The Dynamic Instability of Dispersed Price Equilibria.*

Ratul Lahkar†

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Abstract

We examine whether price dispersion is an equilibrium phenomenon or a cyclical phenomenon. We develop a finite strategy model of price dispersion based on the infinite strategy model of Burdett and Judd (1983). Adopting an evolutionary standpoint, we examine the stability of dispersed price equilibrium under perturbed best response dynamics. We conclude that when both sellers and consumers participate actively in the market, all dispersed price equilibria are unstable leading us to interpret price dispersion as a cyclical process. For a particular case of the model, we prove the existence of a limit cycle.

1 Introduction

Price dispersion, under which different sellers charge different prices for the same homogeneous good is a commonly observed phenomenon. For example, Baylis and Perloff (2002) and Baye and Morgan (2004) have documented price dispersion among internet firms. Similarly, Lach (2002) provides evidence of price dispersion in Israeli product markets. Some recent experimental work by Cason and Friedman (2003), Cason, Friedman, and Wagener (2005), and Morgan, Orzen and Sefton (2006) have also verified the existence of price dispersion. Price dispersion is very puzzling because it seemingly contradicts the "law of one price" of elementary microeconomics. Various models explain price dispersion as an equilibrium phenomenon.[1] The common feature of these models is the presence of heterogeneity among consumers, whether in the number of prices consumers sample before purchasing (Burdett and Judd (1983)), or in search cost (Salop and Stiglitz (1977), Stahl (1989)). Such heterogeneity implies that there are always some consumers who are willing to pay

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†Department of Mathematics and ELSE centre, University College London, Gower Street, London, WC1E 6BT, UK. email: r.lahkar@ucl.ac.uk

[1]Some of these models are Salop and Stiglitz (1977), Varian (1980), Burdett and Judd (1983), Rob (1985), Stahl (1989), Wilde (1992), Benabou (1993).

a price that is not necessarily the lowest prevailing. This prevents the emergence of Bertrand like competition: instead of undercutting each other in attempts to attract more consumers, sellers can earn more by sticking to a higher price and selling to the fraction of consumers who would be willing to buy at that price. We call the resulting mixed equilibrium a dispersed price equilibrium.

While equilibrium analysis is the received approach to solving economic models, it is reasonable to ask whether observed price dispersion might not be an equilibrium phenomenon, but is the manifestation of some disequilibrium occurrence like a price cycle. A price cycle will be discernible as regular fluctuations in the proportion of sellers charging a particular price; and the resulting fluctuation in the average market price called an Edgeworth cycle[2]. Eckert (2003) and Noel (2003) provide evidence of cycles in the prices set by firms. Similarly, the experimental data in Cason and Friedman (2003), Cason, Friedman, and Wagener (2005) also suggest the presence of cycles. Lach (2002) also detects patterns of cyclical behavior similar to that in Cason, Friedman, and Wagener (2005).

In this paper, we examine price dispersion from the standpoint of evolutionary game theory.[3] We analyze a discrete analogue of the Burdett and Judd (1983) model and conclude that all dispersed price equilibria of the model are dynamically unstable. Through simulations, we verify that these dynamics lead naturally to the emergence of disequilibrium behavior in the form of cycles. Our model, therefore, not only rules out equilibrium price dispersion as a robust prediction, but also provides insight into the nature of observed price dispersion.

Evolutionary game theory seems ideal to analyze market situations like price dispersion. Such perpetual disequilibrium behavior is captured naturally by evolutionary game theory. Moreover, the assumption of myopic agents that underpin evolutionary models is not misplaced in this problem. The number of consumers and sellers in a market are large, and each market participant makes many buying or selling decisions. The impact of any single decision on utility obtained by an agent will be very small, particularly if the items involved are items of daily consumption like sugar or coffee as in the study by Lach (2002). Hence, agents are unlikely to expend a substantial amount of reasoning resources in making these decisions.

From a methodological point of view, this paper also addresses the criticism that evolutionary game theory, despite the potentially rich set of predictions it can offer, has found little application in addressing substantive economic problems.[4] It opens up a broad area of economic interest—situations of persistent disequilibrium—where evolutionary game theory can be fruitfully applied.

The discrete price dispersion model we develop is based on the continuous strategy space model

[2]Edgeworth (1925) was the first to theorize on the presence of price cycles. He argued that in the presence of capacity constraints, the Bertrand prediction of prices being driven down to the marginal cost level would not materialize. Instead, sellers would myopically reduce prices by small amounts when there is excess capacity but jump to higher prices when capacity constraints are binding.

[3]Hopkins and Seymour (2002) were the first to introduce evolutionary ideas to the study of price dispersion. We discuss their work at the end of this section.

[4]Some papers related to application of evolutionary game theory are on externality pricing and macroeconomic spillovers (Sandholm (2002, 2005)), natural selection and animal behavior (Maynard Smith (1982), Hofbauer and Sigmund (1988)), congestion in highway and computer networks (Monderer and Shapley (1996), Sandholm (2001)), emergence of residential segregation (Dokumaci and Sandholm (2007)).

of Burdett and Judd (1983). We focus on this discrete analogue to put off the technical complications involved in the evolutionary analysis of continuous strategy games. We have a population of sellers with a strategy set consisting of a finite number of prices. Consumers observe a certain number of prices after paying a search cost for every price they choose to sample and buy at the cheapest price observed. In a mixed equilibrium, consumers are differentiated by whether they observe only one price or two prices. Hence, there is ex post heterogeneity among consumers. Our main conclusion is that all such equilibria are evolutionarily unstable. Hence, observed price dispersion needs to be understood as a perpetual disequilibrium phenomenon like limit cycles. It is hard to prove the existence of limit cycles. But numerical simulations suggest their presence, and help explain the empirical and experimental observations mentioned above.

We conduct our evolutionary analysis using the class of *perturbed best response dynamics*. These dynamics are so called because they are generated by slightly perturbing the payoffs of agents and then allowing agents to optimize against the prevailing social state.[5] The shocks ensure that even as players play nearly optimally with respect to the unperturbed payoffs, they chose mixed strategies that vary continuously with respect to the social state. These features make perturbed best response more consistent with myopic decision making than pure best response since the choice of strategy does not change abruptly when the social state changes. Technically, these shocks ensure that the resulting dynamic is smooth, and hence, can be analyzed using linearization techniques.[6]

This paper is related to the earlier work of Hopkins and Seymour (2002). These authors perform an evolutionary analysis of price dispersion under the class of positive definite adaptive (PDA) dynamics[7] (Hopkins 1999). Their conclusion is that dispersed equilibria are unstable under these dynamics. The conclusions in that paper, however, have certain ambiguities mainly due to the fact that their finite dimensional analysis is employed in the context of a game with infinite strategy sets. In particular, their definitions of payoff functions and their dynamic analysis are not entirely satisfying from a technical point of view.

Our analysis differs from that of Hopkins and Seymour significantly. First, since our dynamic model is explicitly finite dimensional, it is free from the technical ambiguities present in Hopkins and Seymour (2002). By avoiding any admixture of finite and infinite dimensional issues, we ensure that our conclusions have better technical foundations which make them more credible and more easily understandable. Secondly, we go further than Hopkins and Seymour in considering not only instability of equilibrium, but also the presence of cycling which they ignore. Moreover, the two classes of dynamics–PDA dynamics and perturbed best response dynamics–are distinct. Nevertheless, the general approach of Hopkins and Seymour has influenced us a great deal, particularly in

[5]The introduction of shocks implies that the rest points of the perturbed best response dynamics are not Nash equilibria of the game. Instead, rest points coincide with the set of *perturbed equilibria*. Our stability analysis will therefore relate to the stability of perturbed equilibria. For low levels of shocks, however, perturbed equilibria lie very close to Nash equilibria. So, if perturbed equilibria are unstable, we can conclude that society moves away from Nash equilibria as well.

[6]The most well known perturbed best response dynamic is the logit dynamic (Fudenberg and Levine, 1998).

[7]This class of dynamics describes the behavior of agents who imitate successful opponents, and it includes the replicator dynamic as its prototype.

view of results in Hopkins (1999) that show how concepts derived from PDA dynamics can be used to analyze perturbed best response dynamics.

In section 2, we introduce the finite dimensional Burdett and Judd model. In section 3, we discuss perturbed best response dynamics. Section 4 presents some simulations that illustrate our main results. The highlight of these simulations is the emergence of limit cycles. In section 5, we formally prove our results on the instability of dispersed equilibria under perturbed best response dynamics. In section 6, we look at the issue of cycling in a simple case. Section 7 concludes. Most proofs are relegated to the Appendix.

2 Finite Approximation of the Burdett and Judd Model

The Burdett and Judd (1983) price dispersion model is a game with a continuous strategy space. There are a continuum of homogeneous firms, all selling the same good at a price belonging to the set $[0,1]$. We interpret 0 as the cost for the sellers and 1 as the common reservation price of consumers which is known to the sellers. Each firm chooses a price independently. Consumers observe prices set by a certain number of different firms and then buy one unit of the commodity at the lowest observed price provided that price do not exceed 1. If more than one observed firm is charging the minimum price, the consumer randomizes uniformly between them.

To avoid the technical difficulties associated with the evolutionary analysis of a continuous strategy game, we develop a finite analogue of the original Burdett and Judd model. In order to develop this analogue, we construct a sequence of finite approximation $\{S^n\}_{n\in\mathbf{Z}_{++}}$of the strategy set $S=[0,1]$. The set S^n consists of $(n+1)$ prices $\left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$. We will denote $p_i = \frac{i}{n}$. Given the probability distribution x, the notation x_i will refer to the probability of strategy p_i. If we need to emphasize the size of the strategy set, we will use the notations p_i^n and x_i^n.

For consumers, a strategy is to sample a certain number of prices before deciding to purchase at the cheapest price sampled. Hence, consumer behavior can be summarized by the distribution $(y_1, y_2, \cdots y_r)$, with y_m being the proportion of consumers who are sampling m prices. Here, r is a finite number that represents the maximum number of prices any consumer samples.[8]Our main focus is on the general case where the distribution y emerges endogenously. However, in establishing our general result, we will need to examine the case when the distribution y is specified exogenously as means to establish our conclusions about the more general case.

We denote the population of sellers as population 1 and that of consumers as population 2. We also assume that each population is of mass 1, which allows us to identify a population state with a point in the appropriate simplex.

The set $\Delta_1^n = \{x \in \mathbf{R}_+^{n+1} : \sum\limits_{i=0}^{n} x_i = 1\}$ is the set of states in population 1. The set of states in

[8]In the original model, there is no such upper limit. In our case, it is necessary to impose this upper limit in order to define evolutionary dynamics. The imposition of this limit, however, makes no significant difference to equilibrium behavior. In any mixed equilibrium in the model with endogenous consumer behavior, consumers sample either one or two prices.

population 2 is $\Delta_2 = \{y \in \mathbf{R}_+^r : \sum_{m=0}^{r} y_m = 1\}$. The set of social states is thus $\Delta^n = \Delta_1^n \times \Delta_2$. A social state is $(x, y) \in \Delta^n$. Given the social state (x, y), (x_i, y_m) is to be interpreted respectively as the proportion of sellers who are charging price p_i and the proportion of consumers who are sampling m prices.

We now specify the payoff functions of our model. First, we consider the sellers. Let us fix the strategy set S^n of sellers and the distribution y of consumer types. The payoff that a firm receives by charging a price $p_i \in S^n$ depends upon p_i, the distribution y, and the distribution x of prices chosen by the other firms.

Given p_i, x, y and r, the payoff received by a seller is a function $\pi_i : \Delta^n \to \mathbf{R}$ defined by[9]

$$\pi_i(x) = p_i \left[y_1 + \sum_{m=2}^{r} m y_m \left\{ \sum_{k=0}^{m-1} \frac{g_{(k,i)}^m(x)}{k+1} \right\} \right] \tag{1}$$

where

$$g_{(k,i)}^m(x) = \binom{m-1}{k} (x_i)^k \left(\sum_{j>i} x_j\right)^{m-1-k}. \tag{2}$$

The expected mass of consumers who will sample the firm is $\sum_{m=1}^{r} m q_m$.[10]If a consumer samples m firms including the firm in question, $g_{(k)}^m(x)$ is the probability that the price p_i chosen by the firm is the minimum of the m prices and is also chosen by k other firms. Hence, the probability that the consumer will buy from the firm is $\sum_{k=0}^{m-1} \frac{g_{(k)}^m(x)}{k+1}$. Uniform randomization by consumers accounts for division by $k+1$.

Example 2.1 *If the distribution of consumer types is* $\{y_1, y_2\}$, *then*

$$\pi_i(x) = p_i\left(y_1 + 2y_2\left(\sum_{j>i} x_j + \frac{x_i}{2}\right)\right).$$

If the distribution is $\{y_1, y_2, y_3\}$, *then*

$$\pi_i(x) = p_i\left(y_1 + 2y_2\left(\sum_{j>i} x_j + \frac{x_i}{2}\right) + 3y_3\left(\left(\sum_{j>i} x_j\right)^2 + x_i \sum_{j>i} x_j + \frac{x_i^2}{3}\right)\right).$$

We now consider the payoff function of consumers. A strategy for a consumer is now the number of prices that is to be sampled before purchasing. We assume that consumers have to pay a cost $c > 0$ for every price they choose to sample. The parameters c and r are assumed to be common to all consumers. Consumers are therefore a priori homogeneous. If each price quotation is a random draw from the probability distribution p, then the expected cost of purchasing when m prices are

[9]The payoff function differs from the one in Burdett and Judd (1983) in that we have to account for the possibility of sellers choosing equal prices. Burdett and Judd ignore this possibility since in their setting, all mixed equilibrium are absolutely continuous probability measures.

