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Algebraic properties of semi-simple lattices and related groups

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A thesis submitted for the degree of
Doctor of Philosophy.
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25^{th} October, 2005

Abstract

Two abstract theories are developed. The first concerns isomorphism invariants with the same multiplicative properties as the Euler characteristic. It is used to show that the index of a subgroup in a semi-simple lattice is determined by its isomorphism type when that index is finite. This is also proved to be the case for subgroups of finite index in free products of finitely many semi-simple lattices as well as certain non-trivial extensions of $\mathbb{Z}$ by surface groups. In addition, a criterion for the failure of this property is given which applies to a large class of central extensions.

The second development concerns the syzygies of groups. The results of this theory are used to define the cohomology groups of a duality group in terms of morphisms between stable modules in the derived category. The Farrell cohomology of virtual duality groups is also considered.

Acknowledgments

I would like to express my gratitude to Professor F.E.A.Johnson, for his support, encouragement and dedicated instruction throughout the course of my PhD, without which the writing of this thesis would not have been possible. I would also like to remember Dr. K.Carne and Dr. C.Sparrow for their patience and support during my years as an undergraduate, and D.Salisbury and A.Body, whose encouragement and enthusiasm originally stimulated my interest in mathematics. Finally, I would like to thank W.Mannan and T.Edwards for initiating and taking part in many stimulating discussions on topology and sharing the experience of studying at University College.

To Mum, Dad, Nathan and Rossana.

Preface

This thesis is concerned with the algebraic properties of semi-simple lattices and groups related to them. It has arisen out of an attempt to establish further intrinsic properties of semi-simple lattices, with the long term aim of determining an algebraic classification of all such groups. This effort has led firstly, to the development of a theory of "e-invariants" based upon the Euler characteristic, and secondly, to the proof of two theorems on the cohomology of duality groups.

An e-invariant, as defined in this thesis, is a real-valued isomorphism invariant defined on the commensurability class of a group that is multiplicative with respect to covers. The canonical example is the rational Euler characteristic of C.T.C. Wall [41]. The utility of the definition is that e-invariants can be shown to exist in many cases where the traditional Euler characteristic is difficult to calculate or else known to be zero. Here, e-invariants will be shown to exist for all lattices in semi-simple Lie groups with finite centre, the free product of finitely many lattices in linear semi-simple Lie groups of real rank ≥ 2, and certain central extensions of $\mathbb{Z}$ by surface groups.

The theorems proved in the final two chapters simplify cohomology for duality groups by showing that, in all but the top dimension, the cohomology groups of a duality group can be defined as groups of morphisms from syzygies to coefficient modules in the derived category. This reduces the problem of calculating the cohomology of many lattices in semi-simple Lie groups to that of identifying syzygies. A duality theorem for syzygies of duality groups is also proved.

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Chapter 1

Preliminaries on discrete groups

In this thesis, unless explicitly stated otherwise, all rings will have a unit and all ring homomorphisms will be unital.

1.1 Schanuel's lemma, syzygies and the derived category

Let R be a ring and M an R-module. Then a projective (resp. free) resolution of M over R is said to have finite type if and only if all of the projective (resp. free) modules occurring in the resolution are finitely generated. Similarly, a partial projective (resp. free) resolution

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0$$

will be said to have finite type when each of the modules $P_0, \ldots, P_n$ is finitely generated.

The following proposition, known as Schanuel's Lemma, is fundamental to what follows:

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Proposition 1.1.1. *(Schanuel's Lemma)*

Let R be a ring and M an R-module. Then, if

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0$$

and

$$0 \to K' \to P_n' \to \cdots \to P_0' \to M \to 0$$

are partial projective resolutions of M over R,

$$K \oplus P_n' \oplus P_{n-1} \oplus \cdots \cong K' \oplus P_n \oplus P_{n-1}' \oplus \cdots .$$

Proof. See [37] □

Two R-modules K and K' are said to be stably equivalent if there exist finitely generated free R-modules E and E' such that $K \oplus E' \cong K' \oplus E$. This is an equivalence relation on the class of R-modules whose equivalence classes are called stable modules. Now, it is clear from Schanuel's lemma that if

$$0 \to K \to F_n \to \cdots \to F_0 \to M \to 0$$

and

$$0 \to K' \to F_n' \to \cdots \to F_0' \to M \to 0$$

are partial free resolutions of finite type, then K is stably equivalent to K'. Consequently, if

$$0 \to J \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

is a partial free resolution of finite type, then the stable module $\Omega_n(M) = [J]$ is a well-defined isomorphism invariant of M. When it exists, $\Omega_n(M)$ is called the n^{th} syzygy of M. For a finitely generated R-module M, $\Omega_0(M)$ is defined to be the stable class $[M]$.

Proposition 1.1.2. *Let M be an R-module and $n \geq 0$. Then, when defined, $\Omega_n(M)$ depends only on the stable class of M.*

Proof. Fix an integer $n \geq 1$ and let $M' = M \oplus F$, where F is a finitely generated free R-module. Then, if

$$0 \to J \to F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} M \to 0$$

is a partial free resolution of finite type,

$$0 \to J \to F_{n-1} \to \cdots \to F_1 \xrightarrow{\partial_1 \oplus 0} F_0 \oplus F \xrightarrow{\epsilon \oplus \mathrm{id}} M \oplus F \to 0$$

is a partial free resolution of finite type for M'. $[J]$ is therefore the n^{th} syzygy of both M and M'. As $\Omega_0(M)$ is just the stable class $[M]$, this completes the proof. □

1.2 A criterion for the existence of syzygies

Definition 1.2.1. An R-module M is said to have type FP_0 over R if it is finitely generated and type FP_n $(n \geq 1)$ if

1. it has type FP_{n-1} and

2. for any partial projective resolution

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

   of finite type, K is finitely generated.

M will be said to have type FP_∞ if it has type FP_n for all $n \geq 0$.

Lemma 1.2.2. *If*

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0$$

is a partial projective resolution of finite type, then there exists a finitely generated projective module Q and an exact sequence

$$0 \to K \oplus Q \to F_n \to \cdots \to F_0 \to M \to 0$$

in which $F_0, \ldots, F_n$ are free and finitely generated.

Proof. For every finitely generated projective module P there exists a finitely generated projective module Q such that $P \oplus Q$ is free and finitely generated. Let $0 \to K \to P_0 \to M \to 0$ be a short exact sequence in which P_0 is a finitely generated projective module and suppose that Q_0 is a module such that $P_0 \oplus Q_0$ is finitely generated and free. Then the obvious exact sequence

$$0 \to K \oplus Q_0 \to P_0 \oplus Q_0 \to M \to 0$$

is a partial free resolution of finite type. This proves the result for $n = 0$. Now choose $n > 0$, suppose the result holds for $n - 1$ and let

$$0 \to K \to P_n \to \cdots \to P_0 \to M \to 0$$

be a partial projective resolution of finite type. Set $K_1 = \mathrm{Im}(P_n \to P_{n-1})$. Then, by hypothesis, there exists a finitely generated projective Q_{n-1} and an exact sequence

$$0 \to K_1 \oplus Q_{n-1} \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

in which F_i is free and finitely generated for all i. Spliced with the obvious short exact sequence $0 \to K \to P_n \oplus Q_{n-1} \to K_1 \oplus Q_{n-1} \to 0$, this yields an exact sequence

$$0 \to K \to P_n \oplus Q_{n-1} \to F_{n-1} \to \cdots \to F_0 \to M \to 0.$$

Now let Q_n be a projective module such that $F_n = (P_n \oplus Q_{n-1}) \oplus Q_n$ is free and finitely generated. Then, by adding $0 \to Q_n \xrightarrow{\text{id}} Q_n \to 0$ to the sequence above we obtain a free resolution of finite type

$$0 \to K \oplus Q_n \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

as claimed. $\square$

Proposition 1.2.3. *For $n > 0$, $\Omega_n(M)$ is defined if and only if M has type FP_{n-1} and finitely generated if and only if M has type FP_n.*

Proof. $\Rightarrow$) Fix n > 0. Suppose that $\Omega_n(M)$ is defined, and let

$$0 \to J \to F_{n-1} \to \cdots \to F_0 \to M \to 0$$

be a partial free resolution of finite type. We wish to prove that M has type FP_{n-1}. Now, M has type FP_0 if and only if it is finitely generated, so for $n = 1$ this is just the fact that the homomorphism $F_0 \to M$ is surjective. For $n > 1$, let $0 \to K \to P_{n-2} \to \cdots \to P_0 \to M \to 0$ be a partial projective resolution of finite type and suppose that M has type FP_{n-2}. By Schanuel's lemma, there exist finitely generated projective modules P and Q such that

$$K \oplus P \cong \mathrm{Im}(F_{n-1} \to F_{n-2}) \oplus Q.$$

But F_{n-1}, and therefore $\mathrm{Im}(F_{n-1} \to F_{n-2})$, is finitely generated. Thus $K \oplus P$ and, therefore K, is finitely generated, so that M has type FP_{n-1} as claimed.

Now suppose that $\Omega_n(M)$ is finitely generated where $n > 0$, and let

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

be a partial projective resolution of M having finite type. As $\Omega_n(M)$ is finitely generated, so is J. But, by Schanuel's lemma, the finite generation of J implies that of K, which shows that M has type FP_n in this case.

$\Leftarrow$) If M has type FP_{n-1}, then there exists a partial projective resolution of finite type,

$$0 \to J \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

By Lemma 1.2.2, we can assume that P_i is free and finitely generated for all i. So, $\Omega_n(M)$ certainly exists. If, in addition, M has type FP_n, J must be finitely generated, and so $\Omega_n(M)$ is finitely generated also. $\square$

1.3 Derived categories and corepresentability

Let R be a ring. Following F.E.A.Johnson [16], the derived category $Der(R)$, will be defined to be the category of R-modules with morphisms given by

$$Hom_{Der(R)}(M; N) = Hom_R(M; N)/\approx,$$

where $\alpha \approx \beta$ if and only if $\alpha - \beta$ factors through a projective R-module. That is, there exists a projective module P and morphisms $M \to P$, $P \to N$ whose composition is $\alpha - \beta$.

Proposition 1.3.1. *An R-morphism $\alpha : M \to N$ factors through a projective R-module if and only if it factors through a free R-module, which may be taken to be finitely generated whenever M is finitely generated.*

Proof. P is projective if and only if there exists an R-module Q such that $P \oplus Q$ is free over R. An R-morphism $\alpha : M \to N$ that factors through P via $\widetilde{\alpha} : P \to N$, therefore also factors through the free module $F = P \oplus Q$ via $\widetilde{\alpha} \oplus 0$. If M is finitely generated and $q : M \to F$ any R-morphism, then $q(M)$ is a finitely generated submodule of F and as such is spanned by a finite subset of any R-basis of F. So, when M is finitely generated, we may replace F by a finitely generated free submodule. $\square$

The significance of the derived category in the context of syzygies is that, within it, stably equivalent modules are isomorphic. This means that morphisms can be assigned to stable modules. This is not normally possible as there is in general no canonical choice of representative for the stable class.

Proposition 1.3.2. *If S and T are stable modules over a ring R with representatives M and N respectively, then the groups*

$$Hom_{Der(R)}(S;T) = Hom_{Der(R)}(M;N)$$

are well defined up to isomorphism.

Proof. Immediate from the additivity of $Hom_R(M;N)$ in M and N. $\square$

Corollary 1.3.3. *$Hom_{Der(R)}(\Omega_n(M);[N])$ is an isomorphism invariant of M for all $n \geq 0$ and all stable R-modules $[N]$.*

I will often drop the brackets and write $Hom_{Der(R)}(\Omega_n(M);N)$ instead of $Hom_{Der(R)}(\Omega_n(M);[N])$.

1.4 Modules of type FP and FL

A module M is said to have finite projective dimension if it admits a projective resolution of finite length. That is, a resolution of the form

$$0 \to P_n \to \cdots \to P_0 \to M \to 0,$$

where P_i is projective for all i. n is the length of the resolution. The projective dimension of M is the minimum n for which such a resolution exists.

If in addition to having finite projective dimension, a module M has type FP_∞, then it is said to be of type FP. It is well known that this definition is equivalent to the existence of a projective resolution of finite length and type and that the length of the resolution can always be taken to be equal to the projective dimension [12, Chapter VIII, Proposition 6.1].

Now suppose that there exists an exact sequence,

$$0 \to F_m \to \cdots \to F_0 \to M \to 0$$

in which $F_0, \ldots, F_m$ are finitely generated and free. In this case M is said to have type FL. If

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

is a projective resolution of finite type, where n is the projective dimension of M, then it follows from Lemma 1.2.2 that we can construct an exact sequence

$$0 \to P_n \oplus Q \to E_{n-1} \to \cdots \to E_0 \to M \to 0 \tag{1.4.1}$$

in which $E_0, \ldots, E_{n-1}$ are free and finitely generated and Q is finitely generated and projective. Schanuel's Lemma then implies the existence of a finitely generated free module E such that $(P_n \oplus Q) \oplus E$ is free and finitely generated. By taking the direct sum of the exact sequence $0 \to E \xrightarrow{\text{id}} E \to 0$ and the resolution (1.4.1) at the appropriate point, we obtain a free resolution of finite type whose length is equal to the cohomological dimension of M. Thus we have proved:

Proposition 1.4.1. *A module M has type FL and projective dimension n if and only if there exists an exact sequence*

$$0 \to F_n \to \cdots \to F_0 \to M \to 0$$

in which $F_0, \ldots, F_n$ are free and finitely generated.

The principal example of a module of type FL is that of $\mathbb{Z}$ when it is given the structure of a trivial $\mathbb{Z}\Gamma$-module, where Γ is the fundamental group of a compact manifold without boundary having contractible universal cover. Such groups are known to satisfy Poincaré duality, which is a special case of the duality criterion due R.Bieri and B.Eckmann given in the next section.

1.5 Duality groups and their virtual neighbours

A group G is said to satisfy Poincaré duality if and only if, for all G-modules N and all $i \geq 0$, there exist cap product isomorphisms

$$\cap z : H^i(G; N) \to H_{d-i}(G; N),$$

where d is a fixed non-negative integer. In [3], R.Bieri and B.Eckmann defined G to be a duality group if, for all G-modules N and all $i \geq 0$, there exist cap product isomorphisms

$$\cap z : H^i(G; N) \to H_{d-i}(G; \Delta \otimes N),$$

where Δ is a fixed G-module, called the dualising module of G. For a group of type FP there are a number of equivalent definitions:

Theorem 1.5.1. *Let G is a group of type FP. Then the following are equivalent:*

1. *There exists an integer d and a G-module Δ such that*

$$Ext^i_{\mathbb{Z}G}(\Delta; N) \cong Tor^{\mathbb{Z}G}_{d-i}(\mathbb{Z}; \Delta \otimes N)$$

for all G-modules N and all non-negative integers i.

2. There is an integer d such that

$$Ext^i_{\mathbb{Z}G}(\mathbb{Z};\mathbb{Z}G) = \begin{cases} 0 & i \neq d \\ \Delta & i = d \end{cases} \tag{1.5.1}$$

where Δ is $\mathbb{Z}$-torison-free.

3. There exists a class $z \in H_d(G;\Delta)$ such that the cap product map

$$\cap z : Ext^i_{\mathbb{Z}G}(\mathbb{Z};\_) \to Tor^{\mathbb{Z}G}_{d-i}(\mathbb{Z};\Delta \otimes \_)$$

is an isomorphism for all i, where d is the cohomological dimension of G and $\Delta = Ext^d_{\mathbb{Z}G}(\mathbb{Z};\mathbb{Z}G)$.

Proof. [12, Chapter VII, Theorem 10.1]. $\square$

K.S. Brown showed in [11] that a module M has type FP_∞ over a ring R if and only if $Ext^i_R(M,\_)$ commutes with direct limits for all i. As the cap product is natural, while the functors $Tor^{\mathbb{Z}G}_i(\mathbb{Z};\_)$ and $\Delta \otimes \_$ commute with direct limits, this showed that every duality group has type FP.

Now, any group G for which there exist functorial isomorphisms

$$Ext^i_{\mathbb{Z}G}(\mathbb{Z};\_) \cong Tor^{\mathbb{Z}G}_{d-i}(\mathbb{Z};\Delta \otimes \_)$$

necessarily has type FP, satisfies condition 1 of Theorem 1.5.1 and is therefore a duality group. As the cap product is natural, this means that a group G is a duality group if and only if there exists an integer d and G-module Δ for which the above equivalence of functors holds. It is well known however that if R is a ring and M an R-module with no $\mathbb{Z}$-torsion, then, for any R-module N and all $i \geq 0$, there exist functorial isomorphisms

$$Tor^R_i(M,N) \cong Tor^R_i(\mathbb{Z};M \otimes N)$$

(see [12, Chapter III, Proposition 2.2]). Furthermore, it follows from Theorem 1.5.1 and the fact that all duality groups have type FP that the dualising module of a duality group is necessarily $\mathbb{Z}$-torsion-free. So we have shown:

Proposition 1.5.2. *G is a duality group if and only if there exist natural isomorphisms*

$$Ext^i_{\mathbb{Z}G}(\mathbb{Z};\_) \cong Tor^{\mathbb{Z}G}_{d-i}(\Delta;\_)$$

where d is a fixed non-negative integer and Δ a fixed G-module with no $\mathbb{Z}$-torsion.

A virtual duality group is a group containing a duality group with finite index. Such groups are closely related to duality groups and share many of their properties. In particular, they satisfy 1.5.1. This is a consequence of Shapiro's lemma, which is discussed in the next section.

1.6 Shapiro's Lemma

Let R and S be rings and $\theta : R \rightarrow S$ a ring homomorphism. Any right (resp. left) S-module N can be given the structure of a right (resp. left) R-module via θ by defining $n.r = n\,\theta(r)$ (resp. $r.n = \theta(r)\,n$) for all $n \in N$. For a any left R-module M, set $Ind_\theta(M) = S \otimes_\theta M$. $Ind_\theta(M)$ is then a left S-module. It is characterized by the following universal property:

Proposition 1.6.1. *Let M and $\theta : R \rightarrow S$ be as above and suppose that $\alpha : M \rightarrow N$ an R-morphism for some (left) S-module N. Then, there exists a unique S-morphism $\alpha_S : Ind_\theta(M) \rightarrow N$ extending α. That is, $\alpha = \alpha_S \circ i$, where $i : M \rightarrow Ind_\theta(M)$ is the natural inclusion induced from $m \mapsto 1 \otimes_\theta m$.*

Proof. See [12, Chapter III, Section 3]. □

Dual to the concept of the induced module is that of the coinduced module. For M and θ as above, $Coind_\theta(M)$ is defined to be the left S-module $Hom_R(S; M)$. The S-module structure on $Hom_R(S; M)$ is given by $(s \cdot f)(s') = f(s' \cdot s)$.

Proposition 1.6.2. *Let N be a left S-module, M a left R-module and $\alpha : N \to M$ an R-morphism. Then there exists a unique S-morphism $\alpha^S : N \to Coind_\theta(M)$ such that $\alpha = p \circ \alpha^S$, where p is the natural projection $Coind_\theta(M) \to M; f \mapsto f(1)$.*

Proof. See [12, Chapter III, Section 3]. □

Let G be a group and H a subgroup. Given an H-module N, define $Ind_H^G N = Ind_i N$ and $Coind_H^G N = Coind_i N$ where $i : H \hookrightarrow G$ is the natural inclusion. The universal properties of induced and coinduced modules then lead to the following identities of functors, which together are known as Shapiro's Lemma:

Theorem 1.6.3. *Let G be a group and H a subgroup. Then, for any H-module N,*

$$Tor_i^{\mathbb{Z}H}(\mathbb{Z}, N) \cong Tor_i^{\mathbb{Z}G}(\mathbb{Z}, Ind_H^G N)$$

$$Ext^i_{\mathbb{Z}H}(\mathbb{Z}, N) \cong Ext^i_{\mathbb{Z}G}(\mathbb{Z}, Coind_H^G N)$$

Proof. [12, Chapter III, Proposition 6.2]. □

Proposition 1.6.4. *If H has finite index in G, then $Ind_H^G N \cong Coind_H^G N$.*

Proof. [12, Chapter III, Proposition 5.8]. □

Corollary 1.6.5. *If G is a virtual duality group and H a duality group of finite index in G with dualising module Δ and cohomological dimension d, then*

$$Ext^i_{\mathbb{Z}G}(\mathbb{Z}; \mathbb{Z}G) \cong \begin{cases} 0 & i \neq d \\ \Delta & i = d \end{cases}$$

In particular, every virtual duality group of type FP is a duality group.

Proof. $\mathbb{Z}G \cong Ind_H^G(\mathbb{Z}H) \cong Coind_H^G(\mathbb{Z}H)$ and so, by Shapiro's lemma, there exist $\mathbb{Z}$-module isomorphisms

$$Ext^i_{\mathbb{Z}G}(\mathbb{Z};\mathbb{Z}G) \cong Ext^i_{\mathbb{Z}H}(\mathbb{Z};\mathbb{Z}H)$$

for all i. When G has type FP, it therefore satisfies the second criterion of Theorem 1.5.1 and is consequently a duality group. □

Corollary 1.6.6. *If G is a duality group and H a subgroup of finite index in G, then H is a duality group with the same cohomological dimension as G.*

Proof. As every projective $\mathbb{Z}G$-module is projective when regarded as a $\mathbb{Z}H$-module, a projective resolution $\varepsilon : P \to \mathbb{Z}$ over $\mathbb{Z}G$ is also a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}H$. But G has type FP, so H must be a group of type FP also. As

$$Ext^i_{\mathbb{Z}H}(\mathbb{Z};\mathbb{Z}H) \cong Ext^i_{\mathbb{Z}G}(\mathbb{Z};\mathbb{Z}G) \cong \begin{cases} 0 & i \neq d \\ \Delta & i = d \end{cases},$$

Theorem 1.5.1 now shows that H is a duality group of dimension d. □

1.7 Farrell cohomology

In this section, the resolution

$$\cdots \to P_n \to \cdots \to P_0 \xrightarrow{\varepsilon} M \to 0$$

will be dentoed $\epsilon : P \to M$, where P refers to the associated chain complex

$$\cdots \to P_n \to \cdots \to P_0.$$

Now, let G be a group. Then a complete resolution of a G-module M over $\mathbb{Z}G$ is defined to be a pair $(F, \epsilon : P \to M)$, where $\epsilon : P \to M$ is a projective resolution of M over $\mathbb{Z}G$ and F an acylic chain complex of projective G-modules which coincides with P in sufficiently high dimension.

Theorem 1.7.1. *Let G be a group of virtual finite cohomological dimension d. Then,*

1. *if $\epsilon : P \to M$ is a projective resolution of M over $\mathbb{Z}G$, there exists a complete resolution for M of the form $(F, \epsilon : P \to M)$ such that F coincides with P in dimensions greater than or equal to d.*
2. *if $(F, \epsilon : P \to M)$ and $(F', \epsilon' : P' \to M)$ are complete resolutions of M over $\mathbb{Z}G$ and $\theta : P \to P'$ an augmentation preserving chain map, then there exists a chain homotopy equivalence $\widetilde{\theta} : F \to F'$ which coincides with θ in dimensions greater than or equal to d.*

Proof. [12, Chapter X, Proposition 2.1].

□

It follows easily from the projectivity of P and the exactness of $\epsilon' : P' \to M$ that augmentation preserving chain maps $P \to P'$ always exist. This means that F and F' are chain homotopy equivalent whenever $(F, \epsilon : P \to M)$ and $(F', \epsilon' : P' \to M)$ are complete resolutions of M over $\mathbb{Z}G$.

Corollary 1.7.2. *The groups*

$$\widetilde{Ext}^r_{\mathbb{Z}G}(M; N) := H^r(Hom_{\mathbb{Z}G}(F, N))$$

are well-defined invariants of G for all $r \in \mathbb{Z}$, where $(F, \epsilon : P \to M)$ is any complete resolution of M over $\mathbb{Z}G$.

