Essays on Auction Theory

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I, Fabio Michelucci, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

Abstract

The thesis is composed of three essays on auction theory. The focus is mainly on the English auction, which is probably the format most commonly used. In the English auction, a bidder can see his opponents' activity as the price is gradually raised starting from a very low price. When only one bidder is left, such bidder is declared the winner and pays the price at which the last but one bidder exited. The vast majority of the studies assume that the price is raised continuously and that a bidder cannot outbid the current winner by a discrete amount, or to use the terminology employed in the literature to place a jump bid. Such a restriction is often unrealistic and examples of jump bidding can be found in many relevant contexts. In the first two chapters, we analyze settings in which jump bidding is allowed. Previous studies suggest that the possibility of placing a jump bid can be used to signal one's strength and induce the opponents to quit (on average) earlier than they would have otherwise.

In our first paper we follow this approach. In particular, as Fishman (1988), we assume that the purpose of signalling is to discourage the opponents from acquiring finer information regarding their valuations. As opposed to Fishman (1988), we look at an environment where bidders' valuations are determined not only by a private (as he does) but also a common value element. We show that the presence of a common element makes jump bidding harder. Secondly, we prove that when the bidder placing the jump knows his total value but not the relative importance of the common value part, a jump bidding equilibrium can be sustained more often and it yields higher profits than when the bidder knows exactly such value.

In the second paper, we introduce a different and new rational for jump bidding, which involves no signalling or information costs. In an interdependent value setting, where a bidder's value depends on the valuation of the other bidders, it is crucial whether a bidder's exit price is known or not. Jump bidding offers the possibility of hiding the exit value of some opponent, thus affecting the expected value of the remaining bidders and ultimately their bidding behaviour. We illustrate when hiding such information might be profitable. We also show that its effect both on revenues and efficiency is in general ambiguous.

Finally, in the last paper (joint work with Hernando-Veciana) we contribute to the important issue of determining which is the most efficient way to allocate an object among a set of potential buyers. Previous studies provide both conditions under which it is possible to allocate the object to the buyer with the highest willingness to pay (First

Best), and mechanisms that can implement such an allocation. We characterize the most efficient allocation when the First Best cannot be implemented. Then, we study whether the English auction can implement it. While an equilibrium of the English Auction that implements the Second Best exists, it is in general not robust if there are more than two bidders.

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"The miracle isn't that I finished, it's that I had the courage to start", John Bingham (Marathon coach)

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Contents

Chapter 1

Introduction

Auction theory has been one of the most successful and prolific areas of economic theory in the last thirty years. The theoretical guidelines that have been developed have proven to be important in helping practitioners improve the design of the auction mechanisms used in practice. The relevance of the practical applications, together with the fact that the theoretical tools developed can be employed for applications rather than auctions, is probably one of the key factors behind the popularity of this field.

The thesis is composed of three chapters. All chapters focus on frameworks where there is only a single object for sale and the mechanism used is an English auction. In the last chapter we also solve a more general theoretical problem concerning the issue of determining the allocation that maximizes social welfare when it is not possible to achieve the first best.

The single object framework is relatively simpler to analyze compared to the multi-object one. For this reason it has been the first to be analyzed and the one that has received more coverage to date. Among the auction formats, the English auction is probably the one most used in practice and many studies have already pointed out the advantages and disadvantages of this format. In the English auction the seller or the auctioneer announces a starting price, which is then raised until only one of the bidders is still active. That bidder wins the object and pays the price at which his last opponent exited the auction. Such type of format is called open format, referring to the feature that bidders typically see the opponents' activity during the auction.

We believe that even though many previous studies had as focus the single-object English auction, some of its key features have received too little attention. We aim to contribute covering some of these gaps.

In particular, most of the studies have made some restrictive modeling assumptions and/or imposed conditions on bidders' valuations that have enabled to look at "well-behaved" and regular problems and to provide nice clear-cut results in terms of the analysis of the revenues and efficiency properties of the auction and in terms of the comparison with alternative formats. In this type of studies the English auction is often praised as under the circumstances imposed it generates higher revenues and/or provides an efficient allocation. This is generally due to the fact that the English auction being an open format allows the bidders to aggregate extra information in the course of the auction. The conditions that are normally imposed guarantee that such information aggregation happens in a "smooth" way. Conversely, we look at settings where the aggregation of information may not be smooth or bidders may have incentives to affect the information that can be aggregated in the auction. The environments we look at are more complexes. Thus the price we need to pay especially in the first two chapters for looking at less regular problems is that we need to focus on more specific settings.

One of the restrictions normally introduced in the modeling stage is to look at the so-called Japanese version of the English Auction described by Milgrom and Weber (1982). In this variant of the English auction some exogenous device exogenously and continuously raises the price. This greatly simplifies the analysis as it limits the set of possible actions that can be taken by a bidder to the relatively simple decision of staying active or to quit at the current price. In real auctions, however, the price is typically raised either by the bidders or by the seller through discontinuous increments. Such a feature is one of the key elements that distinguish an open format from a sealed one as it allows the bidders to communicate through the bids they place.

In the first two chapters we look at frameworks where bidders can outbid the current winner by a discrete amount or, to use the terminology employed in the literature, to jump bid.

The first chapter follows Fishman's (1988) insight that a bidder may jump bid in order to deter an opponent from (costly) acquiring finer information regarding his valuation. Such information may render the opponent more competitive. Thus, a jump bid can be used to signal one's strength to prevent the opponent from investing resources in acquiring information when the chances of winning with a congruous return would be small. Fishman's paper is set to explain the takeover premium that is often observed in takeover contests. The framework he uses is a two-bidder independent private model in which one of the two bidders knows his valuation, while the other does not. If the former draws a sufficiently high type, he places a pre-emptive bid that forecloses competition from the opponent. We follow closely his setting so that we can derive his model as a special case of ours and exploit it as a benchmark. Our interest is to study whether and to what extent a bidder with finer information over a common value component can use jump bidding to increase his informational rent. We also look at a two-bidder setting but we assume that bidders' valuations are a mix of a private and a common element. Our setting is justified by the observation that in most of the applications the finer information a bidder can achieve is not of a pure private nature, but rather it affects the opponents values as well. In the takeover application for instance a bidder may get a better estimate regarding the quantity or the quality of some assets the target holds. Such information normally contains some common value element. A first result is that when a common value element is present it is harder to profitably jump bid and the profits of the better-informed bidder are reduced compared to the private value case. The larger the relative size of the common value element, the lower the probability of observing a jump bid. The intuition is that now a one-dimensional signal (the price) conveys information about a two-dimensional type (given by the private and common component). A higher price signals a higher value, but it also induces an upward updating of the probability of a better realization of the common value element. The second effect does not deter competition; rather it works in the opposite direction than the one desired by the informed bidder. In fact, a higher bid does not need to more strongly deter the opponent. The second main result lies in the comparison of two different informational settings: in one we assume that the informed bidder gets to know both the private and the common value signal; in the other, that he gets only an aggregate signal about his valuation. We show that informed bidder prefers to receive only the aggregate information, so that more information does harm the bidder in this contest. Finally, we discuss the relation between jump bidding and information acquisition. We argue that the higher profits allowed by jump bidding maybe a necessary rent to leave to a bidder. Otherwise, it may be that no bidder invests in information with a possible loss both in revenues and efficiency.

In the second chapter, we continue to look at a model where jump bidding can arise in equilibrium. Contrary to all previous models in the small literature on jump bidding (with exception of an independent work by Ettinger (2006)), we introduce a new explanation that is not based on a signaling story. The rational we propose is based on a hiding information motivation: a jump may be placed to hide to the opponents some valuable information. The information that can be hidden is the exact exit value of some of the opponents. In an interdependent value setting, the valuation of a bidder depends also on the valuations of the other bidders. By discretely increasing the price, the exact exit value of the opponents that do not match the price that has been called is not observed. Hence, a bidder receives only a coarser information about some opponent's value, which affects his expected valuation and hence his bidding behavior. As mentioned, the English auction is often praised because it allows for information aggregation in the course of the auction and thus, under some circumstances guarantees, both higher revenues and efficiency compared to other commonly used formats. We point out that the bidders may be able to affect the way such information is aggregated in a way that is favorable to them. Not considering such a fact may lead to a wrong conclusion regarding the revenues (or the efficiency level) that is generated by the format. We show that in general the effect of jump bidding on revenues and efficiency is ambiguous.

Finally, in the third chapter we deal with the important question of looking for the allocation that maximizes social welfare. In the second part of the paper we check if such an allocation can be implemented using the English auction. The previous work on efficient auctions has mainly looked for conditions under which it is possible to always allocate the object for sale to the bidder with the highest valuation (First Best) and then looked for mechanism that could achieve such an allocation. The necessary condition for the first best to be implementable is a technical single crossing condition. Essentially, what the condition requires is that the impact of a marginal increase of a bidder's signal on his valuation should be stronger than on the valuation of any other opponent. In our paper we provide interesting applications where such conditions are violated, such as models with an incumbent and models with negative externalities. Our model applies to those settings. Assuming that such a necessary condition holds, Maskin and Dasgupta (1992) develops a mechanism that can implement the First Best in general. Though theoretically very interesting, their mechanism requires the bidders to submit far too much information to be used in practice. Remarkably, with just two bidders the necessary condition mentioned above is also sufficient for the English auction to implement the First Best (Maskin 1992). A stronger condition is needed with more than two bidders (Krishna (2003), Birulin and Izmalkov (2003)). We contribute to the literature on efficient auctions in two directions. First, as mentioned we solve the problem of determining the efficient allocation when the conditions that are needed for the First Best do not hold. That is, we characterize the second best efficient allocation. Interestingly, the second best allocation may involve no selling. Second, we focus on the English auction. A priori there is no reason to believe that the English auction should perform well when the first best cannot be achieved. We show that English auction possesses an equilibrium that implements the second best allocation. However, such equilibrium is in general robust only for two bidders. For more than two bidders, only an equilibrium based on weakly dominated strategies sometimes exists. We suggest that in some cases of interest a two-stage English auction can successfully implement the second best. On the technical side, our work is related to Myerson (1981). We employ the technique he used to characterize the revenue maximizing mechanism in the non-regular case (which is known as ironing technique) to find the allocation that maximizes the expected social surplus.

Chapter 2

Information Acquisition and Pre-emptive bidding

2.1 Introduction

Jump bidding is a widespread phenomenon in open auction formats that consists in placing a substantially higher bid than the one that would be sufficient to replace the current winning bid.

It can be observed in many different applications, which include pure auction settings as well as other competitive settings that can be modeled as auctions and where overbidding a competitor by a large amount is a commonly used procedure.

A well known and economically relevant example of the former ones is the bidding behavior in the UMTS spectrum auctions, of the latter ones the bidding that can be observed during takeover contests.[1]

The standard way to explain bids above the minimum increment is based on a signaling motivation (e.g. Fishman (1988), Avery (1998)). Recently, an alternative explanation has been proposed: Ettinger (2006) and Michelucci (2007a) suggest that a bidder might place a jump bid in order to hide the exact drop-out value of some opponent, thus preventing the remaining active bidders from a precise updating of their valuations. The fact that the opponents are forced to base their expected values on less precise information may favor the bidder placing the jump bid.[2]

[1]See for instance, Börgers and Dustmann (2005) for a discussion on jump bidding in the UK UMTS spectrum auctions, and Fishman (1988) for the takeover application.

[2]In an interdependent value setting not observing the exact exit value of some opponent provides a coarser information with which to update your expected valuation and in turn your bidding behavior.

The present paper follows the standard signaling approach to jump bidding. The objective is to understand if and to what extent a bidder with finer information over a common value component can use a jump bidding strategy to increase his information rent. We are also interested in evaluating how jump bidding affects the incentive to acquire information and in turn the format's revenues and efficiency. We discuss this at the end of the paper.

The closest model to ours is Fishman (1988), which is the first one to suggest that signaling may be used as a pre-emptive device.[3] The key element in pre-emptive models is that one or more bidders can invest to get finer information regarding his (their) valuation(s). Such information may render the opponent(s) more competitive. Since the profitability of acquiring information depends on the chances of winning, a bidder may want to signal to have a high enough valuation to make the acquisition of information unprofitable.[4] The use of such strategies increases the profits of the bidder who places the jump bid at the expenses of the seller's revenues. Furthermore, since the objective of signaling is not to establish oneself as the highest valuation bidder, but rather as a strong enough one, allowing for jump bidding may also decrease efficiency.

Fishman's model is set to explain the bidding premium that is observed in takeover contests, where a traded company is often taken over at a share price which is higher than the market one.[5] The framework is a two-bidder independent private model, in which one of the two bidders (managers) knows his private value (profits realizable with the takeover), whereas the other bidder needs to incur a cost to find out.[6] If the first management assesses a high enough valuation for the takeover, an offer at a premium is made to foreclose competition from the opponent.

We build upon Fishman (1988) work, but we assume that the incumbent has finer information regarding the mix of a private and a common value element. The formulation

Those studies provide interesting applications where a bidder may profitably decide to hide information.

[3]It extends to a bidding setting an idea already present in the literature on limit pricing, see for instance Milgrom and Roberts (1982).

[4]This is similar to what happens in the models motivated by the hiding information insight mentioned above. There, however, no signaling nor costly information acquisition is necessary.

[5]Takeover contests fit nicely a theoretical setting that allows for information acquisition because they may last for a considerably long time, thus giving bidders enough time to acquire extra information. For an auction where the allocation of the object entails only a very small expected profit, a pre-emptive cost could be given by the mental effort/time of assessing more precisely ones own preferences.

[6]The original formulation assumes that also the first bidder needs to incur a cost, but such fact is redundant for the point made by the paper. The ex-ante distribution from which the second bidder's value is drawn is common knowledge.

chosen allows us to derive his model as a special case of ours and thus to use it as a benchmark case. Our setting is motivated by the observation that in most cases the extra information that a bidder can acquire is not entirely of private nature. For instance, in the takeover application a management may get a more precise information about the amount of some assets held by the target, which have a clear market price, but could potentially be used with a different return by the two managements.

A first result is that moving from a pure private value setting to a mix of private and common values decreases the probability of observing a jump. In other words, the introduction of a common value element weakens the incentive to jump bid. The higher the relative size of the common component, the less likely that a jump occurs. To get some intuition for this result, note that in our model, even though a jump signals a higher valuation, it also induces an upward updating of the probability of a better realization of the common value element. Essentially, a one dimensional signal (the price) conveys information regarding a two dimensional component (the bidder's private and common signal) and the information conveyed regarding the common value component does not help to deter the opponent(s) from investing. In fact, it work against it. [7] Increasing the relative size of the common value component makes such undesired effect stronger.

We analyze two different settings: one in which the bidder holding finer information is not aware of the relative importance of the common component in determining his value (partial information); and one in which he is (full information). Our second main insight is that in the full information case it is more difficult to place a jump and the informed bidder gets on average less profits. We show that the qualitative effect of having the informed bidder aware of the size of both the private and the common value component is analogous to the one of increasing the relative size of the common component in the partial information case. This implies that the informed bidder is better off in the partial information case.

Finally, we look a bit closer at how jump bidding affects the incentives to acquire information. Fishman's conclusion is that jump bidding should be banned as anticompetitive as it decreases revenues. However, in drawing such conclusion he ignores the fact that, if the first bidder has to pay to get the finer information, he may never invest were he not allowed to jump. In other words, we argue that the extra profits that jump bidding guarantees may represent a necessary rent to leave to the informed bidder.

Other papers have followed Fishman preemptive insight. Hirshleifer and Png (1989)

[7] In fact, signaling a higher valuation in this setting may even make a jump that was initially preemptive no longer be such.

develops a similar model adding the feature that placing a bid is costly. They show that the takeover target price is maximized when acquiring information bears a strictly positive cost, i.e. when some amount of jump bidding is present in equilibrium. Bhattacharyya (1992) extends Fishman model to allow for two-sided asymmetric information. That is, unlike in Fishman, the second bidder has some initial information that is correlated with the one he can get by investing. The additional uncertainty changes the equilibrium outcome for a set of intermediate types who now use a strictly increasing bidding function that induces partial pre-emption.[8] Bernhardt and Scoones (1993) provides an interesting application of pre-emptive bidding to the labour market.

An alternative rationalization of jump bidding based on a signaling motivation is due to Avery (1998). He introduces a single object affiliated value framework a la Milgrom and Weber (1982) and proposes a pure signaling model that does not need neither costly information acquisition nor costly bidding to generate jump bidding. Essentially, a form of implicit collusion is initiated via jump bidding. Ex-ante symmetric bidders use one or multiple rounds of jumps in order to assess who amongst them is the bidder with the highest valuation; then, in the final round, they compete using asymmetric bidding strategies that favor the designated winner. Jump bidding in this context does not affect efficiency. However, it does increase the profits of the winner at the expense of the revenue of the seller.

The model has the advantage of explaining jump bidding in a more general and well established setting compared to the ones illustrated earlier. However, it has two main drawbacks. First, there is a multiplicity of signaling equilibria. Second, it is not robust to perturbations of the last stage of the game, i.e. it is essential that the bidders who are not "designated" as the winner follow exactly the less aggressive equilibrium strategies they are assigned.[9]

Finally, jump bidding may arise if the act of placing a bid is itself costly. Daniel and Hirshleifer (1997) shows that in that case jump bidding can be an equilibrium even in an independent private value setting where bidders are perfectly informed about their valuations.[10]

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[8] Whereas low types continue to accommodate and high types to fully pre-empt.

[9] We believe that the idea of jump bidding as a way to collusively determine the allocation would fit better a multi-object setting as there a deviation from the accommodating bidding strategy on one item can be punished by a more aggressive bidding on the item that is selected to be assigned to the deviator.

[10] In this regard, let us point out that jump bidding can take place even in a non signaling model. Suppose in fact that in an IPV model it is costly to bid only for one of the bidders (equivalently it could be that the bidder is impatient and wants to avoid the cost of waiting longer to close the auction). Then, such bidder will make multiple jumps each time equating the cost of the bid to the expected foregone

Our work is also related to some literature on information acquisition in auctions. Compte and Jehiel (2002) are interested in revenue ranking sealed bid formats against dynamic formats in the presence of costly information acquisition.[11] Their setting is one of independent private values (IPV), where there is one "special" bidder who can invest to discover his exact valuation at any point in the auction and n symmetric bidders that cannot. These latter base their bidding strategy on their expected valuation. [12] They advocate in favour of dynamic formats as they typically allow to observe the opponents exit values and thus the residual demand. This feature enables to assess the chances of winning when holding a finer information than in a sealed bid setting, and thus provides better incentives to invest, which in turn boosts both revenues and efficiency. [13]

Miettinen (2006a) uses the same setting as Compte and Jehiel (2002), but allows for jumps. He shows that the informed bidder may jump to discourage the uninformed bidder(s) from acquiring information.

Rasmusen (2002) analyses the investment decision aimed at obtaining a finer information in an open ascending price auction with IPV. His aim is to rationalize why bidders may end up spending more than initially planned. He shows that, for some range of the investment cost, it is optimal to invest at a later stage and thus revise the initial estimate. [14]

The paper is structured as follows. Section 2 introduces the model. Section 3 presents and analyses the equilibrium both for the case in which the informed bidder receives only the aggregate information and the one in which he receives information regarding both signals. Section 4 compares the equilibrium behavior in those two settings. Section 5 discusses how jump bidding may affect the incentives for information acquisition. Finally, section 6 comments on the modeling assumptions made and provides final remarks.

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profit of not winning at a price lower than the one posted (the other bidders being infinitely patient use their weakly dominant strategy to drop when their value is reached).

[11]For a related paper, see also Rezende (2001).

[12]In the published version, Compte and Jehiel (2007), all bidders are ex-ante symmetric, the main insights are unchanged.

[13]Miettinen (2006b) investigates the incentive to get a finer information in the Dutch auction. He provides an example in which the Dutch auction outperforms the English auction.

[14]Hence, estimate revisions may be a perfectly rational phenomena, rather than the consequence of some emotional effect at play.

2.2 The Model

2.2.1 The set up

We analyze the following framework. Two players compete in an ascending open English auction with a reserve price, v_0, and no entry costs, for the allocation of an object. One of the them, to which we refer to as F (for First bidder) has an informational advantage in that he knows his valuation for the object and can place a first bid, b_1, before the other bidder, S (for Second bidder), can invest to find out his exact valuation. Bidder S, prior to investing, knows only his expected value for the object. The distribution from which his exact valuation is drawn is common knowledge since the very beginning, as it is the investment cost, c_S. As soon as the auction ends, all the uncertainty is resolved.

We assume that bidders' valuations for the object are given by a mix of a private and common component, in particular we represent them by the value function $v_i = t_i + Q$, where t_i, $i = F, S$ is the private value component and Q is the common value one. The private signal of bidder i is drawn independently of the one of his opponent from a continuous cumulative distribution function with support in $[0, 1]$, i.e. $\tilde{t}_i \sim F_i[0, 1]$.[15] The common value component is taken to have a pointwise distribution; more specifically, we assume that $Q \in \{0, q\}$, with $Pr(Q = 0) = Pr(Q = q) = \frac{1}{2}$ and $q \geq 0$. Some modifications of the basic model are discussed in section VI. Note that we can recover Fishman's framework as a special case of ours setting $q = 0$.

It is probably instructive to think of the two realizations of Q as states of the world, so that we have $Q = 0$, as the bad state and $Q = q$, as the good state. Following this interpretation, we can restate that: if $Q = 0$, players' valuations are taken from a c.d.f with support $[0, 1]$, i.e. $\tilde{v}_i \sim F_i\,[0, 1]$, while if $Q = q$, players valuations are taken from a c.d.f with support $[q, 1 + q]$, i.e. $\tilde{v}_i \sim F_i\,[q, 1 + q]$.[16] Thus, if it is known that v_i belongs to the interval $[0, q]$, also v_j must belong to the same interval; similarly, if v_i belongs to $[1, 1 + q]$, also v_j does. This implies that the random variables $\tilde{v}_i$, $\tilde{v}_j$ are affiliated.[17]

Two different settings are analyzed:

- case a) Bidder F is fully informed regarding both t_F and Q (full information).

[15] We use the tilde notation (e.g. $\tilde{t}_i$, $\tilde{v}_i$) to indicate random variables and the notation without the tilde sign (e.g. t_i, v_i) to indicate a specific realization of the random variable.

[16] Alternatively, one could think of t_i as the expected valuation of the players and of Q as the realization of a common noise on the private signal.

[17] The reader can refer, for instance, to Milgrom and Weber (1982) for the notion of affiliation.

- case b) Bidder F learns only the aggregate information, v_F (partial information).

Notice that for S it makes no difference whether we assume that he observes the aggregate value v_S or the realization of both signals: in either case he has a unique weakly dominant strategy to be active up to v_S.

For both cases we are going to first conjecture that a pre-emptive equilibrium exists and then verify its existence . A pre-emptive equilibrium can exist only if two conditions hold.

- (A1): $0 < c_S \leq c_S^*$.
- (A2): $v_0 \geq v_0^* > 0$.

The first condition says that bidder S investment cost, though strictly positive cannot be too big. Intuitively, if the cost is very high, bidder S would never invest regardless of the initial bid observed. Hence, there would be no reason to jump bid.

The second condition ensures that v_0 is high enough so that the profits from allowing competition are strictly lower than the ones from pre-empting for at least some types of bidder F. To see this point, take the simplest case in which $q = 0$ and $\tilde{t}_i \sim U[0, 1]$ and assume that $v_0 = 0$. Note that if bidder S does not invest, he can still profitably stay active up to a price $p = \frac{1}{2}$. But then it is easy to check that all types of bidder F would rather prefer bidder S to discover his valuation rather than paying with probability one $p = \frac{1}{2}$.[18]

The same restrictions are assumed in Fishman (1988). In particular, he assumes that c_S^* equals bidder S expected profits conditional on the information $\tilde{t}_F \geq v_0$.[19] As for v_0^*, he assumes that such value is given by bidder S expected value. We follow the same restrictions on c_S^* and v_0^*. Since in our model the initial bid conveys some information regarding bidder's S valuation, we assume that v_0^* equals the expected value of the private component plus the common element. Then, S if pre-empted from investing will not be able to profitably bid in the auction.

The difference in terms of equilibrium conditions and outcomes between the partial information case and the full information case emerges only when the pre-emptive bid

[18] With type $t_F = 1$ being indifferent between paying $\frac{1}{2}$ and competing with an informed bidder S.

[19] These are the expected profits in the equilibrium where signaling is not allowed. Note that the less stringent condition that the value c_S^* is the one that equals bidder S profits conditional on not observing a jump could be imposed. We comment further on this in section 5.

does not reveal the state of the world with certainty. We postpone the discussion regarding the equilibrium in which a jump reveals a good state for later. For now it is sufficient to point out that the value of q needs to be not "too large". In particular, the maximum value of q compatible with an equilibrium that leaves some uncertainty regarding the common value is different for the two informational settings. When we discuss the equilibrium for case a) and case b) the relevant restriction indicated below will be assumed to hold:

- case a): $q \leq q^*$.
- case b): $q \leq q^{**}$,

  where $q^{**} > q^*$.

For both cases the following equilibrium structure holds. The initial bid of bidder F takes one of two values, either v_0 or some $p > v_0$. Only the latter one prevents the second bidder from investing and competing. All types of bidder F over a certain threshold place a pre-emptive bid, while all others accommodate competition by bidding v_0. The notion of type, though, is not the same in the two settings. As a matter of fact, in case a) a type is identified by a one-dimensional signal (v_i), while in case b) by a two-dimensional signal (t_i, Q). This implies that in case a) the equilibrium needs to specify a different threshold for each state, whereas only one threshold is necessary for case b).

Once the equilibrium is characterized, we will be interested in two different types of comparative statics. First, we compare our setting with Fishman's one ($q = 0$) and evaluate the effect of increasing the relative size of the common component. Intuitively, a higher value of q could benefit bidder F under a good state as he may be able to pre-empt bidder S even with a lower private type. However, it also becomes more difficult to pre-empt from a bad state. We want to evaluate from an ex-ante perspective how much of the possible increase in rent (due to the higher expected value for the object) bidder F is able to seize and how his profits (as well as the ones of the other players) vary.

Second, we compare the equilibrium in case a) with the one in case b), to understand whether more information advantages or harms bidder F.

2.2.2 Timing

The timing of the game is as follows.

- Stage I: F learns his valuation v_F (either the sum of the signals or both components) and decides which initial bid, b_1, (if any) to place.[20]

- Stage II: S observes b_1 and decides whether to learn his valuation, v_S or not. If he decides to get the information he pays a cost, c_S.

- Stage III: If S has acquired information in stage II, a standard Japanese version of the ascending price auction unfolds.[21]

In equilibrium, F anticipates S decision in the second stage as the latter one is a binary choice that is taken holding no ex-ante private information. The game above can then be seen as a two stage game, where in the first stage the first bidder has the option to end the game immediately with a high offer or to play the Japanese version of the English auction.

We indicate the subgame where the equilibrium actions are given by the pair "pre-emptive bid/no investment" by p (for pre-emptive game) and the subgame characterized by the actions "accommodate/ invest" by c (for competitive game).

Instead of looking at the Japanese auction (where the price is raised continuously), we could allow for jumps to occur at any time. However, assuming a standard Japanese auction is with little loss of generality as such subgame is reached only when both bidders know exactly their valuation. In such case there is little scope for further signaling (or for hiding information as in Michelucci (2007a)) as, given the absence of bidding costs, both bidders have as a weakly dominant strategy to stay active till their value is reached. This avoids considering an enlarged type space.

It may be convenient to illustrate more precisely bidders payoff. Let $d_s \in \{0,1\}$ represent S decision whether to compete or stay out. In particular, let $d_s = 1$ indicate that S competes, while $d_s = 0$ indicate that he does not. Given an initial bid b_1 from F, we have that bidder F profits are:

[20] In equilibrium, a bid is observed provided that $v_F \geq v_0$.

