Endogenous Social Interactions

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Abstract

This thesis comprises three chapters. All chapters have a common theme which at it's base deals with how interactions in social non-market contexts may shape individual and aggregate economic outcomes.

Chapter 1 entitled "Whom Should I Observe?", studies a model of observational learning in the context of a simple learning problem. Players have heterogeneous preferences over outcomes. A player can learn about the underlying outcome distribution by observing one other player. We characterise optimal link formation. The main result is that some players prefer to form a link with a player who experiments with actions that they are not willing to experiment with themselves. This is interpreted as an informational micro-foundation for a preference for diversity. Applications are discussed.

Chapter 2 is entitled "Are Gifts-in-Kind Inefficient?". It is often argued that gifts-in-kind are inefficient transfers (Waldfogel 1993). We study a model of partnership formation, where players have incomplete information about the desirability of the partnership. Prior to the players simultaneously deciding whether to form the partnership, one player gets a signal of the partner's type and can send gifts, which may be either a gift-in-kind or cash. The model has multiple equilibria. Under certain conditions the efficient equilibrium payoffs involves the transfer of gifts-in-kind. The reason is that gifts-in-kind reveals more to the receiver about the giver's beliefs about the receivers type than do other transfers. An evolutionary argument, in the spirit of Kim and Sobel (1995), is given. In the long run the efficient equilibrium is played with positive probability.

Chapter 3 entitled "Revisiting Schelling's Spatial Proximity Model" formalises the model of Schelling (1969, 1971) of interaction in one-dimensional neighbourhoods. We show, via numerical simulations, that the rest points of the adaptive process tends to select neighbourhood configurations which are relatively segregated in the aggregate. We test the robustness of rest points to the introduction of noise in the adaptive process. The long run prediction is that complete segregation occurs. The model is simulated and results show that the wait until the stochastic process reaches the set of segregated states increases rapidly in the size of the population. Variations, with better long run properties, are suggested and analysed. We also analyse a model where residents have a strict preference for integration. Nevertheless the only stochastically stable states are segregated. We test the robustness of this prediction by allowing for heterogeneity in

preferences. Interestingly this turns the prediction on it's head: only integrated states are stable. Schelling's original model is robust to this pertubation.

Acknowledgments

Writing a thesis is in many respects much like giving birth for the first time. Or at least so I imagine it. You have some vague idea that is going to be painful, so at least your acquaintances tells you. You are note quite sure how long it will take. Complications might arise which requires the intervention of an expert. But in the end, for all your breathing exercises eventually you just have to do it. The thesis has matured and is ready to come out. You work hard, there are periods where you just wish it would end and that someone would just come and take your thesis out with a caesarian, times where you even consider an abnormal termination, times of joy and excitement for having understood something you have wondered about for long.

The conception of the idea of ever producing a thesis comes out of the love and interest of a particular topic. You might have encountered this topic many years ago and then suddenly it pops up again and stirs your mind. You take a decision which is going to affect the rest of your life, or at least, given the limited planning period that I have, the foreseeable future. Sometimes things are less well planned. Sometimes opportunities expose themselves and you decide to go with the ride, probably without fully appreciating the "final consequences" that economists are so familiar with. When I look back at how I ended up here it really feels like I am some particle in a stochastic system. Sometimes I get the opportunity to change the speed or the direction I am going, but most of the time it is a matter of trying to control a process that I feel I have very little control over. This is not necessarily bad, the unexpectedness of the ride is really the joy of life. And so research is when it is at its best. You don't really know what to expect, whether you will encounter a dead end, and have to start over again. But given good guidance and some effort on your part you sometimes have the experience of bliss. It is important to remember and savour those moments.

At your disposal for this laborious exercise of giving birth to a thesis you have various people to assist you. Without them you would surely die in labour, and the thesis with you. Some of them are people that you have known all your life, some are people that you met during the first years in university. Most of these people have a role as moral support, they often have no particular experience with the details of giving birth, but they know you well and can see when you need that extra pad on the shoulder, a hug, a word of encouragement and admiration, and maybe a beer (make that two!). Other people you meet later on in the process, they attend the same group as you, themselves preparing for giving birth. Then there is the midwife of the process, what is sometimes referred to as the supervisor. The midwife plays a very particular role. She or he, for indeed it seems that these days there are quite a lot of male midwifes, is abundant with previous experience with delivering a thesis, and will hopefully be there in your times of need, unless she has been urgently called out to a conference or a delivery at home.

At this late hour I am more than a little concerned that I might miss out on a name or two who rightfully deserves a mention. To avoid future law-suits I even toyed with the idea to let people find their own category in the taxonomy I outlined above. This thesis owes a great deal to the advice, guidance, discussions and poignant insights of my supervisor Tilman Börgers. I thank him for what he has taught me both directly and indirectly. I definitely come out a changed person, and I'd like to think that my training has not only produced what is here today, but also a mind which looks at the world differently. I have also benefitted from conversations with numerous other colleagues at various stages during the process. I thank Hans Carlsson for encouraging me to go to UCL and ongoing support. I thank Steffen Huck whom I am working with a the moment, for inspiring conversations and generously granting me the time to finish writing up the thesis. Without proper support from friends, fellow Ph.D.-students and family, etc. I could not have even attempted to complete this thesis. My thanks goes out to all of you for spurring me on, and being supportive. Each single one of you played an important part. My parents, Anni and Kjeld Bøg, and my sister, Pernille, I thank you for your support and love. Members of DONG-klubben also played an integral part (enough said!). Fellow Ph.D.-students Pedro, Cloda, Fabio, Topi, Cristina also were invaluable at various stages of the process. Residents of 73 Raliegh Rd., 2001-2005, were my surrogate family for extended periods. To those of you I forgot, I do apologise, but know that you have a place in my heart.

Something else that I have come to notice is the similarity with which the experience of producing a thesis affects most of us. Thesis production may truly exhibit business cycles movements, periods of low productivity where some outside intervention from the equivalent of the IMF is required, and periods of exuberance where you have enough to go around for everybody. It is during the downturn periods in particular that the

knowledge that you are not the only one having these feelings is most useful. Human beings are in general quite social beings, except for the odd misanthrope, and in the darkest moments it helps to know that you are not alone, that life is cyclical, and things will one way or another find a resolution. To me this realisation came quite late in the process, but once I had confirmed that this was something almost universal I could take a lot of encouragement from that.

