FICTITIOUS PLAY IN AN EVOLUTIONARY ENVIRONMENT

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Abstract. We consider a continuous time version of the fictitious play model in an evolutionary environment. We derive two forms of the continuous time limit, both of which are approximations to a determinate version of the fictitious play updating algorithm. The first is in the form of a first-order partial differential equation ('continuity equation') which we solve explicitly. The dynamic for a distribution of strategies is also derived, which we show that can be written in a form similar to a positive definite dynamic. The asymptotic solution (in the ultra long run) is discussed in detail for 2-player, 2-strategy co-ordination and anti-coordination games, and we show convergence to Nash equilibrium in both cases. The second, and better, approximation (because 2nd-order terms are no-longer neglected) is in the form of a diffusion equation. This is considerably more difficult to analyse. However, we show that it leads to the same asymptotic limit as the 1st-order approximation.

1. Introduction

The fictitious play algorithm was introduced in [1, 15] as a tool for finding Nash equilibria of games. Soon afterward this algorithm was reinterpreted as a model of learning. In the classical case of an n-player game, a player uses a history of past games as a basis for prediction (beliefs) of his opponents' expected actions[1] and responds with a best-reply to the prediction.

The positive results on the convergence of beliefs in the fictitious play algorithm to a Nash equilibrium were established for 2-player zero-sum games ([15]), 2×2 games ([13]), potential games ([14]), games with an interior ESS ([7]) and some classes of super-modular games ([12, 11, 6]).

On the other hand, there are plenty of negative results concerning fictitious play. Even if beliefs do converge to a mixed-strategy Nash equilibrium, the actual game may fail to converge, passing instead through a deterministic cycle involving strategies in the support of the Nash equilibrium. This kind of behavior was noted in [16, 3, 9, 18]. The reason for this is that the best-reply map is not continuous and arbitrarily small changes in empirical frequencies (beliefs) may lead to jumps in actions.

One way to circumvent this problem is to slightly change the game. In [3] game payoffs are randomly perturbed with the consequence that the anticipated behavior of each player is always a mixed strategy. It was then possible to extend results from [13] proving convergence of actual play[2]. Another approach is to change the algorithm itself by adding noise or considering only independent samples from the recent history of play, as in [18]. It is also possible to consider a population of players instead of a single player, in which all players have private information. The dynamic is then an aggregate behavior of the population. This is the approach adopted in [2] and more recently [8].

This paper is concerned with a standard fictitious play algorithm used in an evolutionary scenario in which, in each round, some constant share of a population of players is randomly matched to play a symmetric 2-player game. A history of each player is private, i.e. it is a sequence of opponents' actions observed by a player. We use a setting similar to that introduced in [2], but instead of working within a probabilistic framework we derive two continuous-time partial differential equations which are increasingly close approximations to this dynamic. We solve the simpler, continuity equation approximation analytically and derive the dynamic for the aggregate behavior of the population. Further, we note that the model in [8] is essentially a projection of our model and so we extend the analysis presented there.

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[1]It is straightforward in 2-player games. In n-player games with $n > 2$ a player may build either a joint empirical distribution for all opponents or marginal distributions, one for each opponent. See [4] for further discussion.

[2]The resulting dynamic is called stochastic fictitious play.

Since we work with a limit of a model from [2] we also point to some similarities between these two papers. The second, diffusion approximation cannot be analysed so completely, but, in Appendices, we derive the form of its solution, and present an asymptotic analysis for a 2-player, 2-strategy anti-coordination game.

We introduce our model in Section 2. This section contains a solution of the derived 1st-order partial differential equation. Section 3 contains a discussion of the solution and compares it to the results presented in [2, 8]. In the last section we summarize the main results of the paper.

2. The model

2.1. General setting. The learning process is examined in the context of 2-player symmetric games, $\mathcal{G} = (\{1,2\}, S, A)$ where S is a set of n pure strategies and A is a payoff matrix with a typical element a_{ij}. We are dealing with a population of players, each playing a single pure strategy. The distribution of strategies within the population is described by a vector $\mu = (\mu_1, \dots, \mu_n) \in \triangle_n$ where $\triangle_n = \left\{\mu \in \mathbb{R}^n_+ \;:\; \sum_i \mu_i = 1\right\}$.

The fictitious play algorithm chooses a best reply to a prediction of the opponent's action. The prediction takes the following special form. Each player keeps track of how many times she observes each pure strategy. Let $x = (x_1, \dots, x_n)$ be a vector of non-negative weights. These weights are updated every time a player is selected to play in the following way

$$x_i' = x_i + \begin{cases} 1 & \text{if the observed strategy is } i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with initial values of weights given exogenously. The predicted probability assigned to a pure strategy i given the weight vector x is

$$\gamma_i(x) = \frac{x_i}{\sum_j x_j}, \quad i = 1, \dots, n.$$

The fictitious play algorithm then chooses a strategy that belongs to the set of best replies $\beta(\gamma(x))$. The probability distribution function γ can be used to define the following sets. Let $\Omega = \mathbb{R}^n_+$, and for each pure strategy i, define a subset $\Omega_i \subset \Omega$ by

$$\Omega_i = \{x \in \Omega \;:\; i \in \beta(\gamma(x))\}.$$

It is clear that $\Omega = \bigcup_i \Omega_i$ and that for any generic game, $\Omega_i \cap \Omega_j$ for $j \neq i$ is a subset of a linear subspace of Ω. In particular, it has (Lebesque) measure 0.

As described above, the weights for a player can be represented as a point $x \in \Omega$. On the other hand, the *beliefs* of the population at any time t are represented by a probability distribution over Ω, with probability density function $p(x,t)$. We make the following technical assumption[3].

Condition 2.1. We assume that initially (at $t = 0$) $\mathrm{supp}(p) = \Omega$ and p is continuously differentiable.

Given the probability density p of a distribution over weights, and having defined sets Ω_i, we can calculate shares of the population using particular pure strategies

$$\mu_i = \int_{\Omega_i} p(x)dx, \quad i = 1, \dots, n \tag{2}$$

and since the measure of any set $\Omega_i \cap \Omega_j$ for $i \neq j$ is 0 the vector μ is a probability distribution over pure strategies. Also, because of the Assumption 1, $\mu \in \mathrm{int}(\triangle_n)$ at $t = 0$.

The idea is then to study how the distribution of weights given by a density p changes over time, and consequently how the distribution μ of pure strategies within the population changes over time. The model presented here is essentially the same as the one presented in [2]. Each player may be at one of the possible states and so there is a distribution of players over a state space. This distribution gives in turn a distribution of certain attributes, e.g. of pure strategies within the population, which governs the transition probabilities between states. The exhaustive probabilistic analysis of this class of models[4] was presented in [2] using techniques from [5]. Instead of working within a probabilistic framework we

[3] This is also discussed and assumed in [8, p. 86, 89, 93].

[4] This is a very simplified view of the type of models presented in [2] but the main idea is preserved.

derive here a continuous-time, continuity limit equation and study its solutions. This limit is implicitly assumed in [8][5].

2.2. **Updating process.** Let τ be a small time step and δ a small 'space' step. We consider the following updating process for states $x \in \Omega$:

$$\begin{aligned} p\left(x_1, \ldots, x_n, t+\tau\right) dx = & \,(1-\eta)\, p(x_1, \ldots, x_n, t) dx \\ & + \eta\, \mu_1\, p(x_1 - \delta, \ldots, x_n, t) dx \\ & + \ldots \\ & + \eta\, \mu_n\, p(x_1, \ldots, x_n - \delta, t) dx. \end{aligned} \tag{3}$$

The above updating process is consistent with the following evolutionary scenario. In the short period of time from t to $t+\tau$ a proportion η of a large population of players is randomly matched to play a 2-player symmetric game. Each player observes the strategy of her random opponent and updates her weights x accordingly.

The "mass" interpretation of this scenario is the following. The mass of population at state $x = (x_1, \ldots, x_n)$ at time $t+\tau$ is $p(x_1, \ldots, x_n, t+\tau)dx$. This is a sum of the mass of players not chosen to play in this round, $(1-\eta)\, p(x_1, \ldots, x_n, t)dx$, and the masses of chosen players that have updated their weights, i.e. from the mass of players $p(x_1 - \delta, \ldots, x_n, t)dx$ having weights $(x_1 - \delta, \ldots, x_n)$ at time t the share $\eta\mu_1$ is chosen to play and observes the first pure strategy, and consequently the mass of players $\eta\mu_1\, p(x_1 - \delta, \ldots, x_n, t)dx$ moves to the state $(x_1, \ldots, x_n)$, and so on for each pure strategy. This scenario is called "Story 3" in [8, p. 87].

2.3. **The continuity limit.** To derive a continuous-time limit we expand (3) up to terms of order δ:

$$p(x, t+\tau) = (1-\eta)p(x,t) + \eta \left(p(x,t) - \delta \sum_{i=1}^{n} \mu_i(t)\, \frac{\partial p(x,t)}{\partial x_i} \right). \tag{4}$$

Rearranging (4) leads to:

$$p(x, t+\tau) - p(x,t) = \delta\eta \left(-\sum_{i=1}^{n} \mu_i(t) \frac{\partial p}{\partial x_i}(x,t) \right). \tag{5}$$

We assume that δ is constant (and small) and that $\eta = w\tau$, where w is the rate at which players are matched to play the game (ie. updating by agents is a Poisson process with rate w) Dividing both sides of (5) by τ and taking limit $\tau \to 0$ gives the *continuity equation* (suppressing arguments):

$$\frac{\partial p}{\partial t} + w\delta \sum_{i=1}^{n} \mu_i \frac{\partial p}{\partial x_i} = 0. \tag{6}$$

By rescaling time we may assume that $w\delta = 1$.