[21] See Milgrom and Weber (1982) for the formal description. In equilibrium, a bid by S will be placed only if $v_S \geq \max(v_0, b_1)$.

$$\pi_F(v_F, v_S, b_1, 0) = v_F - b_1$$

$$\pi_F(v_F, v_S, b_1, 1) = \begin{cases} v_F - b_1 & \text{if } v_S \leq b_1 \\ v_F - v_S & \text{if } b_1 < v_S \leq v_F \\ 0 & \text{if } v_S > v_F \end{cases}$$

while bidder S profits are:

$$\pi_S(v_F, v_S, b_1, 0) = 0$$

$$\pi_S(v_F, v_S, b_1, 1) = \begin{cases} v_S - v_F - c_S & \text{if } v_S > v_F \\ -c_S & \text{if } v_S \leq v_F \end{cases}$$

The following section analyses the equilibrium bidding behavior.

2.3 Equilibrium Behavior

We present below the equilibrium analysis both for case a) and case b). In terms of equilibrium actions and equilibrium outcomes the two cases differ only when the pre-emptive bid is placed with positive probability from both states of the world. We refer to such equilibrium as *pooling.*

A different equilibrium arises when the pre-emptive bid can be placed only from the good state. We refer to it as *fully revealing.*[22] All types of bidder F strictly prefer the pooling equilibrium as it guarantees pre-emption for a strictly larger set of types.[23] This will imply that the fully revealing equilibrium can emerge as a "credible" equilibrium outcome only when the pooling one is not available.[24]

For case a), the one-dimensional pre-emptive bid conveys information regarding a two-dimensional type. This means, as anticipated, that while in case b) the equilibrium can be characterized by a unique equilibrium threshold, call it r^{**}, a pair of thresholds call them (r_0^*, r_q^*), is needed in case a).[25] As a matter of fact, even though bidder S can recover from r^{**} that $r_0^{**} = r^{**}$ and $r_q^{**} = r^{**} - q$, what is crucial in case b) is that the difference between those thresholds is equal to q, for any q. Such difference is not equal

[22] We refer to it as fully revealing as the state is revealed with probability one, the private type however is not.

[23] And in equilibrium all types that choose preemption over competition necessarily get higher expected profits under preemption.

[24] We will be more precise about the meaning of "credible", when we discuss uniqueness.

[25] We use r_0^* for the equilibrium threshold of bidder F when $Q = 0$, and r_q^* when $Q = q$.

to q in case a), and we later show that is strictly greater. This will be one of the key point of difference between the two settings.

case a) : Bidder F knows both t_F and Q (full information case)

The following proposition states more formally what already anticipated about the structure of the equilibrium. We assume that in the sub-game in which bidder S invests and finds out his value, he stays active till v_S.[26] Then, the equilibrium is fully characterized by bidder's F initial bid, $b_1(t_F, Q)$, by bidder's S beliefs conditional on observing b_1, $g(b_1)$, and by his investment decision, $d_s(b_1)$. We look at the Perfect Bayesian Equilibrium (PBE), hence the equilibrium triple $(b_1^*(t_F, Q), d_s^*(b_1), g^*(b_1))$ needs to satisfy that: 1) $b_1^*(t_F, Q)$ maximizes bidder F expected payoffs given $d_s^*(b_1)$; 2) $d_s^*(b_1)$ maximizes bidder S expected payoffs given $g^*(b_1)$; 3) $g^*(b_1)$ is consistent with Bayes rule given the equilibrium initial offer $b_1^*(t_F, Q)$. The last part defines consistent beliefs.

Proposition 2.1. *The triple $(b_1^*(t_F, Q), d_s^*(b_1), g^*(b_1))$ constitutes the unique perfect bayesian equilibrium with credible beliefs of the auction game described above, where:*

$$\begin{aligned} b_1^*(t_F, Q) &= \begin{cases} p^* \text{ if } \mathrm{Q} = 0 \text{ and } \mathrm{t_F} \geq \mathrm{r_0^*} \text{ or } \ \mathrm{Q} = \mathrm{q} \text{ and } \mathrm{t_F} \geq \mathrm{r_q^*} \\ v_0 \text{ otherwise} \end{cases} \\ d_s^*(b_1) &= \begin{cases} 0 \text{ if } b_1 = p^* \\ 1 \text{ otherwise} \end{cases} \\ \mathrm{g}^*(\mathrm{b}_1) &= \text{consistent and credible beliefs} \end{aligned}$$

It is easy to construct beliefs that are consistent.[27] The restriction to credible beliefs guarantees uniqueness. Here we prove that a triple satisfying point 1) to 3) exists. We then argue in section 6 that the equilibrium we propose is the unique one with credible beliefs. The fact that there is a unique equilibrium with credible beliefs is already shown in a private value setting by Fishman (1988). No differences arise in our setting. We remand the reader to section 6 for a discussion.

We proceed by construction. First, we prove that the equilibrium pre-emptive bid has to be the same no matter what the state of the world is. Second, we state the conditions that identify the threshold values r_0^* and r_q^* and show how these two values

[26]This is his weakly dominant strategy.

[27]For instance, we could have S believe, after observing $b_1 = p^*$, that with probability $Pr(Q = 0|b_1 = p^*)$, $t_f \geq r_0$ and with probability $Pr(Q = q|b_1 = p^*)$, $t_f \geq r_q$; while $t_F = v_0$, otherwise.

are related. Finally, we make sure that it is not profitable for types of bidder F below the threshold value to pre-empt, thus guaranteeing that is a best reply for S to stay out, when a pre-emptive bid is observed. We remand the discussion regarding "credible beliefs" and uniqueness to section 6.

Lemma 2.1. *The equilibrium pre-emptive bid must be unique independently of bidder F private type and of the state of the world, i.e. different types of bidder F pool on the same bid.*

Proof. Assume, by means of contradiction, that two different pre-emptive bids exist and call them $p^{'}$ and $p^{''}$.[28] Without loss of generality take $p^{''} > p^{'}$.

Since by assumption both bids are pre-emptive, and given that the state of the world is not observable by bidder S , the lower bid, $p^{'}$, guarantees pre-emption at a lower cost. A deviation follows with any bidder F of a type that is supposed to bid $p^{''}$ mimicking the type(s) that is (are) supposed to bid $p^{'}$, which yields the desired contradiction. Therefore, $p^{'} = p^{''} = p^{*}$.

□

In equilibrium bidder F chooses the pre-emptive bid that makes bidder S indifferent between investing and competing or staying out, that is he will push bidder S expected profits net of the investment cost to zero.[29]

In equilibrium, since bidder S knows that bidder F is aware of the state of the world, he can correctly infer that bidder F must be at least type r_0 (r_q) if the state of the world observed is the bad (good) one. A higher bid signals both a higher r_0 and a higher r_q, which holding the probability of the state of the world constant would unambiguously more strongly deter bidder S. However, bidder S also learns something regarding the common value, which as anticipated works against the interest of bidder F.

As a matter of fact, the probability of a bad state conditional on observing an initial bid $p > v_0$ is $Pr(Q = 0|b_1 = p) = \frac{\frac{1}{2}*Pr(\tilde{t}_F \geq r_0)}{\frac{1}{2}*Pr(\tilde{t}_F \geq r_0)+\frac{1}{2}*Pr(\tilde{t}_F \geq r_q)}$; the probability of a good state is $Pr(Q = q|b_1 = p) = 1 - Pr(Q = 0|b_1 = p)$.

Since $r_0 > r_q$, we have that $Pr(Q = 0|b_1 = p) < \frac{1}{2}$, that is placing an initial pre-emptive bid makes the bad state less likely and thus the good state more likely (for which the private threshold signaled is lower).

[28]The same argument excludes that more than two different preemptive bids can be placed.

[29]There exist other equilibria that involve bidder S making strictly negative profits by investing. As anticipated the one we focus is the unique one that is based on credible beliefs.

Conversely, observing $b_1 = v_0$ signals that the bad state is more likely as $Pr(Q = 0|b_1 = v_0) = \frac{\frac{1}{2}*Pr(\tilde{t}_F \in [v_0, r_0])}{\frac{1}{2}*Pr(\tilde{t}_F \in [v_0, r_0]) + \frac{1}{2}*Pr(\tilde{t}_F \in [v_0, r_q])}$ and $r_0 > r_q$.

Thus, differently from Fishman's (1988) pre-emptive model, observing a jump bid conveys relevant information to the second bidder regarding his own valuation.

The expected (gross) profits for bidder S conditional on an initial bid, p, by bidder F and the bad state are $E(\pi_S|Q = 0$ and $b_1 = p) = E(\tilde{t}_S - \tilde{t}_F|\tilde{t}_F \in [r_0, 1], \tilde{t}_S \geq \tilde{t}_F)$. Similarly, his expected (gross) profits conditional on p and the good state are $E(\pi_S|Q = q$ and $b_1 = p) = E(\tilde{t}_S - \tilde{t}_F|\tilde{t}_F \in [r_q, 1], \tilde{t}_S \geq \tilde{t}_F)$.

Recall that we denoted the subgame where S is pre-empted by p and the subgame where competition is allowed by c. The profits for bidder S from staying out are equal to zero and we denote them by $\pi_S^p = 0$. The profits from competing instead are denoted by π_S^c and depend on the pair (r_0, r_q) signaled by bidder F via b_1. The equilibrium p (or equivalently, the equilibrium r) is the one that makes bidder S indifferent between competing or not, i.e. we want to look for a pair (r_0, r_q) such that $\pi_S^c(r_0, r_q) = 0$, where π_S^c are bidder S expected profits from investing and competing with F.

This leads us to the following condition to which we refer to as SIC (bidder S Indifference Condition).

- (SIC): $\pi_S^c(r_0, r_q) = \pi_S^p = 0$ iff $Pr(Q = 0|b_1 = p) * E(\pi_S|Q = 0$ and $b_1 = p) + Pr(Q = q|b_1 = p) * E(\pi_S|Q = q$ and $b_1 = p) = c_S$.

The above expression identifies the set of values of r_0 and r_q that, given the information cost, c_S, make the second bidder indifferent whether investing for information or not entering the competition.

The indifference condition for bidder F then selects which pair from this set is the equilibrium one. If it turns out that no pair (r_0, r_q) such that $r_0 \leq 1$ can be selected it means that only the fully revealing equilibrium exists.

The price p needs to credibly signal a pair (r_0, r_q) as well as guarantee that is not profitable for lower types of bidder F to mimic higher types and place a pre-emptive bid. Since bidder F knows the state of the world we need to look at two different conditions.

The objective is to find an equilibrium initial bid p^* such that, if the bad (good) state is realized, all types $t_F \geq r_0^*$ (r_q^*) prefer to bid p^* and pre-empt the other bidder and all types $t_F < r_0^*$ (r_q^*) prefer to bid v_0 and accommodate competition.

If $Q = 0$, we look for a type r_0 that solves the following equation:

- (FICB): $\pi_F^p(r_0) = r_0 - p = Pr(\tilde{t}_S \leq v_0) * (r_0 - v_0) +$
  $Pr(\tilde{t}_S \in (v_0, r_0]) * E(\tilde{t}_F - \tilde{t}_S|\tilde{t}_F = r_0, \tilde{t}_S \in (v_0, r_0]) = \pi_F^c(r_0)$

where the left hand side of the equality represents bidder F payoff if he decides to pre-empt S and the right hand side the payoff if he decides to compete.

Similarly, if $Q = q$ the indifference condition we look for is the following:

- (FICG): $\pi_F^p(r_q) = r_q + q - p = [Pr(\tilde{t}_S \leq v_0 - q) * (r_q + q - v_0) + Pr(\tilde{t}_S \in (v_0 - q, r_q]) * E(\tilde{t}_F - \tilde{t}_S | \tilde{t}_F = r_q, \tilde{t}_S \in (v_0 - q, r_q]))] = \pi_F^c(r_q)$

The equilibrium values, r_0^*, r_q^*, p^*, stated in Proposition 1 are those that solve the system given by the equations (SIC), (FICB) and (FICG).

Recall that we are assuming that $q \leq q^*$, so that a price that satisfies (FICB) and (FICG) and whose corresponding pair (r_0, r_q) induce bidder S to stay out exists. Call such price p'. To see that (SIC) holds with equality assume by means of contradiction that at p', $\pi_S^c < 0$. Both r_0 and r_q are an increasing and continuos function of p. Similarly, π_S^c is continuos in r_0 and r_q. Further, by assumption (A1) if $p = v_0$, $\pi_S^c > 0$. Hence, there exits some $p \in (v_0, p')$ such that $\pi_S^c = 0$.

One of the characterizing features of the equilibrium in case a) is that $r_0^* - r_q^* > q$. To see why such inequality holds, assume first that p^* induces a pair (r_0^*, r_q^*) such that $r_0^* = r_q^* + q$.

Since in that case type r_0^* in the bad state and type r_q^* in the good state hold the same value for the object, they also make the same profits by placing the pre-emptive bid. However, if bidder S invests, then bidder F with type r_q^* in the good state has less chances to win than type r_0^* in the bad state (as once S gets to know the state only the realization of the private value component matters). Hence, if type r_0^* in the bad state is indifferent between pre-empting and accommodating, type r_q^* in the good state will strictly prefer pre-emption.

Since the left hand side of (FICG) grows faster in r_q^* than the right hand side, any increase in r_q^* from $r_0^* - q$ makes bidder F of type r_q^* even more strongly prefer pre-emption over competition.

By the same argument decreasing r_q^* monotonically decreases the initial imbalance, which proves that if an equilibrium pair (r_0^*, r_q^*) exists, it must be that $r_0^* - r_q^* > q$.

Another result that is easily derivable is an upper bound on q^*, the maximum value of q compatible with a pooling equilibrium. We show that $q^* < 1 - r^0$, where r^0 is the equilibrium threshold when $q = 0$ so that $r_0^* = r_q^* = r^0$.

We argue that at q^*, $r_0^* - r_q^* = 1 - r^0$. The conclusion then follows from $r_0^* - r_q^* > q^*$. Note that the equilibrium price is the lowest that guarantees pre-emption. Since the price is increasing in q (see point i below), the highest q is the one that leaves $t_F = 1$ as the only type to be able to jump under the bad state. In such case, bidder S is virtually sure that the state is good so that the threshold that needs to be signaled under the good state is r^0.[30]

The following lemma shows how the equilibrium parameters are affected by a relative increase of the the common value component, q.

Lemma 2.2. *The effect of a change in q on the equilibrium parameters is the following:*

- *i)* $\frac{\partial p^*}{\partial q} > 0$.
- *ii)* $\frac{\partial r_0^*}{\partial q} > 0$.

- *ii)* $\frac{\partial r_0^*}{\partial q} > 0$.

- *iii)* $\frac{\partial r_q^*}{\partial q} < 0$, *and* $\frac{\partial p^*}{\partial q} < 1$, *for* $q > 0$ *and small enough.*

…m uniform distri-

- *iv)* $\frac{\partial r_q^*}{\partial q} > 0$ *and* $\frac{\partial p^*}{\partial q} > 1$, *for* $q > 0$ *and arbitrarily close to* q^*.
- *v) The ex-ante probability of placing a jump is strictly decreasing in q.*

Proof. i) Suppose that p stays constant as q increases. Condition (FICB) is not affected so that r_0^* is unchanged. Instead, the left hand side of (FICG) becomes greater than the right hand side. To reestablish the equality r_q^* needs to decrease. But then condition (SIC) does not hold anymore. To see that p cannot decrease observe that for any such price the pair (r_0, r_q) signaled with the original level of p was not enough to pre-empt S. The same argument as above excludes that it can be with new value of q.

ii) point i) shows that the price increases with q. Then the left hand side of (FICB) becomes smaller than the right hand side. Thus, r_0^* needs to increase to counteract such imbalance.

iii) To see that for low levels of q, r_q decreases with q, take first $q = 0$. We then have that $r_0 = r_q = r^0$, where r^0 is the level that guarantees pre-emption when $q = 0$. Increase then q marginally to some strictly positive value. We know from ii) that r_0 increases. If r_q were to increase as well, the type signaled would be higher than the one that guarantees pre-emption (r^0), regardless of what the probabilities of the realized state are. Hence, $r_q < r^0$, for any q, which than implies that around $q = 0$, r_q must

[30]Note that r^0 is the threshold that needs to be signaled when there is no uncertainty regarding the state ($q = 0$).

decrease. The second part is implied by the first one. Suppose in fact that p were to increase by more than q. Then the left hand side of FICG would decrease as opposed to a constant right hand side. Hence, r_q would rather increase to reestablish the equality.

iv) This point follows from the fact that for any $q \in (0, q^*)$, $r_q^* < r^0$ (see point iii) and that at q^*, $r_q^* = r^0$. The second part is implied by the first one. Suppose in fact that p were to increase by less than q. Then the left hand side of FICG would increase as opposed to a constant right hand side. Hence, r_q would rather decrease to reestablish the equality.

v) Note that the ex-ante probability of observing a jump is given by $\frac{1}{2} * (1 - r_0^*) + \frac{1}{2} * (1 - r_q^*)$. From point ii) we know that r_0^* is increasing in q. Thus we only need to exclude that the increase in r_0^* could be less than the potential decrease in r_q^*. Suppose they were the same (the same argument applies otherwise). Then, given the uniform distribution, the expected increase in profits for bidder S conditional on the good state would be equal to the expected decrease in profits conditional on the bad state. Since under the jump the good state is more likely, it means that such change in r_0^* and r_q^* is not enough to guarantee pre-emption (the effect on the original probabilities is second order and can be neglected).

□

Point iv) implies the following perhaps slight surprising corollary:

Corollary 2.1. *A marginal increase in q may result in lower expected profits for bidder F even conditional on observing a good state.*

Proof. It follows from r_q^* being increasing in q, for some $q \in (0, q^*)$. All bidders that were not able to pre-empt with the lower level of q are not able to do so with the higher level so that for them the profits are unchanged. However, all types higher than the initial threshold that guarantees pre-emption make strictly higher profits before the increase in q. In fact, the initial threshold type is just indifferent but all higher types enjoy the full rent over the difference between their value and the threshold one, whereas they do not under the equilibrium holding with the higher value of q.

□

Point v) of the lemma above implies the following proposition.

Proposition 2.2. *The bidders and seller's profits vary with the size of the common value component as follows:*

- *The ex-ante expected profits for bidder F are decreasing in q.*

- *The ex-ante expected profits for bidder S are independent of q.*
- *The seller's expected revenues are increasing in q.*

Proof. The first point follows directly if the higher level of q induces both a higher r_0 and a higher r_q (as the set where F can pre-empt is strictly larger and pre-emption guarantees more profits than competition). If instead r_q decreases with q, it follows from point v) and the assumption of uniform distributions (the gains (losses) from having a larger (lower) set of bidders pre-empting rather than allowing competition is just proportional to set of bidder of bidders who can pre-empt).

To see the second point note that by construction an equilibrium jump leaves bidder S indifferent between entering the auction game or not. Thus, for any q we may look at, his profits are the same whether he follows the equilibrium or he always invests regardless of bidder's F action. But if he always invests the expected profits are independent of q.

Finally, the seller's revenues increase as the object in the sub-game analyzed is always sold ($v_F \geq v_0$) and the total surplus that is generated from the auction increases with q (as the expected value for the object is higher). Hence, if bidder's F profits decrease and bidder's S profits are constant, the seller's revenues must increase.

□

Note that the points made above depend crucially on having an exogenous investment cost. We go back to this point later.

case b) : Bidder F knows v_F (partial information case)

This is the scenario where bidder F observes only his value, v_F, but does not know whether the state is good or bad.

To be more precise, given the discreteness of the common value component, a bidder F observing a value $v_F \in [0, q]$ or $v_F \in [1, 1+q]$ is able to infer the state. However, any bidder for which $v_F \in [q, 1]$ cannot make such inference.

Note that, even though bidder F is not aware of the state, he can use Bayes rule to update his beliefs regarding it, by observing $v_F \in [q, 1]$. In general, bidder F retains finer information than bidder S. We want to compare the full information case with the one in which bidder F and bidder S are equally uninformed regarding the state of nature. For case b), we then assume from the beginning that $\tilde{t}_F \sim U[0, 1]$, which guarantees that $Pr(Q = 0|\tilde{v}_F = v_F) = Pr(Q = 0) = \frac{1}{2}$, for any $v_F \in [q, 1]$, as desired.

We again prove the equilibrium by construction by first conjecturing that a pre-emptive equilibrium exists and that all types of bidder F above some threshold do make use of the pre-empive bid, and then showing that this is actually the case. The analysis is largely analogous to case a), we only focus on the differences.

Define the threshold valuation signaled by bidder F by r so that all types $v_F \geq r$ do jump and all types $v_F < r$ do not.

Since bidder S knows that bidder's F threshold type r is unaware of the state of nature, what he infers from the initial bid is that if the state turns out to be bad, bidder F private value component must be at least $r_0 = r$, while if the state is good, bidder F private component must be at least $r_q = r - q$.

In equilibrium the pair (r_0, r_q) will be the one that makes bidder S just indifferent whether to invest and compete or stay out of the game.

Bidder S Indifference Condition for case b) is analogous to the one of case a), except for the fact that $r_0 = r_q + q$.

- (SIC2): $\pi_S^c(r) = \pi_S^p = 0$ iff $Pr(Q = 0|b_1 = p) * E(\pi_S|Q = 0 \text{ and } b_1 = p) + Pr(Q = q|b_1 = p) * E(\pi_S|Q = q \text{ and } b_1 = p) = c_S$,

  where the expressions above are defined as in (SIC), except that $r_0 = r_q + q$

Such pair may not be unique as a higher initial bid not only signals a higher r_0 and r_q but also shifts probabilities from the bad to the good state, under which we have that threshold type is $r_q = r - q < r = r_q$. If there are multiple solutions the equilibrium is given by the one with a lower level of r as it is guarantees higher profits to bidder F for all of his possible types.[31] If the minimum r that solves such equation is strictly greater than one, no pooling equilibrium exists and the next section on the fully revealing equilibrium applies.

For the signal to be credible bidder F with type r needs to be indifferent between pre-empting S or compete with all stronger types strictly preferring pre-emption and all lower types strictly preferring competition.

As opposed to case a), only one indifference condition for bidder F needs to be verified.

The condition we look for, to which we refer to as FIC2, is the following.

- (FIC2): $\pi_F^p(r) = r - p = \frac{1}{2} * [Pr(\tilde{t}_S \leq v_0) * (r - v_0)+$

  $Pr(\tilde{t}_S \in (v_0, r]) * E(\tilde{t}_F - \tilde{t}_S|\tilde{t}_F = r, \tilde{t}_S \in (v_0, r]))] + \frac{1}{2} * [Pr(\tilde{t}_S \leq v_0 - q)*$

[31] There typically exists many equilibria, but the one with lowest r is the only credible one, we postpone the discussion regarding uniqueness for later.

$(r - v_0) + Pr(\tilde{t}_S \in (v_0 - q, r - q]) * E(\tilde{t}_F - \tilde{t}_S | \tilde{t}_F = r - q, \tilde{t}_S \in (v_0 - q, r - q]))] =$
$\frac{1}{2} * \pi^c_{F_0}(r) + \frac{1}{2} * \pi^c_{F_q}(r) = \pi^c_F(r),$

where $\pi^c_{F_0}$ and $\pi^c_{F_q}$ are respectively bidder F profits from allowing competition if the state turns out be bad and if turns out to be good.

Condition (SIC2) determines the minimum threshold level that we define as r^{**} that is high enough to deter bidder S from investing and competing. Given such threshold we can compute the expected profit of a bidder F with such type, $\pi^c_F(r^{**})$. Since $\pi^c_F(r^{**}) = r^{**} - p$, where $r^{**} > \pi^c_F(r^{**}) > 0$, there exists a unique value of p, which we define as $p^{**} = r^{**} - \pi^c_F(r^{**})$, that makes the threshold type, r^{**}, indifferent between placing the pre-emptive bid or accommodating competition.

It is also immediate to see, differentiating with respect to r the left and the right hand side of condition (FIC2), that $\pi^p_F(r)$ grows faster than $\pi^c_F(r)$. Thus, the two values cross only once at p^{**}.[32] We can conclude that p^{**} is the equilibrium pre-emptive price, that all types $v_F \leq r^{**}$ prefer to compete, while all types $v_F > r^{**}$ prefer to pre-empt bidding p^{**}.

Proposition 2.3. *The triple $(b_1^{**}(v_F), d_s^{**}(b_1), g^{**}(b_1))$ constitutes the unique perfect bayesian equilibrium with credible beliefs of the game described above:*

$$b_1^{**}(v_F) = \begin{cases} p^{**} \text{ if } v_F \geq r^{**} \\ v_0 \text{ otherwise} \end{cases}$$
$$d_s^{**}(b_1) = \begin{cases} 0 \text{ if } b_1 = p^{**} \\ 1 \text{ otherwise} \end{cases}$$
$$\mathrm{g}^*(\mathrm{b}_1) = \text{consistent and credible beliefs}$$

As for case a), we can establish a bound on q. We show that $q^{**} \geq 1 - r^0$. Suppose that $q^{**} < 1 - r^0$, and take $r_0^{**} = 1$ and $r_q^{**} = 1 - q^{**}$. We then have that both r_q^{**} and r_0^{**} are greater than r^0, which means that the pair signaled makes the second bidder strictly prefer staying out. Decreasing marginally r_q^{**} would increase q, while still preserving pre-emption. Thus the proposed q^{**} cannot be the maximum q that guarantees the existence of a pooling equilibrium.

The following lemma shows how r_0^{**} and r_q^{**} are affected by an increase in q.

[32] As a matter of fact, any $t_F > r^{**}$ is able to appropriate all of $t_F - p^{**}$ under preemption, while part of such rent is inevitably dissipated by allowing competition.

Lem... *equilibrium parameters is the following:*

- *i)* $\frac{\partial r^{**}}{\partial q} = \frac{\partial r_0^{**}}{\partial q} > 0$.

- *ii)* $\frac{\partial r_q^{**}}{\partial q} < 0$*, for* $q > 0$ *and small enough.*

- *ii)* $\frac{\partial r_q^{**}}{\partial q} < 0$*, for* $q > 0$ *and small enough.*

- *iii)* $\frac{\partial p^{**}}{\partial q} > 0$.

  for the following point, we assume that t_i*,* $i = F, S$ *are taken from uniform distributions. Then:*

- *iv) The ex-ante probability of placing a jump is strictly decreasing in* q*.*

Proof. i) Take some equilibrium pair, (r_0^{**}, r_q^{**}), where $r_0^{**} = r^{**}$. We show that r^{**} needs to increase to keep (SIC) satisfied. If it were to stay constant, r_0^{**} would be constant and $r_q^{**} = r_0^* - q$ would decrease making the good state more likely and the threshold signaled in such event lower, thus signaling a pair of types (r_0, r_q) not high enough to pre-empt S. Since the starting r^{**} was the lowest r to guarantee pre-emption with the original value of q, if we apply the same argument to a lower level of r it is easy to see that the new equilibrium r cannot be lower than the initial one.

ii) See point iii) of lemma 2.

iii) We know from point i) that starting from an equilibrium r and increasing q, r increases. This, as already observed, increases the left hand side of equation SIC by more than the right hand side. Hence, p needs to increase to reestablish the indifference.

iv) See point v) of lemma 2.

□

The same qualitative results as for case a), i.e. proposition 2, follow for case b) and they are not reproduced here.

2.3.1 Fully revealing equilibrium

Before stating the equilibrium conditions for the fully revealing equilibrium we introduce the following lemma.

Lemma 2.4. *The value of* q *for which the fully revealing equilibrium arises is lower in case a) than in case b), i.e.* $q^{**} > q^*$*.*

Proof. Take q^* as the maximum value of q for which a pooling equilibrium exists for case a). We know that the equilibrium thresholds associated with that value of q are such that $r_0^* - r_q^* > q^*$. If the pair (r_0^*, r_q^*) was pre-emptive in case a), it must also be in case b (if SIC is satisfied also SIC2 is). To see that also FIC2 is satisfied simply take $p^{**} = r^{**} - \pi_F^c(r^{**})$, for $r^{**} = r_0^*$. Hence, we have that $q^{**} \geq r_0^* - r_q^* > q^*$, as desired.

□

In the previous section we have illustrated the effect of an increase of q both on the probability of observing a jump and on the players' profits when the value of q was within the range for which the pooling equilibrium is feasible. A natural question that arises is what happens as the type of equilibrium switches to a fully revealing one. This is addressed in the following lemma.