First and foremost producing a thesis is an intellectual journey and a process of self-discovery. I cannot really explain how marvellously privileged it is to be able to dedicate most of your available time to the study of subtle details of an idea that you have had, and which have been shaped and polished in conversations and discussions with your supervisor, friends and colleagues. With privelege comes responsibility. The privilege to let the mind wander, to let it escape from the ongoings of everyday life, and then eventually returning to the "real" world and hopefully have something useful or meaningful to say about how social life is organised, or at least a distilled version of it. To me then doing economic theory has been about creating meaning.

The reader may wonder how the analogy to giving birth entered my head. On the 5th February 2005 my son Malte was born. Observing the 23 hour long ordeal of his mother Carina shall we say planted certain ideas in my head. Thanks for your support and love, you both have a very special place in my heart.

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Chapter 1

Introduction

In recent years economics have according to some started to encroach or colonize the other social sciences. It is certainly true that the methodology of economic analysis is being applied to explain a much wider range of social phenomena than was previously the case. Other mechanisms for allocating resources, apart from the market, is being increasingly recognized. To name but a few the "market" for marriages and social learning and experimentation in networks of say consumers. Psychological phenomena are also attracting increasing attention by economist.

To some extent at least I believe this is due to the success of non-cooperative game theory, as a general methodology for the analysis of strategic interaction.

The chapters contained in this thesis can be viewed as another attempt to expand the domain of economics. In particular they explore such social phenomena as a step toward the analysis of network formation in a heterogenous population, the practice of gift giving as the foundation for building good social interaction, and the aggregate phenomenon of segregation of neighbourhoods.

Much empirical evidence has by now been brought to light that suggests that informal networks of connections between agents, play a major role in determining individual and aggregate economic outcomes. Granovetter (1995 [1975]) was perhaps the first to document empirically that informal contact networks play a crucial role in determining the allocation of jobs. As a more general observation, this is very important lesson. There is usually no market and no direct price for contacts or network connections, instead our choice of what education to take, where to live, what to wear etc. determines to some extent whom we socialise with, form friendships with, and thus from whom we hear about products, jobs, education opportunities, investment opportunities. Information passed through networks can be thought of as either providing information about new choices or actions (such as information about new jobs, products etc.) or more information about choice or actions that the decision maker is already aware of (such as information from neighbours about their experiences with an already known product). The reason why networks are useful is first and foremost that we have incomplete information about what is available and that we face uncertainty over the outcome distribution associated with a particular choice. But there is no reason for why economics should not have anything to say about networks, how they are formed and changes over time in response to the underlying economic conditions.

Chapter 2 explores an information based approach to endogenous link formation. We study a model of observational learning in the context of a simple learning problem. In particular we consider a 2-period bandit problem, where the bandit has either two or three arms. Players have heterogeneous preferences over outcomes. A player can learn about the underlying outcome distribution by observing one other player. We consider two informational settings. One in which the observing player is able to both observe the action and the outcome of the player she is observing, and a setting in which she is only able to observe the action. We characterise whom it is optimal to observe. The main result is that some players prefer to form a link with a player who experiments with actions that they are not willing to experiment with themselves. This is interpreted as an informational micro-foundation for a preference for diversity.

Chapter 3 examines a particular social institution: gift giving. The institution of gift giving is an almost universal cultural institution among human beings. Economic anthropologists have documented the existence of large network based exchange systems such as the Kula Ring in Micronesia studied by Stanislav Malinowski (Malinowski 1978). Mauss (1990) produced a general study of the role that gifts have in human society, noting that gift giving is a "total social phenomenon". In spite of gift giving being such a widespread economic phenomenon only little interest has been taken in it by economists. Camerer (1988) was the first to use game theoretic tools to analyse a gift giving game. Most effort in economics has however been devoted to exploring the efficiency properties of gift giving practices. Waldfogel (1993) argues that gift giving practices around christmas is very inefficient. He estimates a welfare loss between 10% to 33% of the value of the gifts given as valued by the recipients. However Waldfogel's approach is in my view too reductionist. In particular he ignores exactly as pointed out by Mauss that gift giving is a total social phenomenon. It may be that some gifts are given not as a direct transfer from the giver to the receiver, but that gifts are given in order to facilitate social interaction. The chapter studies a model of partnership formation, where players have incomplete information about the desirability of the partnership. Prior to the players simultaneously deciding whether to form the partnership, one player gets a signal of the partner's type and can send gifts, which may be either a gift-in-kind or cash. In general the model has multiple equilibria. We show that under certain conditions the efficient equilibrium payoffs involves the transfer of gifts-in-kind. The reason is that gifts-in-kind reveals more to the receiver about the giver's beliefs about the receivers type than do other transfers, and the additional informational value contained in gifts-in-kind may be sufficient to off-set the inefficiency associated with gifts-in-kind themselves. An evolutionary argument, in the spirit of Kim and Sobel (1995), is given. In the long run the efficient equilibrium is played with positive probability.

In the final chapter we revisit the model neighbourhood interaction of Schelling (1969, 1971). We formalise Schelling's model for one-dimensional residential areas. We then test the robustness of equilibria to persistent shocks via *stochastic stability* (Young 1993), and show that the stochastically stable states are precisely the segregated states. We numerically simulate the model in order to check for the validity of the long run prediction of neighbourhood evoloution. For reasonable noise levels the wait until the segregated states are reached is unacceptably long. We suggest various variations with better long run properties. We then assume that players have a preference for diversity, and show that the segregated states are the only stochastically stable states. Simulations reveal that the problem of waiting times is more severe in this case, and perhaps more worrying for reasonable levels of noise the process spends a significant amount time outside of the segregated states. We then suggest a model with better long run properties. Finally we introduce player heterogeneity to examine the driving force of segregation. We show that with the presence of just one "social activist" who is actively seeking out new opportunities for diverse neighbourhoods to form then the only stable configuration is the fully integrated state. I simulate the model to see how the presence of a few "social activists" can have a positive externality on other players. Interestingly Schelling's original model is robust to this pertubation of the preferences of agents.

Chapter 2

Whom Should I Observe?