Clearly, the equation (6) is only an approximation. Including terms of order δ^2 in the expansion of the right side of the equation (4) gives a better approximation. We discuss this possibility in Section 3.2.

2.4. **Boundary conditions.** We assume there is no probability flux across the boundary of Ω. Writing (6) in vector notation gives

$$\frac{\partial p}{\partial t} + \mu \cdot \nabla p = 0. \tag{7}$$

We integrate (7) over Ω and use conservation of probability and the divergence theorem to obtain:

$$\begin{aligned} \frac{d}{dt} \int_\Omega p\, dV &= 0 = -\int_\Omega \nabla \cdot (\mu p)\, dV \\ &= -\int_{\partial\Omega} (\mathbf{n} \cdot \mu) p\, dA, \end{aligned}$$

[5]In fact the model considered in [8] is a certain projection of our model. However, the limit is never derived explicitly in [8].

where $\mathbf{n}$ is the outward pointing normal on the boundary $\partial\Omega$ of Ω. Let $\partial\Omega_{-i} = \{x \in \partial\Omega \,:\, x_i = 0\}$ and denote any $x \in \partial\Omega_{-i}$ by x_{-i}. It is clear that $\partial\Omega = \bigcup_i \partial\Omega_{-i}$ and the normal $\mathbf{n}$ at $x_{-i} \in \partial\Omega_{-i}$ has i-th coordinate -1 with other elements being 0. The general boundary condition is therefore:

$$\sum_{i=1}^{n} \mu_i(t) \int_{\partial\Omega_{-i}} p(x_{-i}, t)\, dx_{-i} = 0. \tag{8}$$

Thus, if $\mu_i(t) > 0$ we require $p(x_{-i}, t) \overset{a.e.}{=} 0$ on $\partial\Omega_{-i}$. The natural boundary conditions are therefore:

$$p(x_{-i}, t) = 0 \quad \text{for all } x_{-i} \in \partial\Omega_{-i} \text{ and all } t > 0, i = 1, \ldots, n. \tag{9}$$

2.5. **Solutions.** The initial value problem (7) with boundary conditions (9) and specified initial density $p_0(x)$ satisfying Assumption 1 has the following solution.

Proposition 2.2. *The general solution of (7) with initial condition p_0 and boundary conditions (9) is*

$$p(x, t) = p_0(x - c(t)), \tag{10}$$

where $c(t) = (c_1(t), \ldots, c_n(t))$ and $c_i(t) = \int_0^t \mu_i(s)\, ds$.

Proof. It is straightforward to check that (10) is a solution to (7). To deal with boundary conditions we extend p_0 to the whole of $\mathbb{R}^n$ by taking $p_0(x) = 0$ if there is at least one element $x_i < 0$. By definition of μ_i and c_i, we have:

$$\dot{c}_i(t) = \mu_i(t) = \int_{\Omega_i} p_0(x - c(t))\, dx \tag{11}$$

with initial condition $c_i(0) = 0$. The solution of these differential equations specifies $c(t)$, and hence $\mu(t)$, uniquely for all $t \geq 0$. Further, it follows from Assumption 1 that $\dot{c}_i(0) = \mu_i(0) > 0$, and since $\dot{c}_i(t)$ cannot be negative, $c_i(t)$ cannot decrease in t. Hence $c(t) \in \operatorname{int}(\mathbb{R}^n_+)$ for all $t > 0$.
On the boundary component $\partial\Omega_{-i}$ we have $p(x_{-i}, t) = p_0(x_1 - c_1(t), \ldots, -c_i(t), \ldots, x_n - c_n(t)) = 0$ for $t > 0$ since $c(t) \in \operatorname{int}(\mathbb{R}^n_+)$. It follows that the boundary conditions (9) are automatically satisfied. □

Note that (11) implies that $\mu(t) > 0$ for all $t \geq 0$. This is because the set $\{x \in \Omega_i \,:\, x > c(t)\}$ has non-zero Lebesgue measure and p_0 has full support on this set by Condition 2.1.

3. Discussion

3.1. **Strategy distribution.** We are interested in deriving a dynamic for the strategy distribution μ. To do this, we use (7) and the divergence theorem to obtain:

$$\begin{aligned}
\dot{\mu}_i(t) &= \int_{\Omega_i} \frac{\partial p(x,t)}{\partial t}\, dV \\
&= -\int_{\Omega_i} \sum_{j=1}^{n} \mu_j(t) \frac{\partial p(x,t)}{\partial x_j}\, dV \\
&= -\int_{\Omega_i} \nabla \cdot [\mu(t) p(x,t)]\, dV \\
&= -\int_{\partial\Omega_i} \mathbf{n}_i \cdot \mu(t) p(x,t)\, dA, \qquad (12)
\end{aligned}$$

where $\mathbf{n}_i$ is the outward pointing normal to $\partial\Omega_i$. The dynamic (12) says that the rate of change in a share of a population using the i-th pure strategy is equal to the rate of flow of mass of players across the boundary of the region Ω_i corresponding to this strategy[6]. The negative sign before the integral in (12) is due to the normal $\mathbf{n}_i$ pointing outwards, so if the players are leaving the region Ω_i the sign of the integral is positive while the share of a population using the i-th pure strategy is decreasing. Before proceeding further, we illustrate this dynamic with two simple examples.

[6]This is discussed but not explicitly derived in [8, p. 94]

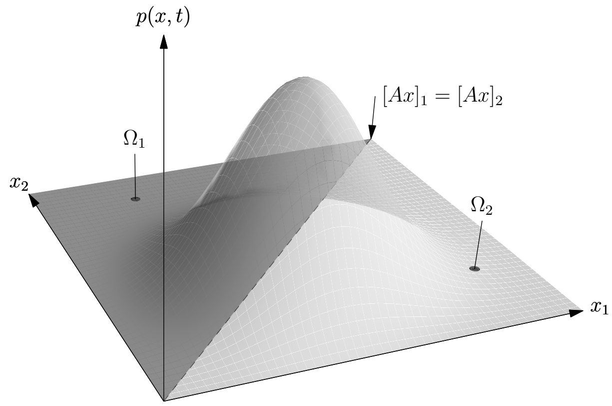

FIGURE 1. Density function p for $n = 2$ and $\beta = 1$ with regions Ω_i, $i = 1, 2$.

Example 3.1. Consider a 2×2 symmetric game with payoff matrix

$$A = \begin{bmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{bmatrix}, \tag{13}$$

where $\alpha_1, \alpha_2 > 0$. There is a unique symmetric mixed strategy Nash equilibrium μ^{NE} with

$$\mu_1^{\mathrm{NE}} = \frac{\alpha_1}{\alpha_1 + \alpha_2} = \frac{1}{1 + \alpha_2/\alpha_1} = \frac{1}{1+\beta}$$

and $\beta > 0$.

The partition of Ω into regions Ω_1 and Ω_2 is given by a ray $r(u) = [u, \beta u]$, $u \in [0, \infty]$, see Fig. 1. Let $s(u)$ denote arc length, so that $ds/du = \sqrt{1+\beta^2}$. The outward pointing normal to $\partial\Omega_1$ along the ray r is given by $\mathbf{n}_1 = [\beta, -1]/\sqrt{1+\beta^2}$. Using formula (12) we obtain

$$\begin{aligned}
\dot{\mu}_1(t) &= -\int_{\partial\Omega_1} \mathbf{n}_1 \cdot \mu(t) p(x,t)\, dA \\
&= -\int_r [\mu_1(t)p(x,t), \mu_2(t)p(x,t)] \cdot \frac{[\beta, -1]}{\sqrt{1+\beta^2}}\, ds \qquad (14) \\
&= -\int_0^\infty [\mu_1(t)p(x,t), \mu_2(t)p(x,t)] \cdot [\beta, -1]\, du \\
&= (\mu_2(t) - \beta\mu_1(t)) \int_0^\infty p(u, \beta u, t)\, du \qquad (15) \\
&= (1 - (1+\beta)\,\mu_1(t)) \int_0^\infty p(u, \beta u, t)\, du, \qquad (16)
\end{aligned}$$

where equality (14) comes from the boundary conditions (9). Since the integral in (16) is always positive, It is clear that the only rest point of this dynamic is the Nash equilibrium μ^{NE}, which is globally asymptotically stable. It is also worth noting that (15) can be rewritten as

$$\begin{aligned}
\mu_1(t) &= (\mu_2(t) - \beta\mu_1(t)) \int_0^\infty p(u, \beta u, t)\, du \\
&= \frac{1}{\alpha_1} (\alpha_1\mu_2(t) - \alpha_2\mu_1(t)) \int_0^\infty p(u, \beta u, t) du \\
&= \frac{1}{\alpha_1} \left[\int_0^\infty p(u, \beta u, t) du \right] ([A\mu]_1 - [A\mu]_2). \qquad (17)
\end{aligned}$$

The dynamic (17) is of the same form as the continuous time version proposed in [8, p. 93, Prop. 5].

