Logit Dynamic for Continuous Strategy Games: Existence of Solutions.*

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August 1, 2007

Abstract

We define the logit dynamic in the space of probability measures for a game with a compact and continuous strategy set. The original Burdett and Judd (1983) model of price dispersion comes under this framework. We then show that if the payoff functions of the game satisfy Lipschitz continuity under the strong topology in the space of signed measures, the logit dynamic admits a unique solution in the space of probability measures. As a corollary, we obtain that logit dynamic generated by the original Burdett and Judd model is well defined.

1 Introduction

In Lahkar (2007), we completed the evolutionary analysis of the finite dimensional Burdett and Judd (1983) price dispersion model under perturbed best response dynamics. We showed that in the most general version of the model in which both sellers and consumers behave strategically, all dispersed price equilibria are unstable under these dynamics. We opted for the finite dimensional evolutionary analysis to avoid technical complications and because the analysis is valuable in its own right. Any game with a continuous strategy space is at best an abstraction of reality. In actual economic situations, the number of strategies will always be finite, even if very large. Our results indicate that in real market situations, we should not expect to see dispersed equilibria as a long run social state. Instead, price dispersion is more likely to be manifested as a disequilibrium phenomenon like cycles.

Nevertheless, we would like to extend our evolutionary analysis to the original continuous strategy Burdett and Judd model. As an academic exercise, this is interesting because it allows us to complete our analysis by providing results on the limit of the sequence of finite dimensional games. Moreover, the infinite dimensional analysis compels us to face new technical challenges which can

*This paper is based on Chapter 4 of my doctoral dissertation (Essays in Evolutionary Game Theory (2007), University of Wisconsin-Madison). I thank Bill Sandholm for his advice and guidance, and the seminar audience in Wisconsin for comments and discussion. All errors remain my own.

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give better insights into the behavior of evolutionary dynamics in the infinite dimensional setting. Furthermore, most problems of economic interest are modeled using continuous strategy spaces, this being a close but convenient approximation of reality. It may not always be possible to have a convenient finite dimensional approximation as we have been able to construct here. In such circumstances, if we are interested in evolutionary models, it will be important to have techniques that allow us undertake the analysis directly in the infinite dimensional setting.

Moreover, once we overcome the technical difficulties, the infinite dimensional analysis can actually be simpler and the results more intuitive. For example, the calculation of Nash equilibrium is much more convenient in the infinite dimensional Burdett and Judd model than in an arbitrarily large finite approximation of Chapter 2. Also, in the finite dimensional Burdett and Judd model, there is no way general way to quantify the number of Nash equilibria. In general, the number of equilibria is quite large. In contrast, for the original continuous strategy model, there is always a unique Nash equilibrium when consumer behavior is exogenous. For the case with endogenous consumer behavior, there are, generically, two Nash equilibria, and one pure equilibria. The infinite dimensional analysis should also allow us to harness the power of calculus and functional analysis to express results and conditions in a simpler and more tractable format. Instead of complicated sums and matrices, we should be able to express conditions for stability of equilibria using a single integral. Arriving at such a goal provides a powerful motivation to undertake the infinite dimensional analysis.

In this chapter, we do some work in extending the evolutionary analysis to the continuous strategy setting. As of now, the results in this chapter are largely of technical importance and is confined to showing existence of unique solution trajectories under the logit dynamic. As in the finite dimensional case, we would like to concentrate on the evolutionary analysis of the Burdett and Judd model under perturbed best response dynamics. However, due to technical difficulties involved in defining the general class of perturbed best response dynamics, we focus only on the logit dynamic with a continuum of strategies. We establish the existence of solutions under this dynamic. Our existence result if for a general continuous strategy game. We establish conditions that the payoff functions of the game should satisfy to ensure that solution trajectories exist. As a corollary, we show that those conditions are satisfied by the payoffs of the Burdett and Judd model. The infinite dimensional analysis is, however, still incomplete since we do not as yet have any stability results in this context. The existence result is, we believe, still sufficiently novel and interesting in its own right to be presented. It lays the groundwork for analyzing the issue of stability as a topic for future research.

In the next section, we describe the Burdett and Judd model. In this model, the strategy set of the sellers is the compact interval $S = [0, 1]$. We interpret the Burdett and Judd model as a population game and identify each probability measure with a population state. We consider both versions of the model—one with exogenous consumer behavior and the other where consumers behave strategically. We then cite Burdett and Judd's theorem identifying the Nash equilibria of the game. The analogy between the equilibria of the original model and our finite approximation of it in Chapter 2 then becomes very clear.

In section 3, we introduce the infinite dimensional logit dynamic. The state space of the dynamic is the set of probability measures on S. The logit dynamic is determined by the logit best response function. Given a population state, the logit best response to that state is an absolute continuous probability measure with positive density everywhere. A rest point of the logit dynamic is a fixed point of the logit best response function and, hence, is a logit equilibrium of the game. In section 4, we establish conditions under which the set of logit equilibria of a continuous strategy game is non-empty. As a corollary, we show that at least one logit equilibrium exists in the Burdett and Judd model. This result follows as a simple consequence of the infinite dimensional extensions of the Brouwer's fixed point theorem.

Section 5 discusses the question of whether the logit dynamic is well defined for continuous strategy games. We provide the conditions on the payoff functions that ensure that starting from any initial population state, a unique solution trajectory in probability measures exists for all future times. The main technical difficulty in showing this is the choice of an appropriate topology on the set of probability measures. The general consensus following Oechssler and Riedel (2002) is that the topology of convergence in distribution (the weak topology) is more appropriate for defining distance between probability measures. This is because this topology respects the natural distance between two pure strategies[1]. However, it is more convenient to prove the existence result under the strong topology under which the distance between two pure strategies is always 2. This topology can be metrized in the vector space of signed measures. We can therefore extend the logit dynamic into this bigger space and show Lipschitz continuity of the logit dynamic. The infinite dimensional version of the Picard-Lindelof theorem then ensures the existence of unique solution trajectories of the dynamic. Existence and uniqueness in the strong topology then implies the same under the weak topology. As a corollary, we obtain the existence result for the Burdett and Judd model.