The groups $\widetilde{H}^r(G; N) = \widetilde{Ext}^r_{\mathbb{Z}G}(\mathbb{Z}; N)$ are called the Farrell cohomology groups of G. It is clear from the definition of $\widetilde{H}^r(G; N)$ that $\widetilde{H}^r(G; N) \cong H^r(G; N)$ for all $r > d$. In dimension d there exists an epimorphism $\widetilde{H}^d(G;N) \to H^d(G;N)$, which we shall now describe explicitly.

Let J and N be G-modules and H a subgroup of finite index having finite cohomological dimension. The transfer map $tr : Hom_{\mathbb{Z}H}(J;N) \to Hom_{\mathbb{Z}G}(J;N)$ is defined by $tr(f)(m) = \Sigma_{g\in E} \quad gf(g^{-1}m)$, where E is a set of right coset representatives for H in G.

If $\epsilon : P \to M$ is a projective resolution of M over $\mathbb{Z}G$, then $\epsilon : P \to M$ is a projective resolution of M over $\mathbb{Z}H$ whose differentials $\partial_n : F_n \to F_{n-1}$ are G-morphisms. In particular, for $f \in Hom_{\mathbb{Z}H}(F_n;N)$, $tr(f \circ \partial_n) = tr(f) \circ \partial_n$ for all n. tr therefore induces morphisms

$$Ext^i_{\mathbb{Z}H}(M;N) \to Ext^i_{\mathbb{Z}G}(M;N); \quad [f] \mapsto [tr(f)],$$

for all i.

Proposition 1.7.3. *Let G be a group of virtual finite cohomological dimension n and H as subgroup of finite index and finite cohomological dimension. Then, for all G-modules M, there exists an exact sequence*

$$Ext^r_{\mathbb{Z}H}(M;N) \xrightarrow{tr} Ext^r_{\mathbb{Z}G}(M;N) \to \widetilde{Ext}^r_{\mathbb{Z}G}(M;N) \to 0$$

Proof. See [12, Chapter X, Section 3.4]. $\square$

Chapter 2

Preliminaries on Lie groups

2.1 Linear algebraic groups

$GL(n, \mathbb{C})$ inherits the structure of an affine variety from the embedding

$$GL(n, \mathbb{C}) \hookrightarrow \mathbb{C}^{n^2+1}; \quad A_{ij} \mapsto (A_{ij}, (det A_{ij})^{-1}).$$

By a linear algebraic group we shall mean a subgroup G of $GL(n, \mathbb{C})$ whose image under this embedding is an affine subvariety of $GL(n, \mathbb{C})$. For any such group, the group operations are necessarily polynomial. G will be said to be defined over a subfield $k \subset \mathbb{C}$ when its image in $\mathbb{C}^{n^2+1}$ is defined over k.

Identify $GL(n, \mathbb{C})$ with its image in $\mathbb{C}^{n^2+1}$. Then, given a ring $B \subset \mathbb{C}^{n^2+1}$, set $G_B = G \cap GL(n, B)$. A subgroup $\Gamma \subset G$ will be called arithmetic if it is commensurable with $G_{\mathbb{Z}}$.

Let $\overline{H}$ denote the closure of a subset $H \subset G$ in the Zariski topology of G. Then, if H is a subgroup of G, so is $\overline{H}$. To see this observe that, since right multiplication is homeomorphic, given $a \in H$, $\overline{H}a = \overline{Ha} = \overline{H}$, so that $\overline{H}H \subset \overline{H}$. Thus, if $a \in \overline{H}$, $a\overline{H} = \overline{aH} \subset \overline{H}$.

Lemma 2.1.1. *Let H_1 and H_2 be subgroups of the linear algebraic group G. Then*

1. *if H_1 normalizes H_2, $\overline{H_1}$ normalizes $\overline{H_2}$*

2. *if H_1 centralizes H_2, then $\overline{H_1}$ centralizes $\overline{H_2}$.*

Proof. (1) Pick $a \in H_1$. Since H_1 normalizes H_2, while the map $x \mapsto axa^{-1}$ is homeomorphic with respect to the Zariski topology, $\overline{H_2} = \overline{aH_2a^{-1}} = a\overline{H_2}a^{-1}$. Thus H_1 normalizes $\overline{H_2}$.

Now let N be the normalizer of $\overline{H_2}$ in G. Since $H_1 \subset N$, to prove that $\overline{H_1}$ normalizes $\overline{H_2}$, it is sufficient to show that N is closed in the Zariski topology of G. Let $f_h(g) = ghg^{-1}$ for all $g \in G$. Then

$$g \in N \iff ghg^{-1} \in \overline{H_2} \text{ for all } h \in \overline{H_2} \iff g \in \bigcap_{h \in \overline{H_2}} f_h^{-1}(\overline{H_2}).$$

As $\overline{H_2}$ is algebraic and f_h is polynomial, this shows that N is Zariski closed.

(2) Pick a $\in H_1$ and set $c_a : x \mapsto axa^{-1}x^{-1}$. Then, as H_1 centralizes H_2, $H_2 \subset c_a^{-1}(1)$. Since c_a is polynomial, this is a Zariski closed subset of G. Thus $\overline{H_2} \subset c_a^{-1}(1)$. This implies that H_1 centralizes $\overline{H_2}$. But H_1 centralizes $\overline{H_2}$ if and only if $\overline{H_2}$ centralizes H_1. So by the same argument, $\overline{H_2}$ centralizes $\overline{H_1}$. ☐

Proposition 2.1.2. *Let H be a subgroup of $GL(n, \mathbb{C})$ and K a subgroup of finite index in H. Then $\overline{K}$ has finite index in $\overline{H}$.*

Proof. Let X be a set of coset representatives for K in H. Then $H = \bigcup_{h \in X} hK$, and, since $\overline{K} \subset \overline{H}$,

$$\overline{H} \subset \bigcup_{h \in X} h\overline{K} \subset \overline{H},$$

so that $\overline{K}$ has finite index in $\overline{H}$. ☐

2.2 Real algebraic groups

If $A \subset GL(n, \mathbb{C})$ is a linear algebraic group, then the set of real points of A, $A_{\mathbb{R}} = A \cap GL(n, \mathbb{R})$, is a closed subgroup of $GL(n, \mathbb{R})$. A group of this form is said to be a real algebraic group. The following theorem is due to G.D.Mostow.

Theorem 2.2.1. *Real algebraic groups have finitely many connected components.*

Proof. See [26]. $\square$

Definition 2.2.2. Let $X \subset GL(n, \mathbb{R})$. Then $\overline{X}_{\mathbb{R}}$ is called the real Zariski closure of X in $GL(n, \mathbb{R})$.

Proposition 2.2.3. *If A and B are subsets of $GL(n, \mathbb{R})$ such that $\overline{A}_{\mathbb{R}} = \overline{B}_{\mathbb{R}}$, then $\overline{A} = \overline{B}$.*

Proof. Let $X = A_{\mathbb{R}}$. Then, since $A \subset X \subset \overline{A}$, $\overline{A} \subset \overline{X} \subset \overline{\overline{A}} = \overline{A}$, Thus $\overline{X} = \overline{A}$, which implies the result. $\square$

Analogous to Proposition 2.1.2, we have:

Proposition 2.2.4. *Let H be a subgroup of $GL(n, \mathbb{R})$ and K a subgroup of finite index in H. Then $\overline{K}_{\mathbb{R}}$ has finite index in $\overline{H}_{\mathbb{R}}$.*

Proof. By Proposition 2.1.2, $\overline{K} \subset \overline{H}$ with finite index. Thus $\overline{K}$ contains the identity component of $\overline{H}$ and $\overline{K}_{\mathbb{R}}$ the identity component of $\overline{H}_{\mathbb{R}}$. But by Theorem 2.2.1, $\overline{H}_{\mathbb{R}}$ has only finitely many connected components. Thus $\overline{K}_{\mathbb{R}}$ has finite index in $\overline{H}_{\mathbb{R}}$ $\square$

2.3 Semi-simple Lie groups

Let $\mathfrak{g}$ be a finite dimensional real Lie algebra. If $\mathfrak{a}, \mathfrak{b} \lhd \mathfrak{g}$ are soluble ideals in $\mathfrak{g}$, then so is $\mathfrak{a} + \mathfrak{b}$. This means there exists a unique maximal soluble ideal in $\mathfrak{g}$. It is called the radical of $\mathfrak{g}$ and denoted $\mathrm{rad}(\mathfrak{g})$.

$\mathrm{rad}(\mathfrak{g})$ contains all the soluble ideals of $\mathfrak{g}$. Consequently, if $\mathrm{rad}(\mathfrak{g}) = 0$, then every soluble ideal in $\mathfrak{g}$ is trivial. Moreover, as the radical is characteristic, if $\mathfrak{b} \lhd \mathfrak{g}$, then $\mathrm{rad}(\mathfrak{b}) \subset \mathrm{rad}(\mathfrak{g})$. In particular, when $\mathrm{rad}(\mathfrak{g}) = 0$, $\mathrm{rad}(\mathfrak{b}) = 0$ for every ideal in $\mathfrak{g}$.

Definition 2.3.1. $\mathfrak{g}$ is said to be semi-simple if and only if $\mathrm{rad}(\mathfrak{g}) = \{0\}$.

We have shown:

Proposition 2.3.2. *If $\mathfrak{g}$ is semi-simple, then so is every ideal in $\mathfrak{g}$.*

It follows from the proposition above that if $\mathfrak{g}$ is a real Lie algebra of finite dimension, then $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is semi-simple. Now, it is classical result (see [31, page 3]) that the corresponding short exact sequence

$$0 \to \mathrm{rad}(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/\mathrm{rad}(\mathfrak{g}) \to 0$$

always splits, so that every finite dimensional real Lie algebra is the semi-direct product of a soluble and a semi-simple Lie subalgebra. This decomposition is known as the Levi decomposition of $\mathfrak{g}$. In terms of Lie groups, this means that every connected Lie group $G = S \cdot R$, where R is the connected soluble Lie subgroup of G corresponding to $\mathrm{rad}(\mathfrak{g})$, S semi-simple and $S \cap R$ is discrete. R is called the radical of G.

We shall now prove a decomposition theorem for semi-simple Lie algebras.

Proposition 2.3.3. *Let $\mathfrak{g}$ be a real semi-simple Lie algebra of finite dimension. Then $\mathfrak{g}$ decomposes as the direct sum of finitely many simple non-abelian ideals.*

Proof. Suppose first that $\mathfrak{g}$ is simple. Then, given $a \in \mathfrak{g}$, $\mathrm{Ker}(\mathrm{ad}(a)) \lhd \mathfrak{g}$, so that $\mathrm{ad}(a)(\mathfrak{g}) = \mathfrak{g}$ or $\{0\}$. However, if $\mathrm{ad}(a)(\mathfrak{g}) = \{0\}$ for all a, then $\mathfrak{g}$ must be abelian. So there exists at least one $a \in \mathfrak{g}$ such that $\mathrm{ad}(a)$ is an automorphism of $\mathfrak{g}$.

Now suppose that $\mathfrak{g}$ is semi-simple. If $\mathfrak{g}$ is not simple, let $\mathfrak{g}_1$ be a minimal ideal in $\mathfrak{g}$. Then $\mathfrak{g}_1$ is simple and, by the argument above, we can pick $a \in \mathfrak{g}_1$ so that the restriction of $\mathrm{ad}(a)$ to $\mathfrak{g}_1$ is an isomorphism. Since $[a, x] \in \mathfrak{g}_1$ for all $x \in \mathfrak{g}$, we have $\mathrm{ad}(a)(\mathfrak{g}) \subset \mathfrak{g}_1$. Thus

$$\mathfrak{g} = \mathrm{Im}(\mathrm{ad}(a)) \dotplus \mathrm{Ker}(\mathrm{ad}(a)) = \mathfrak{g}_1 \dotplus \mathrm{Ker}(\mathrm{ad}(a)).$$

$\mathrm{Ker}(\mathrm{ad}(a))$ is an ideal in $\mathfrak{g}$ of strictly lower dimension. It is semi-simple by the proposition above and so, by induction, decomposes as the sum of finitely many simple non-abelian ideals $\mathfrak{g}_2, \ldots, \mathfrak{g}_k$. As each $\mathfrak{g}_i$ is clearly an ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}_1 \dotplus \cdots \dotplus \mathfrak{g}_k$, this proves the result. □

If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{a} \cap \mathfrak{g}_i$ is an ideal in $\mathfrak{g}_i$ for all i. So, for each i, either $\mathfrak{a} \cap \mathfrak{g}_i = 0$ or $\mathfrak{a} \cap \mathfrak{g}_i = \mathfrak{g}_i$. This means $\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{g}_1) \dotplus \cdots \dotplus (\mathfrak{a} \cap \mathfrak{g}_k)$. In particular, the centre $Z(\mathfrak{g})$ of $\mathfrak{g}$ decomposes as the internal direct sum $Z(\mathfrak{g}) = Z(\mathfrak{g}_1) \dotplus \cdots \dotplus Z(\mathfrak{g}_k)$. The centre of any simple non-abelian ideal is necessarily trivial, so this shows $Z(\mathfrak{g}) = 0$.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and for $i = 1, \ldots, k$, let G_i be the connected Lie subgroup of G corresponding to $\mathfrak{g}_i$. Then $G = G_1 \cdots G_k$. Moreover, since $[\mathfrak{g}_i, \mathfrak{g}_j] = 0$ whenever $i \neq j$, $G_1, \ldots, G_k$ are mutually centralizing.

The groups $G_1, \ldots, G_k$ are known as the simple factors of G. Clearly $G_i \lhd G$ for all i, so that $G_i \cap G_j \lhd G$ for all i and j. As $\mathfrak{g}_i \cap \mathfrak{g}_j = \{0\}$ whenever $i \neq j$, this shows that $G_i \cap G_j$ is a discrete normal subgroup of G for all i and j with

$i \neq j$. However, every discrete normal subgroup of a connected Lie group is necessarily central, so that $G_i \cap G_j \subset Z(G)$ whenever $i \neq j$.

If $Z(G) = \{1\}$, it follows G is the internal direct product $G_1 \circ \cdots \circ G_k$. Since $Z(G) = \mathrm{Ker}(\mathrm{Ad})$, where $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation, $G \cong \mathrm{Ad}(G)$ whenever $Z(G) = \{1\}$. For this reason, semi-simple Lie groups of this type are said to be adjoint.

Adjoint semi-simple Lie groups are a particular example of linear semi-simple Lie groups, which share many of their properties. A Lie group G is said to be linear if there exists a faithful representation $\rho : G \to GL(n, \mathbb{R})$. The following theorem shows that every connected linear semi-simple Lie group is isomorphic to a real algebraic group.

Theorem 2.3.4. *Let G be a connected linear semi-simple Lie group and $\rho : G \to GL(n, \mathbb{R})$ a faithful representation. Then $\rho(G) = \overline{\rho(G)}^o \cap GL(n, \mathbb{R})$, where $\overline{\rho(G)}$ is the Zariski closure of $\rho(G)$ in $GL(n, \mathbb{C})$.*

Proof. See [31, page 10]. □

The Zariski topology on $\overline{G}$ induces a topology on G. This topology turns out to be independent of ρ (see [31, page 10]). It is called the Zariski topology of G.

The closure of a subset $H \subset G$ in the Zariski topology will be called its Zariski closure and denoted $\overline{H}_{\mathbb{R}}$. H will be said to be Zariski dense in G whenever $\overline{H}_{\mathbb{R}} = G$. By Proposition 2.2.3, this is equivalent to requiring that $\rho(G)$ and $\rho(H)$ have the same Zariski closure in $GL(n, \mathbb{C})$. We conclude with the following proposition:

Proposition 2.3.5. *Let G be a linear semi-simple Lie group. Then the centre $Z(G)$ of G is finite.*

Proof. Let $\rho : G \to GL(n, \mathbb{R})$ be a faithful linear representation of G. Identify G with its image under ρ and let $\overline{G}$ be the Zariski closure of G in $GL(n, \mathbb{C})$. Then, as $Z(G)$ centralizes G, $Z(G)$ centralizes $\overline{G}$ by Lemma 2.1.1. Hence $Z(G) \subset Z(\overline{G})$, so that $Z(G) = Z(\overline{G}) \cap GL(n, \mathbb{R})$. As $Z(\overline{G}) = \bigcap_{g \in \overline{G}} f_g^{-1}(1)$, where f_g is the polynomial map $x \mapsto gxg^{-1}x^{-1}$, $Z(\overline{G})$ is Zariski closed in $GL(n, \mathbb{C})$. Thus $Z(G)$ a real algebraic group and as such has only finitely many connected components. Since $Z(G)$ is discrete, this completes the proof. $\square$

Chapter 3

The fundamentals of semi-simple lattices

3.1 Definitions and normalizations

Every locally compact Hausdorff topological group admits a non-zero left invariant Borel measure that is unique up to multiplication by a real number. Such a measure is called a Haar measure, after A.Haar, who first proved its existence (see [28]).

If $\Gamma \subset G$ is a discrete subgroup of the Lie group G, then the quotient space G/Γ admits a unique differentiable structure with respect to which the covering $G \to G/\Gamma$ is smooth (see [44, page 41]). Moreover, G acts transitively by left multiplication on G/Γ. Consequently, G/Γ admits a left-invariant volume form, say ω. This induces a left invariant Borel measure μ on G/Γ.

Now, ω lifts to give a left invariant volume form on G, which in turn determines a unique Haar measure $\widetilde{\mu}$. However ω is determined by its lift to G so that μ is uniquely determined by $\widetilde{\mu}$. We will say that μ is the measure induced by $\widetilde{\mu}$ on G/Γ.

Definition 3.1.1. $\Gamma \subset G$ is a lattice if and only if $\mu(G/\Gamma)$ is finite.

If the quotient space G/Γ is compact, then Γ is clearly a lattice in Γ. Such lattices are said the be cocompact or uniform. Not all lattices however have this type (see [6] and [15]).

Theorem 3.1.2. *If Γ is a lattice in a connected Lie group G, then Γ is finitely generated.*

The first step toward a proof of the finite generation of lattices was taken by C.L.Siegel in [32], who showed that if Γ is a discrete subgroup of a connected Lie group G and there exits a "normal" fundamental domain F for the action of Γ on G, then the set $\{\gamma \in \Gamma : \overline{\gamma F} \cap \overline{F} \neq \phi\}$ generates Γ. This result was subsequently used by A.Borel and Harish-Chandra to show that all arithmetic groups are finitely generated [9].

The restriction on the class of domain meant however, that it was unclear how to apply C.L.Siegel's result without the assumption of arithmeticity. This problem was circumnavigated by A.M.MacBeath in [22], who showed that the assumption of normality could be dropped. This proved the finite generation of cocompact lattices. The non-cocompact case however, proved much harder to deal with.

The finite generation of lattices in a solvable Lie groups was eventually settled by G.D.Mostow in [26]. Then, L.Auslander [42] proved the following:

Theorem 3.1.3. *Let G be a Lie group and R a closed connected soluble normal subgroup. Let $\pi : G \to G/R$ be the natural map. Let H be a closed subgroup of G such that H^0 is soluble. Let U be the closure of $\pi(H)$ in the Euclidean topology of G/R. Then U^0 is soluble.*

L.Auslander's theorem has the following corollary:

Corollary 3.1.4. *Let G be a connected Lie group and $\Gamma \subset G$ a lattice. Let R be the radical of G and S a semi-simple subgroup of G such that $G = S \cdot R$. Assume that the kernel of the action of S on R has no compact factors in its identity component. Then, if $\pi : G \to G/R$ is the natural projection, $\pi(\Gamma)$ is a lattice in G/R and $\Gamma \cap R$ a lattice in R.*

Proof. [31, Corollary 8.28] $\square$

Since a connected Lie group G is isomorphic to a quotient $\widetilde{G}/Z$, where Z is a discrete central subgroup of the universal cover $\widetilde{G}$, to prove the finite generation of a lattice $\Gamma \subset G$, it is sufficient to show that lattices in $\widetilde{G}$ are finitely generated.

Now, every connected and simply-connected Lie group G is isomorphic to a product $G_1 \times G_2$ in which G_1 is connected and semi-simple and $G_2 = SR$, where $R = \mathrm{rad}(G)$, S is semi-simple. Moreover, G_2 can be assumed not to contain any connected normal semi-simple subgroup of G and S to act almost faithfully on R (see [42]).

If Γ is a lattice in G and C the maximal connected compact normal subgroup of G_1, then $\Gamma \cap C$ is finite so that $p(\Gamma)$ is a lattice in G/C, where $p : G \to G/C$ is the natural projection. Since Γ is finitely generated if and only if $p(\Gamma)$ is finitely generated we may therefore assume that G_1 has no compact factors.

Following L.Auslander's theorem therefore, it remained to show only that lattices in connected semi-simple Lie groups are finitely generated. Since any connected compact normal subgroups can be factored out as above, this problem was equivalent to proving the finite generation of lattices in a connected semi-simple Lie group G without compact factors.

When G has real rank 1, this was proved by M.S.Raghunathan in [31, Corollary 13.20]. The finite generation of lattices in semi-simple Lie groups with real rank ≥ 2 was not however settled until D.A.Kazdan introduced his famous T property in [19], which finally proved the finite generation of all lattices.

Our principal objects of study will be lattices in connected semi-simple Lie groups with finite centre, whose finite generation will permit certain normalizations.

Definition 3.1.5. Let L be the class of lattices in connected semi-simple Lie groups with finite centre and L_0 the subclass of torsion-free lattices in L.

Proposition 3.1.6. *Lattices in Γ are virtually torsion-free.*

Proof. Pick $\Gamma \in L$. The group Γ embeds as a lattice in a connected semi-simple Lie group G with finite centre. As $\mathrm{Ad} : G \to \mathrm{Ad}(G) \cong G/Z(G)$ is a covering map of finite degree, $\mathrm{Ad}(\Gamma)$ is discrete in $\mathrm{Ad}(G)$ and the G-invariant volume on $\mathrm{Ad}(G)/\mathrm{Ad}(\Gamma)$ is finite. That is, $\mathrm{Ad}(\Gamma)$ is lattice in $\mathrm{Ad}(G)$.

Now, if $H \subset \mathrm{Ad}(\Gamma)$ has finite index d in $\mathrm{Ad}(\Gamma)$, then $\mathrm{Ad}^{-1}(H)$ has index d in Γ also. Thus, if $\mathrm{Ad}(\Gamma)$ is virtually torsion-free, so is Γ. But $\mathrm{Ad}(G)$, and therefore $\mathrm{Ad}(\Gamma)$, is linear, whilst Selberg's theorem [6] states that every finitely generated linear group is virtually torsion free. This proves the result. $\square$

Proposition 3.1.7. *If $\Gamma \in L_0$, then Γ embeds as a lattice in a connected adjoint semi-simple Lie group without compact factors.*

Proof. Γ is a lattice in a connected semi-simple Lie group G with finite centre. As Γ is torsion-free, $\Gamma \cap Z(G) = \{e\}$. The adjoint map is therefore injective on Γ, which embeds as a lattice in $\mathrm{Ad}(G)$. Let C be the product of the compact simple factors of $\mathrm{Ad}(G)$. Then, as Γ and therefore $\mathrm{Ad}(\Gamma)$ is torsion-free, $\mathrm{Ad}(\Gamma) \cap C = \{e\}$, so that $\mathrm{Ad}(\Gamma)$ embeds as a lattice in the quotient group $\mathrm{Ad}(G)/C$, non of whose simple factors are compact. As $\mathrm{Ad}(G)/C$ is also adjoint, this completes the proof. □

3.2 The Borel density theorem

The Borel density theorem is fundamental to the theory of lattices in semi-simple Lie groups. The statement of the theorem given below is less general than Borel's original result [5], but sufficient for the purposes of this thesis. We shall use it to prove that lattice in L_0 have finite centre.