2.1 Introduction

Many elements of daily economic life involve an element of learning. Through experimentation we learn what consumer goods we like, what projects to invest in, which jobs we like, where we like spending our holidays, etc. In daily interaction with social relations we often ask for and get given advice, or we merely observe how other people choose and we learn from their experiences. Our social acquaintances may have diverse experiences, perhaps due to differing social circumstances, or differences in tastes. At the same time we are often constrained in how many of our acquaintances we can gather information from. Thus we are faced with a choice problem: Whom to observe? Observing people similar to oneself, could lead to better information about choice alternatives that we are already optimistic about. On the other hand observing people who are dis-similar may provide valuable information about choice alternatives that we would not be willing to consider ourselves, in the absence of more information about these choices. The main aim of this paper is to construct a simple set of models, which allow us to study the question whether a decision maker prefers to observe the experiences of someone similar or dis-similar to herself.

Mankind's evolutionary heritage suggests that we are social beings. Dunbar (1996) argues that the human brain is so large relative to our body mass because we must keep track of many social relations (Dunbar estimates that humans can keep track of about 150 social relations). Socialising with others, and learning from observing their behaviour may be an important component in mankind's celebrated adaptability. According to Social Learning Theory (Bandura 1977) people are able to learn from observing others, not simply through imitating or mimicking behaviour, but adapting the observed behaviour to their own circumstances. This suggests that socialising may have an evolutionary advantage to autarky. One of the advantages may be that we through observing what other people do receive information that make us able to make better informed decisions. This is the channel investigated in this paper. Is it an advantage to associate with people who have different tastes? People with different tastes will generally be choosing different alternatives. If such wide-ranging experimentation is important then associating with people that are dis-similar may be an advantage. One of the models presented in this paper will indeed deliver this insight. However the paper also highlights that observability plays a key role when choosing whether to observe similar or dis-similar players. The reason is this: if it becomes more difficult to infer whether you will like a particular alternative from observing the behaviour of others, then the importance of them being like you increases.

Empirical studies have highlighted the importance of social networks for diffusion of knowledge of new products, adoption of new technologies, educational choice, job opportunies, etc. Granovetter (1995 [1975]) studied how a sample of 282 recent job-changers in the Boston suburb of Newton heard about their current job. About 50% of the sample found their new job via referrals from acquaintances, friends and family, as opposed to more formal channels, such as newspaper adverts, etc. Coleman, Katz, and Menzel (1966) studied how adoption of a new medical drug, tetracycline, among physicians in Illonis state, spread via informal network channels: a doctor who knew and had experimented with the drug and had a positive evaluation of the drug relative to similar drugs would recommend it to his colleagues and friends within the profession. Rapid adoption of the drug took place among physicians in the cities in the sample mediated by social networks. Brown and Reingen (1987) summarising earlier evidence find that word-of-mouth was the single most important source of information for 60% of the people sampled, in the choice of automotive diagnostic center and selection of physicians. Foster and Rosenzweig (1995) study how farmers adopt new crop varieties, relying on own experimentation and learning from observing the experiences of their neighbours. Topa (2001) estimates a model of job-contact networks between neighbourhoods in Chicago. Borjas (1995) estimates a model where individual human capital accumulation depends on the average human capital level in the neighbourhood.

In section 2.2 we study players faced with a two-armed bandit problem[1], one *safe* arm with a known distribution of outcomes, and one *risky* arm with an uncertain distribution

[1]See Berry and Fristedt (1985) for a general and comprehensive study of Bandit Problems.

of outcomes. The problem is particularly simple in that conditional on the state the distribution of outcomes on the risky arm is degenerate. Prior to playing the bandit a player can observe one of her contact's (or friends as we shall call them) experiences playing the bandit. The question we shall ask is: Whom among her friends should she observe? We assume that the player observing knows the cardinal preferences of her friends[2]. Learning from the experiences of others is useful in this context only if the environment is relatively stable, i.e. that the outcome distribution does not change too much between observation and play. For simplicity we assume that the outcome distribution is drawn at the start of the game by nature, and remains fixed throughout. Furthermore we assume that a player can only observe one of her friends[3]. We study two different informational settings: one where both the action taken in each period and the outcome is observable, and one where only the action is observable. The latter environment allows the observer to infer whether the player liked the risky outcome or not relative to the safe outcome, but in general she will not be able to infer the outcome itself.

The main insight of this section is that as the environment becomes increasingly less conducive to observation, i.e. when only actions are observable, the importance of associating with players that are similar becomes more important. The reason is that as the environment becomes less and less conducive to observation the player becomes more and more concerned about making inference errors, i.e. making wrong inference from observed behaviour. If players are similar then this becomes less of an issue. Although outcomes are not observable, the observing player can infer her own preference for the unknown outcome from observed behaviour. This is straightforward. A player with the same preferences over outcomes will be making the same decisions as the observing player would, was it her who was experimenting. In this sense the observing player can free-ride on the experimentation of her friends. Note that the argument also applies to players who have the exact opposite preferences over outcomes, since these players will always take the opposite decision of the observing player, but this is sufficient for inference purposes.

In section 2.3 we add one more risky arm to the bandit. The addition of a second risky alternative adds a second insight to the analysis. Some players may be concerned with wide-ranging experimentation, in the sense that ex-ante the player may be willing to

[2]This is equivalent to knowing the payoffs.

[3]While this is clearly a shortcut, in practice people often rely on a limited set of referrals (e.g. Brown and Reingen (1987) report that 82% of receivers of referrals for piano teachers had only one sender).

consider both risky alternatives, but due to the nature of the decision problem, must choose between them. In this case some players may prefer to learn more about a risky alternative which is not their most attractive action ex-ante. The intuition relies on considering the *marginal value of information* of a risky alternative. Acquiring more information on a particular risky alternative is only valuable if it changes the ex-ante ordering of alternatives. The reason for this is the following. If the player favours a particular risky alternative over another risky alternative, and she receives information that her ex-ante preferred alternatives is indeed better, this has no value to her, since she would have learned this by experimenting with the alternative herself. More information about an alternative which is already ranked highest is only valuable if information is revealed that changes the ex-ante ordering, i.e. information which reveals that the alternative is not attractive.

Applications The model can be interpreted as a framework for institutional design. One of the main insights of the model is that some players may benefit from interacting with players that are different from themselves in the sense that they are willing to try out risky alternatives which they are reluctant to try themselves. In this sense we can interpret our results as an informational preference for *diversity.*

Many social institutions involves interaction within groups. From a designer perspective one can then ask whether interaction in heterogeneous groups is desirable to interaction in homogeneous groups? The insights of the model then has some interesting applications. Examples include the composition of student populations in schools, regulation of and social programmes that affect the composition of housing markets (such as e.g. Movement-to-Opportunity Programmes, see also section 2.4 for further discussion of this point), the design of science parks, etc.