Example 3.2. Consider a coordination game with payoff matrix

$$A = \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix},$$

where $\alpha_1, \alpha_2 > 0$. There is a unique symmetric mixed strategy Nash equilibrium μ^{NE} where

$$\mu_1^{\mathrm{NE}} = \frac{\alpha_2}{\alpha_1 + \alpha_2} = \frac{1}{1 + \alpha_1/\alpha_2} = \frac{1}{1+\beta}$$

and $\beta > 0$. The partition of Ω into regions Ω_1 and Ω_2 is again given by a ray $r(u) = [u, \beta u]$, $u \in [0, \infty]$. As before, let $s(u)$ denote arc length, so that $ds/du = \sqrt{1+\beta^2}$. The outward pointing normal for Ω_1 at a ray r is given by $\mathbf{n}_1 = [-\beta, 1]/\sqrt{1+\beta^2}$. Using formula (12) and performing the same calculations as for Example 1 we obtain

$$\begin{aligned} \dot{\mu}_1(t) &= \frac{1}{\alpha_2}\left[\int_0^\infty p(u, \beta u, t)du\right]([A\mu]_1 - [A\mu]_2) \\ &= ((1+\beta)\mu_1(t) - 1)\int_0^\infty p(u, \beta u, t)du. \end{aligned} \tag{18}$$

The sign of $\dot{\mu}_1$ is completely determined by the term $((1+\beta)\mu_1(t) - 1)$. The Nash equilibrium is a rest point of (18) but it is unstable. Depending on initial conditions the share μ_1 either increases or decreases monotonically. Thus, either $\mu_1(t) \to 1$ (and $\mu_2(t) \to 0$) or $\mu_1(t) \to 0$ (and $\mu_2(t) \to 1$) as $t \to \infty$. In either case, it follows from (10) that $\int_0^\infty p(u, \beta u, t)du \to 0$.

We now return to the general case and recover a result from [8, p. 93, Prop. 5]. Let $\Omega_{ij} = \Omega_i \cap \Omega_j$. We have the following proposition.

Proposition 3.3. *The dynamic (12) has the following form:*

$$\dot{\mu}_i = \sum_{j \neq i} g_{ij}(t)\left([A\mu]_i - [A\mu]_j\right), \tag{19}$$

where $g_{ij} = g_{ji} > 0$.

Proof. The boundary $\partial\Omega_i$ is a union of sets Ω_{ij}, $i \neq j$. Any set Ω_{ij} is contained in a set given by an equality $[Ax]_i = [Ax]_j$ and so an outward pointing normal $\mathbf{n}_{ij}$ is given by

$$\mathbf{n}_{ij} = \frac{[a_{j1} - a_{i1}, \ldots, a_{jn} - a_{in}]}{\sqrt{(a_{j1} - a_{i1})^2 + \ldots + (a_{jn} - a_{in})^2}}.$$

Using (12) gives

$$\begin{aligned} \dot{\mu}_i(t) &= -\int_{\partial\Omega_i} \mathbf{n}_i \cdot \mu(t)p(x,t)\, dA \\ &= -\sum_{j\neq i}\int_{\Omega_{ij}} \mathbf{n}_{ij} \cdot \mu(t)p(x,t)\, dA \\ &= \sum_{j \neq i} \frac{\int_{\Omega_{ij}} p(x,t)\, dA}{\sqrt{(a_{j1} - a_{i1})^2 + \ldots + (a_{jn} - a_{in})^2}}\left([A\mu]_i - [A\mu]_j\right)(t) \\ &= \sum_{j\neq i} g_{ij}(t)\left([A\mu]_i - [A\mu]_j\right)(t), \end{aligned} \tag{20}$$

where equation (20) comes from boundary conditions (9) and

$$\begin{aligned} g_{ij}(t) &= g_{ji}(t) = \frac{\int_{\Omega_{ij}} p(x,t)\, dA}{\sqrt{(a_{j1} - a_{i1})^2 + \ldots + (a_{jn} - a_{in})^2}} \\ &= \frac{\int_{\Omega_{ij}} p_0(x - c(t))\, dA}{\sqrt{(a_{j1} - a_{i1})^2 + \ldots + (a_{jn} - a_{in})^2}}, \end{aligned}$$

where $c(t) = (c_1(t), \dots, c_n(t))$ and $c_i(t) = \int_0^t \mu_i(s)\, ds$. □

In fact system (19) can be written in a matrix form $\dot{\mu} = Q^g A\mu$ where matrix Q^g has diagonal entries $Q^g_{ii} = \sum_{j\neq i} g_{ij}$ and off-diagonal entries $Q^g_{ij} = -g_{ij} = -g_{ji}$ for $j \neq i$.
It is clear that the matrix Q^g is symmetric and that $Q^g\mathbf{1} = 0$, where $\mathbf{1} = [1, \dots, 1]$. For any vector $\mu \in \mathbb{R}^n$ we have $\mu^{\mathrm{T}} Q^g \mu = \sum_{j\neq i} g_{ij}\, (\mu_i - \mu_j)^2 \geq 0$ with equality if and only if μ is collinear with $\mathbf{1}$. Also, it follows from (11) that the simplex $\triangle_n$ is forward invariant under the dynamic (12).

Following [8, p. 96]), we call a dynamic satisfying the above conditions a *positive definite dynamic*[7]. Hence, we reestablished Proposition 7 from [8, p. 97].

Proposition 3.4. *The dynamic (12), having the form (19), is a positive definite dynamic.*

3.2. **Diffusion limit.** As mentioned earlier, the continuity limit studied above is only an approximation. It is possible to get a better approximation by including in the expansion of the right side of the equation (3) terms up to order δ^2. This leads to:

$$p(x, t+\tau) = (1-\eta)p(x,t) + \eta \left(p(x,t) - \delta \sum_{i=1}^{n} \mu_i(t)\, \frac{\partial p(x,t)}{\partial x_i} + \frac{\delta^2}{2} \sum_{i=1}^{n} \mu_i(t)\, \frac{\partial^2 p(x,t)}{\partial x_i^2} \right). \tag{21}$$

Rearranging (21) gives:

$$p(x, t+\tau) - p(x,t) = \delta\eta \left(- \sum_{i=1}^{n} \mu_i(t)\, \frac{\partial p(x,t)}{\partial x_i} + \frac{\delta}{2} \sum_{i=1}^{n} \mu_i(t)\, \frac{\partial^2 p(x,t)}{\partial x_i^2} \right). \tag{22}$$

We again assume that δ is constant (but small) and that $\eta = w\tau$. The limit $\tau \to 0$ then gives, after rescaling time (and suppressing arguments) the following *diffusion limit*

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial p}{\partial x_i} = \frac{1}{2}\delta \sum_{i=1}^{n} \mu_i \frac{\partial^2 p}{\partial x_i^2}, \tag{23}$$

where shares μ are defined as before, cf. (2).

It is clear that the continuity approximation (6) "forgets" about players that have not played at least a single round since the support of a solution to (6) moves away from the boundary $\partial\Omega$ at a rate $c(t)$. Also, the shape of the density function p does not change. The diffusion limit (23) is also an approximation, but more accurate. Due to the inclusion of the diffusion term it also allows for changes of the shape of the density function p. As is well known, there are still obvious problems with this approach, cf. [10]. We only note here that the approximation (23) is valid only for large times in the region $|\mathbf{E}p - x| \sim O(\sqrt{t})$.

We require no-flux boundary conditions. Integrating (23) and using the divergence theorem gives

$$\begin{aligned} \frac{d}{dt}\int_\Omega p\, dV &= 0 = \int_\Omega \frac{\partial p}{\partial t} dV \\ &= -\int_\Omega \nabla\cdot(\mu p)\, dV + \frac{1}{2}\delta \int_\Omega \nabla\cdot(\mu * \nabla p)\, dV \\ &= \int_{\partial\Omega} \mathbf{n}\cdot\left(-\mu p + \frac{1}{2}\delta\mu * \nabla p\right) dA, \end{aligned} \tag{24}$$

[7] There is a difference between the usual definition of positive definite dynamic and the one we use following [8]. The matrix Q^g depends through an integral on a density function p which in turn is defined as a shift of an initial density p_0 by a vector c, where $c_i(t) = \int_0^t \mu_i(s)\, ds$. Therefore, the matrix Q^g depends on the whole path of μ up to time t rather than just a value of μ at time t. As a consequence, an initial density function p_0 uniquely determines a solution $p(x,t)$ but different density functions may give the same shares μ. The main reason for this is that we are using richer description of a state of a population, namely a density function p. Instead of assuming that the matrix Q^g is continuously differentiable in μ to get uniqueness of solutions $\mu(t)$ we have uniqueness of solutions on a lower level underlying μ, ie. for an initial density function p_0 we have a unique solution $p(x,t)$.

where $*$ denotes element-wise multiplication and $\mathbf{n}$ is the outward pointing normal on $\partial\Omega$. In particular, (24) is satisfied if

$$\mathbf{n} \cdot \left(-\mu p + \frac{1}{2}\delta\mu * \nabla p\right) = 0 \quad \text{almost everywhere on } \partial\Omega. \tag{25}$$

Specifically, if we let $\Gamma_i = \{x \in \partial\Omega \ : \ x_i = 0\}$ we require the following boundary conditions to hold for each i

$$\mu_i(t)\left[p - \frac{1}{2}\delta\frac{\partial p}{\partial x_i}\right](x_{-i}, t) = 0 \qquad \forall_{x_{-i}\in\Gamma_i},\ t > 0,\ 1 \le i \le n. \tag{26}$$

As before we can derive a dynamic for the shares μ:

$$\begin{aligned}
\dot{\mu}_i &= \int_{\Omega_i} \frac{\partial p}{\partial t} dV \\
&= -\int_{\Omega_i} \nabla \cdot (\mu p)\, dV + \frac{1}{2}\delta \int_{\Omega_i} \nabla \cdot (\mu * \nabla p)\, dV \\
&= \int_{\partial\Omega_i} \mathbf{n}_i \cdot \left(-\mu p + \frac{1}{2}\delta\mu * \nabla p\right) dA,
\end{aligned} \tag{27}$$

where $\mathbf{n}_i$ is the outward pointing normal on $\partial\Omega_i$. If $\delta \to 0$ then (27) gives the continuity limit version (12).