Even though we haven't been able to arrive at stability results, we believe it is plausible to conjecture that positive definiteness would imply instability even in the infinite dimensional game. Hence, we should also expect that dispersed equilibria are unstable in the game with endogenous consumer behavior. This is more so because it can be shown, using ideas from Dasgupta and Maskin (1986), that given any sequence of finite dimensional Nash equilibria, there would exist a subsequence of equilibria that would converge in distribution to a Nash equilibria of the infinite dimensional game. If the entire sequence is unstable, it is reasonable to suppose that the limit will also be unstable. This remains a conjecture because it is by no means obvious. In fact, there are results that show that even strict equilibria can be unstable in infinite games.[2]

[1]If x and y are two pure strategies that are very close to each other, then the two Dirac measures δ_x and δ_y are also close under the weak topology. We explain this in greater detail in Section 5.

[2]Oechssler and Riedel (2001,2002) show instability of strict equilibria in the context of the replicator dynamic. Hofbauer, Oechssler and Riedel (2005) show this for the BNN dynamic.

2 The Burdett and Judd Price Dispersion Model

In this section, we provide a brief description of the Burdett and Judd price dispersion model. In this model, there exist a continuum of firms, all selling a particular good and all facing the same marginal cost of production. As we noted in chapter 2, the strategy set of sellers is $S = [0, 1]$ with 0 being the marginal cost of firms and 1 the common reservation price of consumers. Firms' pricing behavior therefore generates a probability measure P over this set. We will denote the distribution function of the measure P with F. To denote the probability of a single point x, we will use $P(x)$ instead of the more formally accurate $P(\{x\})$. Consumer behavior is summarized by the distribution $(q_1, q_2, \cdots q_r)$ which may be exogenously given or may emerge endogenously

As in the finite case, we model the Burdett and Judd game as a population game with sellers as population 1 and consumers as population 2. Since the sellers' strategy set is a continuum, we are implicitly assuming the sellers' population to be a continuum as well. Population masses are assumed to be 1 so that we can identify population states with probability measures. Let $\mathcal{B}$ be the Borel $\sigma-$algebra on S. The space $(S, \mathcal{B})$ is endowed with the Lebesgue measure. We denote by $\mathcal{M}^e(S, \mathcal{B})$ the set of finite and signed measures on $(S, \mathcal{B})$. We can then define the following two sets.

$$\Delta^1 = \left\{ P \in \mathcal{M}^e(S, \mathcal{B}) : \int_S P(dx) = 1, P(A) \geq 0, \forall A \in \mathcal{B} \right\}$$
$$T\Delta^1 = \left\{ \mu_Z \in \mathcal{M}^e(S, \mathcal{B}) : \int_S \mu_Z(dx) = 0 \right\}. \tag{1}$$

Δ^1 is therefore the set of probability measures on S. We call $T\Delta^1$ the tangent space of Δ^1. We will also use the $(r-1)$ dimensional real sets Δ^2 and $T\Delta^2$ defined earlier in chapter 2, Section 2 as the simplex and tangent space for population 2. We define $\Delta = \Delta^1 \times \Delta^2$ and $T\Delta = T\Delta^1 \times T\Delta^2$. A social state is now a pair $(P, q) \in \Delta$. $P(A)$ denotes the proportion of firms who charge prices in the set A and q_i represents the proportion of consumers who sample i prices before purchasing. For the one population game with exogenous types, $\Delta = \Delta^1$ and $T\Delta = T\Delta^1$. In this case, a social state is $P \in \Delta = \Delta^1$. In defining $T\Delta^1$, we have used the notation μ_Z. Here, Z stands for the distribution function of the signed measure μ_Z.

2.1 Exogenous Consumer Behavior

First, we consider the simpler case of exogenous consumer types. Let $\{q_i\}_{i=1}^r$ be the distribution of consumer types. We now specify the payoff function of producers. The payoff that a firm receives by charging a price $x \in [0, 1]$ will depend on x, the distribution q , and the measure P over the set of prices induced by the pricing behavior of the other firms.

Given the distribution q, the payoff to strategy $x \in S$ is the function $\pi_x^1 : \Delta \to \mathbf{R}$ defined by,

$$\pi_x^1(P) = x \left[q_1 + \sum_{m=2}^{r} m q_m \left\{ \sum_{k=0}^{m-1} \frac{G_{k,x}^m(P)}{k+1} \right\} \right]. \tag{2}$$

where

$$G^m_{k,x}(P) = \binom{m-1}{k} P(x)^k \Big(\int_{y>x} P(dy) \Big)^{m-1-k}$$
$$= \binom{m-1}{k} P(x)^k (1 - F(x))^{m-1-k}. \quad (3)$$

The interpretation of the payoff function is similar to that in the finite dimensional case. If indeed the measure P is absolutely continuous, then the payoff function takes the form.[3]

$$\pi^1_x(P) = x \left[q_1 + \sum_{m=2}^{r} m q_m (1 - F(x))^{m-1} \right]. \quad (4)$$

which is in the form written by Burdett and Judd. Even if P is not absolutely continuous, payoffs are given by (4) almost everywhere since the set of possible discontinuities in F is of measure zero.

If the distribution of consumer type is exogenous, then Burdett and Judd prove that the following three cases exhaust the possibilities for firm equilibria.

Lemma 2.1 *(Burdett and Judd (1983)) If the distribution $\{q_i\}_{i=1}^r$ is exogenous, then the three possible types of Nash equilibria are:*

1. *If $q_1 = 1$, then the unique firm equilibrium is the monopoly price equilibrium. All firms charge the highest available price, which is the reservation price* 1.

2. *If $q_1 = 0$, then the unique firm equilibrium is the competitive price equilibrium. All firms charge the lowest feasible price, which is* 0.

3. *If $0 < q_1 < 1$, the unique firm equilibrium is an absolutely continuous probability measure with compact support $[\underline{x}, 1]$ with $\underline{x} > 0$. The equilibrium payoff is given by*

$$\pi^* = q_1 = \underline{x} \sum_{m=1}^{r} m q_m.$$

One can clearly observe the analogy with the corresponding finite dimensional results. For the proofs of these statements, the reader is referred to the original paper.

2.2 Endogenous Consumer Behavior

As in the finite dimensional case, we now consider the case in which the distribution of consumers aggregate behavior emerges endogenously. Consumers' strategy is the number of prices to be sampled before purchasing. A consumer has to pay a search cost $c > 0$ for every price he chooses

[3] This is the form in which Burdett and Judd define their payoff functions. They therefore ignore the possibility of ties since they show that any dispersed equilibria, the probability measure will be absolutely continuous. To define perturbed best response dynamics, we can also ignore ties since, as we will see, these dynamics do not depend on Lebesgue measure zero sets. But if we want to define other dynamics, say the replicator dynamic, we need the general definition.

to sample. The parameters c and r are the same for all consumers. This being a two population game, the state space is now $\Delta = \Delta^1 \times \Delta^2$ with $T\Delta = T\Delta^1 \times T\Delta^2$ being the tangent space.