Related Literature The literature most closely related to this paper is the models of Bala and Goyal (1998) and Ellison and Fudenberg (1995). In Bala and Goyal (1998) players belong to a fixed network and can observe the actions and outcomes of her neighbours. In Ellison and Fudenberg (1995) a player receives a random sample of fixed size and observes the action and utility of players in her sample. The main distinction between these models and the model presented in this paper is that we assume that the player has some choice over whom she learns from.

Social learning in two-armed bandit problems is studied by Bolton and Harris (1999) and Cripps, Keller, and Rady (2002). Payoff realisations are publically observable. This leads to the usual incentive problems and underprovision of the public good (here experimentation with a single risky alternative). Strategic free-riding is built directly into our model. By choosing to associate with players who are ex-ante willing to experiment, the observing player free-rides on their experimentation. Bolton and Harris (1999) also identifies an *encouragement* effect. This is a dynamic effect: a higher rate of experimentation by a player today, induces a higher rate of experimentation by others tomorrow. This effect does not appear in our model. Since the player being observed does not move again after the observing player has moved, she has no incentives induce more experimentation. The encouragement effect would reappear in a model where players alternated playing the bandit.

Manski (2002a, 2002b) studies social learning, where the current generation tries to learn from observing the experiences of past generations. Current decision making suffers from the selection problem: decision makers today can only observe the outcome realisations of choices actually taken by past generations. Manski shows that some learning is possible as decision makers can sequentially exclude dominated actions. However whereas in Manski decision makers can observe a population sample of past experiences, our focus is on the choice of whom to observe from a group of players. Moreover our decision problem is dynamic, and allows for individual learning.

Finally, there is a literature which studies network formation as a game between players. In Bala and Goyal (2000) a link has a cost which one player must pay, whereas there is a fixed benefit of the link for both players. Bala and Goyal focus on the relation between equilibrium network structures and social efficiency. Goyal and Vega-Redondo (2000) study a related model, but instead of links having a fixed benefit a coordination game is played between linked players. They show that in equilibrium local conformity is obtained. This paper contributes to this literature on endogeneous network formation. It explores an alternate channel for forming links with other players: information transmission between links. This paper studies this problem in the context of heterogeneous preferences. The papers mentioned above characterises the architecture of equilibrium networks. Our model makes no such attempt, and instead seeks to give more structure to the underlying model of incentives to associate.

This paper is organised as follows. The basic model with two actions is presented and analysed in Section 2.2. Section 2.3 looks at a model with three actions. Section 2.4 offers some concluding remarks.

2.2 One Risky Action

2.2.1 Model

The Underlying Decision Problem is a two-armed bandit problem with a risky (r) and a safe (s) action (the *bandit* in what follows). Nature picks a state $\omega \in \Omega = \{1, 2, 3\}$, from a *prior* distribution $\mathbf{p} \in \{q \in \mathbb{R}^3_{++} | \sum_{\omega \in \Omega} q_\omega = 1\}$. The state is not observed by the decision maker. At each time $t = 1, 2$ the decision maker chooses an action $a_t \in \{r, s\}$. If the state is ω and the risky action is chosen then the outcome is x_ω. If the safe action is chosen then the outcome is x_s irrespective of the state. The set of outcomes is $X = \{x_1, x_2, x_3, x_s\}$.

Decision Makers have a vNM-utility function $u : X \to \mathbb{R}$, in the set of utility functions U. If $u \in U$ then u has the following properties: (1) The utility of the safe outcome: $u(x_s) = 0$. (2) There is at least one state ω where the outcome on the risky arm yields greater utility than the safe outcome: $u(x_\omega) > 0$, for some $\omega \in \Omega$, and at least one state ω' where the safe outcome yields greater utility than the outcome on the risky arm: $u(x_{\omega'}) < 0$, for some $\omega' \in \Omega, \omega' \neq \omega$. (3) If two risky outcomes x_ω and $x_{\omega'}$ yield greater utility than the safe outcome then $u(x_\omega) = u(x_{\omega'}) > 0$. Alternatively if the utility of the safe outcome yields greater utility than two risky outcomes x_ω and $x_{\omega'}$ then $u(x_\omega) = u(x_{\omega'}) < 0$. (1) is just a normalisation, (2) is imposed to make r risky for all players, and (3) is made purely for tractability. When this assumption becomes binding for results, we shall comment on how this assumption can be relaxed.

Fix a decision maker $\mathcal{S}$(tatistician) with some utility-function $u \in U$. We assume that $\mathcal{S}$ has a non-empty set of *friends*, which is denoted F. There is a function: $g : F \to U$, which maps a friend to a utility-function. We assume that $\mathcal{S}$ knows g.

We now specify the game.

The Game

Period 0: Nature picks a state of the world (unobserved by the players) from the prior distribution $\mathbf{p}$. The state remains fixed until the end of the game.

Period 1: $\mathcal{S}$ chooses to observe some f among her friends, $f \in F$.

Periods 2 and 3:	f plays the bandit (a 2-period decision problem). $\mathcal{S}$ either observes the realised outcomes and actions *or* only the actions taken by f.
Periods 4 and 5:	Given the information obtained from observing f, $\mathcal{S}$ plays the bandit (a 2-period decision problem).

Players are Bayesian rational. A player's objective is to maximise her expected sum of returns from the bandit. We look for a *sequential* equilibrium of the game. Note the assumption that the state remains fixed throughout the game.

2.2.2 Whom to Observe?

We shall make two different assumptions about what $\mathcal{S}$ can observe: either $\mathcal{S}$ is able to observe both the outcomes and the actions that her friend takes, or she is only able to observe the actions that her friends take.

In each environment we will ask whether there are any friends which allows $\mathcal{S}$ to achieve the first best, in the sense that after having observed the experiences of this friend $\mathcal{S}$ will be able to infer her ex-post optimal action.

The ex-post optimal action maps a utility function and a state into the action which maximises the expected sum of returns,

$$a^*(u,\omega) : U \times \Omega \rightarrow A = \{s, r\}$$

If $\mathcal{S}$ can infer the ex-post optimal action with probability one after having observed the behaviour of a friend $f \in F$ then f is *ideal*:

Definition 1 (Ideal). *A friend $f \in F$ is* ideal *if $\mathcal{S}$ can infer the ex-post optimal action with probability one after having observed how f plays the bandit.*

If $\mathcal{S}$ has an ideal friend then she can achieve the first best.