The model given by (23, 26) and (27) is far more difficult to analyze than the continuity limit, so we restrict further discussion to the class of 2×2 anti-coordination games with a payoff matrix (13) and an appropriate partition of Ω into Ω_1 and Ω_2. We have then the following problem

$$\frac{\partial p}{\partial t} = -\mu_1 \frac{\partial p}{\partial x_1} - \mu_2 \frac{\partial p}{\partial x_2} + \frac{1}{2}\delta\mu_1 \frac{\partial^2 p}{\partial x_1^2} + \frac{1}{2}\delta\mu_2 \frac{\partial^2 p}{\partial x_2^2}, \tag{28}$$

subject to the boundary conditions

$$\left[p - \frac{1}{2}\delta\frac{\partial p}{\partial x_1}\right](0, x_2, t) = 0 \quad \text{and} \quad \left[p - \frac{1}{2}\delta\frac{\partial p}{\partial x_2}\right](x_1, 0, t) = 0, \tag{29}$$

and some initial condition $p_0(x_1, x_2)$ at $t = 0$.

Any solution of the problem (28, 29) with an initial condition $p_0(x_1, x_2)$ can be constructed from the *fundamental solution* (or kernel) $\tilde{p}(x_1, x_2, y_1, y_2, t)$

$$p(x_1, x_2, t) = \int_0^\infty \int_0^\infty \tilde{p}(x_1, x_2, y_1, y_2, t) p_0(y_1, y_2) dy_1 dy_2.$$

The fundamental solution can be shown to have the form (see Appendix A)

$$\tilde{p}(x_1, x_2, y_1, y_2, t) = q(x_1, y_1, c_1(t))\, q(x_2, y_2, c_2(t)), \tag{30}$$

where $c_i(t) = \int_0^t \mu_i(s) ds$, and

$$q(x, y, t) = \exp\left(\frac{1}{\delta}(x - y) - \frac{t}{2\delta}\right)\ r(x, y, \frac{1}{2}\delta t),$$

with

$$\begin{aligned}
r(x, y, \frac{1}{2}\delta t) &= \frac{1}{\sqrt{2\pi\delta t}}\left(\exp\left(-\frac{(x-y)^2}{2\delta t}\right) + \exp\left(-\frac{(x+y)^2}{2\delta t}\right)\right) \\
&- \exp\left(\frac{1}{\delta}(x+y) + \frac{t}{2\delta}\right)\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{2\delta t}}(x+y) + \sqrt{\frac{t}{2\delta}}\right)\right),
\end{aligned}$$

where $\operatorname{erf}(x)$ is the error function $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-z^2)dz$.

For any fundamental solution (30) we can show (see Appendix B) that

$$\mu_1(t) = \int_{\Omega_1} \tilde{p}(x_1, x_2, y_1, y_2 t) dx_1 dx_2 \to \frac{1}{1+\beta} \quad \text{as } t \to \infty.$$

That is, we recovered the same limit as in the continuity case. For the class of 2×2 anti-coordination games the continuity limit and the diffusion limit coincide with the Nash equilibrium.

4. Conclusions

We began with a model of an updating process along the lines of models proposed in [2]. Instead of working within a probabilistic setting we derived a continuous time, continuity limit partial differential equation together with boundary conditions. In this limit, there is a finite incremental rate at which weights are updated. We then solved this equation analytically. This allowed us to derive the explicit form of the dynamic for the proportions μ of a population using particular pure strategies. In particular we showed that our dynamic is a positive definite dynamic. In addition, we obtained an alternative, more complex continuous-time limit, the diffusion limit. Both limiting formulations have a geometrical flavour, in which the best-reply mapping is transformed into a geometrical structure. For the class of 2×2 anti-coordination games both of them converge to the unique Nash equilibrium in the ultra long run.

There are three main contribution of this paper. First, we provide a rigorous derivation for the model proposed in [8]. In the original paper the limit used is never explicitly derived nor discussed. We give a precise assumption on an environment (Poisson movement) and derive the continuity limit. This is the same approximation used in the original paper. Also, we extend the analysis presented in [8] by providing an analytical solution describing the behavior of the density function p.

Once it is clear what kind of approximation is used, we are able to extend the analysis presented in [8] by providing a more accurate approximation, the diffusion limit. This new approximation aims at describing more precisily the shape of the density function p. We solve the diffusion equation and give the analysis in the case of 2×2 anti-coordination games.

Last but not least, we derive our model from the class of models proposed in [2] and, since it is the same model as used in [8], we therefore link these two papers. Consequently, the resulting positive definite dynamics is placed within the context of the much wider class of models considered in [2].

There are several questions left for future studies. The solution of the diffusion limit presented in Appendix A is general. However, the asymptotic properties derived in Appendix B depend crucially on a game studied and the number of strategies. Whether, and if so how, it is possible to extend these results is not clear for the moment and will require more in depth analysis. A more general analysis of this equation is a formidable challenge.

Appendix A. Solutions of the diffusion equation

A.1. **Reduction.** We consider solutions of the following non-linear initial-boundary value problem:

$$\frac{\partial p}{\partial t} = \sum_{i=1}^{n} \left(-\mu_i \frac{\partial p}{\partial x_i} + \frac{1}{2}\delta\mu_i \frac{\partial^2 p}{\partial x_i^2} \right), \tag{31}$$

where

$$\mu_i(t) = \int_{\Omega_i} p(x,t) dV(x) \tag{32}$$

subject to the boundary conditions:

$$\left[p - \frac{1}{2}\delta \frac{\partial p}{\partial x_i} \right] (x_{-i}, t) = 0, \tag{33}$$

and the initial condition:

$$p(x,t) \to p_0(x) \quad \text{as } t \to 0, \tag{34}$$

where $p_0(x)$ is a given (integrable) initial condition. We assume that $p_0(x)$ is a probability density on $\mathbb{R}^n_+$, which implies that $p(x,t)$ is a probability density for all $t \geq 0$.

We construct the (unique) solution to the problem (31, 32, 33, 34), from solutions of the 1-dimensional, linear diffusion equation on the half line $x \geq 0$:

$$\frac{\partial p}{\partial t} = -\frac{\partial p}{\partial x} + \frac{1}{2}\delta \frac{\partial^2 p}{\partial x^2}, \tag{35}$$

with boundary condition

$$\left[p - \frac{1}{2}\delta \frac{\partial p}{\partial x}\right](0,t) = 0. \tag{36}$$

The *fundamental solution* at $y > 0$ for the problem (35, 36) is the unique solution with a δ-function initial condition at y. That is, a function $q(x; y, t)$ which satisfies (35) and (36), and also

$$q(x; y, t) \to \delta[x - y] \quad \text{as } t \to 0. \tag{37}$$

We shall obtain such solutions explicitly later in this Appendix.

Now consider the n-dimensional equation

$$\frac{\partial p}{\partial t} = \sum_{i=1}^{n} \left(-\mu_i \frac{\partial p}{\partial x_i} + \frac{1}{2}\delta \mu_i \frac{\partial^2 p}{\partial x_i^2} \right), \tag{38}$$

in which the $\mu_i(t)$ are fixed, non-negative, bounded functions of t (independent of p), with boundary conditions (33) and initial condition $p_0(x)$. The unique solution of this problem may be constructed from 1-dimensional fundamental solutions of (35, 36). Thus, for $c = (c_1, \ldots, c_n) \in \mathbb{R}^n_+$, define

$$\tilde{p}(x; y, c) = \prod_{i=1}^{n} q(x_i; y_i, c_i). \tag{39}$$

Then the required solution is:

$$p(x,t) = \int_\Omega \tilde{p}(x; y, c(t)) p_0(y) dV(y), \tag{40}$$

with

$$c_i(t) = \int_0^t \mu_i(s) ds. \tag{41}$$

Clearly, $c_i(0) = 0$. For this solution also to satisfy (32), we require:

$$\mu_i(t) = \dot{c}_i(t) = \int_{\Omega_i} p(x,t) dV(x).$$

That is, the $c_i(t)$ must satisfy the system of ordinary differential equations:

$$\dot{c}_i(t) = \int_\Omega \int_{\Omega_i} \tilde{p}(x; y, c(t)) p_0(y) dV(x) dV(y), \qquad c(0) = 0. \tag{42}$$

Since the equations (42) have a unique solution for $c(t)$, it follows that (39, 40, 41) define the unique solution to the problem (31, 32, 33, 34). Note that it follows from (42) that $\sum_i \mu_i(t) = \sum_i \dot{c}_i(t) = 1$, and hence that $\sum_i c_i(t) = t$.

It remains to determine the fundamental solutions $q(x; y, t)$ for the 1-dimensional problem (35, 36).

A.2. **The fundamental solution $q(x; y, t)$.** This problem may be reduced further as follows. Suppose $r(x; , y, t)$ is the fundamental solution at $y > 0$ for the following problem:

$$\frac{\partial r}{\partial t} = \frac{\partial^2 r}{\partial x^2} \quad \text{with} \quad \left[r - \delta \frac{\partial r}{\partial x}\right](0; y, t) = 0. \tag{43}$$

Define

$$q(x; y, t) = r(x; y, \frac{1}{2}\delta t) \exp\left(\frac{1}{\delta}(x - y) - \frac{1}{2\delta} t \right). \tag{44}$$

Then it is easy to check that (44) is a fundamental solution for the problem (35, 36). It therefore remains to determine the fundamental solution $r(x; y, t)$ for the problem (43).

