional, with variables $A = A_1$ and $B = A_2$. We can therefore write these dynamics as

$$\begin{aligned} \dot{A} &= -(u_{22} - u_{12}) + \{(u_{22} - u_{12}) + (u_{11} - u_{21})\} \int_0^1 \left(\frac{\zeta e^B}{1 - \zeta + \zeta e^B} \right) q_0(\zeta) d\zeta, \\ \dot{B} &= -(v_{22} - v_{12}) + \{(v_{22} - v_{12}) + (v_{11} - v_{21})\} \int_0^1 \left(\frac{\xi e^A}{1 - \xi + \xi e^A} \right) p_0(\xi) d\xi, . \end{aligned}$$

with initial condition $A(0) = B(0) = 0$. Using the notation (71) these equations can be written in the form

$$\dot{A} = \lambda_U \{ -\bar{y} + F(e^{-B} | q_0) \}, \quad (78)$$

$$\dot{B} = \lambda_V \{ -\bar{x} + F(e^{-A} | p_0) \}. \quad (79)$$

Equations (78) and (79) therefore constitute an ODE system in \mathbb{R}^2 . In Appendix A.4 we provide a detailed analysis of the solution trajectories of these two equations. We show that as $t \rightarrow \infty$, either $|A(t)|, |B(t)| \rightarrow \infty$ or the solution trajectories exhibit periodic orbits. Given these properties of $A(t)$ and $B(t)$, we can state the following proposition.

Proposition 9.1 *Let $p_0(x)$ and $q_0(y)$ be the initial density functions for populations 1 and 2 respectively. Then,*

1. *Either: Both $p(x, t)$ and $q(y, t)$ converge to mass points on 0 or 1 as $t \rightarrow \infty$. That is, $p(x, t) \rightarrow \delta(x - 1)$ or $\delta(x)$, and $q(y, t) \rightarrow \delta(y - 1)$ or $\delta(y)$.*
2. *Or: The trajectories of $p(x, t)$ and $q(y, t)$ are periodic.*

Proof. In Appendix A.4, we show that under the dynamics (78), (79), $(|A(t)|, |B(t)|) \rightarrow (\infty, \infty)$, or the trajectories $A(t)$ and $B(t)$ describe a closed orbit in the (A, B) -plane. By Corollary 6.3, if $A(t) \rightarrow \infty$, $p(x, t) \rightarrow \delta(x - 1)$ and if $A(t) \rightarrow -\infty$, $p(x, t) \rightarrow \delta(x)$. Similarly, $q(y, t) \rightarrow \delta(y - 1)$ if $B(t) \rightarrow \infty$ or $\delta(y)$ if $B(t) \rightarrow -\infty$.

On the other hand, if $A(t)$ and $B(t)$ exhibit periodic motion, it follows from (53) that the trajectories of $p(x, t)$ and $q(y, t)$ are periodic. ■

Proposition 9.1 implies that $p(x, t)$ and $q(y, t)$ will never converge to probability measures whose means are the mixed strategy Nash equilibrium. This conclusion evokes the well known result that in 2×2 asymmetric games, a mixed strategy Nash equilibrium is never stable under the replicator dynamics (Selten, 1980). However, unlike in 2×2 symmetric games, the convergence behaviour of the means under the Replicator continuity equation does not necessarily replicate the convergence behaviour of the state variables under the Replicator dynamics (74), (75). We present an example that establishes this fact. Thus, we shall construct a game in which the Replicator dynamic converges to the Nash equilibrium $(1, 1)$ from given initial conditions (x_0, y_0) . However, under the replicator continuity equation, and with appropriate initial distributions satisfying $(\langle x \rangle_0, \langle y \rangle_0) = (x_0, y_0)$, the density functions over mixed strategies, $(p(x, t), q(y, t))$, converge to $(\delta(x), \delta(y))$. Hence, for means, $(\langle x \rangle_t, \langle y \rangle_t) \rightarrow (0, 0)$.

9.3 A class of examples

As discussed in section 9.1, we assume that all payoff differences are positive, so that $0 < \bar{x}, \bar{y} < 1$ and λ_U and λ_V are both positive.

We assume population 2 is initially homogeneous, in the sense that all agents use a common mixed strategy y_0 with $y_0 \neq 0, 1, \bar{y}$. However, we assume that population 1 consists initially of two

types, agents who use a mixed strategy a_0 , with $0 < a_0 < \bar{x}$, and agents who use a mixed strategy a_1 , with $\bar{x} < a_1 < 1$. The population proportions of these agents are $1 - \alpha$ and α , with $0 < \alpha < 1$. Thus, the initial distributions of the two populations are $p_0(x) = (1 - \alpha)\delta(x - a_0) + \alpha\delta(x - a_1)$ for population 1, and $q_0(y) = \delta(y - y_0)$ for population 2. The initial means of the two populations are therefore

$$x_0 = \langle x \rangle_0 = (1 - \alpha)a_0 + \alpha a_1, \quad (80)$$

$$y_0 = \langle y \rangle_0. \quad (81)$$

Note that $x_0 = \bar{x}$ when $\alpha = \alpha^*$, where

$$\alpha^* = \frac{\bar{x} - a_0}{a_1 - a_0}, \quad (82)$$

and that $x_0 < \bar{x}$ for $\alpha < \alpha^*$, and $x_0 > \bar{x}$ for $\alpha > \alpha^*$.

The Distributional Replicator dynamics associated to these initial densities are:

$$\dot{A} = \lambda_U \left\{ -\bar{y} + \frac{y_0 e^B}{1 - y_0 + y_0 e^B} \right\}, \quad (83)$$

$$\dot{B} = \lambda_V \left\{ -\bar{x} + (1 - \alpha) \frac{a_0 e^A}{1 - a_0 + a_0 e^A} + \alpha \frac{a_1 e^A}{1 - a_1 + a_1 e^A} \right\}. \quad (84)$$

We wish to compare these dynamics to those associated with the Replicator dynamics having initial condition (x_0, y_0) . From Corollary 5.2, this is equivalent to the Distributional Replicator dynamics associated to the initial densities $(\delta_{x_0}, \delta_{y_0})$. That is:

$$\dot{A} = \lambda_U \left\{ -\bar{y} + \frac{y_0 e^B}{1 - y_0 + y_0 e^B} \right\}, \quad (85)$$

$$\dot{B} = \lambda_V \left\{ -\bar{x} + \frac{x_0 e^A}{1 - x_0 + x_0 e^A} \right\}. \quad (86)$$

Initial conditions for both sets of dynamics are $A(0) = B(0) = 0$.

We aim to show that there are situations in which the means of the distributions determined by these dynamics exhibit radically different asymptotic behaviours. A general method of constructing such examples is described in Appendix A.5. We show that, for fixed $(\bar{x}, \bar{y}) \in (0, 1)$, $\alpha \in (0, \alpha^*)$ and $y_0 \in (\bar{y}, 1)$, positive constants λ_U and λ_V can be chosen so that the means of the trajectories of the two dynamics converge to different Nash equilibria.

Here we refer to the numerical example illustrated in Figure 2, which shows a trajectory of the Replicator dynamics (74)-(75) that converges to the Nash equilibrium $(1, 1)$, and a trajectory of the means associated to the Distributional Replicator dynamics (83)-(84), starting from the same initial condition, which converges to the Nash equilibrium $(0, 0)$.