[10]The expected number of m price samplers who sample a particular firm is $m y_m$. Hence, the expected measure of consumers who will sample a firm is $y_1 + \sum_{m=2}^{r} m y_m$.

observed is given by the function $C_m : \Delta_1^n \to \mathbf{R}$ defined by,

$$C_m(x) = mc + m\sum_{i=0}^{n} p_i x_i \left\{ \sum_{k=0}^{m-1} \frac{g^m_{(k,i)}(x)}{k+1} \right\} \tag{3}$$

with $g^m_{(k,i)}(x)$ defined in (2). The interpretation of the cost function is as follows. Suppose a consumer is randomly sampling m prices. If one of the prices he observes is p_i, then $g^m_{(k)}(x)$ is the probability that p_i is the minimum of the m prices and that the consumer has observed k other equal prices. Uniform randomization leads to division by $(k+1)$. We multiply by m since p_i can be observed in any of the m draws. Hence, $mx_i \sum_{k=0}^{m-1} \frac{g^m_{(k,i)}(x)}{k+1}$ represents the probability of paying p_i.

Consumers' payoff is the negative of (3). It is important to note that the cost function is independent of consumers' aggregate behavior given by the distribution y. This fact will have important consequences for the stability properties of mixed equilibria.

2.1 Equilibria with Fixed Consumer Types

In deriving the Nash equilibria of our model, we follow the general strategy in Burdett and Judd (1983). We first derive equilibria by fixing the distribution $\{y_i\}_{i=1}^r$ of consumer types. The results derived in this case then allows us to solve the model in the more general case when consumer search behavior is endogenous.

We now fix y and characterize the Nash equilibria of the game with payoff function (1). First, let us consider the case $0 < y_1 < 1$. This case is important because it is the presence of some uninformed consumers that prevents prices from falling to the competitive level. On the other hand, if all consumers are uninformed, then 1 is a dominant strategy. Hence, for dispersed price equilibria to emerge, this condition must be satisfied. [11]

We now show that for n sufficiently large, all equilibria are mixed equilibria. This follows from the following lemma, which shows that as n gets large, the probability attached by any Nash equilibrium on any single price must go to zero.

Lemma 2.2 *Let the type distribution* $\{y_1, y_2, \cdots y_r\}$ *satisfy* $0 < y_1 < 1$. *Let* $\overline{x}^n$ *be a Nash equilibrium of the game with strategy set* S^n. *Then, for all strategies* p_i^n, $\overline{x}_i^n \to 0$ *as* $n \to \infty$.

Proof. In the Appendix. ■

The intuition behind this result is as follows. As the number of prices increase, the difference between any two successive prices goes to zero. Hence, if the weight on any price remains bounded

[11] If $y_1 \in (0,1]$, then when n is large, there are positive prices that are dominated. In fact, any price that is less than $\underline{p} = y_1(\sum_{m=1}^{r} my_m)^{-1}$ is dominated by price 1. The lowest possible payoff obtained from charging 1 is y_1 whereas the highest possible payoff from any price p_i is $p_i \sum_{i=1}^{r} my_m$. This gives us $\underline{p}$. For positive dominated strategies, we require $\underline{p} \geq \frac{1}{n}$ which implies $n \geq (\sum_{m=1}^{r} my_m) y_1^{-1}$.

away from zero, any seller charging that price can deviate to the price immediately below that. While the two prices are nearly the same, the probability of being the minimum price sampled increases significantly. In the appendix, we show that this intuition works for all prices except the first two positive prices. But for n large, these prices are dominated by 1 and so can be ignored.

We therefore conclude that for n sufficiently large, the only Nash equilibria are mixed strategy Nash equilibria if $0 < y_1 < 1$.[12]

We now consider the two special cases $y_1 = 0$ and $y_1 = 1$. The only Nash equilibria in these cases are pure strategy equilibria.

Lemma 2.3 *1. Let $y_1 = 1$. Then, for all n, the only Nash equilibrium is $x_n^n = 1$, i.e. all firms charge the highest price 1.*

2. Let $y_1 = 0$. Then, for any n, there are always two pure strategy Nash equilibria. One Nash equilibrium is $x_0^n = 1$, i.e., all firms charge price 0. Another Nash equilibrium is $x_1^n = 1$. In the particular case where $y_2 = 1$, $x_2^n = 1$ is also a Nash equilibrium. Moreover, for all n, there exist no other Nash equilibria.

Proof. In the Appendix. ■

Part (1) follows because 1 is then the dominant price. The proof of part (2) is somewhat tedious but the intuition is largely that of Bertrand competition. For the special case where $y_2 = 1$, $x_2^n = 1$ is a non strict Nash equilibrium.

The results in this subsection are very analogous to the corresponding result (Lemma 2) in the original Burdett and Judd (1983) model. They find that if $y_1 = 1$, the only equilibrium is the monopoly equilibrium whereas if $y_1 = 0$, the only equilibrium is the competitive equilibrium. If $0 < y_1 < 1$, the unique equilibrium is an absolutely continuous probability measure with compact and connected support. One significant difference is that in the finite case, there may be more than one mixed strategy equilibrium.

2.2 Equilibrium with Endogenous Consumer Behavior

We can now characterize the equilibria of the complete model in which consumer behavior emerges endogenously. First, we present the following lemma that has important implications for the characterization of Nash equilibria. In this lemma, we show that the cost function defined in (3) is convex in the number of prices a consumer chooses to observe, provided that the strategy distribution in the population of sellers is mixed.

Lemma 2.4 *Let the population 1 state x be mixed. Let F be the distribution function of x. Hence, $F_i = \sum_{j \le i} x_j$. Then, the cost function $C_m(x)$ is strictly convex in m.*

[12] Let $\lceil a \rceil$ the smallest integer strictly larger than a. If $n > \frac{\lceil L \rceil}{\underline{p}}$ where $L = 1 + y_1 (\sum_{m=2}^{r} (m-1)\, y_m)^{-1}$, then there is no pure strategy equilibrium in the game with strategy set S^n

Proof. It can be shown through some tedious manipulation that

$$C_m(x) = mc + \frac{1}{n}\sum_{i=0}^{n}(1-F_i)^m \tag{4}$$

For any number $b \in (0,1)$, $(1-b)^m$ is strictly convex in m. Hence, as long as the distribution x is not a pure strategy, $C_m(x)$ will be strictly convex in m. ■

A feature of (4) is that the price term p_i does not appear in it. This is because prices are uniformly spaced due to which their effect is incorporated in the $\frac{1}{n}$ term. The convexity of the cost function implies that it is minimized at either a unique integer m^* or two successive integers m^* and m^*+1.

Let us fix the strategy size n. We can now establish certain facts about Nash equilibrium when consumer behavior is endogenous. These results are analogous to the infinite dimensional case. We first show that monopoly pricing is always an equilibrium irrespective of the number of prices available. Apart from this, there will exist no other pure strategy equilibrium for any n. Next, we argue that at any mixed equilibrium, consumers will sample either only one price or two prices.

Theorem 2.5 *In the game with endogenous consumer behavior.*

1. $\{x_n^n = 1, y_1 = 1\}$ *is always a Nash equilibrium. This is the monopoly equilibrium.*

2. *For all n, the only other Nash equilibria are mixed equilibria in which both producers and consumers randomize.*

3. *In any mixed equilibrium, $0 < y_1 < 1$ and $y_1 + y_2 = 1$. Consumers sample at most two prices.*

Proof. We prove each statement in turn.

1. If all sellers are charging the highest price, then the cost minimizing strategy for consumers is to sample just one price. On the other hand, if all consumers are searching just once, then sellers' profits are maximized by charging the highest price.

2. We first rule out equilibria in which $y_i = 1$ for $i > 1$. If $y_i = 1$ for $i > 1$, then by Lemma 2.3, the only possible equilibria are pure equilibria where firms charge either the zero price, or p_1^n or p_2^n. Since all firms charge the same price, the cost minimizing strategy for consumers is uniquely $y_1 = 1$. But then sellers will charge the highest price and we are back to the monopoly equilibrium. Next, suppose consumers randomize but all producers charge a single price. Then, all consumers will deviate to sampling just one price.

3. At any mixed equilibrium, we must have $0 < y_1 < 1$. If $y_1 = 0$, then by lemma 2.3, all sellers charge the same price. But then, all consumers sample just once. If $y_1 = 1$, then all firms charge the highest price and we have the monopoly equilibrium. Hence $0 < y_1 < 1$ which implies that sampling one price is one of the cost minimizing strategies of the consumers.

Since sellers play a mixed strategy at the Nash equilibrium, the cost function is strictly convex. This implies that the only other cost minimizing strategy is to sample two prices. Thus, $y_1 + y_2 = 1$.

This completes the proof. ■

Example 2.6 *Consider the game with* $n = 5$, $r = 3$, *and* $c = 0.07$. *Hence the strategy set of sellers is* $S = \left\{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\right\}$. *Consumers observe a maximum of three prices. The monopoly equilibrium is a pure equilibrium. Apart from this, there are nine mixed equilibria, all having the characteristic* $0 < y_1 < 1$ *and* $y_1 + y_2 = 1$. *We list the mixed equilibria in the appendix. One particular mixed equilibrium we will focus on to illustrate our results on instability is* $x^* = (0, 0, 0, 0.4684, 0.4176, 0.1140)$ *and* $y^* = (0.6680, 0.3319, 0)$.

Theorem 2.5 have exact counterparts in the infinite case (see Theorem 2 in Burdett and Judd (1983)). The only pure equilibrium in the infinite game is the monopoly equilibrium. Any mixed equilibria is characterized by $0 < y_1 < 1$ and $y_1 + y_2 = 1$. For the infinite dimensional case, it is possible to go further and show that there may be zero, one or two mixed equilibria, depending on c. In the finite case, we don't have such an exact result on the number of equilibria that can exist.

3 Perturbed Best Response Dynamics

We now consider the dynamic analysis of our price dispersion model. We will model dynamic behavior in the two populations by using perturbed best response dynamics.[13]Formally, these dynamics are derived by requiring agents to optimize against payoffs after they have been subjected to some perturbations. These shocks can be interpreted to mean actual payoff noise, or mistakes agents make in perceiving payoffs or in implementing pure best responses. Since perturbed best response has appealing behavioral properties, they have been used in the experimental literature as a tool to rationalize noisy experimental data (Cheung and Friedman (1997), Camerer and Ho (1999), Battalio et al. (2001)).

Our objective is to analyze the stability properties of dispersed price equilibria under perturbed best response dynamics. In order to motivate these dynamics, we focus on the one population price dispersion game with fixed consumer behavior. Since agents are myopic, their perceived payoffs on which they base their decisions are always a function of the current social state. Hence, the underlying payoff function that defines the dynamic is $\pi : \Delta^n \to \mathbf{R}^{n+1}$ with $\pi_i(x)$ defined in (1). The discussion that follows is, however, more general and applies to any population game. The derivation of the dynamics can also be readily extended to multipopulation cases.

[13]This dynamic model can be provided microfoundations by appealing to the model of revision protocols in Sandholm(2006c). In this model, agents myopically change their behavior in response to the present social state whenever they receive opportunities to revise their strategies. The resulting process of social change can then be summarized using an evolutionary dynamic.

We can write the evolutionary dynamic as the ordinary differential equation $\dot{x} = V(x)$[14]where $x \in \Delta^n$ and $V(x)$ is the vector of change in social state x. Thus. $V_i(x)$ will indicate the direction and magnitude of change in the proportion of agents playing strategy x_i at the social state x. To be admissible as an evolutionary dynamic, we require that from each initial condition $x_0 \in \Delta^n$, there must exist a unique solution trajectory $\{x_t\}_{t \in [0,\infty)}$ with $x_t \in \Delta^n$, for all $t \in [0, \infty)$.

We now give a brief description of the derivation of perturbed best response dynamics. The prototypical perturbed best response dynamic, the logit dynamic, was introduced by Fudenberg and Levine (1998). Since then, a number of authors including Benaim and Hirsch (1999), Hofbauer and Hopkins (2005), and Hofbauer and Sandholm (2002, 2005) have studied these dynamics in more general form.

We call $v : int(\Delta^n) \to \mathbf{R}$ an admissible deterministic perturbation if the second derivative of v at x, $D^2 v(x)$ is positive definite for all $x \in int(\Delta^n)$ and if $|\nabla v(x)| \to \infty$ whenever $x \to bd(\Delta^n)$. In words, v is admissible if it is convex and becomes infinitely steep at the boundary of the simplex. We may interpret v as a "control cost function" associated with implementing any particular mixed strategy. The cost becomes large whenever the agent puts too little probability on any pure strategy.

Given the payoff function π and population state x, we define the perturbed payoff to mixed strategy $q \in int(\Delta^n)$ as $q'\pi(x) - \eta v(q)$.

The perturbed best response to x, $\widetilde{B}(x)$ can then be obtained as the solution to the maximization exercise

$$\widetilde{B}(x) = \underset{q \in int(\Delta^n)}{\operatorname{argmax}} \; q'\pi - \eta v(q) \tag{5}$$

The properties of convexity and steepness of the control cost function are crucial for determining the key characteristics of the perturbed best response function. Convexity of the cost function ensures that the perturbed best response to every population state is unique. Steepness implies that the perturbed best response is a fully mixed strategy.[15]Moreover, $\widetilde{B}(x)$ is differentiable with respect to x. In terms of these three properties- uniqueness, complete mixture and smoothness- the perturbed best response differs critically from the actual best response. Nevertheless, if the perturbation factor η is small, then $\widetilde{B}(x)$ puts most of the weight on the actual best response to x.

State x is a *perturbed equilibrium* of the population game π if it is a fixed point of the perturbed best response function, i.e. if $x = \widetilde{B}(x)$. Given a particular η, the set of perturbed equilibria and Nash equilibria will differ for most games. However, if x^* is a Nash equilibrium, then, typically, for small η, there will be an associated perturbed equilibrium $\widetilde{x}_\eta$ such that $\lim_{\eta \to 0} \widetilde{x}_\eta = x^*$.

From the perturbed best response function, we now define the perturbed best response dynamic

[14]To be strictly accurate, we should write $V(x)$ as $V_\pi(x)$ to indicate the dependence of the dynamic on the payoff function. However, since the underlying game is usually clear from the context, we will dispense with the extra notation.