A.3. **The fundamental solution** $r(x;y,t)$**.** We begin with fundamental solutions of the Dirichlet boundary problem on $\mathbb{R}_+$:

$$\frac{\partial P}{\partial t} = \frac{\partial^2}{\partial x^2}, \qquad P(0,t) = 0. \tag{45}$$

Denote the fundamental solution at $y > 0$ by $P(x;y,t)$. This is given explicitly by ([17, p. 113]):

$$P(x;y,t) = \frac{1}{\sqrt{4\pi t}} \left(\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right). \tag{46}$$

A solution of (45) with general initial condition $P_0(x)$ is obtained from fundamental solutions by:

$$P(x,t) = \int_0^\infty P(x;y,t)P_0(y)dy. \tag{47}$$

More generally, we prove the following proposition relating solutions of the problems (43) and (45).

Proposition A.1. *Let $r_0(x)$ be a given (differentiable) initial condition for the problem (43). Let $P(x,t)$ be the solution to the Dirichlet problem (45) with initial condition*

$$P_0(x) = r_0(x) - \delta r_0'(x). \tag{48}$$

Then

$$r(x,t) = \frac{1}{\delta} \int_x^\infty \exp\left(\frac{1}{\delta}(x-\xi)\right) P(\xi,t)d\xi \tag{49}$$

is the solution to the problem (43) with initial condition $r_0(x)$.

Proof. From (49) we have

$$\begin{aligned}
\frac{\partial r}{\partial t}(x,t) &= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} \frac{\partial P}{\partial t}(\xi,t)d\xi \\
&= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} \frac{\partial^2 P}{\partial \xi^2}(\xi,t)d\xi \quad \text{from (45)} \\
&= \frac{1}{\delta}\left[\mathrm{e}^{(x-\xi)/\delta}\frac{\partial P}{\partial \xi}(\xi,t)\right]_x^\infty + \frac{1}{\delta^2}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} \frac{\partial P}{\partial \xi}(\xi,t)d\xi \qquad (50) \\
&= -\frac{1}{\delta}\frac{\partial P}{\partial x}(x,t) + \frac{1}{\delta^2}\left[\mathrm{e}^{(x-\xi)/\delta}P(\xi,t)\right]_x^\infty + \frac{1}{\delta^3}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta}P(\xi,t)d\xi \\
&= -\frac{1}{\delta}\left(P + \delta\frac{\partial P}{\partial x}\right)\Big|_{(x,t)} + \frac{1}{\delta^2}r(x,t), \qquad (51)
\end{aligned}$$

where (50) follows from integrating by parts. Again, from (49):

$$\begin{aligned}
\frac{\partial r}{\partial x}(x,t) &= -\frac{1}{\delta}P(x,t) + \frac{1}{\delta}, \qquad (52) \\
\frac{\partial^2 r}{\partial x^2}(x,t) &= -\frac{1}{\delta}\frac{\partial P}{\partial x}(x,t) + \frac{1}{\delta^2}\left(-P + r\right)\Big|_{(x,t)} \\
&= -\frac{1}{\delta}\left(P + \delta\frac{\partial P}{\partial x}\right)\Big|_{(x,t)} + \frac{1}{\delta^2}r(x,t). \qquad (53)
\end{aligned}$$

Since (51) and (53) are equal, it follows that (49) is a solution of (43).

For the boundary condition, we have, from (45) and (52):

$$\left[r - \delta\frac{\partial r}{\partial x}\right](0,t) = r(0,t) - \delta\left(-\frac{1}{\delta}P(0,t) + \frac{1}{\delta}r(0,t)\right) = 0.$$

Hence, the required boundary condition (43) is satisfied.

Finally, for the initial condition, from (49) we have:

$$
\begin{aligned}
r(x,0) &= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} P(\xi,0)d\xi \\
&= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} (r_0(\xi) - \delta r_0'(\xi))d\xi \quad \text{using (48)} \\
&= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} r_0(\xi)d\xi - \left[\mathrm{e}^{(x-\xi)/\delta} r_0(\xi)\right]_x^\infty - \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} r_0(\xi)d\xi \qquad (54)\\
&= r_0(x),
\end{aligned}
$$

where (54) comes from integrating by parts. This proves the proposition. □

Now use (47) to express the solution (49) with initial condition (48) in terms of the fundamental solutions (46):

$$
\begin{aligned}
P(x,t) &= \int_0^\infty P(x;y,t)(r_0(y) - \delta r_0'(y))dy \\
&= \int_0^\infty P(x;y,t) r_0(y)dy - \delta\left[P(x;y,t)r_0(y)\right]_0^\infty \\
&\quad + \delta\int_0^\infty \frac{\partial P}{\partial y}(x;y,t) r_0(y)dy \\
&\quad \text{integrating by parts and using } P(x;0,t) = P(x;\infty,t) = 0, \text{ cf. (46)} \\
&= \int_0^\infty \left(P - \delta\frac{\partial P}{\partial y}\right)\Big|_{(x;y,t)} r_0(y)dy. \qquad (55)
\end{aligned}
$$

We wish to find fundamental solutions for the problem (43), and so we require δ-function initial conditions in (55): i.e. $r_0(x;y) = \delta[x-y]$. Substituting this in (55), we obtain the corresponding form for the solution of (45):

$$
\begin{aligned}
Q(x;y,t) &= \int_0^\infty \left(P + \delta\frac{\partial P}{\partial z}\right)\Big|_{(x;y,t)} \delta[z-y]dz \\
&= P(x;y,t) + \delta\frac{\partial P}{\partial y}(x;y,t). \qquad (56)
\end{aligned}
$$

Substituting (56) into (49) now gives the required fundamental solution for the problem (43):

$$
\begin{aligned}
r(x;y,t) &= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} Q(\xi;y,y)d\xi \\
&= \frac{1}{\delta}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} P(\xi;y,t)d\xi + \frac{\partial}{\partial y}\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} P(\xi;y,t)d\xi \\
&= \frac{1}{\delta}\left(1 + \delta\frac{\partial}{\partial y}\right)\int_x^\infty \mathrm{e}^{(x-\xi)/\delta} P(\xi;y,t)d\xi. \qquad (57)
\end{aligned}
$$

It therefore remain to calculate the integral in (57) explicitly using (46). This is straightforward, and yields the required fundamental solution:

$$
\begin{aligned}
r(x;y,t) &= \frac{1}{\sqrt{4\pi t}}\left(\exp\left(-\frac{(x-y)^2}{4t}\right) + \exp\left(-\frac{(x+y)^2}{4t}\right)\right) \\
&\quad - \frac{1}{\delta}\exp\left(\frac{1}{\delta}(x+y) + \frac{t}{\delta^2}\right)\left(1 - \mathrm{erf}\left(\frac{1}{\sqrt{4t}}(x+y) + \frac{\sqrt{t}}{\delta}\right)\right), \qquad (58)
\end{aligned}
$$

where $\mathrm{erf}(\xi)$ is the error function, $\mathrm{erf}(\xi) = \frac{2}{\sqrt{\pi}}\int_0^\xi \exp(-z^2)dz$.

Appendix B. Asymptotic solution for the anti-coordination game

B.1. **Preliminary considerations.** We consider the fundamental solution (39) for $n = 2$. Recall that $c_i(t) = \int_0^t \mu_i(s)ds$ for $i = 1, 2$. Since $0 \le \mu_i(t) \le 1$, we have $0 \le c_i(t) \le t$, and $\mu_1(t) + \mu_2(t) = 1$ implies that $c_1(t) + c_2(t) = t$. It will be more convenient to work with the variables:

$$z_i(t) = \sqrt{\frac{c_i(t)}{2\delta}}. \tag{59}$$

In this section we shall show the following:

$$c_i(t) > 0 \quad \text{for all}\, t > 0, \tag{60}$$

and

$$z_i(t) \sim \sqrt{\frac{t}{2\delta}} \cdot \bar{\omega}_i \quad \text{as}\, t \to \infty, \tag{61}$$

where $\bar{\omega}_i$ are non-negative constants satisfying $\bar{\omega}_1^2 + \bar{\omega}_2^2 = 1$.

B.1.1. *Proof of (60).* We consider the differential equation (42) associated to the fundamental solution at $(y_1, y_2) \in \Omega$. First note that $q(x; y, s) > 0$ for all (x, y) and $s > 0$. For the anti-coordination game, we have:

$$\begin{aligned}
\mu_1(t) = \dot{c}_1(t) &= \int_{\Omega_1} q(x_1; y_1, c_1(t))\, q(x_2; y_2, c_2(t)) dx_1 dx_2 \\
&= \int_0^\infty \left(\int_{\beta x_1}^\infty q(x_2; y_2, c_2(t)) dx_1 \right) q(x_1; y_1, c_1(t)) dx_1 \qquad (62) \\
&= \int_0^\infty \left(\int_0^{x_2/\beta} q(x_1; y_1, c_1(t)) dx_1 \right) q(x_2; y_2, c_2(t)) dx_2. \qquad (63)
\end{aligned}$$

First note that $\dot{c}_i(t) = \mu_i(t) \ge 0$ implies that $c_i(t)$ cannot decrease. Suppose $\mu_2(0) = 1$. Then $\dot{c}_2(0) > 0$, and hence $c_2(t) > 0$ for all $t > 0$. On the other hand, if $c_1(t) = 0$ for $t \ge 0$, then $q(x_1; y_1, c_1(t)) = \delta\,[x_1 - y_1]$ for $t \ge 0$, and hence, from (62):

$$\mu_1(t) = \int_{\beta y_1}^\infty q(x_2; y_2, c_2(t)) dx_2. \tag{64}$$

However, since $q(x; y, s) > 0$ for all (x, y) and $s > 0$, and $\int_0^\infty q(x; y, s)dx = 1$, it follows from (64) that $\mu_1(t) = \dot{c}_1(t) > 0$ for $t > 0$, which yields a contradiction. Similarly, if $\mu_1(0) = 1$, then $c_1(t) > 0$ for all $t > 0$. In this case:

$$\mu_2(t) = \int_0^\infty \left(\int_{x_2/\beta}^\infty q(x_1; y_1, c_1(t)) dx_1 \right) q(x_2; y_2, c_2(t)) dx_2. \tag{65}$$

Hence, if $\mu_2(t) = 0$ for $t \ge 0$, then $q_2(x_2; y_2, c_2(t)) = \delta\,[x_2 - y_2]$, and (65) gives:

$$\mu_2(t) = \int_{y_2/\beta}^\infty q(x_1; y_1, c_1(t)) dx_1,$$

which is strictly positive since $q(x_1; y_1, c_1(t)) > 0$ for $t > 0$. Again we have a contradiction. We have shown therefore that $0 < \mu_i(t) < 1$ for $t > 0$ and $i = 1, 2$, from which follows that $c_i(t) > 0$. This proves (60).