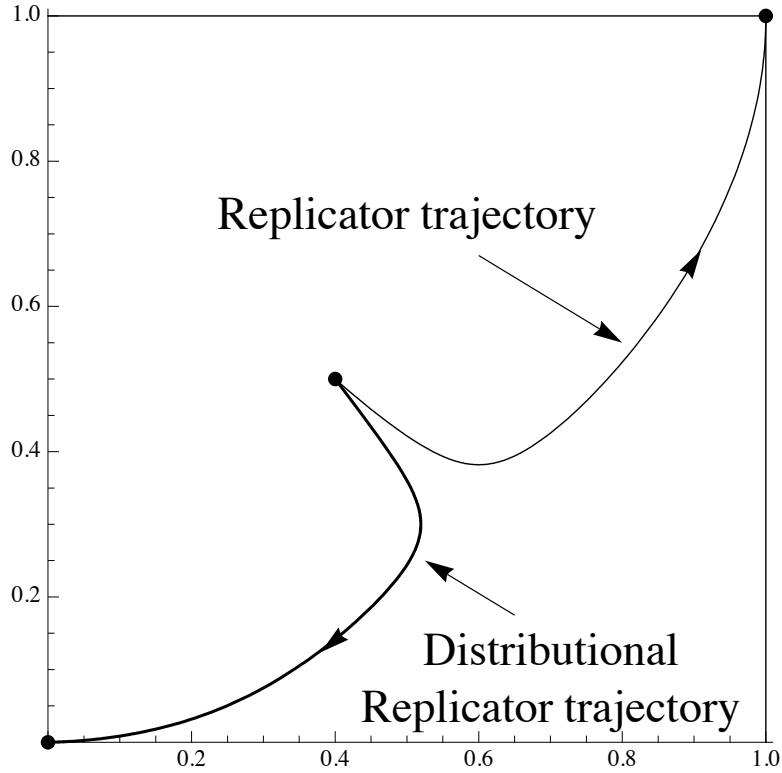


Figure 2: Trajectories of the means $(\langle x \rangle_t, \langle y \rangle_t)$, starting from a common initial condition, $(\langle x \rangle_0, \langle y \rangle_0) = (x_0, y_0)$, for the two dynamics (83)-(84) (thick curve) and (85)-(86) (thin curve). These trajectories converge to the Nash equilibria $(0, 0)$ and $(1, 1)$, respectively. Parameters are: $(\bar{x}, \bar{y}) = (0.6, 0.3)$, $a_0 = 0.2$, $a_1 = 0.8$, $y_0 = 0.5$, $\alpha = \frac{1}{2}\alpha^* = 0.33$, with corresponding $x_0 = 0.4$, and $\lambda_V = 100$, $\lambda_U = 120$.

10 Discussion and Conclusion

The motivation behind this paper was to study the evolution of mixed strategies in population games. Traditional ODE techniques can handle only the evolution of finite dimensional variables, whereas the state variables in the problems we consider here are probability density functions. Hence, it was necessary to introduce evolutionary dynamics that are partial differential equations and develop methods for solving these equations. In order for these equations to be meaningful as a description of social behaviour, we required them to be generated by some plausible individual behavioural rules. Finally, to justify this effort, our analysis needed to provide insights into social behaviour that contrasted with those obtained from a more conventional analysis of pure strategy evolution.

To meet these objectives, we introduced the general (non-linear) continuity equations for population games as the PDE system required for our purpose (section 3). These equations are applicable for any plausible mixed strategy updating rule. By showing that reinforcement based learning rules can be extended from learning theory and thereby generate the replicator continuity equations, we have been able to provide credible microfoundations to our evolutionary dynamics (section 4). Although the resulting equations cannot be solved explicitly (any more than can the classical replicator dynamics), we have proposed a general solution method using Liouville’s formula and an associated finite-dimensional ODE system that we call ‘distributional dynamics’, which can be applied to all finite normal form games. Finally, in our application of these techniques to 2×2 asymmetric games, we have constructed a class of examples for asymmetric games in which the replicator continuity equations lead to very different predictions about the observed social state from that of the classical replicator dynamics.²³

This class of examples for 2×2 asymmetric games illustrates the importance of the levels of sophistication we assign to agents in our evolutionary models of learning in games for the conclusions we obtain from these models. Conventionally, at each time step each agent is assumed to be primed to play only a particular pure strategy. Which pure strategy gets played is then merely a function of which agent from the population gets chosen to play. While these assumptions may be perfectly justified in biological models of evolution, they seem excessively naive in models of human interaction. One possible justification for such assumptions is that they lead to relatively simple, analytically tractable models, which, by stripping away the extraneous complexities of special-case scenarios, can be used to gain valuable intuitive insight into generic aspects of social learning. In this light, such models are not meant to be construed as descriptions of specific realities, and in particular cannot be used predictively.²⁴

Nevertheless, that agent’s play mixed strategies does not necessarily assume a high degree of cognitive sophistication. In particular, we do suppose that agents consciously use randomizing devices as part of a rational calculation. We can assume instead that agents make their decisions

²³The possibility of the divergence in behaviour under the classical replicator dynamics and the replicator continuity dynamics also holds for 2-player symmetric games with more than two strategies. We have been able to construct examples for 2-player, 3-strategy symmetric games in which convergence under the two dynamics is to different equilibria. However, we do not present such examples here.

²⁴A similar perspective on the function and value of models has been forcefully argued by Rubinstein – e.g. Rubinstein (2001, 2006).

within a stochastic environment (which may be stationary, as in section 2) that offers them “cues” that they use to condition their choice of action in the game. How this conditioning takes place depends on the agent’s *behavioural disposition*, conceived simply as a function that converts environmental cues into actions. Thus, it is the stochastic environment that acts as a randomizing device, and this, together with the agent’s disposition, generates a (pre-play) mixed strategy that characterizes her response to whatever is the state of the world when she is called upon to play. It is this disposition that is updated by reinforcement in response to payoff information (see section 4). In this interpretation, agents are of very limited cognitive capacity, and respond automatically to whatever “instruction” the environment provides. Of course, though many of the cues that condition the agents action may be processed subconsciously, she may nevertheless tell herself elaborate stories about why her action is the only “rational” response to the situation in which she finds herself.

It may be argued that even if we allow agents to play mixed strategies, the law of large numbers ensures that the mean of the mixed strategy distribution over the population will be identical to the proportion of agents playing different pure strategies. Since it is only the mean strategy that can be observed, allowing for mixed strategies has no observational consequences at an aggregate level. Our class of examples for 2×2 asymmetric games invalidates this argument. We have been able to show unambiguously that it is possible for all agents to play one pure strategy equilibrium in the long run by using only pure strategies under the classical replicator dynamic but to converge to another pure equilibria by playing mixed strategies under the replicator continuity dynamics. In order, therefore, to decide which particular approach—pure or mixed strategy— would be more relevant to model any particular situation, it is necessary to make appropriate assumptions about the nature of behavioural flexibility that agents may exhibit in that situation.

It should be possible to use the continuity equation approach to analyze mixed strategy evolution in other types of player-matching schemes than the simple pairwise-matching scheme discussed here. In this paper, a player interacts with a potentially different partner in each round of the game. However, the theory has a straightforward extension to the case in which some fixed proportion of agents are matched in each round. Alternatively, one may fix the population into matched pairs of players at the beginning, and allow these pairs to interact repeatedly using some learning protocol. The change in the distribution of mixed strategies in the populations can then be studied using a continuity equation.²⁵ Or one can study a more realistic scenario of a combination of the two matching schemes—where players play with a fixed partner for a certain number of periods and then change partners. Such problems can form a substantial research agenda for the future.

In this paper we have analyzed generic 2×2 asymmetric games (i.e. games with simple, isolated Nash equilibria— see section 9 and appendix A.4–A.5). An important future application area of the mixed strategy approach will be to non-generic games. For example, the ‘mini ultimatum game’ analyzed in Binmore, Gale and Samuelson (1995), is a non-generic 2×2 asymmetric game in which there exists both an isolated Nash equilibrium (the subgame perfect equilibrium) and a connected component of Nash equilibria. Given initial distributions over mixed strategies in the proposer and responder populations, the continuity dynamics would be expected to lead to an asymptotic

²⁵In fact, for a two population, 2×2 asymmetric game in which one member of each pair is chosen from each population, the associated continuity equation is of classical, linear form — cf. section 2.

probability weight on the subgame perfect equilibrium, together with a conditional distribution over the connected component. It would then be possible to make a prediction about the probability of any particular Nash equilibrium in the connected component in the long run. This can therefore act as a selection mechanism within the Nash equilibrium component. How the form of this asymptotic distribution depends on the initial distribution is a substantial question for the future, both in this and similar contexts.

A Appendix

A.1 Proof of Liouville's formula

We prove the formula (45) giving the unique solution to the initial value problem specified by (43).