[15]The method we described generates $\widetilde{B}(x)$ using deterministic perturbation of the payoffs. The traditional method of deriving the perturbed best response function is by adding stochastic perturbations to the payoffs. However, Hofbauer and Sandholm (2002) show that the deterministic perturbation method is the more general technique.

as follows

$$\dot{x} = \widetilde{B}(x) - x$$

The precise form of the perturbed best response function will depend on the control cost function $v(x)$ and the perturbation factor η. Hence, the dynamic is also a function of $v(x)$ and η. Clearly, rest points of the dynamic coincide with the set of perturbed equilibria.

The next step in our analysis will be to analyze the stability properties of these dynamics. For most evolutionary dynamics, a Nash equilibrium is a rest point. An unstable rest point therefore means that the corresponding Nash equilibrium is also unstable. However, for perturbed best response dynamics, rest points are not Nash equilibria. Hence, any stability result for these dynamics will refer to stability of perturbed equilibria rather than Nash equilibria. Our interest, however, is primarily on the dynamics when η is small. Since typically, in such a situation, a perturbed equilibrium lies very close to a Nash equilibrium, stability of perturbed equilibria is sufficient to inform us whether the corresponding Nash equilibrium is a credible long run prediction.

The prototypical perturbed best response dynamic is the logit dynamic obtained from the logit best response function. The logit best response function can be obtained by specifying $v(q) = \sum\limits_{x_i \in S^n} q_i \log q_i$. This gives us the function

$$\widetilde{B}_i(x) = \frac{\exp(\eta^{-1}\pi_i(x))}{\sum\limits_{x_j \in S^n} \exp(\eta^{-1}\pi_j(x))}$$

4 Simulations

Following the description of the perturbed best response dynamics, we can now examine the stability properties of dispersed price equilibria. We will, however, defer a formal analysis of this question until the next section. In this section, we present two simulations that illustrate the stability properties of dispersed equilibria. We consider two numerical examples and simulate solution trajectories using the logit dynamic. We will see that dispersed equilibria are unstable in these games under the logit dynamic. Instead, the solution trajectories converge to limit cycles, thereby implying that the dispersed equilibria are not a valid explanation of observed price dispersion. We will then evaluate the time average of these limit cycles to see whether the credibility of the Nash equilibria prediction can be partially restored.

4.1 A Game with Four Strategies

We first discuss the simplest possible case that can induce a dispersed price equilibrium. We look at a game with just four prices. We assume that there are only two types of consumers, $\{y_1, y_2\}$ exogenously given. We fix y_1 at 0.85 and set the sellers' strategy set as $S = \left\{0, \frac{10}{12}, \frac{11}{12}, 1\right\}$. Since the prices are not uniformly spaced, this example does not fall within the framework specified above. The particular value of y_1 and the asymmetric strategy set are motivated by two concerns. The first is to provide a pictorial representation of the instability of equilibria and the location of a limit cycle in the entire state space.[16] Hence, we cannot go beyond four strategies. The second is to rule out pure equilibria. This concern excludes strategy sets of size two or three. Even with four strategies, if the prices are uniformly spaced, we will have pure equilibria. The particular numbers chosen are to ensure that our objectives are met.

A broader conceptual reason why we choose to run a simulation with a two-type consumer behavior case is that the stability properties of this simple case has a crucial bearing on the corresponding properties in the more general case with endogenous consumer behavior. Since any mixed equilibrium in Theorem 2.5 has consumers sampling either one or two prices, any instability result in the two-type fixed consumer behavior case can be used to show instability of dispersed equilibria in the more general case. In fact, this is the strategy we will employ in the next section when we formally demonstrate instability of dispersed equilibria.

The payoff function of the game is given by

$$\pi_i\left(x\right) = p_i(y_1 + 2y_2(\frac{x_i}{2} + \sum_{j>i} x_j)) \tag{6}$$

for $p_i \in S$. The unique Nash equilibrium of the game is $x^* = (0, 0.4536, 0.2022, 0.3443)$. We now show that the corresponding perturbed equilibrium is unstable[17]under the logit dynamic for small η.

For this example, we fix η at a relatively small value of 0.001. The perturbed equilibrium is then $\widetilde{x} = (\sim 0, 0.4571, 0.2032, 0.3397)$ with ~ 0 indicating a value that is positive but extremely close to zero. As we would expect, $\widetilde{x}$ is very close to the Nash equilibrium. The most straightforward way to verify the instability of the perturbed equilibrium for this relatively simple case is to linearize the logit dynamic around the perturbed equilibrium. If the real part of even a single eigenvalue of $DV\left(\widetilde{x}\right)$ is positive, we can conclude that $\widetilde{x}$ is unstable. The eigenvalues of $DV\left(\widetilde{x}\right)$ are $\{1.9513 \pm 24.3474i,\ -1\}$.[18] Hence, we may conclude that the equilibrium is unstable. This method of direct linearization is of course cumbersome to implement in the general case when the number of prices will be large. Instead, in the next section, we will use techniques based on Hopkins (1999) to arrive at a similar conclusion using a more analytical approach. We will revisit this example then to illustrate those techniques.

In Figure 1,[19] we plot the four solution trajectories with initial conditions being the four monomorphic states. On can clearly see that the solution trajectories converge to a limit cycle. In Section 6, we will conclude that as $\eta \to 0$, the limit cycle converges to the triangle with

[16]Since consumer behavior is fixed,this is a one population game with state space Δ_1^3.

[17]Strictly speaking, the simulation only reveals the convergence of trajectories to a limit cycle. However, Proposition 5.6 imply that the perturbed equilibrium is unstable. This instability result follows from the positive definiteness of the game, a notion we discuss in the Appendix.

[18]The reason why we have three eigenvalues instead of four is because the movement of the population is restricted to three directions. We discuss this issue in greater detail in the next subsection.

[19]This figure has been created using the Dynamo software by Sandholm and Dokumaci (2006).

vertices $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(0, \frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, and $(0, \frac{4}{9}, \frac{1}{9}, \frac{1}{9})$. Simulations from other initial points suggest that this is globally attracting limit cycle. Therefore, under the logit dynamic, this limit cycle is a more credible long run prediction than the Nash equilibrium.

Long-run price dispersion, in this example, is therefore a cyclical phenomenon. We cannot expect to see period by period Nash equilibrium behavior on the part of sellers. Nevertheless, it might still be possible to rescue the equilibrium prediction partially if the time average of the limit cycle is close to the logit equilibrium. This would imply that the average of social states over a long period of time would resemble the equilibrium distribution. This may lead to the erroneous conclusion that play has indeed converged to a Nash equilibrium, since cycles in mixed strategies would be difficult to observe in a laboratory experiment. It might even be regarded as a partial justification of Nash equilibria. Hence, it would be fruitful to examine whether such an error can occur.

The time average of the limit cycle may be computed as

$$\overline{x} = \lim_{T \to \infty} \int_0^T x(t)dt$$

where $x(0)$ is the initial point of any solution trajectory that converges to the limit cycle. Numerical estimation reveals that $\overline{x} = (\sim 0, 0.4999, 0.2075, 0.2926)$. As can be seen in Figure 1, this time average is sufficiently close to the perturbed equilibrium to be actually mistaken as convergence to equilibrium, particularly if noisy experimental data is used to make such judgements. This is not surprising since the limit cycle also lies near the equilibrium. However, the next example illustrates that the time average of a limit cycle can be significantly different from any dispersed equilibrium.

4.2 A Two Population Game with Six Prices

We now consider an example which is more illustrative of our general model with both sellers and consumers behaving strategically. In particular, we provide a simulation of solution trajectories of the game in Example 2.6 under the logit dynamic. We have $n = 5$, $r = 3$ and $c = 0.07$.

In figures 2 and 3, we plot solution trajectories for the logit dynamic with $\eta = 0.001$ starting from two different initial points. In Figure 2, we plot trajectories with the initial point being the Nash equilibrium (x^*, y^*) where $x^* = (0, 0, 0, 0.4684, 0.4176, 0.1140)$ and $y^* = (0.6680, 0.3319, 0)$. In Figure 3, the initial point is $x(0) = (\frac{1}{6}, \frac{1}{6}, \cdots, \frac{1}{6})$ and $y(0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Note that we have only plotted the trajectories for some of the variables since the other variables fluctuate at levels that are positive but indistinguishable from zero.

It is clear that from both initial points, trajectories converge to price cycles. However, the two limit cycles are different. Let us look at the limit cycle in Figure 2, which we call **LC1**. Here, if we consider seller behavior, it is only prices $\frac{3}{5}$ and 1 that are charged by significant proportions of the population: x_3 fluctuates between about 0.7 and 0.8, while x_5 fluctuates between about 0.2 and 0.3. The population shares of the prices remain at levels very close to zero, but nevertheless

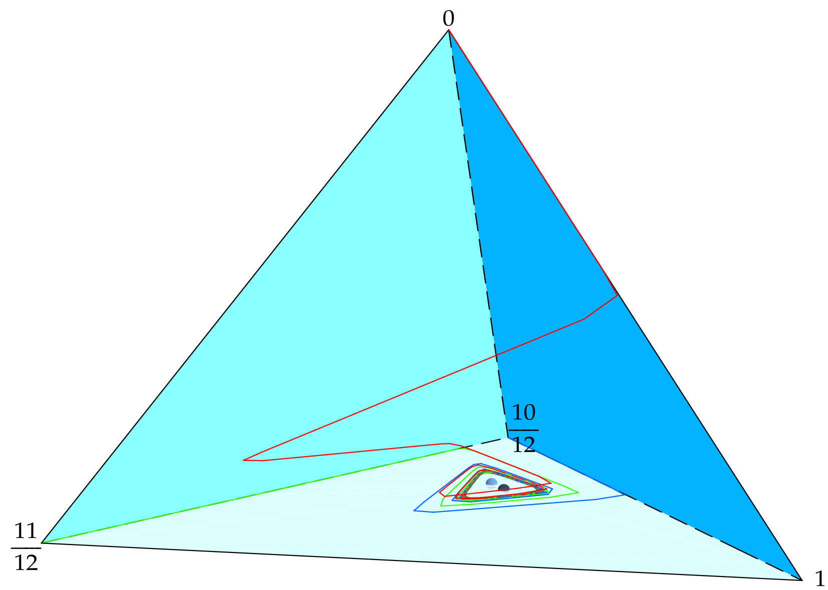

Figure 1: Solution Trajectories from the four monomorphic states in the 4-strategy game. The perturbed equilibrium of the game is $(\sim 0, 0.4571, 0.2032, 0.3397)$ while the time average of the limit cycle is $(\sim 0, 0.4999, 0.2075, 0.2926)$, for $\eta = 0.001$. The darker dot inside the limit cycle is the perturbed equilibrium while the lighter dot is the time average.

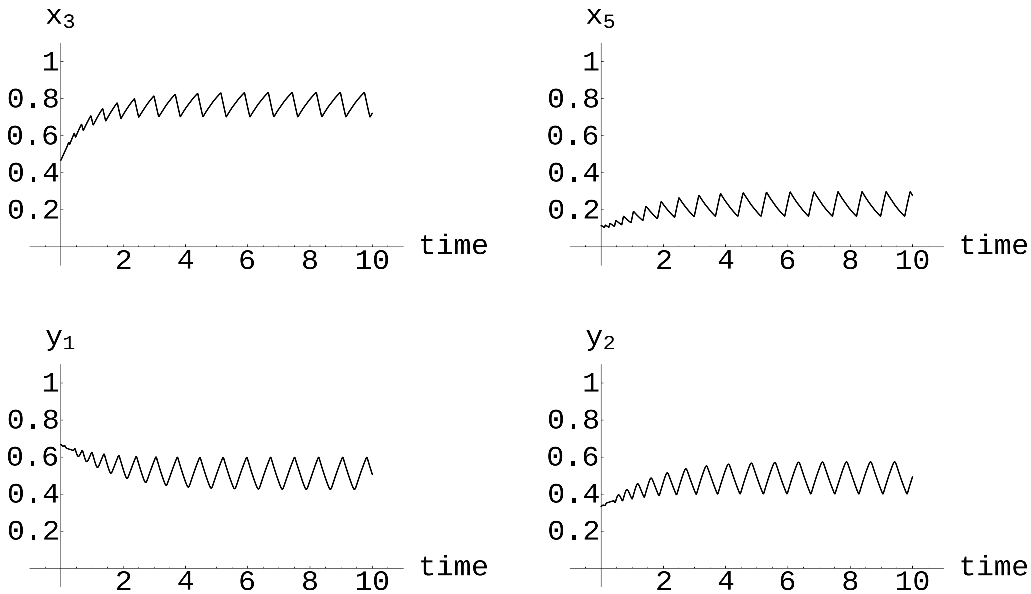

Figure 2: Solution trajectories under the logit (0.001) dynamic in the game with 6 prices, $r = 3$, and $c = 0.07$. The initial point is a Nash equilibrium.

positive. Hence, we can loosely term the support of the cycle to be prices $\{\frac{3}{5}, 1\}$. For the cycle in Figure 3, which we call **LC2**, the support consists of $\{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$.

There are eight other mixed equilibria in this game, all listed in the appendix. We ran simulations of the solution trajectories of the same dynamic from each of these equilibria as initial points. We found that trajectories from all but one of the mixed equilibria [20]exhibit the same cycle as in Figure 3. From the one equilibrium that is the exception, the limit cycle is the one in Figure 2. Similarly, simulations from other points reveal that most trajectories converge to the cycle in Figure 3. The simulations could not detect any other limit cycle. Hence, it appears that the likelihood of the emergence of cycle in Figure 3 is much higher than that in Figure 2. We also note that even though some of the mixed equilibria puts positive probability on price $\frac{1}{5}$, the weight on this price declines to zero in both the limit cycles. Hence, even the fact that these equilibria put positive weight on price $\frac{1}{5}$ is misleading as a prediction. Finally, we note that the monopoly equilibrium is a strict equilibrium. Hence, the perturbed equilibrium corresponding to this equilibrium is locally stable.