Lemma 2.5. *In general, in case b), a marginal increase in q that switches the equilibrium from a pooling to a fully revealing one causes a discontinuous change both in the ex-ante probability of observing a jump, in the pre-emptive price and in the players' profits. Such a discontinuity is not observed in case a).*

Proof. For case b), unless we are in the non generic case for which $q^{**} = 1 - r^0$, we have that $q^{**} > 1 - r^0$, which means that the threshold r_0^{**} associated with q^{**} must be strictly less than one. To see why note that when $r_0^{**} = 1$, the only possible values of r_q that guarantee pre-emption are greater or equal than r^0, but in that case $1 - r_0^{**} < q^{**}$. We then have that the valuation signaled in a neighborhood of q^{**} changes from $r^{**} = r_0^* < 1$ in the pooling equilibrium to $r^0 + q^{**} > 1$ in the fully revealing one. Since all the other variables we are interested in (price, profits, ex-ante probability of winning) are continuos functions of r^{**}, such marginal change in q also discontinuously affects those variables.

For case a), instead we already noticed that at q^*, $r_0^* = 1$ and $r_q^* = r^0$ so that no discontinuity arises.

□

The fully revealing equilibrium differs from the pooling one as the threshold value signaled is such that $v_F > 1$. We denote the the corresponding private type signaled by r^q. Note that it must be that $r^q = r^0$ as in both cases the state is perfectly known.

The value r^q is the level of r solves the following condition.

- (SIC3): $\pi_S^c(r) = \pi_S^p = 0$, iff $E(\tilde{t}_S - \tilde{t}_F | \tilde{t}_F \in [r, 1], \tilde{t}_S \geq \tilde{t}_F) = c_S$

As for the pooling equilibrium, for such threshold to be credible bidder F holding such type needs to be effectively indifferent among pre-empting or competing with all higher (lower) types strictly preferring pre-emption (competing). The condition we look for is the following.

- (FIC3): $\pi_F^p(r) = r + q - p = [Pr(\tilde{t}_S \leq v_0 - q) * (r - v_0) +$
  $Pr(\tilde{t}_S \in (v_0 - q, r - q]) * E(\tilde{t}_F - \tilde{t}_S | \tilde{t}_F = r - q, \tilde{t}_S \in (v_0 - q, r - q]))] = \pi_F^c(r)$

The same arguments used for the pooling equilibrium ensure that an equilibrium r and equilibrium p exists. Call the equilibrium p in the fully revealing case p^q.

We can then state the following.

Proposition 2.4. *The triple $(b_1^{***}(t_F, Q), d_s^{***}(b_1), g^{***}(b_1))$ constitutes the unique perfect bayesian equilibrium with credible beliefs of the auction game described above:*

$$b_1^{***}(t_F, Q) = \begin{cases} p^q \text{ if } \mathrm{Q} = \mathrm{q} \text{ and } \mathrm{t_F} \geq \mathrm{r^q} \\ v_0 \text{ otherwise} \end{cases}$$

$$d_s^{**}(b_1) = \begin{cases} 0 \text{ if } b_1 = p^q \\ 1 \text{ otherwise} \end{cases}$$

$$\mathrm{g}^{**}(\mathrm{b}_1) = \text{consistent and credible beliefs}$$

We also have the following lemma.

Lemma 2.6. *The effect of a change of q on the equilibrium parameters is the following:*

- *i)* $\frac{\partial r^q}{\partial q} = 0$.
- *ii)* $\frac{\partial p^q}{\partial q} = 1$.

Proof. The first point follows from the fact that (SIC3) is independent of q. Given such fact, everything else but q and p is fixed in (FIC3). Hence, to keep the equilibrium indifference condition holding, all the increase in q must be fully absorbed by an increase in the price.

□

A corollary result is the following.

Corollary 2.2. *The bidders and sellers' profits vary with the size of the common value component as follows:*

- *The ex-ante expected profits of bidder F are independent of q.*
- *The ex-ante expected profits of bidder S are independent of q.*
- *The sellers expected revenues are increasing in q.*

Proof. Bidder's F ex-ante expected profits depend only on r^q. Since r^q is independent of q, so are the profits. Bidder's S ex-ante profits are independent of q for the same reason as in the pooling equilibrium. All the increase in the object value given by the increase in q is then fully appropriated by the seller. Hence, the third point.

□

2.4 More vs. less information (case a vs. case b)

The following lemma illustrates how the pair (r_0^*, r_q^*), (r_0^{**}, r_q^{**}) are related. It will be key in assessing the comparative statics analysis that follows.

Lemma 2.7. *The threshold pairs (r_0^*, r_q^*), (r_0^{**}, r_q^{**}) are related as follows.*

- *i) $r_0^* > r_0^{**}$*
- *ii) for any $q^{'} \leq q^*$, such that (r_0^*, r_q^*) are the equilibrium thresholds signaled in case a), there exists a level of q, $q^{''} \leq q^{**}$, such that (r_0^{**}, r_q^{**}) are the equilibrium thresholds signaled in case b) and $r_0^{**} = r_0^*$, $r_q^{**} = r_q^*$, $r_0^{**} - r_q^{**} = q^{''}$.*

Proof. i) Suppose $r_0^* \leq r_0^{**}$. Either $r_q^* \geq r_q^{**}$ or $r_q^* < r_q^{**}$. The first case cannot hold true as it implies $r_0^* - r_q^* \leq q$.

To see why the second cannot hold either, note that by assumption the pair (r_0^*, r_q^*) pre-empts bidder S from competing in case a) and that $r_0^* - r_q^* > q$. But then take $r_0^{'} = r_0^*$ and $r_q^{'} = r_0^* - q$. If the pair (r_0^*, r_q^*) was pre-emptive, the pair $(r_0^{'}, r_q^{'})$ must also be as it differs only because of $r_q^{'} > r_q^*$, which makes the pair even "more pre-empting" as a higher value of r_q both shifts the probabilities towards the bad state and makes competition tougher in the good state. Hence, by continuity there exists a pair $(r_0^{''}, r_q^{''})$, with $r_0^{''} < r_0^{**}$ and such that pre-emtion is still ensured, which contradicts that (r_0^{**}, r_q^{**}) is the lowest pair inducing pre-emption for case b).

ii) Note first that for any given value of r_0 there exists at most one value of r_q for which (SIC) and (SIC2) are satisfied. This is so as we have seen that a lower level of r_q strictly increases bidder S ex-ante expected profits and vice versa a higher level strictly decreases them. Moreover, both r_0^* and r_0^{**} are a continuos increasing function of q. So any particular value of r_0 greater than r^0 and lower than the maximum value compatible with a pooling equilibrium forms an equilibrium pair (r_0, r_q) for some value $q^{'}$ for case a) and for some $q^{''} < q^{'}$ for case b).

□

Point ii) of the above lemma leads us to the following proposition.

Proposition 2.5. *If bidders private value are taken from uniform distributions, then case a) and case b) can be unambiguously ranked as follows. For any given $q \leq q^{**}$, the ex-ante probability of observing a jump and the ex ante expected profits of bidder F are higher in case b) than case a). The ex-ante expected profits for bidder S are invariant; finally, the seller's profits are higher in case a).*

Proof. Note that once we know that a pair (r_0, r_q) is an equilibrium one, we can recover from that all other variables we are interested in. Point ii) above tells us that the threshold pair and hence all other variables of interest for case b), for some value q', are the same as the one for case a), for some $q'' < q'$. The conclusion then follows because all relevant variables at q' and q'' can be unambiguously ranked as shown in Proposition 2. □

2.5 Jump Bidding and Information Acquisition

In this section we discuss the effect of jump bidding on the incentive to acquire information and in turn on the format's revenues and efficiency.

It is interesting to analyze the incentives to acquire information both for the first bidder and the second bidder. Here we assume that both need to invest to find out their values. As before, the first bidder can do so before the second bidder (call his investment cost c_F). Fishman (1988) argues that the possibility of jump bidding should be banned as it reduces the seller's revenues. He also points out that efficiency is not affected. Such conclusions are based on the implicit assumption that the investment cost is low enough so that both bidders invest when jump bidding is not allowed.

However, not only the first bidder expected profits are higher under the jump bidding equilibrium, but also and the second bidder's profits conditional on not observing a jump are. This means that the investment cost may be low enough to allow the first bidder to participate in the auction if jump bidding is allowed, but not if it is forbidden (and similarly for the second bidder). Hence, the finer information that is brought to the second bidder in the jump bidding equilibrium can potentially increase efficiency and revenues. We first focus on efficiency.

2.5.1 Efficiency

We distinguish two cases: the one in which the investment cost is low enough so that a bidder would invest also if jump bidding is forbidden, and the one in which if jump bidding is not allowed the bidder does not invest.

Let us start with the first case and look at how bidder's S participation decision affects efficiency.

We argue that efficiency is not affected in the first case. To see why note that bidder's S expected profits if he competes coincide with the expected gain in social surplus (given by the increase in value of always allocating to the highest valuation bidder). Further, the jump bid is constructed as to equate bidder's S investment cost to his expected profits from investing and competing. Hence, the social costs of misallocating the object is cancelled out by the social gain of avoiding a sunk cost (bidder's S investment cost).

Let us then move to the second case. It is easy to see that the jump bidding equilibrium increases efficiency as under no jump bidding bidder S is always out of the game, while with jump bidding he enters when it is strictly profitable for him to do so (and we noted that his private interest coincides with the social one).

A similar analysis holds true for bidder F. The same arguments used above shows that jump bidding does not affect efficiency if the investment cost is in the range implied by the first case. As for the second case, if the first bidder does not participate there is a clear loss in efficiency. Note that to be precise on the consequences of bidder F non participation, we need to specify whether bidder F is then out of the game or he can participate after the other bidder has potentially invested and placed a first bid. We might have scenarios in which none of the bidders is willing to invest first as the expected profits are too low, which would leave the object unsold.

We have shown that for some ranges of bidder's S profits efficiency is unchanged, while for others strictly increases. We can conclude that jump bidding, if anything, is beneficial as far as efficiency is concerned.

2.5.2 Revenues

The effect of jump bidding on revenues depends on the value of the investment cost we look at. As a matter of fact, in those scenarios where efficiency is increased also revenues are (the participation of an additional bidder clearly raises revenues). However, when the bidders' investment costs are low enough that they both invest when jump bids are prohibited, the seller's revenues are strictly lower under jump bididing. This point is already pointed out in Fishman (1988).

To see why first note that no difference arises if the first bidder is below the equilibrium threshold as in both cases the second bidder invests and competes. The equilibrium threshold type is indifferent between pre-empting or not. Under the first case he wins with probability one, under the second with probability less than one. This means that the equilibrium jump bid is such that the expected foregone profits of winning at an average price lower than the pre-emptive one, it is equal to the expected gain of winning at the pre-empt price instead of loosing when the second bidder has a higher valuation. Note that the expected gains (losses) for the threshold type of the first bidder are expected losses (gains) in terms of revenues for the seller. Hence, if the seller were to know that the first bidder type is the threshold one, his expected revenues would be the same with or without jump bidding. However, for all higher types of bidder F, the seller experiences a decrease in revenues.

Finally, notice that in the analysis above we have not considered the effect of a lower or higher common value component. However, it can simply be recovered from the analysis made earlier. The higher q, the lower the ex-ante expected profits for bidder F and the less the scenarios (values of c_F) for which there may be an improvement in efficiency and revenues. Conversely, a higher value of q does not affect the expected profits of bidder S conditional on not observing a jump and so there is no change in efficiency due to that aspect.

2.6 Remarks on the model

We discuss in this section some of the modeling assumptions made in the paper and address related issues. We start discussing the uniqueness of our equilibrium.

2.6.1 Uniqueness

We stated that the equilibrium provided is the unique one compatible with credible beliefs. We remand to Fishman (1988) for the formal details and we provide instead a more informal discussion here. The formal argument is based on an adaptation of the equilibrium refinement proposed in Grossman and Perry (1986).

It is easy to see that there are many other potential equilibrium candidates. In fact, they are all Perfect Bayesian Equilibria, but they are based on non credible beliefs. They resemble the one we presented in the sense that they share the same structure. However, they all involve a higher equilibrium pre-emptive price. Recall, in fact, that we took the minimum price that enabled to equate bidder's S expected profits to his investment cost. So for any such p, we have that $p > p^*$. This higher price needs to be chosen so that the pair of (r_0, r_q) signaled makes bidder S willing to stay out. Assuming that we are in case a), this means that $r_0 > r_0^*$ and $r_q > r_q^*$. Choosing bidder's S out of equilibrium beliefs so that bidder S is induced to compete against any lower initial bid, any such equilibrium can be easily constructed.[33]

Those equilibria are not credible in that they do not pass the following credibility check that restricts bidder's S out of equilibrium beliefs. Suppose that instead of observing p, S observes a lower bid, for instance, our p^*. Then he should ask himself, which types of bidder F could have possibly preferred to pre-empt at p^*, rather than allowing competition. We know by our construction that optimal response for bidder S, had he made such introspection, it is to stay out. But then bidder F would rather bid p^* then p. It is easy to see that p^* is the unique price that is compatible with such type of beliefs.

2.6.2 Choice of value function

We have assumed a mix of a private and a common component adopting an additive separable specification for the bidders' value functions. The common value component has been chosen to take a point-wise distribution with a low and high realization being equally likely. The fact that the two values are assumed equally likely does not affect the analysis. A higher rather than a lower probability of a good state would simply increase the bid that is needed to pre-empt S, but would not change the qualitative features of the equilibrium.

The assumption that the common value can only take two values is also with little loss of generality. The one thing that would change were we to assume a continuous distribution, it is that the fully revealing equilibrium would disappear. However, increasing the maximum value that the common value component can take (i.e. enlarging its support) has an analogous effect to an increase in the value of q in the model presented. To have an intuition for this, note that the continuos distribution can be taken as the limit of a discrete one with an arbitrarily large number of states. Adding an extra state to our model requires considering an extra threshold private value type, but does not affect the workings of our equilibrium.

2.6.3 No ex-ante information

The fact that bidder S holds no private information is made to simplify the analysis. We wanted to study the effect of jump bidding when a bidder had a better information regarding a common value component in the simplest possible setting. The main

[33]The same would hold in case b).

insights would go through allowing for some initial private information. Adding private information would give rise to some partial pre-emption as illustrated in a pure private value model by Bhattacharyya (1992).

An additional complication would instead arise having the first bidder invest holding already some private information. In such a case, if the acquisition of information acquisition is observable, the act of investing can signal something about the private value of bidder F and therefore limit the need for jump bidding. Vice versa, if the information is covered (not observable), jump bidding may serve also to signal that the initial private value was high.

2.6.4 $n > 2$ bidders

Another direction in which the model could be extended is to have the better informed bidder having an informational advantage with respect to a set of $n - 1$ symmetric bidders rather than just one. In this case the informed bidder needs to signal a high enough value such that the expected profits for any of his opponents conditional on having a value higher than the informed bidder and the other n-2 initially symmetric bidders just equals the investment cost. Note that having $n > 2$ bidders also allows to weaken assumption (A1). That assumption was made to ensure that accommodating would not result to be always preferred to pre-empting. Having more bidders decreases the expected profits of accommodating, and therefore decreases the value of v_0 required.

2.6.5 Uncertainty over c_S

In this paper we have assumed that the investment cost of the second bidder, c_S was common knowledge. We then have discussed how efficiency and revenues could differ (whether or not allowing for jumps) if different value of c_S were considered (section 5). What if bidder's F only knows the underlying distribution of c_S but not its exact realization? The effect of bidder's S holding private information over his investment cost on bidder's F bidding is similar to the one of him holding private information over his valuation. If the investment cost is bounded away from zero there will still be a fraction of bidder's F type that will place a fully pre-emptive bid. However, intermediate types of bidder F will pre-empt bidder S only if the investment cost that this latter one observes is above some threshold. Those intermediate types do not pool on the same initial bid, rather they use a strictly increasing bid function. The uncertainty over c_S, weakens the incentive to jump bid. To see why note that the threshold type that is able to jump bid needs to increase compared to the case we studied. The reason is that here, by jumping, he fully reveals his value, while when there is no uncertainty with respect of c_S, he can pool with higher types. Another important consequence is that bidder's S expected profits are no longer independent of c_S. This is likely to affect our analysis regarding the format' s revenues and efficiency properties. A careful analysis of this is left for future research.

Chapter 3

Hiding information in open auctions

3.1 Introduction

Open auction formats allow bidders to observe one another's bidding behavior in the course of the auction. Thanks to this dynamic feature bidders can learn extra pieces of information compared to sealed formats and can update their expected valuations for the object on sale accordingly.

The possibility of aggregating extra information affects bidders' strategic behavior and ultimately the efficiency and revenue generating properties of a format.

The standard literature either implicitly or explicitly assumes that bidders cannot alter the aggregation of information or analyzes settings where it is simply not profitable.

In particular, it is often assumed that the price is continuously raised by some exogenous device that only allows to stay active or to drop-out but not to call a price. Such format is known in the literature as Japanese (or clock) auction. This assumption greatly simplify the analysis as it restricts a bidder's options to the fairly simple choice of deciding whether to stay active or to quit at any given price. However, the dynamic aspect of the open format is no longer fully captured.

If we look at practical application, we can verify that in the vast majority of open formats the price is not raised in a continuous fashion, but rather discontinuously either by the auctioneer or through price calls made by the bidders.

The present paper covers the second possibility by looking at a modification of the Japanese auction that allows to call a price. When a bidder stops the clock to call a price, we say that such bidder has placed a *jump bid*. The possibility of calling a price provides bidders with an instrument to affect the quality of the information that is transmitted during the auction. In fact, if some bidder does not match a price call, only the coarser information that his drop-out value is within some price interval will be available. In an interdependent value setting the precision of the information regarding the opponents' exit values influences the expected valuations of the active bidders and therefore their exit decisions.

Since open auctions are very commonly used and they often allow to raise the price discontinuously, it is of great economic importance to understand how such possibility can be used in a strategic way.

Moreover, the analysis of open auctions is often relevant beyond those mechanisms that can be formally described as auctions. An example that is often given in the literature is the one of takeover battles, but most economic environments in which agents compete in price do also fit.

One of the typical situations we would like to represent is the following. Suppose that a novice bidder is competing with a set of more experienced bidders. To fix ideas on a simple economic problem, imagine that the novice bidder is competing for the allocation of a second hand car and that the other buyers are car dealers.

The novice bidder really likes the car while the car dealers care only about the resale price and have heterogeneous estimates regarding this latter value.

His valuation is not the highest, however, if some repairs are needed as those are more costly to him. The key assumption is that the active presence of many car dealers at high prices signals to the novice bidder that the car is in good condition and allows him to infer that he is the buyer with the highest valuation. Here is where the possibility of calling a price can be exploited. One of the expert bidders that has a high valuation for the car may place a fairly high bid at an initial stage to hide the drop-out values of other expert bidders. The idea is that the jump pre-empts the novice bidder from acquiring finer information and leaves him uncertain about the real quality of the car, thus affecting his bidding strategy. We formalize this intuition and show when it is actually profitable in equilibrium to place a jump and why.

We also show that allowing bidders to call a price does not always have an anticompetitive effect as the above scenario might suggest. As a matter of fact, the possibility of calling a price makes the strategic environment so rich that the effect not only on revenues but also on efficiency is in general ambiguous. We highlight some details of the environment that may lead to a certain outcome rather than another.

There are two main related strand of the literature to this paper. On the one hand, there are many papers pointing out the advantages and disadvantages of using an open ascending price auction rather then a sealed format. Our paper stresses that the possibility of calling a price is a crucial factor determining the performance of an open auction. Therefore when deciding which format to opt for, such a possibility should be carefully evaluated.

On the other hand, ours is not the first work that allows bidders to call a price and place jump bids. Yet, the previous models in the small literature on jump bidding provide a rational for such phenomena based on signaling rather than "hiding" information.[1]

A priori it is not clear whether the fact that bidders receive extra information during the auction should be beneficial for the seller's revenues or for the format efficiency.

The first strand of literature clarifies the effect of receiving finer information on revenues and efficiency in some subclasses of auction games.

Milgrom and Weber (1982) symmetric setting with affiliated valuations revenue ranks English Auction (EA), Second Price Auction (SPA) and First Price Auction (FPA) in this order.[2] As soon as some asymmetries are introduced however the comparison becomes ambiguous.

From a social welfare viewpoint, it has been shown (Krishna (2003), Birulin and Izmalkov (2003)) that if some technical *single crosssing conditions* are satisfied full efficiency is achievable by means of a standard Japanese auction.[3] When those conditions do not hold though the Japanese auction is in general not efficient.[4]

A strategic issue that is relevant in some applications[5] is that in an open format a bidder can invest in order to get a finer estimate about his valuation as the auction unfolds, while in a sealed bid format such decision needs to be taken once and for all at the start. Compte and Jehiel (2007) show, in a private value setting, that an open format by endowing bidders with more information regarding the strength of competition induces more information acquisition that in turn, under general circumstances, translates into higher revenues and efficiency.

[1]See also an independent work by Ettinger (2006).

[2]Milgrom and Weber (1982) consider the Japanese version of the English Auction, where jumps are not allowed.

[3]Maskin (1992) provides the result for the $n = 2$ case. There though the Japanese auction is strategically equivalent to the sealed bid SPA so that the distinction between open and sealed format as far as information aggregation is concerned is not as crucial as for $n > 2$.

[4]See Hernando-Veciana and Michelucci (2007) regarding when the Japanese auction is second best efficient.

[5]In particular, auctions or competitive situations that last sufficiently long. An example of this kind that is often referred to in the literature is the one of takeover contests.

Allowing bidders to observe each others' behavior has also drawbacks. It is generally argued, for instance, that open formats are more subject to collusion as a colluding cartel can easily observe when one of its members breaks the agreement and consequently can punish such deviator.

Furthermore, open formats leave a *weak* bidder with little chance to win. If there are participation costs this leads to low entry to the auction, which in turn suppresses revenues.

The second line of research that is related to ours is the one which considers models that allow for jump bids.

Two main rationalizations of jump bidding have been advocated. Both are based on a signaling motivation.

The first one was suggested by Fishman (1988).[6] He presents a two-bidder independent private value model in which one of the two bidders has an informational advantage in that he is able to costly discover his valuation prior to the start of the auction, while the other bidder does not. If the first bidder's value is above some critical threshold, a jump bid that pre-empts the second bidder from investing and competing is placed.[7] The effect of a jump in this setting is anticompetitive and reduces the seller's revenue. Essentially this model introduces an entry deterrence scenario, which requires costly information acquisition to work. This intuition fits very nicely some applications. However, it does not provide a general theory of jump bidding.

The other justification has been proposed by Avery (1998). He shows in a symmetric model with affiliated valuations that jump bidding can be employed by bidders to select during a first stage who among them is the strongest bidder.[8] The signaling induces asymmetric bidding behavior in the second stage of the game with a strong bidder committing to a more aggressive strategy than a weak bidder. Such equilibrium behavior can be viewed as the author points out as a form of implicit collusion.

The model however cannot predict which of the many signaling equilibria that can constructed should be the one that is most "reasonably" played. Further, even admitting that the bidders can somehow coordinate on using the same system of messages, it is not clear to us what provides the credible commitment device to the asymmetric strategies in the unfolding of the game.

[6]See also for related works Hirshleifer and Png (1989) and Bhattacharyya (1992) and Bernhardt and Scoones (1993).

[7]This model is in private values. For a model that looks at the incentive to jump when the finer information involves a common value element see Michelucci (2007b).

[8]See for instance Milgrom and Weber (1982) for the concept of affiliation.

We find the signaling explanation more convincing in a multi-object auction as there bidders could coordinate on which object to focus their collusively efforts in order to split the "pie of profits". The adoption of signals conveying information of the type "if you do not bother me on this object, I will not bother you on the one you are interested in" could potentially provide a robust collusive device.

Conversely, in a one object setting a bidder is ultimately going to take the whole pie. Why should he be given the chance of enjoying higher profits?

In an affiliated value setting the information provided by a jump induces bidders to revise their expected valuation upwards. Why should bidders revise their exit values downwards? As a matter of fact, Proposition 5.4 in Avery (1998) states that the equilibrium is not robust to perturbations[9] of the second stage of the game.[10]

The explanation we propose does not suffer from the issues raised above as it is based on a different rational.[11]

In an open auction when valuation are interdependent a bidder constantly revises the estimate for his own valuation as well as for the maximum exit value of his opponents (conditional on him being the winner). The estimate revision is based on the observation of who is still active and who is not at any given price.

As more information gets aggregated, the ratio of those two values changes.

In particular a bidder may anticipate that the information that is going to be aggregated is going to boost the expected maximum exit value of his opponents much more than his own valuation.

A jump cannot prevent the other bidders from acquiring further information, but it does provide them with a coarser information. A bidder may benefit from all bidders (including himself) aggregating a coarser information as this may in expected terms increase his valuation relatively to the second highest one.

Hence, rather than to scare his opponents as in Avery (1998), here the jump prevents them from getting the finer information they would need in order to be competitive.

Also in Fishman (1988) there is some pre-emption of information. In his model

[9]It is not robust to small trembles, i.e. if the strong type erroneously quits at lower prices it is strictly optimal for the weaker bidder to stay active till the value at which he would break even given his updated expectation regarding the value of the object. Such value is strictly greater than the one prescribed in the proposed equilibrium.

[10]Furthermore, in some applications the bidders are firms that subsequently compete in the same market. As pointed out by the literature on financial externalities, a firm that does not manage to get the resource on sale for itself could nonetheless try to make the winner pay the highest possible price to more severely affect its financial resources.

[11]An independent work by Ettinger (2006) also presents two examples characterized by this motivation.

though a bidder is pre-empted from buying information. No information acquisition is needed in our model. More fundamentally, there it is the signaling that induces pre-emption of information and not the mere act of placing a jump.

Compared with the standard setting in which discontinuous increases in the price are not allowed, the game where jump bidding is allowed is characterized by larger strategy sets. This makes the setting truly dynamic, more realistic as far as many applications are concerned, but also far richer and complex from a strategic point of view. The cost we need to pay for not looking at a "well behaved" environment is that we cannot have a framework as general as Avery (1998) and have clear cut predictions. Our objective though is precisely to start shading some light on what may happen when we consider a dynamic environment and to stress how apparently small differences from one setting to another may completely overturn the predictions in terms of revenues and efficiency.

We use the following structure. In section 2 we present the general setting and the auction rules. In section 3 we analyze why jumps may arise in equilibrium and how they affect the aggregation of information. In section 4 we introduce a few more strategically complex scenarios to illustrate the rich strategic implications that can arise once bidders are allowed to call a price. Section 5 presents a completely symmetric setting. Finally, section 6 concludes.

3.2 Auction setting

3.2.1 Environment

In the present paper we analyze a modification of the Japanese Auction (JA), which aims to capture the truly dynamic features of the English Auction (EA) that cannot be represented when adopting the standard JA format.

A set N of $i := 1..n$ bidders is present at the start of the auction. No further entry takes place after the auction has started and the decision to exit the auction is irreversible. Bidders valuations are interdependent, i.e. $v_i(t_i, t_{-i})$, where v_i is bidder i's value function, $t_i \in T_i$ is the private signal bidder i is endowed with at the start of the auction and $t_{-i} \in T_{-i}$ represents the vector of the other bidders' signals, i.e. it is a short cut notation for the vector $(t_j)_{j \neq i}$. While the information about t_i is private to bidder i, the value functions v_i as well as the cumulative distribution functions, $F_i(t_i)$, from which the signals t_i are independently drawn, are common knowledge among the bidders.

We assume to depart the least possible from the standard assumptions made in the auction literature that the function $v_i(.)$ is strictly increasing in t_i and weakly increasing in t_j for any t_j. This assumption if anything imposes some regularity to the setting that works against the natural arise of jumps. We show in the last section that in such "standard" setting no jump can arise in equilibrium when we restrict to symmetric settings.[12]

Moreover, even though a hiding information motivation and a signaling motivation may coexist and often will, we do not consider such possibility here as we want to isolate the effect that a jump may be used to pre-empt the acquisition of finer information. Further, the point that jump bidding can be used to signal a strong type has already been made elsewhere. This is achieved by assuming that after a jump has hidden some information any bidder bids up to his expected value given the information that can be gathered exclusively by observing which bidders are active at any given price p, that is we exclude that a bidder may quit before his break even price is reached as a result of some other bidder signaling to have a higher valuation.[13]This is essentially a restriction on bidders' beliefs.

Finally, we assume that there are no bidding or entry costs. Again allowing for a signaling explanation or for bidding costs would only make it easier to construct equilibria where a jump can profitably be used.