Proof. We begin by writing the continuity equation as

$$\frac{\partial p}{\partial t} + \nabla p \cdot X + p \nabla \cdot X = 0. \quad (87)$$

Consider a pure-time function of the form $h(t) = p(x(t), t)z(t)$, where $p(x, t)$ is a solution of (87).

Then,

$$\frac{dh}{dt} = z(t) \frac{\partial p}{\partial t} + z \nabla p \cdot \frac{dx}{dt} + p \frac{dz}{dt} = z \left\{ \frac{\partial p}{\partial t} + \nabla p \cdot \frac{dx}{dt} + p \frac{1}{z} \frac{dz}{dt} \right\}.$$

Thus, $h(t) = \text{constant}$ defines a solution of the continuity equation (87) provided:

$$\frac{dx(t)}{dt} = X(x(t), t), \quad \frac{1}{z(t)} \frac{dz(t)}{dt} = [\nabla \cdot X](x(t), t).$$

That is, $x(t)$ is a solution of the dynamical system (44) and

$$z(t) = z_0 \exp \left\{ \int_0^t [\nabla \cdot X](x(s), s) ds \right\}.$$

The required solution $p(x, t)$ therefore satisfies:

$$p(x(t), t) = h_0 \exp \left\{ - \int_0^t [\nabla \cdot X](x(s), s) ds \right\}. \quad (88)$$

for some constant h_0 . When $t = 0$, we require $p(x, 0) = p_0(x)$. Hence,

$$p(x(0), 0) = p_0(x(0)) = h_0. \quad (89)$$

For fixed $t \geq 0$ and any $0 \leq s \leq t$, we take $x(s) = x_{t,s}(x)$, the solution of the characteristic ODE (44) that passes through x at time t . Then $x(t) = x_{t,t}(x) = x$, and substituting in (88) and (89) we obtain the required solution to the initial value problem (43):

$$p(x, t) = p_0(x_{t,0}(x)) \exp \left\{ - \int_0^t [\nabla \cdot X](x_{t,s}(x), y(s), s) ds \right\}. \quad (90)$$

This proves Liouville's formula. ■

A.2 Proof of Proposition 5.1

Proof. Make the change of variable:

$$\xi = x_{t,0}(x). \quad (91)$$

This has inverse

$$x = x_{0,t}(\xi). \quad (92)$$

Thus $dV(\xi) = |J_t(\xi; x)|dV(x)$, where the Jacobian is:

$$J_t(\xi; x) = \det \left(\frac{\partial \xi_i}{\partial x_j} \right).$$

Now generalize this to define:

$$J_{t,s}(x) = \det \left(\frac{\partial x_{t,s}(x)_i}{\partial x_j} \right).$$

Then $J_t(\xi; x) = J_{t,0}(x)$, and $J_{t,t}(x) = 1$. Next, observe that, by definition of the trajectories $x_{t,s}(x)$, we have

$$\frac{d}{ds} \left[\frac{\partial x_{t,s}(x)_i}{\partial x_j} \right] = \frac{\partial}{\partial x_j} \left[\frac{dx_{t,s}(x)_i}{ds} \right] = \frac{\partial}{\partial x_j} [X_i(x_{t,s}(x), s)] = \sum_{k=1}^n \frac{\partial X_i}{\partial x_k}(x_{t,s}(x), s) \frac{\partial x_{t,s}(x)_k}{\partial x_j}. \quad (93)$$

Let $J_{t,s}^{(i)}(x)$ be the determinant of the matrix obtained from $J_{t,s}(x)$ by taking the time derivatives with respect to s of the entries in the i -th row, as in (93), but leaving the other rows unchanged.

Let $[J_{t,s}(x)]_{i,j}$ be the ij -th minor of $J_{t,s}(x)$.²⁶ Then:

$$\begin{aligned} \frac{dJ_{t,s}(\xi; x)}{ds} &= \sum_{i=1}^n J_{t,s}^{(i)}(x) \\ &= \sum_{i=1}^n \sum_{j=1}^n (-1)^{i+j} \frac{d}{ds} \left[\frac{\partial x_{t,s}(x)_i}{\partial x_j} \right] [J_{t,s}(x)]_{i,j} \quad \text{expanding } J_{t,s}^{(i)}(x) \text{ by the } i\text{-th row} \\ &= \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (-1)^{i+j} \frac{\partial X_i}{\partial x_k}(x_{t,s}(x), s) \frac{\partial x_{t,s}(x)_k}{\partial x_j} [J_{t,s}(x)]_{i,j} \quad \text{using (93)} \\ &= \sum_{i=1}^n \sum_{k=1}^n (-1)^{i+k} \frac{\partial X_i}{\partial x_k}(x_{t,s}(x), s) \left\{ \sum_{j=1}^n (-1)^{k+j} \frac{\partial x_{t,s}(x)_k}{\partial x_j} [J_{t,s}(x)]_{i,j} \right\} \\ &= \sum_{i=1}^n \sum_{k=1}^n (-1)^{i+k} \frac{\partial X_i}{\partial x_k}(x_{t,s}(x), s) \delta_{ik} J_{t,s}(x). \end{aligned}$$

The last equality holds because, for $k \neq i$, the expression in $\{\}$ is the determinant of an $n \times n$ matrix whose i -th and k -th rows are identical, and hence this determinant is zero. We therefore have:

$$\frac{dJ_{t,s}(x)}{ds} = J_{t,s}(x) \sum_{i=1}^n \frac{\partial X_i}{\partial x_i}(x_{t,s}(x), s) = J_{t,s}(x) [\nabla \cdot X](x_{t,s}(x), s).$$

²⁶That is, the determinant of the $(n-1) \times (n-1)$ -matrix obtained from $J_{t,s}(x)$ by deleting the i -th row and the j -th column.

Integrating this from $s = 0$ to $s = t$ and recalling that $J_{t,t}(x) = 1$ and $J_{t,0}(x) = J_t(\xi; x)$, gives:

$$-\ln |J_t(\xi; x)| = \int_0^t [\nabla \cdot X](x_{t,s}(x), s) ds,$$

and hence:

$$|J_t(\xi; x)| = \exp \left\{ - \int_0^t [\nabla \cdot X](x_{t,s}(x), s) ds \right\}.$$

Finally, since $dV(\xi) = |J_t(\xi; x)| dV(x)$, substitution of (91) and (92) into (45) and (46) yields the required formula (47). ■

A.3 Proof of Proposition 6.2

For the pseudo-Replicator vector field $X(x, t) = R^1(x)y(t)$ on the simplex $\Delta_1 \subset \mathbb{R}^{n_1}$, we have $\sum_{i=1}^{n_1} x_i = 1$ and $\sum_{i=1}^{n_1} X_i = 0$. Hence, the independent components are x_i and X_i for $1 \leq i \leq n_1 - 1$. We therefore take the state space to be the projection of Δ_1 into \mathbb{R}^{n_1-1} defined by:

$$\Omega_1 = \left\{ (x_1, \dots, x_{n_1-1}) \in \mathbb{R}^{n_1-1} : 0 \leq x_i \leq \sum_{i=1}^{n_1-1} x_i \leq 1 \right\}. \quad (94)$$

Then, if $(x_1, \dots, x_{n_1-1}) \in \Omega_1$, the associated point $x \in \Delta_1$ is $x = (x_1, \dots, x_{n_1-1}, x_{n_1})$ with $x_{n_1} = 1 - \sum_{i=1}^{n_1-1} x_i$. Generally x denotes a point in Δ_1 , but relevant operations often involve only the independent components, i.e. the associated point in Ω_1 .