Theorem 3.2.1. *(Borel Density) Let Γ be a lattice in a linear semi-simple Lie group G without compact factors. Then Γ is Zariski dense in G.*

Corollary 3.2.2. *With Γ and G as above, $C(\Gamma) \subset Z(G)$, where $C(\Gamma)$ is the centralizer of Γ in G and $Z(G)$ the centre of G.*

Proof. Fix $g \in G$. Then, $g \in C(\Gamma) \iff g\gamma g^{-1} = \gamma$ for all $\gamma \in \Gamma \iff \Gamma \subset C(g)$, the centralizer of g. But $C(g) = f^{-1}(1)$ where f is the polynomial map $h \mapsto ghg^{-1}h^{-1}$. Thus $C(g)$ is a Zariski closed subset of G containing Γ. But Γ is Zariski dense in G, and so in fact $G \subset C(g)$. Thus $g \in Z(G)$. This proves the result. □

Corollary 3.2.3. *Lattices in L_0 have finite centre.*

Proof. Pick Γ in L_0. Then Γ can be embedded as a lattice in an adjoint semi-simple Lie group G without compact factors. As the adjoint representation is faithful G is linear. So, by the corollary above, $Z(\Gamma) \subset C(\Gamma) \subset Z(G)$. But G is adjoint and thus $Z(G)$ is trivial. Hence $Z(\Gamma) = \{1\}$. □

3.3 A structure theorem for groups in L

Two groups A and B are said to be commensurable if there exists a group Δ together with embeddings $j_1 : \Delta \hookrightarrow A$ and $j_2 : \Delta \hookrightarrow B$, such that the subgroups $j_1(\Delta)$ and $j_2(\Delta)$ have finite index in A and B respectively. An infinite group Γ is said to be irreducible if it is not commensurable with the direct product of two infinite groups. The concept of irreducibility is important in the classification of infinite groups. In the theory of lattices in semi-simple Lie groups however, this concept has a more natural formulation.

Let Γ be a lattice in a connected linear semi-simple Lie group G whose simple factors $G_1, \ldots, G_n$ are all non-compact. For each subset J of $\{1, \ldots, n\}$, define $G_J = \Pi_{j \in J} G_j$ and let $p_J : G \to G/G_{J'}$ be the natural projection, where J' is the complement of J in $\{1, \ldots, n\}$.

Theorem 3.3.1. *Γ is irreducible if and only if $\pi_J(\Gamma)$ is non-discrete for every subset J of $\{1, \ldots, n\}$.*

The proof depends on the following:

Theorem 3.3.2. *Let Γ be a lattice in a connected linear semi-simple Lie group G without compact factors. Let H and H' be closed normal subgroups of G such that $G = HH'$ and $H \cap H'$ is discrete. Let π and π' be the natural projections onto G/H' and G/H respectively. Then, the following are equivalent:*

1. *$\pi'(\Gamma)$ is discrete in G/H*

2. *$H \cap \Gamma$ is a lattice in H*

3. *$\pi(\Gamma)$ is discrete in G/H*

4. *$H' \cap \Gamma$ is a lattice in H'*

5. $(H \cap \Gamma)(H' \cap \Gamma)$ has finite index in Γ

Proof. [31, Corollary 5.19].

□

proof of Theorem 3.3.1.
Suppose that $\pi_J(\Gamma)$ is discrete for some $J \subset \{1, \ldots, n\}$. Set $J' = \{1, \ldots, n\} - J$. Then, by the theorem above, $(\Gamma \cap G_J)(\Gamma \cap G_{J'})$ has finite index in Γ and Γ is reducible.

Conversely, suppose that Γ is commensurable with the product $A \times B$ of two infinite groups. As a subgroup of finite index in a lattice is lattice of the same type, we may assume without loss of generality that Γ embeds with finite index in $A \times B$. Identify Γ with its image in $A \times B$. Then there exist injections $A/\Gamma \cap A \hookrightarrow (A \times B)/\Gamma$ and $B/\Gamma \cap B \hookrightarrow (A \times B)/\Gamma$. So, the groups $\Gamma \cap A$ and $\Gamma \cap B$ have finite index in A and B respectively and $(\Gamma \cap A)(\Gamma \cap B)$ has finite index in Γ. Thus $G = \overline{\Gamma}_{\mathbb{R}} = \overline{(\Gamma \cap A)(\Gamma \cap B)}_{\mathbb{R}}$. But $(\Gamma \cap A)(\Gamma \cap B)$ normalizes $\Gamma \cap A$ and $\Gamma \cap B$, and so, by Proposition 2.1.1, $(\overline{\Gamma \cap A})^0_{\mathbb{R}}$ and $(\overline{\Gamma \cap B})^0_{\mathbb{R}}$ are normal in G. Each of these groups is therefore a product of simple factors from G. Moreover, as A and B are mutually centralizing, they commute. Since the simple factors of G are non-abelian this shows that $(\overline{\Gamma \cap A})^0_{\mathbb{R}}$ and $(\overline{\Gamma \cap B})^0_{\mathbb{R}}$ have no common factor. Hence $(\overline{\Gamma \cap A})^0_{\mathbb{R}} \cap (\overline{\Gamma \cap B})^0_{\mathbb{R}}$ is a discrete normal subgroup of G and therefore contained in $Z(G)$, which is finite. This implies that the projection of Γ on to either group is discrete. □

By reformulating of the concept of irreducibility in this way, it is possible to show that every lattice in L is commensurable with the direct product of finitely many irreducible lattices from the same class. The proof of this fact depends on the concept of compatible partitions.

Let Γ be a lattice in a connected linear semi-simple Lie group G as above. Then, a partition $\{J_1, \ldots, J_m\}$ of $\{1, \ldots, n\}$ is said to be compatible with Γ if $\pi_J(\Gamma)$ is discrete in G_J for each $J \in \{J_1, \ldots, J_m\}$. The relation,

$$\{J_1, \ldots, J_m\} \leq \{K_1, \ldots, K_n\} \iff m \geq n \text{ and for each } r, J_r \subset K_s \text{ for some } s.$$

is a partial order on the set of compatible partitions. As only finitely many partitions exist, minimal partitions compatible with Γ can certainly be found. If $\{J_1, \ldots, J_m\}$ is such a partition, then, for each i, the projection of Γ on to any product of simple factors from G_{J_i} is non-discrete. In this case therefore, $\pi_{J_i}(\Gamma)$ is an irreducible lattice in G_{J_i}.

The following proposition shows that the intersection of distinct sets from two minimal partitions must be empty, so that, in fact, there is only one such partition. This fact will be used to prove the structure theorem for L.

Proposition 3.3.3. *Let Γ be a lattice in a connected semi-simple Lie group G with finite centre. Let H_1 and H_2 be closed connected normal subgroups of G such that $G = H_1H_2$. Set $K = H_1 \cap H_2$ and for $j = 1, 2$ let A_j be the unique closed connected normal subgroup of H_j such that $K \cap A_j$ is discrete and $H_j = KA_j$. Let $\pi_1 : G \to G/A_2$ and $\pi_2 : G \to G/A_1$ be the natural projections. Then, if $\pi_j(\Gamma)$ is discrete in H_j for $j = 1, 2$, the image of Γ is discrete under the natural projection $G \to G/A_1A_2$.*

Proof. There is a commutative diagram:

$$\begin{tikzcd} & G \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ G/A_2 \arrow[dr, "p_1"'] & & G/A_1 \arrow[dl, "p_2"] \\ & G/A_1A_2 & \end{tikzcd}$$

Since $G = H_1A_2$, whilst $H_1 \cap A_2$, together with the image of Γ under the natural projection $G \to G/A_2$ is discrete, Theorem 3.3.2 shows that $(\Gamma \cap H_1)(\Gamma \cap A_2)$ has finite index in Γ. It is therefore sufficient to prove that the image of $(\Gamma \cap H_1)(\Gamma \cap A_2)$ is discrete in G/A_1A_2. Now, as $q_1(\Gamma \cap A_2)$ is trivial,

$$p_1q_1((\Gamma \cap H_1)(\Gamma \cap A_2)) = p_1q_1(\Gamma \cap H_1) = p_2q_2(\Gamma \cap H_1).$$

Moreover, $q_2(\Gamma \cap H_1) \subset q_2(\Gamma) \cap H_1/A_1$ is discrete in G/A_1. But, $H_1/A_1 \cong K/K \cap A_1$ while $G/A_1A_2 = KA_1A_2/A_1A_2 \cong K/K \cap A_1A_2$. So, by naturality, the restriction of p_1 to H_1/A_1 corresponds to the finite covering map $K/K \cap A_1 \to K/K \cap A_1A_2$ and this implies that $p_2q_2(\Gamma \cap H_2)$ is discrete in G/A_1A_2. Hence the result. $\square$

Theorem 3.3.4. *Let Γ be a lattice in L. Then Γ is commensurable with a direct product $\Gamma_1 \times \cdots \times \Gamma_k$ of irreducible lattices from L. Moreover, the groups $\Gamma_1, \ldots, \Gamma_k$ are uniquely defined up to commensurability.*

Proof. Let Δ be a torsion-free subgroup of finite index in Γ. Then, by Proposition 3.1.7, Δ embeds as a lattice in a connected adjoint semi-simple Lie group G without compact factors. Suppose that G has n simple factors and let $\{J_1, \ldots, J_k\}$ be the unique minimal partition of $\{1, \ldots, n\}$ compatible with Δ. Then $\Delta_j = \Delta \cap G_{J_j}$ is an irreducible lattice in G_{J_j} for each j. Moreover, by Theorem 3.3.2, the product $\Delta_1 \cdots \Delta_k \cong \Delta_1 \times \cdots \times \Delta_k$ is contained in Δ with finite index, so that Γ is commensurable with $\Delta_1 \times \cdots \times \Delta_k$.

Suppose now that $\Gamma \sim \Gamma_1 \times \cdots \times \Gamma_l$, where Γ_j is irreducible for each j. Let H be a group that embeds in $\Gamma_1 \times \cdots \times \Gamma_l$ and Γ with finite index. Then, as $\Gamma_j/\Gamma_j \cap H$ injects into $(\Gamma_1 \times \cdots \times \Gamma_l)/H$, $H \cap \Gamma_j$ has finite index in Γ_j for each j, and $\Gamma' = (H \cap \Gamma_1) \cdots (H \cap \Gamma_l)$ has finite index in Γ. As the groups $H \cap \Gamma_j$ are mutually centralizing, $(\overline{H \cap \Gamma_j})^0_{\mathbb{R}}$ is a connected normal subgroup of G for each j, and therefore a product of simple factors of G. That is, $(\overline{H \cap \Gamma_j})^0_{\mathbb{R}} = G_{J_j}$ for some $J_j \subset \{1, \ldots n\}$.

Suppose that $G_{J_p} \cap G_{J_q} \neq \{1\}$ for some pair p, q. Then, since $\Gamma' \cap G_{J_j} = \Gamma_j \cap H$ for all j,

$$\Gamma' \cap (G_{J_p} \cap G_{J_q}) = (\Gamma' \cap G_{J_p}) \cap (\Gamma' \cap G_{J_q}) = (\Gamma_p \cap H) \cap (\Gamma_q \cap H),$$

which is non-trivial if and only if $p = q$. Thus $\{J_1, \ldots, J_l\}$ is a partition of $\{1, \ldots, n\}$. As Γ_j and therefore $H \cap \Gamma_j$ is irreducible for each j, this partition must be minimal. But there exists only one such partition. So, $l = k$ and there exists a bijection $\sigma \in S_k$ such that $\Gamma_j \sim \Delta_{\sigma(j)}$ for all j. $\square$

Chapter 4

E-invariants and the SFC property

Let Γ be a group of type FP and $\varepsilon : P \to \mathbb{Z}$ a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ having finite type. Then, as P_i is finitely generated as a $\mathbb{Z}\Gamma$-module, $Hom_{\mathbb{Z}\Gamma}(P_i, \mathbb{Z})$ and its subquotient $H^i(\Gamma, \mathbb{Z})$ are finitely generated abelian groups for all i. Consequently every group Γ of type FP has a well-defined Euler characteristic

$$\chi(\Gamma) = \Sigma_{i=0}(-1)^i \mathrm{rank}(H^i(\Gamma; \mathbb{Z})).$$

Now, it is well known that in this case,

$$\chi(\Delta) = [\Gamma : \Delta]\chi(\Gamma)$$

whenever Δ is a subgroup of finite index in Γ (see [12, Chapter IX, Theorem 6.3]). This fact is expressed by saying that χ is multiplicative with respect to covers (on groups of type FP). In [41], C.T.C.Wall used this result to extend the definition of the Euler characteristic to groups containing a subgroup of type FP with finite index. He showed that if A is a group and Γ a subgroup of finite index in A having type FP, then the quantity

$$\chi_{\mathbb{Q}}(A) = \frac{1}{[A : \Gamma]}\chi(\Gamma)$$

is independent of the choice of Γ and therefore an isomorphism invariant of A.

Of course this also means that $\chi_{\mathbb{Q}}(A)$ is a defined for any group A that is commensurable with a fixed group Γ having type FP. In this way, C.T.C.Wall extended the definition of the Euler characteristic from a particular group of type FP to the whole of its commensurability class.

It turns out that C.T.C.Wall's argument is entirely general and that any isomorphism invariant that is multiplicative with respect to covers can be extended to define an isomorphism invariant on the commensurability class of a group with the same property. This is the content of Theorem 4.1.6.

Definition 4.0.5. A non-zero isomorphism invariant that is defined on the commensurability class of a group and multiplicative with respect to covers will be called an e-invariant.

This term will also be used to describe isomorphism invariants which are multiplicative with respect to covers but have only been defined on subgroups of finite index within a particular group.

Now, if v is an e-invariant for a group Γ and Δ_1 and Δ_2 isomorphic subgroups of finite index in Γ so that $v(\Delta_1) = v(\Delta_2)$, then, since $v(\Gamma) \neq 0$,

$$[\Gamma : \Delta_1] = v(\Delta_1)/v(\Gamma) = v(\Delta_2)/v(\Gamma) = [\Gamma : \Delta_2].$$

This shows that the index of a subgroup of finite index in Γ only depends on its isomorphism type.

Definition 4.0.6. A group Γ will be called strongly finitely cohopfian or SFC, if the index of a subgroup of finite index in Γ is determined by its isomorphism type.

We have shown,

Proposition 4.0.7. *If a group admits and e-invariant, then it is strongly finitely cohopfian.*

The aim of this chapter is to prove that the SFC property is in fact a property of the commensurability class and that possession of this property is a necessary and sufficient condition for the commensurability class of a group to admit an e-invariant.

4.1 Duality in the SFC property and the existence of e-invariants

In the following, an embedding of groups $\varphi : \Delta \hookrightarrow \Gamma$ will be called cofinite if $\varphi(\Delta)$ has finite index in Γ.

Definition 4.1.1. Γ will be said to have the finite index property if, given any two cofinite embeddings $i_1 : H \hookrightarrow \Gamma$ and $i_2 : H \hookrightarrow \Gamma$, $[\Gamma : i_1(H)] = [\Gamma : i_2(H)]$.

Definition 4.1.2. Γ will be said to have the finite coindex property if, for cofinite embeddings $j_1 : \Gamma \to \Gamma_1$ and $j_2 : \Gamma \to \Gamma_2$, the existence of an isomorphism $\Gamma_1 \cong \Gamma_2$ implies that $[\Gamma_1 : j_1(\Gamma)] = [\Gamma_2 : j_2(\Gamma)]$.

Lemma 4.1.3. *(Duality) Let Γ be a group with the finite index property. Then Γ also has the finite coindex property.*

Proof. Let $j_1 : \Gamma \hookrightarrow \Gamma_1$ and $j_2 : \Gamma \hookrightarrow \Gamma_2$ be cofinite embeddings and suppose that $\varphi : \Gamma_1 \to \Gamma_2$ is an isomorphism. Then $j_2(\Gamma) \subset \Gamma_2$ and $\varphi j_1(\Gamma) \subset \Gamma_2$ with finite index. Set $H = j_2(\Gamma) \cap \varphi j_1(\Gamma)$. Then H has finite index in $j_2(\Gamma)$ and $\varphi j_1(\Gamma)$ and

$$[\Gamma_1 : j_1(\Gamma)][j_1(\Gamma) : \varphi^{-1}(H)] = [\Gamma_1 : \varphi^{-1}(H)] = [\Gamma_2 : H] = [\Gamma_2 : j_2(\Gamma)][j_2(\Gamma) : H].$$

So, to show that $[\Gamma_1 : j_1(\Gamma)] = [\Gamma_2 : j_2(\Gamma)]$, it is sufficient to prove that

$$[j_1(\Gamma) : \varphi^{-1}(H)] = [j_2(\Gamma) : H].$$

Now, $[j_1(\Gamma) : \varphi^{-1}(H)] = [\varphi j_1(\Gamma) : H]$. Moreover, $\varphi j_1(\Gamma) \cong \Gamma \cong j_2(\Gamma)$. Pick an isomorphism $\theta : \varphi j_1(\Gamma) \to j_2(\Gamma)$. Then, as Γ has the finite index property,

$$[j_2(\Gamma) : \theta(H)] = [j_2(\Gamma) : H].$$

So $[j_1(\Gamma) : \varphi^{-1}(H)] = [\varphi j_1(\Gamma) : H] = [j_2(\Gamma) : \theta(H)] = [j_2(\Gamma) : H]$. Hence the result. □

Theorem 4.1.4. *A group Γ has the finite index property if and only if every subgroup of finite index in Γ has the finite coindex property.*

Proof. $\Leftarrow$) Let H be a group and suppose that there exist cofinite embeddings $i_1 : H \hookrightarrow \Gamma$ and $i_2 : H \hookrightarrow \Gamma$. Then H, being isomorphic to a subgroup of finite index in Γ, must have the finite coindex property by hypothesis. But this means $[\Gamma : i_1(H)] = [\Gamma : i_2(H)]$, so that Γ has the finite index property.

$\Rightarrow$) If Γ has the finite index property, then so does any subgroup H of finite index in Γ. But H then has the finite coindex property by Lemma 4.1.3 above. Hence the result. □

Clearly a group Γ has the finite index property if and only if it is SFC.

Corollary 4.1.5. *If a group Γ has the strong finite cohopfian property, then so does any group commensurable with Γ.*

Proof. Let B be a group commensurable with Γ. Any group that embeds with finite index in Γ is certainly SFC, so it is sufficient to show that B is SFC whenever there exists a cofinite embedding $\Gamma \hookrightarrow B$.

Identify Γ with its image in B. By Theorem 4.1.4, B has the SFC property if and only if every subgroup of finite index has the finite coindex property. So, let H be a subgroup of finite index in B and suppose that $j_1 : H \hookrightarrow \Gamma_1$ and $j_2 : H \hookrightarrow \Gamma_2$ are cofinite embeddings, where $\Gamma_1 \cong \Gamma_2$. Then, as $\Gamma \cap H$ has the finite coindex property,

$$[\Gamma_1 : j_1(\Gamma \cap H)] = [\Gamma_2 : j_2(\Gamma \cap H)].$$

But, for $r = 1, 2$, $[\Gamma_r : j_r(\Gamma \cap H)] = [\Gamma_r : j_r(H)][j_r(H) : j_r(\Gamma \cap H)]$, while $[j_1(H) : j_1(\Gamma \cap H)] = [H : \Gamma \cap H] = [j_2(H) : j_2(\Gamma \cap H)]$. Consequently $[\Gamma_1 : j_1(H)] = [\Gamma_2 : j_2(H)]$, which completes the proof. □

Let Γ be SFC and pick a non-zero real number c. Define $v(\Gamma) = c$. Now suppose that Γ' is a group commensurable with Γ, so that there exists a group Δ and cofinite embeddings $j : \Delta \hookrightarrow \Gamma$ and $k : \Delta \hookrightarrow \Gamma'$. Define

$$v(\Gamma') = \frac{[\Gamma : j(\Delta)]}{[\Gamma' : k(\Delta)]}\, c.$$

Theorem 4.1.6. *$v(\Gamma')$ is independent of the choice of Δ, j and k and therefore depends only on the isomorphism type of Γ'. Moreover $v(A) = [\Gamma' : A]v(\Gamma')$ whenever A is a subgroup of finite index in Γ', so that v is an e-invariant defined on the commensurability class of Γ.*

Proof. Let Δ_1 and Δ_2 be groups such that there exist cofinite embeddings

$$j_1 : \Delta_1 \hookrightarrow \Gamma \text{ and } k_1 : \Delta_1 \hookrightarrow \Gamma'$$

$$j_2 : \Delta_2 \hookrightarrow \Gamma \text{ and } k_2 : \Delta_2 \hookrightarrow \Gamma'$$

Set $H = j_1(\Delta_1) \cap j_2(\Delta_2)$. Then H is contained in Γ with finite index and

$$[\Gamma : j_1(\Delta_1)][j_1(\Delta_1) : H] = [\Gamma : H] = [\Gamma : j_2(\Delta_2)][j_2(\Delta_2) : H].$$

Let $\psi_1 : j_1(\Delta_1) \to k_1(\Delta_1)$ and $\psi_2 : j_2(\Delta_2) \to k_2(\Delta_2)$ be isomorphisms. Then $[k_1(\Delta_1) : \psi_1(H)] = [j_1(\Delta_1) : H]$ and $[k_2(\Delta_2) : \psi_2(H)] = [j_2(\Delta_2) : H]$. So,

$$[\Gamma : j_1(\Delta_1)][k_1(\Delta_1) : \psi_1(H)] = [\Gamma : j_2(\Delta_2)][k_2(\Delta_2) : \psi_2(H)].$$

Multiplying both sides by $[\Gamma' : k_1(\Delta_1)]$, we deduce:

$$[\Gamma' : \psi_1(H)][\Gamma : j_1(\Delta_1)] = [\Gamma' : k_1(\Delta_1)][\Gamma : j_2(\Delta_2)][k_2(\Delta_2) : \psi_2(H)]$$

But H is SFC and so $[k_2(\Delta_2) : \psi_2(H)] = [k_2(\Delta_2) : \psi_1(H)]$. Therefore,

$$[\Gamma' : k_2(\Delta_2)][\Gamma : j_1(\Delta_1)] = [\Gamma' : k_1(\Delta_1)][\Gamma : j_2(\Delta_2)],$$

which completes the proof. □

Corollary 4.1.7. *A group Γ is SFC if and only if it admits an e-invariant.*

Proof. Immediate. □

4.2 The relation to rigidity

In this section, an irreducible lattice in a connected semi-simple Lie group with finite centre will be shown to be SFC. This result, which makes use of the Mostow rigidity theorem [27], subsumes an earlier result due to A.Borel, who originally initiated investigation into the relationship between the index of a subgroup in a semi-simple lattice and its isomorphism type, proving the following theorem in [7]:

Theorem 4.2.1. *Let G be a linear algebraic group defined over $\mathbb{R}$ and Γ a lattice in $G_{\mathbb{R}}$ such that $H^1(\Delta; Ad) = 0$ for all subgroups Δ of finite index in Γ. Then Γ is not isomorphic to a proper subgroup of finite index.*

Here $H^*(\_;\mathrm{Ad})$ denotes cohomology with local coefficients in the sense of N.Steenrod [35, pages 151-166]. The local system in question is constructed in a canonical way from the representation Ad a described in [31, pages 105 - 107].

Now, A.Weil had proved in [43] that if Γ is a discrete cocompact subgroup of a connected linear semi-simple Lie group whose non-compact factors all have dimension > 3, then $H^1(\Gamma;\mathrm{Ad}) = 0$. Any such group $G \subset GL(n,\mathbb{R})$ is however, just the set of real points of the identity component of its Zariski closure in $GL(n,\mathbb{R})$. As A.Borel observed, the above theorem therefore applies to any cocompact lattice Γ in a connected linear semi-simple Lie group whose non-compact factors all have dimension > 3. In addition, using the results of M.S.Raghunathan (see [29] and [30]), A.Borel was able to show that Theorem 4.2.1 holds for a large class of arithmetic lattices in semi-simple Lie groups that includes all arithmetic lattices in connected linear algebraic groups defined and simple over $\mathbb{Q}$ that are not locally isomorphic to $SL(n,\mathbb{C})$. A linear algebraic group that is defined over $\mathbb{Q}$ is said to be $\mathbb{Q}$-simple or simple over $\mathbb{Q}$ if it contains no proper connected normal subgroup defined over $\mathbb{Q}$.