B.1.2. *Proof of (61).* Because $c_1(t)$ and $c_2(t)$ are positive for $t > 0$ and monotonically increasing, so are $z_1(t)$ and $z_2(t)$, cf. (59). Also, $c_1(t) + c_2(t) = t$ implies $z_1^2(t) + z_2^2(t) = t/(2\delta)$. Write

$$z_i(t) = \sqrt{\frac{t}{2\delta}} \cdot \omega_i(t). \tag{66}$$

Then $\omega_i(t)$ is positive for $t > 0$, and $\omega_1^2 + \omega_2^2 = 1$. Hence, we may write

$$\omega_1(t) = \sin(\theta(t)) \quad \text{and} \quad \omega_2(t) = \cos(\theta(t)), \tag{67}$$

with $\theta(t) \in (0, \pi/2)$. In particular, $\omega_i(t)$ is a bounded function. In addition,

$$\dot{z}_i(t) = \frac{1}{2\sqrt{2\delta t}} \cdot \omega_i(t) + \sqrt{\frac{t}{2\delta}} \cdot \dot{\omega}_i(t) < \frac{1}{2\sqrt{2\delta t}} + \sqrt{\frac{t}{2\delta}} \cdot \dot{\omega}_i(t).$$

Suppose $\dot{z}_1(t)$ and $\dot{z}_2(t)$ are bounded away from zero as $t \to \infty$. That is, suppose $\dot{z}_1(t), \dot{z}_2(t) \geq \epsilon > 0$ for all sufficiently large t. Then, since $1/\sqrt{2\delta t} \leq \epsilon$ for sufficiently large t, we have:

$$\dot{\omega}_i(t) \geq \sqrt{\frac{2\delta}{t}} \cdot \frac{1}{2}\epsilon,$$

for all sufficiently large t. Thus, both $\dot{\omega}_1(t)$ and $\dot{\omega}_2(t)$ are positive for large t. However, from (67):

$$\dot{\omega}_1(t) = \cos(\theta(t)) \cdot \dot{\theta}(t), \quad \text{and} \quad \dot{\omega}_2(t) = -\sin(\theta(t)) \cdot \dot{\theta}(t),$$

and hence $\dot{\omega}_1(t)$ and $\dot{\omega}(t)$ are either both zero or have opposite signs. This gives a contradiction, and we conclude that $\dot{z}_i(t) \to 0$ as $t \to \infty$ for at least one i. This implies that $\dot{\omega}_i(t) \to 0$.

Suppose that $\dot{z}_2(t) \to 0$ as $t \to \infty$. Then $\dot{\omega}_2(t) \to 0$, and hence either $\dot{\theta}(t) \to 0$ or $\theta(t) \to 0$. If $\dot{z}_1(t) \geq \epsilon > 0$, then $\dot{\omega}_1(t)$ is positive for all sufficiently large t (as above), and hence so is $\dot{\theta}(t)$. But this means that $\theta(t)$ is monotonically increasing for all sufficiently large t, which implies that $\theta(t) \to 0$ is not possible. We conclude therefore that $\dot{\theta}(t) \to 0$, and hence that $\theta(t) \to \bar{\theta} \leq \pi/2$. A similar argument shows that if $\dot{z}_1(t) \to 0$ and $\dot{z}_2(t) \geq \epsilon > 0$, then $\theta(t) \to \bar{\theta} \geq 0$.

Finally, suppose that $\dot{z}_i(t) \to 0$ for $i = 1, 2$. Then $\dot{\omega}_i(t) \to 0$ for $i = 1, 2$, and the only possibility is $\dot{\theta}(t) \to 0$. Since $\theta(t)$ is bounded, it follows that $\theta(t) \to \bar{\theta} \in [0, \pi/2]$.

We conclude that in all cases $\theta(t) \to \bar{\theta} \in [0, \pi/2]$ as $t \to \infty$. We have therefore shown that

$$z_i(t) \sim \bar{\omega}_i \sqrt{\frac{t}{2\delta}} \quad \text{as}\, t \to \infty,$$

for constants $\bar{\omega}_i \in [0, 1]$ satisfying $\bar{\omega}_1^2 + \bar{\omega}_2^2 = 1$.

B.2. **Asymptotic properties of the fundamental solution.**

B.2.1. *The 1-dimensional fundamental solution.* From (44) and (58), the 1-dimensional fundamental solution is:

$$q(x; y, t) = \exp\left(\frac{1}{\delta}(x - y) - \frac{t}{2\delta}\right) \cdot r(x; y, \frac{1}{2}\delta t), \tag{68}$$

where

$$\begin{aligned} r(x; y, \frac{1}{2}\delta t) = {} & \frac{1}{\sqrt{2\pi\delta t}} \left(\exp\left(-\frac{(x-y)^2}{2\delta t}\right) + \exp\left(-\frac{(x+y)^2}{2\delta t}\right)\right) \\ & - \frac{1}{\delta} \exp\left(\frac{1}{\delta}(x+y) + \frac{t}{2\delta}\right)\left(1 - \operatorname{erf}\left(\frac{1}{\sqrt{2\delta t}}(x+y) + \sqrt{\frac{t}{2\delta}}\right)\right). \end{aligned} \tag{69}$$

We consider the asymptotics of (68, 69) as $t \to \infty$.

Write:

$$k = \frac{x+y}{2\delta}, \quad \lambda = \frac{x-y}{x+y}, \quad z = \sqrt{\frac{t}{2\delta}}. \tag{70}$$

Then

$$\begin{aligned}
\sqrt{\frac{\pi\delta t}{2}}\cdot r(x;y,\frac{1}{2}\delta t) &= \frac{1}{2}\left(\exp\left(-\frac{(\lambda k)^2}{z^2}\right)+\exp\left(-\frac{k^2}{z^2}\right)\right)\\
&\quad -\sqrt{\pi}z\exp\left(-\frac{k^2}{z^2}\right)\exp\left(\left(\frac{k}{z}+z\right)^2\right)\left(1-\operatorname{erf}\left(\frac{k}{z}+z\right)\right)\\
&= \frac{1}{2}\left(\exp\left(-\frac{(\lambda k)^2}{z^2}\right)+\exp\left(-\frac{k^2}{z^2}\right)\right)\\
&\quad -\frac{z}{\frac{k}{z}+z}\exp\left(-\frac{k^2}{z^2}\right)\cdot\sqrt{\pi}\left(\frac{k}{z}+z\right)\exp\left(\left(\frac{k}{z}+z\right)^2\right)\left(1-\operatorname{erf}\left(\frac{k}{z}+z\right)\right)\\
&\sim \frac{1}{2}\left(\exp\left(-\frac{(\lambda k)^2}{z^2}\right)+\exp\left(-\frac{k^2}{z^2}\right)\right)-\frac{z}{\frac{k}{z}+z}\exp\left(-\frac{k^2}{z^2}\right)\quad\text{as }\frac{k}{z}+z\to\infty \tag{71}\\
&= \frac{1}{2}\exp\left(-\frac{(\lambda k)^2}{z^2}\right)+\frac{1}{2}\exp\left(-\frac{k^2}{z^2}\right)\left(\frac{k-z^2}{k+z^2}\right).
\end{aligned}$$

In the step (71), we have used the fact that the function:

$$H(\xi)=\sqrt{\pi}\xi\exp\left(\xi^2\right)\left(1-\operatorname{erf}\left(\xi\right)\right) \tag{72}$$

has the following properties:

$$\begin{aligned}
&H(0)=0,\\
&H(\xi)=\text{is monotonically increasing for}\,\xi\geq 0,\\
&H(\xi)\to 1\,\text{as}\,\xi\to\infty.
\end{aligned}$$

In (71) we have taken $\xi=\frac{k}{z}+z$.