We can once again calculate the time averages of the two limit cycles. The time average corresponding **LC1** is $(\overline{x}_1, \overline{y}_1)$ where $\overline{x}_1 = (\sim 0, \sim 0, \sim 0, 0.7726, \sim 0, 0.2274)$ and $\overline{y}_1 = (0.5085, 0.4915, \sim 0)$. The time average corresponding to **LC2** is $(\overline{x}_2, \overline{y}_2)$ with $\overline{x}_2 = (\sim 0, \sim 0, 0.7205, 0.0393, 0.1718, 0.0684)$ and $\overline{y}_2 = (0.2712, 0.7287, \sim 0)$.

It is of interest to note that $\overline{x}_1$ is very close to x^9, where x^9 is the distribution in the population

[20]This particular Nash equilibrium is $x^9 = (0, 0, 0, 0.7739, 0, 0.2261)$; $y^9 = (0.5522, 0.4398)$

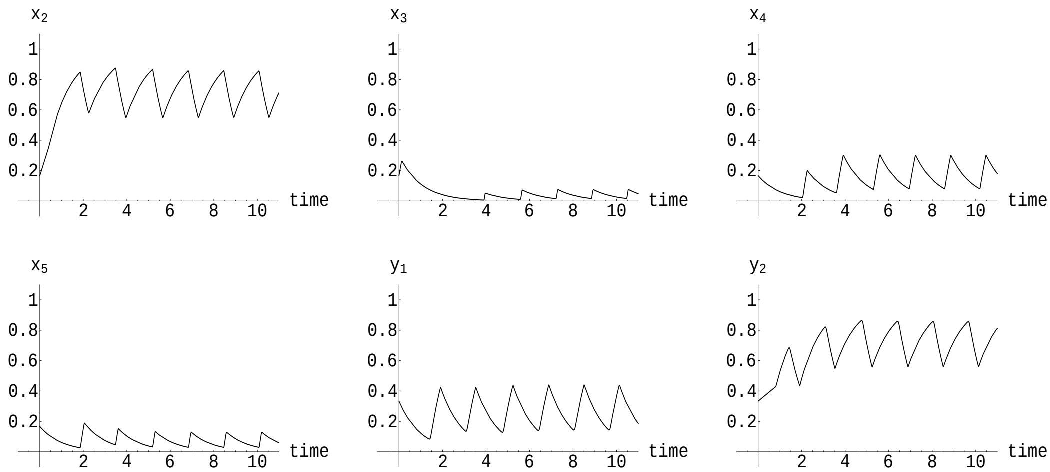

Figure 3: Solution trajectories under the logit (0.001) dynamic in the game with 6 prices, $r = 3$, and $c = 0.07$. $x(0) = (\frac{1}{6}, \frac{1}{6}, \cdots, \frac{1}{6})$ and $y(0) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

of sellers at the Nash equilibrium (x^9, y^9) listed in footnote 20. However, $\overline{y}_1$ is still very significantly different from y^9. Hence, none of the mixed Nash equilibrium prediction is borne out by the time average of **LC1**. The time average of **LC2** is even more drastically different from any of the mixed Nash equilibria. None of the Nash equilibria even have a support of four prices, unlike the time average of **LC2**.

The cyclical fluctuation in the distribution x imply a cycle in the average price over time. The average price at a particular time is $\overline{p}(t) = \sum_{i=0}^{n} p_i x_i(t)$. Since solution trajectories converge to a limit cycle, $\overline{p}(t)$ will also exhibit a regular cycle. This cycle is the Edgeworth cycle that has been observed in the experiments and market data mentioned in the Introduction.

An important characteristic of these Edgeworth cycles is that the upward phase of the cycles is much steeper than the downward phase. The following story of firm behavior might explain this feature of these cycles. Upon reaching the highest level of the cycle, firms slowly start undercutting their rivals. Hence, the proportion of firms charging lower prices increases. This phase continues till a certain minimum level is reached. Upon reaching that level, most firms then increase their prices very significantly. This raises the average price sharply and the cycle continues. Such a pattern of firm behavior has been noted by Noel (2003) in his analysis of the Toronto retail gasoline market.

These limit cycles are of importance not only in the context of these examples but also more generally. To our knowledge, these are the first examples of evolutionary limit cycles in an economic model. Conceptually, they provide a new way of thinking about the long run consequences of economic interaction.

In general, it is very difficult to prove the identity or the existence of limit cycles. The fact that there may be a number of them makes this an even more difficult task. The only way to verify their presence would be to run simulations with specific numerical examples. In fact, long run

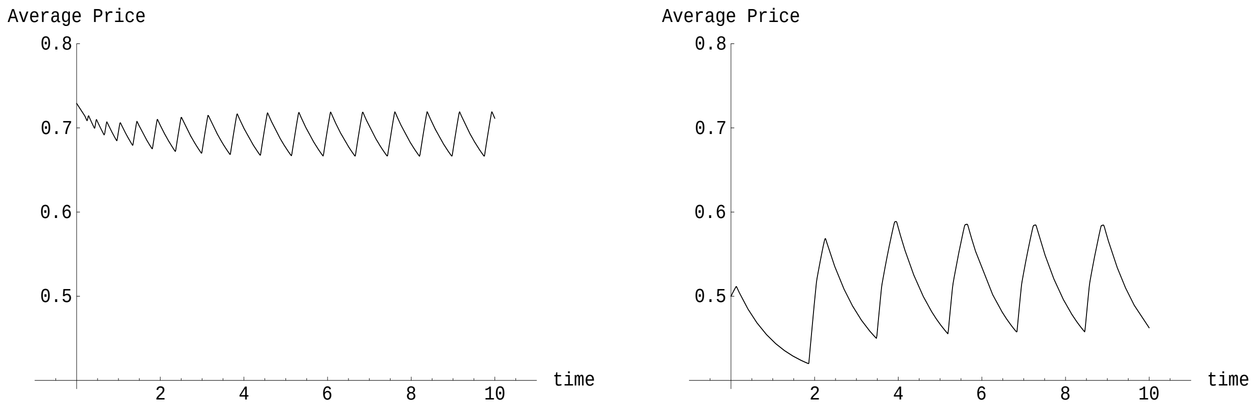

Figure 4: In the left panel, we plot $\overline{x}(t)$ with initial point same as in figure 2. In the right panel, the initial point is the one in figure 3.

disequilibrium behavior may take even more complicated forms like strange or chaotic attractor, where trajectories would exhibit elaborate dependence on initial conditions.

In one particularly simple case, however, we can prove the existence of perpetual disequilibrium. This is the case when consumer behavior is given by the exogenous distribution $\{y_1, y_2\}$, $0 < y_1 < 1$ and we impose a further technical condition called quasi-monocyclicity. We do this in Section 6.

5 Instability of Dispersed Price Equilibria

5.1 Instability with Exogenous Types

We now look more generally at the issue of stability of perturbed equilibrium. We fix the strategy set size n beforehand. Our focus is on the stability properties of the perturbed equilibria corresponding to the dispersed equilibria characterized in Theorem 2.5. However, we will first establish instability results for dispersed equilibria in the one population game with consumer behavior exogenously restricted to observing either one or two prices. Hence, we fix the distribution $\{y_1, y_2\}$. We further assume that $0 < y_1 < 1$, and that n is large enough, to ensure the existence of mixed Nash equilibria. Since we are considering the one population case, our state space is Δ_1^n and the tangent space is $T\Delta_1^n$.[21] However, for the rest of this subsection, we will dispense with the subscript and superscript in referring to the state space and the tangent space.

To determine the stability properties of rest points, we use the standard techniques of linearizing the dynamic around the rest points. Given the control cost function $v(x)$ and the dynamic $\dot{x} = V(x)$, let $\widetilde{x}$ be a perturbed equilibrium, and hence a rest point of the dynamic. By Taylor's theorem, if we consider the dynamic at a point $\widetilde{x} + z$ in the neighborhood of $\widetilde{x}$, then $V(\widetilde{x} + z) \approx$

[21]The tangent space is the set of feasible directions of motion of the population. Formally, $T\Delta_1^n = \left\{ z \in \mathbf{R}^{n+1} : \sum_{i=0}^{1} z_i = 0 \right\}$

$V(\widetilde{x}) + DV(\widetilde{x})\,z = DV(\widetilde{x})\,z$ where $DV(\widetilde{x})$ is the Jacobian $DV(x) : T\Delta \to T\Delta$ evaluated at $\widetilde{x}$. The non-linear dynamic $V(\widetilde{x} + z)$ can therefore be approximated by a linear differential equation $DV(\widetilde{x})\,z$ in a neighborhood of the rest point. Standard results then imply that if even a single eigenvalue of $DV(\widetilde{x})$ has a positive real part, then the rest point will be unstable. Now, since $V(x){=}\widetilde{B}(x) - x$

$$DV(\widetilde{x}) = D\widetilde{B}(\widetilde{x}) - I$$

where $D\widetilde{B}(\widetilde{x}) : T\Delta \to T\Delta$ is the Jacobian of $\widetilde{B}$ evaluated at $\widetilde{x}$, and I is the $(n+1)$ dimensional identity matrix. So, to determine stability of $\widetilde{x}$, it is sufficient to determine the eigenvalues of $D\widetilde{B}(\widetilde{x})$. If the real parts of all the eigenvalues of $D\widetilde{B}(\widetilde{x})$ are less than one, then the rest point $\widetilde{x}$ will be locally stable. If, on the other hand, even one eigenvalue of $D\widetilde{B}(\widetilde{x})$ has real part greater than one, then $\widetilde{x}$ will be unstable.

While determining eigenvalues, we need to bear in mind that any change in population state must leave the total population mass unchanged. Hence, from a given state x, the only possible directions in which the population can move are those that are in the tangent space.[22] So the stability of an equilibrium $\widetilde{x}$ is determined by the n eigenvalues of $DV(\widetilde{x})$ or $D\widetilde{B}(\widetilde{x})$ that refer to the tangent space.

To summarize the above discussion, the operators $DV(\widetilde{x})$ and $D\widetilde{B}(\widetilde{x})$ are defined from $T\Delta$ to $T\Delta$ and hence have n eigenvalues. If even one eigenvalue of $DV(\widetilde{x})$ has a positive real part, or equivalently, if any eigenvalue of $D\widetilde{B}(\widetilde{x})$ has real part greater than 1, then the equilibrium $\widetilde{x}$ is unstable.

A result by Hopkins (1999) simplifies the task determining the eigenvalues of $D\widetilde{B}(x)$ considerably. Hopkins' result shows that the matrix $D\widetilde{B}(x)$ may be written as the product of two matrices Q and $D\pi(x)$.

Before stating the result, we define the notion of a positive definite game.

Definition 5.1 *A population game with payoff function $\pi : \Delta_1^n \to \mathbf{R}^{n+1}$ is positive definite on $T\Delta_1^n$ if*

$$zD\pi(x)\,z > 0, \forall x \in \Delta_1^n, z \in T\Delta^n, z \neq 0 \tag{7}$$

Positive definiteness of a game implies that if a small group of players switch from strategy i to strategy j, then the marginal improvement in the payoff of strategy j resulting from the switch exceeds the improvement in the payoff of i. This property is known as "self-improving externalities". We discuss positive definiteness in greater detail in the appendix.

Similarly, we say a matrix Q is positive definite with respect to $T\Delta_1^n$ if

$$zQz > 0, \forall z \in T\Delta_1^n, z \neq 0.$$

Now, we state the result from Hopkins (1999) in the following lemma.

[22]This is the reason why the domain of $DV(x)$ and $D\widetilde{B}(\widetilde{x})$ is $T\Delta$ and not $\mathbf{R}^{n+1}$.

Lemma 5.2 *(Hopkins, 1999): We may write the operator* $D\widetilde{B}(x) : T\Delta \rightarrow T\Delta$ *as*

$$D\widetilde{B}(x) = \frac{1}{\eta} Q(x) D\pi(x)$$

where $Q(x)$ *is a symmetric matrix positive definite with respect to* $T\Delta(x)$*. Furthermore,* $Q\mathbf{1} = 0$.

For example, in the logit dynamic, $Q(x)$ is a $(n+1)\times(n+1)$ matrix where $Q_{ii} = \widetilde{B}_i(x)(1-\widetilde{B}_i(x))$ and $Q_{ij} = -\widetilde{B}_i(x)\widetilde{B}_j(x)$, $i \in \{0, 1, \cdots, n\}$.

The following lemma then permits us to determine the sign of the eigenvalues of $D\widetilde{B}(x)$ if the game is positive or negative definite at x. This lemma appears in Hofbauer and Sigmund (1988, p. 129) as an exercise. Sandholm (2006a, Lemma A.4) provides a proof. The proof is actually for positive definiteness on $T\Delta$. But it can be readily adapted to positive definiteness on $T\Delta(x)_0$[23]

Lemma 5.3 *(Hofbauer and Sigmund ,1988) Suppose* $Q(x)$ *is a symmetric positive definite matrix with respect to* $T\Delta(x)_0$, $Q\mathbf{1} = 0$, *and* π *is a positive definite game. Then all eigenvalues of* $Q(x) D\pi(x) : T\Delta(x)_0 \rightarrow T\Delta(x)_0$ *have positive real parts. If, on the other hand,* $D\pi(x)$ *is negative definite, then all the eigenvalues have negative real parts.*

Before going to the instability results, we define a regular equilibrium as in Hofbauer and Hopkins (2005) (Van Damme (1987) calls this a quasi-strict equilibria).

Definition 5.4 *Let* x^* *be a partially mixed equilibrium. We say that* x^* *is a regular equilibrium if* $\pi_i(x^*) > \pi_j(x^*)$, *for all* $i \in \text{supp}(x^*)$, $j \notin \text{supp}(x^*)$.

Hence, x^* is a regular equilibrium if the Nash equilibrium payoff is strictly greater than the payoff of any pure strategy not in the support of the Nash equilibrium. It is well known that almost all equilibria in generic simultaneous move games are regular.

Our stability results apply only for a class of perturbed best response dynamics that satisfy a certain technical condition stated in Assumption 5.5 below. The condition relates to the limiting behavior of the Q matrix and is necessary to define the operator $\lim_{\eta\rightarrow 0} Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta)$. This assumption is necessary because the mixed Nash equilibria of our game do not have complete support. While it would be very difficult to prove that this condition holds in general, it is not very stringent and is satisfied by the logit dynamic. Moreover, we conjecture that our results can be proved even without using the assumption. However, the condition greatly simplifies the proofs.