3.2.2 Auction Rules

The rules of the auction are as follows. The price starts from a very low value and it is increased at a constant pace by an exogenous devise such as a clock.[14] Bidders are considered as active only if they are currently pressing a button. At any point in time, i.e. at any price p indicated by the clock at a specific instant of time, each bidder i faces a decision problem with three alternatives: exit at p by releasing the button, remain active by keep pressing the button and, finally, call a price. The identity of the bidders who quit is publicly revealed so that a bidder knows exactly against whom he

[12]The fact that the signals are unidimensional is a restriction. Our main insights should nonetheless hold in the more complex multi-dimensional setting.

[13]This excludes that a jump could be used with a signaling motivation, so if it is used it must certainly be due to the hiding motivation. In general, though, we should take care of the fact that a jump bid by bidder i, as a side effect of hiding information, may convey information regarding his type that makes the jump less profitable. Such tradeoff is not present in the settings we look at as in most of them the signal of the bidder who jumps does not affect the opponents' valuations.

[14]The starting price is low enough so that all bidders wish to participate. We normalize this value to zero.

is still competing. Using his third option a bidder can interrupt the exogenous price increase and the clock is then stopped at the price indicated at that time and then reset at the price that has been called. In case more than one bidder simultaneously stop the clock, the identity of the bidder to whom the right of calling the price is assigned is randomly selected by the auctioneer. We refer to the k-th jump arising in equilibrium as J_k^i, $k : 1..K$, where i is the identity of the bidder placing the k-th jump.[15] Such jump is defined by the pair (p_k^*, p_k^{**}), where p_k^* is the price at which the clock is stopped and $p_k^{**} > p_k^*$ is the price that is called. We can then represent bidder i's decision at p by $a_i(p) \in \{exit, active, p_k^{**}\}$. The jumps are ordered so that $k'' > k'$ iff $p_{k''}^* > p_{k'}^{**}$. Let $k(p)$ instead be the cardinality of the set of jumps that took place up to price p. After a jump J_k^i all the bidders that were active at price p_k^* need to independently decide whether they want to be active also at price p_k^{**}; the identity of the bidders who do not match the jump is publicly revealed. The auction ends either when a price is called and no other bidder matches it or when in the continuous price increase phase the last but one bidder quits. In the first case the winning bid is given by the price that was called, in the second by the the price at which the last but one bidder exited.

3.3 Analysis of Jumps

There is a general wisdom that price jumps are used by bidders as anticompetitive instruments that typically reduce efficiency and revenues and advantage only the bidder which makes use of them. We show that this is not true in general. As letting more information aggregate may not always be a good idea, similarly altering the precision of the information that bidders receive may not always have negative effects. We also illustrate that even if no jumps occur in equilibrium, the equilibrium outcome can be severely affected by the mere possibility of calling a price. Conversely, if a jump is observed in equilibrium, we cannot infer that the bidder who placed the jump is better off than in the game where jumps are not allowed. We introduce a few of these more strategically sophisticated settings later on, for now we start with the simpler scenario presented in the introduction.

[15] The act of calling a price is publicly observable, hence the identity of the bidder who places the jump is common knowledge.

3.3.1 A very stylized example

Let us start by modelling in the most stylized way the situation described in the introduction.

Recall that a novice bidder was assumed to compete against a group of car dealers for the allocation of a second hand car and that the active presence of many expert car dealers would signal him that the car is of good quality (scenario under which he holds the highest valuation).[16] The following value functions capture the main ingredients of such scenario.

- $v_{C_1} = t_{C_1}$, $t_{C_1} \in \{\frac{1}{4}, \frac{4}{5}\}$, $Pr(t_{C_1} = \frac{1}{4}) = Pr(t_{C_1} = \frac{4}{5}) = \frac{1}{2}$
- $v_{C_2} = t_{C_2}$, $t_{C_2} \in \{\frac{2}{5}, \frac{1}{2}\}$, $Pr(t_{C_2} = \frac{2}{5}) = Pr(t_{C_2} = \frac{1}{2}) = \frac{1}{2}$
- $v_N = t_N$, if $(t_{C_1}, t_{C_2}) \neq (\frac{4}{5}, \frac{1}{2})$; $v_N = t_N + 1$, otherwise; $t_N = 0$

Note that in the event that $t_{C_1} = \frac{4}{5}$ the first car dealer (C_1) holds a higher valuation than the second car dealer (C_2), but potentially a lower valuation than the novice bidder (N).

In such scenario C_1 has an incentive to hide the exact drop out value of C_2. To see why note that C_1 anticipates that if he does not stop the clock his profits are equal to $\pi_{C_1} = \frac{1}{2} * (\frac{4}{5} - \frac{2}{5}) + \frac{1}{2} * 0 = \frac{1}{5}$ (as half of the times he looses against N). Instead, by stopping the clock at any price strictly lower than $\frac{2}{5}$ and calling a price of $\frac{1}{2}$, his profits are equal to $\pi'_{C_1} = 1 * (\frac{4}{5} - \frac{1}{2}) = \frac{3}{10} > \frac{1}{5} = \pi_{C_1}$. This is so as the novice bidder remains uncertain about the quality of the car and he is forced to drop out when the price reaches his expected valuation, i.e. at $E(\tilde{v}_N | \tilde{t}_N = 0, \tilde{t}_{C_2} \in \{\frac{2}{5}, \frac{1}{2}\}) = \frac{1}{2}$. C_1 then wins with probability one and pays $\frac{1}{2}$, which makes the jump bid profitable.[17]

Before exploring when and why placing a jump may be profitable, let us have a closer look at how the possibility of calling a price affects the aggregation of information.

3.3.2 Information Aggregation and Jumps

The underlying assumption in this paper is that once the auction has started the only additional information to the private one that a bidder receives at the start is given by observing who is still active and who is not.

In other words we exclude that a jump can be used to communicate the bidder's own type to the opponents. An implication of this is that even though the overall information

[16] The focus on the car market is purely illustrative.

[17] We use the tilde sign to denote random variables.

publicly available includes also the jump bids that were matched by all of the bidders active at the time of the jump, those bids do not affect the equilibrium behavior.

The information that is publicly available at price p is assumed to be common knowledge and we represent it using the vector $h(p)$, where $h(p) \in H(p)$ can also be viewed as the realized history up to price p, out of the set of all possible histories $H(p)$. To be more precise $h(p)$ is a vector with n entries, $h_i(p)$, $i : 1..n$, where $h_i(p)$ represents the information regarding bidder i that is publicly available at p.

Each entry $h_i(p)$ consists of three items, $(J_i(p), J_k^j, d_i(p))$. The first item is a vector containing all the jumps placed by bidder i up to price p, i.e. $J_k^i \in J_i(p)$ iff $k \leq k(p)$; $J_i(p) = \emptyset$, if bidder i has yet to place a jump.

The second entry records the jump by some bidder $j \neq i$ that is not matched by bidder i that consequently exits the auction (if bidder i is still active or has dropped while the price was being raised continuously such entry displays the value $\emptyset$).

To the third entry we assign value p if bidder i has not dropped yet at price p; value p_i if bidder i has dropped at some price $p_i < p$; finally, value p_k^*, if all that is known is that bidder i was active at p_k^* but did not match the price call p_k^{**} made by bidder j. Clearly, $d_i(p)$ is strictly increasing until bidder i drops out and then stays constant. Taking $\overline{p}$ as an upper bound on bidders' highest possible valuation and assuming that no rational bidder would stay active over $\overline{p}$, we can define d_i over the closed interval $[0, \overline{p}]$. Then, $d_i(\overline{p})$ identifies bidder i's drop-out value. Also, let $d_m \equiv max_i\,(d_i(\overline{p}))$ be the highest drop-out value (or in other words the realized winning price).

We denote bidder's i optimal strategy by s_i^*. In general a strategy s_i needs to specify an action $a_i(p)$ at any price and for any possible realized type t_i and history $h(p)$, i.e. $a_i(p)$ establishes a mapping from $T_i \times H(p)$ into $\{exit, active, p_k^{**}\}$. All bidders' actions are taken simultaneously and the resulting new information produced is immediately updated in h as follows.

If $a_i(p) = quit$, then for all $p' > p$, $d_i(p') = p < p'$; if $a_i(p) = active$, then $d_i(p') = p'$, for all $p' > p$ until a different information becomes available; finally, if $a_i(p) = p_k^{**}$, then J_k^i is stored in the first entry of $h_i(p')$ for all $p' > p$.

Conversely, if p is the price that was called by some bidder $j \neq i$ and bidder's i decision is *exit* then $d_i(p) = p_k^*$, for $p \geq p_k^{**}$; if instead bidder's i decision is *active*, then $d_i(p) = p$, until a new information becomes available.

Note that $H(p)$ records simply who is still active and who is not at price p, what is known of the drop-out values of the bidders who already quitted and all previous histories of jumps. A different object is the information regarding each others valuations that the bidders can infer from such vector.

With this respect, let $D^i_j(t_i, h(p), s_i) \equiv E(d_j(\overline{p})|\tilde{t}_i = t_i, h(p), s_i)$, be the expected drop-out value of bidder j evaluated at p, by bidder i, given his private information, the information publicly available, and his strategy s_i. As a matter of fact, the expected drop-out value of bidder j depends also on the information that he is able to aggregate during the auction and such information can be manipulated by bidder i through s_i. The vector $\left(D^i_j(t_i, h(p), s_i)\right)_{j \neq i}$ represents the relevant information that bidder i can infer regarding his opponents at price p. Bidder i in particular is interested in the expected maximum exit value of his opponents, which we denote by $D^i_m(t_i, h(p), s_i)$. In fact, among all possible strategies he should pick s^*_i as the one that maximizes at any given p the expected difference between his value and the maximum exit value of his opponents (conditional on the latter being lower than former). This should start to give some ideas about the complexity we may run into, were we to analyze a fully general set-up. We do not tackle such very complicated dynamic game here, instead we focus our attention to a few selected settings. Even restricting to very simplified scenarios, the implication that we obtain are quite rich.

To clarify the effect of a jump on the vector $H(p)$, suppose that bidder i is contemplating to jump from p^*_k to p^{**}_k and assume for simplicity that if i does not place the jump no one else will before the price reaches p^{**}_k. The jump affects only those entries for which $d_i(p^*_k) = p^*_k$. The value that is assigned to some of those entry at p^{**}_k if the price is raised continuously and some bidder $j \neq i$ quits a price $p_j \in (p^*_k, p^{**}_k)$ is $d_j(p^{**}_k) = p_j < p^{**}_k$. Conversely, if a jump is placed only the coarser information J^i_k could become available.

What we want to point out is that after the jump, the entry $h_j(p^{**})$ will not necessarily record J^i_k but could rather record $d_j(p^{**}_k) = p^{**}_k$.[18] Similarly, if in the continuous price increase case we had that $d_j(p^{**}_k) = p^{**}_k$, with the jump we could have that J^i_k is recorded in the second component of h_j.

This is to say that the jump by affecting the structure of the information recorded in $H(p)$, it affects the information that can be inferred from it and in turn who is going to be active at a any given price.

3.3.3 Profitable Jumps: Costs Vs. Gains

When is the case that the gains from a jump outweigh its costs? What factors do influence such comparison? In this section we identify a couple of typical scenarios

[18] An illustration of this can be found in the setting that follows.

where the arise of a jump is more likely.

We start from a setting where no jumps arise in equilibrium. Then we show which variations of such basic set up can lead to an environment which is instead favorable to the arise of jumps.

In the example we propose, one of the bidders (bidder 2) always holds the highest value and wins the auctions. The example could be easily be made more realistic to allow for all bidders winning the object with some strictly positive probability. This formulation is just meant to pinpoint in the easiest possible way the key aspects we want to capture. Also, the use of pointwise distributions is made to render the equilibrium analysis easier, but it is to necessary to reach our results. The same considerations hold for the examples that follow.

- $v_1 = t_1$, $t_1 = 2$;
- $v_2 = t_2$, $t_2 = \{\frac{1}{2}, 1\}$; $Pr(t_2 = \frac{1}{2}) = Pr(t_2 = 1) = \frac{1}{2}$
- $v_3 = t_3$, if $t_2 = \frac{1}{2}$; $v_3 = t_3 + \frac{5}{2}$, if $t_2 = 1$; $t_3 = \frac{1}{2}$

Of the three bidders, bidder 1 and bidder 2 have private values and have a weakly dominant strategy to stay active up to their valuation regardless of whether the price is raised in a continuous or discontinuous fashion. Conversely, bidder's 3 valuation depends on bidder's 2 value so that his exit value differers whether or not he gets to know the exact realization of it.

If the information "$t_2 = 1$" is aggregated the valuation of bidder 3 increases substantially compared to the one of bidder 1, with the latter ending up loosing the auction. We need to check whether bidder 1 might have an incentive to pre-empt bidder 3 from getting such information.

The least costly way bidder 1 can do so is to jump from a value "slightly" lower than $\frac{1}{2}$ to 1. This induces bidder 3 to stay active until his expected valuation given the information available, i.e. $E(\tilde{v}_3|\tilde{t}_2 \in \{\frac{1}{2}, 1\}) = \frac{3}{2}$. Under the jump, bidder 1 wins $\frac{1}{2}$ with certainty, while without it he wins only with probability $\frac{1}{2}$, but a much higher sum ($\frac{3}{2}$), which guarantees a higher expected profit. Hiding information is therefore not profitable.

Even though the expected drop-out value of bidder 3 is the same in both scenarios, the probability that bidder 1 wins is affected, which in turn means that the expected price he pays conditional on winning is different.

It is useful in order to understand which factors may favor the arise of a jump to compare its costs with its benefits.

Let us from now on indicate by $J = (p^*, p^{**})$ a generic jump that a bidder might contemplate placing.

It seems natural to define the cost of a jump in the following way.

Definition 3.1. *The cost for bidder i of placing a jump J, $c_i(J)$, is the foregone profit of not being able to win at a price $p \in (p^*, p^{**})$.*

In the setting above it is easy to verify that $c_1(J) = \frac{1}{2}(\frac{3}{2} - \frac{1}{2}) = \frac{1}{2}$, where $J = (\frac{1}{2}, 1)$. It is to be pointed out that in general the price paid in the foregone opportunity scenario can be strictly higher than the price called (here for instance it is equal to $\frac{3}{2} > 1 = p^{**}$).

Definition 3.2. *The potential gains for bidder i of placing a jump J, $g_i(J)$, are given by expected increase in profits in the event that the price paid without the jump would have been higher than p^{**}.*

We can verify that $g_1(J) = \frac{1}{2}(\frac{1}{2} - 0) = \frac{1}{4} < c_1(J)$, where $J = (\frac{1}{2}, 1)$. We refer to $g_1(J)$ as "potential" gains as the change in profits conditional to the event that the price paid without the jump would have been higher than p^{**} does not have to be positive. A jump though will be placed only if such value is positive and outweighs the costs.

The next two settings provide simple variations of the one above. Let us start by adding an extra bidder (bidder 4).

- $v_1 = t_1$, $t_1 = 2$;
- $v_2 = t_2$, $t_2 = \{\frac{1}{2}, 1\}$; $Pr(t_2 = \frac{1}{2}) = Pr(t_2 = 1) = \frac{1}{2}$
- $v_3 = t_3$, if $t_2 = \frac{1}{2}$; $v_3 = t_3 + \frac{5}{2}$, if $t_2 = 1$; $t_3 = \frac{1}{2}$
- $v_4 = t_4$, $t_4 = \frac{3}{2}$

The only bidder that might place a jump is still bidder 1. The optimal jump to consider is the same as before and so are the profits he can make using such jump, i.e. $\pi_1' = \frac{1}{2}$.

Instead, if the jump J is not placed, the event that is favorable to bidder 1 happens with probability $\frac{1}{2}$ as in the original setup, but the presence of bidder 4 limits the amount of profits that can be made to $2 - \frac{3}{2} = \frac{1}{2}$. The expected profits realized in the game where J is not placed are then $\pi_1 = \frac{1}{2} * \frac{1}{2} = \frac{1}{4} < \frac{1}{2} = \pi_1'$. We should expect a jump to arise in equilibrium. Indeed we have that $g_1(J_1^1) = \frac{1}{2}(\frac{1}{2}-0) = \frac{1}{4} > c_1(J_1^1) = 0$, where $J_1^1 = (\frac{1}{2}, 1)$.

We move to the second variation. Here two extra bidders are added: bidder 4 is symmetric to bidder 2 and bidder's 5 interdependence with bidder's 4 value is symmetric to bidder's 3 interdependence with bidder's 4 value.

- $v_1 = t_1$, $t_1 = 2$;
- $v_2 = t_2$, $t_2 = \{\frac{1}{2}, 1\}$; $Pr(t_2 = \frac{1}{2}) = Pr(t_2 = 1) = \frac{1}{2}$
- $v_3 = t_3$, if $t_2 = \frac{1}{2}$; $v_3 = t_3 + \frac{5}{2}$, if $t_2 = 1$; $t_3 = \frac{1}{2}$
- $v_4 = t_4$, $t_4 = \{\frac{1}{2}, 1\}$; $Pr(t_4 = \frac{1}{2}) = Pr(t_2 = 1) = \frac{1}{2}$
- $v_5 = t_5$, if $t_4 = \frac{1}{2}$; $v_5 = t_5 + \frac{5}{2}$, if $t_4 = 1$; $t_5 = \frac{1}{2}$

Again, the optimal jump for bidder 1 is the same so that his profits from placing a jump are also unchanged. With such jump neither bidder 3 nor bidder 5 can discover their exact valuation and stay active until their expected valuation. Since $E(\tilde{v}_3|\tilde{t}_2 \in \{\frac{1}{2}, 1\}) = E(\tilde{v}_5|\tilde{t}_2 \in \{\frac{1}{2}, 1\}) = \frac{3}{2}$, this leaves again bidder 1 with a sure profit of $\pi_1^{'} = \frac{1}{2}$.

The profits from not jumping are instead lower than in the original setting as the event where bidder 1 can profit is now "$t_2 = \frac{1}{2}$" $\bigwedge$ "$t_4 = \frac{1}{2}$" (rather than just "$t_2 = \frac{1}{2}$"), which happens with a lower probability. The profits in the event of not placing the jump J are then $\pi_1 = \frac{1}{4} * \frac{3}{2} = \frac{3}{8} < \frac{1}{2} = \pi_1^{'}$, so that in equilibrium a jump is placed.

We can also verify that $g_1(J_1^1) = \frac{3}{4}(\frac{1}{2} - 0) = \frac{3}{8} > c_1(J_1^1) = \frac{1}{4}(\frac{3}{2} - \frac{1}{2}) = \frac{1}{4}$.

The jump has become profitable both because its cost has gone down and the deriving gains gone up.

The above setting points out not just that the same jump can affect the bidding behavior of severable bidders, but more importantly that if the hidden finer information that would have been relevant is not the same for all of them, then the more likely it is that a jump can profitably be used. The reason is that the probability that one of those bidders could have observed a favorable information becomes higher. Essentially because $max\left(E(\tilde{v}_3|\tilde{t}_2 \in \{\frac{1}{2}, 1\}), E(\tilde{v}_4|\tilde{t}_2 \in \{\frac{1}{2}, 1\})\right) < E\left(max(\tilde{v}_3, \tilde{v}_4)|\tilde{t}_2 \in \{\frac{1}{2}, 1\}\right)$.

To sum up this section suggests the following:

1. The more uncorrelated the amount of information a jump hides, the more profitable the jump as the lower the cost and the higher the gains deriving from it.

2. The more intense is competition the lower the probability that the bidder who jumped could have won at a price lower than the one called.

3.4 A Strategically Complex Environment

In the previous section we introduced a few simple examples where only one bidder had an incentive to jump. Here we look at how the possibility that someone else might jump affects the strategic interaction. We show that a bidder may be induced to jump by the anticipation of someone else hiding some information later on; viceversa, a bidder may also be induced to quit earlier than he would otherwise do.

Non obvious results are nonetheless raised also by the scenario where only one bidder is allowed (or has an incentive) to jump.

In section 4.2, for instance, we are able to show that being allowed to jump when all other bidders are not, need not to be a strategic advantage. As a matter of fact, a bidder could be willing to pay the seller (or the auctioneer) not to be allowed to jump.

Another interesting point is that even if a bidder's expected profits were to increase (when allowed to jump), it does not follow that the seller's profits are reduced. In fact, as well illustrated by Ettinger (2006), while the seller's profits conditional on the bidder who jumps being the winner decrease, the profits conditional on some other bidder winning may increase. The second effect can outweigh the first.

In general, however, it is true that the more the bidders who have an incentive to place one or more jumps, the more exponentially complex the dynamic game becomes and the richer the strategic implications that may derive.

To keep things as simple as possible here we restrict our attention to settings where figuring out the optimal jump is immediate.

3.4.1 A Pareto worse equilibrium

Let us refer to the game where jumps are not allowed and the price is raised continuously as the C game. Vice versa, we refer to the game where all bidders are allowed to jump as the J game.

We provide first an illustration where the equilibrium in the J game can be ranked as pareto worse than the one in the C game. In particular, the setting allows us to make the following points:

1. A bidder can be induced to jump by the anticipation that another bidder may strategically hide some relevant information (via a jump) later on in the auction.
2. Even though in equilibrium jump bids are used, all bidders would be better off in the C game.
3. Both revenues and efficiency are higher in the C game. Even stronger, the J equilibrium can be ranked as pareto worse than the C equilibrium as everybody is worse off.

- $v_1 = t_1$; $t_1 \in \{0, \frac{9}{10}\}$, $Pr(t_1 = 0) = Pr(t_1 = \frac{9}{10}) = \frac{1}{2}$
- $v_2 = t_2$, if $t_1 = 0$; $v_2 = t_2 + \frac{3}{2}$, if $t_1 = \frac{9}{10}$; $t_2 = 0$
- $v_3 = t_3$; $t_3 \in \{0, \frac{9}{10}\}$, $Pr(t_3 = 0) = Pr(t_3 = \frac{9}{10}) = \frac{1}{2}$
- $v_4 = t_4$, if $t_3 = 0$; $v_4 = t_4 + \frac{3}{2}$, if $t_3 = \frac{9}{10}$; $t_4 = 0$
- $v_5 = t_5$; $t_5 \in \{\frac{1}{2}, \frac{4}{5}\}$, $Pr(t_5 = \frac{1}{2}) = Pr(t_5 = \frac{4}{5}) = \frac{1}{2}$
- $v_6 = t_6$, if $t_5 = \frac{1}{2}$; $v_6 = t_2 + 2$, if $t_5 = \frac{4}{5}$; $t_6 = 0$

Let us start with the analysis of the C game. Bidders $j : 1, 3, 5$ know their valuation and stay active up to that price. The valuation of a bidder $j + 1$ depends on the value of bidder j. It is important to notice that bidder 2 and 4 can find out about their exact valuation before bidder 6 can, by staying active over $p = 0$. In equilibrium they quit immediately if respectively bidder 1 or bidder 3 quits. Vice versa, they stay active till the price equals $\frac{3}{2}$ if $t_1 = \frac{9}{10}$ or $t_3 = \frac{9}{10}$ respectively. In this latter scenario bidder 6 can find out his value without incurring any loss as either bidder 2, bidder 4 or both are active over the price of $\frac{1}{2}$. Bidder 6 then quits at price $\frac{1}{2}$ if bidder 5 does so and stays active until price equals 2, otherwise. If instead both bidders 1 and 3 quit at $p = 0$ also bidders 2 and 4 will. Nonetheless, the same strategy for bidder 6 remains optimal as $\frac{1}{2} * (0 - \frac{1}{2}) + \frac{1}{2} * (2 - \frac{4}{5}) = \frac{7}{20} > 0$. The revenues for the seller are $R^C = \frac{1}{4} * (\frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{4}{5}) + \frac{1}{2}(\frac{1}{2} * \frac{1}{2} + \frac{1}{2} * \frac{3}{2}) + \frac{1}{4}(\frac{1}{2} * \frac{3}{2} + \frac{1}{2} * \frac{3}{2}) = \frac{83}{80}$. The expected value of the winner (efficiency) instead is $E^C = \frac{1}{4} * (\frac{1}{2} * 0 + \frac{1}{2} * 2) + \frac{1}{2}(\frac{1}{2} * \frac{3}{2} + \frac{1}{2} * 2) + \frac{1}{4}(\frac{1}{2} * \frac{3}{2} + \frac{1}{2} * 2) = \frac{25}{16}$.

We turn now to the J game. Again bidder 1, 3 and 5 know their value and stay active up to that price. We first focus on bidder 2 (bidder 4 is symmetric) and check if and when he can find profitable to jump. There are only two possibilities: either he jumps before he finds out his exact value or after. In the first case the jump is not profitable as his ex ante expected value equals $\frac{3}{4}$ which is less than the value at which he should jump in order to hide some relevant information. Bidder 2 can get to know whether his value is 0 or $\frac{3}{2}$ at no cost as he can quit immediately if $t_1 = 0$. In the subgame where he finds out that his value equals $\frac{3}{2}$ and that bidder 4 has quitted, we need to compare the profits he can get from calling a price with the ones from letting the price increase without a stop (if bidder 4 also found out that his value is $\frac{3}{2}$ the comparison is irrelevant as the profits would be zero in both cases). If no price is called bidder's 2 profits are $\frac{1}{2} * (\frac{3}{2} - \frac{4}{5}) + \frac{1}{2} * 0 = \frac{7}{20}$. Instead by hiding information stopping the clock at any price in $(0, \frac{1}{2})$ and calling the price of $\frac{4}{5}$ the profits are $\frac{3}{2} - 1 = \frac{1}{2} > \frac{7}{20}$ (as bidder 6 in this case is not able to infer his exact valuation and bids according to his expected one) so that a jump is indeed profitable. Note that under bidder's 2 jump (or bidder 4) bidder 6 never wins. We now look for bidder's 6 optimal strategy. In equilibrium bidder 6 anticipates that if at least one of bidder 2 and bidder 4 finds out that his value is $\frac{3}{2}$, he is going to loose the auction. That happens with probability $\frac{3}{4}$. With probability $\frac{1}{4}$ instead he is going compete with bidder 5 only, which yields him an expected profit of $\frac{1}{4}(\frac{1}{2} * (0 - \frac{1}{2}) + \frac{1}{2} * (2 - \frac{4}{5})) = \frac{7}{80}$. The only other option bidder 6 has is to jump at the very start of the auction. In this case he needs to jump to the value of $\frac{9}{10}$ so that bidder 2 and 4 are not able to find out their exact valuations. Since bidders 2 and 4 expected values are lower than $\frac{9}{10}$, they cannot match the jump. In such case, bidder 6 always wins and makes a profit of $1 - \frac{9}{10} = \frac{1}{10} > \frac{7}{80}$. In equilibrium we should expect the auction to clear immediately with bidder 6 calling such price. This gives to the seller revenues $R^J = \frac{9}{10} < \frac{83}{80} = R^C$. Efficiency also drops as the expected value of the winner is $E^J = \frac{1}{2} * 0 + \frac{1}{2} * 2 = 1 < \frac{25}{16} = E^C$ (bidder 6 is always the winner). Also, the only bidder that is able to make some profit is bidder 6 and the profits he makes are lower than in the C game as he gets $\frac{1}{10}$ instead of $\frac{11}{40}$. Interestingly, allowing bidders to call a price in this setting makes everybody worse off including the bidder placing the jump.

In the next section we analyze settings where the bidders may be active at prices where conditional on winning they would make a loss. This happens as the expected profits that they are able to make at higher prices more than compensate such losses. We find such settings particularly interesting as there the aggregation of information allowed by open auctions can potentially be very powerful in boosting revenues and efficiency. At the same time though we are able to show that a standard open auction that does not allow for price calls sometimes creates perverse incentives that may impede the aggregation of information. Allowing for jumps can sometimes surprisingly increase both revenues and efficiency.

3.4.2 Jumps, Information Aggregation and a Free-Rider Problem

Open formats enable to learn extra information during the course of the auction. In general a bidder may be active at prices where conditional on winning he makes a loss. This happens if he anticipates that with some probability he may profitably win at higher prices and that such expected gains outweigh the expected losses.

If competition is strong the expected cost of being active can be extremely low. Even though the first intuition may be that this should induce more bidders to be active for longer, thus improving the information they hold and resulting in higher revenues and efficiency, in general this does not have to be the case.