Let $L_{ij}(x) = x_i(\delta_{ij} - x_j)$. Then, from (49) we can write the divergence of X on Ω_1 as:

$$\begin{aligned} \nabla \cdot X(x, t) &= \sum_{i=1}^{n_1-1} \left\{ \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{n_1}} \right\} X_i(x, t) && x \in \Delta_1 \\ &= \sum_{i=1}^{n_1-1} \left\{ \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{n_1}} \right\} \left\{ \sum_{j=1}^{n_1} L_{ij}(x) c_j(t) \right\} \\ &= \sum_{j=1}^{n_1} \left\{ \sum_{i=1}^{n_1-1} \left\{ \frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_{n_1}} \right\} L_{ij}(x) \right\} c_j(t) \end{aligned}$$

Also:

$$\begin{aligned} \frac{\partial}{\partial x_i} [L_{ij}(x)] &= (1 - x_i)\delta_{ij} - x_j, && 1 \leq j \leq n_1 - 1, \\ \frac{\partial}{\partial x_{n_1}} [L_{ij}(x)] &= 0, && 1 \leq j \leq n_1 - 1, \\ \frac{\partial}{\partial x_i} [L_{in_1}(x)] &= -x_{n_1}, \\ \frac{\partial}{\partial x_{n_1}} [L_{in_1}(x)] &= -x_i. \end{aligned}$$

Hence,

$$\begin{aligned}
\nabla \cdot X(x, t) &= \sum_{i,j=1}^{n_1-1} \{(1-x_i)\delta_{ij} - x_j\} c_j(t) + \sum_{i=1}^{n_1-1} (x_i - x_{n_1}) c_{n_1}(t) \\
&= \sum_{i=1}^{n_1-1} (1-x_i) c_i(t) - (n_1-1) \sum_{j=1}^{n_1-1} x_j c_j(t) + \left(\sum_{i=1}^{n_1-1} x_i \right) c_{n_1}(t) - (n_1-1) x_{n_1} c_{n_1}(t) \\
&= \sum_{i=1}^{n_1-1} c_i(t) - n_1 \sum_{j=1}^{n_1-1} x_j c_j(t) + \left(\sum_{i=1}^{n_1-1} x_i \right) c_{n_1}(t) + x_{n_1} c_{n_1}(t) - n_1 x_{n_1} c_{n_1}(t) \\
&= \sum_{i=1}^{n_1-1} c_i(t) - n_1 \sum_{i=1}^{n_1} x_i c_i(t) + \left(\sum_{i=1}^{n_1} x_i \right) c_{n_1}(t) \\
&= \sum_{i=1}^{n_1-1} c_i(t) - n_1 \sum_{i=1}^{n_1} x_i c_i(t) + c_{n_1}(t) \\
&= \sum_{i=1}^{n_1} c_i(t) - n_1 \sum_{i=1}^{n_1} x_i c_i(t) \\
&= \{e^1 - n_1 x\} \cdot c(t),
\end{aligned}$$

where $e^1 = \sum_{i=1}^{n_1} e_i^1 \in \mathbb{R}^{n_1}$ is the vector all of whose entries are 1. That is:

$$\nabla \cdot X(x, t) = \nabla \cdot [L(x)c(t)] = (e^1 - n_1 x) \cdot c(t).$$

It now follows that, if $x_{t,s}(x)$ are the solution trajectories of the pseudo-Replicator equations (51), then we obtain

$$[\nabla \cdot X](x_{t,s}(x), s) = \{e^1 - n_1 x_{t,s}(x)\} \cdot c(s) = e^1 \cdot c(s) - n_1 \sum_{i=1}^{n_1} \frac{x_i c_i(s) e^{C_i(s,t)}}{x \cdot e^{C(s,t)}}.$$

Thus

$$\begin{aligned}
\int_0^t [\nabla \cdot X](x_{t,s}(x), s) ds &= e^1 \cdot \int_0^t c(s) ds - n_1 \sum_{i=1}^{n_1} \int_0^t \frac{x_i e^{C_i(s,t)}}{x \cdot e^{C(s,t)}} c_i(s) ds \\
&= e^1 \cdot C(t) - n_1 \int_0^t \frac{d}{ds} \left[\ln \left(x \cdot e^{C(s,t)} \right) \right] ds \\
&= e^1 \cdot C(t) - n_1 \ln \left[x \cdot e^{C(t,t)} \right] + n_1 \ln \left[x \cdot e^{C(0,t)} \right] \\
&= e^1 \cdot C(t) + n_1 \ln \left[x \cdot e^{-C(t)} \right],
\end{aligned}$$

because $C(t, t) = 0$, $C(s, t) = C(s) - C(t)$ and $e^1 \cdot x = 1$. We therefore have:

$$\exp \left\{ - \int_0^t [\nabla \cdot X](x_{t,s}(x), s) ds \right\} = \left(\frac{1}{x \cdot e^{-C(t)}} \right)^{n_1} \exp \{ -e^1 \cdot C(t) \}.$$

Substituting in Liouville's formula (45), it now follows that the solution of the continuity equation associated to a pseudo-Replicator vector field (49) is given by (53). ■

A.4 Dynamics of $A(t)$ and $B(t)$ in 2×2 asymmetric games

We analyze the Distributional Replicator dynamics (78) and (79). These dynamics are:

$$\dot{A} = \lambda_U \{-\bar{y} + F(e^{-B} | q_0)\}, \quad A(0) = 0, \quad (95)$$

$$\dot{B} = \lambda_V \{-\bar{x} + F(e^{-A} | p_0)\}, \quad B(0) = 0. \quad (96)$$

Our objective is to show that under this dynamics, either $|A(t)|, |B(t)| \rightarrow \infty$ as $t \rightarrow \infty$ or $A(t)$ and $B(t)$ exhibit cyclical motion around a rest point. This will complete the proof of Proposition 9.1.

Recall from the definition (71) that $F(e^{-A} | p_0)$ is monotonically increasing in A with $F(e^{-A} | p_0) \rightarrow 0$ as $A \rightarrow -\infty$ and $F(e^{-A} | p_0) \rightarrow 1$ as $A \rightarrow \infty$. Moreover, for $A = 0$, $F(e^{-A} | p_0) = \langle x \rangle_0$.

We first consider the case where \bar{x} or \bar{y} does not lie between 0 and 1. Suppose $\bar{y} > 1$. The monotonicity properties of the F function imply that $\text{sign } \dot{A} = -\text{sign } \lambda_U$, which implies $A(t) \rightarrow -\text{sign}(\lambda_U)\infty$ as $t \rightarrow \infty$. Similarly, if $\bar{y} < 0$, then $A(t) \rightarrow \text{sign}(\lambda_U)\infty$. As $A(t) \rightarrow \infty$, $B(t) \rightarrow \text{sign}(\lambda_V(1 - \bar{x}))\infty$, and as $A(t) \rightarrow -\infty$, $B(t) \rightarrow -\text{sign}(\lambda_V\bar{x})\infty$. The roles of $A(t)$ and $B(t)$ are reversed if either $\bar{x} < 0$ or $\bar{x} > 1$.

In sum, if either \bar{x} or \bar{y} lies outside the interval $[0, 1]$, then $|A(t)|, |B(t)| \rightarrow \infty$. For the generic games we are considering, the only other possible outcomes occur when $0 < \bar{x}, \bar{y} < 1$. In this case, (\bar{x}, \bar{y}) is the mixed strategy Nash equilibrium.

If $0 < \bar{y} < 1$, then $-\bar{y} + F(e^{-B} | q_0)$ has indeterminate sign, and there is a unique, finite \bar{B} such that $-\bar{y} + F(e^{-\bar{B}} | q_0) = 0$. Similarly, if $0 < \bar{x} < 1$, there is a unique, finite \bar{A} such that $-\bar{x} + F(e^{-\bar{A}} | p_0) = 0$. That is, there is a unique equilibrium (\bar{A}, \bar{B}) of the system (95), (96) determined by,

$$F(e^{-\bar{A}} | p_0) = \bar{x}, \quad (97)$$

$$F(e^{-\bar{B}} | q_0) = \bar{y}. \quad (98)$$

In order to analyze the local stability properties of the rest point, we linearize the system around the rest point. The Jacobean matrix at equilibrium is

$$J(\bar{A}, \bar{B}) = \begin{pmatrix} \frac{\partial \dot{A}}{\partial A} & \frac{\partial \dot{A}}{\partial B} \\ \frac{\partial \dot{B}}{\partial A} & \frac{\partial \dot{B}}{\partial B} \end{pmatrix} = \begin{pmatrix} 0 & -\lambda_U F'(e^{-\bar{B}} | q_0) e^{-\bar{B}} \\ -\lambda_V F'(e^{-\bar{A}} | p_0) e^{-\bar{A}} & 0 \end{pmatrix}. \quad (99)$$

Since $\text{Trace } J = 0$, this equilibrium can only be neutrally stable. Since $F'(k | p) < 0$, it follows that $\text{Det } J > 0$ if λ_U and λ_V have opposite signs. The eigenvalues are then purely imaginary. In this case, (\bar{A}, \bar{B}) is a centre, and trajectories are closed orbits around it. Since the initial conditions for (95) and (96) are $A(0) = B(0) = 0$, it follows that $(A(t), B(t))$ maps out a bounded, closed trajectory in the (A, B) -plane passing through $(0, 0)$.