We now formally state the assumptions under which our stability results will be based. With a little abuse of notation, we write $\lim_{\eta\rightarrow 0} Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta)$ as $Q(x^*)D\pi(x^*)$. Hence, for the logit dynamic, $Q_{ii}(x^*) = x_i^*(1 - x_i^*)$ and $Q_{ij}(x^*) = -x_i^* x_j^*$.

Assumption 5.5 *We assume the following.*

[23] $T\Delta(x)_0$ is the subspace of the tangent space in which movement is restricted to the support of x. Formally, $T\Delta^n(x)_0 = \{z \in T\Delta^n : z_i = 0 \text{ if } x_i = 0\}$. Since π is a positive definite game, $D\pi(x)$ is positive definite on $T\Delta(x)_0$.

1. *x^* is a regular equilibrium.*

2. *Let x^* be a partially mixed equilibrium and let $\{\widetilde{x}_\eta\}$ be the sequence of corresponding perturbed equilibria. Then $\lim_{\eta\to 0} Q(\widetilde{x}_\eta) = Q(x^*)$ exists.*

Part 2 of the assumption ensures that the operator $Q(x^*)D\pi(x^*)$ is well defined.

We now consider the equilibria in the price dispersion games with exogenous consumer behavior $\{y_1, y_2\}$, $0 < y_1 < 1$. Since the game has dominated strategies, any mixed equilibrium will have less than complete support. We show that all mixed equilibria in this game are unstable. This result is based on the following lemma about the positive definiteness of the game.

Lemma 5.6 *Consider the price dispersion game 1 with exogenous consumer types $\{y_1, y_2\}, 0 < y_1 < 1$. The resulting finite game is positive definite.*

Proof. In the Appendix. ■

We can now state our result about the stability of dispersed equilibria in this game.

Proposition 5.7 *Let $\pi(x)$ be the price dispersion game with $n+1$ prices. Let $\{y_1, y_2\}, 0 < y_1 < 1$ be the exogenous distribution of consumer types. Let x^* be a mixed equilibrium. Let $\widetilde{x}_\eta$ be the perturbed equilibrium corresponding to x^* with perturbation level η. If the perturbed best response dynamic satisfies part 2 of Assumption 5.5, then there exists $\eta^* > 0$ such that for all $\eta < \eta^*$, the equilibrium $\widetilde{x}_\eta$ is unstable.*

Proof. In the Appendix. ■

The intuition behind this result is as follows. The property of "self-improving externalities" implies that near a mixed equilibrium, if a small group of sellers deviate to another strategy, then this creates an incentive for other sellers to do likewise. The population, therefore, tends to move away from an equilibrium. It is this tendency that leads to the instability of equilibria.

Hence, for each mixed Nash equilibrium, we can find an η small enough such that the corresponding perturbed equilibria is unstable. Since the number of Nash equilibria, and hence perturbed equilibria, is generically finite, we can find an η small enough such that all the perturbed equilibria will be unstable.

We should also note that this proposition is not saying that there exists some η^* such that for all $\eta < \eta^*$, perturbed equilibria will be unstable for all n. Whether this is true or not remains an open question. For the purposes of the above proposition, it is critical that we fix n beforehand and then look at the equilibria of the game corresponding to that particular n.

We revisit the example in Section 4.1. The game in that example is positive definite. Recall that the Nash equilibrium in that game is $x^* = (0, 0.4536, 0.2022, 0.3443)$ and the corresponding perturbed equilibrium is $\widetilde{x} = (\sim 0, 0.4571, 0.2032, 0.3397)$ for $\eta = 0.001$. As an operator on $T\Delta(x^*)_0$, $Q(x^*)\, D\pi(x^*)$ has two eigenvalues, $\{0.00296 \pm 0.0246i\}$. As an operator on $T\Delta(x^*)$,

$Q(x^*) D\pi(x^*)$ has the same two eigenvalues and a zero eigenvalue. For small η, the eigenvalues of $Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta)$ are very close to those of $Q(x^*) D\pi(x^*)$. For $\eta = 0.001$, these eigenvalues are $\{0.00295 \pm 0.0243i, \sim 0\}$. Dividing by η and subtracting 1, we obtain the eigenvalues of $DV(\widetilde{x}_\eta)$ as $\{1.9513 \pm 24.3474i, -1\}$. Since the complex eigenvalues have positive real parts, we conclude that $\widetilde{x}$ is unstable.

5.2 Instability with Endogenous Types

We now consider the general model of price dispersion with endogenous consumer behavior. Since this is a two population game, the evolutionary dynamic must specify motion in both populations. Given $(x, y) \in \Delta = \Delta_1 \times \Delta_2$, we will denote the corresponding vector of change in social state as $V(x, y) \in T\Delta = T\Delta_1 \times T\Delta_2$ where $T\Delta_2$ is the tangent space of population 2. The payoff function for population 1 is given by (1) and of population 2 by the negative of (3). For simplicity, we assume that both populations face the same perturbation factor η. The control cost function v can, however, differ between the two populations. Hence, the perturbed best response dynamics at a population state $(x, y) \in \Delta$ are given by

$$\begin{aligned} V^1(x, y) &= \dot{x} = \widetilde{B}^1(x, y) - x \\ V^2(x, y) &= \dot{y} = \widetilde{B}^2(x, y) - y \end{aligned}$$

where $\widetilde{B}^1(x, y)$ and $\widetilde{B}^2(x, y)$ are the perturbed best response functions of populations 1 and 2 respectively. Clearly, a perturbed equilibrium $(\widetilde{x}, \widetilde{y})$ of the game is a rest point of the dynamic.

The stability of an equilibrium is once again determined by the eigenvalues of $DV(x, y)$ evaluated at the rest point $(\widetilde{x}, \widetilde{y})$. As in the single population case, $DV(\widetilde{x}, \widetilde{y}) = D\widetilde{B}(\widetilde{x}, \widetilde{y}) - I$ where I is now the $(n+1+r)\times(n+1+r)$ identity matrix. Since any change in the social state must leave the mass in both populations unchanged, we need to view both $DV(x, y)$ and $D\widetilde{B}(x, y)$ as operators from $T\Delta$ to $T\Delta$. Hence, the stability properties of $(\widetilde{x}, \widetilde{y})$ is determined by the $n+(r-1)$ eigenvalues that refer to $T\Delta$. The equilibrium $(\widetilde{x}, \widetilde{y})$ is unstable if at least one eigenvalue of $D\widetilde{B}(\widetilde{x}, \widetilde{y})$ is greater than one.

The results of Hopkins (1999) apply to multipopulation game. Hence, in order to determine the eigenvalues of the Jacobian $D\widetilde{B}(x, y)$, we apply Lemma 5.2 and write $D\widetilde{B}(x, y)$ as

$$\begin{aligned} D\widetilde{B}(x, y) &= \frac{1}{\eta}\begin{pmatrix} Q^1(x) & 0 \\ 0 & Q^2(y) \end{pmatrix}\begin{pmatrix} D_x\pi(x, y) & D_y\pi(x, y) \\ -D_xC(x) & -D_yC(x) \end{pmatrix} \\ &= \frac{1}{\eta}Q(x, y)\, D(x, y) \end{aligned}$$

Let us now look at each of the two matrices on the right hand side. The first matrix is a block diagonal matrix with $Q^1(x, y)$ and $Q^2(x, y)$ being both square matrices of dimensions $n+1$ and r respectively, and both being symmetric and positive definite with respect to $T\Delta_1(x)_0$ and $T\Delta_2(x)_0$ respectively.

Two characteristics of the second matrix are of importance in determining the stability properties of perturbed equilibria. The first is that at a mixed Nash equilibrium, consumers sample sample either only one price or two prices. Hence $D_x\pi\left(x, y^*\right)$ is positive definite on $T\Delta_1$ by Lemma 5.6. The second critical fact is that consumers payoffs are independent of the distribution y. Hence, $D_y C\left(x, y\right) = 0$, at all population states (x, y).

We now show that given the strategy size n, a perturbed equilibrium $(\widetilde{x}, \widetilde{y})$ corresponding to a mixed equilibrium will be unstable. Since a mixed equilibrium has less than complete support, we continue to assume that Assumption 5.5 holds. We assume that both parts of the assumption holds separately for Q^1 and Q^2, so that in particular

$$\lim_{\eta\to 0} Q^1(\widetilde{x}_\eta) = Q^1(x^*), \ \ \lim_{\eta\to 0} Q^2(\widetilde{x}_\eta) = Q^2(x^*). \tag{8}$$

Proposition 5.8 *Consider the two population price dispersion game. Let $(\widetilde{x}_\eta, \widetilde{y}_\eta)$ be a perturbed equilibrium of this game corresponding to a regular mixed strategy Nash equilibrium (x^*, y^*) with perturbation level η. If the perturbed best response dynamic satisfies (8), then there exists $\eta^* > 0$ such that for all $\eta < \eta^*$, the equilibrium is unstable.*

Proof. In the Appendix. ■

Since the number of Nash equilibria is generically finite, we can find an η small enough that all perturbed equilibria corresponding to dispersed equilibria are unstable for perturbation levels smaller than that η^*.

Example 5.9 *We illustrate Proposition 5.8 with Example 2.6. We consider the mixed Nash equilibrium (x^*, y^*) where $x^* = (0, 0, 0, 0.4684, 0.4176, 0.1140)$ and $y^* = (0.6680, 0.3319, 0)$. We will demonstrate instability under the logit dynamic. Let $\eta = 0.001$. The corresponding perturbed equilibrium is $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} = (\sim 0, \sim 0, \sim 0, 0.4621, 0.4278,\ 0.1100)$, and $\widetilde{y} = (0.6689, 0.3310, \sim 0)$.*

First, we consider the operator $Q^1(x^*)D_x\pi(x^*, y^*)$ restricted to $T\Delta^1(x^*)_0$. Holding y^* fixed, $\pi(x, y^*)$ is a positive definite game. Hence, $D_x\pi(x^*, y^*)$ is positive definite with respect to $T\Delta_1$. $Q^1(x^*)$ is positive definite on $T\Delta^1(x^*)_0$. By Lemma 5.3, the eigenvalues of $Q^1(x^*)D_x\pi(x^*, y^*) : T\Delta^1(x^*)_0 \to T\Delta^1(x^*)_0$ have positive real parts. Since x^* has three strategies in its support, $Q^1(x^*)D_x\pi(x^*, y^*)$ restricted to $T\Delta^1(x^*)_0$ has two eigenvalues, namely, $0.0116 \pm 0.0392i$. Hence, its trace is 0.0232.

Since $D_y C(x^*) = 0$, the trace of $Q\left(x^*, y^*\right) D\left(x^*, y^*\right) : T\Delta^1(x^*)_0 \times\ T\Delta^2(y^*)_0 \to T\Delta^1(x^*)_0 \times T\Delta^2(y^*)_0$ is also 0.0232. Hence, at least one of its three eigenvalues has positive real part. The three eigenvalues are $\{-0.0054 \pm 0.0684i, 0.0340\}$. As an operator from $T\Delta$ to $T\Delta$, $Q\left(x^*, y^*\right) D\left(x^*, y^*\right)$ has the same three eigenvalues along with five zero eigenvalues corresponding to the five unused strategies at the Nash equilibrium. The three corresponding eigenvalues of $Q(\widetilde{x}_\eta, \widetilde{y}_\eta)D(\widetilde{x}_\eta, \widetilde{y}_\eta)$ are then $\{-0.0060 \pm 0.0672i,\ 0.0349\}$, which also has five other eigenvalues close to zero. Hence, $DB\left(\widetilde{x}, \widetilde{y}\right)$ has eigenvalues with real parts $\{-6.0269, 34.9928\}$. The real parts of the corresponding eigenvalues of $DV\left(\widetilde{x}, \widetilde{y}\right)$ are $\{-7.0269, 33.9928\}$. Hence, $(\widetilde{x}, \widetilde{y})$ is an unstable rest point.

5.3 Discussion

We have established the instability of dispersed price equilibria in our general model. Along the way, we have also shown that if we fix consumers to sample either one or two prices, then all dispersed price equilibria are unstable. Hence, we rule out mixed equilibria as a credible explanation of price dispersion in these models. How then do we explain price dispersion observed in real world markets and in laboratory experiments?

Since all dispersed equilibria are unstable, solution trajectories will either settle down around a pure strategy Nash equilibria, or exhibit some form of long run disequilibrium behavior. In our general model, we do have one pure strategy equilibrium—the monopoly equilibrium. Moreover, it is a strict equilibrium. Hence, the perturbed equilibrium corresponding to the strict equilibrium will be locally stable. However, it will not be globally stable, as can be seen from the simulations in the two-population game in Section 4.

Hence, we need to invoke disequilibrium attractors like limit cycles or chaotic attractors to explain long run price dispersion. As already noted, identifying or showing the existence of such attractors is very difficult; except in one case. This is the case when consumer behavior is specified by an exogenous distribution $\{y_1, 1 - y_1\}$. We address this issue in the next section.

What can we say of the stability properties of equilibria in models in which consumers are allowed to observe more than two prices but their type distribution is still exogenous? Thus, the exogenous distribution is $\{y_1, y_2, \cdots, y_m\}$, $m > 2, 0 < y_1 < 1$. From Lemma 2.2, we know that for n large enough, all equilibria will be mixed. Now, it can be shown that if y_2 is large enough, then all mixed equilibria are unstable. A high y_2 implies that at a mixed Nash equilibrium x^*, the game is positive definite on the subspace of the tangent space where the movement of the population is confined to the support of the Nash equilibrium.[24]It turns out that this restricted notion of positive definiteness is sufficient to make the perturbed equilibria corresponding to x^* unstable for small perturbations. The proof involves the use of Lemma 5.3 and the argument is much the same as in the proof of Lemma 5.6. We do not formally present the proof since the method involved in showing positive definiteness is extremely tedious and does not add substantially to our main result on the instability of dispersed equilibria in the general model with endogenous consumer behavior.