For producers, the payoff function continues to be (2) with the domain of the function being $\Delta = \Delta^1 \times \Delta^2$. We now specify the cost function of consumers. If each price quotation is a random draw from the probability measure P, then the expected cost of purchasing when m prices are observed is given by the function $C_m : \Delta^1 \to \mathbf{R}$ defined by

$$C_m(P) = mc + m\int_0^1 x\left\{\sum_{k=0}^{m-1}\frac{G_k^m(P)}{k+1}\right\}P(dx).$$

with $G_k^m(P)$ defined in (3). The interpretation of the cost function is similar to that in the finite case. The cost function is independent of consumers' aggregate behavior $q \in \Delta^2$.

Once again, our definition of the cost function is more general than that of Burdett and Judd. They define the cost function for only such P that are absolutely continuous. In this case, $C_m(P)$ simplifies to

$$mc + m\int_0^1 x(1 - F(x))^{m-1}dF(x).$$

The payoff function of a consumer is the negative of the cost function. Thus,

$$\pi_m^2(P) = -C_m(P). \tag{5}$$

Burdett and Judd assert that the cost function is convex in m. This is however, not clear from the basic definition of the cost function. However, using integration by parts and through some tedious algebraic manipulation, we can write the cost function in the following form.

$$C_m(P) = mc + \int_0^1 (1 - F(x))^m\, dx.$$

By an argument similar to that in the finite dimensional case, it becomes clear that the function is indeed convex in m as long as the measure P is not a pure strategy. The fact that $C_m(P)$ is convex in m implies that there exists a unique integer m^* or there exists two integers m^* and m^*+1 that minimizes the expected cost of purchase.

The following theorem characterizes Nash equilibria in the game. As we remarked earlier, the analogy with the finite dimensional case (Theorem 2.5 in chapter 2) is very clear.

Theorem 2.2 *(Burdett and Judd (1983))*

1. *The monopoly situation* $(P(1) = 1, q_1 = 1)$ *is always a Nash equilibrium.*

2. *Any other Nash equilibria is a mixed equilibria with both producers and consumers randomizing.*

3. Depending upon the cost level c, there may be zero, one or two mixed equilibria. If (P^*, q^*) *is a mixed Nash equilibrium and* F^* *is the distribution function of* P^**, then* $0 < q_1^* < 1$ *and* $q_1^* + q_2^* = 1$ *and*

$$F^*(x) = 1 - \frac{q_1^*(1-x)}{2(1-q_1^*)x}.$$

Thus, F^* *is a continuous distribution function increasing monotonically over its support* $[\underline{x}, 1]$ *with*

$$\underline{x} = \frac{q_1^*}{2-q_1^*}.$$

For formal proofs, the reader is referred to the original paper.

3 The Logit Dynamic

To describe behavior dynamics in infinite dimensional games, we need to define evolutionary dynamics in the space of probability measures Δ. Let us consider a one population game with payoff function $\pi_x^1 : \Delta^1 \to \mathbf{R}$ for strategy $x \in S = [0, 1]$. For the one population Burdett and Judd model, $\pi_x^1(P)$ is given by (2). As in the finite dimensional case, we can view evolutionary dynamics as the differential equation $\dot{P}(A) = V(P)(A)$. Here, $A \subseteq S$ and $V(P)(A)$ gives the direction and magnitude of change in the proportion of agents playing strategies in A. To be admissible as an evolutionary dynamic, we require that from every initial condition $P(0) \in \Delta$, there must exist a solution trajectory, preferably unique, $\{P_t\}_{t \in [0,\infty]}$ with $P_t \in \Delta$, for all $t \in [0, \infty]$. Since we require that This second property is called forward invariance. To ensure forward invariance, we require that $V(P)(S) \in T\Delta$ so that the mass of the population remains fixed at 1 at all time.

Much of the literature on infinite dimensional evolutionary dynamics has focused on the replicator dynamic of Taylor and Jonker (1978). The infinite dimensional version of the dynamic has been studied by Bomze (1990, 1991), Cressman (2005), Cressman and Hofbauer (2005), Cressman, Hofbauer and Riedel (2005) and Oechssler and Riedel (2001,2002). The continuous version of the Brown-von Neumann-Nash (Brown and von Neumann, 1950) dynamic has also been studied—see Hofbauer, Oechssler and Riedel (2005).

In this section, we introduce the infinite dimensional logit dynamic. For the purpose of this general discussion, we consider a one population game with a compact strategy set S. Given $x \in S$,the payoff to x is given by $\pi_x^1 : \Delta \to \mathbf{R}$. Given $A \subseteq S$, the logit dynamic is given by

$$\dot{P}(A) = V(P)(A) = L_\eta(P)(A) - P(A). \tag{6}$$

where $L_\eta : \Delta \to \Delta$ is the logit best response function. $L_\eta(P)$ is defined by

$$L_\eta(P)(A) = \int_A \frac{\exp(\eta^{-1}\pi_x^1(P))}{\int_S \exp(\eta^{-1}\pi_y^1(P))dy} dx. \tag{7}$$

For the Burdett and Judd model, π^1_x is given by (2). We note that the logit best response function does not depend on what happens to the payoff function on a set of measure zero. Hence, to define $L_\eta(P)$ for the Burdett and Judd model, we only need to consider payoffs at those prices x where the distribution function F is continuous. At such prices, payoffs are given by (4).

The density function of $L_\eta(P)$ is given by

$$l_\eta(P)(x) = \frac{\exp(\eta^{-1}\pi^1_x(P))}{\int_S \exp(\eta^{-1}\pi^1_y(P))dy}.$$

The function $l_\eta(P)$ has been also obtained by Mattsson and Weibull (2002). Their interest in it was, however, more decision theoretic than its applicability as a continuous strategy evolutionary dynamic.

We call a fixed point of $L_\eta(P)$ a *logit equilibrium* of the population game π. Clearly, the set of rest points of the dynamic coincide with the set of fixed points of the logit best response function.