The case where λ_U and λ_V have the same sign deserves more consideration. In this case, $\text{Det } J < 0$, and the eigenvalues are real but of opposite sign. Hence, (\bar{A}, \bar{B}) is an unstable saddle node. It suffices to consider the case $\lambda_U, \lambda_V > 0$. If $\lambda_U, \lambda_V < 0$ then the roles of A and B are

interchanged. We can have four (generic) cases.

- i)* $\langle x \rangle_0 > \bar{x}, \langle y \rangle_0 > \bar{y},$
- ii)* $\langle x \rangle_0 > \bar{x}, \langle y \rangle_0 \leq \bar{y},$
- iii)* $\langle x \rangle_0 \leq \bar{x}, \langle y \rangle_0 > \bar{y},$
- iv)* $\langle x \rangle_0 < \bar{x}, \langle y \rangle_0 < \bar{y}.$

We will show that, under mild assumptions, $|A(t)|, |B(t)| \rightarrow \infty$ as $t \rightarrow \infty$ in each of these cases. This is straightforward for cases (i) and (iv). Indeed, following the argument in the proof of lemma 8.1, we have

$$\begin{aligned}\dot{A}(0) &= \lambda_U (-\bar{y} + \langle y \rangle_0) \\ \dot{B}(0) &= \lambda_V (-\bar{x} + \langle x \rangle_0).\end{aligned}$$

Consider case (i). If $\langle x \rangle_0 > \bar{x}$ and $\langle y \rangle_0 > \bar{y}$, then $\dot{A}(0) > 0$ and $\dot{B}(0) > 0$. The monotonicity properties of $F(e^{-A} | p_0)$ and $F(e^{-B} | q_0)$ then imply that these initial conditions are reinforced as t increases. Hence, both $A(t)$ and $B(t) \rightarrow \infty$. Similarly, in case (iv), both $A(t)$ and $B(t) \rightarrow -\infty$.

Cases (ii) and (iii) are more complex and a proof is given in subsection A.4.2. First, we require a more in depth analysis of the Distributional dynamics. This analysis will also be required in Appendix A.5 to construct the class of examples discussed in section 9.3.

A.4.1 Analysis of Distributional dynamics

The system (95)-(96) may be represented in Hamiltonian form:

$$\dot{A} = \frac{\partial H}{\partial B}, \quad \dot{B} = -\frac{\partial H}{\partial A},$$

where

$$H(A, B) = -\lambda_U \bar{y} B + \lambda_U \int_0^B F(e^{-r} | q_0) dr + \lambda_V \bar{x} A - \lambda_V \int_0^A F(e^{-s} | p_0) ds. \quad (100)$$

Thus, the solution trajectory of the system (95)-(96) with general initial conditions (A_0, B_0) at $t = 0$ is the curve in the (A, B) -plane given by $H(A, B) = H(A_0, B_0)$.

Now note that, from the definition (71) of F ,

$$\int_0^X F(e^{-r} | p) dr = \int_0^1 \left[\int_0^X \frac{z}{z + (1-z)e^{-r}} dr \right] p(z) dz = \int_0^1 \ln(1 - z + ze^X) p(z) dz$$

We can therefore write $H(A, B)$ in the form:

$$H(A, B) = \lambda_V y_1(A | p_0) - \lambda_U y_2(B | q_0), \quad (101)$$

where

$$y_i(X | p) = \bar{w}_i X - \int_0^1 \ln(1 - z + ze^X) p(z) dz \quad (102)$$

$$= -(1 - \bar{w}_i)X - \int_0^1 \ln(z + (1 - z)e^{-X}) p(z) dz, \quad (103)$$

for $i = 1, 2$, with $\bar{w}_1 = \bar{x}$ and $\bar{w}_2 = \bar{y}$.

We are concerned with the trajectory with initial conditions $(A_0, B_0) = (0, 0)$. This trajectory is $H(A, B) = H(0, 0) = 0$. The required solution trajectory of (95)-(96) is therefore:

$$\lambda_U y_1(A | p_0) = \lambda_V y_2(B | q_0). \quad (104)$$

To analyze the case in which λ_U and λ_V have the same sign, and (\bar{A}, \bar{B}) is an unstable saddle node, first consider the eigenvalue problem for a 2×2 matrix of the general form as (99). An eigenvalue μ with eigenvector $(v, w)^T$ satisfies

$$\begin{pmatrix} -\mu & R \\ S & -\mu \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} -\mu v + R w \\ S v - \mu w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $w = \alpha\mu$ and $v = \alpha R$, and $v = \beta\mu$, $w = \beta S$, and hence

$$\frac{v}{w} = \frac{R}{\mu} = \frac{\mu}{S},$$

which gives $\mu^2 = RS$, the required characteristic equation for the eigenvalues. The two eigenvalues are $\mu = \pm\sqrt{RS}$, and the corresponding eigenvectors are:

$$v_{\pm} = \alpha \begin{pmatrix} R \\ \pm\mu \end{pmatrix} = \beta \begin{pmatrix} \pm\mu \\ S \end{pmatrix}.$$

The eigenvector defining the unstable manifold passing through (\bar{A}, \bar{B}) is v_+ , and that defining the stable manifold is v_- . For the matrix (99), we have

$$\mu = \sqrt{\lambda_U \lambda_V F'(e^{-\bar{A}} | p_0) F'(e^{-\bar{B}} | q_0) e^{-(\bar{A} + \bar{B})}}, \quad v_{\pm} = \begin{pmatrix} \lambda_U \left| F'(e^{-\bar{B}} | q_0) \right| e^{-\bar{B}} \\ \pm\mu \end{pmatrix}. \quad (105)$$

Note that, when $\lambda_U > 0$, v_+ points into the positive quadrant of the (A, B) -plane centred at (\bar{A}, \bar{B}) . See Fig 3.

The seperatrices. These are the trajectories defining the stable and unstable manifolds, meeting at the equilibrium (\bar{A}, \bar{B}) , with v_- and v_+ defining tangents at (\bar{A}, \bar{B}) . The seperatrices are defined by the solution trajectories of (95)-(96) given by $H(A, B) = \bar{H}$, with $\bar{H} = H(\bar{A}, \bar{B})$. These curves are illustrated in Fig 3.