Compte and Jehiel (2004) provide a private value model where such intuition holds true. With some probability bidders may receive a better estimate about their private valuation at some exogenously determined random time. If there are enough bidders in the auction it may be profitable for some of them "to wait and see" for favorable information. Such possibility raises both efficiency and revenues. A similar insight is also present in Compte and Jehiel (2007); there a bidder may stay active beyond his initial expected valuation to observe the strength of the competition. Only in the event that competition is not intense, a bidder would invest to get to know his exact valuation.

Here we provide two illustrative settings. The first one provides an interdependent value scenario where the private information that bidders hold is aggregated in a very desirable way thanks to the possibility of "wait and see". Allowing bidders to call a price instead causes both efficiency and revenues to drop.

The second setting provides a new insight. The information there fails to aggregate precisely because the low cost of staying active if there are other competitors active may lead to an instance of the free rider-problem that results in no bidder ultimately be willing to acquire information, where acquiring information in our case simply means stay active in order to be able to update the estimate of your valuation. The possibility of jump bidding here allows the bidder with the ex-ante higher valuation to hide the piece of information causing such free-rider problem and to profitably win the auction. This boosts both efficiency and revenues.

We start with the scenario where the aggregation of information is very smooth.

Such setting also illustrates the following points:

1. The anticipation of a future jump may induce a bidder to quit earlier than he would in the C game.

2. Even though no jump is observed in equilibrium, the equilibrium outcome in the J game substantially differs from the one in the C game.

- $v_1 = t_1$, $t_1 = \{\frac{3}{4}, \frac{4}{5}, 1\}$, $Pr(t_1 = \frac{3}{4})) = Pr(t_1 = \frac{4}{5})) = Pr(t_1 = 1) = \frac{1}{3}$
- $v_2 = t_2$, if $t_1 = \frac{3}{4}$; $v_2 = t_2 + \frac{5}{4}$, otherwise; $t_2 = 0$
- $v_3 = t_3$, if $t_1 \in \{\frac{3}{4}, \frac{4}{5}\}$; $v_3 = t_3 + 2$, otherwise; $t_3 = 0$

The key feature of this setting is that both for bidder 2 and for bidder 3 winning the object at a low price (at $p = \frac{3}{4}$) entails a big loss as it means that bidder 1 signal was low and consequently that they both value the object zero. It can be easily checked that in a sealed bid SPA this fact implies that both bidder 2 and bidder 3 would always bid zero. Then, bidder 1 would win regardless of his signal at a price of zero.

Conversely, if bidders are not allowed to jump the Japanese auction allows the information to be aggregated in a very desirable way. Essentially what happens is that the open format allows bidder 2 and bidder 3 to "share" the risk of winning at a low price. Note that if only one of them were present in the auction, he would not find profitable to stay active at low prices and he would rather quit immediately. What is crucial is that the two bidders can also "split" the benefits of being active at higher prices in a way that allows both to recover the expected losses.

The bidders' strategies are as follows. Bidder 1 stays active until his private value is reached. Bidder 2 quits as soon as bidder 1 quits if that happens at a price less or equal than $\frac{3}{4}$, and stays active until the price reaches $\frac{5}{4}$, otherwise. Bidder 3 quits as soon as bidder 1 quits if that happens at a price less or equal than $\frac{4}{5}$, and stays active until the price reaches 2, otherwise. This way bidder 2 and bidder 3 "share" an expected loss of $\frac{1}{3} * \frac{1}{2} * \frac{3}{4} = \frac{1}{8}$ each in the event that $t_1 = \frac{3}{4}$. The benefits more than compensate: bidder 2, if $t_1 = \frac{4}{5}$, gets an expected profit of $\frac{1}{3} * (\frac{5}{4} - \frac{4}{5}) = \frac{3}{20}$ and bidder 3, if $t_1 = 1$, gets an expected profit of $\frac{1}{3} * (2 - \frac{5}{4}) = \frac{1}{4}$. The revenues are then $R^C = \frac{1}{3} * \frac{3}{4} + \frac{1}{3} * \frac{4}{5} + \frac{1}{3} * \frac{5}{4} = \frac{14}{15}$ which is substantially more than zero as in the SPA. Similarly, the expected value of the winner (measuring efficiency) is $E^C = \frac{1}{3} * 0 + \frac{1}{3} * \frac{5}{4} + \frac{1}{3} * 2 = \frac{13}{12}$ which is bigger than $\frac{1}{3} * \frac{3}{4} + \frac{1}{3} * \frac{4}{5} + \frac{1}{3} * 1 = \frac{17}{20}$ in the SPA. The ability of open formats to aggregate information in the course of the auction boosts both revenue and efficiency. What happens if we allow for jumps?

In that case the smooth sharing of costs and benefits is unattainable. Recall that bidder 2 and 3 can be active at low prices only if they do so jointly. Note that if $t_1 \neq \frac{3}{4}$, bidder 2 is going to learn that $v_2 = \frac{5}{4}$, while bidder 3 is still uncertain regarding his exact value. Bidder 2 can "hide" such information to bidder 3 by calling a price equal to 1 when the current price is still in $(\frac{3}{4}, \frac{4}{5})$. In such event bidder 2 makes a sure profit of $\frac{1}{4}$ (bidder 3 would stay active until $E(\tilde{v}_3|\tilde{t}_1 \in \{\frac{4}{5}, 1\}) = 1$), while if he lets the price increase continuously he can win only in the event that $t_1 = \frac{4}{5}$, which yields $\frac{1}{2} * (\frac{5}{4} - \frac{4}{5}) = \frac{9}{40} < \frac{1}{4}$. Bidder 2 cannot commit not to call such a price. But then bidder 3 anticipating that bidder's 2 jump will pre-empt him from winning in the only event in which it is profitable, he is no longer willing to stay active over $t_1 = \frac{3}{4}$. Since bidder's 3 presence is necessary for bidder 2, the equilibrium outcome is that they both quit the auction immediately. This brings zero revenues and it inefficiently allocates the object always to bidder 1 as in the SPA.

It is important to point out that without being fully aware of the implications of allowing bidders to call a price, the seller may be wrongly induced to believe that the bidders' valuations were very low. The setting just presented also implies the following result.

Proposition 3.1. *A bidder may be willing to pay not to be allowed to call a price even in the event that he were the only bidder granted such option.*

Proof. Take the setting above where bidder 2 is the only bidder allowed to jump. Since in the J game he never wins, he would be willing to pay up to the expected profits he makes in the C game to restrict his strategies space to the choice of quitting or stay active alone.

□

We turn to the second setting. This setting shows that:

1. In the C game perverse incentives may impede the aggregation of information.

2. The enlarged strategy set of the J game may alleviate such problem and bring higher revenues and efficiency.

- $v_1 = t_1$, $t_1 = \{\frac{9}{10}, 1\}$, $Pr(t_1 = \frac{9}{10})) = Pr(t_1 = 1) = \frac{1}{2}$

- $v_2 = t_2$, $t_2 = \frac{4}{5}$, if $t_1 = \frac{9}{10}$;$v_2 = t_2 + \frac{1}{2}$, if $t_1 = 1$

- $v_3 = t_3$, $t_3 = 0$, if $t_1 = \frac{9}{10}$;$v_3 = t_3 + \frac{9}{5}$, if $t_1 = 1$

The framework is similar to the previous one in so far as both bidder 2 and bidder 3 may have an incentive to "wait and see". Differently from the previous setting though one of the bidders (bidder 2) has an ex-ante value higher than bidder 1. This means that if bidder 2 were the only bidder competing with bidder 1, he would profitably be active over the price of $\frac{9}{10}$ to be able to infer the realization of t_1. The same is not true for bidder 3.

Bidder 3 could potentially benefit from the active presence of bidder 2 over the price of $\frac{9}{10}$. The problem is that if both bidders are active at that price the expected losses are shared but the expected gains are not. In fact, if bidder 3 infers that $t_1 = 1$, he always wins against bidder 2. But then bidder 2 prefers to stay active only until $\frac{4}{5}$ to avoid incurring in a loss. In turn if bidder 2 quits before the price reaches $\frac{9}{10}$ (else said he does not "wait and see"), so will bidder 3, as the latter one has an expected value lower than bidder 1. Hence, no aggregation of information is possible.

The equilibrium strategies are the following. Bidder 1 quits at his privately known value; bidder 2 quits when the price reaches $\frac{4}{5}$; bidder 3 quits as soon as bidder 2 quits if that happens for a price less or equal than $\frac{9}{10}$ and stays active until $\frac{9}{5}$, otherwise. The auction clearly performs very poorly as bidder 1 always wins for a price of $\frac{4}{5}$, which implies that both revenues and efficiency would be higher if bidder 3 were excluded from the competition.

Let us now look at the J game. Bidder's 2 ex-ante expected value is bigger then the one of both bidder 1 and bidder 3. While in the C game bidder 2 cannot do anything to prevent bidder 3 from free-riding on his presence, in the J game he can affect bidder's 2 bidding behavior by hiding the relevant piece of information. He can do that by stopping the clock at any price lower than $\frac{9}{10}$ and calling a price of 1. Doing so he wins at that price making an expected profit of $\frac{1}{20}$ (his expected valuation is $\frac{21}{20}$), which is better than in the C game where he never wins. Revenues go up from $R^C = \frac{4}{5}$ in the C game to $R^J = 1$ in the J game. Similarly, the expected value of the winner (efficiency) increases from $E^C = \frac{1}{2} * \frac{9}{10} + \frac{1}{2} * 1 = \frac{19}{20}$ in the C game to $E^J = \frac{1}{2} * \frac{4}{5} + \frac{1}{2} * \frac{13}{10} = \frac{21}{20}$ in the J game.

This allows us to conclude the section with the following result.

Proposition 3.2. *Allowing bidders to call a price can decrease or increase revenues. The result is sensitive to the specific setting analyzed. The same conclusion holds regarding the format's efficiency.*

Proof. We only need to provide examples where all those possibilities are covered. The setting in section 4.1 proves that revenues and efficiency can drop. The setting above proves that they can increase.

□

3.5 Symmetric settings

The reader may wonder whether some asymmetries in the setting are needed for jumps to arise in equilibrium. The answer is that asymmetries are not needed and the same insights could be delivered starting from symmetric settings. Asymmetric settings were simply more flexible in allowing us to generate examples that could convey in the easiest possible way the main insights.

The following example introduces a completely symmetric setting in which those bidders receiving the highest signal have an incentive to strategically jump bid in order to hide the drop-out values of bidders with lower signals.

Here we relax the assumption that bidders valuations are strictly increasing in their own type. Proposition 4 shows that under the standard valuation structure assumed in the literature, no jump is placed in equilibrium when a symmetric value model is assumed.

- $i : 1, ..4$ Bidders.
- Any bidder i receives a unidimensional independently drawn signal $t_i \in T_i$, where $T_i \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$, and $Pr(t_i = 0) = Pr(t_i = \frac{1}{4}) = Pr(t_i = \frac{1}{2}) = Pr(t_i = \frac{3}{4}) = Pr(t_i = 1) = \frac{1}{5}$.
- $v_i(t_i, t_{-i}) = 1$, if $t_i = \frac{1}{2}$ and $t_j > 0$ for any $j \neq i$;
- $v_i(t_i, t_{-i}) = t_i$, otherwise.

In the setting above any type $t_i \neq \frac{1}{2}$ knows his exact valuation as in a private value model. Conversely, type $t_i = \frac{1}{2}$ needs to revise his expected valuation according to the information he receives during the auction.

What is not standard is that if the vector of the opponents' signals is strictly positive, a bidder of type $t_i = \frac{1}{2}$ holds a value that is higher than the one of type $t_i = \frac{3}{4}$ and equal to the one of type $t_1 = 1$, i.e. as anticipated, $v_i(t_i, t_{-i})$ is not strictly increasing in t_i, $\forall t_i$, $\forall t_{-i}$.

To get an economic interpretation for this setting, recall the example of the novice bidder and the car dealers presented in the introduction. The same interpretation can be given here once we assume that there is ex-ante uncertainty regarding who is expert and who is not. Bidders are then ex-ante symmetric in that they privately know if they are novice or expert but they are not aware about the others' "types".

In particular, the example proposed fits such scenario if we think of bidders' types other than $t_i = \frac{1}{2}$ as experienced bidders, and of bidders with type $t_i = \frac{1}{2}$ as novice bidders.

The former are not influenced by the opponents' estimates for the object , the latter ones are.

This is so as the interpretation was that the novice bidder really likes the car, but he is afraid it might be a bad bargain unless he observes that everybody attaches a fairly high value to it (in this parametrization this is given by the event "all expert bidders have a value strictly higher than zero").

We show in the next proposition that such favorable information for type $t_i = \frac{1}{2}$ can profitably be hidden by a competitor of type $t_j = 1$ calling a price of $\frac{1}{4}$ at the very beginning of the auction.

Proposition 3.3. *The following strategies constitute a sequential equilibrium of the modified Japanese auction game, where the bidders' value functions are as described above:*

- *if $t_i \in \{0, \frac{1}{4}, \frac{3}{4}\}$, stay active until t_i.*
- *if $t_i = \frac{1}{2}$, stay active until t_i, if for some $j \neq i$, and $p \in (0, \frac{1}{2})$, $d_j(p) = 0$; stay active until the price reaches 1, if $d_j(p) = p_j > 0$, $\forall j \neq i$, for some $p > 0$; stay active until your expected value conditional on t_i and on the public information $H(p)$, otherwise.*
- *if $t_i = 1$ stop the clock at the very start of the auction and call a price of $\frac{1}{4}$, i.e. place a jump $J_1^i = (0, \frac{1}{4})$, then stay active until the price reaches $p = 1$.*

Proof. We first prove that bidders with types $t_i \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ cannot profitably jump. Then, we will show that the optimal jump for bidder $t_i = 1$ is from the value of 0 to the value of $\frac{1}{4}$, i.e. $J_1^i = (0, \frac{1}{4})$.

Note first that a jump affects an opponent j behavior only if he is of type $t_j = \frac{1}{2}$ and it creates uncertainty on whether all other opponents $k \neq \{i, j\}$ are of type $t_k > 0$

or not. The less costly way to create such an uncertainty is to bid $\frac{1}{4}$ as this is the lowest value at which the smallest type larger than $t_k = 0$ would also drop.

A type $t_i = \frac{1}{2}$ can only win against opponents with types $t_j \in [0, \frac{1}{4}]$, unless he finds out that $t_j > 0$, $\forall t_j$. These types have a private valuation and their bidding is not influenced by a jump, which would be then unprofitable. A similar argument excludes that types $t_i = 0$ and $t_i = \frac{1}{4}$ can profitably jump. A type $t_i = \frac{1}{2}$ can also win against types $t_j = \frac{3}{4}$ and type $t_j = 1$, but he certainly does not find profitable to hide to himself information that can render him more competitive ($t_j > 0$, $\forall j$) or make him quit at the right price ($t_j = 0$, for some j).

As for type $t_i = \frac{3}{4}$, the most "effective" jump he can place is from a value of 0 to $\frac{1}{4}$ as placing a jump $J = (0, p^{**})$, with $p^{**} \in (\frac{1}{4}, \frac{3}{4})$ would be more costly but would not affect type $t_j = \frac{1}{2}$ bidding behavior with respect to the jump proposed. We show that even such candidate jump cannot be profitable.

The price he pays (when winning) is affected by a jump only if at least one of the opponents is of type $t_j = \frac{1}{2}$ and at least one of type 0 or $\frac{1}{4}$

The jump is not profitable in the subcases where only one of the two other bidders is of type 0 or $\frac{1}{4}$, as bidder $t_j = \frac{1}{2}$ would bid up to his expected valuation that in such scenario equals to $\frac{3}{4}$ (so that bidder i would break even in expected terms as opposed to gaining positive profits without the jump). Finally, in the four subcases where two of the opponents are either 0 or $\frac{1}{4}$, bidder $t_j = \frac{1}{2}$ would bid up to $\frac{5}{8}$. Without the jump bid in three of the four cases $t_j = \frac{1}{2}$ would bid $\frac{1}{2}$ with a gain of $\frac{1}{8}$ and in the remaining case he would bid up to 1 and bidder i would forego a profit of $\frac{1}{8}$ obtainable with the jump. Also in this scenario then the jump is not profitable.

Finally, lets check that indeed type $t_i = 1$ can profitably jump.[19] The candidate jump is again $J_1^i = (0, \frac{1}{4})$. The same reason as above excludes any jump higher than that but lower than $\frac{3}{4}$. It remains to exclude a jump $J = (0, \frac{3}{4})$ (as a jump above $\frac{3}{4}$ would be more costly but add no gain).

Lets us condition on the event that no opponent of bidder i is of type $t_j = 1$, as in that case bidder i makes zero profits in all circumstances. To prove that the jump J is not profitable, note that his expected gain are bounded above by $(\frac{37}{64} * \frac{9}{16}) * \frac{1}{4}$ where the first product gives the probability of the event "at least one bidder has drawn type $t_j = \frac{1}{2}$ and the other two opponents have drawn a type $t_j > 0$" (which is exactly the event

[19] If there is more than one bidder with $t_i = 1$, the identity of the bidder to whom the jump is attributed is randomly selected.

in which jumping is profitable) and $\frac{1}{4}$ is an upper bound on the gains. The expected losses are instead bounded below by such number as the first product indicates also the probability of the event "at least one opponent has drawn type $t_j = 0$ and the maximum type of the other two opponents is some $t_j \leq \frac{1}{2}$", under which bidder i foregoes the possibility of winning for a price equal or lower than $\frac{1}{2}$ and the losses are bounded below by $\frac{1}{4}$.

We now instead verify that the candidate jump is profitable.

Again the jump alters the price paid by bidder i only when at least one of the opponents is of type $\frac{1}{2}$ and at least one of type 0 or $\frac{1}{4}$ and none of type 1. One of the opponents can then be "freely" chosen to be of type 0, $\frac{1}{4}$, $\frac{1}{2}$ or $\frac{3}{4}$. In the first three cases the expected price paid by bidder i is the same with or without the jump. Without the jump i faces the uncertain distribution of $\frac{1}{2}$ and 1, [20] while with the jump he pays the expected value of that.

In the last case (one bidder being of type $t_j = \frac{3}{4}$), the jump becomes strictly profitable as now in the uncertain distribution bidder i can no longer benefit from the event "at least one bidder has signal 0" which for the previous cases was leading i to win at a price of $\frac{1}{2}$.[21] These gains exceed the cost of the jump which is given by the fact of foregoing the possibility of winning the object at price zero when all opponents have drawn a signal $t_j = 0$. It is easy to verify that the expected profits without the jump are $\frac{70}{125} * 0 + \frac{32}{125} * \frac{1}{4} + \frac{15}{125} * \frac{1}{2} + \frac{7}{125} * \frac{3}{4} + \frac{1}{125} * 1 = \frac{87}{500}$ and $\frac{65}{125} * 0 + \frac{38}{125} * \frac{1}{4} + \frac{4}{125} * \frac{3}{8} + \frac{10}{125} * \frac{1}{2} + \frac{8}{125} * \frac{3}{4} = \frac{88}{500}$ with the jump.

The above proves that only type $t_i = 1$ jumps and what his optimal jump is. It is immediate to verify that the strategies of the remaining types are of equilibrium as they simply require them to stay active until their privately known value or their expected values given the information gathered during the auction.

□

The presence of a type $t_i = \frac{3}{4}$ in the setting proposed is important in making the jump profitable. Note with this regard the similarities with the first setting that made the arise of a jump profitable in section 3.3. The type of insight provided is essentially the same (a more competitive environment under some circumstances can make the arise of a jump more likely) but the equilibrium analysis even in a fairly basic set up is far

[20]The first event has probability $\frac{1}{2}$ if only one bidder drops after the jump and probability $\frac{3}{4}$ if two bidders drop. The second event of course has the residual probability.

[21]The uncertain distribution is now over the values $\frac{3}{4}$ and 1.

more complex.

The next proposition shows that in a symmetric setting with "standard" value functions no jumps arise in equilibrium.

It is well known that in the symmetric setting if the standard Japanese version of the English auction is assumed there exists a unique symmetric equilibrium where bidders drop-out values are fully revealing of their types. In such equilibrium the types of the bidders who drop become common knowledge and the remaining bidders use a bidding strategy that prescribes them to exit at the value that is obtained plugging in the value function the signals of the bidders who have already dropped and assigning to the types of the bidders that are still active the same type as their own (for details and a formal description, see for instance Krishna (2002) or the original paper by Milgrom and Weber (1982)). We show that such equilibrium remains the unique symmetric equilibrium even if bidders are allowed to jump.[22] Hence, this result renders more robust the predictions for the standard symmetric Japanese version of the EA provided by the literature. Such result holds under the assumption made through out the paper that excludes that a jump can affect the expected valuation of a bidder or his beliefs through signaling.

Proposition 3.4. *Assume that $v_i(t_i, t_{-i})$ is strictly increasing in t_i for any profile of the opponents' signals, t_{-i}, and weakly increasing in t_j for any j, then no jump (aimed at hiding information) can arise in the equilibrium of the symmetric value model.*

Proof. □

Suppose that the conditions stated above are satisfied and that some jump bid is placed. We show that any such jump is not profitable and that the allocation and the price paid by the winner are the same as in the model where jumps are not allowed.

The strategy set is nonetheless enlarged and thus a strategy needs to specify how the bidder would react if some other bidder were to place a jump.

Let us assume that bidder i instead of following the action prescribed by the "standard" equilibrium strategy without jumps places the first jump, J_1^i, from p_1^* to p_1^{**}; we limit our analysis to the first jump only as a similar argument excludes that further jumps may emerge.

Up to the price reaches p_1^* there is a unique symmetric bidding strategy that bidders can follow and it is the one described in the papers cited above. Using such strategy a bidder quits at a price at which he would break even were all the remaining active

[22] Except the obvious complication that the bidding behavior in case a bidder deviates and places a jump needs to be specified.

bidders to quit at the same time as him. Those studies show that such break even condition can be constructed.

Note also that, once the jump $J_1^i = (p_1^*, p_1^{**})$ is placed, there must exist a threshold type (if any) that is just indifferent between matching the jump or not (with all bidders with a lower type quitting the auction). This is so as valuations are symmetric and strictly increasing in t_i for any signal profile t_{-i}.

Denote the set of bidders who decide to exit after observing the jump by E, and by $\overline{E}$ the cardinality of the set. It then becomes publicly known that for any bidder $e \in E$, $t_e \in [\underline{t}_e, \overline{t}_e]$, where $\underline{t}_e$ ($\overline{t}_e$) is the inferred lower (upper) bound for t_e given that e was active at p_1^*, but not at p_1^{**}. Denote instead the active bidders by the set A, and by $\overline{A}$ the cardinality of the set. The bidders who dropped prior to the jump than belong to the set $N - E - A$ and without loss of generality we can order them to be bidders 1 to $\overline{N} - \overline{E} - \overline{A}$.

Compared to the scenario without jumps we have that after the jump is placed only only the coarser information $t_e \in [\underline{t}_e, \overline{t}_e]$ becomes regarding any $e \in E$.

Hence, we need to define what is the optimal action for a bidder that decides to be active after observing J_1^i for all possible histories that may unfold after that event.

Our claim is that any bidder $a \in A$ uses the symmetric bidding strategy $b_a^A = E(\tilde{v}_a | \tilde{t}_1 = t_1, \tilde{t}_2 = t_2, ..., \tilde{t}_{\overline{N}-\overline{E}-\overline{A}} = t_{\overline{N}-\overline{E}-\overline{A}}; \tilde{t}_e \in [\underline{t}_e, \overline{t}_e], e \in E; \tilde{t}_a = t_a, a \in A)$, to determine when to drop in case the remaining active bidders (including himself) are A and where t_a is the type of bidder a for whom we are evaluating the strategy.

Given the assumptions made, the bidding strategy is strictly increasing, hence it fully reveals the type of the bidders who quit. If no other jump follows, bidders drop in order (from the lowest type to the highest) at the price prescribed by the equilibrium strategy. The fact that no bidders can profitably deviate from such strategy follows from the same arguments as in the English auction without jumps to which we remand the reader.

The allocation that results is that the bidder with the highest type wins, and the price he pays is the expected value of the bidder with the second highest type, which is $E(\tilde{v}_a | \tilde{t}_1 = t_1, \tilde{t}_2 = t_2, ..., \tilde{t}_{\overline{N}-\overline{E}-\overline{A}} = t_{\overline{N}-\overline{E}-\overline{A}}; \tilde{t}_e \in [\underline{t}_e, \overline{t}_e], e \in E; \tilde{t}_a = t_a, a \in A)$, for $A = 2$.

Without the jump the price paid would have been

$E(\tilde{v}_a | \tilde{t}_1 = t_1, \tilde{t}_2 = t_2, ..., \tilde{t}_{\overline{N}-\overline{E}-\overline{A}} = t_{\overline{N}-\overline{E}-\overline{A}}; \tilde{t}_e, e \in E; \tilde{t}_a = t_a, a \in A | \tilde{t}_e \in [\underline{t}_e, \overline{t}_e])$, for $A = 2$. The two values are clearly the same. In other words, the gains from the jump are $G(J_1^i) = 0$.

However, any candidate jump with strictly positive probability is not matched by some bidder,[23] which given symmetry means that with strictly positive probability all bidders exit after the jump so that $C(J_1^i) > 0$. Hence, placing a jump cannot be profitable.

[23]As a jump that is matched by all bidders does not affect the bidding behavior given the assumptions made in this paper.

3.6 Conclusions

We have analyzed a truly dynamic version of the Japanese auctions where bidders are allowed to stop the continuous price increase to call a price at any point during the auction.

We have looked at how the possibility of calling a price affects the way information is aggregated and shown that bidders may have an incentive to alter the aggregation of information by placing jump bids to hide the drop-out value of some of their opponents.

There is a general wisdom that jumps are anticompetitive. We show that the environment is so strategically rich that many things can happen. In particular, in situations where the aggregation of information would otherwise not be "smooth", the bigger flexibility allowed by the larger strategy sets can alleviate an instance of the free-rider problem and improve the auction performance both in terms of revenues and efficiency.

This analysis brings powerful implications as it shows that the possibility of placing jump bids affects severely (though in general ambiguously) both revenues and efficiency.

In particular, when evaluating the advantages and disadvantages of open versus sealed bid format great care needs to be placed on whether the setting could be favorable to jump bids.

Chapter 4

Second Best Efficiency and English auctions

4.1 Introduction

In this paper we study mechanisms that maximize the expected social surplus generated by the sale of an indivisible unit subject to the buyers' incentive compatibility constraints. This has been a fundamental question in the auction literature since Vickrey (1961) seminal paper.

Previous papers that have addressed this question, e.g. Maskin (1992), Maskin (2000), Dasgupta and Maskin (2000), Jehiel and Moldovanu (2001), have focused on conditions that guarantee that the incentive compatibility constraints are not binding. For instance, when buyers have interdependent values, unidimensional type spaces, independent types and we consider Bayesian incentive compatibility constraints, Dasgupta and Maskin (2000) has shown that the condition takes the form of a particular single crossing condition. This condition basically says that if it is efficient to allocate the good to one buyer for a given profile of types, it must also be efficient to allocate the good to this buyer when we increase her type keeping constant the types of the other buyers.[1]

We analyse a simple extension of the framework described in the paragraph above to allow for externalities. However, we depart from the existing auction literature and study efficiency when the incentive compatibility constraints are binding, i.e. when the single crossing condition does not hold.

Maskin (2000) gives a realistic example in which the single crossing condition may

[1] Assuming that a buyer type's space is ordered according to her value, this is, higher types correspond to higher values.

fail. Suppose the sale of the right to drill for oil between two wildcatters. The first one has a high marginal cost and a low fixed cost whereas the second one has a low marginal cost and a high fixed cost. In this case, it is efficient to allocate the good to the first wildcatter if there is little oil and to the second one if there is much oil. Thus, the single crossing condition will fail if the amount of oil is exclusive information of the first wildcatter.

We argue in the paper that a similar argument applies with some generality to auctions with a (potentially) inefficient incumbent and some fixed costs. We also argue that the single crossing condition may fail naturally in the one-dimensional reduced form of models with multi-dimensional types. In particular, this may happen when the multidimensional type contains information about common and private components of the value and/or about an allocative externality.