We shall also be interested in the question of whom $\mathcal{S}$ should observe if she cannot achieve the first best. This may occur for two reasons: either because the first best is not achievable, or because $\mathcal{S}$ has no friends which allow her to achieve the first best.

We now solve the game.

How Friends Play the Bandit A strategy of a friend is $\sigma \in \{s, r\}$, and denotes the action to be taken in the first period, and when play in the second period is sequentially rational. If $\sigma = r$ then the player is *experimenting*. The worth of a strategy, σ, is denoted

$W(\sigma)$. An optimal strategy, σ^*, maximises the worth of the bandit among all possible strategies:

$$\sigma^* \in \arg\max_{\sigma} W(\sigma)$$

Fix a utility function $u \in U$. Let $\mu \equiv \sum_{\{\omega\in\Omega|u(x_w)>0\}} p_\omega$, be the prior probability that the risky arm gives an outcome which is preferred to the safe outcome. Then we have:

$$\sigma^* = \begin{cases} r & \text{if} \quad \bar{u} \geq \frac{-\underline{u}(1-\mu)}{2\mu} \\ s & \text{otherwise} \end{cases}$$

where $\bar{u}$ is the utility of outcome(s) that are preferred to the safe outcome, and $\underline{u}$ is the utility of outcome(s) which are not preferred to the safe outcome. Note that if the player is indifferent between *experimenting* and choosing the safe alternative, then we assume that she experiments.

Observable Actions and Outcomes

Partition the set of friends into experimenting friends: $F(r) = \{f \in F|\sigma^*(f) = r\} \subseteq F$, and friends who are not experimenting $F(s) = F \setminus F(r)$.

$\mathcal{S}$ has Experimenting Friends If $\mathcal{S}$ has friends who are experimenting then these friends are ideal as the following proposition shows:

Proposition 1 (1^{st} Best). *Suppose Outcomes and Actions are Observable. If $\mathcal{S}$ has experimenting friends, $F(r) \neq \emptyset$, then any friend $f \in F(r)$ is ideal.*

Proof. Suppose a friend f is experimenting. Conditional on the state being ω the outcome is x_ω for sure. Since outcomes are observable to $\mathcal{S}$ she learns from observing f whether she likes the outcome on the risky action relative to the safe outcome with probability one in any state of the world. □

Remark 1. *The result relies on the assumption that outcomes are non-stochastic conditional on the state, and that the state remains fixed throughout the game.*

Remark 2. *From knowing g $\mathcal{S}$ can deduce the optimal strategy of her friends. Conditional on a friend being willing to experiment, her individual taste does not play a role.*

$\mathcal{S}$ has no Experimenting Friends If $\mathcal{S}$ does not have any friends who are experimenting, then the result is:

Proposition 2 (2^{nd} Best). *Suppose Outcomes and Actions are Observable. If S has no experimenting friends, $F = F(s)$, then there are no benefits to observation.*

Proof. If a friend f is not experimenting, then she chooses the safe arm in both periods. Therefore S receives no new information about the state of the risky alternative. □

Observable Actions

We now focus on the case where S has a non-empty set of friends who are experimenting, i.e. $F(r) \neq \emptyset$.

We partition the set of experimenting friends into friends that are *preference-homophile* (p-homophile): that is friends who like the same outcome(s) on the risky arm as S (relative to the safe outcome) or who do not like the outcome whenever S likes it (and vice versa), $F(r, \sim)$. Friends who are contained in the complement: $F(r, \nsim) = F(r) \setminus F(r, \sim)$ are *preference-heterophile* (p-heterophile)[4].

When only actions are observable S can only attempt to infer from f's decision to stay with or switch away from the risky action, whether she will like the outcome. Thus she is concerned about not making inference errors. Observing experimenting friends with different preferences over outcomes will generate different information structures, but all have this generic form:

	ω	ω'	$\tilde{\omega}$
t_1	1	0	0
t_2	0	$\frac{p_{\omega'}}{p_{\omega'}+p_{\tilde{\omega}}}$	$\frac{p_{\tilde{\omega}}}{p_{\omega'}+p_{\tilde{\omega}}}$

where ω, ω' and $\tilde{\omega}$ are the possible states of the world. A signal realisation is whether f stayed with or switched away from the risky action in period 2 of her decision problem. An entry is the posterior belief about a state of the world, conditional on a signal realisation $t \in \{t_1, t_2\}$.

S has p-Homophile Friends The following proposition establishes that if S has friends that are p-homophile, then these friends are ideal. Moreover p-heterophile players are not ideal:

Proposition 3 (1^{st} Best). *Suppose only Actions are Observable. A friend $f \in F$ is ideal if and only if f is experimenting, $f \in F(r)$, and f is* p-homophile, *i.e.* $f \in F(r, \sim)$.

[4]The terms *homophile* and *heterophile* are used by Rogers (1983). Two people who are homophile share some observed social characteristics, such as education, age and gender. If they do not share these characteristics then they are heterophile

Proof. It follows from propositions 1 and 2 that a necessary condition for a player to be ideal, is that she is experimenting. Next note that from the action taken in the second period $\mathcal{S}$ can infer whether f likes the outcome of the risky action relative to the safe outcome.

Now consider p-homophile friends. If f likes the same outcomes as $\mathcal{S}$ then f's second period action is identical to the ex-post optimal action: $a^* = a_2(f)$. If f does not like the outcome whenever $\mathcal{S}$ likes it (and vice versa) then the ex-post optimal action is identical to the action which is not played by f in the second period of f's decision problem: $a^* = A \setminus a_2(f)$. This establishes that such players are *ideal.*

Friends who are p-heterophile are not ideal. Since f and $\mathcal{S}$ are p-heterophile, there is at least one state ω such that $a^*(\mathcal{S}, \omega) = a^*(f, \omega)$, and another state $\omega' \neq \omega$ such that $a^*(\mathcal{S}, \omega') \neq a^*(f, \omega')$. Wlog. assume that $a^*(f, \omega) = a^*(f, \omega') = r$. Both ω and ω' has positive prior probability. Since $\mathcal{S}$ only observes actions she cannot infer from observing $a_2(f) = r$ whether the state is ω or ω', but $a^*(\mathcal{S}, \omega) \neq a^*(\mathcal{S}, \omega')$. Hence $\mathcal{S}$ cannot infer the ex-post optimal action with probability one. □

The intuition for the result is that when players are p-homophile then f's action to either stay with or switch away from the risky action in period 2 is sufficient to infer the ex-post optimal action. For players with the same ordinal preferences the action taken by f in period 2 is perfectly positively correlated with the ex-post optimal action. For players with the opposite ordinal preferences the action taken by f in period 2 is perfectly negatively correlated with the ex-post optimal action. This does not hold for p-heterophile players. Note that S's knowledge of f's cardinal preferences matters for identifying the optimal strategy of f. Conditional an optimal strategy only ordinality matters for inference.