It now follows that:

$$r(x;y,\frac{1}{2}\delta t)\sim\frac{1}{\sqrt{\pi}}\cdot\frac{1}{z}\cdot\frac{1}{2\delta}\left\{\exp\left(-\frac{(\lambda k)^2}{z^2}\right)+\exp\left(-\frac{k^2}{z^2}\right)\left(\frac{k-z^2}{k+z^2}\right)\right\}. \tag{73}$$

Hence, from (68) and (70):

$$\begin{aligned}
q(x;y,t) &= \exp\left(2\lambda k-z^2\right)r(x;y,\frac{1}{2}\delta t)\\
&\sim \frac{1}{\sqrt{\pi}}\cdot\frac{1}{z}\cdot\frac{1}{2\delta}\left\{\exp\left(-\frac{(\lambda k)^2}{z^2}+2\lambda k-z^2\right)+\exp\left(-\frac{k^2}{z^2}+2\lambda k-z^2\right)\left(\frac{k-z^2}{k+z^2}\right)\right\}\\
&= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{z}\cdot\frac{1}{2\delta}\left\{\exp\left(-\left(\frac{\lambda k}{z}-z\right)^2\right)+\exp\left(-\left(\frac{\lambda k}{z}-z\right)^2-(1-\lambda^2)\frac{k^2}{z^2}\right)\left(\frac{k-z^2}{k+z^2}\right)\right\}\\
&- \frac{1}{\sqrt{\pi}}\cdot\frac{1}{z}\cdot\frac{1}{2\delta}\exp\left(-\left(\frac{\lambda k}{z}-z\right)^2\right)\left\{1+\exp\left(-(1-\lambda^2)\frac{k^2}{z^2}\right)\left(\frac{k-z^2}{k+z^2}\right)\right\}. \tag{74}
\end{aligned}$$

Recall that $k=(x+y)/(2\delta)$, and $\lambda=(x-y)/(x+y)=1-2y/(x+y)=1-y/(\delta k)$. Substituting for λ in (74), we obtain:

$$\begin{aligned}
q(x;y,t) &\sim \frac{1}{\sqrt{\pi}}\cdot\frac{1}{z}\cdot\frac{1}{2\delta}\exp\left(-\left(\frac{k}{z}-\left(\frac{y}{\delta z}+z\right)\right)^2\right)\left\{1+\exp\left(-\frac{2y}{\delta z^2}k+\frac{y^2}{\delta^2 z^2}\right)\left(\frac{k-z^2}{k+z^2}\right)\right\}\\
&= \frac{1}{\sqrt{\pi}}\cdot\frac{1}{z}\cdot\frac{1}{2\delta}\left[\exp\left(-\left(\frac{k}{z}-\left(\frac{y}{\delta z}+z\right)\right)^2\right)+\exp\left(-\frac{2y}{\delta}\right)\exp\left(-\left(\frac{k}{z}-z\right)^2\right)\left(\frac{k/z-z}{k/z+z}\right)\right]. \tag{75}
\end{aligned}$$

Now note that:

$$\int_K^\infty\exp\left(-\left(ak-b\right)^2\right)dk=\frac{\sqrt{\pi}}{2}\cdot\frac{1}{a}\left(1+\operatorname{erf}\left(b-aK\right)\right).$$

Setting $\ell = k/z$, from (70) we have $dx = 2\delta dk = (2\delta z)d\ell$, and we obtain:

$$\frac{1}{\sqrt{\pi}} \cdot \frac{1}{z} \cdot \frac{1}{2\delta} \int_{k=K}^{\infty} \exp\left(-\left(\frac{k}{z} - \left(\frac{y}{\delta z} + z\right)\right)^2\right) dx = \frac{1}{2}\left(1 + \operatorname{erf}\left(z + \frac{y}{\delta z} - \frac{K}{z}\right)\right) \to 1 \text{ as } z \to \infty.$$

This is true for any finite $K \geq K_0 = y/2\delta$. Since we know that

$$\int_{x=0}^{\infty} q(x;y,t)dx = \int_{k=K_0}^{\infty} q(x;y,t)dx = 1,$$

it follows that this part of the expression (75) accounts for the total probability mass as $t \to \infty$ (equivalently, $z \to \infty$).

To deal with the remaining part of (75), take $\ell = k/z - z$, so that $dx = 2\delta z d\ell$. Then, from (75):

$$\begin{aligned} I(K,z) &= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{z} \cdot \frac{1}{2\delta} \exp\left(-\frac{2y}{\delta}\right) \int_{k=K}^{\infty} \exp\left(-\left(\frac{k}{z} - z\right)^2\right) \left(\frac{k/z - z}{k/z + z}\right) dx \\ &= \frac{1}{\sqrt{\pi}} \cdot \exp\left(-\frac{2y}{\delta}\right) \int_{K/z-z}^{\infty} \frac{\ell}{\ell + 2z} \mathrm{e}^{-\ell^2} d\ell. \end{aligned}$$

A straightforward estimate now shows that, for any finite $K \geq K_0$:

$$-\frac{1}{2\sqrt{\pi}} \cdot \exp\left(-\frac{2y}{\delta}\right) \cdot \frac{1}{K/z + z} < I(K,z) \leq \frac{1}{2\sqrt{\pi}} \cdot \exp\left(-\frac{2y}{\delta}\right) \cdot \frac{1}{K/z + z} \exp\left(-\left(z - \frac{K}{z}\right)^2\right), \tag{76}$$

and hence $I(K,z) \to 0$ as $z \to \infty$.

In summary, we have shown that

$$\int_{k=K}^{\infty} q(x;y,t)dx \sim \frac{1}{2}\left[1 + \operatorname{erf}\left(z + \frac{y}{\delta z} - \frac{K}{z}\right)\right] \quad \text{as } z \to \infty. \tag{77}$$

B.2.2. *The 2-dimensional fundamental solution.* Now consider the 2-dimensional situation described in Appendix A. Take $x = x_i$, $y = y_i$ and $z = z_i = \sqrt{c_i(t)/2\delta}$ for $i = 1,2$. For fixed $x_1 \geq 0$, we have the region $\Omega_1 = \{(x_1, x_2) : 0 \leq \beta x_1 \leq x_2 < \infty\}$. Thus:

$$k_2 = \frac{x_2 + y_2}{2\delta} \geq K_2 = \frac{\beta x_1 + y_2}{2\delta} = \beta\frac{x_1 + y_1}{2\delta} + \frac{y_2 - \beta y_1}{2\delta} = \beta k_1 + \frac{y_2 - \beta y_1}{2\delta}.$$

From (62), we require to find the asymptotic limit as $t \to \infty$ of:

$$\begin{aligned} \mu_1(t) &= \int_{\Omega_1} q(x_1; y_1, c_1(t))\, q(x_2; y_2, c_2(t)) dx_1 dx_2 \\ &= \int_0^{\infty} \left[\int_{\beta x_1}^{\infty} q(x_2; y_2, c_2(t)) dx_2\right] q(x_2; y_2, c_2(t)) dx_1 \\ &= \int_0^{\infty} \left[\int_{k_2=K_2}^{\infty} q(x_2; y_2, c_2(t)) dx_2\right] q(x_2; y_2, c_2(t)) dx_1 \\ &\sim \int_0^{\infty} \left[1 + \operatorname{erf}\left(z_2 + \frac{y_2}{\delta z_2} - \frac{K_2}{z_2}\right)\right] q(x_2; y_2, c_2(t)) dx_1 \quad \text{by (77)} \\ &= \frac{1}{2} + \frac{1}{2}\int_0^{\infty} \operatorname{erf}\left(z_2 + \frac{y_2}{\delta z_2} - \frac{K_2}{z_2}\right) q(x_1; y_1, c_1(t)) dx_1 \\ &= \frac{1}{2} + \frac{1}{2}\int_{k_1 = y_1/(2\delta)}^{\infty} \operatorname{erf}\left(z_2 + \frac{\beta y_1 + y_2}{2\delta z_2} - \beta\frac{k_1}{z_2}\right) q(x_1; y_1, c_1(t)) dx_1 \quad \text{as } z_2 \to \infty. \end{aligned} \tag{78}$$

To evaluate the integral, note that the erf term is bounded between -1 and 1. It therefore follows from (76) that the second summand in (75) makes zero contribution to the outcome as $z_1 \to \infty$. It therefore suffices to evaluate:

$$
\begin{aligned}
J_1 &= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{z_1} \cdot \frac{1}{2\delta} \int_{k_1=y_1/(2\delta)}^{\infty} \operatorname{erf}\left(z_2 + \frac{\beta y_1 + y_2}{2\delta z_2} - \beta \frac{k_1}{z_2}\right) \exp\left(-\left(\frac{k_1}{z_1} - \left[\frac{y_1}{\delta z_1} + z_1\right]\right)^2\right) dx_1 \\
&= \frac{1}{\sqrt{\pi}} \cdot \frac{1}{z_1} \int_{y_1/(2\delta)}^{\infty} \operatorname{erf}\left(z_2 + \frac{\beta y_1 + y_2}{2\delta z_2} - \beta \frac{k_1}{z_2}\right) \exp\left(-\left(\frac{k_1}{z_1} - \left[\frac{y_1}{\delta z_1} + z_1\right]\right)^2\right) dk_1,
\end{aligned}
$$

noting that $dx_1 = 2\delta dk_1$. Set $\ell - k_1 - y_1/(2\delta)$. Then:

$$
J_1 = -\frac{1}{\sqrt{\pi}} \cdot \frac{1}{z_1} \int_0^{\infty} \operatorname{erf}\left(\beta \frac{\ell}{z_2} - \left[z_2 + \frac{y_2}{2\delta z_2}\right]\right) \exp\left(-\left(\frac{\ell}{z_1} - \left[z_1 + \frac{y_1}{2\delta z_1}\right]\right)^2\right) d\ell.
$$

Thus, $\mu_1 \sim \frac{1}{2}(1 + J_1)$ as $z_1, z_2 \to \infty$. Since $y_i/(2\delta z_i) \to 0$, we can ignore these terms to obtain:

$$
\begin{aligned}
J_1 &\sim -\frac{1}{\sqrt{\pi}} \cdot \frac{1}{z_1} \int_0^{\infty} \operatorname{erf}\left(\beta \frac{\ell}{z_2} - z_2\right) \exp\left(-\left(\frac{\ell}{z_1} - z_1\right)^2\right) d\ell \\
&= -\frac{1}{\sqrt{\pi}} \int_0^{\infty} \operatorname{erf}\left(\beta \frac{z_1}{z_2} \ell - z_2\right) \exp\left(-(\ell - z_1)^2\right) d\ell \\
&= -\frac{1}{\sqrt{\pi}} \int_{-z_1}^{\infty} \operatorname{erf}\left(\beta \frac{z_1}{z_2} m - z_2 + \beta \frac{z_1^2}{z_2}\right) \exp\left(-m^2\right) dm. \qquad (79)
\end{aligned}
$$

That is, asymptotically, J_1, and hence μ_1, is independent of (y_1, y_2).