We provide two types of results. First, we characterize the *second best* allocation when the single crossing condition fails. This allocation is not trivial, in fact, we show that it may be second best efficient to not allocate the good with positive probability even when it is first best to always allocate the good. Regretfully, our characterization is not complete in this last case.

Second, we analyse whether the second best can be implemented as the equilibrium of an English auction. We start noting a negative result when the second best allocation implies no selling, even if we allow for reserve prices or entry fees. Thus, we add the constraint that the good is always sold. In this case, we can always find an equilibrium that implements the second best. However, the existence of this equilibrium when there are three or more bidders seems to be a consequence of the usual multiplicity of equilibria in English auctions. To illustrate the non-robustness of this equilibrium, we study the case in which the single crossing condition fails for one bidder with a weakly dominant strategy. We show that the second best allocation cannot be implemented in an equilibrium in which this bidder uses her weakly dominant strategy.[2] This is because the information that this bidder quits may induce a "rush" with positive probability, i.e. all the remaining bidders may quit simultaneously. We find that a two-stage English auction may solve this problem.

The most closely related papers are those we have already quoted by Maskin (1992), Maskin (2000), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001). Krishna and Perry (1998) and Williams (1999) also characterize conditions for efficient Bayesian

[2]We do not include the formal proof as we are currently revising this part of our work. The same is true for the remark on the two-stage English auction made below.

implementation[3] with independent types.

Independence of types is probably a not very realistic assumption when there is a common value component in bidders' uncertainty and two or more bidders have private information about it. Note, however, that when we withdraw this assumption, the results of Cremer and McLean (1985) imply that the efficient allocation can be Bayesian implemented in general.[4] But, the mechanism requires arbitrarily large payments that seem unrealistic. More recently, McLean and Postlewaite (2004) have shown that this critic does not apply when agents are "informationally small". However, we expect agents to be "informationally small" in general only when they are sufficiently many.

From a different perspective, Mezzetti (2004) has shown that Bayesian implementation of the efficient outcome can always be achieved in a two-stage mechanism if the mechanism can condition on the realized outcome-payoffs. However, as Jehiel and Moldovanu (2003) (the design of a private industry) have already pointed out Mezzetti's mechanism displays no incentives to reveal truthfully in the second stage, and requires that payments are made when all information is available which makes it sensitive to renegotiation and moral hazard.

Some of the above papers, in particular Maskin (1992), Maskin (2000), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001), have also studied the set of implementable allocations when bidders have multidimensional private information. They show that an implementable allocation cannot depend on the type beyond a particular one-dimensional reduction. In general, the efficient allocation does not correspond to this one-dimensional reduction and thus it is not implementable. They, however, note that we can always define a constrained efficient allocation that maximizes expected social surplus subject to the one-dimensional reduction.

Although our second-best analysis is in principle unrelated to the problem described in the previous paragraph because we assume a one-dimensional type space, we note that our results may be useful in the efficiency analysis based on the one-dimensional reduced types. The reason is that there are no a priori arguments that ensure that the one-dimensional reduction satisfy the single crossing condition and thus the constrained efficient allocation may not be implementable.

Finally, we want to note another branch of the literature that has looked at the implementation of the efficient allocation through realistic auction formats, and in par-

[3]We discuss the literature on ex post implementation in the Conclusions.

[4]Neeman (2004) and Heifetz and Neeman (2006) also cast some doubts on the genericity of these results.

ticular for single unit sales through an English auction. Maskin (1992), Krishna (2003), Birulin and Izmalkov (2003) give necessary and sufficient conditions and note an important difference between two bidders and more than two bidders: in the former case the single crossing condition plus a natural non-decreasing condition are necessary and sufficient, whereas the latter case requires more restrictive conditions.[5] We also show that there is a difference between these two cases but because of a different reason. In fact, the necessary and sufficient conditions for the English auction to implement the efficient allocation are the same under our assumptions for any number of bidders.

On the technical side, our work is related to Myerson (1981). We use the techniques he used to characterize the revenue maximizing mechanism in the non-regular case, what he calls ironing technique, to find the allocation that maximizes the expected social surplus.

The rest of the paper is organized as follows. Section 2 introduces the formal model. Sections 3 defines the First Best allocation. Section 4 introduces some motivating examples where the First Best cannot be achieved. Section 5 contains the characterization of the Second Best efficient allocation. Section 6 shows when the English Auction can be effectively employed to implement the Second Best efficient allocation, discussing the robustness of the result both for the two bidders' case and the generic n bidders' case. Section 7 deals with the one dimensional reduction of multidimensional type models. Section 8 concludes.

4.2 The Model

One unit of an indivisible good is put up for sale to a set N of n bidders. Let $s = (s_1, ..., s_n) \in \mathbb{R}^n$ be a vector where s_i corresponds to the realization of an independent random variable with distribution F_i with strictly positive density in all the support $S_i \subset \mathbb{R}$. Bidder $i \in N$ observes privately s_i and gets a von Neumann-Morgenstern utility $v_i(s) - p$ if she gets the good for sale at price p, and gets utility $-e_j(s_j) - p$ if bidder j, $j \neq i$, gets the good and i pays a price p. Thus, e_j denotes a negative externality[6] produced by j on each of the other bidders. To make the analysis simpler, we assume that the seller neither derives utility from getting the good nor suffers any externality.

We assume additive separability of the bidders' value functions plus a symmetry assumption on the common value component. Formally, $v_i(s) = t_i(s_i) + \sum_{i \in N} q_j(s_j)$ for

[5] Other papers that address this question are Izmalkov (2003), Dubra (2006), and Echenique and Manelli (2007).

[6] Note that we also allow for $e_j(s_j) < 0$ and thus for positive externalities.

any $i \in N$, where $t_i(s_i)$ (t_i stands for taste) is the private value and $\sum_{i \in N} q_j(s_j)$ (q_j stands for quality) is the common value. We also assume that $v_i(s) - (n-1)e_i(s_i) \geq 0$ $\forall s$ and that $v_i(s)$ is a strictly increasing function of s_i, i.e. that $\phi_i(s_i) \equiv t_i(s_i) + q_i(s_i)$ is strictly increasing. Finally, we assume that $h_i(s_i) \equiv t_i(s_i) - (n-1)e_i(s_i)$ is measurable and at any point either right-continuous or left-continuous.

Our additive separability assumptions are restrictive. We can interpret them as first order approximations. The independency assumption may sound unrealistic for a model with common values. Note, however, that this critic only applies when two or more players have private information about the common value whereas this is not the case in two of our motivating examples, see Sections 4.4.1 and 4.4.2. Moreover, there are other real life examples in which the independency assumption is reasonable, see Bergemann and Välimäki (2002). Finally, that ϕ_i is an increasing function is without loss of generality since we can always reorder the set S_i to satisfy this assumption. However, the fact that ϕ_i is strictly increasing is restrictive. Typically, we would expect this assumption to be violated when the original type space was multidimensional, see Footnote 9. The analysis of this case is done in Section 4.7.

4.3 First Best Efficiency

We call an *allocation* to a measurable function $p : S \rightarrow [0,1]^n$, where $S \equiv \prod_{i=1}^n S_i$, such that $\sum_{i=1}^n p_i(s) \leq 1$ for any $s \in S$. We are interested in the set of allocations that can be implemented in an auction mechanism. By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair of measurable functions (p, x) where p is an allocation and $x : S \rightarrow^n$ a payment function. In the direct mechanism (p, x), each bidder announces a type, and $p_i(s)$ denotes the probability that i gets the good and $x_i(s)$ her transfers to the auctioneer when the vector of announced types is $s \in S$.

The expected utility of bidder i with type s_i that reports s_i' when all the other bidders report truthfully is equal to:

$$U_i(s_i, s_i') \equiv Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x),$$

where[7]

$$Q_i(s_i', p) \equiv \int_{S_{-i}} p_i(s_i', s_{-i}) f_{-i} ds_{-i},$$

and,

$$\Psi_i(s_i', p, x) \equiv \int_{S_{-i}} \left(\sum_{j \neq i} \left(p_i(s_i', s_{-i}) q_j(s_j) - e_j(s_j) p_j(s_i', s_{-i})\right) - x_i(s_i', s_{-i}) \right) ds_{-i},$$

where $S_{-i} \equiv \prod_{j \neq i} S_j$ and $f_{-i}(s_{-i}) = \prod_{j \neq i} f_j(s_j)$.

Thus, we say that an allocation $p : S \to [0,1]^n$ is feasible if there exists a direct mechanism (p, x) that satisfies the following Bayesian incentive compatibility constraint:

$$U_i(s_i, s_i) = \sup_{s_i' \in S_i} \{U_i(s_i, s_i')\},$$

for all $s_i \in S_i$ and $i \in N$.

The following lemma characterizes the feasible allocation using a standard argument in mechanism design, see for instance Myerson (1981), Rochet (1985) and McAfee and McMillan (1988):

Lemma 4.1. *An allocation p is feasible if and only if $Q_i(p, s_i)$ is weakly increasing in s_i for all $s_i \in [0,1]$ and $i \in N$.*

Proof. We first prove the "only if"-part. Suppose a feasible allocation p. Then, there exists a direct mechanism (p, x) for which:

$$\begin{aligned} V_i(s_i) \equiv U_i(s_i, s_i) &\geq U_i(s_i, s_i') \\ &= Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x) \\ &= Q_i(s_i', p)\phi_i(s_i') + \Psi_i(s_i', p, x) + Q_i(s_i', p)(\phi_i(s_i) - \phi_i(s_i')) \\ &= V_i(s_i') + Q_i(s_i', p)(\phi_i(s_i) - \phi_i(s_i'), \end{aligned}$$

for all $s_i, s_i' \in S_i$, $i \in N$. Hence, we have that for $s_i > s_i'$, $V_i(s_i) \geq V_i(s_i')$, and hence,

$$Q_i(s_i', p) \leq \frac{V_i(s_i) - V_i(s_i')}{\phi_i(s_i) - \phi_i(s_i')},$$

and applying the same inequality with the roles of s_i and s_i' interchanged,

$$Q_i(s_i, p) \geq \frac{V_i(s_i) - V_i(s_i')}{\phi_i(s_i) - \phi_i(s_i')}.$$

[7]With some abuse of notation, we denote by $p_i(s_i, s_{-i})$ and $p_j(s_i, s_{-i})$ the function p_i and p_j, respectively, evaluated at a vector whose l-th component is equal to the l-th component of s_{-i} if $l < i$, it is equal to s_i if $l = i$ and it is equal to the $l-1$-th component of s_{-i} if $l > i$. We adopt the same convention for $x_i(t_i, t_{-i})$.

Thus, $Q_i(s_i, p) \geq Q_i(s_i', p)$ as desired.

To prove the "if"-part, suppose an allocation p for which $Q_i(., p)$ is increasing, for all $i \in N$. Note first that by assumption $\phi_i(.)$ is a strictly increasing function, and thus invertible in $[\phi_i(0), \phi_i(1)]$. Let $\tilde{V}_i(y) \equiv \int_{\phi_i(0)}^{y} Q_i(\phi_i^{-1}(\tilde{y}), p)\, d\tilde{y}$ for $y \in [\phi_i(0), \phi_i(1)]$ and,

$$x_i(s) \equiv Q_i(s_i, p)\phi_i(s_i) + \int_{S_{-i}} \sum_{j \neq i} \left(p_i(s_i, \tilde{s}_{-i}) q_j(\tilde{s}_j) - e_j(s_j) p_j(s_i, \tilde{s}_{-i})\right) f_{-i}(s_{-i})\, d\tilde{s}_{-i} - \tilde{V}_i(\phi_i(s_i)),$$

for any $s \in S$. This means that $\Psi_i(s_i, p, x) = \tilde{V}_i(\phi_i(s_i)) - Q_i(s_i, p)\phi_i(s_i)$, for any $s_i \in S_i$, and hence that $\tilde{V}_i(\phi_i(s_i)) = U_i(s_i, s_i)$ for the direct mechanism (p, x). We shall show that this direct mechanism satisfies the Bayesian incentive compatibility constraints. To see why, note that for any $s_i, s_i' \in S_i$

$$\begin{aligned}
U_i(s_i, s_i) &= \tilde{V}_i(\phi_i(s_i)) \\
&\geq \tilde{V}_i(\phi_i(s_i')) + Q_i(\phi_i^{-1}(\phi_i(s_i')), p)\left(\phi_i(s_i) - \phi_i(s_i')\right) \\
&= Q_i(s_i', p)\phi_i(s_i') + \Psi_i(s_i', p, x) + Q_i(s_i', p)\left(\phi_i(s_i) - \phi_i(s_i')\right) \\
&= Q_i(s_i', p)\phi_i(s_i) + \Psi_i(s_i', p, x) \\
&= U_i(s_i, s_i'),
\end{aligned}$$

where the inequality is a consequence of $\tilde{V}_i$ being a convex function and $Q_i(\phi_i^{-1}(y), p) \in \partial\tilde{V}_i(y)$ by definition of $\tilde{V}_i$. □

We use the following natural definition:

Definition 4.1. *We say that an allocation p is first best efficient when $p_i(s) > 0$ only if:*

$$v_i(s) - (n-1)e_i(s_i) = \max\{v_j(s) - (n-1)e_j(s_j)\}_{j=1}^{n},$$

and $\sum_{j=1}^{n} p_j(s) = 1$, for all $s \in S$. This is equivalent to say that an allocation p is first best efficient when $p_i(s) > 0$ only if:

$$h_i(s_i) = \max\{h_j(s_j)\}_{j=1}^{n},$$

and $\sum_{j=1}^{n} p_j(s) = 1$, for all $s \in S$.

We adapt the following definition to our framework:[8]

[8]The single crossing condition is usually formulated with the following alternative condition:

$$v_i(s) - (n-1)e_i(s_i) \geq \max\{v_j(s) - (n-1)e_j(s_j)\}_{j \neq i}$$

Definition 4.2. *We say that the single crossing condition is satisfied for bidder i if for any* $s, s' \in S$ *such that* $s'_i > s_i$ *and* $s_j = s'_j$ *for* $j \neq i$, *it is verified that*

$$v_i(s) - (n-1)e_i(s_i) > \max\{v_j(s) - (n-1)e_j(s_j)\}_{j\neq i}$$

implies that

$$v_i(s') - (n-1)e_i(s'_i) \geq \max\{v_j(s') - (n-1)e_j(s'_j)\}_{j\neq i}.$$

Our additive separability assumptions allow for a condition simpler to check in applications. Note that it basically says that $h_i(.)$ must be an increasing function at any point at which its value may determine the first best efficient allocation.

Lemma 4.2. *The single crossing condition for bidder i is satisfied if and only if for any* $s_i, s'_i \in S_i$ *such that* $s_i < s'_i$:

$$h_i(s_i) > h_i(s'_i) \text{ implies } \{s_{-i} \in S_{-i} : \max\{h_j(s_j)\}_{j\neq i} \in (h_i(s'_i), h_i(s_i))\} = \emptyset.$$

Proof. The definition of single crossing condition is equivalent to say that for any $i \in N$, $s \in S$ and $s'_i \in S_i$ such that $s'_i > s_i$:

$$h_i(s_i) > \max\{h_j(s_j)\}_{j\neq i} \text{ implies } h_i(s'_i) \geq \max\{h_j(s_j)\}_{j\neq i}. \tag{4.1}$$

This is equivalent to say that for any $i \in N$, $s \in S$ and $s'_i \in S_i$ such that $s'_i > s_i$,

$$h_i(s'_i) < \max\{h_j(s_j)\}_{j\neq i} \text{ implies } h_i(s_i) \leq \max\{h_j(s_j)\}_{j\neq i}. \tag{4.2}$$

The above two conditions are equivalent to say that for any $i \in N$, $s \in S$ and $s'_i \in S_i$ such that $s'_i > s_i$:

$$h_i(s_i) > \max\{h_j(s_j)\}_{j\neq i} > h_i(s'_i) \text{ implies } h_i(s'_i) \geq \max\{h_j(s_j)\}_{j\neq i} \geq h_i(s_i). \tag{4.3}$$

Since the consequence of the last condition cannot hold true, this last condition is equivalent to the condition in the lemma. □

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implies that

$$v_i(s') - (n-1)e_i(s'_i) > \max\{v_j(s') - (n-1)e_j(s'_j)\}_{j\neq i},$$

for any $s'_i > s_i$ and $s'_j = s_j$ for $j \neq i$.

This alternative condition is sufficient for feasibility of the first best. If we a add differentiability, it also implies the single crossing condition of Dasgupta and Maskin (2000), the pairwise single crossing condition, the average crossing condition and the cyclical crossing condition of Krishna (2003) and the generalized single crossing condition of Birulin and Izmalkov (2003). We have used our alternative definition to get also a necessary condition.

Proposition 4.1. *A necessary condition for the first best to be feasible is that the single crossing condition is satisfied for all bidders. Moreover, this condition is also sufficient if the set $\{s_i : h_i(s_i) = k\}$ has zero probability for any $i \in N$ and $k \in \mathbb{R}$.*

Proof. In the proof let $\overline{J}_i(s_i) \equiv \{s_{-i} : \max\{h_j(s_j)\}_{j\neq i} \leq h_i(s_i)\}$ and $\underline{J}_i(s_i) \equiv \{s_{-i} : \max\{h_j(s_j)\}_{j\neq i} < h_i(s_i)\}$. Note that for any efficient allocation p^*, $Q_i(s_i, p^*)$ is weakly greater than the probability of $\underline{J}_i(s_i)$ and weakly higher than the probability of $\overline{J}_i(s_i)$.

We start with the necessary part. We prove it by contradiction. Suppose that the single crossing condition does not hold. Then, by Lemma 4.2 there exists a bidder $i \in N$ with types $s_i' > s_i$ and a vector $s_{-i} \in S_{-i}$ such that $\max\{h_j(s_j)\}_{j\neq i} \in (h_i(s_i'), h_i(s_i))$. By either right or left continuity of the $h_j(.)$'s, there exists an open set $O \subset S_{-i}$ such that $\max\{t_j(s_j')\}_{j\neq i} \in (t_i(s_i'), t_i(s_i))$ for any $s_{-i}' \in O$. Thus, $\overline{J}_i(s_i') \cap O = \emptyset$ and $\overline{J}_i(s_i') \cup O \subset \underline{J}_i(s_i)$. Thus, by the above arguments we have that $Q_i(s_i', p^*) + \rho(O) \leq Q_i(s_i, p^*)$ for any efficient allocation p^* and where $\rho(O)$ denotes the probability of O. Since O is open $\rho(O) \neq 0$, which implies a violation of the feasibility conditions of Lemma 4.1.

For the sufficient part, note that the assumption that the set $\{s_i : h_i(s_i) = k\}$ has zero probability for any $i \in N$ and $k \in$ implies that for any first best efficient allocation p^*, $Q_i(s_i, p^*)$ is equal to the probability of both the set $\overline{J}_i(s_i)$ and the set $\underline{J}_i(s_i)$. Thus, the sufficient part follows since $\underline{J}_i(s_i) \subset \overline{J}_i(s_i')$ by Lemma 4.2, and then the condition of Lemma 4.1 is satisfied as desired. □

That a version of our single crossing condition is sufficient for feasibility of the first best is well known. The necessary part is a consequence of the additive structure of our model. Dasgupta and Maskin (2000) have also proved that a single crossing condition is necessary for a more demanding definition of feasibility of the first best.

4.4 Economic Applications

In this section, we provide some economic models in which the single crossing condition will typically fail. A setting that allows for generalizations of the following examples can be found in Section 4.7.

4.4.1 An Incumbent's Model

We now formalizes the wildcatters' example mentioned in the Introduction. Suppose the sale of a license to become a monopolist of a market with an inverse demand function

$P(Q) = 1 - \frac{Q}{s_1}$ where s_1 is equal to the realization of a random variable with distribution function F_1 with a density on the support $[\underline{s}, \overline{s}]$, $0 < \underline{s} < \overline{s}$.

We assume a set of firms N. One of the firms, that we call the *incumbent* and denote by $1 \in N$, has a constant marginal cost c_1, whereas each of the other firms, the *entrants*, has the same constant marginal cost c, $c < c_1 < 1$. We also assume that the incumbent has zero setup costs and that each of the entrants i has a setup cost $-s_i$, which is equal to the realization of a random variable with an independent distribution F_i with density and support $[-\underline{s}\frac{(1-c)^2}{4}, 0]$. The assumption on the support of s_i ensures that the entrants always find it profitable to buy the good at zero price, and that there is a set of types that value the license higher than any type of the incumbent. We assume that the value of each s_i is private information to bidder i.

The above example correspond to: $t_1(s_1) = h_1(s_1) = s_1\left(\frac{(1-c_1)^2}{4} - \frac{(1-c)^2}{4}\right)$, $q_1(s_1) = s_1\frac{(1-c)^2}{4}$, $e_1(s_1) = 0$, and $t_j(s_j) = h_j(s_j) = s_j$, $q_j(s_j) = 0$ and $e_j(s_j) = 0$, for $j \neq 1$. The single crossing condition is violated for bidder 1. To see why note that $h_1(\underline{s}) > h_1(\overline{s})$, $h_j(-\underline{s}\frac{(1-c)^2}{4}) < h_1(\underline{s})$ and $h_j(0) > h_1(\overline{s})$, $j \neq 1$. Hence, by continuity there exists an $s_j \in [-\underline{s}\frac{(1-c)^2}{4}, 0]$ such that $h_j(s_j) \in (h_1(\overline{s}), h_1(\underline{s}))$. Thus, the single crossing condition fails for bidder 1 by application of Lemma 4.2 to $s_i = \underline{s}$ and $s_i' = \overline{s}$, and $i = 1$.

4.4.2 An Insider's Model

In this example, we consider the sale of a good, say a painting, to some bidders whose value depends on their private taste and on the fact that the painting may be a copy or an original. We shall assume that only one of the bidders, *the insider* (perhaps an expert dealer), knows whether the painting is original.

Formally, assume that the value of the good to each bidder is equal to the sum of a private value τ_i and a common value ρ. Assume by now that each τ_i follows an independent and identical distribution G with density in the support $[\underline{t}, \overline{t}]$. Assume also that ρ takes value 0 (the painting is a copy) with probability α, and value $\overline{q}$ (the painting is original) with probability $1 - \alpha$. Suppose that each bidder observes privately her private value and that the insider, say bidder 1, also observes privately the common value.

For $\overline{q} + \underline{t} > \overline{t}$, we can make this example fit into our assumptions.[9] In particular,

[9]Note that in principle it is not direct how to fit the case $\overline{q} + \underline{t} \leq \overline{t}$ in our model. The reason is that in this case there exists two possible types of bidder 1, say s_1 and s_1', for which $(t(s_1), q(s_1)) \neq (t(s_1'), q(s_1'))$ and $\phi_1(s) = \phi_1(s')$, which violates our assumption that ϕ_1 must be strictly increasing. This case, however, may be analyzed using the framework of Section 4.7.

let s_1 be drawn according to a distribution $F_1(s_1) = \alpha G(s_1)$ if $s_1 < \overline{q} + \underline{t}$ and $F_1(s_1) = \alpha + (1-\alpha)G(s_1 - \overline{q})$ otherwise, and let s_i, $i \neq 1$ be drawn according to $G(.)$. Thus, we can make $\rho = 0$ and $\tau = s_1$ if $s_1 < \overline{q} + \underline{t}$ and $\rho = \overline{q}$ and $\tau = s_1 - \overline{q}$ otherwise. Then, the above example can be described in terms of our model by $t_1(s_1) = s_1$ and $q_1(s_1) = 0$ if $s_1 \in [\underline{t}, \overline{t})$ and $t_1(s_1) = s_1 - \overline{q}$ and $q_1(s_1) = \overline{q}$ if $s_1 \in [\overline{q} + \underline{t}, \overline{q} + \overline{t}]$; and by $t_i(s_i) = s_i$ and $q_i(t_i) = 0$, for $i \neq 1$. Moreover, $e_i(s_i) = 0$ for any i and s_i.

It is easy to verify that the single crossing condition is violated for bidder 1. To see why, apply Lemma 4.2 to $s_1 = \overline{t} - \epsilon$ and $s_1' = \underline{t} + \overline{q} + \epsilon$ for $\epsilon > 0$ and small enough.

4.4.3 A Model with Negative Externalities

Suppose n local markets, each with a unit mass of consumers with reservation value 1 for the consumption of the good. Suppose also n firms. Each firm starts with a branch in a local market. At the beginning, no two firms have a branch at the same local market. Firms can also open new branches at a fixed cost $C < 1$. Firms serve any local market in which they have a branch at a marginal cost c.

Suppose that a seller puts up for sale a technology that reduces the marginal costs of firm i by an amount s_i. Suppose that s_i is drawn from an independent distribution F_i with support $[0, 1]$. If only one firm serves a market, its profits are equal to $1 - c$. When more than one firm serves a local market, we assume an outcome consistent with Bertrand competition: the firm with the lowest marginal cost serves the market at a price equal to the second lowest marginal cost. In case of more than one firm with the lowest marginal cost, we assume that they split equally the demand at a price equal to their common marginal cost. Thus, a firm finds it profitable to open a branch in each of the other markets only if she has won the technology and the reduction in the marginal cost is sufficiently large, in particular, if and only if $s_i \geq C$.

Hence, in this model, for any i, $t_i(s_i) = s_i$ and $e_i(s_i) = 0$ if $s_i \leq C$, and $t_i(s_i) = s_i + (n-1)(s_i - C)$ and $e_i(s_i) = 1 - c$, otherwise. Note also that $q_i(s_i) = 0$ for any i and s_i. As consequence, $h_i(s_i) = s_i$ for $s_i \leq C$ and $h_i(s_i) = s_i - (n-1)(C + 1 - c - s_i)$, otherwise. In this case, the single crossing condition is violated for any bidder. To see why, apply Lemma 4.2 to $s_i = C - \epsilon$ and $s_i = C + \epsilon$, and $s_j \in (C - \epsilon, C)$ for all $j \neq i$, and $\epsilon > 0$ and small enough.

4.5 Second Best Efficiency

In light of Proposition 4.1, it is natural to define second best efficiency.

Definition 4.3. *We say that an allocation is second best efficient if it maximizes*

$$\int_S \sum_{i=1}^{n} (v_i(s) - (n-1)e_i(s_i))\, p_i(s) f(s)\, ds,$$

subject to p feasible.

Certainly, the set of second best allocations includes the first best allocation when the single crossing condition is satisfied.

It also turns out to be useful to define the following concept.

Definition 4.4. *We say that an allocation is second best efficient subject to always selling if it maximizes*

$$\int_S \sum_{i=1}^{n} (v_i(s) - (n-1)e_i(s_i))\, p_i(s) f(s)\, ds,$$

subject to p feasible and $\sum_{i=1}^{n} p_i(s) = 1$ *for any* $s \in S$. *An equivalent characterization is that it maximizes,*

$$\int_S \sum_{i=1}^{n} h_i(s_i) p_i(s) f(s)\, ds,$$

subject to p feasible and $\sum_{i=1}^{n} p_i(s) = 1$ *for any* $s \in S$.

To simplify the notation in what follows, we shall assume without loss of generality that each F_i is uniform on $[0,1]$.[10] To see why this assumption is without loss of generality note that if F_i were not uniform, we could always re-scale the vector of signals $\tilde{s}_i \equiv F_i(s_i)$ and value functions $\tilde{v}_i(\tilde{s}) \equiv \tilde{t}_i(\tilde{s}_i) + \sum_{i\in N} \tilde{q}_j(\tilde{s}_j)$ and $\tilde{e}_j(\tilde{s}_j)$ where $\tilde{t}_i(\tilde{s}_i) \equiv t_i(F_i^{-1}(\tilde{s}_i))$, $\tilde{q}_j(\tilde{s}_j) \equiv q_j(F_j^{-1}(\tilde{s}_j))$ and $\tilde{e}_j(\tilde{s}_j) \equiv e_j(F_j^{-1}(\tilde{s}_j))$. Note that each of the new signals $\tilde{s}_i$'s has a uniform distribution on $[0,1]$.[11]

To characterize the second best, we apply the "ironing technique" whose details were first described by Myerson (1981). Let $H_i(s_i) \equiv \int_0^{s_i} h_i(\tilde{s}_i)\, d\tilde{s}_i$ for all $i \in n$ and $s_i \in [0,1]$, and let $G_i(s_i) : [0,1] \to \mathbb{R}$ be the convex hull of the function H_i (i.e. the highest convex function on $[0,1]$ such that $G_i(s_i) \leq H_i(s_i)$ for all $s_i \in [0,1]$.) Formally:

$$G_i(s_i) = \min\left\{wH_i(r_1) + (1-w)H_i(r_2) : w, r_1, r_2 \in [0,1] \text{ and } wr_1 + (1-w)r_2 = s_i\right\}.$$

Lemma 4.3. *Properties of* G_i*:*

[10] Lehmann (1988) was the first one showing that there is no loss of generality in assuming that signals have a uniform marginal distribution.