The intuition behind the result is not very involved. Let us consider the meaning of positive definiteness and imagine y_1 is very high. Then, consumers possess very little information about the market. Consequently, if some seller reduces price, then other sellers do not face the incentive to do likewise. On the other hand, if y_i, where i is some large number, is high, then there would be too much competition in the market. Hence, sellers would not match an increase in price by some other seller with a similar increase. A high y_2 implies an appropriate mixture of information and ignorance on the part of consumers to ensure positive definiteness.

[24]Formally, the game is positive definite on $T\Delta^n(x^*)_0 = \{z \in T\Delta^n : z_i = 0 \text{ if } x_i^* = 0\}$

6 Cycling in the Exogenous Game with Two Consumer Types

In this section, we examine in greater detail the case of exogenous consumer type distribution $\{y_1, y_2\}$, $0 < y_1 < 1$. The objective is to analytically prove the existence of disequilibrium attractors. We show that if the game satisfies a further condition called quasi-monocyclicity, then there exists a globally attracting limit cycle under the *best response dynamic*. We can then show that for small η, a perturbed best response dynamic will also have an attractor near the limit cycle under the best response dynamic. The attractor under the perturbed best response dynamic can be a limit cycle or a strange attractor.

Let us consider the game with strategy space S^n. The strategy size n is assumed to be sufficiently large that there is no pure strategy Nash equilibrium. For the purpose of this section, it will be helpful to express the payoff function (1) in the following equivalent form. Given the population state p, the payoff to price x_i is

$$\widehat{\pi}_i(x) = \pi_i(x) - \sum_{j=1}^{n} p_j x_j = p_i(y_1 + 2y_2(\frac{x_i}{2} + \sum_{j>i} x_j)) - \sum_{j=1}^{n} p_j x_j \tag{9}$$

where $\pi_i(x)$ is given by (1) with $m = 2$. The best response dynamic (Gilboa and Matsui (1991)) takes the form

$$\dot{x} \in BR(x) - x \tag{10}$$

where $BR(x)$ is the best response to population state x. The best response dynamic is actually a differential inclusion. Hofbauer (1995) studies these dynamics and proves that at least one solution from each initial point is guaranteed. Any rest point of the best response dynamic is a Nash equilibrium.

It has been shown in Benaim, Hofbauer, and Hopkins (2005) (Proposition 2) that all mixed Nash equilibria of a positive definite game are unstable under the best response dynamic. We now consider the existence of a limit cycle. Formally, a limit cycle is a locally attracting closed or invariant solution trajectory without a rest point.

Given the finite game with n prices, we define the function $W : \Delta^n \to \mathbf{R}$ by

$$W(x) = \max_{x_i \in S^n} \widehat{\pi}_i(x) \tag{11}$$

as a Lyapunov function. Gaunersdorfer and Hofbauer (1995) use this function to identify the limit cycle of the bad Rock-Paper-Scissor game, a positive definite game. Here, we show that if 9 satisfies quasi-monotonicity, then the game contains a unique almost globally attracting limit cycle with characteristic $W(x) = 0$ for any x in the limit cycle.

First, we define *monocyclic games*, a concept introduced in Hofbauer (1995). A two-player symmetric normal form game A with N strategies is called monocyclic if

1. $a_{ii} = 0$

2. $a_{ij} > 0$ for $i \equiv j + 1 \pmod N$ and $a_{ij} < 0$ otherwise.

To see the relevance of monocyclicity to our game, we note that the game with payoff function (9) has an equivalent normal form representation

$$\begin{array}{c} p_i y_1 - p_j, \text{ if } i > j; \\ p_i(y_1 + y_2) - p_j = 0, \text{ if } i = j \\ p_i(y_1 + 2y_2) - p_j, \text{ if } i < j \end{array} \tag{12}$$

The normal form representation may be described as follows. Let us suppose that two sellers are randomly matched to play the pricing game. Let us denote the seller charging p_i as i. Then, if $p_i > p_j$, seller i gets only those consumers who sample just once. If $p_i = p_j$, then both sellers share equally the consumers who sample twice. Finally, if i's price is lower, then he gets all the consumers sampling twice. The subtraction by p_j is a normalization device to ensure that the diagonal elements of the normal form are zeros.

Since our model has dominated strategies, the monocyclicity condition will not be satisfied for the entire game. Let $S^{ud} \subset S^n$ be the set of strategies that are undominated by strategy 1. We now define a restricted notion of monocyclicity we call *quasi-monocyclicity.*

Definition 6.1 *We call the game defined by strategy set S^n and payoff function (9) quasi-monocyclic if its equivalent normal form (12) satisfies the monocyclicity condition on the strategy set S^{ud}.*

The intuition behind the quasi-monocyclicity condition is as follows. Let $p_j \in S^{ud}$. Suppose the population state is e_j, that is, the entire population is playing strategy p_j. The quasi-monotonicity condition requires that the payoff to strategy p_j should be less than the strategy immediately preceding it, but be more than the payoffs of all the other strategies in S^{ud} . Here precedence is in the modular sense. The strategy immediately preceding the lowest undominated strategy is 1. It is, however, not easy to provide a condition that ensures quasi-monotonicity in this game.

Example 6.2 *The game with $S = \{0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$, $y_1 = 0.45$ and $y_2 = 0.55$ is a quasi-monocyclic game. Prices 0 and $\frac{1}{5}$ are dominated by 1. The normal form equivalent of the game satisfies the monocyclicity conditions on $S^{ud} = \{\frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1\}$. This game has three Nash equilibria: (0, 0, 0.6364, 0.0909, 0.2727, 0); (0, 0, 0, 3388, 0.4876, 0, 0.1736); (0, 0, 0, 0.5105, 0.2587, 0.1259, 0.1049).*

The game in Section 4.1 is also a quasi-monocyclic game. Monocyclicity is satisfied on $S^{ud} = \{\frac{10}{12}, \frac{11}{12}, 1\}$.

To prove the existence of a limit cycle in a quasi-monocyclic game. we need the following lemma.

Lemma 6.3 *Consider the finite dimensional game with strategy space S^n with consumer types $\{y_1, y_2\}$, $0 < y_1 < 1$. Let x^* be a Nash equilibrium of the game. Then $W(x^*) < 0$. Let δ_i be the pure strategy that puts probability 1 on the pure strategy x_i. Then, for n sufficiently large, $W(\delta_i) > 0$.*

Proof. In the Appendix. ■

We now assume that n is sufficiently large so that Lemma 6.3 is satisfied. By the convexity of Δ^{ud} and the continuity of $W(x)$ we can be assured that there exists a set $W^0 = \left\{x \in \Delta^{ud} : W(x) = 0\right\}$. Moreover, by Lemma 6.3, W^0 is disjoint from the set of Nash equilibria.

The following proposition establishes the existence of an almost globally attracting limit cycle in our model under the best response dynamic. The proof relies on a result in Benaim, Hofbauer and Hopkins (2005).

Proposition 6.4 *Consider the finite price dispersion game with consumer types given exogenously by the distribution* $\{y_1, y_2\}$, $0 < y_1 < 1$. *Suppose the game satisfies the condition of quasi-monotonicity. Then* Δ^n *contains a closed orbit under the best response dynamic. Furthermore, from a dense, open and full measure set of initial conditions, the best response dynamics converge to this closed orbit. Moreover, for any state* x *in the limit cycle,* $W(x) = 0$.

Proof. The set $\Delta^{ud} = \{x \in \Delta^n : x_i = 0 \text{ if } p_i \notin S^{ud}\}$ is invariant under the best response dynamic. We now consider an initial point $x(0) \in \Delta^{ud}$. Proposition 1 in Benaim, Hofbauer and Hopkins (2005) then implies the existence of a limit cycle in Δ^{ud} that attracts trajectories from a dense, open and full measure set of initial conditions in Δ^{ud}. Moreover, since $W^0 \in \Delta^{ud}$, the same proposition in Benaim, Hofbauer and Hopkins (2005) implies $W(x) = 0$, for any x in the limit cycle. To complete the argument, we note that if $x(0) \notin \Delta^{ud}$, then solution trajectories will converge to Δ^{ud}. ■

In terms of the original payoff function π, $W^0 = \{x \in \Delta^n : \max_{x_i \in S^n} \pi_i(x) = \sum_{j=1}^{n} p_j x_j\}$. In order to understand W^0, we invoke the intuition provided by Gaunersdorfer and Hofbauer (1995) in explaining the emergence of a limit cycle in the bad Rock-Paper-Scissors game. We note that in terms of the original payoff function, $\pi_j(e_j) = p_j$. At any Nash equilibrium x^*, we have, by Lemma (6.3)

$$\pi(x^*) < \sum_{j \in S^n} x_j^* \pi_j(e_j)$$

This condition means that at the Nash equilibrium, the population will benefit from splitting itself into a number of different subpopulations, this number being equal to the number of strategies in the support of the equilibrium. This causes the population to move away from the equilibrium towards W^0.

Note that we are not claiming that solutions converge to the limit cycle from all initial conditions. In fact this will not be true since the mixed Nash equilibria will not have full support. Hence they will be saddle points and there will be some initial conditions from which solution trajectories will converge to some Nash equilibrium.

For the simple four-price example we examined in Section 4.1, we can set $W(x) = 0$ and identify the limit cycle as the triangle with vertices $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $(0, \frac{2}{3}, \frac{1}{6}, \frac{1}{6})$, and $(0, \frac{4}{9}, \frac{1}{9}, \frac{1}{9})$. For this case, the limit cycle coincides with the set W^0.

What can we now say about any possible limit cycle under perturbed best response dynamics? Intuitively, since the perturbed best response converges pointwise to the best response, it is plausible to believe that if the best response dynamic contains a limit cycle, then for low levels of perturbation, perturbed best response dynamics will also have an attractor near the attractor of the best response dynamic. This intuition can potentially breakdown at points with multiple best responses since the best response dynamic will be multivalued at such points. However, the quasi-monocyclicity property and the invariance of Δ^{ud} implies that if the initial point $x(0) \in \Delta^{ud}$, the best response dynamic will spend a measure zero amount of time at such states. Hence, we can ignore such points and argue that the limit of solution trajectories under perturbed best response dynamics as the level of perturbation goes to zero will be the trajectory under the best response dynamic. We can then apply a well know theorem about attractors of dynamical systems (Theorem A.1 in Hofbauer and Sandholm (2006b))and conclude that a perturbed best response dynamic will have an attractor near the limit cycle of the best response dynamic.

Details of the argument are in the appendix. Here, we provide a statement of the proposition. Given a class of control cost function $v(x)$ and level of perturbation η, let Φ_η be the semi-flow of the corresponding perturbed best response dynamic. Thus, $x(t) = \Phi^t_\eta(x(0))$ is the solution at time t starting from initial point $x(0)$. Let Φ be the semi-flow corresponding to the best response dynamic. Hence, a limit cycle SP is an attractor for Φ with basin of attraction $B(SP)$ that includes almost all of Δ^n. In the appendix, we show that Φ^t is the limit of Φ^t_η as $\eta \to 0$ for every time period $t > 0$. We can then state the following proposition.

Proposition 6.5 *For each level of perturbation $\eta > 0$, there exists an attractor SP_η of Φ_η with basin $B(SP_\eta)$such that the map $\eta \mapsto SP_\eta$ is upper hemicontinuous and the map $\eta \mapsto B(SP_\eta)$ is lower hemicontinuous.*

Proof. In the Appendix. ■

This proposition is sufficient for us to conclude that for η sufficiently low, there will be an attractor SP_η of the corresponding perturbed best response dynamic near the set SP. However, it is difficult to actually identify the attractor SP_η analytically. In fact, it is not even clear that the attractor in question will be a limit cycle. It can even be a chaotic attractor.

We can summarize the above discussion in the following proposition.

Proposition 6.6 *Consider the finite price dispersion game with consumer types given exogenously by the distribution $\{y_1, y_2\}$, $0 < y_1 < 1$. Suppose the game satisfies the condition of quasi-monotonicity. Let SP be the globally attracting closed orbit under the best response dynamic. Then, for η sufficiently small, a perturbed best response dynamic will contain a global attractor near SP.*

Proof. The proof follows from Propositions 6.4 and 6.5. ■

We ran numerical simulations for the game with 6 prices in Example 6.2 under the logit dynamic with $\eta = 0.001$. Simulations suggest the presence of a unique limit cycle that is globally attracting. In Figure 5 we plot the trajectory converging to the limit cycle from the initial point

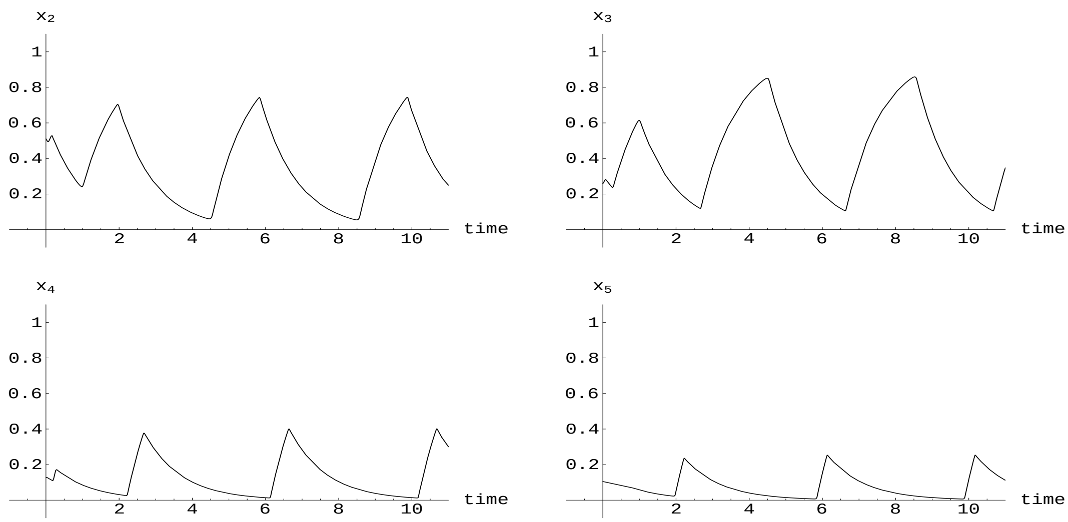

Figure 5: Solution trajectories under the logit (0.001) dynamic in the game with 6 prices, $y_1 = 0.45$. The initial point is a Nash equilibrium.