$\mathcal{S}$ has only p-Heterophile Friends Suppose $\mathcal{S}$ can only choose among friends who are experimenting but which are p-heterophile. It follows from proposition 3 that no such players are ideal. $\mathcal{S}$ no longer receives a signal realisation (f's second period action) which is perfectly correlated with the ex-post optimal action, in every state of the world, instead in two states she receives a noisy (but informative) signal realisation.

In other words there are two states, ω' and $\tilde{\omega}$, for which f will take the same action in period two, but for $\mathcal{S}$ the ex-post optimal action is not the same in both states.

Before stating the result observe that, given the prior $\mathbf{p}$, if the action taken by $\mathcal{S}$ in the first period of her decision problem is independent of the signal realisation, then learning through observation has no value to her. Therefore we restrict attention to players for whom observation matters for some possible information structure.

Definition 2 (Observation Matters). *Observation matters to $\mathcal{S}$ if there is at least one information structure where her period one action is not independent of the signal she receives.*

This means that there must be some information which $\mathcal{S}$ could recieve, which would make her change action. A more technical interpretation is that, for a fixed prior, we have imposed an upper and a lower bound on the utility of outcomes on the risky arm. We can now state the proposition.

Proposition 4 (2^{nd} Best). *Suppose only Actions are Observable, and that* Observation Matters *to $\mathcal{S}$. If $\mathcal{S}$ can only observe $f \in F(r)$, such that no friends are p-homophile, i.e. $\forall f : f \in F(r, \not\sim)$, then $\mathcal{S}$ optimally observes f such that the prior probability of being able to infer the ex-post optimal action with probability one is maximised.*

Proof. Let the signal realisation where the ex-post optimal action is known with probability one be t_1, and the signal realisation when it's not be t_2.

Wlog. assume that the action taken after t_1 is r (identical to the ex-post optimal action), and the corresponding state is ω. Since observation matters there is another state ω' with support in t_2 such that the ex-post optimal action is r but the action taken after t_2 is s. When t_2 is received and the state is ω' then $\mathcal{S}$ makes an *inference error*, denoted $\epsilon_{\omega'}$, since the ex-post optimal action is r (and learning stops).

If t_1 is received in state ω then the probability of making an inference error, $\epsilon_{\omega'}$, is:

$$\epsilon_{\omega'} = \frac{p_{\omega'}}{p_{\omega'} + p_{\tilde{\omega}}}$$

where $a^*(\mathcal{S}, \tilde{\omega}) = s$, and if t_1 is received in state ω' then

$$\epsilon_{\omega} = \frac{p_{\omega}}{p_{\omega} + p_{\tilde{\omega}}}$$

Clearly $\epsilon_\omega \geq \epsilon_{\omega'}$ iff $p_\omega \geq p_{\omega'}$. Hence the information structure where t_1 is received in state ω is optimal iff $p_\omega \geq p_{\omega'}$. □

Remark 3. *Note that the argument is only complete because the* cost *of inference error are identical in states ω and ω' (parametrised by $u(x_\omega) = u(x_{\omega'})$).*

Remark 4. *Note that Blackwell's Sufficiency Criterion for Information Structures (see e.g. Kihlstrom (1984)) (which is an incomplete ordering) does not order the information structures.*

We now relax the assumption that the utility of the two different outcomes which are preferred to the safe outcome are identical.

The Marginal Value of Information The *marginal value of information* is defined as the marginal increase in the value of the bandit as a result of receiving a particular information structure. As above assume that the ex-post optimal action in the states $\omega, \omega' \in \Omega$ is r, and s in state $\tilde{\omega}$. Assume that in the absence of learning through observation the optimal strategy is $\sigma^* = r$. Also assume that *observation matters* for $\mathcal{S}$ in both possible information structures. Then the marginal value of information if t_1 is received in state ω is:

$$MV_\omega = -p_{\omega'}(2u(x_{\omega'})) - p_{\tilde{\omega}}u(x_{\tilde{\omega}}) \geq 0$$

and if it is received in state ω':

$$MV_{\omega'} = -p_\omega(2u(x_\omega)) - p_{\tilde{\omega}}u(x_{\tilde{\omega}}) \geq 0$$

hence $MV_\omega \geq MV_{\omega'}$ if:

$$p_{\omega'}u(x_{\omega'}) \leq p_\omega u(x_\omega)$$

The marginal value of information in this case is that $\mathcal{S}$ may switch behaviour to the safe action, when she does not know the state with probability one. The benefit of doing so is that $x_{\tilde{\omega}}$ is avoided, the cost increases proportionally to the likelihood of the state (in the case where $u(x_\omega) = u(x_{\omega'})$), and to the expected value of the state (when $u(x_\omega) \neq u(x_{\omega'})$).

Remark 5. *Results are easily extended to any finite set of outcomes.*

2.3 Two Risky Actions

2.3.1 Model

The Underlying Decision Problem is a three-armed bandit problem with two risky (r_1 and r_2) and a safe (s) action (referred to as the *bandit*). Nature picks a state $\omega = (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 = \{1,2\} \times \{1,2\}$, from a prior distribution, $\mathbf{p} = (p_1, p_2)$, where $p_i \in \{q \in \mathbb{R}^2_{++} | \sum_{\omega \in \Omega_i} q_\omega = 1\}$, $i = 1, 2$. We assume that the arms are independent. The state chosen by nature is unobserved by the decision maker.

At each time $t = 1, 2$ the decision maker chooses an action $a_t \in \{r_1, r_2, s\}$. If arm r_i is chosen and the state of arm r_i is ω, then the outcome is $x_{i\omega}$, $i = 1, 2$. If the safe action is chosen then the outcome is x_s independent of the state. The set of outcomes is $X = \{x_{11}, x_{12}, x_{21}, x_{22}, x_s\}$.