One can clearly notice the analogy with the finite dimensional case. Just as the finite dimensional logit best response function puts positive probability on all strategies, the logit density function takes positive value over all $x \in S$. Hence, all sets of positive Lebesgue measure receive positive probability under the logit best response measure.

In the finite dimensional case, the logit dynamic has some technical and behavioral features that make it preferable to the best response dynamic. For example, the logit dynamic is a smooth differential equation that makes it amenable to analysis by using standard techniques, whereas the best response dynamic is a differential inclusion that is difficult to analyze. In the infinite dimensional case, this advantage of the logit dynamic is even more striking: the logit best response is always well defined whereas the proper best response may not even exist. For example, in the Burdett and Judd game with exogenous consumer types, suppose $x \in S$ is not dominated by strategy 1.[4] Suppose the population state P is $P(x) = 1$. Then, any seller has an incentive to deviate to a strategy immediately below x. But there is no best response. However, the logit best response is well defined since the density function exists at all points in S. Moreover, it also manages to capture the intuition of the best response. For η sufficiently small, the logit best response will assign most of the probability mass on a small interval immediately to the left of x. Thus, let A be any connected interval that contains x and some strategies below x. Then $\lim_{\eta \to 0} \int_A l_\eta(P)(x)dx = 1$.

We now define the logit dynamic for a two population game. To ensure consistency with the general setting of the Burdett and Judd model with endogenous consumer behavior, we will assume that population 1 has a continuous strategy space $[0, 1]$ while population 2 has a discrete strategy set $\{0, 1, \dots, r\}$. In this framework, the state space is $\Delta = \Delta^1 \times \Delta^2$ with Δ^2 being the $(r - 1)$ dimensional simplex.The function $\pi^1_x : \Delta \to \mathbf{R}$ denotes the payoff to strategy $x \in [0, 1]$ of population 1. Similarly, $\pi^2_m : \Delta \to \mathbf{R}$ is the payoff to $m \in \{0, 1, \dots, r\}$ of population 2. For

[4] Given exogenous consumer behavior $(q_1, q_2, \cdots q_m)$, strategy x is dominated by 1 if $x < q_1^{-1}\left(\sum_{m=1}^{r} mq_m\right)$.

the Burdett and Judd model, these payoffs are given by (2) and (5) respectively. Assuming, for simplicity, that the perturbation factor η is the same for both populations, the logit dynamic at population state $(P, q) \in \Delta$ is given by the ordinary differential equation on Δ ,

$$V^1(P,q)(A) = \dot{P}(A) = L^1_\eta(P,q)(A) - P(A)$$
$$V^2_m(P,q) = \dot{q}_m = L^2_\eta(P,q)(m) - q_m. \tag{8}$$

where $L^1_\eta : \Delta \to \Delta^1$ is the logit best response function for the population of sellers defined by (7). $L^2_\eta : \Delta \to \Delta^2$ is the logit best response function for the population of consumers given by

$$L^2_\eta(P,q)(m) = \frac{\exp(\eta^{-1}(\pi^2_m(P,q)))}{\sum\limits_{i=1}^{r} \exp(\eta^{-1}(\pi^2_i(P,q)))}$$

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4 Existence of Logit Equilibrium

In this section, we consider the first of the two technical questions we address in this chapter–the existence of logit equilibria in a continuous strategy game. Since our ultimate interest is on the stability analysis of logit equilibria, this is fundamental issue that needs to resolved to ensure that the evolutionary analysis of infinite dimensional games using the logit dynamic is even meaningful. Our proof of existence of equilibria is for the one population game with strategy space $S = [0, 1]$ and payoff function π^1_x. This result can, however, be easily extended to the Burdett and Judd framework with two populations where the second population has a finite strategy set.

We now state the main result on the existence of logit equilibria. Let $LE(\pi)$ be the set of logit equilibria of the game. The proof of the theorem is a straightforward application of the Schauder fixed point theorem which is an extension of the Brouwer's fixed point theorem to infinite dimensional settings.

Theorem 4.1 *Consider the one population game with strategy set $S = [0, 1]$ and payoff function $\pi^1_x : \Delta^1 \to \mathbf{R}$ for $x \in [0, 1]$. Then, if the payoff function π^1_x is continuous with respect to the topology of convergence in distribution on Δ for almost all x, the set $LE(\pi)$ is non-empty.*

Proof. First, we impose the topology of convergence in distribution on Δ^1. The set Δ^1 is compact under this topology of weak convergence. Now, let $P \in \Delta$ and let the sequence P^n converge in distribution to P. Since π^1_x is continuous with respect to the topology of convergence in distribution on Δ for almost all x, we have $\lim_{n\to\infty} \pi^1_x(P^n) = \pi^1_x(P)$ almost everywhere. So, $\lim_{n\to\infty} \exp(\eta^{-1}\pi^1_x(P^n)) = \exp(\eta^{-1}\pi^1_x(P))$ almost everywhere. The logit best response function is therefore continuous in this topology. The existence of a logit equilibrium then follows from the Schauder fixed point theorem.[5] ■

[5] The Schauder fixed point theorem requires $\mathcal{M}^e(S, \mathcal{B})$ to be a Banach space. To fulfil this condition, we can impose

To establish existence of equilibria in the Burdett and Judd model, we only need to verify that the continuity condition of Theorem (4.1) is satisfied by the payoff function (2). This is established in the following corollary.

Corollary 4.2 *The set of logit equilibria* $LE(\pi)$ *of the Burdett and Judd model with payoff function (2) is nonempty.*

Proof. We only need to verify that the payoff function π_x^1 is continuous with respect to the topology of convergence in distribution on Δ for almost all x. Let the sequence P^n converge in distribution to P. Since $P^n \xrightarrow{d} P$, the distribution function of P^n will converge pointwise to the distribution function F of P on a set of full measure, the possible exceptions being the zero measure set of discontinuities in F. Hence, $\lim_{n\to\infty} \pi_x^1(P^n) = \pi_x^1(P)$ almost everywhere. The result now follows from Theorem (4.1). ■

5 Existence and Uniqueness of Solutions for the Logit Dynamic in the Burdett and Judd Model

Our discussion about the logit dynamic till now has not considered the fundamental question of whether the dynamic is well defined or not in continuous strategy games. Unless we can show that a forward invariant solution trajectory , preferably unique, exists from every initial population state, we cannot use it to model evolutionary behavior for such games. In the finite dimensional case, it is easy to show the existence and uniqueness of solutions by appealing to the Picard-Lindelof Theorem, at least as long as payoff functions are smooth. In the infinite dimensional case, however, the matter is not so easily resolved as we have to resolve some fundamental topological issues before defining the dynamic. In this subsection, we provide some comments on the technical issues and problems involved in answering this question. We then provide the conditions under which the logit dynamic admits unique solutions in continuous strategy games. In establishing this, a personal communication from Frank Riedel has been immensely helpful. Finally, we show that these conditions are satisfied by the Burdett and Judd model. The result can be easily extended to games with populations having only continuous strategy spaces.