$(0, 0, 0.5105, 0.2587, 0.1259, 0.1049)$ which is a Nash equilibrium. The corresponding perturbed equilibrium is $(\sim 0, \sim 0, 0.5150, 0.2548, 0.1301, 0.1001)$ We plot only the support of the limit cycle, (x_2, x_3, x_4, x_5). The time average of the limit cycle is $(\sim 0, \sim 0, 0.3292, 0.4736, 0.1256, 0.0716)$ which is very different from even the equilibrium that has the same support.

We also plot the Edgeworth cycle of $\overline{p}(t)$ in the figure 6. Given our conjecture based on the numerical simulation that the limit cycle of $x(t)$ is unique, $\overline{p}(t)$ will also have a unique limit cycle.

7 Conclusion

In this paper, we have analyzed the question of price dispersion from an evolutionary standpoint. In order to avoid technical complications, we have constructed a finite dimensional model of price dispersion based on the original Burdett and Judd (1983) model. We have focused our attention on the mixed equilibria of the model and have analyzed their stability properties under perturbed best response dynamics. Building on the theoretical work of Hopkins (1999), we have found that mixed equilibria in our model are unstable under these dynamics. Intuitively, instability arises due to positive definiteness of the game around a mixed Nash equilibrium.

Given these instability results, we view observed price dispersion as a long run disequilibrium phenomenon. We show through numerical simulation that the disequilibrium phenomenon can be expected to take the form of limit cycles that attract solution trajectories of the perturbed best response dynamics. In these cycles, both the proportion of firms charging a particular price and the average market price keeps fluctuating in a regular manner. For a simple case, we have established the existence of a long run disequilibrium state. In general, it is very difficult to characterize or prove the existence of disequilibrium attractors of a dynamic. Sunch attractors may be limit

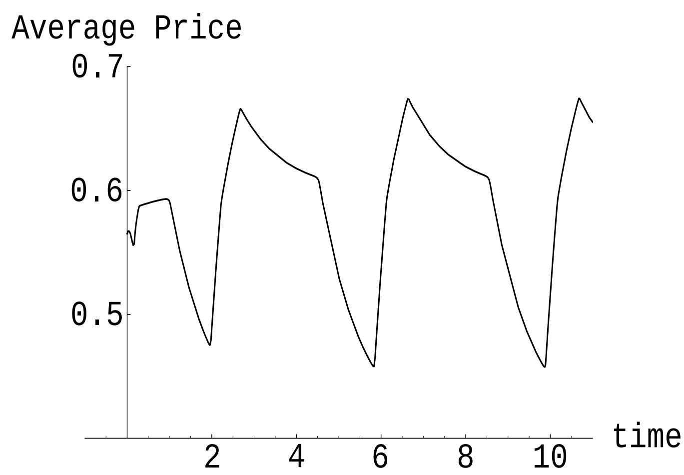

Figure 6: Trajectory of $\overline{x}(t)$ for the 6-price game in Example 6.2.

cycles or chaotic. The detailed investigation of such attractors can be a potentially rich avenue for research.

This paper illustrates the general principle that certain economic phenomenon may not be explained by invoking traditional equilibrium concepts, particularly when empirical and experimental evidence runs contrary to the equilibrium prediction of economic theory. Perpetual disequilibrium is captured naturally by evolutionary game theory. By exploiting this aspect of the evolutionary approach, we believe that this paper has made a major methodological contribution that should lead to further work on the application of evolutionary game theory in economics.

In a companion paper (Lahkar (2007)), we extend the analysis to the original continuous strategy game of Burdett and Judd. We have shown that the infinite dimensional logit dynamic is well defined in the Burdett and Judd model. Establishing stability results in the infinite dimensional context, however, remains a challenge. The interest in this question is not merely technical, but also practical since most economic situations of interest are naturally modeled as having continuous strategy spaces.

Among other research questions in this area, we can try to generalize the results established here to other evolutionary dynamics and other price dispersion models. Finally, one can use evolutionary game theory to try and investigate other possible economic issues. For example, perturbed best response can be a attractive way to study markets which are subject to rapid change, either due to changes in technology or in consumer tastes. In such situations, firms may not have exact knowledge of demand and supply conditions, and so would be prone to making mistakes in recognizing payoffs or in implementing best response. Perturbed best response can take into account such possibilities.

Appendix

A Nash equilibria

Proof. **Proposition 2.2**

Suppose there exists some price such that $\overline{x}_i^n > \varepsilon > 0$. By making n sufficiently large, the price that is immediately lower than p_i can be brought arbitrarily close to p_i. Let us denote this price by p_{i-}. The payoff from p_{i-} is at least

$$p_{i-}\left[y_1 + \sum_{m=2}^{r} m y_m (\sum_{j\geq i} \overline{x}_j^n)^{m-1}\right]$$

On the other hand, the payoff from p_i is given by (1). Let us denote $\sum_{j>i} \overline{x}_j^n$ by G. By applying the binomial formula, it can be established than,

$$\begin{aligned}&\pi_{i-}(x) - \pi_i(x) \geq y_1 (p_{i-} - p_i) \\ &+ \sum_{m=2}^{r}\left[\sum_{k=0}^{m-1} \binom{m-1}{k} (\overline{x}_i^n)^k G^{m-1-k}\left(p_{i-} - \frac{p_i}{k+1}\right)\right]\end{aligned} \tag{13}$$

As $n \to \infty$, $y_1 (p_{i-} - p_i) \nearrow 0$. Now, for all prices and for all n, $p_{i-} > \frac{p_i}{3}$ except when $i = 1$. Also, $p_{i-} > \frac{p_i}{2}$ for all prices except the zero price and the first two positive prices. However, the first two positive prices are dominated for all n sufficiently large and hence, can be ignored. Hence, for all m, the part of the above expression inside the square bracket will be positive if

$$\left({}^{m-1}C_1\right)(\overline{x}_i^n)\, G^{m-2}\left(p_{i-} - \frac{p_i}{2}\right) \geq G^{m-1}(p_{i-} - p_i)$$

Since $G \leq 1$, $\overline{x}_i^n > \varepsilon$, and $m \geq 2$, for the above expression to hold, it is sufficient that

$$\left(p_{i-} - \frac{p_i}{2}\right)\varepsilon \geq (p_{i-} - p_i) = \frac{1}{n} \tag{14}$$

Now, let $\underline{p}$ be the lowest price in the support of the corresponding continuous game. This price is greater than zero since $0 < y_1 < 1$. Hence, $\lim_{n\to\infty}\left(p_{i-} - \frac{p_i}{2}\right)\varepsilon \geq \left(\underline{x} - \frac{x}{2}\right)\varepsilon > c > 0$, for some c. So, for n sufficiently large (14) holds. Hence, as $n \to \infty$, the expression inside the square bracket in (13) remains bounded away from zero whereas $y_1 (p_{i-} - p_i)$, while being negative, goes to zero. So, for n sufficiently large, $\pi_{i-}(x) - \pi_i(x)$ will be positive. Hence, $\overline{x}_i^n$ cannot be a Nash equilibrium. ■

Proof. **Lemma 2.3**

1. This is obvious. The payoff to any strategy p_i is $p_i y_1$. Hence, the highest price dominates all other prices.

2. If $x_0^n = 1$, then $\pi_i^n = 0$ for all prices. If $x_1^n = 1$, then $\pi_1^n = p_1^n$. But since $y_1 = 0$, $\pi_i^n = 0$, for all other prices p_i^n. Hence, these two pure strategy Nash equilibria always exist. For any

price p_i^n greater than p_1^n, if $x_i^n = 1$, then

$$\pi_i^n = \sum_{m=2}^{r} p_i^n y_m = p_i^n, \text{ whereas}$$
$$\pi_{i-}^n = p_{i-}^n \sum_{m=2}^{r} m y_m = \sum_{m=2}^{r} m p_{i-}^n y_m$$

where p_{i-}^n is the price immediately lower than p_i^n. If $p_i^n > p_1^n$, then $m p_{i-}^n \geq p_i^n$ with the equality holding only for $m = 2$ and $p_i^n = p_2^n$. Hence, if $r \geq 3$, then $\pi_{i-}^n > \pi_i^n$ for all $i > 1$ and so, there can be no other pure equilibria. Now, consider the special case where $r = 2$. Hence, $y_2 = 1$. Then, if $x_2^n = 1$, $\pi_2^n = p_2^n$, $\pi_1^n = 2p_1^n$ and $\pi_i^n = 0$, for all other i. Since $p_2^n = 2p_1^n$, $x_2^n = 1$ is a Nash equilibrium for the case $r = 2$.

Next, we rule out the possibility of any mixed equilibria. Suppose p_H^n is the highest price in the support of a mixed equilibrium $\overline{x}^n$. The payoff to p_H^n is

$$\pi_H^n = p_H^n \left(\sum_{m=2}^{r} m y_m \frac{(\overline{x}_H^n)^{m-1}}{m} \right) = \sum_{m=2}^{r} y_m p_H^n (\overline{x}_H^n)^{m-1}$$

Let p_{H-}^n be the price that is immediately lower than p_H^n. Since price 0 cannot be a part of a mixed strategy, $p_{H-}^n \geq \frac{1}{n}$. The payoff to p_{H-}^n is

$$\begin{aligned}\pi_{H-}^n &= p_{H-}^n \left[\sum_{m=2}^{r} m y_m \left\{ \sum_{k=0}^{m-1} \binom{m-1}{k} \left(\overline{x}_{H-}^n\right)^k (\overline{x}_H^n)^{m-1-k} \frac{1}{k+1} \right\} \right] \\ &\geq p_{H-}^n \left[\sum_{m=2}^{r} m y_m (\overline{x}_H^n)^{m-1} \right] = \sum_{m=2}^{r} m y_m p_{H-}^n (\overline{x}_H^n)^{m-1}\end{aligned}$$

$m p_{H-}^n > p_H^n$ except for the case where $m = 2$ and $p_H^n = p_2^n$. Hence, if $r \geq 3$, $\pi_{H-}^n > \pi_H^n$ which shows that $\overline{x}^n$ cannot be a Nash equilibrium. If $r = 2$ and $p_H^n > p_2^n$, then too $\pi_{H-}^n > \pi_H^n$. The only case that remains is where $r = 2$ and $p_H^n = p_2^n$. Hence, $y_2 = 1$,

$$\pi_1^n = p_1^n \left(2 \left(\frac{\overline{x}_1^n}{2} + \overline{x}_2^n \right) \right), \ \pi_2^n = p_2^n \left(2 \left(\frac{\overline{x}_2^n}{2} \right) \right)$$

Since $p_2^n = 2p_1^n$, these payoffs can only be equal if $\overline{x}_2^n = 1$. Hence, this special case reduces to the case of the pure strategy equilibrium $x_2^n = 1$ when $y_2 = 1$. ■

B Mixed Equilibria of Example 2.6.

The mixed equilibria of the game are listed below. The first set of numbers refer to seller behavior while the second set refers to consumer behavior.

$$\begin{array}{ll}
x^1 = (0, 0, 0, 0.4684, 0.4176, 0.1140) & y^1 = (0.6680, 0.3319, 0) \\
x^2 = (0, 0, 0.8084, 0, 0.1496, 0.0416) & y^2 = (0.4201, 0.5799, 0) \\
x^3 = (0, 0, 0.2673, 0.6485, 0, 0.084) & y^3 = (0.5037, 0.4963, 0) \\
x^4 = (0, 0.42202, 0.5413, 0, 0, 0.0367) & y^4 = (0.2585, 0.7415, 0) \\
x^5 = (0, 0.8035, 0, 0.179, 0, 0.0174) & y^5 = (0.2171, 0.7829, 0) \\
x^6 = (0, 0.7738, 0, 0.2262, 0, 0) & y^6 = (0.215, 0.785, 0) \\
x^7 = (0, 0, 0.7738, 0, 0.2262, 0) & y^7 = (0.4363, 0.5637, 0) \\
x^8 = (0, 0, 0.8651, 0, 0, 0.1349) & y^8 = (0.3472, 0.6528, 0) \\
x^9 = (0, 0, 0, 0.7739, 0, 0.2261) & y^9 = (0.5622, 0.4398, 0)
\end{array}$$

C Positive Definite Games

We now discuss the notion of a positive definite game which has been crucial to us in determining the stability properties of mixed equilibria. For our purpose, it is enough to define the concept for a one population game. Let us have a game with state space Δ^n, tangent space $T\Delta^n$ and payoff function $\pi : \Delta^n \to \mathbf{R}^{n+1}$ is positive definite at $x \in \Delta^n$ if

$$zD\pi(x)z > 0, \text{ for all } z \in T\Delta^n, z \neq 0. \tag{15}$$

If (15) is satisfied for all $x \in \Delta^n$, we say that the game is *positive definite*.

Example C.1 *The canonical example of a positive definite game is a symmetric two player coordination game with positive diagonal elements and zero non-diagonal elements. Let us consider the following three strategy coordination game with strategy set* $S = \{1, 2, 3\}$.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

The tangent space for this game is $T\Delta = \{z \in \mathbf{R}^3 : \sum_{i=0}^{3} z_i = 0\}$. *Given the population state* $x, \pi_i(x) = ix_i$. $D\pi(x) = C$. *Hence,* $zD\pi(x)z = \sum_{i=1}^{3} iz_i^2 > 0$ *if* $z \neq 0$. *Thus, this game is positive definite.*

One way to interpret the positive definiteness condition is through the notion of "self-improving externalities" which is analogous to the notion of "self-defeating externalities" introduced in Hofbauer and Sandholm (2006a) in connection with negative definite games, or "stable" games. Condition (15) is equivalent to the condition $\sum_{i\in S^n} z_i(D\pi_i(x)z) > 0$ where $D\pi_i(x)z$ is the directional derivative of $\pi_i(x)$ in the direction z. The vector z describes the change in population state when a small group of agents revise strategies at state x. $D\pi_i(x)z$ then represents the marginal impact of this strategy revision on the payoffs of those agents currently playing i. If we weigh these payoff changes with the changes in the population weight of each strategy, then condition (15) says that the aggregate effect should be positive. Intuitively, self improving externalities mean that if a small group of players are switching from strategy i to strategy j, then the marginal improvement of the payoff of strategy j resulting from the switch exceeds the improvement of the payoff of i.