Decision Makers have a vNM-utility function $u : X \to \mathbb{R}$, in the set of utility functions U. Furthermore any $u \in U$ satisfy the following properties: (1) The utility of the safe outcome: $u(x_s) = 0$. (2) On each risky arm, r_i, there is one outcome which is preferred to the safe outcome, and one which is not: $\underline{u}_i \equiv u(x_{i\omega}) < 0 < u(x_{i\omega'}) \equiv \bar{u}_i$, for some $\omega, \omega' \in \Omega_i$, $\omega \neq \omega'$, $i = 1, 2$

Fix a decision maker $\mathcal{S}$ with some utility function $u \in U$. We assume that $\mathcal{S}$ has a non-empty set of *friends*, which is denoted F. There is a function $g : F \to U$, which maps a friend to a utility function in U. We assume that $\mathcal{S}$ knows g.

The Game is identical to the game studied in the section with two actions, we repeat it here for convenience.

Period 0: Nature picks a state of the world (unobserved by the players) from the prior distribution $\mathbf{p}$. The state remains fixed until the end of the game.

Period 1: $\mathcal{S}$ chooses to observe some f among her friends, $f \in F$.

Periods 2 and 3: f pla$\mathcal{S}$ chooses to observe some f among her friends, $f \in F$. $\mathcal{S}$ either observes the realised outcomes and actions *or* only the actions taken by f.

Periods 4 and 5: Given the information obtained from observing f, $\mathcal{S}$ plays the bandit (a 2-period decision problem).

Players are Bayesian rational. A player's objective is to maximise her expected sum of returns from the bandit. Again we look for a *sequential* equilibrium of the game. Note the assumption that the state remains fixed throughout the game.

2.3.2 Whom to Observe?

As in the section with two actions we shall be asking whether there are friends which achieve the first best. We shall also characterise whom should be observed if the first best is not achievable.

How Friends Play the Bandit A strategy $\sigma \in \{s, r_1, r_2\}$ denotes the arm to be tried in the first period, and when play is sequentially rational. The worth of a strategy, σ, is denoted, $W(\sigma)$. The optimal strategy σ^* is the strategy which maximises the worth of the bandit among all possible strategies:

$$\sigma^* \in \arg\max_{\sigma} W(\sigma)$$

Let μ_i be the prior probability that the outcome on arm r_i is preferred to the safe outcome, $i = 1, 2$. We characterise optimal strategies in the following lemma.

Let a period 2 history, h, be the outcome induced by a period 1 action, $h \in X$. By sequential rationality each possible history induces an optimal action to be taken in period 2, $\alpha(h) : X \to A$. Since period 2 is the final period, the optimal action is simply the action that maximises the myopic payoff in period 2.

Fix a utility function $u \in U$. The only non-trivial histories are histories where the period 1 action was to experiment with a risky arm, r_i, $i = 1, 2$, but the outcome was worse than the safe outcome. Denote these two histories h_i if the realised outcome was $x_{i\omega}$ in period 1 and $u(x_{i\omega}) < 0$, $i = 1, 2$.

Given the history h_1 it is optimal to play r_2 in period 2 iff:

$$\mu_2 \bar{u}_2 + (1 - \mu_2)\underline{u}_2 \geq 0 \Leftrightarrow \bar{u}_2 \geq -\frac{(1 - \mu_2)}{\mu_2}\underline{u}_2 \equiv \kappa_2$$

otherwise s is the optimal action.

Equivalently the optimal action is r_1 after history h_2 iff:

$$\bar{u}_1 \geq -\frac{(1 - \mu_1)}{\mu_1}\underline{u}_1 \equiv \kappa_1$$

and s otherwise. All other continuations are trivial.

We can now derive optimal strategies.

Lemma 1. *The optimal strategy σ^* is given by:*

1. *If $\bar{u}_1 < \kappa_1$ and $\bar{u}_2 < \kappa_2$:*

$$\sigma^* = \begin{cases} s & \text{if} \quad \bar{u}_i < \frac{\kappa_i}{2},\ i = 1, 2. \\ r_1 & \text{if} \quad \bar{u}_1 \geq \frac{\kappa_1}{2} \text{ and } \bar{u}_2 \leq \frac{2\mu_1\bar{u}_1 - \mu_1\kappa_1 + \mu_2\kappa_2}{2\mu_2} \\ r_2 & \text{otherwise} \end{cases}$$

2. *If $\bar{u}_1 \geq \kappa_1$ and $\bar{u}_2 < \kappa_2$:*

$$\sigma^* = \begin{cases} r_1 & \text{if} \quad \bar{u}_2 \leq \frac{(1+\mu_2)\mu_1\bar{u}_1 - \mu_2(\mu_1\kappa_1 - \kappa_2)}{2\mu_2} \\ r_2 & \text{otherwise} \end{cases}$$

3. If $\bar{u}_1 < \kappa_1$ *and* $\bar{u}_2 \geq \kappa_2$*:*

$$\sigma^* = \begin{cases} r_1 & \text{if} \quad \bar{u}_2 \leq \frac{\mu_1(2\bar{u}_1 - \kappa_1 + \mu_2\kappa_2)}{\mu_2(1+\mu_1)} \\ r_2 & \text{otherwise} \end{cases}$$

4. If $\bar{u}_1 \geq \kappa_1$ *and* $\bar{u}_2 \geq \kappa_2$*:*

$$\sigma^* = \begin{cases} r_1 & \text{if} \quad \bar{u}_2 \leq \frac{\bar{u}_1\mu_1(1+\mu_2) - \mu_1\mu_2(\kappa_1 - \kappa_2)}{\mu_2(1+\mu_1)} \\ r_2 & \text{otherwise} \end{cases}$$

The following example illustrates optimal strategies for a given prior distribution over outcomes, and a given subset of utility functions.

Example 1. *Lemma 1 is illustrated in figure 2.1 for utility functions of the form:* $u_{11}.u_{21} > 0$ *and* $u_{12} < u_{22} < 0$ *is fixed. Also* $p_{11} = p_{21} = \mu$.