In a later chapter, I will show that any lattice in a connected semi-simple Lie group with finite centre has the SFC property (Theorem 5.2.2). This result extends both A.Borel's original theorem and the theorem derived from the Mostow Rigidity Theorem in the next section.

4.2.1 The SFC property of irreducible lattices in L

In [27], G.D.Mostow showed that if Γ_j is an irreducible cocompact torsion-free lattice in an adjoint semi-simple Lie group $G_j \neq PSL(2,\mathbb{R})$ without compact factors $(j = 1, 2)$, then every isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ extends to a unique

Lie group isomorphism $\widetilde{\varphi} : G_1 \to G_2$. This phenomenon is known as (Mostow) rigidity. The hypotheses of G.D.Mostow's original theorem were subsequently weakened by G.A.Margulis, G.Prasad and M.S.Raghunathan to include non-cocompact lattices (see [27, pages 8 - 9]). The full theorem is therefore:

Theorem 4.2.2. *For $j = 1, 2$, let Γ_j be an irreducible torsion-free lattice in a connected adjoint semi-simple Lie group G_j without compact factors and not isomorphic to $PSL(2, \mathbb{R})$. Then, every isomorphism $\Gamma_1 \to \Gamma_2$, extends uniquely to a Lie group isomorphism $G_1 \to G_2$.*

A couple (G, Γ) satisfying the hypotheses of the full rigidity theorem will be called a rigid couple. The following proposition shows that once a Haar measure $\widetilde{\mu}$ has been fixed on G, the volume $\mu(G/\Gamma)$ depends only on the isomorphism type of Γ, where μ is the G-invariant Borel measure on G/Γ induced by $\widetilde{\mu}$.

Proposition 4.2.3. *Let (G, A) and (G, B) be rigid couples and suppose that $A \cong B$, then $\mu(G/A) = \mu(G/B)$.*

Proof. Let $\varphi : A \to B$ be an isomorphism and $\widetilde{\varphi} : G \to G$ the unique Lie group isomorphism extending φ. Let ω be the volume form on G inducing $\widetilde{\mu}$. Then, it follows from the definition of μ that

$$\mu(G/B) = \widetilde{\mu}(V) = \int_V \omega,$$

where V is a fundamental domain for the action of B on G. But $U = \widetilde{\varphi}^{-1}(V)$ is a fundamental domain for the action of A, while $\widetilde{\varphi}^*\omega = det(L\widetilde{\varphi})\omega$, where L is the Lie functor. So in fact

$$\mu(G/B) = \int_V \omega = |det(L\widetilde{\varphi})| \int_U \omega = |det(L\widetilde{\varphi})| \, \mu(G/A)$$

It is therefore sufficient to show $|det(L\widetilde{\varphi})| = 1$. To see this, first observe that, as G is semi-simple, $\mathrm{Ad}(G) = \mathrm{Aut}(\mathfrak{g})^0$, where $\mathfrak{g}$ is the Lie algebra of G. As connected semi-simple Lie groups only admit trivial characters, this implies $det(\theta) = 1$ for all $\theta \in Aut(\mathfrak{g})^0$. Let $\{E_i\}$ be a basis for $\mathfrak{g}$. We may identify $End(\mathfrak{g})$ with $GL(n^2, \mathbb{R})$ via $\alpha \mapsto \alpha_{ij}$, where $\alpha(E_i) = \alpha_{ji}E_j$ for all i,j. Let C_{ijk} be real numbers such that $[E_i, E_j] = C_{ijk}E_k$ for all i,j,k. Then $\alpha_{ij} \in GL(n, \mathbb{R})$ corresponds to an element of $Aut(\mathfrak{g})$ if and only if

$$\alpha_{ip}\alpha_{jq}C_{pqr}E_r = [\alpha(E_i), \alpha(E_j)] = \alpha[E_i, E_j] = C_{ijk}\alpha(E_k) = C_{ijk}\alpha_{rk}E_r,$$

so that $\alpha_{ip}\alpha_{jq}C_{pqr} = C_{ijk}\alpha_{rk}$. But this mean that $Aut(\mathfrak{g})$ is an algebraic subgroup of $GL(n^2, \mathbb{R})$ defined over $\mathbb{R}$ and therefore has only finitely many connected components. $\{1, -1\}$ contains all finite subgroups of $\mathbb{R}$, and any real-valued character of $\mathrm{Aut}(\mathfrak{g})$ must therefore take values in this set, which implies that $|det(L\widetilde{\varphi})| = 1$. □

Corollary 4.2.4. *Let Γ be an irreducible torsion-free cocompact lattice in a connected adjoint semi-simple Lie group G without compact factors. Then, if μ is a non-zero G-invariant Borel measure on G/Γ, the mapping*

$$f_\mu : \Delta \mapsto \mu(\Delta),$$

defined on subgroups of finite index in Γ, is an e-invariant for Γ.

Proof. The proposition above shows that f_μ is an isomorphism invariant. We must show that it is multiplicative with respect to covers. Let Δ be a subgroup of finite index d in Γ. Then there exists a $d-1$ covering map $G/\Delta \to G/\Gamma$ defined by $g\Delta \mapsto g\Gamma$, so that $\mu(G/\Delta) = d \cdot \mu(G/\Gamma)$. Hence the result. □

Theorem 4.2.5. *If $\Gamma \in L$ is irreducible, then it has the SFC property.*

Proof. As lattices in L are virtually torsion-free by Proposition 3.1.6, while the SFC property is a property of the commensurability class (Corollary 4.1.5), we may assume that Γ is torsion-free. In this case however, Proposition 3.1.7 shows that Γ embeds as a lattice in a connected adjoint semi-simple Lie group without compact factors. The only case that is excluded by the theorem above is therefore that of lattices in $PSL(2,\mathbb{R})$ (which are necessarily irreducible since $PSL(2,\mathbb{R})$ is simple). Now, $PSL(2,\mathbb{R})$ acts transitively on the upper half-plane $\mathbb{H}$ by Möbius transformations. The stabilizer of i is the compact subgroup $PSO(2)$, so that $PSL(2,\mathbb{R})/PSO(2) \cong \mathbb{H}$. But Γ is discrete and torsion-free, and therefore acts freely and properly discontinuously on $PSL(2,\mathbb{R})/PSO(2)$ by left multiplication. The double coset space $X_\Gamma = \Gamma\backslash PSL(2,\mathbb{R})/PSO(2)$ is a smooth manifold covered by $\mathbb{H} = PSL(2,\mathbb{R})/PSO(2)$ whose fundamental group isomorphic to Γ. Thus X_Γ is a $K(\Gamma,1)$ space and $\chi(\Gamma) = \chi(X_\Gamma)$.

Now, when Γ is cocompact, X_Γ, being the image of the compact space $\Gamma\backslash PSL(2,\mathbb{R})$ under the continuous map $\Gamma g \mapsto \Gamma g PSO(2)$, is a compact manifold. X_Γ is therefore the fundamental group of compact surface without boundary. But by Corollary 3.2.3, $Z(\Gamma) = \{1\}$, so that Γ is non-abelian. Thus X_Γ has genus > 1 and $\chi(\Gamma) = \chi(X_\Gamma) \neq 0$. As the Euler characteristic is multiplicative on covers, this shows that Γ is SFC.

If Γ is not cocompact, then its cohomological dimension must be strictly less than the dimension of X_Γ (see [12, Chapter VII, Proposition 8.1]). Γ, being non-trivial, therefore has cohomological dimension 1. It is a celebrated theorem of J.R.Stallings [34] and R.G.Swan [38] that every group of cohomological dimension 1 is free. As Γ is finitely generated and non-abelian, it is therefore isomorphic to the free group on r generators, for some $r > 1$. So $\chi(\Gamma) = 1 - r \neq 0$, which shows that Γ is SFC. $\square$

Chapter 5

Direct products of groups and the SFC property

If Γ_1 and Γ_2 are groups, and $\Gamma_1 \times \Gamma_2$ is SFC, then so are Γ_1 and Γ_2. For, if A and B are isomorphic subgroups of finite index in Γ_1, then $A \times \Gamma_2$ and $B \times \Gamma_2$ are isomorphic subgroups of finite index in $\Gamma_1 \times \Gamma_2$, so that

$$[A : \Gamma_1] = [A \times \Gamma_2 : \Gamma_1 \times \Gamma_2] = [B \times \Gamma_2 : \Gamma_1 \times \Gamma_2] = [B : \Gamma_1].$$

In this chapter it will be shown that, for a suitably strong notion of irreducibility, the converse is also true. The resulting product theorem will then be used to show that all lattices in connected semi-simple lattices with finite centre are SFC (Theorem 5.2.2). Finally, counter examples will be given to show that for an arbitrary extension $1 \to K \to G \to Q \to 1$ in which Γ is SFC, both K and Q can fail to possess the strong finite cohopfian property.

As a non-zero Euler characteristic is an e-invariant, were it the case that $\chi(\Gamma) \neq 0$ whenever $\Gamma \in L$, Theorem 5.2.2 would be well-known. However, though always defined for such Γ, in many cases $\chi(\Gamma)$ is either zero or else still unknown. For example, if Γ is a cocompact lattice in L then any torsion-free subgroup Δ of finite index in Γ satisfies Poincaré duality. So, if the cohomological dimension of Δ is odd, then $0 = \chi(\Delta) = \chi_{\mathbb{Q}}(\Gamma)$. In addition, certain non-cocompact lattices are known to have zero Euler characteristic. In particular, $\chi(SL(n, \mathbb{Z})) = 0$ for all $n \geq 3$ (see [12, page 256]).

5.1 The product theorem

Definition 5.1.1. An infinite group H will be called strongly irreducible if it is not commensurable with a group that can be written as the product of two infinite mutually centralizing subgroups.

Theorem 5.1.2. *The direct product of finitely many strongly irreducible torsion-free SFC groups has the strong finite cohopfian property.*

We begin with the following lemma:

Lemma 5.1.3. *Suppose that $H = H_1 \circ \cdots \circ H_n$ and $K = K_1 \circ \cdots \circ K_n$ are internal direct products of strongly irreducible torsion-free groups. Then, if K is contained in H with finite index, there exists a permutation $\sigma \in S_n$ such that H_j contains $K_{\sigma(j)}$ with finite index for all $j \in \{1, \ldots n\}$.*

Proof. Let $\pi_i : H \to H_i$ be the natural projection onto the i^{th} factor. Then $\pi_i(K) = \pi_i(K_1) \cdots \pi_i(K_n)$ has finite index in H_i. As $\pi_i(K_1), \ldots, \pi_i(K_n)$ are mutually centralizing subgroups of the strongly irreducible group H_i, there exists $\sigma(i) \in \{1, \ldots, n\}$ such that $\pi_i(K) = \pi_i(K_{\sigma(i)})$ and $\pi_i(K_j) = \{1\}$ for $j \neq \sigma(i)$.

Now, suppose that j is fixed and that $K_j \neq K_{\sigma(i)}$ for $i = 1, \ldots, n$. Then $\pi_i(K_j) = \{1\}$ for all i so that $K_j = \{1\}$, which is a contradiction, since K_j is assumed to be infinite. The map

$$\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}; i \mapsto \sigma(i),$$

is therefore onto and hence bijective. So, $K_{\sigma(i)} \neq K_{\sigma(j)}$ for $i \neq j$. In particular, $\pi_j(K_{\sigma(i)}) = \{1\}$ whenever $i \neq j$, which implies that $K_{\sigma(i)} \subset H_i$. Thus $\pi_i(K) = \pi_i(K_{\sigma(i)}) = K_{\sigma(i)}$, which shows that $K_{\sigma(i)}$ has finite index in H_i. $\square$

proof of Theorem 5.1.2. Let $(\Gamma_1, \dots, \Gamma_n)$ be a sequence of strongly irreducible torsion-free groups with the SFC property. Set $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. For each $i \in \{1, \dots, n\}$ do the following: if for some $j \neq i, \Gamma_i$ is commensurable with, but not isomorphic to Γ_j, replace Γ_i by Γ_j in the sequence. Now relabel the sequence as $(\Gamma'_1, \dots, \Gamma'_n)$. Then $\Gamma' = \Gamma'_1 \times \cdots \times \Gamma'_n \sim \Gamma$, and, as Γ is SFC if and only if Γ' has the SFC property by Corollary 4.1.5, we may assume that $\Gamma = \Gamma'$. i.e. if Γ_i is commensurable with Γ_j for some j then $\Gamma_i \cong \Gamma_j$.

For each $i \in \{1, \dots, n\}$, identify Γ_i with its image in Γ under the canonical embedding $\Gamma_i \hookrightarrow \Gamma_1 \times \cdots \times \Gamma_n$. Then Γ is the internal direct product $\Gamma_1 \circ \cdots \circ \Gamma_n$. Let A and B be isomorphic subgroups of finite index in Γ and, for $i \in \{1, \dots, n\}$, set $A_i = \Gamma_i \cap A$, $B_i = \Gamma_i \cap B$. Then $A' = A_1 \circ \cdots \circ A_n$ and $B' = B_1 \circ \cdots \circ B_n$ have finite index in A and B respectively.

Pick an isomorphism $\varphi : A \to B$. Then $\varphi(A') = \varphi(A_1) \circ \cdots \circ \varphi(A_n)$ is a product of strongly irreducible torsion-free groups contained in Γ with finite index. By the lemma above, there exists a permutation $\sigma \in S_n$ such that $\varphi(A_i) \subset \Gamma_{\sigma(i)}$ with finite index for all i. But this means that $\Gamma_i \sim \varphi(A_i) \sim \Gamma_{\sigma(i)}$ for $i = 1, \dots, n$. By our assumption, there must therefore exist isomorphisms $\Gamma_i \cong \Gamma_{\sigma(i)}$ for all i. As Γ_i and therefore A_i is SFC, duality in the SFC property (Lemma 4.1.3) now implies that $[\Gamma_i : A_i] = [\Gamma_{\sigma(i)} : \varphi(A_i)]$. However, $\varphi(A_i) \subset \Gamma_{\sigma(i)} \cap B = B_{\sigma(i)}$. So, as φ is an isomorphism and σ is bijective, $\varphi(A_i) = B_{\sigma(i)}$. Thus $\varphi(A') = B'$, $[A : A'] = [B : B']$ and $[\Gamma_i : A_i] = [\Gamma_{\sigma(i)} : B_{\sigma(i)}]$ for all i. As σ is a permutation, this shows that $[\Gamma : A'] = [\Gamma : B']$. Hence

$$[\Gamma : A] = [\Gamma : A']/[A : A'] = [\Gamma : B']/[B : B'] = [\Gamma : B].$$

□

5.2 The SFC property of lattices in L

The hypothesis of Theorem 4.2.5 can be extended using Theorem 5.1.2 above to include all lattices in L, and not just those which are irreducible.

Theorem 5.2.1. *If Γ is an irreducible lattice in a connected semi-simple Lie group with finite centre then Γ is strongly irreducible.*

Proof. Suppose that Γ embeds as a subgroup of finite index in a group $H = H_1H_2$, where H_1 and H_2 are mutually centralizing infinite subgroups of H. Then, as $H_j/(\Gamma \cap H_j)$ injects into the finite set H/Γ, $\Gamma \cap H_j$ has finite index in H_j for $j = 1, 2$. Pick $h \in H$. Then $h = h_1h_2$, where $h_1 \in H_1$ and $h_2 \in H_2$. Moreover,

$$h_1h_2(\Gamma \cap H_1)(\Gamma \cap H_2) = h_1(\Gamma \cap H_1)\, h_2(\Gamma \cap H_2),$$

so that $[H : (\Gamma \cap H_1)(\Gamma \cap H_2)] \leq [H_1 : \Gamma \cap H_1][H_2 : \Gamma \cap H_2]$. Thus $(\Gamma \cap H_1)(\Gamma \cap H_2)$ is contained in Γ with finite index. Consequently, it is sufficient to show that Γ itself cannot be decomposed as the product of two mutually centralizing infinite subgroups.

Now, as strong irreducibility is a property of the commensurability class, we may further assume that Γ is torsion-free, so that Γ embeds as a lattice in a connected adjoint semi-simple Lie group G, without compact factors (Proposition 3.1.7).

Identify G with its image in $GL(n, \mathbb{C})$ under the adjoint representation and suppose that $\Gamma = \Gamma_1\Gamma_2$ where Γ_1 and Γ_2 are infinite mutually centralizing subgroups of Γ. Pick $\gamma \in \Gamma$. The mapping $x \mapsto \gamma x \gamma^{-1}$ is polynomial with polynomial inverse and therefore a homeomorphism of G onto itself in the Zariski topology. So, for $j = 1, 2$,

$$\overline{\Gamma_j} = \overline{\gamma\Gamma_j\gamma^{-1}} = \gamma\overline{\Gamma_j}\gamma^{-1},$$

which shows that Γ normalizes both $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$.

Γ is Zariski dense in G, so that $\overline{\Gamma} = \overline{G}$ and by Lemma 2.1.1, $\overline{G}$ normalizes $\overline{\Gamma_j}$ for $j = 1, 2$. However, $G = (\overline{G})^{\circ}_{\mathbb{R}} \subset GL(n, \mathbb{R})$. Thus $(\overline{\Gamma_j})_{\mathbb{R}}$ is normal in G and $(\overline{\Gamma_j})^{\circ}_{\mathbb{R}}$, being characteristic in $(\overline{\Gamma_j})_{\mathbb{R}}$, is a closed connected normal subgroup of G.

Since Γ_1 and Γ_2 are mutually centralizing, it also follows from Lemma 2.1.1 that $(\overline{\Gamma_1})^{\circ}_{\mathbb{R}}$ and $(\overline{\Gamma_2})^{\circ}_{\mathbb{R}}$ centralize eachother, so that $(\overline{\Gamma_1})^{\circ}_{\mathbb{R}} \cap (\overline{\Gamma_2})^{\circ}_{\mathbb{R}}$ is abelian. As the simple factors of G are all non-abelian, this shows $(\overline{\Gamma_1})^{\circ}_{\mathbb{R}} \cap (\overline{\Gamma_2})^{\circ}_{\mathbb{R}} = \{1\}$. Now, for $j = 1, 2$, $(\overline{\Gamma_j})_{\mathbb{R}}$ is a real algebraic group, and as such has only finitely many connected components. Consequently $\Delta_j = (\overline{\Gamma_j})^{\circ}_{\mathbb{R}} \cap \Gamma_j$ is a subgroup of finite index Γ_j for $j = 1, 2$. Moreover, $\Delta_1 \cap \Delta_2 \subset (\overline{\Gamma_1})^{\circ}_{\mathbb{R}} \cap (\overline{\Gamma_2})^{\circ}_{\mathbb{R}} = \{1\}$, so that $\Delta_1 \cdot \Delta_2 \cong \Delta_1 \times \Delta_2$. As $x_1 x_2 \Delta_1 \cdot \Delta_2 = x_1 \Delta_1 \cdot x_2 \Delta_2$ for all $x_1 \in \Gamma_1$, $x_2 \in \Gamma_2$, $\Delta_1 \cdot \Delta_2$ has finite index in Γ. This contradicts the irreducibility of Γ. Hence the result. $\square$

Theorem 5.2.2. *If $\Gamma \in L$ then Γ has the SFC property.*

Proof. As we have seen, Γ admits a torsion-free subgroup of finite index, say Γ', that embeds as a lattice in an connected adjoint semi-simple Lie group. The irreducible factors of Γ' are also lattices in adjoint semi-simple Lie groups and by the above theorem, strongly irreducible. Thus Γ', and therefore Γ, is commensurable with a direct product of finitely many strongly irreducible torsion-free SFC groups. This product has the SFC property by Theorem 5.1.2. As the SFC property is a property of the commensurability class, this shows that Γ is SFC. $\square$

5.3 SFC group extensions with non-SFC factors

The aim of this section of to show that a group G arising as an extension

$$1 \to K \to G \to Q \to 1,$$

can possess the SFC property even when K or Q fail to do so. This will be demonstrated by two examples: non-trivial extensions of $\mathbb{Z}$ by oriented surface groups of genus > 1, and Stallings fibrations constructed via pseudo-Anosov diffeomorphisms.

5.3.1 The Stallings fibration

Let X be a compact orientable surface without boundary and φ a diffeomorphism of X. The quotient space $E(\varphi) = X \times I/\sim$ is then a smooth closed 3-manifold, where $\sim$ identifies $(x, 0)$ with $(\varphi(x), 1)$ for all $x \in X$. Let $[x, t]$ denote the equivalence class of (x, t). The map $[x, t] \mapsto t$ then defines a fibration $E(\varphi) \to S^1$ with fibre X, called a Stallings fibration, after J.R.Stallings, who proved the following theorem in [33]:

Theorem 5.3.1. *Let M be a compact irreducible orientable 3-manifold without boundary for which there exists a short exact sequence of groups*

$$1 \to K \to \pi_1(M) \to \mathbb{Z} \to 0,$$

in which K is finitely generated. Then M is a locally trivial fibre bundle over S^1 whose fibre is a closed orientable surface X such that $\pi_1(X) \cong K$. Moreover, the short exact sequence $1 \to \pi_1(X) \to \pi_1(M) \to \pi_1(S^1) \to 0$ arising from the long homotopy exact sequence of M is isomorphic to the extension above.

Now set $\Sigma = \pi_1(X)$ and let

$$1 \to \Sigma \to \Gamma \to \mathbb{Z} \to 0$$

be a short exact sequence. It follows easily from the Lyndon-Hochschild-Serre spectral sequence for Γ that $\operatorname{rank}(H^2(\Gamma;\mathbb{Z})) = 2g-1$, where g is the genus of Σ. Since Σ is determined upto isomorphism by its genus, this shows that the isomorphism type of Σ is determined by Γ.

The following proposition shows that when $g > 1$, Σ is characteristic in Γ.

Proposition 5.3.2. *If $1 \to \Sigma \to \Gamma_j \to \mathbb{Z} \to 0$ is an extension for $j = 1, 2$, then any isomorphism $\theta : \Gamma_1 \to \Gamma_2$ induces a commutative diagram,*

$$\begin{array}{ccccccc} 1 \to & \Sigma & \to & \Gamma_1 & \to & \mathbb{Z} & \to 0 \\ & \theta' \downarrow & & \downarrow \theta & & \downarrow \theta'' & \\ 1 \to & \Sigma & \to & \Gamma_2 & \to & \mathbb{Z} & \to 0 \end{array}$$

Proof. We must show that $\theta(\Sigma) = \Sigma$. Set $\Gamma = \Gamma_2$, $\Sigma_1 = \Sigma$ and $\Sigma_2 = \theta(\Sigma)$. Suppose that $\Sigma_1 \neq \Sigma_2$. Then, as Σ_j is a normal subgroup of Γ for $j = 1, 2$, $H = \Sigma_1 \cdot \Sigma_2$ is also a normal subgroup in Γ. Suppose that $H \subset \Gamma$ with finite index. Then, if p_j is the projection $\Gamma \to \Gamma/\Sigma_j$, $p_j(H) \cong \mathbb{Z}$ for $j = 1, 2$ and there exists a short exact sequence

$$1 \to \Sigma_2 \to H \xrightarrow{p_2} \mathbb{Z} \to 0.$$

This means that there exists $\sigma = \sigma_1\sigma_2 \in H$ such that σ_1 is not contained in $\Sigma_1 \cap \Sigma_2$ and $p_2(\sigma)$ has infinite order. As $p_2(\sigma) = p_2(\sigma_1\sigma_2) = p_2(\sigma_1)$, this means $p_2(\sigma_1)$ has infinite order also. So, there must exist a short exact sequence

$$1 \to \Sigma_1 \cap \Sigma_2 \to \Sigma_1 \to A \to 0.$$

in which A is finitely generated and abelian ($[\Sigma_1, \Sigma_1] \subset [\Gamma, \Gamma] \subset \Sigma_1 \cap \Sigma_2$, so that $\Sigma_1/\Sigma_1 \cap \Sigma_2)$ is abelian). But this implies that $2 - 2g = \chi(\Sigma_1) = 0$, a contraction.