Our discussion of the existence question is for the two population framework described in Section 3. Thus population 1 has a continuous strategy space $[0,1]$ while population 2 has a discrete strategy set $\{0,1,\ldots,r\}$. The functions of the two populations are π_x^1 and π_m^2 respectively, as described earlier. The logit dynamic itself is given by (8).

We noted earlier that an evolutionary dynamic needs to satisfy forward invariance in Δ and admit at least one solution trajectory in Δ. For the logit dynamic, forward invariance follows from the fact that $V^1(P,q)(S) = \sum_{m=1}^{r} V_m^2(P,q) = 0$ so that solution trajectories that start in Δ will

the BL (bounded Lipschitz) norm (Shiryaev (1995)) on $\mathcal{M}^e(S,\mathcal{B})$. This norm metrizes convergence in distribution on Δ.

remain in Δ at all future times. To show the existence of a unique solution trajectory is, however, a much more challenging task.

Before we begin our proof of existence of solutions, we have to resolve a very important topological question: how do we define a neighborhood of a probability measure in Δ^1. In the finite dimensional case, the particular choice of topology is not of much consequence because all norms are equivalent. In the infinite dimensional case, however, the structure of the neighborhood of a probability measure can depend on the choice of topology.

The two most prominent candidates for the topology on Δ^1 are the *strong topology* and the *weak topology*. The strong topology is the topology induced by the variational norm on $\mathcal{M}^e(S, \mathcal{B})$, given by $\|\mu\| = \sup_{x\in S} |\int_S f d\mu|$ where f is a measurable function $f : S \to \mathbf{R}$ such that $\sup_{x\in S} |f(x)| \leq 1$. If P, Q are two probability measures, then the distance between them under this norm is (Shiryaev (1995), p. 360)

$$\|P - Q\| = 2 \sup_{A\in\mathcal{B}} |P(A) - Q(A)|.$$

The appendix in Oechssler and Riedel (2001) presents the variational norm in great detail. They note that in the finite dimensional case, the strong topology is equivalent to the topology induced by the pointwise convergence of probabilities.

Oechssler and Riedel (2002) argues for an alternative topology on Δ^1, particularly for the case where the strategy space is a compact interval of the real line. This is the weak topology or the topology induced by convergence in distribution. On Δ^1, this topology can be metrized by the Prohorov metric (Shiryaev, 1995).

$$\rho(P, Q) = \inf\{\varepsilon > 0, Q(A) \leq P(A^\varepsilon) + \varepsilon \text{ and } P(A) \leq Q(A^\varepsilon) + \varepsilon, \forall A \in \mathcal{B}\}.$$

The main reason why Oechssler and Riedel (2002) prefer the weak topology is that it respects the natural distance in $\mathbf{R}$. Hence, if x and y are two points in S that are very close to one another, the two Dirac measures δ_x and δ_y will also be close under the weak topology. Intuitively, this is because the distribution functions of δ_x and δ_y will be close to one another. Thus, apart from small intervals around x and y, the two distribution functions will be virtually identical over the rest of S. This is unlike the case in the strong topology where the distance between two pure strategies is always 2. Moreover, under the weak topology, Δ^1 is a compact set, unlike under the strong topology.

Given the arguments in Oechssler and Riedel (2002), it seems the weak topology is a more appropriate choice in analyzing economic problems like the Burdett and Judd model. Hence, we will use the weak topology to define a neighborhood in Δ^1. However, trying to resolve the question of existence and uniqueness of solutions by using the weak topology leads to some intractable problems. In particular, there is no known norm that metrizes the weak topology in $\mathcal{M}^e(S, \mathcal{B})$. The Prohorov metric and the BL metric only works with respect to Δ^1. It therefore becomes impossible to do analysis with the weak topology. For instance, there is no way to define the notion of Lipschitz continuity in $\mathcal{M}^e(S, \mathcal{B})$ with the weak topology.

In contrast, the variational norm metrizes the strong topology in $\mathcal{M}^e(S,\mathcal{B})$. $\mathcal{M}^e(S,\mathcal{B})$ is therefore a Banach space under the strong topology. In particular, an infinite dimensional version of the Picard-Lindelof theorem holds true. The Lipschitz continuity of the underlying dynamic is sufficient to ensure existence and uniqueness. Hence, we will use the strong topology to establish of existence and uniqueness of solutions under the logit dynamic. This, however, is sufficient to establish the result for the weak topology because existence and uniqueness under the strong topology is a stronger result than existence and uniqueness under the weak topology.

Since we have defined the logit dynamic on $\Delta^1 \times \Delta^2$, we need to define Lipschitz continuity on $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$. For this, we first define a norm on $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$. Let $P \in \mathcal{M}^e(S,\mathcal{B})$ and $q \in \mathbf{R}^r$. Then, we define the norm $\|(P,q)\|$ as

$$\|(P,q)\| = \max |\|P\|, \|q\|_1| . \tag{9}$$

where $\|P\|$ is the variational norm and $\|q\|_1 = \sum_{i=1}^{r} |q_i|$. Hence, given two points in $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$, (P,q) and (Q,r), the distance between them is

$$\|(P,q) - (Q,r)\| = \max |\|P-Q\|, \|q-r\|_1| .$$

We now present the result about the existence of unique solution trajectories for the logit dynamic for our two-population game setting. First, we define the notion of uniform Lipschitz continuity of the payoff function π_x^1 with respect to the norm (9). For this, we need to extend π_x^1 and π_m^2 to all of $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$ in the obvious manner. We then denote

$$(\mathcal{M}R)_2 = \{(P,q) \in \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r : \|(P,q)\| \leq 2\} .$$

Definition 5.1 *The payoff function* $\pi_x^1 : \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r \to \mathbf{R}$ *is Lipschitz continuous on* $(\mathcal{M}R)_2$*, uniformly in* $x \in S$*, if there exists a constant* K *such that*

$$|\pi_x(P,q) - \pi_x(Q,r)| \leq K \|(P,q) - (Q,r)\|$$

for all (P,q)*,* $(Q,r) \in (\mathcal{M}R)_2$*,* $x \in S$*.*

We can now state our main result of this chapter.