[11] To see why, note that the probability of $\{\tilde{s}_i \leq z\}$ for $z \in [0,1]$ is equal to the probability of $\{F_i(s_i) \leq z\}$ which is equal to the probability of $\{s_i \leq F_i^{-1}(z)\}$ and thus, it is equal to $F_i(F_i^{-1}(z)) = z$.

- *G_i is convex.*
- *$G_i(0) = H_i(0)$ and $G_i(1) = H_i(1)$.*
- *$G_i(t_i) \leq H_i(t_i)$ for all $t_i \in [0,1]$.*
- *If $G_i(t_i) < H_i(t_i)$ in an open interval, then G_i is linear in the same open interval.*

Proof. All the properties in the proposition follow from the application of standard mathematical arguments that we do not reproduce to the definition of convex hull. □

As a convex function G_i is differentiable except at countably many points, and its derivative is a non-decreasing function. We define $g_i : [0,1] \rightarrow$ to be the differential of G_i completed by right-continuity.

Proposition 4.2. *An allocation p^* such that $\sum_{i=1}^n p_i^*(s) = 1$ for all s, is second best efficient subject to always selling if $\forall i \in N$:*

(i) $p_i^*(s) > 0$ only if $g_i(s_i) = \max\{g_j(s_j)\}_{j\in N}$ a.e.;

(ii) $Q_i(., p^*)$ is constant in any open interval in which $g_i(.)$ is constant.[12]

Moreover, if $\max_{i\in N} g_i(s_i) + \sum_{j=1}^n q_j(s) \geq 0$ for any $s \in S$, then p^* is also second best efficient.

Proof. The second best maximizes:

$$\int_S \sum_{i=1}^n (v_i(t) - (n-1)e_i(s))\, p_i(s)\, ds = \int_S \sum_{i=1}^n \left(h_i(s_i) + \sum_{j=1}^n q_j(s_j) \right) p_i(s)\, ds. \tag{4.4}$$

Next note that using integration by parts we can show that,

$$\begin{aligned}
\int_S (h_i(s_i) - g_i(s_i))\, p_i(s) ds = \int_{S_i} (h_i(s_i) - g_i(s_i))\, Q_i(s_i, p)\, ds_i = \\
[(H_i(s_i) - G_i(s_i))Q_i(s_i, p)]_{s_i=0}^{s_i=1} - \int_{S_i} (H_i(s_i) - G_i(s_i))\; Q_i(ds_i, p) = \\
- \int_{S_i} (H_i(s_i) - G_i(s_i))\; Q_i(ds_i, p),
\end{aligned} \tag{4.5}$$

where the last step follows from item 2 of Lemma 4.3.

[12]There always exists an allocation that satisfies both properties. For instance, $p_i(t) = 0$ if $g_i(t_i) \neq \max_{j\in N} g_j(t_j)$, and otherwise, $p_i(t) = \frac{1}{m(t)}$, where $m(t)$ denotes the cardinality of $\{k \in N : g_k(t_k) = \max_{j\in N} g_j(t_j)\}$. The monotonicity of $g_i(t_i)$ ensures that the allocation is feasible.

Consequently, we can write Equation (4.4) as,

$$\int_S \sum_{i=1}^n \left(g_i(s_i) + \sum_{j=1}^n q_j(s_j) \right) p_i(s)\, ds + \sum_{i=1}^n \int_{S_i} (G_i(s_i) - H_i(s_i))\, Q_i(ds_i, p). \qquad (4.6)$$

Thus, if p^* satisfies (i), it maximizes the first integral among all the allocations p for which $\sum_{i=1}^n p_i(s) = 1$ for all s. Similarly, if p^* satisfies (i) and $\max_{i \in N} g_i(s_i) + \sum_{j=1}^n q_j(s_j) \geq 0$, p^* maximizes the first integral among all allocations. Next, note that the second integral is non-positive since $Q_i(.,p)$ is increasing for any p feasible by Lemma 4.1, and $G_i(s_i) \leq H_i(s_i)$, by item 3 of Lemma 4.3. Moreover, note that this second integral is equal to zero for any p that satisfies (ii). This is because item 4 of Lemma 4.3 implies that $g_i(.)$ is constant, and so $Q_i(.,p)$, in any open interval in which $G_i(s_i) < H_i(s_i)$. Consequently, an allocation p^* that satisfies (ii) maximizes the second integral. □

However, if $\max_{i \in N} g_i(s_i) + \sum_{j=1}^n q_j(s_j) < 0$ in an open interval, the second best allocation may imply that the good is unsold for some bidders' types. To see why note the following example:

Example 4.1. *$N = \{1, 2\}$, $v_i(s) = s_i + 2s_j$ and $e_i(s_i) = 0$ for $i, j \in \{1, 2\}$ and $i \neq j$.*

Proposition 4.3. *Any second best allocation of Example 1 satisfies $p_1(s) + p_2(s) = 0$ for $s_1 + s_2 \leq 1/4$ a.e.*

Proof. For our example, $h_i(s_i) = -s_i$, $H_i(s_i) = -\frac{s_i^2}{2}$, and thus, $G_i(s_i) = -\frac{s_i}{2}$ and $g_i(s_i) = -\frac{1}{2}$. Hence, the corresponding Equation (4.6) to Example 1 is:

$$\int_{[0,1]^2} \sum_{i=1,2} \left(-\frac{1}{2} + 2s_1 + 2s_2 \right) p_i(s)\, ds + \sum_{i=1,2} \int_0^1 \left(-\frac{s_i}{2} + \frac{s_i^2}{2} \right) Q_i(ds_i, p). \qquad (4.7)$$

The first integral is maximized by any allocation that satisfies a.e. $p_1(s_1, s_2) + p_2(s_1, s_2) = 0$ if $s_1 + s_2 < 1/4$, and $p_1(s_1, s_2) + p_2(s_1, s_2) = 1$ otherwise, whereas the second integral is maximized by any allocation such that $Q_i(.,p)$, $i = 1, 2$, is constant. An example of an allocation that satisfies these two conditions is the following:[13]

- $p_1(s) = p_2(s) = 0$ if $s_1 + s_2 \leq 1/4$.
- $p_1(s) = \frac{17}{48} + \frac{2}{3}s_2$ and $p_2(t) = 1 - p_1(s)$, if $s_1 \geq 1/4$, $s_2 < 1/4$.

[13]Regretfully, we cannot always find the maximum of Equation (4.6) by maximizing both integrals simultaneously and thus a more complex analysis is required. We illustrate this claim in the appendix with a symmetric and an asymmetric example.

- $p_1(s) = 1 - \left(\frac{17}{48} + \frac{2}{3}s_1\right)$ and $p_2(s) = 1 - p_1(s_1, s_2)$, if $s_2 \geq 1/4$, $s_1 < 1/4$.
- $p_1(s) = p_2(s) = 1/2$ otherwise.

Note that $Q_1(s_1) = Q_2(s_2) = \frac{31}{64}$ for any $s_1, s_2 \in [0,1]$. □

To get a superficial intuition of the result note that in the example it is efficient to allocate the good to bidder 2 when her signal is low and bidder 1's signal is high. Similarly, it is efficient to allocate the good to Bidder 1 when her signal is low and bidder 2's signal is high. However, this allocation is difficult to implement because both bidders have very little incentives to report truthfully when their signal is high. They know that by reporting a high signal they increase the probability that the good goes to the other bidder. Thus, no selling if both bidders report low makes it easier to induce truth-telling when signals are high. Moreover, the cost in terms of social surplus is small since the value of both bidders is close to zero when both signals are low.

Finally, it is interesting to remark that if there exists a bidder for which $g_i = h_i$, the condition in Proposition 4.2 that ensures that it is second best efficient to always sell is always verified. To see why, note that $g_i = h_i$ implies that $g_i(s_i) + \sum_{j=1}^{n} q_j(s_j) = h_i(s_i) + \sum_{j=1}^{n} q_j(s_j)$ which is equal to $v_i(s) - (n-1)e_i(s_i)$ and thus non-negative.[14] It is easy to see that $h_i = g_i$ for $i \neq 1$ in the examples in Sections 4.4.1 and 4.4.2. More generally, this condition is verified by any example in which no more than one bidder has private information about the common value and/or induces an allocative externality on the other bidders.

4.6 English Auction

In this section we analyze whether the second best can be implemented with an English auction. In particular, we assume the model of the English auction described by Krishna (2003). This auction model is a variation of the Japanese auction proposed by Milgrom and Weber (1982) in which the identity of the bidders is observable.

We introduce two additional assumptions. The first one is a simplification, we assume that the functions h_i's are continuous. This assumption also implies:

Lemma 4.4. *The functions g_i's are continuous. Moreover:*

- $g_i(s_i) = h_i(s_i)$ *if* $G_i(s_i) = H_i(s_i)$ *and* $s_i \in (0,1)$.

[14]Recall that by assumption $v_i(s) - (n-1)e_i(s_i) \geq 0$.

- $g_i(0) \leq h_i(0)$ *with strict inequality only if* $G_i(\epsilon) < H_i(\epsilon)$ *for any* $\epsilon > 0$ *small enough.*

- $g_i(1) \geq h_i(1)$ *with strict inequality only if* $G_i(1-\epsilon) < H_i(1-\epsilon)$ *for any* $\epsilon > 0$ *small enough.*

Proof. The function g_i cannot be discontinuous at points in an open interval in which $G_i(s_i) = H_i(s_i)$ by continuity of h_i, or at points in an open interval in which $G_i(s_i) \neq H_i(s_i)$ by item 4 of Lemma 4.3. Take now a point $s_i^* \in (0,1)$ in which to the left $H_i(s_i) = G_i(s_i)$ and to the right $H_i(s_i) > G_i(s_i)$ (the other case is symmetric). Then the left derivative of G_i is equal to $h_i(s_i^*)$ and the right derivative is bounded above by $h_i(s_i^*)$. Moreover, by the convexity of G_i the left derivative of G_i must be less than or equal to the right derivative. As a consequence, G_i is differentiable at s_i^* and its differential $g_i(s_i^*)$ is equal to $h_i(s_i^*)$. Continuity at 0 and 1 together with the last two items of the lemma are direct consequences of items 2 and 3 in Lemma 4.3 and the boundedness of h_i. □

The other assumption is that $\zeta_i(s_i) \equiv q_i(s_i) + e_i(s_i)$ is non-decreasing. This assumption ensures that our proposed bidding functions are increasing. This assumption translates to our set-up a similar assumption used in the study of first best efficiency in English auctions, Maskin (2000), Krishna (2003) and Birulin and Izmalkov (2003), and that basically requires that the bidders' values are non-decreasing functions of the bidders' types.

We start remarking that in general there is very little hope that the English auction can implement the second best when it implies no selling. To see why, recall that in Example 1, second best requires no selling if $s_1 + s_2 \leq 1/4$. However, an English auction with an entry fee and/or a reserve price can only ensure no selling for sets of types $\{(s_1, s_2) \in [0,1]^2 : s_1, s_2 \geq \underline{s}\}$ for some $\underline{s} \in [0,1]$. A more general analysis of this question remains open.

We shall see that our conclusions depend on whether the number of bidders is equal to two or it is larger than two. We start with the former case.

4.6.1 The Two-Bidder Case

Suppose in this subsection that $n = 2$. We shall show that in this case, the English auction implements the second best efficient allocation. To prove so, we follow three steps. First, we propose a bid function for each bidder; second, we prove that they allocate the good according to the second best allocation; and finally, we show that they are an equilibrium of the English auction.

We start with some auxiliary definitions. Let $\underline{s}_i^j, \overline{s}_i^j : [0,1] \to [0,1]$, $i \neq j$, be such that $\underline{s}_i^j(s_j)$ and $\overline{s}_i^j(s_j)$ are equal to the minimum and maximum s_i, respectively, that solve $g_i(s_i) = g_j(s_j)$, if $g_i(s_i) \in [g_j(0), g_j(1)]$, this is when our equation has a solution. In case the equation has no solution, we let $\underline{s}_i^j(s_j)$ and $\overline{s}_i^j(s_j)$ be both equal to zero if $g_j(s_j) < g_i(0)$ and equal to one if $g_j(s_j) > g_i(1)$.

Let $b_1^*(s_1) \equiv v_1^e(s_1, \overline{s}_2^1(s_1))$, and $b_2^*(s_2) \equiv v_2^e(\underline{s}_1^2(s_2), s_2)$ where $v_i^e(s_i, s_j) \equiv v_i(s_i, s_j) + e_j(s_j)$ (this is $v_i^e(s_i, s_j) = \phi_i(s_i) + \zeta_j(s_j)$). Thus, we propose that bidder 1 (respectively bidder 2) bids her maximum willingness to pay to obtain the object when the alternative is that the good goes to the other bidder conditional on the hypothetical event that the signal of the other bidder is equal to $\overline{s}_2^1(s_1)$ (respectively $\underline{s}_1^2(s_2)$.) To understand the intuitive meaning of this hypothetical event note first the following auxiliary result:

Lemma 4.5.

- $b_1^*(s_1) \geq b_2^*(s_2)$ *if and only if* $g_1(s_1) \geq g_2(s_2)$.
- $b_1^*(s_1) < b_2^*(s_2)$ *if and only if* $g_1(s_1) < g_2(s_2)$.

Proof. We only prove the first item. The second one is simply the negation of the first one.

We first show the "if" part. Consider first the case $g_1(0) > g_2(1)$. By monotonicity of the bid functions, we only need to show that $g_1(0) > g_2(1)$ implies that $b_1(0) \geq b_2(1)$. Note that it is easy to see that $b_1(0) - b_2(1) = \phi_1(0) - \zeta_1(0) + (\phi_2(1) - \zeta_2(1)) = h_1(0) - h_2(1)$, which is greater than $g_1(0) - g_2(1)$ by Lemma 4.4, and thus non-negative as desired. Consider now the case $g_1(0) \leq g_2(1)$. In this case, $g_1(s_1) \geq g_2(s_2)$ implies that there exists $s_1' \leq s_1$ and $s_2' \geq s_2$ such that $g_1(s_1') = g_2(s_2')$, $s_1' = \underline{s}_1^2(s_2')$ and $s_2' = \overline{s}_2^1(s_1')$. Thus:

$$
\begin{aligned}
b_1^*(s_1) - b_2^*(s_2) \geq b_1^*(s_1') - b_2^*(s_2') = \\
[\phi_1(s_1') - \zeta_1(\underline{s}_1^2(s_2'))] - [\phi_2(s_2') - \zeta_2(\overline{s}_2^1(s_1'))] = \\
[\phi_1(\underline{s}_1^2(s_2')) - \zeta_1(\underline{s}_1^2(s_2'))] - [\phi_2(\overline{s}_2^1(s_1')) - \zeta_2(\overline{s}_2^1(s_1'))] = \\
h_1(\underline{s}_1^2(s_2')) - h_2(\overline{s}_2^1(s_1')),
\end{aligned}
$$

which by application of Lemma 4.4 and the definition of $\underline{s}_1^2$ and $\overline{s}_2^1$ is weakly greater than $g_1(\underline{s}_1^2(s_2')) - g_2(\overline{s}_2^1(s_1'))$. This last expression is equal to zero as a consequence of the properties that define s_1' and s_2'.

We prove the "only if" part by contradiction. We shall show that $g_2(s_2) > g_1(s_1)$ implies that $b_2(s_2) > b_1(s_1)$. The proof is similar to the "if" part. The case $g_2(0) > g_1(1)$ is symmetric to the case $g_1(0) > g_2(1)$ above. In the case $g_2(0) \leq g_1(1)$, $g_2(s_2) > g_1(s_1)$ implies that there exists a decreasing sequence $\{s_{2,m}\}$ starting at s_2 and an increasing sequence $\{s_{1,m}\}$ starting at s_1 with respectively limits s_2' and s_1' that satisfy $g_2(s_2') = g_1(s_1')$, $s_2' = \underline{s}_2^1(s_1')$ and $s_1' = \overline{s}_1^2(s_2')$. Note that along the sequence $g_2(s_{2,m}) > g_1(s_{1,m})$ and hence, $s_{2,m} \geq \overline{s}_2^1(s_{1,m})$ and $s_{1,m} \leq \underline{s}_1^2(s_{2,m})$. Using these properties and the monotonicity of the bid functions we have that:

$$b_2^*(s_2) - b_1^*(s_1) > \lim_{m\to\infty} [b_2^*(s_{2,m}) - b_1^*(s_{1,m})] =$$
$$\lim_{m\to\infty} \left[\left(\phi_2(s_{2,m}) - \zeta_2(\overline{s}_2^1(s_{1,m}))\right) - \left(\phi_1(s_{1,m}) - \zeta_1(\underline{s}_1^2(s_{2,m}))\right) \right] =$$
$$\lim_{m\to\infty} \left[\left(h_2(s_{2,m}) + \zeta_2(s_{2,m}) - \zeta_2(\overline{s}_2^1(s_{1,m}))\right) - \left(h_1(s_{1,m}) + \zeta_1(s_{1,m}) - \zeta_1(\underline{s}_1^2(s_{2,m}))\right) \right] \geq$$
$$\lim_{m\to\infty} [h_2(s_{2,m}) - h_1(s_{1,m})] = h_2(\underline{s}_2^1(s_1')) - h_1(\overline{s}_1^2(s_2')),$$

which by the same arguments as in the "if" part is non-negative as desired. □

Thus, $b_2^*(s_2) \in (b_1^*(s_1 - \epsilon), b_1^*(s_1 + \epsilon)]$ for $\epsilon > 0$ is equivalent to $s_2 \in (\overline{s}_2^1(s_1 - \epsilon), \overline{s}_1^2(s_1 + \epsilon)]$, and consequently, the hypothetical event on which bidder 1 conditions to compute her bid is the limit of $b_2^*(s_2) \in (b_1^*(s_1 - \epsilon), b_1^*(s_1 + \epsilon)]$ when ϵ goes to zero.

Since both bid functions b_1^* and b_2^* are strictly increasing ties happen with zero probability. Thus, Lemma 4.5 implies that both conditions in Proposition 4.2 are satisfied and as a consequence:

Corollary 4.1. *The allocation induced by* (b_1^*, b_2^*) *is second best efficient conditional on always selling.*

Finally, next proposition shows that the proposed bid functions are in fact an equilibrium:

Proposition 4.4. *The bid functions* (b_1^*, b_2^*) *characterise a Bayesian Nash equilibrium of the English auction.*

Proof. We only show that Bidder 1 finds it optimal to bid according to b_1^* when Bidder 2 plays b_2^*. The corresponding proof for Bidder 2 is symmetric. Let $u_1(s_1, b)$ be the expected utility of Bidder 1 when she has a private type s_1, submits a bid b, and Bidder 2 uses the bid function b_2^*. We only show that Bidder 1 does not have incentives to deviate downwards, i.e. $u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) \geq 0$ for $b < b_1^*(s_1)$. The analysis of incentives to deviate upwards, i.e. $b > b_1^*(s_1)$, is symmetric. Downward deviations only affect payoffs when Bidder 1 wins with $b_1^*(s_1)$ and loses with b. If we ignore ties, this happens when $b_2^*(S_2) \in (b, b_1^*(s_1)]$. In this case, if Bidder 1 wins, she gets a good with value $v_1(s_1, s_2)$ and pays Bidder 2's bid and if bidder 1 loses she suffers a negative externality equal to $e_2(s_2)$. Thus, the change in utility when bidder 1 wins is equal to $v_1^e(s_1, s_2) - v_2^e(s_2, \underline{s}_1^2(s_2)) = \phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2)$. By Lemma 4.5, $b_2^*(s_2) \leq b_1^*(s_1)$ is equivalent to $g_2(s_2) \leq g_1(s_1)$, and thus to $s_2 \leq \overline{s}_2^1(s_1)$, and $s_1 \geq \underline{s}_1^2(s_2)$. Let $\tau_2(b)$ be the infimum of $\{s_2 \in [0,1] : b_2^*(s_2) > b\}$ if it exists, and zero otherwise. Then:

$$
\begin{aligned}
&u_1(s_1, b_1^*(s_1)) - u_1(s_1, b) = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left(\phi_1(s_1) + \zeta_2(s_2) - \phi_2(s_2) - \zeta_1(\underline{s}_1^2(s_2))\right) ds_2 = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} [\phi_1(s_1) - \zeta_1(\underline{s}_1^2(s_2))] - [\phi_2(s_2) - \zeta_2(s_2)]\, ds_2 \geq \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} [\phi_1(\underline{s}_1^2(s_2)) - \zeta_1(\underline{s}_1^2(s_2))] - [\phi_2(s_2) - \zeta_2(s_2)]\, ds_2 = \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} \left[h_1(\underline{s}_1^2(s_2)) - h_2(s_2)\right] ds_2 \geq \\
&\int_{\tau_2(b)}^{\overline{s}_2^1(s_1)} (g_2(s_2) - h_2(s_2))\, ds_2
\end{aligned}
$$

where we have used that ϕ_1 is strictly increasing and $s_1 \geq \underline{s}_1^2(s_2)$ in the first inequality and that $h_1(\underline{s}_1^2(s_2)) \geq g_2(s_2)$, by Lemma 4.4 and the definition of $\underline{s}_1^2$, in the second inequality.

Finally, we argue that the last expression is non-negative. This integral is equal to

$$\left[G_2(\overline{s}_2^1(s_1)) - H_2(\overline{s}_2^1(s_1))\right] + \left[H_2(\tau_2(b)) - G_2(\tau_2(b))\right].$$

The second difference is non-negative by the third item of Lemma 4.3. We next argue that the first one is equal to zero. If $\overline{s}_2^1(s_1)$ is either zero or one, this is because of the second item of Lemma 4.3; otherwise, it is because $G_2(\overline{s}_2^1(s_1)) < H_2(\overline{s}_2^1(s_1))$ would imply that g_2 is constant around $\overline{s}_2^1(s_1)$ by item 4 of Lemma 4.3, which is a contradiction with the definition of $\overline{s}_2^1(s_1)$. □

We can thus conclude from Corollary 4.1 and Proposition 4.4,

Corollary 4.2. *The English auction implements the second best conditional on always selling when there are two bidders.*

A careful reader may worry that the above bid functions are identity dependent. We shall argue that in general this is not a problem. First, let $K_i \equiv \{k \in : \exists(\underline{s}, \overline{s}) \neq \emptyset, g_i(s_i) = k, \forall s_i \in (\underline{s}, \overline{s})\}$ and note the following lemma:

Lemma 4.6. *If $K_1 \cap K_2 = \emptyset$, then $\underline{s}_j^i(s_i) = \overline{s}_j^i(s_i)$ almost everywhere.*

Proof. Since any increasing function can be discontinuous in at most countably many points, it is sufficient to show that $\underline{s}_j^i$ is discontinuous at any point $s_i \in [0,1]$ for which $\underline{s}_j^i(s_i) < \overline{s}_j^i(s_i)$. First, note that g_j is constant and equal to $g_i(s_i)$ in $(\underline{s}_j^i(s_i), \overline{s}_j^i(s_i))$ by definition of $\underline{s}_j^i$ and $\overline{s}_j^i$. Since $K_1 \cap K_2 = \emptyset$, this means that g_i is not constant at s_i but strictly increasing. Hence, $\overline{s}_j^i(s_i - \epsilon) < \underline{s}_j^i(s_i + \epsilon)$ for $\epsilon > 0$ and small. We can, thus, deduce that $\underline{s}_j^i(s_i) < \overline{s}_j^i(s_i)$ implies that $\underline{s}_j^i$ is discontinuous at s_i. □

Since asymmetries only arise when $\underline{s}_j^i(s_i) \neq \overline{s}_j^i(s_i)$, asymmetries only occur in a zero measure set of types, and thus, could be removed without upsetting the equilibrium, when $K_1 \cap K_2 = \emptyset$. But, we expect $K_1 \cap K_2 = \emptyset$ to hold generically since the sets K_i's are countable.[15]

Note, however, that the symmetric model, i.e. when $\phi_1 = \phi_2$, and $\zeta_1 = \zeta_2$, is not generic but deserves special interest and verifies that $K_1 \cap K_2 \neq \emptyset$. Moreover, if the single crossing condition fails, the symmetric equilibrium is not second best efficient subject to always selling. To see why recall Example 1 and note that its symmetric equilibrium is characterised by the bid function $b(s_i) = 3s_i$. Thus, the bidder with higher type wins. However, this allocation is precisely the one that minimizes the expected value of the winning bidder, at least among the allocations that always allocate the good to one of the bidders. It is easy to see that our proposed second best equilibrium in this example is characterised by $b_1^*(s_1) = s_1 + 2$ and $b_2^*(s_2) = s_2$.

As we shall show next by means of an example, even when $K_1 \cap K_2 = \emptyset$ the problems of multiplicity of equilibria in the English auction are more severe when the single crossing condition fails than when it holds:

Example 4.2. $N = \{1, 2\}$, $v_1(s_1, s_2) = s_1 + 2s_2$ *and* $v_2(s_1, s_2) = s_2 + 2s_1 + .1$.

In the above example $h_1(s_1) = -s_1$ and $h_2(s_2) = -s_2 + .1$. Thus, $g_1(s_1) = -\frac{1}{2}$ and $g_2(s_2) = -\frac{2}{5}$, and consequently, the second best allocation is to give the good to Bidder 2 for any realization of the bidders' types. However, it is easy to see that the following

[15] A simple argument is as follows. Let $g_i^{-1}(k) \equiv \min\{s_i \in [0,1] : g_i(s_i) = k\}$. It is easy to see that if g_i is constant and equal to k in an open interval, then the function g_i^{-1} is discontinuous at k by definition. Finally, note Note that the set of the discontinuities of g_i^{-1} must be countable since g_i^{-1} is increasing.

strategies constitute an equilibrium: $b_1(s_1) = s_1 + 2$ and $b_2(s_2) = s_2 + .1$. Moreover, we are not able to find an iterative elimination of weakly dominated strategies that eliminates neither this equilibrium nor the one we have proposed for implementation of the second best. Thus we do not expect that the refinement used by Chung and Ely (2001) to get uniqueness works when the single crossing condition fails.

We would like to conclude the section with a positive result regarding the implementation of the second best when there are only two bidders. Suppose that one of the two bidders' valuations (say the one of bidder i) depends only on his own signal (i.e. $\zeta_j(s_j) = 0$). Then any equilibrium in non-weakly dominated strategies implements the second best. In any such equilibrium bidder i is active up to his value. It can be shown that any strategy that leads bidder j to win for a set of type different from the one prescribed by the second best yields strictly less profits for bidder j.[16]

4.6.2 The case $n > 2$

We follow the same steps as in the previous section to prove that there exists an equilibrium of the English auction that implements the second best.

Let,

$$\tilde{g}_i(s_i) \equiv g_i(s_i) + \int_0^{s_i} \mathbf{1}_{\{g_i'(\tilde{s}_i)=0\}}\, d\tilde{s}_i + \sum_{l<i} \int_0^{\underline{t}_l^i(s_i)} \mathbf{1}_{\{g_l'(\tilde{s}_l)=0\}}\, d\tilde{s}_l + \sum_{l>i} \int_0^{\overline{s}_l^i(s_i)} \mathbf{1}_{\{g_l'(\tilde{s}_l)=0\}}\, d\tilde{s}_l.$$

The function $\tilde{g}_i(s_i)$ is constructed to ensure that (i) $\tilde{g}_i'(s_i) > 0$ for all $s_i \in [0,1]$ and that (ii) $g_i(s_i) > g_j(s_j)$ implies that $\tilde{g}_i(s_i) > \tilde{g}_j(s_j)$ for all $i, j \in N$, and $s_i, s_j \in [0,1]$.