Unfortunately, the integral in (79) has no closed-form solution in general. However, we shall show that J_1, and hence μ_1 has a unique asymptotic limit as $t \to \infty$.

B.3. **Solutions for the anti-coordination game.** We know from (60, 61) that $c_1(t)$ and $c_2(t)$ are both positive and non-decreasing, with $c_1(t) + c_2(t) = t$. Hence $c_i(t) \to \infty$ for at least one i. In particular, if $\bar{\omega}_1$ and $\bar{\omega}_2$ are both positive then both $c_1(t)$ and $c_2(t) \to \infty$ as $t \to \infty$.

On the other hand, $\bar{\omega}_1 = 1$ and $\bar{\omega}_2 = 0$ only if $\theta(t) \to 0$ in (67). In this case, (61) implies that $c_2(t) \to \infty$ and $c_1(t) \to \bar{c}_1 \leq \infty$. If $\bar{c}_1 < \infty$, then $\mu_1(t) = \dot{c}_1(t) \to 0$. However, from (78):

$$
\begin{aligned}
\mu_1(t) &\sim \int_0^{\infty} \left[\int_{\beta x_1}^{\infty} q(x_2; y_2, c_2(t)) dx_2\right] q(x_1; y_1, \bar{c}_1) dx_1 \\
&\sim \frac{1}{2} + \frac{1}{2} \int_0^{\infty} \operatorname{erf}\left(z_2(t) + \frac{y_2}{2\delta z_2(t)} - \beta \frac{x_1}{2\delta z_2(t)}\right) q(x_1; y_1, \bar{c}_1) dx_1 \to 1 \quad \text{since}\, z_2(t) \to \infty \,\text{as}\, t \to \infty.
\end{aligned}
$$

This is a contradiction, and we conclude that $c_i(t) \to \infty$ for both $i = 1, 2$. A similar argument yields the same conclusion if $\bar{\omega}_1 = 1$ and $\bar{\omega}_2 = 0$. We conclude that both $c_1(t)$ and $c_2(t) \to \infty$ as $t \to \infty$. It therefore follows that the asymptotic approximation $\mu_1 = \frac{1}{2}(1 + J_1)$,with J_1 given by (79), is valid.

From (61) we have:

$$
\frac{z_1}{z_2} \to \frac{\bar{\omega}_1}{\bar{\omega}_2} \quad \text{as}\, t \to \infty. \qquad (80)
$$

Substituting in (79) gives:

$$
J_1 \sim -\frac{1}{\sqrt{\pi}} \int_{-z_1}^{\infty} \operatorname{erf}\left(\frac{\bar{\omega}_1}{\bar{\omega}_2} m + \left(\beta \frac{\bar{\omega}_1}{\bar{\omega}_2} - 1\right) z_2\right) \mathrm{e}^{-m^2} dm.
$$

If $\bar{\omega}_1/\bar{\omega}_2 > 1/\sqrt{\beta}$, then the erf term $\to 1$ as t (and hence both z_1 and z_2) $\to \infty$. Thus, $J_1 \to -1$, and hence $\mu_1 \to 0$ and $\mu_2 \to 1$. However, $\bar{\omega}_1/\bar{\omega}_2 > 1/\sqrt{\beta}$ implies that $\bar{\omega}_1 > 0$, and hence $c_1(t) \sim t\bar{\omega}_1^2$ as $t \to \infty$. Thus, $\mu_1(t) = \dot{c}_1(t) \sim \bar{\omega}_1^2 > 0$, and we we obtain a contradiction. A similar argument in the case $\bar{\omega}_1/\bar{\omega}_2 < 1/\sqrt{\beta}$ also yields a contradiction.

It remains to consider the case:

$$
\frac{\bar{\omega}_1}{\bar{\omega}_2} = \frac{1}{\sqrt{\beta}}. \qquad (81)
$$

Then $\beta z_1^2/z_2 - z_2$ in (79) has an indeterminate limit as $t \to \infty$. Suppose that

$$\beta \frac{z_1^2}{z_2} - z_2 \to \bar{\omega} \quad \text{as}\, t \to \infty, \tag{82}$$

where $\bar{\omega}$ can take any value between $-\infty$ and ∞. Recalling that $z_1^2 + z_2^2 = t/(2\delta)$, this gives, on substituting for z_1^2, a quadratic in z_2:

$$(\beta+1) z_2^2 + \bar{\omega} z_2 - \beta \frac{t}{2\delta} \sim 0,$$

and hence

$$z_2(t) \sim \frac{1}{2(\beta+1)} \left(-\bar{\omega} + \sqrt{\bar{\omega}^2 + 2\frac{t}{\delta}\beta(\beta+1)} \right).$$

Recall that z_2 must be positive, so only the positive root is relevant. It now follows that

$$c_2(t) = 2\delta z_2^2 \sim \frac{\delta}{(\beta+1)^2} \left(\bar{\omega}^2 + \frac{t}{\delta}\beta(\beta+1) - \bar{\omega}\sqrt{\bar{\omega}^2 + \frac{2t}{\delta}\beta(\beta+1)} \right),$$

and hence

$$\mu_2(t) = \dot{c}_2(t) \sim \frac{\beta}{\beta+1} \left(1 - \frac{\bar{\omega}}{\sqrt{\bar{\omega}^2 + 2t\beta(\beta+1)/\delta}} \right) \to \frac{\beta}{\beta+1} \quad \text{as}\, t \to \infty.$$

We have therefore shown that, for any finite $\bar{\omega}$,

$$\mu_1(t) \to \frac{1}{\beta+1}, \quad \mu_2(t) \to \frac{\beta}{\beta+1} \quad \text{as}\, t \to \infty. \tag{83}$$

Thus, 83 is the only possible asymptotic solution. However, because $\mu_1 \sim (1 + J_1)/2$, to show that (83) actually yields a consistent solution we have to show that $\bar{\omega}$ can be chosen so that $J_1 \to -(\beta-1)/(\beta+1)$ as $t \to \infty$. We shall show that there is a unique value of $\bar{\omega}$ for which this holds.

Substituting from (81) and (82) into (79), we obtain:

$$-J_1 \to K_1(\bar{\omega}) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \operatorname{erf}\left(\frac{1}{\sqrt{\beta}} m + \bar{\omega} \right) \mathrm{e}^{-m^2} dm \quad \text{as}\, t \to \infty. \tag{84}$$

Thus:

$$\begin{aligned} \frac{\partial K_1}{\partial \bar{\omega}} &= \frac{2}{\pi} \int_{-\infty}^{\infty} \exp\left(-\left[\frac{1}{\sqrt{\beta}} m + \bar{\omega} \right]^2 - m^2 \right) dm \\ &= \frac{1}{\sqrt{\pi}} \cdot \sqrt{\frac{\beta}{\beta+1}} \exp\left(-\frac{\beta}{\beta+1} \bar{\omega}^2 \right). \end{aligned}$$

If follows that $K_1(\bar{\omega})$ is a monotonically increasing function of $\bar{\omega}$. Further, it is clear that $K_1 \to \pm 1$ as $\bar{\omega} \to \pm\infty$ (see Fig. 2). Since $-1 < (\beta-1)/(\beta+1) < 1$ for any $\beta > 0$, it follows that there is unique, finite value $\bar{\omega}^\star = \bar{\omega}^\star(\beta)$ such that $K_1(\bar{\omega}^\star) = (\beta-1)/(\beta+1)$. This value therefore defines the unique consistent solution. This completes the proof.

Remark B.1. The analysis given in this Appendix has concerned the special case with δ-function initial conditions. At the cost of some notational complexification, it is straightforward to generalize these arguments to solutions with arbitrary initial conditions, (40, 41, 42).

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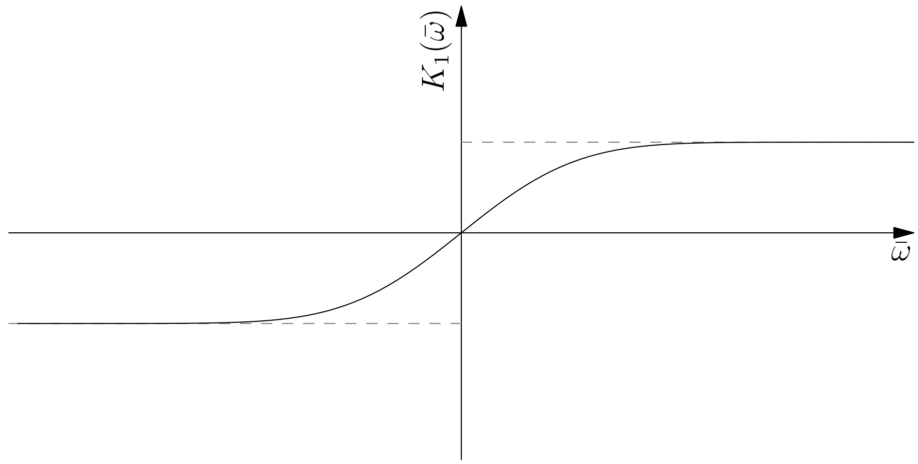

FIGURE 2. Graph of the function $K_1(\bar{\omega})$ given by (84). This increases monotonically from -1 to $+1$ as $\bar{\omega}$ increases from $-\infty$ to $+\infty$.

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