Theorem 5.2 *Let* $\pi_x^1 : \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r \to \mathbf{R}$ *be Lipschitz continuous on* $(\mathcal{M}R)_2$*, uniformly in* $x \in S$*. Let* $\pi_m^2 : \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r \to \mathbf{R}$ *be Lipschitz continuous on* $(\mathcal{M}R)_2$*, for all* m*. Then, from each initial condition* $(P,q) = (P(0),q(0)) \in \Delta$*, there exists a unique solution* $(P(t),q(t)) \in \Delta$ *of the ordinary differential equation (8) for all time* $t \in [0,\infty]$*.*

We now present a brief discussion of the way we prove Theorem (5.2). The details are given in the appendix. The proof is founded on the infinite dimensional version of the Picard-Lindelof theorem. Given $(P,q) \in \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$, suppose we have a dynamic $\widetilde{V}$ on $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$ given by the ordinary differential equation,

$$\begin{aligned} \dot{P} &= \widetilde{V}^1(P,q) \\ \dot{q} &= \widetilde{V}^2(P,q). \end{aligned} \tag{10}$$

The Picard-Lindelof theorem for infinite dimensional dynamics is stated below.

Theorem 5.3 *Suppose $\widetilde{V}^1(P,q)$ and $\widetilde{V}^2(P,q)$ are bounded and satisfies a global Lipschitz continuity condition: $\exists K > 0$ such that $\forall (P,q),(Q,r) \in \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$,*

$$\left\| (\widetilde{V}^1(P,q), \widetilde{V}^2(P,q)) - (\widetilde{V}^1(Q,r), \widetilde{V}^2(Q,r)) \right\| \leq K \left\| (P,q) - (Q,r) \right\|.$$

Then a unique solution of the differential equation (10) exists for all time $t \in [0,\infty]$.

A proof of Theorem 5.3 appears in Zeidler (1986) (Corollary 3.9).

We now consider existence and uniqueness of solutions specifically under the logit dynamic. To prove these results, it is first necessary to extend the logit dynamic (8) to all of $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$. This is because V^1 and V^2 defined in (8) are neither bounded or globally Lipschitz continuous on $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$. So, we cannot apply Theorem 5.3 directly. Instead, we construct two auxiliary functions $\widetilde{V}^1(P,q)$ and $\widetilde{V}^2(P,q)$ defined on all of $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$ and which coincides with $V^1(P,q)$ and $V^2(P,q)$ in Δ. We then show that the dynamics defined by these functions satisfy the global Lipschitz continuity condition of Theorem 5.3 as long as the payoff functions π_x^1 and π_m^2 satisfy the conditions in Theorem (5.2). This then implies that from every initial point in $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$, $\widetilde{V}$ will have a unique solution. Since $\widetilde{V}$ coincides with V in Δ and V is forward invariant in Δ, we can conclude that solution trajectories to these dynamics starting in Δ do not leave Δ. Hence, from every initial point $(P(0),q(0)) \in \Delta$, the logit dynamic defined by π_x^1 and π_m^2 admits a unique solution in Δ. The details of the proof are given in the appendix.

As a corollary, we can now show that the logit dynamic is well defined for the Burdett and Judd model. The proof is for the case with endogenous consumer behavior. But it can be easily applied to the one population game with exogenously defined consumer types.

Corollary 5.4 *Consider the logit dynamic (8) defined by the payoff function (2) and (5) of the Burdett and Judd model. Then, from every initial condition $(P(0),q(0)) \in \Delta$, there exists a unique solution $(P(t),q(t)) \in \Delta$ of the dynamic for all time $t \in [0,\infty]$.*

Proof. The corollary follows from Theorem (genexist) as long as the conditions about the Lipschitz continuity of π_x^1 and π_m^2 are satisfied. We verify these conditions in the Appendix. ■

6 Conclusion

The objective of this chapter has been to extend the evolutionary analysis in chapter 2 to the original Burdett and Judd model of price dispersion with continuous strategy space using perturbed best response dynamics. Due to technical difficulties, we have limited ourselves to the logit dynamic. We have introduced the logit dynamic for continuous strategy games in a naturally analogous fashion to the finite dimensional logit dynamic. Given a game, we have defined the set of logit equilibria of the game to be equal to the set of rest points of the logit dynamic induced by the payoffs of the game. We have provided conditions that ensure the existence of logit equilibria in continuous strategy games. We have shown that the Burdett and Judd model fulfils these conditions. We have then resolved the important technical question of whether the logit dynamic generated by continuous strategy games is well defined. We have shown that from every initial population state, the logit dynamic generates a unique solution trajectory of population states for all forward time provided the payoff functions of the game satisfy a certain Lipschitz continuity condition. This result establishes the existence of solutions in the Burdett and Judd model as a corollary.

The main technical contribution of this chapter is therefore introducing the logit dynamic and resolving the question of existence and uniqueness of solution trajectories. The analysis is, however, incomplete since we haven't been able to establish results about the stability of equilibria of infinite dimensional games. Certain questions have to be resolved before we can examine such results. If we are to be guided by the finite dimensional analysis, we need to first arrive at a definition for positive definite games for general nonlinear payoff functions[6]. The difficulty in this is that there are certain technical problems in defining the notion of derivative appropriately for infinite dimensional payoff functions. We then need to find an appropriate method to show stability or instability of equilibria. Again, the difficulty here is that there is no analogue of the linearization techniques we used in the finite strategy case.

We therefore leave the study of stability properties of the logit dynamic as a subject for future research. The resolution of this question is important as it will open the way for the analysis of a host of economic models using evolutionary and learning models. Another important research question is providing microfoundations to infinite dimensional evolutionary dynamics. For evolutionary dynamics to have credibility as a description of aggregate behavior in society, it is necessary that it be based on the behavior of individual agents. The revision protocol model in Sandholm (2006c) provides such microfoundations for finite dimensional evolutionary dynamics. There is, however, no obvious mathematical machinery that will provide such a foundation for infinite dimensional dynamics. This problem has also hampered the development of learning models for continuous strategy games.