This shows that H is contained in Γ with infinite index. But then, for $j = 1, 2$,

$$\Gamma/H \cong (\Gamma/\Sigma_j)/(H/\Sigma_j) \cong \mathbb{Z}/(H/\Sigma_j)$$

Thus H/Σ_j is finite for $j = 1, 2$. So, $\Sigma_1 \cap \Sigma_2 \subset H$ with finite index. So $\Gamma/\Sigma_1 \cap \Sigma_2 \cong \mathbb{Z}$ and there exists a short exact sequence

$$1 \to \Sigma_1 \cap \Sigma_2 \to \Gamma \to \mathbb{Z} \to 0.$$

But the genus of $\Sigma_1 \cap \Sigma_2$ is determined by Γ and so $\chi(\Sigma_1 \cap \Sigma_2) = \chi(\Sigma_1)$, which is impossible unless $\Sigma_1 = \Sigma_2$. $\square$

It follows from the above proposition that if $\theta : \Gamma_1 \to \Gamma_2$ is an isomorphism, then there exist sections $s_j : \mathbb{Z} \to \Gamma_j$ such that

$$\theta s_1(1) = s_2(\pm 1).$$

So, if φ_j is the operator homomorphism of the extension

$$1 \to \Sigma \to \Gamma_j \to \mathbb{Z} \to 0,$$

then $[\theta'] \cdot \varphi_1(1) \cdot [\theta']^{-1} = \varphi_2(\pm 1)$, where θ' is the restriction of θ to Σ. This shows that, up to isomorphism, Γ is uniquely determined by the couple $(g, C(\varphi))$, where $C(\varphi)$ is the set of elements in $\mathrm{Out}(\Sigma)$ conjugate to either $\varphi(1)$ or $\varphi(-1)$. We shall write $\Gamma = \Gamma(g, C(\varphi))$.

Now, if $E(\varphi)$ is an orientated Stallings fibration, then the image of 1 under the canonical homomorphism $\alpha : \pi_1(S^1) \to \mathrm{Out}(\pi_1(\Sigma))$ induced by the outer action of $\pi_1(S^1)$ on the fibre Σ of $E(\varphi)$, is just $[\varphi]$, where $[f]$ denotes the element determined by a diffeomorphism f under the canonical homomorphism $\pi_0(Diff(\Sigma)) \to \mathrm{Out}(\pi_1(\Sigma))$. However, α is the operator homomorphism of the extension

$$1 \to \pi_1(\Sigma) \to \pi_1(E(\varphi)) \to \pi_1(S^1) \to 0$$

arising from the long exact homotopy sequence of $E(\varphi)$. Thus $\pi_1(E(\varphi)) \cong \Gamma(g, C[\varphi])$, so that, if $E(\varphi_1)$ and $E(\varphi_1)$ are Stallings fibrations constructed from the same surface X, then $\pi_1(E(\varphi_1)) \cong \pi_1(E(\varphi_1))$ if and only if $C[\varphi_1] = C[\varphi_2]$.

For a compact orientable surface X without boundary and having genus $>$ 1, F.Waldhausen [40] has shown that $\pi_0(Diff(X)) \cong \mathrm{Out}(\pi_1(X))$, so in fact $[\varphi_1] = [\varphi_2]$ if and only if φ_1 is isotypic to φ_2. According to W.P.Thurston [39], a diffeomorphism $f \in Diff(\Sigma)$ corresponds to an element of infinite order in $\mathrm{Out}(\pi_1(X))$ if and only if is isotopic to a pseudo-Anosov diffeomorphism. As $\mathrm{Out}(\pi_1(X))$ is known to have non-zero virtual cohomological dimension (see [13]), diffeomorphisms of this type certainly exist. The following theorem, also due to W.P.Thurston [25], shows that if φ is such a diffeomorphism, then $\pi_1(E(\varphi))$ is a lattice in L:

Theorem 5.3.3. *Let X be a closed orientable surface of genus > 1 and φ a diffeomorphism of X. Then, the Stallings fibration $E(\varphi)$ admits a complete hyperbolic metric if and only if φ is isotopic to a pseudo-Anosov diffeomorphism. That is, there exists a Riemannian metric on $E(\varphi)$ with respect to which it is complete Riemannian manifold of constant negative curvature.*

Proof. See [25, Theorem 3.9]. □

Corollary 5.3.4. *If φ is isotopic to a pseudo-Anosov diffeomorphism, then $\pi_1(E(\varphi))$ is isomorphic to a cocompact lattice in the simple Lie group $O(3,1)$.*

The corollary is an entirely standard deduction from the theorem above. However, the proof is often omitted from texts on group theory and has therefore been included below.

Proof. Let $\pi : E \to E(\varphi)$ be the universal covering. Then, in the Riemannian structure induced by π, E is also a complete Riemannian manifold of constant negative curvature (see [20, pages 176 and 202]). Moreover, with respect to this structure, π is a local isometry and so the canonical action of $\pi_1(E(\varphi))$ on E by covering transformations is isometric. $\Gamma = \pi_1(E(\varphi))$ therefore embeds as a discrete subgroup of $I(E)$, the isometry group of E.

It is a classical theorem that a complete simply connected Riemannian manifold M of constant curvature k and is isometric to the space form $L(k)$ [20, Chapter VI, Theorem 7.10]. That is, the hypersurface

$$\{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_n^2 + rt^2 = r\},$$

where $r = 1/k$ and n is the dimension of M. The metric on $L(k)$ is given by restriction of the form $dx_1^2 + \cdots + dx_n^2 + rdt^2$ and its isometry group $I(k)$ consists of those linear transformations of $\mathbb{R}^{n+1}$ that leave the quadratic form $x_1^2 + \cdots + x_n^2 + rt^2$ invariant (see [44, Theorem 2.4.4]).

Now, $I(k) \cong O(n,1)$ whenever $k < 0$, so that $I(E) \cong O(3,1)$. Moreover, $I(r)$ acts transitively on $L(k)$ with compact stabilizers [20, Chapter VI, Theorem 3.4] so that $E \cong O(3,1)/K$ for some compact subgroup of K of $O(3,1)$. Thus $E(\varphi) \cong \Gamma\backslash O(3,1)/K$. As $\Gamma\backslash O(3,1)$ is a fibre bundle over $\Gamma\backslash O(3,1)/K$ with fibre K, this shows that Γ is cocompact in $O(3,1)$. Hence the result. $\square$

As $O(3,1)$ is simple, this shows that $\pi_1(E(\varphi))$ has the strong finite cohopfian property whenever φ is a isotopic to a pseudo-Anosov diffeomorphism. However, $\pi_1(E(\varphi))$ is given as an extension,

$$1 \to \pi_1(\Sigma) \to \pi_1(E(\varphi)) \to \mathbb{Z} \to 0$$

and $\mathbb{Z}$ is certainly not SFC.

5.3.2 The groups $\Gamma(g, c)$

Let Γ be a group arising as a central extension of $\mathbb{Z}$ by the fundamental group Σ^g of a closed orientable surface with genus $g > 1$, so that there exists a short exact sequence

$$0 \to \mathbb{Z} \to \Gamma \to \Sigma^g \to 1. \tag{5.3.1}$$

As $Z(\Sigma^g) = \{1\}$, $\Sigma^g \cong \Gamma/Z(\Gamma)$ depends only on the isomorphism type of Γ, so that g is an isomorphism invariant of Γ.

Now, the extension 5.3.1 is classified by a class $c \in H^2(\Sigma^g, \mathbb{Z}) \cong \mathbb{Z}$. Since $\mathrm{Aut}(\mathbb{Z}) \cong \{\pm 1\}$, up to sign, c depends only on the isomorphism type of Γ. So, for fixed g, c determines a unique group $\Gamma = \Gamma(g, c)$ upto isomorphism for each $c \geq 0$. We will show that when $c > 0$, $\Gamma(g, c)$ is SFC, thereby proving that an extension of a non-SFC group by one with the strong finite cohopfian property can still be SFC. This example also shows that SFC groups can have non-trivial centres.

Let Δ be a subgroup of finite index in Γ. Then $\Delta \cap Z(\Gamma)$ has finite index in $Z(\Gamma)$. Consequently $\Delta \cap Z(\Gamma) \cong \mathbb{Z}$. Moreover, the inclusion $\varphi : \Delta \to \Gamma$ induces an injection $\Delta/\Delta \cap Z(\Gamma) \hookrightarrow \Sigma^g$, so that Δ is given as an extension

$$1 \to \mathbb{Z} \to \Delta \to \Sigma^h \to 1 \tag{5.3.2}$$

for some h. This extension is clearly central, so that $\Delta \cong \Gamma(h, d)$, where $d \in H^2(\Sigma^h; \mathbb{Z})$ classifies 5.3.2. Moreover, there exists a commutative diagram

$$\begin{array}{ccccccc}
0 \to & \mathbb{Z} & \to & \Delta & \to & \Sigma^h & \to 1 \\
& \varphi' \downarrow & & \downarrow \varphi & & \downarrow \varphi'' & \\
0 \to & \mathbb{Z} & \to & \Gamma & \to & \Sigma^g & \to 1
\end{array}$$

in which all of the downward arrows are inclusions.

Theorem 5.3.5. *If $c > 0$, then $\Gamma(g, c)$ has the SFC property .*

Proof. Since surface groups of genus > 1 have the SFC property and are determined up to isomorphism by their genus, the index $[\Sigma^g : \varphi''(\Sigma^h)]$ depends only on h (g being given). We shall show that the index $[Z(\Gamma) : Z(\Delta)]$ is determined by the value of d. Since $[\Gamma; \Delta] = [\Sigma^g : \Sigma^h][Z(\Gamma) : Z(\Delta)]$, this will prove the theorem.

Let $s : \Sigma^g \to \Gamma$ be a section and

$$c_s : \Sigma^g \times \Sigma^g \to \mathbb{Z}; (g_1, g_2) \mapsto s(g_1)s(g_2)s(g_1g_2)^{-1}$$

the corresponding cocycle (which represents c). By commutativity, $s(\varphi''(x)) \subset \mathrm{Im}\varphi$ for all $x \in \Sigma^h$. In addition, φ is injective, so that s defines a section $t : \Sigma^h \to \Delta$ such that

$$\varphi' t = s \circ \varphi''.$$

Let d_t be the corresponding cocycle (which represents d). Then,

$$\varphi' d_t = c_s \circ (\varphi'' \times \varphi'').$$

However, $[c_s \circ (\varphi'' \times \varphi'')] = (\varphi'')^* c$, where $(\varphi'')^*$ is the induced map $H^2(\Sigma^g; \mathbb{Z}) \to H^2(\Sigma^h; \mathbb{Z})$. But this is just the map defined by $1 \mapsto [\Sigma^g : \Sigma^h]$. Similarly, $[\varphi' d_t] = (\varphi')_* d$, where $(\varphi')_*$ is defined by $1 \mapsto [Z(\Gamma) : Z(\Delta)]$. Thus

$$[Z(\Gamma) : Z(\Delta)]d = [\Sigma^g : \Sigma^h]c,$$

so that $[Z(\Gamma) : Z(\Delta)]$ depends only on the isomorphism type of Δ. $\square$

Chapter 6

E-invariants and the theory of central extensions

Let G be a group with centre Z, and suppose that both Z and $H^2(G/Z;\mathbb{Z})$ are finitely generated. Then, modulo torsion, the characteristic class of the extension

$$0 \to Z \to G \to G/Z \to 1$$

can be regarded as a matrix with entries in $\mathbb{Z}$. The rank of this matrix turns out to be an isomorphism invariant of G, and closely related to the SFC property. Before any theorems can be proved however, some definitions are required.

Let $M_{mn}(\mathbb{Q})$ be the group of $m \times n$ matrices with rational entries and $e_{ij} = \delta_{i\,k}\delta_{j\,l}$ the canonical basis of $M_{mn}(\mathbb{Q})$. For $x \in M_{mn}(\mathbb{Q})$, define the numbers x_{ij} by $x = x_{ij}e_{ij}$. Set $c_i = \{e_{1i}, \ldots, e_{mi}\}$, and take $\mathbb{Q}c_i$ to be the subgroup of $M_{mn}(\mathbb{Q})$ generated by the elements of c_i, so that $M_{mn}(\mathbb{Q})$ is the internal direct sum

$$\mathbb{Q}c_1 \dotplus \cdots \dotplus \mathbb{Q}c_n \cong \mathbb{Q}^m \oplus \cdots \oplus \mathbb{Q}^m = (\mathbb{Q}^m)^n.$$

Similarly, let $r_i = \{e_{i1}, \ldots, e_{in}\}$, and take $\mathbb{Q}r_i$ to be the subgroup of $M_{mn}(\mathbb{Q})$

generated by the elements of r_i, so that

$$M_{mn}(\mathbb{Q}) = \mathbb{Q}r_1 \dotplus \cdots \dotplus \mathbb{Q}r_m \cong \mathbb{Q}^n \oplus \cdots \oplus \mathbb{Q}^n = (\mathbb{Q}^n)^m.$$

Definition 6.0.6. A group homomorphism $t : M_{mn}(\mathbb{Q}) \to M_{mn}(\mathbb{Q})$ will be said to have type L if $t : X \mapsto PX$ for some $m \times m$ matrix P and type R if $t : X \mapsto XQ$ for some $n \times n$ matrix Q.

If t is given by $X \mapsto PX$, and X_i are the columns of the matrix $X \in M_{mn}(\mathbb{Q})$, so that $X = (X_1, \ldots, X_n)$, it is elementary to see that $PX = (PX_1, \ldots, PX_n)$. t therefore corresponds to the transformation $e_{ji} \mapsto P_{kj}e_{ki}$ applied to each of the subgroups $\mathbb{Q}c_i$. Similarly, if t is given by the map $X \mapsto XQ$, then t corresponds to the transformation $e_{ij} \mapsto e_{ik}Q_{jk}$ applied to each of the subgroups $\mathbb{Q}r_i$.

6.1 Matrix invariants and central extensions of groups

Let Σ be a group for which $H^2(\Sigma;\mathbb{Z})$ is finitely generated, so that there exists an isomorphism $j : H^2(\Sigma;\mathbb{Q}) \to \mathbb{Q}^m$ for some non-negative integer m. Then, given a finitely generated abelian group A and an isomorphism $i : A \otimes \mathbb{Q} \cong \mathbb{Q}^n$, define $\theta(i,j)$ to be the composition

$$H^2(\Sigma; A \otimes \mathbb{Q}) \xrightarrow{i_*} H^2(\Sigma;\mathbb{Q}^n) \xrightarrow{\pi} H^2(\Sigma;\mathbb{Q})^n \xrightarrow{j\oplus\cdots\oplus j} (\mathbb{Q}^m)^n \cong M_{mn}(\mathbb{Q}),$$

where $\pi : H^2(\Sigma;\mathbb{Q}^n) \cong H^2(\Sigma;\mathbb{Q})^n$ is the natural isomorphism, and i_* is the induced map $i_* : [f] \mapsto [i(f)]$.

Definition 6.1.1. An isomorphism $H^2(\Sigma; A \otimes \mathbb{Q}) \to M_{mn}(\mathbb{Q})$ will be called admissible when it is equal to $\theta(i,j)$ for some pair of isomorphisms $i : A \otimes \mathbb{Q} \to \mathbb{Q}^n$ and $j : H^2(\Sigma;\mathbb{Q}) \to \mathbb{Q}^m$.

Theorem 6.1.2. *Let G be a group, Z its centre and c the characteristic class the canonical extension*

$$0 \to Z \to G \to G/Z \to 1.$$

Suppose that Z and $H^2(G/Z;\mathbb{Z})$ are finitely generated and let

$$\theta : H^2(G/Z; Z \otimes \mathbb{Q}) \to M_{mn}(\mathbb{Q})$$

be an admissible isomorphism. Then, if $c_{\mathbb{Q}}$ denotes the image of c under the natural map $H^2(G/Z;Z) \to H^2(G/Z;Z\otimes\mathbb{Q})$, rank$(\theta(c_{\mathbb{Q}}))$ is independent of the choice of isomorphism θ and therefore an isomorphism invariant of G. Similarly, when rank$(Z) =$ rank$(H^2(G/Z;\mathbb{Z}))$, so that $\theta(c_{\mathbb{Q}})$ is square, $|det(\theta(c_{\mathbb{Q}}))|$ is an isomorphism invariant of G.

When the hypotheses of the above theorem hold, $\text{rank}(G) = \text{rank}(\theta(c_{\mathbb{Q}}))$ will be called the rank of G and $\det(G) = |\det(\theta(c_{\mathbb{Q}}))|$ its determinant. The proof of their existence is based upon the following:

For $j = 1, 2$, let $0 \to A_j \to G_j \to \Sigma_j \to 1$, be a central extension in which A_j and $H^2(\Sigma_j;\mathbb{Z})$ are finitely generated. Let $\varphi : G_1 \to G_2$ be a homomorphism and suppose that $\varphi(A_1) \subset A_2$, so that φ induces maps $\varphi'' : \Sigma_1 \to \Sigma_2$ and $\varphi' : A_1 \to A_2$. Now define

$$\varphi_* : H^2(\Sigma_1 : A_1 \otimes \mathbb{Q}) \to H^2(\Sigma_1 : A_2 \otimes \mathbb{Q})$$

$$\varphi^* : H^2(\Sigma_2 : A_2 \otimes \mathbb{Q}) \to H^2(\Sigma_1 : A_2 \otimes \mathbb{Q})$$

to be the natural maps induced by φ' and φ'' respectively.

Lemma 6.1.3. *Suppose $A_1 \otimes \mathbb{Q} \cong A_2 \otimes \mathbb{Q}$ and φ'' is an isomorphism. Then φ^* and φ_* induce group homomorphisms $M_{mn}(\mathbb{Q}) \to M_{mn}(\mathbb{Q})$ defined over $\mathbb{Z}$ with respect to all admissible isomorphisms, where $n =$ rank(A_j) and $m =$*

rank$(H^2(\Sigma_j;\mathbb{Z}))$ *for* $j = 1,2$. *Moreover, the maps determined by* φ^* *and* φ_* *have type type* L *and* R *respectively.*

Proof. For $r = 1,2$, choose isomorphisms $i^r : A_r \otimes \mathbb{Q} \to \mathbb{Q}^n$ and $j_r : H^2(\Sigma_r;\mathbb{Q}) \to \mathbb{Q}^m$. Then φ^* induces a homomorphism of $M_{mn}(\mathbb{Q})$ defined by the commutative diagram:

$$\begin{array}{ccccccc}
H^2(\Sigma_2;A_2\otimes\mathbb{Q}) & \stackrel{(i^2)_*}{\to} & H^2(\Sigma_2;\mathbb{Q}^n) & \to & H^2(\Sigma_2;\mathbb{Q})^n & \stackrel{j_2\oplus\cdots\oplus j_2}{\to} & (\mathbb{Q}^m)^n \cong M_{mn}(\mathbb{Q}) \\
\varphi^*\downarrow & & (\varphi'')^*\downarrow & & [(\varphi'')^*]^n\downarrow & & \downarrow \\
H^2(\Sigma_1;A_2\otimes\mathbb{Q}) & \stackrel{(i^1)_*}{\to} & H^2(\Sigma_1;\mathbb{Q}^n) & \to & H^2(\Sigma_1;\mathbb{Q})^n & \stackrel{j_1\oplus\cdots\oplus j_1}{\to} & (\mathbb{Q}^m)^n \cong M_{mn}(\mathbb{Q})
\end{array}$$

The map $M_{mn}(\mathbb{Q}) \to M_{mn}(\mathbb{Q})$ is clearly defined by a linear map $x \mapsto Px$ applied to each of the subspaces $\mathbb{Q}c_i$ and therefore has type L. Moreover, as it is induced by the isomorsphism $H^2(\Sigma_2;A_2) \to H^2(\Sigma_1;A_2); [f] \to [f\circ(\varphi''\times\varphi'')]$ via the map $H^2(\Sigma_2;A_2)\otimes\mathbb{Q} \cong H^2(\Sigma_2;A_2\otimes\mathbb{Q})$, φ^* is also defined over $\mathbb{Z}$.

The automorphism of $M_{mn}(\mathbb{Q})$ induced by φ_* with respect to $\theta(i,j)$ is defined by the commutative diagram:

$$\begin{array}{ccccccc}
H^2(\Sigma_1;A_1\otimes\mathbb{Q}) & \to & H^2(\Sigma_1;\mathbb{Q}^n) & \to & (\mathbb{Q}^n)^m \cong M_{nm}(\mathbb{Q}) & \stackrel{T}{\to} & M_{mn}(\mathbb{Q}) \\
\varphi_*\downarrow & & \alpha_*\downarrow & & \downarrow & & \downarrow \\
H^2(\Sigma_1;A_2\otimes\mathbb{Q}) & \to & H^2(\Sigma_1;\mathbb{Q}^n) & \to & (\mathbb{Q}^n)^m \cong M_{nm}(\mathbb{Q}) & \stackrel{T}{\to} & M_{mn}(\mathbb{Q})
\end{array}$$

It is clear from the diagram that the map $M_{nm}(\mathbb{Q}) \to M_{nm}(\mathbb{Q})$ is given by a linear map $x \mapsto Q^Tx$ applied to each of the spaces $\mathbb{Q}(r_i)$, so that the homomorphism $M_{mn}(\mathbb{Q}) \to M_{mn}(\mathbb{Q})$ induced by φ_* is given by right matrix multiplication $X \to XQ$. As φ_* is induced by the homomorphism $H^2(\Sigma_1;A_1) \to H^2(\Sigma_1;A_2); [x] \mapsto [\varphi'(f)]$, the map $M_{mn}(\mathbb{Q}) \to M_{mn}(\mathbb{Q})$ determined by φ_* is also defined over $\mathbb{Z}$. □

Proof of theorem 6.1.2

For $j = 1, 2$, let G_j be group such that $H^2(G_j; \mathbb{Z})$ and $Z_j = Z(G_j)$ are finitely generated. Suppose that $\varphi : G_1 \to G_2$ is an isomorphism. Then there exists a commutative diagram:

$$
\begin{array}{llllllllll}
\varepsilon_1) & 0 \to & Z_1 & \to & G_1 & \to & G_1/Z_1 & \to & 1 \\
 & & \varphi' \downarrow & & \varphi \downarrow & & \varphi'' \downarrow & & \\
\varepsilon_2) & 0 \to & Z_2 & \to & G_2 & \to & G_2/Z_2 & \to & 1
\end{array}
$$

in which all of the downward arrows are isomorphisms. If c_j is the characteristic class of ε_j for $j = 1, 2$, then by naturality,

$$\varphi^*((c_2)_{\mathbb{Q}}) = \varphi_*((c_1)_{\mathbb{Q}}).$$

But φ^* and φ_* induce isomorphisms of type L and R respectively. So, given admissible isomorphisms $\theta_j : H^2(G_j/Z_j; Z_j \otimes \mathbb{Q}) \to M_{mn}(\mathbb{Q})$ $(j = 1, 2)$, there must exist invertible matrices P and Q such that

$$P\theta_1(c_1)_{\mathbb{Q}} = \theta_2(c_2)_{\mathbb{Q}}Q.$$

Thus $\mathrm{rank}(\theta_1(c_1)_{\mathbb{Q}}) = \mathrm{rank}(\theta_1(c_2)_{\mathbb{Q}})$. Similarly, when $m = n$, we may take determinants to obtain,

$$det(P)det(\theta_1(c_1)_{\mathbb{Q}}) = det(\theta_2(c_2)_{\mathbb{Q}})det(Q).$$

However, as the morphisms induced by φ^* and φ_* are defined over $\mathbb{Z}$, P and Q lie in $GL_n(\mathbb{Z})$. Consequently $|det(P)| = |det(Q)| = 1$ and $|det(\theta_1((c_1)_{\mathbb{Q}}))| = |det(\theta_2((c_1)_{\mathbb{Q}}))|$.