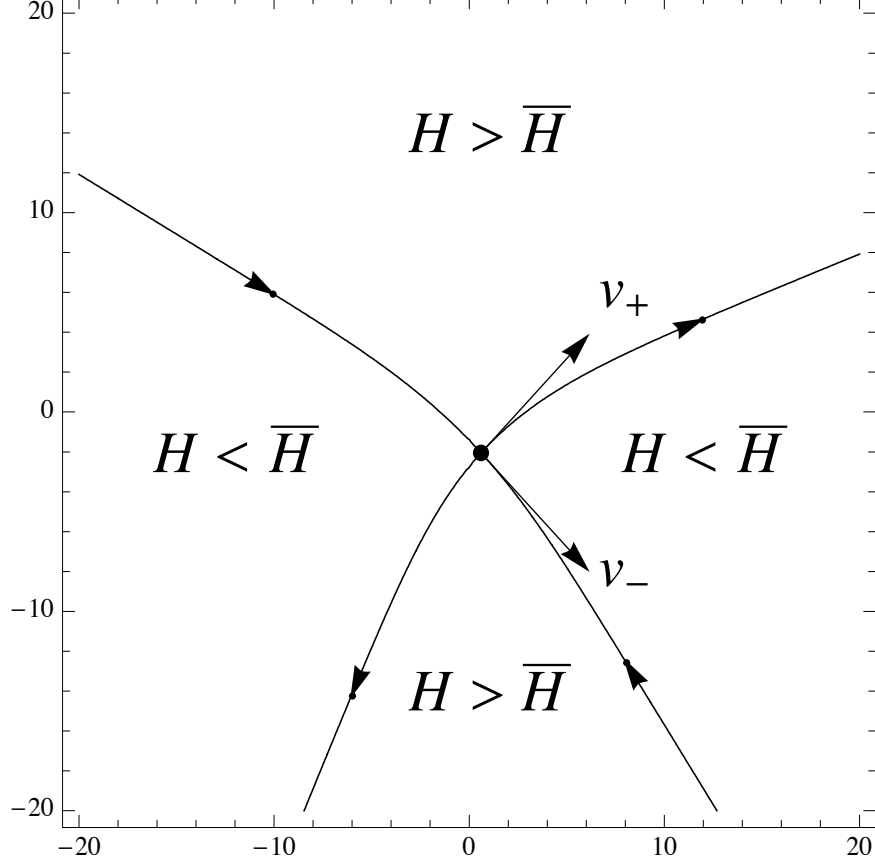


Figure 3: The separatrices defined by $H(A, B) = \bar{H}$. These meet at the equilibrium point (\bar{A}, \bar{B}) . The unstable manifold is tangent at (\bar{A}, \bar{B}) to the eigenvector v_+ , given by (105), and the stable manifold is tangent at (\bar{A}, \bar{B}) to the eigenvector v_- . These define the directions of motion of the solutions of (95)-(96) along these curves. The regions $H(A, B) > \bar{H}$ and $H(A, B) < \bar{H}$ are indicated. This example has $p_0(x) = q_0(y) = 1$, both defining uniform distributions. Other parameters are: $(\bar{x}, \bar{y}) = (0.6, 0.2)$, $\lambda_U = 1, \lambda_V = 0.8$.

From (100), we have:

$$\frac{\partial H}{\partial A} = \lambda_V \bar{x} - \lambda_V \int_0^1 \frac{ze^A}{1-z+ze^A} p_0(z) dz, \quad \frac{\partial H}{\partial B} = -\lambda_U \bar{y} + \lambda_U \int_0^1 \frac{ze^B}{1-z+ze^B} q_0(z) dz,$$

$$\frac{\partial^2 H}{\partial A^2} = -\lambda_V \int_0^1 \frac{z(1-z)e^A}{(1-z+ze^A)^2} p_0(z) dz, \quad \frac{\partial^2 H}{\partial B^2} = \lambda_U \int_0^1 \frac{z(1-z)e^B}{(1-z+ze^B)^2} q_0(z) dz,$$

and $\partial^2 H / \partial A \partial B = 0$. It follows that (\bar{A}, \bar{B}) is a saddle point for the surface $H(A, B)$ when λ_U, λ_V have the same sign. Thus, if $\lambda_U, \lambda_V > 0$, then (\bar{A}, \bar{B}) is a local maximum with respect to variation in A and a local minimum with respect to variation in B . The regions of the (A, B) -plane satisfying $H < \bar{H}$ and $H > \bar{H}$ are separated by the separatrices (along which $H(A, B) = \bar{H}$), with H increasing from \bar{H} in the B -direction, and H decreasing from \bar{H} in the A -direction. The situation is illustrated in Fig 3.

The solution trajectory of of (95)-(96) we are interested is $H(A, B) = H(0, 0) = 0$. How this

trajectory behaves depends on which region of the (A, B) -plane contains the origin $(0, 0)$. Thus, if $\bar{H} < 0$, then $(0, 0)$ lies in one of the two regions $H > \bar{H}$, and if $\bar{H} > 0$, it lies in one of the two regions $H < \bar{H}$ (Fig 3). There are several (generic) cases²⁷

$$\begin{aligned}
i) \quad \langle x \rangle_0 > \bar{x}, \quad \langle y \rangle_0 > \bar{y}, \quad &\Rightarrow \quad \bar{A} < 0, \quad \bar{B} < 0, \\
ii) \quad \langle x \rangle_0 > \bar{x}, \quad \langle y \rangle_0 \leq \bar{y}, \quad &\Rightarrow \quad \bar{A} < 0, \quad \bar{B} \geq 0, \\
iii) \quad \langle x \rangle_0 \leq \bar{x}, \quad \langle y \rangle_0 > \bar{y}, \quad &\Rightarrow \quad \bar{A} \geq 0, \quad \bar{B} < 0, \\
iv) \quad \langle x \rangle_0 < \bar{x}, \quad \langle y \rangle_0 < \bar{y}, \quad &\Rightarrow \quad \bar{A} > 0, \quad \bar{B} > 0.
\end{aligned} \tag{106}$$

Cases (i) and (iv) have already been considered. For cases (ii) and (iii), \bar{A} and \bar{B} have opposite signs, and four possible subcases can arise, depending on which region of the (A, B) -plane the origin lies (see Fig 3):

$$\begin{aligned}
i) \quad \bar{A} > 0, \quad \bar{B} < 0 \text{ with } \bar{H} < 0 \quad &\Rightarrow \quad (A(t), B(t)) \rightarrow (\infty, \infty), \\
ii) \quad \bar{A} > 0, \quad \bar{B} < 0 \text{ with } \bar{H} > 0 \quad &\Rightarrow \quad (A(t), B(t)) \rightarrow (-\infty, -\infty), \\
iii) \quad \bar{A} < 0, \quad \bar{B} > 0 \text{ with } \bar{H} > 0 \quad &\Rightarrow \quad (A(t), B(t)) \rightarrow (\infty, \infty), \\
iv) \quad \bar{A} < 0, \quad \bar{B} > 0 \text{ with } \bar{H} < 0 \quad &\Rightarrow \quad (A(t), B(t)) \rightarrow (-\infty, -\infty).
\end{aligned} \tag{107}$$

We prove these asymptotic properties in the next subsection.

A.4.2 Asymptotic properties of the trajectories $A(t), B(t)$

It remains to show that, under mild assumptions, $|A(t)|, |B(t)| \rightarrow \infty$ as $t \rightarrow \infty$ under the dynamics (95)-(96). Specifically, we require that $p_0(x)$ has no mass points at $x = 0$ or $x = 1$, and similarly for $q_0(y)$.

Suppose that $p_0(x)$ has no mass point at $x = 0$. More precisely, suppose that:

$$\lim_{\varepsilon \rightarrow 0} \int_0^\varepsilon \ln(z) p_0(z) dz = 0.$$

In this case, it follows that $\int_0^1 \ln(z + (1-z)e^{-A}) p_0(z) dz$ is negative and decreasing in A , with

$$\int_0^1 \ln(z + (1-z)e^{-A}) p_0(z) dz \rightarrow -L_0(p_0) \quad \text{as } A \rightarrow \infty,$$

where $L_0(p_0) = -\int_0^1 \ln(z) p_0(z) dz$ is a finite, positive constant.

Similarly, suppose that $p_0(x)$ has no mass point at $x = 1$. More precisely, suppose that

$$\lim_{\varepsilon \rightarrow 0} \int_{1-\varepsilon}^1 \ln(1-z) p_0(z) dz = 0.$$

²⁷These implications follow from the definitions of \bar{A}, \bar{B} in (97) and (98), together with the fact that $F(e^{-X} | p)$ is increasing in X from 0 when $X = -\infty$ to 1 when $X = \infty$, and that $F(1 | p) = \langle z \rangle$, the mean of z with respect to the density $p(z)$.

In this case, it follows that $\int_0^1 \ln(1 - z + ze^A) p_0(z) dz$ is negative and decreasing in $-A$, with

$$\int_0^1 \ln(1 - z + ze^A) p_0(z) dz \rightarrow -L_1(p_0) \quad \text{as } A \rightarrow -\infty,$$

where $L_1(p_0) = -\int_0^1 \ln(1 - z) p_0(z) dz$ is a finite, positive constant.