Finally, the application of evolutionary ideas to other economic models can be a fruitful area of research. One possible application is in explaining price cycles in "switching" models. There are duopoly models (Narasimhan (1988), Raju, Srinivasan, and Lal (1990)) in which two stores have

[6]Hofbauer, Oechssler and Riedel (2005) definite the notion of negative definiteness for games with linear payoff functions.

loyal customers and compete for "switchers" by setting prices. The Nash equilibrium is mixed in this game. However, experiments by Choi and Messinger (2005) reveal predictable cycles in which for both firms, prices start high initially, decline gradually and then rise abruptly.. This raises the possibility that these cycles can be explained using learning models like stochastic fictitious play.

7 Appendix

7.1 Logit Dynamic is well defined (Proof of Theorem 5.2)

Given $(P,q) \in \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$, we define the two auxiliary functions as follows.

$$\widetilde{V}^1(P,q) = [2 - \|(P,q)\|]_+ V^1(P,q)$$
$$\widetilde{V}^2(P,q) = [2 - \|(P,q)\|]_+ V^2(P,q)$$

where $V^1(P,q)$ and $V^2(P,q)$ are as defined in (8).

We need to show that the function $\tilde{V}(P,q)$ is Lipschitz continuous. Formally, we want to show

$$\left\|(\widetilde{V}^1(P,q),\widetilde{V}^2(P,q)) - (\widetilde{V}^1(Q,r),\widetilde{V}^2(Q,r))\right\| \leq K\,\|(P,q)-(Q,r)\|$$

or

$$\max\left|\left\|\widetilde{V}^1(P,q)-\widetilde{V}^1(Q,r)\right\|,\left\|\widetilde{V}^2(P,q)-\widetilde{V}^2(Q,r)\right\|_1\right| \leq K\max\left|\|P-Q\|,\|q-r\|_1\right| \tag{11}$$

for some constant K independent of (P,q), (Q,r).

In showing Lipschitz continuity, we note that the logit dynamic is not affected by payoffs on a measure zero set.[7] Hence, given (P,q), (Q,r), we can afford to consider only those prices at which both P and Q are continuous. Hence, instead of (2), we can use (4) to define sellers' payoff. Doing this considerably simplifies our proof of Lipschitz continuity.

To show (11), we need to distinguish between three cases. First, if both$\|(P,q)\|$, $\|(Q,r)\| \geq 2$, then both $\widetilde{V}(P,q)$ and $\widetilde{V}(Q,r)$ are both zero and there is nothing to show. Next, we consider $\|(P,q)\| \geq 2 > \|(Q,r)\|$. Hence, $\widetilde{V}^1(P,q) = \widetilde{V}^2(P,q) = 0$. Hence, the left hand side of (11) is

$$\begin{aligned}&\max\left|\left\|\widetilde{V}^1(Q,r)\right\|,\left\|\widetilde{V}^2(Q,r)\right\|_1\right| \\ &= [2-\|(Q,r)\|]\max\left|\left\|V^1(Q,r)\right\|,\left\|V^2(Q,r)\right\|_1\right|\end{aligned}$$

Given any $(Q,r) \in \mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$, the logit best response function to it is a probability measure.

[7] Irrespective of whether there is a mass point on some $x \in S$, $\dot{P}(x) = -P(x)$ under the logit dynamic.

Hence, $\|L(Q,r)\| = 1$. Hence,

$$\left\|V^1(Q,r)\right\| = 1 + \|Q\| \leq 1 + \max\left|\|Q\|, \|r\|_1\right| \leq 3 \tag{12}$$
$$\left\|V^2(Q,r)\right\|_1 = 1 + \|r\|_1 \leq 1 + \max\left|\|Q\|, \|r\|_1\right| \leq 3$$

Hence,

$$\max\left|\left\|\widetilde{V}^1(Q,r)\right\|, \left\|\widetilde{V}^2(Q,r)\right\|_1\right| \leq 3[2 - \|(Q,r)\|] \leq 3[\|(P,q)\| - \|(Q,r)\|] \leq 3\,\|(P,q) - (Q,r)\|$$

So, we obtain

$$\left\|(\widetilde{V}^1(P,q), \widetilde{V}^2(P,q)) - (\widetilde{V}^1(Q,r), \widetilde{V}^2(Q,r))\right\| \leq 3\,\|(P,q) - (Q,r)\|$$

Next, we consider the case where both $\|(P,q)\|,\ \|(Q,r)\| \leq 2$. Continuing from (11), we can write

$$\begin{aligned}\left\|\widetilde{V}^1(P,q) - \widetilde{V}^1(Q,r)\right\| &= \left\|[2 - \|(P,q)\|]V^1(P,q) - [2 - \|(Q,r)\|]V^1(Q,r)\right\| \\ &\leq (2 - \|(P,q)\|)\left\|V^1(P,q) - V^1(Q,r)\right\| + \left\|V^1(Q,r)\right\|\left|\|(P,q)\| - \|(Q,r)\|\right| \\ &\leq (2 - \|(P,q)\|)\left\|V^1(P,q) - V^1(Q,r)\right\| + 3\,\|(P,q) - (Q,r)\| \\ &\leq 2\left\|V^1(P,q) - V^1(Q,r)\right\| + 3\,\|(P,q) - (Q,r)\|\end{aligned}$$

where $\left\|V^1(Q,r)\right\| \leq 3$ by (12). Similarly,

$$\left\|\widetilde{V}^2(P,q) - \widetilde{V}^2(Q,r)\right\|_1 \leq 2\left\|V^2(P,q) - V^2(Q,r)\right\|_1 + 3\,\|(P,q) - (Q,r)\|$$