□

6.2 A criterion for the failure of the SFC property

Theorem 6.2.1. *Let*

$$0 \to A_1 \oplus A_2 \to \Gamma \to \Sigma \to 1$$

be a central extension classified by $c \in H^2(\Sigma; A_1 \oplus A_2) \cong H^2(\Sigma; A_1) \oplus H^2(\Sigma; A_2)$ *and suppose that* A_1 *and* A_2 *are finitely generated with rank*$(A_1) > 0$*. Then, if the projection of* c *onto* $H^2(\Sigma; A_1)$ *is zero,* Γ *is not SFC.*

Proof. Choose a section $s : \Sigma \to \Gamma$, so that

$$f : \Sigma \times \Sigma \to A_1 \oplus A_2; (\sigma_1, \sigma_2) \mapsto s(\sigma_1)s(\sigma_2)s(\sigma_1\sigma_2)^{-1}$$

is a cocycle representing c. Then multiplication in Γ is given by

$$a_1 s(\sigma_1) \cdot a_2 s(\sigma_2) = (a_1 + a_2 + f(\sigma_1, \sigma_2))s(\sigma_1\sigma_2)$$

Let $p_1 : H^2(\Sigma; A_1 \oplus A_2) \to H^2(\Sigma; A_1)$ be the natural projection so that $p_1(c) = 0$. Then, without loss of generality we may assume $Im(f) \subset A_2$. But this means

$$\Gamma_2 = \{as(\sigma) : a \in A_2, \sigma \in \Sigma\}$$

is a subgroup of Γ. Now, if B is subgroup of A_1, then $B\Gamma_2$ is also subgroup in Γ, for if $b, b' \in B$ and $as(\sigma), a'(\sigma') \in \Gamma_2$, then

$$(b + a)s(\sigma)(b' + a')s(\sigma') = (b + b' + [a + a' + f(\sigma, \sigma')])s(\sigma\sigma') \in B\Gamma_2.$$

It is clear moreover, that if $B, B' \leq \Gamma_1$, then any isomorphism $\theta : B \to B'$ extends to an isomorphism from $B\Gamma_2 \to B'\Gamma_2$. But $\text{rank}(A_1) > 1$, and so A_1 admits an isomorphism onto a proper subgroup of finite index, say B, which extends to an isomorphism $\Gamma \cong A_1\Gamma_2 \to B\Gamma_2$. Thus Γ is not SFC. $\square$

In the light of the above, given a central extension, $0 \to Z \to G \to \Sigma \to 1$, it is natural to ask when a direct sum decomposition $\widetilde{A} \dotplus \widetilde{B}$ of $M_{mn}(\mathbb{Q})$ is induced by a decomposition of Z. That is, when do there exist subgroups A and B of Z and an admissible isomorphism $\theta : H^2(\Sigma; Z \otimes \mathbb{Q}) \to M_{mn}(\mathbb{Q})$ such that

1. $Z = A \dotplus B$ and
2. there is a commutative diagram,

$$\begin{tikzcd}
H^2(\Sigma; A \otimes \mathbb{Q} \dotplus B \otimes \mathbb{Q})) \arrow[rr, "\pi_A \oplus \pi_B"] \arrow[dr, "\theta"'] & & H^2(\Sigma; A \otimes \mathbb{Q}) \oplus H^2(\Sigma; B \otimes \mathbb{Q}) \arrow[dl, "\widetilde{\theta}"] \\
& M_{mn}(\mathbb{Q}) &
\end{tikzcd}$$

with $\widetilde{\theta} \circ \pi_A(H^2(\Sigma; A \otimes \mathbb{Q})) = \widetilde{A}$ and $\widetilde{\theta} \circ \pi_B(H^2(\Sigma; B \otimes \mathbb{Q})) = \widetilde{B}$, where π_A and π_B are the natural projections and $\widetilde{\theta}$ is induced by the same underlying isomorphisms as θ.

Lemma 6.2.2. *Let Σ be a group such that $H^2(\Sigma; \mathbb{Q}) \cong \mathbb{Q}^m$ and Z a torsion-free abelian group of rank n, so that $H^2(\Sigma; Z \otimes \mathbb{Q}) \cong M_{mn}(\mathbb{Q})$. Then, a decomposition $M_{mn}(\mathbb{Q}) = \widetilde{A} \dotplus \widetilde{B}$ is induced by a direct sum decomposition of Z if and only if there exists a type R automorphism t defined over $\mathbb{Z}$ such that*

$$t(\mathbb{Q}c_1 \oplus \cdots \oplus \mathbb{Q}c_a) = \widetilde{A}$$

$$t(\mathbb{Q}c_{a+1} \oplus \cdots \oplus \mathbb{Q}c_n) = \widetilde{B},$$

where a is the rank of $\widetilde{A}$. Since the choice of isomorphism $H^2(\Sigma; \mathbb{Q}) \cong \mathbb{Q}^m$ is arbitrary, this shows that the image of a induced decomposition under any type L automorphism defined over $\mathbb{Z}$ is also induced.

Proof. Suppose that the decomposition $M_{mn}(\mathbb{Q}) = \widetilde{A} \dot{+} \widetilde{B}$ is induced from a decomposition $Z = A \dot{+} B$ via $\theta = \theta(i,j)$, so that there exists commutative diagram:

$$\begin{array}{ccc} H^2(\Sigma; A\otimes\mathbb{Q} \dot{+} B\otimes\mathbb{Q})) & \xrightarrow{\pi_A \oplus \pi_B} & H^2(\Sigma; A\otimes\mathbb{Q}) \oplus H^2(\Sigma; B\otimes\mathbb{Q}) \\ \theta \searrow & & \swarrow \widetilde{\theta} \\ & \widetilde{A} \dot{+} \widetilde{B} & \end{array}$$

with $\widetilde{\theta} \circ \pi_A(H^2(\Sigma; A\otimes\mathbb{Q})) = \widetilde{A}$ and $\widetilde{\theta} \circ \pi_B(H^2(\Sigma; B\otimes\mathbb{Q})) = \widetilde{B}$. Choose an isomorphism $\varphi : Z \to Z$ such that $i \circ (\varphi(A)\otimes\mathbb{Q}) = \mathbb{Q}\{e_1, \ldots, e_a\}$ and $i \circ (\varphi(B)\otimes\mathbb{Q}) = \mathbb{Q}\{e_{a+1}, \ldots, e_n\}$, where $\{e_1. \ldots, e_n\}$ is the standard basis of $\mathbb{Q}^n$. Then there exists a commutative diagram,

$$\begin{array}{ccccc} H^2(\Sigma; Z\otimes\mathbb{Q}) & \to & H^2(\Sigma; A\otimes\mathbb{Q}) \oplus H^2(\Sigma; B\otimes\mathbb{Q}) & \xrightarrow{\widetilde{\theta}} & \widetilde{A} \dot{+} \widetilde{B} \\ \varphi_* \downarrow & & \downarrow & & \downarrow t_\varphi \\ H^2(\Sigma; Z\otimes\mathbb{Q}) & \to & H^2(\Sigma; \mathbb{Q}^a) \oplus H^2(\Sigma; \mathbb{Q}^b) & \to & M_{mn}(\mathbb{Q}) \end{array}$$

in which the automorphism t_φ, being induced by φ_*, has type R and is defined over $\mathbb{Z}$. Moreover, $t_\varphi(\widetilde{A}) = \mathbb{Q}c_1 \dot{+} \cdots \dot{+} \mathbb{Q}c_a$ and $t_\varphi(\widetilde{B}) = \mathbb{Q}c_{a+1} \dot{+} \cdots \dot{+} \mathbb{Q}c_n$. Thus, t_φ^{-1}, which is also has defined over $\mathbb{Z}$ is an automorphism of type R with the required properties.

Now suppose that there exists an isomorphism q of type R defined over $\mathbb{Z}$ such that

$$q(\mathbb{Q}c_1 \dot{+} \cdots \dot{+} \mathbb{Q}c_a) = \widetilde{A}$$

$$q(\mathbb{Q}c_{a+1} \dot{+} \cdots \dot{+} \mathbb{Q}c_n) = \widetilde{B}.$$

q is induced by a matrix map $q' : \mathbb{Q}^n \to \mathbb{Q}^n; x \mapsto xQ$, where $Q \in GL(n, \mathbb{Z})$. Set $(\pi_i)_* : [g] \mapsto [\pi_i(g)]$, where π_i is the natural projection. Then, given $[f] \in H^2(\Sigma; \mathbb{Q}^n)$, $[f] = [\pi_i(f)e_i]$ and so, $q'_*[f] = [q'(\pi_i(f)e_i)] = [\pi_i(f)Q_{ij}e_j]$, which shows that q is induced by q'_*. That is, there exists a commutative diagram

$$\begin{array}{ccccc} H^2(\Sigma;\mathbb{Q}^n) & \to & H^2(\Sigma;\mathbb{Q})^n & \stackrel{j\oplus\cdots\oplus j}{\to} & (\mathbb{Q}^m)^n \cong M_{mn}(\mathbb{Q}) \\ \downarrow & & \downarrow & & q\downarrow \\ H^2(\Sigma;\mathbb{Q}^n) & \to & H^2(\Sigma;\mathbb{Q})^n & \stackrel{j\oplus\cdots\oplus j}{\to} & (\mathbb{Q}^m)^n \cong M_{mn}(\mathbb{Q}) \end{array}$$

where $j : H^2(\Sigma;\mathbb{Q}) \to \mathbb{Q}^m$ is an (arbitrary) isomorphism. So, if $A = q'(\mathbb{Q}e_1 + \cdots + \mathbb{Q}e_a)$ and $B = q'(\mathbb{Q}e_{a+1} + \cdots + \mathbb{Q}e_a)$, then the decomposition $M_{mn}(\mathbb{Q}) = \widetilde{A} \oplus \widetilde{B}$ is induced by the decomposition $Z = A \dotplus B$, where $\widetilde{A} = q(\mathbb{Q}c_1 \oplus \cdots \oplus \mathbb{Q}c_a))$ and $\widetilde{B} = q(\mathbb{Q}c_{a+1} \oplus \cdots \oplus \mathbb{Q}c_n))$. Since q' is defined over $\mathbb{Z}$, this shows that the decomposition $M_{mn}(\mathbb{Q}) = \widetilde{A} \oplus \widetilde{B}$ is induced from a decomposition of $\mathbb{Z}^n$ and therefore of Z. □

Theorem 6.2.3. *Let*

$$0 \to Z \to \Gamma \to \Sigma \to 1$$

be a central extension classified by $c \in H^2(\Sigma; Z)$ in which Z and $H^2(\Sigma;\mathbb{Z})$ are finitely generated, $H^2(\Sigma;\mathbb{Z})$ is torison-free and rank$(Z) > 1$. Then Γ fails to have the strong finite cohopfian property whenever rank$(Z) >$ rank$(c_{\mathbb{Q}})$. In particular, Γ is not SFC when rank$(Z) >$ rank$(H^2(\Sigma;\mathbb{Z}))$.

Proof. We will show that when $n = \text{rank}(Z) > \text{rank}(c_{\mathbb{Q}})$, there exists a decomposition $Z = A \dotplus B$ such that rank$(A) > 1$ and $\pi(c) = 0$, where π is the natural projection $H^2(\Sigma; Z) \to H^2(\Sigma; A)$. Γ will then fail to be SFC by Theorem 6.2.1.

Suppose first that c is a torsion element and let $T(Z)$ be the torsion subgroup of Z. Then there exists a torsion-free subgroup M of Z such that $Z = M \dotplus T(Z)$ and $H^2(\Sigma; Z) \cong H^2(\Sigma; M) \oplus H^2(\Sigma; T(Z))$. As $H^2(\Sigma; T(Z))$ consists entirely of torsion elements, while $H^2(\Sigma;\mathbb{Z})$ torsion free, $H^2(\Sigma; T(Z)) =$

$T(H^2(\Sigma; Z))$, the torsion subgroup of $H^2(\Sigma; Z)$. In particular, $c \in H^2(\Sigma; T(Z))$. As $\text{rank}(M) = \text{rank}(Z) > 1$, this shows Γ fails to be SFC.

Now suppose that c has infinite order. Let $\theta : H^2(\Sigma; Z \otimes \mathbb{Q}) \to M_{mn}(\mathbb{Q})$ be an admissible isomorphism and set $C = \theta(c_{\mathbb{Q}})$. Then $\text{rank}(C) < n$ by hypothesis. But this means there is a non-zero endomorphism $\alpha : \mathbb{Q}^n \to \mathbb{Q}^n$ defined over $\mathbb{Z}$ such that $\alpha(C_i) = 0$ for each column C_i of C. So, by the rank-nullity theorem (for $\mathbb{Z}^n$), there exists a subgroup $B \subset \mathbb{Q}^n$ defined over $\mathbb{Z}$ such that $\mathbb{Q}^n = A \dotplus B$, where $A = Ker(\alpha)$. Moreover, as $\text{rank}(C) < n$, $\text{rank}(B) > 1$.

Let q be an automorphism of $\mathbb{Q}^n$ defined over $\mathbb{Z}$ such that $q(\mathbb{Z}\{e_1, \ldots, e_a\}) = A$ and $q(\mathbb{Z}\{e_{a+1}, \ldots, e_n\}) = B$. Then, if $Q \in GL(n, \mathbb{Z})$ is the matrix such that $q : x \mapsto xQ$ for all $x \in \mathbb{Q}^n$, the automorphism

$$t_u : X \mapsto XQ$$

of $M_{mn}(\mathbb{Q})$ operates as $x \mapsto xQ$ on $\mathbb{Q}c_i$, so that $C_i \in t_u(\{e_1 i, \ldots, e_a i\})$ for all i. That is, $C \in t_u(\mathbb{Q}c_1 + \cdots \mathbb{Q}c_a)$. However, it follows from Lemma 6.2.2 that the decomposition

$$M_{mn}(\mathbb{Q}) = t_u(\mathbb{Q}c_1 + \cdots \mathbb{Q}c_a) \oplus t_u(\mathbb{Q}c_{a+1} + \cdots \mathbb{Q}c_n)$$

is induced. Consequently Γ fails to have the SFC property by Theorem 6.2.1. $\square$

Corollary 6.2.4. *Let Γ be given by a central extension $0 \to \mathbb{Z}^n \to \Gamma \to \Sigma \to 1$ with classifying class c, where Σ is a surface group of genus $g > 1$. Then, Γ is SFC if and only if $n = 1$ and $c \neq 0$.*

Proof. Immediate consequence of Lemma 5.3.5 and Theorem 6.2.1. $\square$

6.3 Central extensions by lattices in L

If Σ is a lattice in a connected semi-simple Lie group with finite centre, then $Z(\Sigma) = \{1\}$ (Corollary 3.2.3). Thus, in a central extension of the form

$$0 \to Z \to \Gamma \to \Sigma \to 1,$$

$Z(\Gamma) = Z$. Cocompact lattices have type FL, so for such Σ, $H^2(\Sigma; \mathbb{Z})$ is finitely generated. This means that the rank of Γ is well-defined whenever Σ is cocompact and Z is finitely generated.

We have already seen that when Σ is an oriented surface group with genus $g > 1$, Γ has the SFC property if and only if

1. $\text{rank}(\Gamma) = \text{rank}(Z)$ and

2. the characteristic class $c \neq 0$.

A slighter weaker result can be shown to hold for all cocompact lattices in L. Recall that a group Γ is said to be FC if every cofinite embedding $\Gamma \to \Gamma$ is an automorphism.

Theorem 6.3.1. *Let*

$$0 \to \mathbb{Z} \to \Gamma \to \Sigma \to 1,$$

be a central extension classified by the class c, where Σ is a cocompact lattice in L. Then Γ is FC if and only if rank$(\Gamma) \neq 0$.

For $0 \neq c \in \mathbb{Z}^n$, let $hcf(c)$ be the highest common factor of the non-zero coefficients of c. When $c = 0$ define $hcf(c) = 0$. By convention, the highest common factor will be taken to be a non-negative integer.

Lemma 6.3.2. *$c \in \mathbb{Z}^m$ is basic (that is, a member of a basis for $\mathbb{Z}^m$) if and only if $hcf(c) = 1$.*

Proof. $c \in \mathbb{Z}^m$ is basic $\iff$ the extension $0 \to \mathbb{Z}c \to \mathbb{Z}^m \to \mathbb{Z}^m/\mathbb{Z}c \to 0$ splits $\iff$ $\mathbb{Z}^n/\mathbb{Z}c$ is torsion free. Now, $(y + \mathbb{Z}c) \in \mathbb{Z}^n/\mathbb{Z}c$ is a torsion element if and only if there exist integers k and l such that $ky_i = lc_i$ for all i. So, if $hcf(c) = 1$, it follows that $k|l$, and so $y \in \mathbb{Z}c$. Thus $\mathbb{Z}^m/\mathbb{Z}c$ is torsion free. Conversely, if $h = hcf(c) \neq 1$, then $y = c/h$ is a torsion element. Thus c is basic if and only if $hcf(c) = 1$ as claimed. $\square$

Corollary 6.3.3. *If $P \in GL_m(\mathbb{Z})$, then $hcf(Pc) = hcf(c)$.*

Proof. $c = h.c'$, where c' is basic and $h = hcf(c)$. If $P \in GL_m(\mathbb{Z})$, then Pc' is basic. Thus $hcf(Pc) = hcf(hP(c')) = |h|\, hcf(P(c')) = |h|\, hcf(c') = hcf(c)$. $\square$

Proof of theorem 6.3.1.

Set $m = \mathrm{rank}(H^2(\Sigma; \mathbb{Z}))$ and suppose that $\varphi : \Gamma \hookrightarrow \Gamma$ is a cofinite embedding. Then φ induces a commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \mathbb{Z} \to & \Gamma \to & \Sigma & \to 1 \\ \varphi' \downarrow & & \downarrow \varphi & \downarrow \varphi'' & \\ 0 \to & \mathbb{Z} \to & \Gamma \to & \Sigma & \to 1 \end{array}$$

in which all the downward arrows are inclusions and φ' and φ'' are cofinite. Thus

$$[\Gamma : \varphi(\Gamma)] = [\Sigma : \varphi''(\Sigma)][\mathbb{Z} : \varphi'(\mathbb{Z})].$$

Let c be the characteristic class of the extension $0 \to \mathbb{Z} \to \Gamma \to \Sigma \to 1$. Then, by the naturality of φ_* and φ^*,

$$\varphi_*(c_{\mathbb{Q}}) = \varphi^*(c_{\mathbb{Q}}).$$

Let $C \in \mathbb{Z}^m \subset \mathbb{Q}^m$ be a vector corresponding to $c_{\mathbb{Q}}$ under some admissible isomorphism. Since Σ is FC, φ'' is an isomorphism. So, φ^* and φ_* induce automorphisms of $M_{m1}(\mathbb{Z})$ defined over $\mathbb{Z}$ of type L and R respectively. In particular, there exists $P \in GL_m(\mathbb{Z})$ and $Q \in M_{(1,1)}(\mathbb{Z}) \cong \mathbb{Z}$ such that $PC = CQ$. Hence

$$hcf(C) = hcf(PC) = hcf(CQ) = |det(Q)|\, hcf(C).$$

But, $\text{rank}(\Gamma) = \text{rank}(C) \neq 0 \iff C \neq 0 \iff hcf(C) \neq 0$. Thus $|det(Q)| = 1$. As $|det(Q)| = [\mathbb{Z} : \varphi'(\mathbb{Z})]$, while $[\Sigma : \varphi''(\Sigma)] = 1$ this shows that $[\Gamma : \varphi(\Gamma)] = 1$ and that φ is an bijective. $\square$

A similar result holds when Γ has a well defined determinant:

Theorem 6.3.4. *Let Γ be a central extension of $\mathbb{Z}^n$ by a cocompact lattice $\Sigma \in L$ and set $m = rank(H^2(\Sigma; \mathbb{Z}))$. Then, if $m < n$, Γ is not FC. If $m = n$, Γ is FC if an only if $det(\Gamma) \neq 0$.*

Proof. The fact that a G fails to be FC if $m < n$ is a consequence of Theorem 6.2.3 above. Suppose therefore that $m = n$ and let $\varphi : \Gamma \hookrightarrow \Gamma$ be a cofinite embedding. As before, φ induces a commutative diagram:

$$\begin{array}{ccccccc}
0 \rightarrow & \mathbb{Z} \rightarrow & \Gamma \rightarrow & \Sigma & \rightarrow 1 \\
\varphi' \downarrow & & \downarrow \varphi & \downarrow \varphi'' & \\
0 \rightarrow & \mathbb{Z} \rightarrow & \Gamma \rightarrow & \Sigma & \rightarrow 1
\end{array}$$

in which all the downward arrows are inclusions and φ' and φ'' are cofinite, so that

$$[\Gamma : \varphi(\Gamma)] = [\Sigma : \varphi''(\Sigma)][\mathbb{Z}^n : \varphi'(\mathbb{Z}^n)].$$

Let c be the characteristic class of the extension and C the matrix corresponding to $c_{\mathbb{Q}}$ under some admissible isomorphism. Then, since φ'' is an automorphism of Σ while φ' is injective, there exist $P \in GL_n(\mathbb{Z})$ and $Q \in$

$M_n(\mathbb{Z}) \cap GL_n(\mathbb{Q})$ such that $PC = CQ$. Taking determinants gives,

$$det(P)det(C) = det(C)det(Q).$$

Now, $|det(P)| = 1$ and $det(\Gamma) = det(C) \neq 0$, so that $|det(Q)| = 1$. But $[\mathbb{Z}^n : \varphi'(\mathbb{Z}^n)] = |det(Q)|$, and so φ is bijective. $\square$

Chapter 7

Free products of groups and the SFC property

The aim of this chapter is to show that free products of certain generic classes of Poincaré duality groups and irreducible lattices in semi-simple Lie groups with finite centre possess the strong finite cohopfian property. We shall begin by extending the results of A.G.Kurosh on free products of groups.

7.1 Kurosh's theorem

A group G is said to be the (internal) free product of subgroups $\{H_\alpha : \alpha \in A\}$ if it is isomorphic to their free product. This is written as $G = \coprod_{\alpha \in A} H_\alpha$.

Theorem 7.1.1. *(Kurosh's Theorem) Let G be a group and suppose $G = \coprod_{\alpha \in A} H_\alpha$. Then, if H is a subgroup of G,*

$$H = \coprod_{\alpha \in A} \Big(\coprod_{\beta \in B_\alpha} H_{\alpha\beta} \Big) * F,$$

where

1. $H_{\alpha\beta} = H \cap x_{\alpha\beta} H_\alpha x_{\alpha\beta}^{-1}$, *and* $\{x_{\alpha\beta} : \beta \in B_\alpha\}$ *is a set of set of double-coset representatives of* $H\backslash G/H_\alpha$

2. F *is free*

3. if H *has finite index* d *in* G, *then* $d = \textit{rank}(F) + \Sigma_{\alpha\in A}|H\backslash G/H_\alpha| - 1$.