For example, if $p_0(x)$ is the density of a beta distribution, $p_0(x) = \beta(a, b)^{-1} x^{a-1} (1-x)^{b-1}$ with $a, b > 0$, then

$$\begin{aligned} L_0(p_0) &= \frac{\Gamma'(a+b)}{\Gamma(a+b)} - \frac{\Gamma'(a)}{\Gamma(a)}, \\ L_1(p_0) &= \frac{\Gamma'(a+b)}{\Gamma(a+b)} - \frac{\Gamma'(b)}{\Gamma(b)}. \end{aligned}$$

In general, we have shown that the following boundedness conditions hold:

$$-L_0(p_0) < \int_0^1 \ln(z + (1-z)e^{-A}) p_0(z) dz \leq 0 \quad \text{for all } A \geq 0, \quad (108)$$

$$-L_1(p_0) < \int_0^1 \ln(1 - z + ze^A) p_0(z) dz \leq 0 \quad \text{for all } A \leq 0. \quad (109)$$

Similar bounds can be obtained under similar assumptions for $q_0(y)$.

Now consider the case discussed in subsection A.4.1 in which $\lambda_U, \lambda_V > 0$ and the equilibrium (\bar{A}, \bar{B}) , given by (97) and (98), is an unstable saddle node - see Fig 3. We show that all trajectories are asymptotic to straight lines as $t \rightarrow \infty$.

From (101), (102) and (103), we have:

$$\begin{aligned} y_1(A | p_0) &= \bar{x}A - \int_0^1 \ln(1 - z + ze^A) p_0(z) dz \\ &= -(1 - \bar{x})A - \int_0^1 \ln(z + (1-z)e^{-A}) p_0(z) dz. \end{aligned}$$

Then the above discussion implies that $y_1(A | p_0) \sim -(1 - \bar{x})A + L_0(p_0)$ as $A \rightarrow \infty$, and $y_1(A | p_0) \sim \bar{x}A + L_1(p_0)$ as $A \rightarrow -\infty$. It follows that, for $0 < \bar{x} < 1$, we have $y_1(A | p_0) \rightarrow -\infty$ as $|A| \rightarrow \infty$. Similarly, $0 < \bar{y} < 1$ implies that $y_2(B | q_0)$ is asymptotic to a straight line, with $y_2(B | q_0) \rightarrow -\infty$ as $|B| \rightarrow \infty$.

Now consider a trajectory of the form

$$H(A, B) = \lambda_V y_1(A | p_0) - \lambda_U y_2(B | q_0) = H_0,$$

with $\lambda_U, \lambda_V > 0$ and H_0 a constant. Then $\lambda_U y_2(B | q_0) = \lambda_V y_1(A | p_0) - H_0 \rightarrow -\infty$ as $|A| \rightarrow \infty$. Thus, $y_2(B | q_0) \rightarrow -\infty$ as $|A| \rightarrow \infty$, from which it follows that $|B| \rightarrow \infty$ as $|A| \rightarrow \infty$. This implies that the trajectory $H(A, B) = H_0$ is asymptotic to a straight line of the form:

$$C_U B - C_V A = D,$$

for C_U, C_V non-zero constants, as $A, B \rightarrow \pm\infty$. The parameters of these asymptotic lines are given by:

	C_U	C_V	D	
<i>i</i>)	$(A, B) \rightarrow (+\infty, +\infty)$	$\lambda_U(1 - \bar{y})$	$\lambda_V(1 - \bar{x})$	$H_0 - \lambda_V L_0(p_0) + \lambda_U L_0(q_0)$
<i>ii</i>)	$(A, B) \rightarrow (-\infty, +\infty)$	$\lambda_U(1 - \bar{y})$	$\lambda_V \bar{x}$	$H_0 + \lambda_V L_1(p_0) + \lambda_U L_0(q_0)$
<i>iii</i>)	$(A, B) \rightarrow (+\infty, -\infty)$	$\lambda_U \bar{y}$	$\lambda_V(1 - \bar{x})$	$H_0 - \lambda_V L_0(p_0) - \lambda_U L_1(q_0)$
<i>iv</i>)	$(A, B) \rightarrow (-\infty, -\infty)$	$\lambda_U \bar{y}$	$\lambda_V \bar{x}$	$H_0 + \lambda_V L_1(p_0) - \lambda_U L_1(q_0)$

(110)

This analysis completes the justification of the asymptotic properties stated in (107).

A.5 A class of examples for 2×2 asymmetric games

For the initial densities $p_0(x) = (1 - \alpha)\delta(x - a_0) + \alpha\delta(x - a_1)$ and $q_0(y) = \delta(y - y_0)$, as discussed in section 9.3, it follows from (102) that

$$y_1(A | p_0) = \bar{x}A - \{(1 - \alpha)\ln(1 - a_0 + a_0e^A) + \alpha\ln(1 - a_1 + a_1e^A)\}, \quad (111)$$

$$y_2(B | q_0) = \bar{y}B - \ln(1 - y_0 + y_0e^B). \quad (112)$$

The equilibrium \bar{B}_0 of (83) is given by:

$$\bar{B}_0 = \ln \left[\frac{(1 - y_0)\bar{y}}{(1 - \bar{y})y_0} \right], \quad (113)$$

and

$$y_2(\bar{B}_0 | \delta_{y_0}) = \ln \left[\left(\frac{\bar{y}}{y_0} \right)^{\bar{y}} \left(\frac{1 - \bar{y}}{1 - y_0} \right)^{1 - \bar{y}} \right]. \quad (114)$$

Note that this is always positive for $y_0 \neq \bar{y}$. Similarly, the equilibrium \bar{A}_0 associated with the initial density $\delta(x - x_0)$ is

$$\bar{A}_0 = \ln \left[\frac{(1 - x_0)\bar{x}}{(1 - \bar{x})x_0} \right]. \quad (115)$$

Observe that this is positive for $x_0 < \bar{x}$. As in (101), we can define

$$\bar{H}_0 = H(\bar{A}_0, \bar{B}_0) = \lambda_V y_1(\bar{A}_0 | \delta_{x_0}) - \lambda_U y_2(\bar{B}_0 | \delta_{y_0}). \quad (116)$$

This is the difference of two positive constants, and can therefore be either positive or negative. In particular, with all other parameters fixed, we can choose $\lambda_U > 0$ so that \bar{H}_0 is negative.

The equilibrium \bar{A} for (84) is given by a solution of

$$(1 - \alpha) \frac{a_0 e^A}{1 - a_0 + a_0 e^A} + \alpha \frac{a_1 e^A}{1 - a_1 + a_1 e^A} = \bar{x}, \quad (117)$$

The left hand side is a monotonically increasing function of A , equal to 0 when $A = -\infty$, and equal to 1 when $A = \infty$. There is therefore a unique solution $\bar{A} = \bar{A}(\alpha)$ of (117). Note that $\bar{A} = 0$ if and

only if $\alpha = \alpha^*$, and $\bar{A}(\alpha) > 0$ for $\alpha < \alpha^*$, and $\bar{A}(\alpha) < 0$ for $\alpha > \alpha^*$.

Now define

$$\bar{H} = H(\bar{A}, \bar{B}_0) = \lambda_V y_1(\bar{A} | p_0) - \lambda_U y_2(\bar{B}_0 | \delta_{y_0}). \quad (118)$$

We wish to consider the relationship between \bar{H} and \bar{H}_0 . To this end, note that

$$y_1(A | p_0) - y_1(A | \delta_{x_0}) = \ln(1 - x_0 + x_0 e^A) - \{(1 - \alpha) \ln(1 - a_0 + a_0 e^A) + \alpha \ln(1 - a_1 + a_1 e^A)\}$$

is strictly positive for $0 < \alpha < 1$, since $\ln(\cdot)$ is a strictly concave function. Thus, we can write

$$\begin{aligned} \bar{H} - \bar{H}_0 &= \lambda_V \{y_1(\bar{A} | p_0) - y_1(\bar{A}_0 | \delta_{x_0})\} \\ &= \lambda_V \{y_1(\bar{A} | p_0) - y_1(\bar{A}_0 | p_0)\} + \lambda_V \{y_1(\bar{A}_0 | p_0) - y_1(\bar{A}_0 | \delta_{x_0})\} \\ &> \lambda_V \{y_1(\bar{A} | p_0) - y_1(\bar{A}_0 | p_0)\} \end{aligned} \quad (119)$$

for $0 < \alpha < 1$. We shall show that this is strictly positive for $0 < \alpha < \alpha^*$.