Denote by $m \leq n$ the number of active bidders, define the following strategies for $\epsilon > 0$ small enough:[17]

- If $m \geq 2$, then $b_i^{*,m}(s_i) \equiv \epsilon \tilde{g}_i(s_i)$, for all $i \in [0,1]$ and $s_i \in [0,1]$.
- If $m = 2$, then $b_i^{*,2}(s_i|\{(k,p_k)\}) \equiv v_i(\hat{s})$, where $\{(k,p_k)\}$ is the set identities and prices at which the other bidders have quit, and $\hat{s} \in^n$ is a vector whose i-th component is equal to s_i, whose j-th component, where j is the identity of the other active bidder, is equal to $\underline{s}_i^j(s_i)$ if $i > j$ and to $\overline{s}_i^j(s_i)$ if $i < j$, and whose k-th, $k \neq i, j$ component is equal to $\tilde{g}_k^{-1}(p_k/\epsilon)$, where p_k is the price at which bidder k quit.

[16]We are currently formalizing the proof of this fact.

[17]If ϵ is small enough, our proposed strategies are strictly monotone. Otherwise, it could happen that $b_i^{*,3}(s_i) > b_i^{*,2}(s_i|b_{-i,j})$ for some values.

We next provide an auxiliary lemma.

Lemma 4.7. *For any $i > j$ and $s_i, s_j \in [0,1]$,*

- $b_i^{*,m}(s_i) \geq b_j^{*,m}(s_j)$ *if and only if* $g_i(s_i) \geq g_j(s_j)$, *for* $m > 2$.
- $b_i^{*,m}(s_i) < b_j^{*,m}(s_j)$ *if and only if* $g_i(s_i) < g_j(s_j)$, *for* $m > 2$.
- $b_i^{*,2}(s_i|b_{-i,j}) \geq b_j^{*,2}(s_j|b_{-i,j})$ *if and only if* $g_i(s_i) \geq g_j(s_j)$.
- $b_i^{*,2}(s_i|b_{-i,j}) < b_j^{*,2}(s_j|b_{-i,j})$ *if and only if* $g_i(s_i) < g_j(s_j)$.

Proof. The first two items are a direct consequence of the definition of $\tilde{g}_i$. The third item, and thus the fourth which is its negation, have the same proof as Lemma 4.5. Simply note that for $i > j$,

$$b_i^{*,2}(s_i|b_{-i,j}) - b_j^{*,2}(s_j|b_{-i,j}) =$$
$$\phi_i(s_i) + \zeta_j(\overline{s}_i^j(s_i)) + \sum_{k\neq i,j} q_k([b_k^{*,m}]^{-1}(b_k)) - \phi_j(s_j) - \zeta_i(\underline{s}_i^j(s_j)) - \sum_{k\neq i,j} q_k([b_k^{*,m}]^{-1}(b_k)) =$$
$$\left[\phi_i(s_i) - \zeta_i(\underline{s}_i^j(s_j)\right] - \left[\phi_j(s_j) - \zeta_j(\overline{s}_i^j(s_i))\right].$$

□

Again, since the bid functions are strictly increasing ties happen with probability zero and thus Lemma 4.7 implies that both conditions in Proposition 4.2 are satisfied and as a consequence:

Corollary 4.3. *The allocation induced by $\{b_i^{*,m}\}$ is second best efficient.*

Finally, we show that our proposed strategies constitute an equilibrium.

Proposition 4.5. *The strategies $\{b_i^{*,m}\}_{i\in I}$ and $\{b_i^{*,2}\}_{i\in I}$ constitute a Perfect Bayesian equilibrium of the English auction.*

Proof. We shall assume that all the bidders but one, Bidder i, follow the proposed strategies. For any vector of bidders' types, let j be the bidder whose type s_j is such that $\tilde{g}_j(s_j) = \max_{k\neq i}\{\tilde{g}_k(s_k)\}$, i.e. is highest among the other bidders' types re-scaled by the functions $\tilde{g}_l$. We denote this vector by $s_{-i,j}$. We shall show that Bidder i does not have incentives to deviate in the more stringent case in which she observes the vector of bidders' types but Bidder j's.

Let $u_i(s_i, s_j, b, s_{-i,j})$ be the expected utility of Bidder i with type s_i, when she quits at price b, the other bidders follow the proposed bid functions and $s_{-i,j}$ is the vector of all the other bidders' types but Bidder j. Assume in what follows that $i > j$ and consider only downward deviations. The other cases have a symmetric analysis. Downward deviations are payoff relevant conditional on $s_{-i,j}$ if the proposed strategy implies that Bidder i remains active after all bidders $k \neq j$ have quit. In this case, if Bidder i wins she gets a good that she values:

$$t_i(s_i) + \sum_{k \in N} q_k(s_k).$$

and pays a price equal to Bidder j's bid, which when all the other bidders follow the proposed strategy is equal to:

$$t_j(s_j) + q_i(\underline{s}_i^j(s_j)) + \sum_{k \neq i} q_k(s_k) + \zeta_i(\underline{s}_i^j(s_j)).$$

Whereas if i loses, she gets utility $-e_j(s_j)$.

Thus, the difference in utility between winning and losing is equal to $\left[\phi_i(s_i) - \zeta_i(\underline{s}_i^j(s_j)\right] - [\phi(s_j) - \zeta(s_j)]$. The remaining steps are similar to the proof of Proposition 4.4 where the only difference is that we use,

$$s_j \leq \tau_j(b_i^{*,2}(s_i)) \Leftrightarrow b_j^{*,2}(s_j|b_{-i,j}) \leq b_i^{*,2}(s_i|b_{-i,j}) \Leftrightarrow g_j(s_j) \leq g_i(s_i). \tag{4.8}$$

which is a consequence of Lemma 4.7. □

Corollary 4.4. *The English auction implements the second best when there are three or more bidders.*

We argue that the generalization of the $n = 2$ equilibrium to the $n > 2$ case presented above does sometimes lead to equilibria that are based on weakly dominated strategies. A simple case of interest where this holds true is when one of the bidders' values (say the one of bidder i) is independent of the other bidders' signals (i.e $\zeta_j(s_j) = 0$, for any $j \neq i$). Recall that we argued that under this condition all equilibria in non weakly dominated strategy implement the second best, when $n = 2$. To see what might go wrong with $n > 2$, suppose that bidder i plays her weakly dominant strategy to be active up to her value. Suppose also that the single crossing condition is violated for this bidder. This latter fact implies that some set of bidders $j \neq i$ would rather loose against bidder i if her signal is low and vice versa win if her signal is high. Thus, it can be the case that a set of bidders $j \neq i$ is active when conditional on winning they would make a loss. More precisely, they will be active whenever the expected profits of winning when i has a high value outweigh the expected losses of winning when bidder i quits "early". The problem that might arise is that when bidder's i drop-out value reveals that her signal is low all bidders want to quit simultaneously. The object is than allocated though some lottery.[18] The allocation that results is typically inefficient and does not allow to respect the conditions for the second best stated in proposition 4.2.[19]

The above result questions the desirability of using the English auction to achieve an efficient outcome in the above setting. As a positive remark, we note that a two-stage English auction can be used to implement the second best and that its equilibrium survives the iterated elimination of weakly dominated strategies. More precisely, we propose to run a first round in which all bidders other than i bid in an English auction. The winner of such auction is awarded the right to compete with i in the second stage.[20]

4.7 Multidimensional Type Models

In this section we extend our analysis to a family of problems with a multidimensional type space. We shall show that under certain assumptions the analysis of these models can be done on an equivalent model with a one-dimensional type space. This analysis allows the extension of the models in Section 4.4.

Suppose that bidder i's private information is a three dimensional vector $\hat{s}_i = (\hat{t}_i, \hat{q}_i, \hat{e}_i)$ that it is drawn according to an independent distribution $\hat{F}_i$ with support in a bounded set $\hat{S}_i \subset \mathbb{R}^3$. We shall assume that this distribution is such that the induced distribution that generates $\hat{t}_i + \hat{q}_i$, say $\overline{F}_i$, has a strictly positive density $\overline{f}_i$ in all the support $\overline{S}_i \subset \mathbb{R}$. Denote by $\hat{S} = \prod_{i=1}^{n} \hat{S}_i$ and by $\overline{S} = \prod_{i=1}^{n} \overline{S}_i$. We assume that bidder i gets utility $\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - b$ if she wins and pays b, and utility $-\hat{e}_j - b$ if $j \neq i$ wins and bidder i pays b.

By the revelation principle, there is no loss of generality in restricting to direct mechanisms. A *direct mechanism* is a pair $(\hat{p}, \hat{x})$ where $\hat{p} : \hat{S} \to [0,1]^n$ and $\hat{x} : \hat{S} \to \mathbb{R}^n$ such that $\sum_{i=1}^{n} \hat{p}_i(\hat{s}) \leq 1$ and where $\hat{p}_i(\hat{s})$ denotes the probability that bidder i gets the good and $\hat{x}_i(\hat{s})$ denotes the payments of i to the auctioneer when the announced vector of types is equal to $\hat{s}$. We shall refer to $\hat{p}$ as an allocation*.

The expected utility of bidder i with type $\hat{s}_i$ that reports $\hat{s}'_i$ when all the other bidders

[18]Ties in any "standard" English auction are resolved by some device that randomizes the allocation.

[19]We are currently formalizing the proof of such argument. We also have examples where equilibria that are in non-weakly dominated strategies and implement the second best exist.

[20]The proof of this fact is under revision.

report truthfully is equal to:

$$\hat{U}_i(\hat{s}_i,\hat{s}'_i)\equiv\hat{Q}_i(\hat{s}'_i,\hat{p})(\hat{t}_i+\hat{q}_i)+\hat{\Psi}_i(\hat{s}'_i,\hat{p},\hat{x}),$$

where

$$\hat{Q}_i(\hat{s}_i,\hat{p})\equiv\int_{\hat{S}_{-i}}\hat{p}_i(\hat{s}_i,\hat{s}_{-i})\,d\hat{F}_{-i}(d\hat{s}_{-i}),$$

and,

$$\hat{\Psi}_i(\hat{s}_i,\hat{p},\hat{x})\equiv\int_{\hat{S}_{-i}}\left(\hat{p}_i(\hat{s}_i,\hat{s}_{-i})\sum_{j\neq i}\hat{q}_j-\hat{x}_i(\hat{s}_i,\hat{s}_{-i})-\sum_{j\neq i}\hat{p}_j(\hat{s}_i,\hat{s}_{-i})\hat{e}_j\right)d\hat{F}_{-i}(d\hat{s}_{-i}),$$

for $\hat{F}_{-i}(\hat{s}_{-i})\equiv\prod_{j\neq i}\hat{F}_j(\hat{s}_j)$ and $\hat{S}_{-i}\equiv\prod_{j\neq i}\hat{S}_j$.

Thus, we say that an allocation* $\hat{p}:S\to[0,1]^n$ is *feasible** if there exists a direct mechanism $(\hat{p},\hat{x})$ that satisfies the following Bayesian incentive compatibility constraint:

$$\hat{U}_i(\hat{s}_i,\hat{s}_i)=\sup_{\hat{s}'_i\in\hat{S}_i}\{\hat{U}_i(\hat{s}_i,\hat{s}'_i)\},$$

for all $\hat{s}_i\in\hat{S}_i$ and $i\in N$.

We shall show that we can study second best efficiency in the model of this section, using the results in the model of Section 4.2:

Definition 4.5. *We call the uni-dimensional equivalent to a model as in Section 4.2 in which for all $i\in N$ and $s_i\in\overline{S}_i$:*

$$S_i=\overline{S}_i,\ F_i(s_i)=\overline{F}_i(s_i),$$

$$t_i(s_i)=\int_{\hat{S}_i(s_i)}\hat{t}_i\,\frac{\hat{F}_i(d\hat{s}_i)}{\overline{f}_i(s_i)},\ q_i(s_i)=\int_{\hat{S}_i(s_i)}\hat{q}_i\,\frac{\hat{F}_i(d\hat{s}_i)}{\overline{f}_i(s_i)},\ \text{and } e_i(s_i)=\int_{\hat{S}_i(s_i)}\hat{e}_i\,\frac{\hat{F}_i(d\hat{s}_i)}{\overline{f}_i(s_i)},$$

where $\hat{S}_i(s_i)\equiv\{\hat{s}_i\in\hat{S}_i:\hat{t}_i+\hat{q}_i=s_i\}$.

Definition 4.6. *We call the uni-dimensional version of an allocation* $\hat{p}$ to a function $p:\overline{S}\to[0,1]^n$ where*

$$p_i(s)=\int_{\hat{S}_1(s_1)}\cdots\int_{\hat{S}_n(s_n)}\hat{p}(s)\,\frac{\hat{F}_n(d\hat{s}_n)}{\overline{f}_n(s_n)}\cdots\frac{\hat{F}_1(d\hat{s}_1)}{\overline{f}_1(s_1)}.$$

Lemma 4.8. *The allocation* $\hat{p}$ is feasible* if and only if its uni-dimensional version p is feasible and $\hat{Q}_i(\hat{s}_i,\hat{p})\in[Q_i(t_i+q_i,p)^-,Q_i(t_i+q_i,p)^+]$ for any $\hat{s}_i\in\hat{S}_i$ and $i\in N$.*[21]

[21] We denote by $Q_i(x_0,p)^-$ and $Q_i(x_0,p)^+$ the limits

$$\lim_{\substack{x\to x_0\\ x<x_0}}Q_i(x,p)\quad\text{and}\quad\lim_{\substack{x\to x_0\\ x>x_0}}Q_i(x,p)$$

respectively. To avoid problems at the infimum and supremum of $\overline{S}_i$, we shall adopt the convention that $Q_i(\inf\overline{S}_i,p)^-=Q_i(\inf\overline{S}_i,p)$ and $Q_i(\sup\overline{S}_i,p)^+=Q_i(\sup\overline{S}_i,p)$. We adopt the same notation and conventions for the functions Q_i in the proof of the lemma.

Proof. By definition of the uni-dimensional version p and Lemma 4.1, the lemma is equivalent to the following statement:

> The allocation* $\hat{p}$ is feasible* if and only if there exists a vector of increasing functions $Q_i(x)$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i, \hat{p}) \in [Q_i(t_i + q_i)^-, Q_i(t_i + q_i)^+]$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$.

We prove the following equivalent statement, see Rockafellar (1970):

> The allocation* $\hat{p}$ is feasible* if and only if there exists a set of increasing convex functions $\hat{v}_i : \overline{S}_i \to \mathbb{R}_+$, $i \in N$, such that $\hat{Q}_i(\hat{s}_i, \hat{p}) \in \partial\hat{v}(\hat{t}_i + \hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$ and $i \in N$.

We first prove the "only if"-part. Suppose a feasible* allocation* $\hat{p} : \hat{S} \to [0,1]^n$, and let $V_i(\hat{s}_i) \equiv \hat{U}_i(\hat{s}_i, \hat{s}_i)$. Then,

$$\begin{aligned} V_i(\hat{s}_i) &\geq \hat{U}_i(\hat{s}_i, \hat{s}'_i) \\ &= \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}) \\ &= \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}'_i + \hat{q}'_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}) + \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}'_i - \hat{q}'_i) \\ &= V_i(\hat{s}'_i) + \hat{Q}_i(s'_i, \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}'_i - \hat{q}'_i), \end{aligned}$$

for all $\hat{s}_i, \hat{s}'_i \in \hat{S}_i$, $i \in N$, and some $\hat{x} : \hat{S} \to \mathbb{R}^n$.

The above inequality applied twice, one with the roles of $\hat{s}_i$ and $\hat{s}'_i$ interchanged, to any two vectors $\hat{s}_i, \hat{s}'_i \in \hat{S}_i$ such that $\hat{t}_i + \hat{q}_i = \hat{t}'_i + \hat{q}'_i$, implies that $V_i(\hat{s}_i) = V_i(\hat{s}'_i)$. Consequently, there exists a function $v_i : \overline{S}_i \to \mathbb{R}$ such that $V_i(\hat{s}_i) = v_i(\hat{t}_i + \hat{q}_i)$ for any $\hat{s}_i \in \hat{S}_i$. Moreover, v_i is convex because V_i is convex. Note that V_i must be convex because it is equal to the maximum of some linear functions by the incentive compatibility constraint. Finally, note that the above inequality together with the definition of v_i implies that $v_i(y) \geq v_i(\hat{t}_i + \hat{q}_i) + \hat{Q}_i(\hat{s}_i, \hat{p})(y - (\hat{t}_i + \hat{q}_i))$ for any y as desired.

To prove the "if"-part, suppose a function $\tilde{v}$ that satisfies the conditions of the lemma for an allocation* $\hat{p}$, and let $\hat{x} : \hat{S} \to \mathbb{R}^n$ be such that $\hat{\Psi}_i(\hat{s}_i, \hat{p}, \hat{x}) = \hat{v}_i(\hat{t}_i + \hat{q}_i) - (\hat{t}_i + \hat{q}_i)\hat{Q}_i(\hat{s}_i, \hat{p})$ for any $i \in N$. We shall show that the direct mechanism $(\hat{p}, \hat{x})$ satisfies the Bayesian incentive compatibility constraints. To see why, note that for any $\hat{s}_i, \hat{s}'_i \in \hat{S}_i$:

$$\begin{aligned} V_i(\hat{s}_i) = \tilde{v}_i(\hat{t}_i + \hat{q}_i) &\geq \tilde{v}_i(\hat{t}'_i + \hat{q}'_i) + \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}'_i - \hat{q}'_i) \\ &= \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}'_i + \hat{q}'_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}) + \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i - \hat{t}'_i - \hat{q}'_i) \\ &= \hat{Q}_i(\hat{s}'_i, \hat{p})(\hat{t}_i + \hat{q}_i) + \hat{\Psi}_i(\hat{s}'_i, \hat{p}, \hat{x}) = \hat{U}_i(\hat{s}_i, \hat{s}'_i), \end{aligned}$$

where the inequality is a consequence of $\hat{Q}_i(\hat{s}'_i, \hat{p}) \in \partial\tilde{v}_i(\hat{t}'_i + \hat{q}'_i)$. □

Now, we can state the main result of this section:

Proposition 4.6. *An allocation* $\hat{p}^*$ is a solution to the problem,*

$$\max \int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - (n-1)\hat{e}_i \right) \hat{p}_i(\hat{s}) \, \hat{F}(d\hat{s}),$$

subject to $\hat{p}$ feasible and $\sum_{i \in N} \hat{p}_i(\hat{s}) = 1$ for all $\hat{s} \in \hat{S}$, if and only if its uni-dimensional version p^* is second best efficient subject to always selling for the uni-dimensional equivalent model.*

Proof. Take any $\hat{p}^*$ feasible and such that $\sum_{i \in N} \hat{p}_i(\hat{s}) = 1$ for all $\hat{s} \in \hat{S}$, and denote by p^* its one-dimensional version. Then we can deduce the lemma from the following sequence of algebraic transformations:

$$\int_{\hat{S}} \sum_{i=1}^{n} \left(\hat{t}_i + \sum_{j=1}^{n} \hat{q}_j - (n-1)\hat{e}_i \right) \hat{p}_i(\hat{s}) \, \hat{F}(d\hat{s}) =$$

$$\int_{\hat{S}} \sum_{i=1}^{n} (\hat{t}_i - (n-1)\hat{e}_i) \, \hat{p}_i(\hat{s}) \, \hat{F}(d\hat{s}) + \int_{\hat{S}} \sum_{i=1}^{n} \hat{p}_i(\hat{s}) \sum_{j=1}^{n} \hat{q}_j \, \hat{F}(d\hat{s}) =$$

$$\sum_{i=1}^{n} \int_{\hat{S}_i} (\hat{t}_i - (n-1)\hat{e}_i) \, \hat{Q}_i(\hat{s}_i, \hat{p}) \, \hat{F}_i(d\hat{s}_i) + \int_{\hat{S}} \sum_{j=1}^{n} \hat{q}_j \, \hat{F}(d\hat{s}) =$$

$$\sum_{i=1}^{n} \int_{\overline{S}_i} \int_{\hat{S}_i(s_i)} (\hat{t}_i - (n-1)\hat{e}_i) \, \hat{Q}_i(\hat{s}_i, \hat{p}) \, \frac{\hat{F}_i(d\hat{s}_i)}{\overline{f}_i(s_i)} \overline{f}_i(s_i) \, ds_i + \sum_{j=1}^{n} \int_{\hat{S}} \hat{q}_j \, \hat{F}(d\hat{s}) =$$

$$\sum_{i=1}^{n} \int_{\overline{S}_i} \int_{\hat{S}_i(s_i)} (\hat{t}_i - (n-1)\hat{e}_i) \, Q_i(\hat{s}_i, \hat{p}) \, \frac{\hat{F}_i(d\hat{s}_i)}{\overline{f}_i(s_i)} \overline{f}_i(s_i) \, ds_i + \sum_{j=1}^{n} \int_{\hat{S}_j} \hat{q}_j \, \hat{F}_j(d\hat{s}_j) =$$

$$\sum_{i=1}^{n} \int_{\overline{S}_i} (t_i(s_i) - (n-1)e_i(s_i)) \, Q_i(s_i, p) \overline{f}_i(s_i) \, ds_i + \sum_{j=1}^{n} \int_{s_j \in S_j} \int_{\overline{S}_j(s_j)} \hat{q}_j \, \frac{\hat{F}_j(d\hat{s}_j)}{\overline{f}_j(s_j)} \overline{f}_j(s_j) \, ds_j =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} (t_i(s_i) - (n-1)e_i(s_i)) \, p(s) \overline{f}(s) ds + \sum_{j=1}^{n} \int_{\overline{S}_j(s_j)} q_j(s_j) \, \overline{f}_j(s_j) \, ds_j =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} (t_i(s_i) - (n-1)e_i(s_i)) \, p(s) \overline{f}(s) ds + \sum_{j=1}^{n} \int_{\overline{S}_j(s_j)} q_j(s_j) \sum_{i=1}^{n} p_i(s) \, \overline{f}_j(s_j) \, ds_j =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} (t_i(s_i) - (n-1)e_i(s_i)) \, p(s) \overline{f}(s) ds + \sum_{i=1}^{n} \int_{\overline{S}} \left(\sum_{j=1}^{n} q_j(s_j) \right) p_i(s) \, \overline{f}(s) \, ds =$$

$$\sum_{i=1}^{n} \int_{\overline{S}} \left(t_i(s_i) + \sum_{j=1}^{n} q_j(s_j) - (n-1)e_i(s_i) \right) p(s) \overline{f}(s) ds,$$

where we have used: in step 3, that $\sum_{j=1}^{n} \hat{p}(\hat{s}) = 1$; in step 5, independency of the $\hat{F}_i$'s and that $\hat{Q}_i(\hat{s}_i, \hat{p}) = Q_i(s_i, p)$ a.e. by Lemma 10 and because $Q_i(s_i, p)$ is increasing and thus continuous a.e.; and in step 7, that $\sum_{i=1}^{n} p_i(s) = 1$. □

4.8 Conclusions

In this paper, we have studied mechanisms that maximize the expected social surplus deriving from the sale of a (single-unit) object subject to Bayesian incentive compatibility constraints. An alternative approach for future research is the equivalent analysis under ex post incentive compatible constraints. This alternative is specially interesting since Bergemann and Morris (2005) has shown it to be equivalent to robust implementation.

Note, however, that the study of second best efficiency subject to ex post incentive compatible constraints presents one additional difficulty. Second best efficiency requires trading off between different inefficient allocations. Under Bayesian implementation the common prior assumption provides natural weights for the expected social surplus maximization. But this is not possible under ex post implementation. One alternative is to assume some ad hoc weights π, and define π-second best efficiency as the allocation that maximizes expected surplus defined according to π and subject to ex post incentive compatibility constraints. In this sense, it is remarkable that the allocations derived from our analysis satisfy the necessary conditions provided by Bikhchandani, Chatterji, and Sen (2006) for ex post implementability, and thus we would expect that our analysis characterizes π-second best efficiency.

A more appealing approach related to the analysis of robust implementation by Bergemann and Morris (2005) is to characterize mechanisms that maximize expected surplus subject to Bayesian incentive compatibility constraints for any common prior. Our results about English auctions provide a very small step in this direction since the English auction implements the second best independently of the prior. Note, however, that an assumption mantained in our analysis is that types are independent across bidders, and thus we can only derive that the English auction implements the second best for any common prior which is independent across bidders.

Chapter 5

Appendix

We illustrate in this appendix the claim in Footnote 13. We propose two examples. The first one is an asymmetric modification of Example 1: $v_1(s_1, s_2) = s_1 + \epsilon + 2s_2$ and $v_2(s_1, s_2) = s_2 + 2s_1$ with $\epsilon > 0$ and small. Now, $h_1(s_1) = -s_1 + \epsilon$, $H_1(s_1) = -\frac{s_1^2}{2} + \epsilon s_1$, and thus, $G_1(s_1) = -\frac{s_1}{2} + \epsilon s_1$ and $g_1(s_1) = -\frac{1}{2} + \epsilon$. Similarly, $h_2(s_2) = -s_2$, $H_2(s_2) = -\frac{s_2^2}{2}$, and thus, $G_2(s_2) = -\frac{s_2}{2}$ and $g_2(s_2) = -\frac{1}{2}$. As a consequence, Equation (4.6) becomes,

$$\int_{[0,1]^2} \left(\left(-\frac{1}{2} + 2s_1 + 2s_2 + \epsilon \right) p_1(s) + \left(-\frac{1}{2} + 2s_1 + 2s_2 \right) p_2(s) \right) ds + \sum_{i=1,2} \int_0^1 \left(-\frac{s_i}{2} + \frac{s_i^2}{2} \right) Q_i(ds_i, p).$$

The first integral is maximized by $(p_1(s), p_2(s)) = (0, 0)$ a.e. if $s_1 + s_2 < \frac{1}{4} - \frac{\epsilon}{2}$, and $(p_1(s), p_2(s)) = (1, 0)$ a.e. otherwise. This allocation does not maximize the second integral since $Q_1(s_1, p)$ is strictly increasing in s_1 for $t_s \in [0, \frac{1}{4} - \frac{\epsilon}{2}]$.

The second example is a symmetric modification of Example 1: $v_i(s) = s_i + (k+1)s_j$. $k \geq 1$. Now, $h_i(s_i) = -ks_i$, $H_i(s_i) = -k\frac{s_i^2}{2}$, and thus, $G_i(s_i) = -k\frac{s_i}{2}$ and $g_i(s_i) = -\frac{k}{2}$. As a consequence, Equation (4.6) after integrating by parts the second integral becomes,

$$\int_{[0,1]^2} \left(-\frac{k}{2} + (k+1)(s_1 + s_2) \right) (p_1(s) + p_2(s))\, ds - \sum_{i=1,2} k \int_0^1 Q_i(s_i, p) s_i (1 - s_i)\, ds_i\, Q_i(ds_i, p).$$

The maximization of the first integral requires that $(p_1(s), p_2(s)) = (0, 0)$ if $s_1 + s_2 < \frac{k}{2(k+1)}$, and otherwise, that $p_1(s) + p_2(s) = 1$, a.e. For k sufficiently large any allocation that satisfies these conditions does not maximize the second integral. To see why, note that in the limit as k tends to infinity the first condition implies: A) $Q_i(s_i, p) \leq \frac{s_i^2}{2}$ and the second condition either B1) $\sum_{i=1,2} Q_i(1/2, p) >)$ or B2) $\forall i$, $Q_i(x, p) = 0$ if $x < 1/2$

and $Q_i(x,p) \geq \frac{(x-1/2)^2}{2}$ otherwise. We shall show that an allocation that satisfies either A) and B1) or A) and B2) implies that the second integral is strictly negative. We start with A) and B1), let $\mu_i \equiv Q_i(1/2,p)$, then,

$$\int_0^1 Q_i(s_i,p)s_i(1-s_i)\,ds_i \leq$$
$$\int_0^{1/2} \min\{\frac{s_i^2}{2},\mu_i\}s_i(1-s_i)\,ds_i + \int_{1/2}^1 \mu_i s_i(1-s_i)\,ds_i < \int_0^1 \mu_i s_i(1-s_i)\,ds_i = 0.$$

Next, consider A) and B2),

$$\int_0^1 Q_i(s_i,p)s_i(1-s_i)\,ds_i = \int_{1/2}^1 Q_i(s_i,p)s_i(1-s_i)\,ds_i < 0.$$

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