Proof. See [14, Chapter 14, Theorem 10]. □

As there exist surjections $H\backslash G \to H\backslash G/H_\alpha$ for all α, this shows that if $G \cong G_1 * \cdots * G_n$ and H has finite index d in G, then

$$H \cong (\coprod_{i=1}^{n} (H_1^{(i)} * \cdots * H_{p_i}^{(i)})) * F,$$

where F is free, $d = \text{rank}(F) + (p_1 + \cdots + p_n) - 1$ and for each j, $H_j^{(i)} = H \cap x_j G_i x_j^{-1}$ for some $x_j \in G$. Moreover, as H and therefore $x_j H x_j^{-1}$ has finite index in G, while $G_j/(x_j H x_j^{-1} \cap G_j)$ injects into $G/x_j H x_j^{-1}$, $H_j^{(i)}$ is isomorphic to a subgroup of finite index in G_j for all i and j, namely $x_j H x_j^{-1} \cap G_j$.

7.2 Indecomposable groups

In [21], Kurosh defined a group to be indecomposable if it cannot be written as a free product in a non-trivial way. He went on to show that when a group is isomorphic to a free product of indecomposable groups, the factors that occur in the decomposition are unique up to isomorphism. This leads to the fact that every non-trivial direct product of groups is indecomposable. The aim of this section is to show that this result still holds when the definition of indecomposability is extended to the commensurability class.

Definition 7.2.1. A group will be defined to be decomposable if it is commensurable with a non-trivial free product of groups and indecomposable otherwise.

Suppose that H is a group for which there exists a collection of proper subgroups $\{H_\alpha : \alpha \in A\}$ such that $H = \coprod^*_{\alpha \in A} H_\alpha$. Let B be the collection $\{\alpha \in A : H_\alpha \text{ is free}\}$. Then

$$H = \coprod_{\alpha \in A \setminus B} H_\alpha * \coprod_{\beta \in B} H_\beta.$$

That is, H is the free product of the free group $F = \coprod_{\beta \in B} H_\beta$ and the proper subgroups $\{H_\alpha : \alpha \in A \setminus B\}$, none of which are free.

A couple of the form $\{\{H_\alpha : \alpha \in A \setminus B\}, F\}$ is said to be a free-product decomposition of H. Two decompositions $\{\{H_\alpha : \alpha \in A\}, F\}$ and $\{\{H_{\alpha'} : \alpha' \in A'\}, F'\}$ are defined to be isomorphic if

1. $F \cong F'$
2. There exists a bijection $\theta : A \to A'$ such that $H'_{\theta(\alpha)}$ is conjugate to H_α in G for all $\alpha \in A$

A refinement of a free product decomposition $\{\{H_\alpha : \alpha \in A\}, F\}$ of H, is a decomposition $\{\{H_{\alpha\beta} : \alpha \in A,\ \beta \in B_\alpha\}, F\}$ such that $H_\alpha = \coprod_{\beta \in B_\alpha} H_{\alpha\beta}$ for all α.

Lemma 7.2.2. *Any two free-product decompositions of a group H have isomorphic refinements.*

Proof. See [21, Chapter XI, Section 35]. □

Proposition 7.2.3. *A decomposition $\{\{H_\alpha : \alpha \in A\}, F\}$ of G into in indecomposable groups $\{H_\alpha : \alpha \in A\}$ is unique up to isomorphism.*

Proof. If $\{\{H'_\alpha : \alpha \in A'\}, F'\}$ is another free product decomposition of G, then $\{\{H'_\alpha : \alpha \in A'\}, F'\}$ and $\{\{H_\alpha : \alpha \in A\}, F\}$ have isomorphic refinements. But the groups $\{H_\alpha : \alpha \in A\}$ are indecomposable and cannot therefore be decomposed as free products. So $\{\{H'_\alpha : \alpha \in A'\}, F'\}$ must be isomorphic to $\{\{H_\alpha : \alpha \in A\}, F\}$ □

The following corollary, though not essential for the remaining discussion of the SFC property has been included for general interest.

Corollary 7.2.4. *If an infinite group G is commensurable with a free product $\coprod_{\alpha \in A} G_\alpha$ of indecomposable groups G_α, no two of which are commensurable, then the groups G_α are uniquely defined up to commensurability.*

Proof. Suppose $\coprod_{\alpha \in A} G_\alpha \sim \coprod_{\alpha' \in A'} G'_{\alpha'}$, where the groups $G_{\alpha'}$ are also indecomposable and pairwise incommensurable. Then, there exists a group H that embeds with finite index in both products. By Kurosh's theorem, H admits a free product decomposition

$$H = \coprod_{\alpha \in A} (\coprod_{\beta \in B_\alpha} H_{\alpha\beta}) * F,$$

in which $H_{\alpha\beta}$ is isomorphic to a subgroup of finite index in H_α for all $\alpha \in A$ and $\beta \in B_\alpha$. However, by the corollary above, up to isomorphism, this must also be the free product decomposition induced by $\coprod_{\alpha' \in A'} G'_{\alpha'}$. In particular, given $\beta \in B_\alpha$, there exists $\sigma(\alpha) \in A'$ such that $H_{\alpha\beta} \sim G'_{\sigma(\alpha)}$. But $G_\alpha \sim H_{\alpha\beta}$ and the groups G_α are pairwise incommensurable, so $\alpha \mapsto \sigma(\alpha)$ defines a map $\sigma : A \to A'$ such that $G_\alpha \sim G'_{\sigma(\alpha)}$ for all α. By the same argument, there exist a map $\lambda : A' \to A$ such that $G'_{\alpha'} \sim G_{\lambda(\alpha')}$ for all α'. But then $G_{\lambda\sigma(\alpha)} \sim G'_{\sigma(\alpha)} \sim G_\alpha$, which shows that $\lambda \circ \sigma = id_A$. Similarly, $\sigma \circ \lambda = id_{A'}$. This proves the result. □

7.3 The strong irreducibility of decomposable groups

R.Baer and F.Levi proved in [1] that no group can be simultaneously isomorphic to a free product and a direct product of groups in a non-trivial way. This result is extended in the following theorem.

Theorem 7.3.1. *Let G be a decomposable group. Then G is strongly irreducible.*

The proof depends of the following lemma, which is a very slight generalization of Baer-Levi theorem:

Lemma 7.3.2. *Let G be a group such that $A * B = G = CD$, for subgroups A, B, C, D, where C and D are mutually centralizing and infinite and $A * B$ denotes the internal free product. Then at least one of the groups $\{A, B, C, D\}$ is trivial.*

Proof. Suppose that A,B,C and D are non-trivial. If $1 \neq x \in A \cap C$, then, since D centralizes x, $D \subset A$. But, as C centralizes D, C must also lie in A. Thus $G = CD = A$, which implies that B is trivial, a contraction. So $A \cap C = \{1\}$. Now, C is normal in G. Thus $xAx^{-1} \cap C \cong A \cap x^{-1}Cx = A \cap C = \{1\}$ for all $x \in G$. Similarly, $xBx^{-1} \cap C = \{1\}$ for all $x \in G$. So, according to Kurosh's Theorem, C must be free.

Now, exactly the same argument applies to D which is therefore also free. Let $p : G \to G/C$ be the natural projection. Then, as $A \cap C = \{1\}$, p maps A injectively into D which is therefore free. Similarly, B is free so that $G = A * B$ a free group. But, as $G = CD$, and C , D are infinite and mutually centralizing, the maximal abelian rank of G must be ≥ 2, a contradiction. Hence the result. $\square$

Proof of Theorem 7.3.1.

G is commensurable with a non-trivial free product H. By Kurosh's theorem, any group that embeds with finite index in a non-trivial free product is also isomorphic to non-trivial free-product, so that we may assume $H \subset G$.

Now suppose that G and therefore H is strongly reducible, so that H is commensurable with a product H_1H_2 where H_1 and H_2 are mutually centralizing infinite groups. Let J be a normal subgroup of H that embeds with finite index in both H and H_1H_2. For $j = 1, 2$, there exists a commutative diagram

$$\begin{array}{ccccc} H_j \cap J & \hookrightarrow & H_j & \rightarrow & H_j/H_j \cap J \\ \downarrow & & \downarrow & & \downarrow \\ J & \hookrightarrow & H_1H_2 & \rightarrow & H_1H_2/J \end{array}$$

in which all the downward pointing arrows are inclusions. This shows that $H_j \cap J$ has finite index in H_j for $j = 1, 2$, so that $(H_1 \cap J)(H_2 \cap J)$ has finite index in H_1H_2. $(H_1 \cap J)(H_2 \cap J)$ therefore also has finite index in J, and by Kurosh's theorem is isomorphic to a non-trivial free product. But the groups $(H_1 \cap J)$ and $(H_2 \cap J)$ are mutually centralizing, contradicting Proposition 7.3.2.

□

This generalizes Kurosh's second result and moreover, suggests a duality between irreducibility and indecomposability for infinite groups. The converse of Theorem 7.3.1, is however false. We will see in the next section that a lattice in a connected semi-simple Lie group with finite centre and real rank ≥ 2 is indecomposable. An irreducible lattice of this type is therefore both irreducible and indecomposable, showing that indecomposability does not imply reducibility.

7.4 A theorem on the SFC property of free products

Definition 7.4.1. A group Γ will be said to have property (F) if $H_1(\Delta;\mathbb{Z})$ is finite for every subgroup $\Delta \subset \Gamma$ of finite index.

Theorem 7.4.2. *The free product of finitely many indecomposable and finitely generated groups with property (F) has the strong cohopfian property.*

Proof. Let $\Gamma_1, \ldots, \Gamma_n$ be finitely generated groups with property (F) and suppose that H is a subgroup of finite index d in $\Gamma_1 * \cdots * \Gamma_n$. If $\Gamma_1 * \cdots * \Gamma_n$ has no subgroups of finite index, it will obviously possess the strong finite cohopfian property, so we may assume $d > 1$. By Kurosh's theorem,

$$H = (\coprod_{i=1} (\Gamma_1^{(i)} * \cdots * \Gamma_{p_i}^{(i)})) * F', \tag{7.4.1}$$

where $\Gamma_1^{(i)}, \ldots, \Gamma_{p_i}^{(i)}$ are isomorphic to subgroups of finite index in Γ_i for $i = 1, \ldots, n$, F' is free and $\mathrm{rank}(F') = d - (p_1 + \cdots + p_n) + 1$. The groups $\Gamma_1, \ldots, \Gamma_n$ are indecomposable by hypothesis, and so $\Gamma_j^{(i)}$ is indecomposable for all i and j. Equation 7.4.1 is therefore a decomposition of H into indecomposable factors. By Theorem 7.1.1, this decomposition is unique up to isomorphism and consequently the sum $p_1 + \cdots + p_n$ depends only on the isomorphism type of H.

Since the groups $\Gamma_1, \ldots, \Gamma_n$ have property (F) by hypothesis, $H_1(\Gamma_j^{(i)};\mathbb{Z})$ is finite for all i and j. Thus $\mathrm{rank}(F') = \mathrm{rank}(H_1(H : \mathbb{Z}))$. Consequently, d depends only on the isomorphism type of H, so that $\Gamma_1 * \cdots * \Gamma_n$ has the strong finite cohopfian property. $\square$

Corollary 7.4.3. *For $i = 1, \ldots, n$, let Γ_i be the free product of finitely many* *indecomposable finitely generated groups with property (F). Then, if Γ_i is also torsion-free for all i, $\Gamma_1 \times \cdots \times \Gamma_n$ is SFC.*

Proof. Γ_i is strongly irreducible for all i by Theorem 7.3.1, so that $\Gamma_1 \times \cdots \times \Gamma_n$ is SFC by Theorem 5.1.2. □

7.5 Free products of lattices and Poincaré duality groups

The following theorems show that groups from the two classes below are indecomposable and have property (F), so that Theorem 7.4.2 applies:

L_1) Poincaré duality groups Γ of cohomological dimension ≥ 2 such that $H_1(\Gamma; \mathbb{Z})$ is finite.

L_2) Irreducible lattices in connected semi-simple Lie groups having finite centre and real rank ≥ 2.

The following theorem is well known. A proof has been included for completeness.

Theorem 7.5.1. *If Γ is a duality group of cohomological dimension $d \geq 2$ and $\Gamma \cong \Gamma_1 * \Gamma_2$, then either Γ_1 or Γ_2 is trivial.*

Proof. Given any Γ module M, the Mayer-Vietoris sequence for the free product of two groups (see [12, page 178]) gives rise to a natural map

$$H^i(\Gamma; M) \to H^i(\Gamma_1; M) \oplus H^i(\Gamma_2; M)$$

that is an isomorphism for $i > 1$ and an epimorphism when $i = 1$.

Now, it follows from a theorem of R.Bieri [2] that, as Γ has type FP, so do Γ_1 and Γ_2. As $\mathbb{Z}\Gamma$ is a flat $\mathbb{Z}\Gamma_j$ module for $j = 1, 2$, this implies

$$H^i(\Gamma_j; \mathbb{Z}\Gamma) \cong H^i(\Gamma_j; \mathbb{Z}\Gamma_j) \otimes_{\mathbb{Z}\Gamma_j} \mathbb{Z}\Gamma = \mathrm{Ind}_{\Gamma_j}^{\Gamma} H^i(\Gamma_j; \mathbb{Z}\Gamma_j)$$

for all i (see [12, page 198]). Consequently, for each $i > 0$ there exists a natural map

$$H^i(\Gamma;\mathbb{Z}\Gamma) \longrightarrow \mathrm{Ind}_{\Gamma_1}^{\Gamma} H^i(\Gamma_1;\mathbb{Z}\Gamma_1) \oplus \mathrm{Ind}_{\Gamma_2}^{\Gamma} H^i(\Gamma_2;\mathbb{Z}\Gamma_2)$$

that is an isomorphism for $i > 1$ and an epimorphism when $i = 1$. This shows that $H^i(\Gamma_j;\mathbb{Z}\Gamma_j) = 0$ for all $i < d$. Moreover, as $H^d(\Gamma;\mathbb{Z}\Gamma)$ is $\mathbb{Z}$-torsion free, so is $H^d(\Gamma_j;\mathbb{Z}\Gamma_j)$ and Γ_j is a duality group for $j = 1, 2$.

Set $D = H^d(\Gamma;\mathbb{Z}\Gamma)$ and $D_j = H^d(\Gamma_j;\mathbb{Z}\Gamma_j)$ for $j = 1, 2$. Then $D = \mathrm{Ind}_{\Gamma_1}^{\Gamma} D_1 \oplus \mathrm{Ind}_{\Gamma_2}^{\Gamma} D_2$. By Shapiro's Lemma $H_i(\Gamma; \mathrm{Ind}_{\Gamma_j}^{\Gamma} D_j) \cong H_i(\Gamma_j; D_j)$ for all i. So

$$\begin{aligned}
\mathbb{Z} \cong H^0(\Gamma;\mathbb{Z}) &\cong H_d(\Gamma; D) \\
&\cong H_d(\Gamma_1; \mathrm{Ind}_{\Gamma_1}^{\Gamma} D_1) \oplus H_d(\Gamma_2; \mathrm{Ind}_{\Gamma_2}^{\Gamma} D_2) \\
&\cong H_d(\Gamma_1; D_1) \oplus H_d(\Gamma_2; D_2) \\
&\cong H^0(\Gamma_1;\mathbb{Z}) \oplus H^0(\Gamma_2;\mathbb{Z}) \cong \mathbb{Z}^2,
\end{aligned}$$

which is a contradiction. Hence the result. □

The above theorem shows that groups from L_1 are indecomposable. However, it also applies to groups in the class L_2. A celebrated theorem of G.A.Margulis [24] states that given a lattice Γ in a connected linear semisimple Lie group G of real rank ≥ 2, there exists a reductive linear algebraic group $R \subset GL(n, \mathbb{C})$ defined over $\mathbb{Q}$ together with an epimorphism $\theta : R_{\mathbb{R}} \to G$ such that

1. $\mathrm{Ker}(\theta)$ is compact
2. $\theta(R_{\mathbb{Z}}) \sim \Gamma$.

A.Borel and J.P.Serre proved in [10] that every arithmetic subgroup of a reductive linear algebraic group defined over $\mathbb{Q}$ is a duality group. As every group of type FP that is commensurable with a duality group is itself a duality group (Corollary 1.6.5 and 1.6.6), groups in the class L_2 are themselves duality groups. It only remains to show therefore that groups from the classes L_1 and L_2 have property (F).

Proposition 7.5.2. *If $\Gamma \in L_1$, then Γ has property (F).*

Proof. It is a theorem of F.E.A.Johnson and C.T.C.Wall [18] that if G is torsion free and H a subgroup of finite index in G, then G is a Poincaré duality group if and only if H is. So, let Δ be a subgroup of finite index in Γ, and suppose that Γ has cohomological dimension d. Then, Δ is a Poincaré duality group by the Johnson-Wall theorem and has the same cohomological dimension as Γ by Shapiro's Lemma. Thus $H_1(\Delta;\mathbb{Z}) \cong H^{d-1}(\Delta;\mathbb{Z})$. We will show that $H^{d-1}(\Delta;\mathbb{Z})$ is finite.

The intersection N of the conjugates of Δ is a normal subgroup of finite index in Γ. If $H_1(N;\mathbb{Z}) \cong N/[N,N]$ is finite, $\Delta/[\Delta,\Delta]$ must be finite also and we may therefore assume that $\Delta = N$ so that Δ is normal in Γ. In this case there exists a short exact sequence,

$$1 \to \Delta \to \Gamma \to Q \to 1,$$

in which Q is finite. Let $\{E^r, d^r\}$ be the corresponding Lyndon-Hochschild-Serre spectral sequence. Then $\{E^r, d^r\}$ is a third quadrant spectral sequence such that

$$E_2^{p,q} \cong H^p(Q; H^q(\Delta;\mathbb{Z})) \xRightarrow{p} H^{p+q}(\Gamma;\mathbb{Z}).$$

Thus, for all $r \geq 0$, there exist groups $K^{p,q}$ such that

$$0 \subset K^{r,0} \subset K^{r-1,1} \subset \cdots \subset K^{1,r-1} \subset K^{0,r} = H^r(\Gamma;\mathbb{Z}),$$

where $K^{p,q}/K^{p+1,q-1} \cong E_\infty^{p,q}$ for all p and q.

Since Q is finite, given any Q-module A and $m > 0$, $H^m(Q; A)$ consists entirely of elements of finite order [23, Chapter IV, Proposition 5.3]. So, as Δ has type FP, $H^p(Q; H^q(\Delta;\mathbb{Z}))$ is finite whenever $p > 0$. Thus, $\operatorname{rank}(H^r(\Gamma;\mathbb{Z}) \cong K^{0,r}) = \operatorname{rank}(E_\infty^{0,r})$ for all $r \geq 0$. Now, the differential d^r has bidegree $(r, 1-r)$ and $E_r^{p,q}$ is finite whenever $p > 0$. So $E_\infty^{0,r} \cong E_{r+2}^{0,r}$ and $\operatorname{rank}(E_2^{0,r}) = \operatorname{rank}(E_{r+2}^{0,r}) = \operatorname{rank}(E_\infty^{0,r})$. But, $E_2^{0,r} \cong H^r(\Delta;\mathbb{Z})$ and so

$$\operatorname{rank}(H^r(\Delta;\mathbb{Z})) > 0 \Rightarrow \operatorname{rank}(E_\infty^{0,r}) > 0 \Rightarrow \operatorname{rank}(H^r(\Gamma;\mathbb{Z})) > 0.$$

Taking $r = d-1$ we deduce that if $H^{d-1}(\Gamma;\mathbb{Z})$ is finite, then so is $H^{d-1}(\Delta;\mathbb{Z})$

□

The fact that groups in L_2 have property (F) is a consequence of the following theorem of D.A.Kazdan [19]:

Theorem 7.5.3. *Let Γ be an irreducible lattice in a connected semi-simple Lie group with finite centre and real rank ≥ 2. Then $H^1(\Gamma;\mathbb{Z})$ is finite.*

As we have seen, if G is a group containing a subgroup H of finite index for which $H_1(H;\mathbb{Z})$ is finite, then $H_1(G;\mathbb{Z})$ is finite also. Consequently, to prove that lattices in L_2 have property (F), we need only consider the torsion-free case.

Now, any torsion-free lattice in a connected semi-simple Lie group G with finite finite centre embeds as a lattice in $\operatorname{Ad}(G)$. As the real rank of a semi-simple Lie group is determined by its Lie algebra, $\operatorname{Ad}(G)$ has the same real rank as G. But $\operatorname{Ad}(G)$ is linear, so if $\Gamma \in L_2$ then Kazdan's theorem applies, and $H^1(\Gamma;\mathbb{Z})$ must be finite. However, a subgroup Δ of finite index in Γ is also a lattice in $\operatorname{Ad}(G)$, so that $H^1(\Delta;\mathbb{Z})$ is also finite. Thus Γ has property (F).

Chapter 8

Two theorems on the corepresentability of groups

In [16, pages 78, 132], F.E.A.Johnson gave the first explicit statement of the Corepresentability Theorem, which demands the existence of epimorphisms

$$\rho_r : Ext^r_{\mathbb{Z}G}(M; N) \longrightarrow Hom_{Der(\mathbb{Z}G)}(\Omega_r(M); N)$$

for any finite group G and $\mathbb{Z}G$ lattices M and N, where $r \geq 2$. In this chapter, the corepresentability of infinite groups is considered. I will prove the existence of epimorphisms ρ_r whenever M is an $\mathbb{Z}G$-module of type FP_{r-1} and show, in addition, that ρ_r is an isomorphism if and only if $Ext^r_{\mathbb{Z}G}(\mathbb{Z}; \mathbb{Z}G) = 0$. When G is a virtual duality group of dimension d, so that $Ext^d_{\mathbb{Z}G}(\mathbb{Z}; \mathbb{Z}G) \neq 0$, ρ_d will be shown to factor through the natural projection $H^d(G; N) \longrightarrow \widetilde{H}^d(G; N)$, yielding an isomorphism $\widetilde{H}^d(G; N) \cong Hom_{Der(\mathbb{Z}G)}(\Omega_n(\mathbb{Z}); N)$.

Now, while it is not known whether every lattice in L is a duality group, the results of A.Borel and J.P.Serre [10] and G.A.Margulis [24] show that with the possible exception of non-arithmetic lattices in semi-simple Lie groups of real rank 1, this is in fact the case. It is in this way that the results of the remaining two chapters relate to groups in L.

8.1 A proof of the corepresentability theorem

Proposition 8.1.1. *Let R be a ring and M an R-module of type FP_{n-1} over R, where $n \geq 1$. Then, for all $1 \leq r \leq n$ and R-modules N, there exist epimorphisms*

$$\rho_r : Ext^r_R(M; N) \to Hom_{Der(R)}(\Omega_r(M); N). \tag{8.1.1}$$

Moreover, ρ_r is natural in N for all r.

Proof. Fix $1 \leq r \leq n$ and let $0 \to K \to F_n \to \cdots \to F_0 \to M \to 0$ be a partial free resolution of M over R having finite type. Set $J = \mathrm{Ker}(F_{r-1} \to F_{r-2})$, so that $[J] = \Omega_r(M)$ (here F_{-1} is taken to mean M). A map $f \in Hom_R(F_r; N)$ is a cocycle if and only if $f \circ (F_{r+1} \to F_r) = 0$, and any such map will therefore factor through $\mathrm{Im}(F_{r+1} \to F_r)$.

$$\begin{tikzcd}
\cdots \arrow[r] & F_{r+1} \arrow[r] & F_r \arrow[r] \arrow[d] & F_{r-1} \arrow[r] & \cdots \\
& & N & &
\end{tikzcd}$$

But, by exactness, $\mathrm{Im}(F_{r+1} \to F_r) = \mathrm{Ker}(F_r \to F_{r-1})$, so that f induces a map $\widetilde{f} : J = \mathrm{Im}(F_r \to F_{r-1}) \to N$. Clearly any such map also determines a cocycle in $Hom_R(F_r; N)$, so that the correspondence $f \mapsto \widetilde{f}$ identifies the cocycles in $Hom_R(F_r; N)$ with the group $Hom_R(J; N)$.