Hence, the proof will be complete if we can show that $V^1(P,q)$ and $V^2(P,q)$ are Lipschitz continuous on the bounded subset of $\mathcal{M}^e(S,\mathcal{B}) \times \mathbf{R}^r$ where $\|(P,q)\| \leq 2$. To show this, it is sufficient to show that the logit best response functions $L^1(P,q)$ and $L^2(P,q)$ are Lipschitz continuous on this set. Let us consider $L^1(P,q)$ first. Since it is a probability measure, we need to show that

$$\sup_{A\subseteq S}\left|\frac{\int_A e^{\pi_x(P,q)}dx}{\int_S e^{\pi_x(P,q)}dx} - \frac{\int_A e^{\pi_x(Q,r)}dx}{\int_S e^{\pi_x(Q,r)}dx}\right| \leq K_1\,\|(P,q) - (Q,r)\|$$

for some K_1 independent of (P,q), (Q,r). Here, without loss of generality, we have taken $\eta = 1$. For this, we only need to show that $e^{\pi_x(P,q)}$ is Lipschitz. For, then $\int_A e^{\pi_x(P,q)}$ will also be Lipschitz, for all $A \subseteq S$. Moreover, since payoffs are bounded on a bounded set, the denominator of $L^1(P,q)$ will be bounded away from zero. Since the ratio of two Lipschitz continuous functions is Lipschitz as long as the denominator stays away from zero, we can conclude that the Lipschitz continuity of $e^{\pi_x(P,q)}$ will imply the Lipschitz continuity of $L^1(P,q)$. Now, the exponential function is Lipschitz on bounded sets. Hence, the Lipschitz continuity of $\pi_x(P,q)$, uniformly in x, is sufficient for the

Lipschitz continuity of $e^{\pi_x(P,q)}$.

In sum, the Lipschitz continuity of $\pi_x(P,q)$, uniformly in x, is sufficient for the Lipschitz continuity of $\widetilde{V}^1(P,q)$. By a similar argument, the Lipschitz continuity of $C_m(P,q)$, uniformly in m, is sufficient for the Lipschitz continuity of $\widetilde{V}^2(P,q)$. Theorem 5.3 and the forward invariance of the logit dynamic then implies Theorem 5.2.

7.2 Proof of Corollary 5.4

By Theorem 5.2, it is sufficient to prove the Lipschitz continuity of $\pi^1_{x\cdot}$ and $\pi^2_{m\cdot}$ We show this in the form of two lemmas. Lemma 7.1 shows the uniform Lipschitz continuity of $\pi^1_{x\cdot}$ Lemma 7.2 shows the Lipschitz continuity of π^2_m, or equivalently of the cost function C_m. Before presenting the lemmas, we note that the logit dynamic is not affected by payoffs on a measure zero set.[8] Hence, given (P,q), (Q,r), we can afford to consider only those prices at which both P and Q are continuous. Hence, instead of (2), we can use (4) to define sellers' payoff. Doing this considerably simplifies our proof of Lipschitz continuity.

Lemma 7.1 *Let* $\|(P,q)\|$, $\|(Q,r)\| \leq 2$. *Then, for all* x,

$$|\pi_x(P,q) - \pi_x(Q,r)| \leq K_2 \|(P,q) - (Q,r)\|$$

Proof. By our earlier remark, it is sufficient to consider those x at which both P and Q are continuous. Hence,

$$\begin{aligned}
&|\pi_x(P,q) - \pi_x(Q,r)| \\
&= \left| x[(q_1 - r_1) + \sum_{m=2}^{r} m\{q_m P(x,1)^{m-1} - r_m Q(x,1)^{m-1}\}] \right| \\
&= \left| x[(q_1 - r_1) + \sum_{m=2}^{r} m\{q_m P(x,1)^{m-1} - q_m Q(x,1)^{m-1} + q_m Q(x,1)^{m-1} - r_m Q(x,1)^{m-1}\}] \right| \\
&= \left| x[(q_1 - r_1) + \sum_{m=2}^{r} m\{q_m (P(x,1)^{m-1} - Q(x,1)^{m-1}) + Q(x,1)^{m-1}(q_m - r_m)\}] \right| \\
&\leq \left| x[(q_1 - r_1) + \sum_{m=2}^{r} m\{q_m (P(x,1) - Q(x,1))(P(x,1) + Q(x,1))^{m-2} + Q(x,1)^{m-1}(q_m - r_m)\}] \right| \\
&\leq \left| x[(q_1 - r_1) + \sum_{m=2}^{r} m\{2(P(x,1) - Q(x,1))4^{m-2} + 2^{m-1}(q_m - r_m)\}] \right| \\
&\leq |q_1 - r_1| + \sum_{m=2}^{r} m\{2^{2m-3} |P(x,1) - Q(x,1)| + 2^{m-1} |q_m - r_m|\} \\
&\leq [\|q - r\|_1 + \sum_{m=2}^{r} m\{2^{2m-3} \|P - Q\| + 2^{m-1} \|q - r\|_1\}] \\
&\leq (1 + \sum_{m=2}^{r} m(2^{2m-3} + 2^{m-1})) \max |\|P - Q\|, \|q - r\|_1| = K_2 \|(P,q) - (Q,r)\|
\end{aligned}$$

[8] Irrespective of whether there is a mass point on some $x \in S$, $\dot{P}(x) = -P(x)$ under the logit dynamic.

with $K_2 = (1 + \sum_{m=2}^{r} m(2^{2m-3} + 2^{m-1}))$ and we use the fact that $x \leq 1$. ■

Next, we look at the Lipschitz continuity of the consumers' cost function.

Lemma 7.2 *Let $\|(P,q)\|, \|(Q,r)\| \leq 2$. Let F and G be the distribution functions of P and Q respectively. Then, for all m,*

$$|C_m(P) - C_m(Q)| \leq K_3 \|(P,q) - (Q,r)\|$$

Proof. We use the simpler version of the cost function.

$$|C_m(P) - C_m(Q)| = \left| \int_0^1 (1 - F(x))^m dx - \int_0^1 (1 - G(x))^m dx \right|$$
$$\leq \left| \int_0^1 [(1 - F(x)) - (1 - G(x))][(1 - F(x)) + (1 - G(x))]^{m-1} dx \right|$$
$$\leq 4^{m-1} \int_0^1 |F(x) - G(x)| \, dx \leq 4^{m-1} \|P - Q\| \leq 4^{m-1} \max |\|P - Q\|, \|q - r\|_1|$$
$$= K_3 \|(P,q) - (Q,r)\|$$

with $K_3 = 4^{r-1}$. ■

$V^1(P,q)$ and $C_m(P,q)$ are therefore Lipschitz in the bounded set where $\|(P,q)\| \leq 2$. If we denote the Lipschitz constants as $\overline{K}_2$ and $\overline{K}_3$ respectively, the Lipschitz continuity of the dynamic $\widetilde{V}$ in this set will be given by the constant $2(\overline{K}_2 + \overline{K}_3) + 6$.

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