We have also used a restricted notion of positive definiteness in which the game is positive definite at some p only with respect to some subspace of $T\Delta^n$. Let us denote the support of p by supp(p). Let $T\Delta^n(p)_0$ be a subspace of the tangent space defined as

$$T\Delta^n(p)_0 = \{z \in T\Delta^n : z_i = 0 \text{ if } i \notin \text{supp}(p)\} \tag{16}$$

Then, we say that the game is positive definite with respect to $T\Delta^n(p)_0$ at p if $zD\pi\,(p)\,z > 0$ for all $z \in T\Delta^n(p)_0$. Note that if p has full support, then $T\Delta^n(p)_0 = T\Delta^n$

D Positive Definiteness in the Finite Game

Proof. **Proposition 5.6**,

The payoff to price p_i is

$$\pi_i\,(x) = p_i\left[y_1 + 2y_2\left\{\frac{x_i}{2}_i + \sum_{j>i} x_j\right\}\right]$$

Hence, for $z \neq 0$,

$$D\pi_i\,(x)\,z = p_i\left[2y_2\left\{\frac{z_i}{2}_i + \sum_{j>i} z_j\right\}\right]$$

Hence,

$$zD\pi\,(x)\,z = -2p_0y_2Z_0^2 + p_0y_2z_0^2 - 2y_2\sum_{i=1}^{n} p_iZ_i(Z_i - Z_{i-1}) + y_2\sum_{i=1}^{n} p_iz_i^2$$

because $z_0 = Z_0$ and $z_i = (Z_i - Z_{i-1})$ for all $i > 0$.

Now

$$Z_i(Z_i - Z_{i-1}) = \frac{1}{2}(Z_i^2 - Z_{i-1}^2) + \frac{z_i^2}{2}$$

and $Z_0^2 = \frac{1}{2}Z_0^2 + \frac{1}{2}z_0^2$. So, we can rewrite $zD\pi(x)$ as

$$zD\pi(x)z = -2p_0y_2(\frac{1}{2}Z_0^2 + \frac{1}{2}z_0^2) + p_0y_2z_0^2$$
$$-2y_2\sum_{i=1}^{n}p_i(\frac{1}{2}(Z_i^2 - Z_{i-1}^2) + \frac{z_i^2}{2}) + y_2\sum_{i=1}^{n}p_iz_i^2$$
$$= -p_0y_2Z_0^2 - y_2\sum_{i=1}^{n}p_i(Z_i^2 - Z_{i-1}^2) = -y_2\sum_{i=0}^{n}Z_i^2(p_i - p_{i+1})$$

where we make use of the fact that $Z_n^2 = 0$. Since $p_i - p_{i+1} < 0$ and $Z_i^2 > 0$, we conclude $zD\pi(x)z > 0$.

E Instability of Dispersed Price Equilibria

Proof. **Proposition 5.7** We first consider the operator $Q(x^*)D\pi(x^*)$. By assumption, x^* is a regular equilibrium. Hence, we can regard $Q(x^*)D\pi(x^*)$ as an operator from $T\Delta(x^*)_0$ to $T\Delta(x^*)_0$. Since π is a positive definite game, $D\pi(x^*)$ is positive definite on $T\Delta(x^*)_0$. As an operator on $T\Delta(x^*)_0$, $Q(x^*)$ is positive definite. Let the cardinality of supp(x^*) be k. Hence, by Lemma 5.3, all the $(k-1)$ eigenvalues of $Q(x^*)D\pi(x^*) : T\Delta(x^*)_0 \to T\Delta(x^*)_0$ have positive real parts. Now, we consider the eigenvalues of $Q(x^*)D\pi(x^*) : T\Delta \to T\Delta$. If λ_1 is an eigenvalue of $Q(x^*)D\pi(x^*) : T\Delta(x^*)_0 \to T\Delta(x^*)_0$, then it is also an eigenvalue of $Q(x^*)D\pi(x^*) : T\Delta \to T\Delta$. Hence, at least one eigenvalue of $Q(x^*)D\pi(x^*) : T\Delta \to T\Delta$ has a positive real part. Let this eigenvalue be $\overline{\lambda}$ with real part $\overline{\lambda}^R > 0$.

Now, we consider $Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta) : T\Delta \to T\Delta$. Part 2 of Assumption 5.5 implies that for small η, the eigenvalues of $Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta)$ are close to the eigenvalues of $Q(x^*)D\pi(x^*)$. Hence, $Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta)$ has an eigenvalue $\widetilde{\lambda}_\eta$ such that $\lim_{\eta\to 0}\widetilde{\lambda}_\eta = \overline{\lambda}$. Denoting the real part of $\widetilde{\lambda}_\eta$ by $\widetilde{\lambda}_\eta^R$, we conclude that sufficiently small η, $\frac{\widetilde{\lambda}_\eta^R}{\eta} > 1$. But $\frac{\widetilde{\lambda}_\eta^R}{\eta}$ is the real part of an eigenvalue of $\frac{1}{\eta}Q(\widetilde{x}_\eta)D\pi(\widetilde{x}_\eta)$. This completes the proof. ■

Proof. **Proposition 5.8** We first consider the operator $Q(x^*, y^*)D(x^*, y^*)$ and show that it has at least one positive eigenvalue.

By our discussion preceding the statement of this proposition, $D_x\pi(x, y^*)$ is equal to the Jacobian of the payoff function of the one population game with an exogenous consumer type distribution being $\{y_1^*, y_2^*\}$. Hence, $D_x\pi(x, y^*)$ is positive definite with respect to $T\Delta_1$. By assumption, (x^*, y^*) is a regular equilibrium. Hence, we can regard $Q^1(x^*)D_x\pi(x^*, y^*)$ as an operator from $T\Delta^1(x^*)_0$ to $T\Delta^1(x^*)_0$. As an operator on $T\Delta^1(x^*)_0$, $Q^1(x^*)$ is positive definite. Let the cardinality of supp(x^*) be k. Hence, by lemma 5.3, all the $(k-1)$ eigenvalues of $Q^1(x^*)D_x\pi(x^*, y^*)$ will have positive real parts. Hence, the trace of $Q^1(x^*)D_x\pi(x^*, y^*)$ is positive.

Next, we consider $Q(x^*, y^*)D(x^*, y^*) : T\Delta^1(x^*)_0\times T\Delta^2(y^*)_0 \to T\Delta^1(x^*)_0\times T\Delta^2(y^*)_0$. Since $D_yC(x^*, y^*) = 0$, the trace of $Q(x^*, y^*)D(x^*, y^*)$ is equal to the trace of $Q^1(x^*)D_x\pi(x^*, y^*)$, the latter regarded as an operator on $T\Delta^1(x^*)_0$. Hence, the trace of $Q(x^*, y^*)D(x^*, y^*)$ must also be positive. But this means that $Q(x^*, y^*)\, D(x^*, y^*)$ has at least one eigenvalue with positive real part. Let this eigenvalue be $\overline{\lambda}$ with real part $\overline{\lambda}^R > 0$.

If λ_1 is an eigenvalue of $Q(x^*, y^*)\, D(x^*, y^*)$ as an operator on $T\Delta^1(x^*)_0 \times T\Delta^2(y^*)_0$, then it is also an eigenvalue of $Q(x^*, y^*)\, D(x^*, y^*) : T\Delta \to T\Delta$, which therefore has an eigenvalue with positive real part.

$Q(x^*, y^*)\, D(x^*, y^*)$, as an operator on $T\Delta$, therefore has an eigenvalue $\overline{\lambda}$ with real part $\overline{\lambda}^R > 0$. Part 2 of Assumption 5.5 implies that for small η, the eigenvalues of $Q(\widetilde{x}_\eta, \widetilde{y}_\eta)D(\widetilde{x}_\eta, \widetilde{y}_\eta) : T\Delta \to T\Delta$ are close to the eigenvalues of $Q(x^*, y^*)\, D(x^*, y^*)$. Hence, by an argument similar to that in proposition 5.7, we can conclude that $\frac{1}{\eta}Q(\widetilde{x}_\eta, \widetilde{y}_\eta)D(\widetilde{x}_\eta, \widetilde{y}_\eta)$ has an eigenvalue greater than one if η is sufficiently small. This completes the proof. ■

F Cycling

First, we provide the equivalent normal form representations of the games in Example 6.2. Here, the strategies are arranged in descending order, with the payoff of the highest price first. The normal form representation of the 4-strategy game (also the game (6)) is,

$$\begin{pmatrix} 0 & -0.0667 & 0.0167 & 0.85 \\ 0.0542 & 0 & -0.0542 & 0.7791 \\ -0.0417 & 0.0417 & 0 & 0.7083 \\ -1 & -0.9167 & -0.8333 & 0 \end{pmatrix}$$

The normal form of the 6-strategy is

$$\begin{pmatrix} 0 & -0.35 & -0.15 & 0.05 & 0.25 & 0.45 \\ 0.24 & 0 & -0.24 & -0.04 & 0.16 & 0.36 \\ -0.07 & 0.13 & 0 & -0.13 & 0.07 & 0.27 \\ -0.38 & -0.18 & 0.02 & 0 & -0.02 & 0.18 \\ -0.69 & -0.49 & -0.29 & -0.09 & 0 & 0.09 \\ -1 & -0.8 & -0.6 & -0.4 & -0.2 & 0 \end{pmatrix}$$

To prove lemma 6.3, we denote the normal form representation of $\widehat{\pi}(x)$ as $\widehat{A}$. Thus, $\widehat{\pi}(x) = \widehat{A}x$

Proof. **Lemma 6.3**

Consider a mixed equilibrium x^*. Let $\widehat{A}$ be the normal form matrix of the game. By positive definiteness of the game, $(x - x^*)\widehat{A}\,(x - x^*) > 0$, $\forall x \neq x^*$. Let x be such that if $x_i^* = 0$, then $x_i = 0$. Then, $(x - x^*)\widehat{A}\,(x - x^*) = (x - x^*)\widehat{A}\,x > 0$. Take $x = e_j$ for some j in the support of x^*. Since the diagonal elements of $\widehat{A}$ are zero, $e_j\widehat{A}e_j = 0$ which implies $x^*\widehat{A}e_j < 0$. This implies $x^*\widehat{A}x^* < 0$. Hence, $W(x^*) < 0$.

Let x_i be price $\frac{i}{n}$. The payoff from δ_i is 0. On the other hand, the payoff from x_{i-1} given δ_i is $x_{i-1} + x_{i-1}y_2 - x_i$. Since $x_i - x_{i-1} = \frac{1}{n}$, it can easily be shown that if $i > \frac{1}{y2} + 1$, then $\widehat{\pi}_{i-1}(\delta_i) > \widehat{\pi}_i(\delta_i)$. For such prices, $W(\delta_i) > 0$. For prices less than $\frac{1}{y_2} + 1$, we need to make n sufficiently large such that $\frac{\frac{1}{y_2}+1}{n} < \underline{x}$. This ensures that such prices are dominated by 1. Then, $W(\delta_i) = y_1 > 0$. ■

F.1 Proof of Proposition 6.5

To prove proposition 6.5, we first need the following lemma.

Lemma F.1 *Let $x(0) \in \Delta^{ud}$. Given time $t > 0$, $\lim_{\eta \to 0} \Phi^t_\eta(x(0)) = \Phi^t(x(0))$.*

Proof. Let us fix time $t > 0$ and initial point $x(0) \in \Delta^{ud}$. We need to show that for any $\delta > 0$, we can find a perturbation level $\eta > 0$ such that starting from $x(0)$,

$$\|x_\eta(t) - x(t)\| \leq \delta \tag{17}$$

Given $x(0)$, we have

$$\|x_\eta(t) - x(t)\| \leq \int_0^t \left\| \widetilde{B}(x_\eta(s))) - B(x(s))) \right\| ds + \int_0^t \|x_\eta(s) - x(s)\| \, ds$$

where $x_\eta(t)$ and $x(t)$ are the solution trajectories under the perturbed best response dynamic and the best response dynamic respectively.

By the discussion preceding the statement of Proposition 6.5, $B(x(s))$ is single-valued for almost all $s \in [0, t]$. At all such points where $B(x(s))$ is single-valued, $\widetilde{B}(x(s))$ converges pointwise to $B(x(s))$. This, and the fact that both integrands on the right hand side are bounded [25] implies that both the integrals on the right hand side can be taken arbitrarily close to zero. Hence, given the initial point $x(0)$, we can find a perturbation level η_0 such that (17) is satisfied. By the compactness of Δ^{ud}, we can find η such that (17) is satisfied for all initial points. ■

This lemma along with Theorem A.1 in Hofbauer and Sandholm (2006b) which we present below implies Proposition 6.5.

Theorem F.2 *(Hofbauer and Sandholm, 2006b, Theorem A.1) Let A be an attractor for Φ with basin $B(A)$. Then for each small enough $\eta > 0$ there exists an attractor A_η of Φ_η with basin $B(A_\eta)$, such that the map $\eta \mapsto A_\eta$ is upper hemicontinuous and the map $\eta \mapsto B(A_\eta)$ is lower hemicontinuous.*

[25] $\left\| \widetilde{B}(p_\eta(s)) - B(p(s)) \right\| \leq 2$ and $\|p_\eta(s) - p(s)\| \leq 2$

Lemma F.1 and Theorem F.2 imply that for any solution trajectory with $x(0) \in \Delta^{ud}$, the attractors under the best response and the perturbed best response dynamic lie close to each other. To extend the argument to any initial condition, we note that under both dynamics, if $x(0) \notin \Delta^{ud}$, solution trajectories converge to Δ^{ud}.

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