The coboundaries in $Hom_R(F_r; N)$ are those cocycles which factor through the map $F_r \to F_{r-1}$ and therefore correspond to morphisms from $J \to N$ which factor through the inclusion $J \hookrightarrow F_{r-1}$. As, F_{r-1} is a projective R-module, any such map will be mapped to zero under the canonical projection

$Hom_R(J;N) \to Hom_{Der(R)}(J;N)$, which therefore factors through the quotient map $Hom_R(J;N) \to Ext_R^r(M;N)$ to yield an epimorphism,

$$\rho_r : Ext_R^r(M;N) \to Hom_{Der(R)}(J;N) \cong Hom_{Der(R)}(\Omega_r(M);N).$$

This map is clearly natural in N. $\square$

Proposition 8.1.2. *Let M have type FP_n. Then, if $1 \leq r \leq n$, ρ_r is an isomorphism if and only if $Ext_R^r(M;R) = 0$.*

Proof. Since $Hom_{Der(R)}(\Omega_r(M);E)$ is zero for any free R-module E, necessity is obvious. To prove sufficiency we must show that, in the notation above, any morphism from $J \to N$ which factors through a projective is in fact a coboundary (i.e. factors through the inclusion map $J \hookrightarrow F_{r-1}$). As M has type $\mathrm{FP_n}$ and $r \leq n$, J is finitely generated. So, by Proposition 1.3.1, if $f : J \to F_{r-1}$ factors through a projective, is must factor through a finitely generated free R-module E. That is, there exists a commutative diagram:

$$\begin{array}{ccc} J & \hookrightarrow & F_{r-1} \\ f\downarrow & \searrow & q \\ N & \leftarrow & E \end{array}$$

Now, q corresponds to a cocyle representing an element of $Ext_R^r(M;E)$. However, the functor $Ext_R^r(M;\_\,)$ is additive and $Ext_R^r(M;R) = 0$ by hypothesis, so $Ext_R^r(M;E) = 0$. Thus q is a coboundary and therefore factors through $J \hookrightarrow F_{r-1}$. But this means f also factors through $J \hookrightarrow F_{r-1}$, which completes the proof. $\square$

8.2 Corepresentability and Farrell cohomology

Let R and S be rings.

Proposition 8.2.1. *If $\alpha : R \to S$ is a ring homomorphism and P a projective R-module, then $Ind_\alpha P = S \otimes_\alpha P$ is a projective S-module.*

Proof. Suppose there is a mapping problem of S-modules,

$$\begin{array}{ccc} Ind_R^S P & & \\ g \downarrow & & \\ A & \underset{p}{\leftarrow} & B \end{array}$$

where p is onto. Let $i : P \to Ind_\alpha(P)$ be the natural inclusion (defined by $p \mapsto p \otimes 1$) and set $f = g \circ i$. Then, since P is a projective R-module, there exists an R-morphism $\widetilde{f} : P \to B$ such that $f = p \circ \widetilde{f}$. By the characteristic property of $Ind_\alpha(P)$, there exists an S-morphism $\widetilde{g} : Ind_\alpha(P) \to B$ such that $\widetilde{g} \circ i = \widetilde{f}$ as an R-morphism. $p \circ \widetilde{g}$ is then an S-morphism from $Ind_\alpha(P)$ to A extending f. But g is the unique S-morphism $Ind_\alpha(P) \to A$ extending f, so that $g = p \circ \widetilde{g}$ and $Int_R^S P$ is a projective S-module as claimed. □

Let G be a group and H a subgroup of finite index. For G-modules J and N, the transfer map $tr : Hom_{\mathbb{Z}H}(J; N) \to Hom_{\mathbb{Z}G}(J; N)$ is defined by $tr(f)(m) = \Sigma_{g \in E} \; gf(g^{-1}m)$, where E is a set of right coset representatives for H in G.

It follows immediately from the definition of tr that if G is also a G-module and $\varphi : P \to N$ a $\mathbb{Z}G$-linear map, then $tr(\lambda \circ \varphi) = tr(\lambda) \circ \varphi$ for any $\mathbb{Z}H$-linear map $J \to P$.

Proposition 8.2.2. *tr induces a homomorphism*

$$tr : Hom_{Der(\mathbb{Z}H)}(J; N) \to Hom_{Der(\mathbb{Z}G)}(J; N); \quad [f] \mapsto [tr(f)].$$

Proof. Let $f : J \to N$ be an H-morphism and suppose that f factors through a projective H-module P via morphisms $q : J \to P$ and $r : P \to N$. By the characteristic property of $Ind_H^G P$, there exists a G-morphism $\widetilde{f} : Ind_H^G P \to N$ such that the following diagram commutes:

$$\begin{array}{ccc} J & \xrightarrow{f} & N \\ q\downarrow & \nearrow & \uparrow \tilde{f} \\ P & \underset{i}{\rightarrow} & Ind_H^G P \end{array}$$

Set $\psi = i \circ q$. Then $f = \widetilde{f} \circ \psi$ and, as $\widetilde{f}$ is a G-morphism, $tr(f) = tr(\widetilde{f} \circ \psi) = \widetilde{f} \circ tr(\psi)$. But $Ind_H^G P$ is a projective G-module, so that $tr(f) \sim 0$, which proves the result. □

Let $\epsilon : P \to M$ be a projective resolution of M over $\mathbb{Z}G$. Then $\epsilon : P \to M$ is a projective resolution of M over $\mathbb{Z}H$ whose differentials $\partial_n : F_n \to F_{n-1}$ are G-morphisms. In particular, for $f \in Hom_{\mathbb{Z}H}(F_n; N)$, $tr(f \circ \partial_n) = tr(f) \circ \partial_n$ for all n. tr therefore induces a morphism

$$Ext^n_{\mathbb{Z}H}(M; N) \to Ext^n_{\mathbb{Z}G}(M; N); \quad [f] \mapsto [tr(f)].$$

This map clearly commutes with the maps ρ_n of 8.1.1.

Proposition 8.2.3. *Let $f : J \to N$ be a G-morphism. Then, if f factors through a finitely generated free $\mathbb{Z}G$ module, $f = tr(\widetilde{f})$ for some H-morphism $\widetilde{f}$.*

Proof. Since $Hom_{\mathbb{Z}G}(\_\ ; N)$ and tr are additive, it is sufficient to consider the case when f factors through $\mathbb{Z}G$. For $\lambda \in Hom_{\mathbb{Z}H}(\mathbb{Z}H; N)$, define a map $\widetilde{\lambda} \in Hom_{\mathbb{Z}H}(\mathbb{Z}G; N)$ by

$$\widetilde{\lambda}(g) = \begin{cases} \lambda(g) & g \in H \\ 0 & g \notin H \end{cases}$$

Let $\{g_1, \ldots, g_d\}$ be a set of left coset representatives for H in G. Then, for $g = g_i h \in G$

$$tr(\widetilde{\lambda})(g) = \Sigma_j\, g_j \widetilde{\lambda}(g_j^{-1} g_i h) = g_i \widetilde{\lambda}(h) = g_i \lambda(h).$$

Thus $tr(\widetilde{\lambda}) = Ind_H^G(\lambda)$, where for each $\mathbb{Z}H$ morphism $\alpha : \mathbb{Z}H \to N$, $Ind_H^G(\alpha)$ is the unique $\mathbb{Z}G$ morphism making the following diagram commute:

$$\begin{tikzcd} \mathbb{Z}H \arrow[r, hook] \arrow[d, "\alpha"'] & \mathbb{Z}G \arrow[dl, "Ind_H^G(\alpha)"] \\ N & \end{tikzcd}$$

As every $\mathbb{Z}G$-linear map $\alpha : \mathbb{Z}G \to N)$ satisfies $f = Ind_H^G(f|_{\mathbb{Z}H})$, this shows Ind_H^G and therefore tr is surjective.

Now, suppose $f = \psi \circ \varphi$ for $\mathbb{Z}G$-morphisms $\varphi : J \to \mathbb{Z}G$ and $\psi : \mathbb{Z}G \to N$. Then $\psi = tr(\psi_1)$ for some H-morphism ψ_1. Let f_1 be the H-morphism $\psi_1 \circ \varphi$. Then, since φ is $\mathbb{Z}G$-linear, $tr(f_1) = tr(\psi_1 \circ \varphi) = tr(\psi_1) \circ \varphi = f$, which completes the proof. □

Theorem 8.2.4. *Let G be a group of virtual finite cohomological dimension $d \geq 1$ having type FP_{d-1}. Then, for any coefficient module N,*

$$\widetilde{H}^d(G; N) \cong Hom_{Der(\mathbb{Z}G)}(\Omega_d(\mathbb{Z}); N),$$

where $\widetilde{H}^d(G; N)$ denotes the Farrell cohomology in dimension d.

Proof. Let $0 \to J \to F_d \to \cdots \to F_0 \to \mathbb{Z} \to 0$ be a partial free resolution of finite type over $\mathbb{Z}G$ and H a torsion-free subgroup of finite index in G. Then, since H has cohomological dimension d, J is a projective H-module and $Hom_{Der(\mathbb{Z}H)}(J; N) = 0$. Moreover, as there exists a commutative diagram,

$$\begin{tikzcd} H^d(H; N) \arrow[r, "tr"] \arrow[d, "\rho_d"] & H^d(G; N) \arrow[d, "\rho_d"] \\ Hom_{Der(\mathbb{Z}H)}(J; N) \arrow[r, "tr"'] & Hom_{Der(\mathbb{Z}G)}(J; N) \end{tikzcd}$$

the canonical map $\rho_d : H^d(G; N) \to Hom_{Der(\mathbb{Z}G)}(J; N)$ factors through the quotient $H^d(G; N)/tr(H^d(H; N))$. By Proposition 8.2.3, $f : J \to N$ represents zero in $Hom_{Der(\mathbb{Z}G)}(J, N)$ only if $f \in \mathrm{Im}(tr)$. So the induced map

$H^d(G; N)/tr(H^d(H; N)) \to Hom_{Der(\mathbb{Z}G)}(J, N)$ is in fact an isomorphism. Since $H^d(G; N)/tr(H^d(H; N)) \cong \widetilde{H}^d(G; N)$, this completes the proof. $\square$

Chapter 9

A theorem concerning the syzygies of duality groups

This section introduces the concept of a duality module, which extends that of a duality group, viewing $\mathbb{Z}$ is a duality module over the ring $\mathbb{Z}G$ whenever G is a duality group. By extending a result of F.E.A. Johnson on the representability of $\Omega_1(M)$, I will show that if G is a duality group of type FL and cohomological dimension $d \geq 3$, then the map

$$J \mapsto J^* = Hom_{\mathbb{Z}G}(J, \mathbb{Z}G)$$

is a bijection from $\Omega_r(\mathbb{Z})$ to $\Omega_{d-r}(\mathbb{Z})$ for all $2 \leq r \leq d$. The identity,

$$Ext^r_{\mathbb{Z}G}(\Delta; N) \cong Tor^{\mathbb{Z}G}_{n-r}(\mathbb{Z}; N),$$

will be deduced as a corollary.

9.1 Duality modules and their syzygies

Fix a ring R. An R-module M will be said to be a duality module if, for some fixed $n \geq 0$,

$$Ext^r_R(M;R) = \begin{cases} 0 & r \neq n \\ \Delta_M & r = n \end{cases}.$$

Δ_M is called the dualising module of M. The follow proposition, due to F.E.A.Johnson will be used to show that, for any duality module M and all $k \geq 2$, every element $K \in \Omega_k(M)$ can be explicitly realized as $\mathrm{Ker}(F_n \to F_{n-1})$, where $\varepsilon : F \to M$ is a free resolution of finite length and type.

Proposition 9.1.1. *Let M be a finitely generated module over R such that $Ext^1(M;R) = 0$. Then, for each $J \in \Omega_1(M)$, there exists a short exact sequence $0 \to J \to S \to M \to 0$, where S is finitely generated and stably free.*

Proof. See [17]. □

Corollary 9.1.2. *If M is an R module of type FP_n for some fixed $n \geq 2$ and if $Ext^r(M;R) = 0$ whenever $1 \leq r \leq n$, then, for each $J \in \Omega_n(M)$, there exists an exact sequence $0 \to J \to F_{n-1} \to \cdots \to F_0 \to M \to 0$ in which $F_0, \ldots, F_{n-1}$ are finitely generated and free over R.*

Proof. The proof is by induction on n. Pick $J \in \Omega_2(M)$ and $K \in \Omega_1(M)$. Then, since $Ext^1(M;R) = 0$, there exists a short exact sequence

$$0 \to K \to S_0 \to M \to 0$$

in which S_0 is stably free and finitely generated by the above proposition. However, since $\Omega_2(M) = \Omega_1(K)$ and $Ext^1(K;R) = Ext^2(M;R) = 0$, Proposition 9.1.1 also implies the existence of a short exact sequence

$$0 \to J \to S_1 \to K \to 0,$$

where S_1 is stably free and finitely generated. Splicing these two sequences together gives an exact sequence:

$$0 \to J \to S_1 \to S_0 \to M \to 0. \tag{9.1.1}$$

Now let E_1 and E_0 be finitely generated free modules such that $S_j \oplus E_j$ is finitely generated and free for $j = 0, 1$. Then, by taking the direct sum of 9.1.1 with $0 \to E_0 \oplus E_1 \xrightarrow{\text{id}} E_0 \oplus E_1 \to 0$, it is possible to construct an exact sequence

$$0 \to J \to F_1 \to F_0 \to M \to 0$$

in which both F_0 and F_1 are free and finitely generated. This proves the result for $n = 2$.

Now pick $n > 2$ and fix $J \in \Omega_n(M)$. Pick and $K \in \Omega_{n-1}(M)$. Then, since $\Omega_{n-1}(M) = \Omega_1(K)$ and $Ext^1(K; R) = Ext^{n-1}(M; R) = 0$, Proposition 9.1.1 implies the existence of a short exact sequence $0 \to J \to S \to K \to 0$ in which S is stably free. Let F be a finitely generated free module such that $F_{n-1} = S \oplus F$ is free. Then, adding on the exact sequence $0 \to F \xrightarrow{\text{id}} F \to 0$ gives an exact sequence,

$$0 \to J \to F_{n-1} \to K \oplus F \to 0 \tag{9.1.2}$$

However, $K \oplus F \sim K \in \Omega_{n-1}(M)$. So, by induction, there exists an exact sequence

$$0 \to K \oplus F \to F_{n-2} \to \cdots \to F_0 \to M \to 0 \tag{9.1.3}$$

in which the modules $F_0, \ldots, F_{n-2}$ are free and finitely generated. Splicing 9.1.2 and 9.1.3 now gives the desired result. $\square$

Theorem 9.1.3. *Let M be a duality module of type FL and projective dimension $n \geq 3$. Then, for all $2 \leq k \leq n-1$, $J \in \Omega_k(M)$ if and only if* *there exists a finite free resolution* $0 \to F_n \to \cdots \to F_0 \to M \xrightarrow{\epsilon} 0$ *with* $Ker(F_{k-1} \to F_{k-2}) \cong J$.

Proof. $\Rightarrow$) trivial. $\Leftarrow$ Fix $J \in \Omega_k(M)$. Then, by the corollary above, we can construct an exact sequence

$$0 \to J \to A_{k-1} \to \ldots \to A_0 \to M \to 0 \tag{9.1.4}$$

with $A_0, \ldots, A_{k-1}$ are free and finitely generated over R. By Proposition 1.4.1, there exists an exact sequence $0 \to B_n \to \ldots \to B_0 \to M \to 0$ in which B_i is free and finitely generated for all i. Setting $D_k = \mathrm{Ker}(B_{k-1} \to B_{k-2})$, we may split this sequence to obtain exact sequences,

$$0 \to B_n \to \cdots \to B_k \to D_k \to 0$$

$$0 \to D_k \to B_{k-1} \to \cdots \to B_k \to M \to 0.$$

Now, by Schanuel's Lemma, $J \sim D_k$. J therefore has the same projective dimension as D_k over R and is also FL. In particular, it admits a free resolution of finite type and length $n-k$:

$$0 \to A_n \to \cdots \to A_k \to J \to 0.$$

Splicing this sequence with 9.1.4 gives the desired resolution for M. $\square$

Recall that an involution on a ring R is a ring homomorphism $i : R \to R^{opp}$ satisfying $i \circ i = \mathrm{id}_R$. If such a map exists, then for any left R-module M, the group $Hom_R(M; R)$ has a left R-module structure defined by $(r \cdot f)(m) = f(m)i(r)$ for all $m \in M$. If an R-morphism $\alpha : R^n \to R^m$ is represented by an $m \times n$ matrix A with respect to the standard basis, then the dual map $\alpha^* : (R^m)^* \to (R^n)^*; f \mapsto f \circ \alpha$ is represented by the matrix $i(A)^T$, where $i(A)_{ij} = i(A_{ij})$. This means the double dual α^{**} is also represented by A, so that α^{**} corresponds to α under the canonical isomorphism $R \mapsto R^{**}; r \mapsto \epsilon_{i(r)}$, where ϵ_s denotes the evaluation map $f \mapsto f(s)$ for all $s \in R$.

Theorem 9.1.4. *Let M be a duality module of type FL over an involutive ring R having projective dimension $n \geq 3$. Then,*

1. *the dualising module Δ_M is a duality module of type FL and projective dimension n over R with dualising module M.*
2. *for all $2 \leq k \leq n-1$, the duality map $\delta_M : J \mapsto J^*$ is a bijection $\Omega_k(M) \to \Omega_{n+1-k}(\Delta_M)$.*

Proof. Fix $2 \leq k \leq n-1$ and pick $J \in \Omega_k(M)$. By Theorem 9.1.3, there exists a finite free resolution

$$0 \to F_n \to \cdots \to F_0 \to M \xrightarrow{\epsilon} 0, \tag{9.1.5}$$

such that $J \cong \mathrm{Im}(F_k \to F_{k-1})$. For $r \geq 1$, set $J_r = \mathrm{Im}(F_r \to F_{r-1})$. Then there exist short exact sequences,

$$0 \to J_1 \to F_0 \to M \to 0$$

$$0 \to J_2 \to F_1 \to J_1 \to 0$$

$$\cdots$$

$$0 \to F_n \to F_{n-1} \to J_{n-1} \to 0.$$

Dualising them yields the following, which are again exact:

$$0 \to M^* \to F_0^* \to J_1^* \to Ext^1_R(M; R) \to 0$$

$$0 \to J_1^* \to F_1^* \to J_2^* \to Ext^1_R(J_1; R) \to 0$$

$$\cdots$$

$$0 \to J_{n-1}^* \to F_{n-1}^* \to F_n^* \to Ext^1(J_{n-1}; R) \to 0.$$

$M^* \cong Ext^0(M; R) = 0$ and $Ext^1(J_r; R) \cong Ext^{r+1}(M; R)$ for all $r \geq 1$. Splicing these sequences together therefore gives the long exact sequence,

$$0 \to F_0^* \to \cdots \to F_n^* \to \Delta_M \to 0. \tag{9.1.6}$$

As F_r is finitely generated and free, $F_r \cong F_r^*$ for all r, so that the above sequence above is in fact a finite free resolution of Δ_M. This shows that Δ_M has type FL. In addition, $J^* = J_k^* = \mathrm{Ker}(F_k^* \to F_{k+1}^*)$, which means $J^* \in \Omega_{n+1-k}(\Delta_M)$ and δ_M is a map from $\Omega_k(M) \to \Omega_{n+1-k}(\Delta_M)$ as claimed.

Now dualize again. As dualisation is self inverse for maps between finitely generated free R-modules, dualising 9.1.6 just gives back the original resolution 9.1.5 of M. Thus $J \cong J^{**}$, and the duality map $\delta_{\Delta_M} : \Omega_{n-k+1}(\Delta_M) \to \Omega(M)$ is a left inverse for δ_M. We now prove that Δ_M is a duality module of projective dimension n over R with dualising module M. As precisely the same arguments then apply to Δ_M, this will show δ_M is a left inverse to δ_{Δ_M}, showing that δ_M is indeed a bijection.

Now, dualising 9.1.6 yields the short exact sequences,

$$0 \to \Delta_M^* \to F_n^{**} \to K^* \to Ext^1(\Delta_M; R) \to 0$$

$$0 \to K^* \to F_{n-1}^{**} \to J_{n-1}^{**} \to Ext^1(K; R) \to 0$$

$$\cdots$$

$$0 \to J_3^{**} \to F_2^{**} \to J_2^{**} \to Ext^1(J_3^*, R) \to 0$$

$$0 \to J_2^{**} \to F_1^{**} \to F_0^{**} \to Ext^1(J_2^*, R) \to 0$$

where $K = \mathrm{Im}(J_{n-1}^* \to F_n^*)$. The composition $F_r^{**} \to J_r^{**} \to F_{r-1}^{**}$ is equivalent to $F_r \to F_{r-1}$, while the composition $F_n^{**} \to K^* \to F_{n-1}^{**}$ is just $F_n \to F_{n-1}$.

Thus $Ext^1(K, R) = 0$ and $Ext^1(J_r^*; R) = 0$ for $r = 2, \ldots, n-1$ by the exactness of 9.1.5. As $J_r^* \in \Omega_{n+1-r}(\Delta_M)$ for all such r, and $K \in \Omega_1(\Delta_M)$, this implies $Ext^1(\Delta_M; R) = \cdots = Ext^{n-1}(\Delta_M; R) = 0$. In addition, $Ext^0(\Delta_M; R) = \Delta_M^* = 0$ and $Ext^n(J_2^*; R) \cong Ext^n(\Delta_M, R) \cong M$, so that Δ_M is a duality module with projective dimension n over R and dualising module M as claimed. This completes the proof. □

9.2 Identities for Ext and Tor

Theorem 9.1.4 leads to certain cohomological identities. In particular,

Corollary 9.2.1. *Let M be a duality module of type FL and projective dimension $n \geq 3$ over an involutive ring R. Then for $r = 0, \ldots, n$ and all R-modules N*

$$Ext^r(M; N) \cong Tor_{n-r}(\Delta_M; N) \tag{9.2.1}$$

and

$$Ext^r(\Delta_M; N) \cong Tor_{n-r}(M; N) \tag{9.2.2}$$

Remark 9.2.2. This result is a straight forward generalization of the equivalence $Ext^r_{\mathbb{Z}G}(\mathbb{Z}; N) \cong Tor^{\mathbb{Z}G}_{n-r}(\Delta; N)$ for a duality group G with dualising module Δ. The point here is that, as Δ_M is also duality module over R and has dualising module M, 9.2.2 must also be true.

Proof. Take a finite free resolution $0 \to F_n \to \cdots \to F_0 \to M \to 0$ for M over R. Then, as we have seen, $0 \to F_0^* \to \cdots \to F_n^* \to \Delta_M \to 0$ is a finite free resolution of Δ_M over. As F_r is free and finitely generated over R, there exists a natural isomorphism $Hom_R(F_r; N) \cong F_r^* \otimes N$ for all r. Consequently, the cocomplex

$$0 \to Hom_R(F_0; N) \to \cdots \to Hom_R(F_n; N) \to 0$$

is equivalent to

$$0 \to F_0^* \otimes N \to \cdots \to F_n^* \otimes N \to 0,$$

proving (i). By the theorem above, Δ_M is a duality module of type FL and projective dimension $n \geq 3$ with dualising module M, so 9.2.2 is just 9.2.1 with M replaced by Δ_M. $\square$

According to our definition, a group G is a duality group if and only if, given the trivial G-module structure, $\mathbb{Z}$ is a duality module over $\mathbb{Z}G$. Now the group ring $\mathbb{Z}G$ has an involution defined by $g \mapsto g^{-1}$. So, if in addition, G has type FL and cohomological dimension $n \geq 3$, Theorem 9.1.4 implies that $Ext^r_{\mathbb{Z}G}(\Delta; N) \cong Tor^{\mathbb{Z}G}_{n-r}(\mathbb{Z}; N)$, for all $r \in \{0, \ldots, n\}$ where Δ is the dualising modules of G.

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