Given this claim, it now follows from (119) that $\bar{H} > \bar{H}_0$ for any α in the range $0 < \alpha < \alpha^*$. Given such an α , and fixed $\lambda_V > 0$ and (\bar{x}, \bar{y}) , it follows from (116) and (118), and the fact that $y_2(\bar{B}_0 | \delta_{y_0})$ is a positive constant, that we can choose $\lambda_U > 0$ so that $\bar{H}_0 < 0$ and $\bar{H} > 0$. For α in the given range we have $\langle x \rangle_0 = x_0 < \bar{x}$. If, in addition, we choose $y_0 > \bar{y}$, then it follows from (106) (iii) that $\bar{A}_0, \bar{A} > 0$ and $\bar{B}_0 < 0$. Hence, since $\bar{H}_0 < 0$, it follows from (107) (i) that $(A_0(t), B_0(t)) \rightarrow (\infty, \infty)$, and since $\bar{H} > 0$, it follows from (107) (ii) that $(A(t), B(t)) \rightarrow (-\infty, -\infty)$ as $t \rightarrow \infty$. Thus, the two trajectories, the first associated with the Replicator dynamics, and the second with the Distributional Replicator dynamics, converge to opposite equilibria. This establishes the claims of section 9.3.

Proof of Claim. Consider

$$\begin{aligned} \frac{\partial y_1}{\partial A}(A | p_0) &= \bar{x} - (1 - \alpha) \frac{a_0 e^A}{1 - a_0 + a_0 e^A} - \alpha \frac{a_1 e^A}{1 - a_1 + a_1 e^A} \\ \frac{\partial^2 y_1}{\partial A^2}(A | p_0) &= -(1 - \alpha) \frac{a_0(1 - a_0)e^A}{(1 - a_0 + a_0 e^A)^2} - \alpha \frac{a_1(1 - a_1)e^A}{(1 - a_1 + a_1 e^A)^2} \end{aligned}$$

Clearly, $\frac{\partial^2 y_1}{\partial A^2}(A | p_0) < 0$ for all A . Since, by definition, $\frac{\partial y_1}{\partial A}(\bar{A} | p_0) = 0$, it follows that $A = \bar{A}$ is a non-degenerate global maximum of $y_1(A | p_0)$. Also, $\frac{\partial y_1}{\partial A}(A | p_0)$ decreases monotonically from \bar{x} to $-(1 - \bar{x})$ as A increases from when $-\infty$ to ∞ .

Since \bar{A} is a global maximum for $y_1(A | p_0)$, we need only show that $\bar{A}_0(\alpha) \neq \bar{A}(\alpha)$ for any α in the range $0 < \alpha < \alpha^*$. It then follows that $y_1(\bar{A} | p_0) - y_1(\bar{A}_0 | p_0) > 0$, as claimed. To this end, it suffices to show that $\frac{\partial y_1}{\partial A}(\bar{A}_0 | p_0) \neq 0$. Consider

$$\begin{aligned} \frac{\partial y_1}{\partial A}(\bar{A}_0 | p_0) &= \bar{x} - (1 - \alpha) \frac{a_0}{a_0 + (1 - a_0)e^{-\bar{A}_0}} - \alpha \frac{a_1}{a_1 + (1 - a_1)e^{-\bar{A}_0}} \\ &= \bar{x} - (1 - \alpha) \frac{a_0(1 - x_0)\bar{x}}{a_0(1 - x_0)\bar{x} + (1 - a_0)x_0(1 - \bar{x})} - \alpha \frac{a_1(1 - x_0)\bar{x}}{a_1(1 - x_0)\bar{x} + (1 - a_1)x_0(1 - \bar{x})} \\ &= \bar{x} \left\{ 1 - \frac{(1 - \alpha)a_0(1 - x_0)}{a_0(1 - x_0)\bar{x} + (1 - a_0)x_0(1 - \bar{x})} - \frac{\alpha a_1(1 - x_0)}{a_1(1 - x_0)\bar{x} + (1 - a_1)x_0(1 - \bar{x})} \right\} \end{aligned}$$

To evaluate the expression in $\{\}$, consider

$$\begin{aligned}
N &= (1 - \alpha)a_0 \{a_1(1 - x_0)\bar{x} + (1 - a_1)x_0(1 - \bar{x})\} + \alpha a_1 \{a_0(1 - x_0)\bar{x} + (1 - a_0)x_0(1 - \bar{x})\} \\
&= a_0 a_1 (1 - x_0)\bar{x} + \{(1 - \alpha)a_0(1 - a_1) + \alpha a_1(1 - a_0)\} x_0(1 - \bar{x}) \\
&= a_0 a_1 (1 - x_0)\bar{x} + (x_0 - a_0 a_1)x_0(1 - \bar{x}) \quad \text{using (80)} \\
&= a_0 a_1 \{(1 - x_0)\bar{x} - x_0(1 - \bar{x})\} + x_0^2(1 - \bar{x}) \\
&= a_0 a_1 (\bar{x} - x_0) + x_0^2(1 - \bar{x}).
\end{aligned}$$

which is positive for $\alpha < \alpha^*$ – see (82). Now consider

$$\begin{aligned}
M &= \{a_1(1 - x_0)\bar{x} + (1 - a_1)x_0(1 - \bar{x})\} \{a_0(1 - x_0)\bar{x} + (1 - a_0)x_0(1 - \bar{x})\} \\
&= a_0 a_1 (1 - x_0)^2 \bar{x}^2 + (1 - a_0)(1 - a_1)x_0^2(1 - \bar{x})^2 + \{a_1(1 - a_0) + a_0(1 - a_1)\} x_0(1 - x_0)\bar{x}(1 - \bar{x}) \\
&= a_0 a_1 \{(1 - x_0)^2 \bar{x}^2 + x_0^2(1 - \bar{x})^2 - 2x_0(1 - x_0)\bar{x}(1 - \bar{x})\} \\
&\quad - (a_0 + a_1) \{x_0^2(1 - \bar{x})^2 - x_0(1 - x_0)\bar{x}(1 - \bar{x})\} + x_0^2(1 - \bar{x})^2 \\
&= a_0 a_1 \{(1 - x_0)\bar{x} - x_0(1 - \bar{x})\}^2 + (a_0 + a_1)x_0(1 - \bar{x}) \{(1 - x_0)\bar{x} - x_0(1 - \bar{x})\} + x_0^2(1 - \bar{x})^2 \\
&= a_0 a_1 (\bar{x} - x_0)^2 + (a_0 + a_1)x_0(1 - \bar{x})(\bar{x} - x_0) + x_0^2(1 - \bar{x})^2
\end{aligned}$$

Hence, setting $L = M - N(1 - x_0)$, we have

$$\begin{aligned}
L(\alpha) &= -\{a_0 a_1 - (a_0 + a_1)x_0 + x_0^2\} (1 - \bar{x})(\bar{x} - x_0) \\
&= (a_1 - x_0)(x_0 - a_0)(1 - \bar{x})(\bar{x} - x_0)
\end{aligned}$$

Thus $L(0) = L(\alpha^*) = 0$, and $L(\alpha) > 0$ for $0 < \alpha < \alpha^*$. It therefore follows that

$$\frac{\partial y_1}{\partial A}(\bar{A}_0 | p_0) = \bar{x} \frac{L(\alpha)}{M} > 0, \quad \text{for } 0 < \alpha < \alpha^*.$$

This shows that $\bar{A}_0(\alpha) < \bar{A}(\alpha)$ for $0 < \alpha < \alpha^*$, as required. ■

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