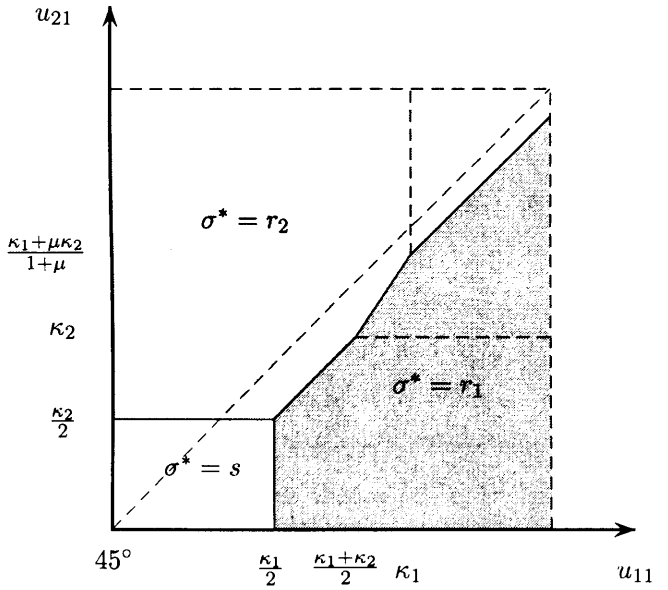


Figure 2.1: Optimal Strategies as characterised by lemma 1 in (u_{11}, u_{21})-space. Drawn for fixed $u_{12} < u_{22} < 0$, $u_{11}.u_{21} > 0$ and prior $p_{11} = p_{21} = \mu$. *Yellow*: Experimentation with r_2. To the right of vertical dashed line: willing to experiment with r_1 in period 2 if r_2 reveals x_{22} in period 1. *Lightgray*: Experimentation with r_1. Above horizontal dashed line willing to experiment with r_2 in period 2 if r_1 is reveals x_{12} in period 1.

Observable Outcomes and Actions

Friends Who are Ideal The next proposition shows how ideal friends play the bandit:

Proposition 5 (1^{st} Best). *Suppose Outcomes and Actions are Observable, and that S' preferences satisify* $u(x_{i\omega_i}) \equiv \bar{u}_i > \bar{u}_j \equiv u(x_{j\omega_j}) > 0$, $i, j = 1, 2$, $i \neq j$. *A friend f* *is ideal if-and-only-if* $\sigma^*(f) = r_i$ *and after history* $h = \{x_{i\omega_i'}\}$, $\omega_i' \neq \omega_i$, f *takes action* $\alpha(h) = r_j$.

Proof. Wlog. assume that the most preferred outcome of $\mathcal{S}$ is on arm r_1 and let this outcome be $x_{1\omega_1}$. Hence for $\mathcal{S}$: $\bar{u}_1 > u(x_{2\omega_2}) \equiv \bar{u}_2 > 0 \equiv u(x_s)$.

Consider observing a friend, f, such that $\sigma^* = r_1$ and after history $h = \{x_{1\omega_1'}\}$, $\omega_1' \neq \omega_1$ the optimal action for f is $\alpha(x_{1\omega_1'}) = r_2$. Such a friend is ideal by the following argument. If f plays r_2 in period 2, then $\mathcal{S}$ learns the state of arm r_2, and hence whether the outcome on r_2 is preferred to the safe outcome. Furthermore she knows that the outcome on r_1 is not $x_{1\omega_1}$ (otherwise f would not have switched arm in period 2). Thus she learns that the ex-post optimal action. If f also plays r_1 in period 2, then $\mathcal{S}$ knows that the ex-post optimal action is r_1 (otherwise f would have switched arm in period 2), and $\bar{u}_1 > \bar{u}_2$. Hence $\mathcal{S}$ learns the ex-post optimal action with probability one.

Now we establish that only such friends are ideal. Clearly any f such that $\sigma^* = s$ are not ideal.

Consider friends such that $\sigma^* = r_1$, but $\alpha(x_{1\omega_1'}) = s$. Such friends are not ideal, since after $h = \{x_{1\omega_1'}\}$ $\mathcal{S}$ only learns that $a^*(\mathcal{S}) \in \{s, r_2\}$. A friend with optimal strategy $\sigma^* = r_2$, but $\alpha(h) = s$ for some history $h \in \{x_{2\omega_2}, x_{2\omega_2'}\}$ is not ideal since after history h either $a^*(\mathcal{S}) = \{s, r_1\}$ or $a^*(\mathcal{S}) \in \{r_1, r_2\}$.

Now consider a friend f such that $\sigma^* = r_2$ and $\alpha(h) = r_1$ for some $h \in \{x_{2\omega_2}, x_{2\omega_2'}\}$. First suppose $h = \{x_{2\omega_2}\}$. When the realised outcome on r_2 is $x_{2\omega_2'}$ (as revealed by f's behaviour) then $\mathcal{S}$ learns that $a^*(\mathcal{S}) \in \{s, r_1\}$. Then suppose $h = \{x_{2\omega_2'}\}$. When the realised outcome is $x_{2\omega_2}$ then $\mathcal{S}$ learns that $a^*(\mathcal{S}) \in \{r_1, r_2\}$. Finally consider a friend f, such that $\sigma^* = r_1$ but after $h = \{x_{1\omega}\}$ f optimally plays $\alpha(h) = r_2$. Then after observing that the outcome is $x_{1\omega'}$ on r_1 then $\mathcal{S}$ learns that $a^*(\mathcal{S}) \in \{s, r_2\}$. □

The intuition for the result is straigthforward. $\mathcal{S}$ will only need to learn the state of both arms (in order to be able to infer the ex-post optimal action), in states of the world where the ex-post optimal action is not on the arm that f experiments with in the first period. Thus f should be willing to switch to the other risky arm in states where $\mathcal{S}$ herself would want to switch away.

Remark 6. *Note that the characterisation in proposition 5 does not rely on the prior distribution.*

Remark 7. *Note that by restricting the space of outcomes on each risky arms to two, we lose the distinction between* p-homophile *and* p-heterophile. *In essence all friends* *are, conditional on a particular arm,* p-homophile. *Note however that if a friend is* ideal *it is no longer sufficient that she is* p-homophile *she must have the same ordering as S on the risky alternative she experiments with in period 1.*

Example 2. *The characterisation of ideal friends given in proposition 5, is illustrated in figure 2.2. for the same utility function and prior distribution as in example 1. We have superimposed the characterisation on figure 2.1 characterisering optimal strategies with no learning through observation.*

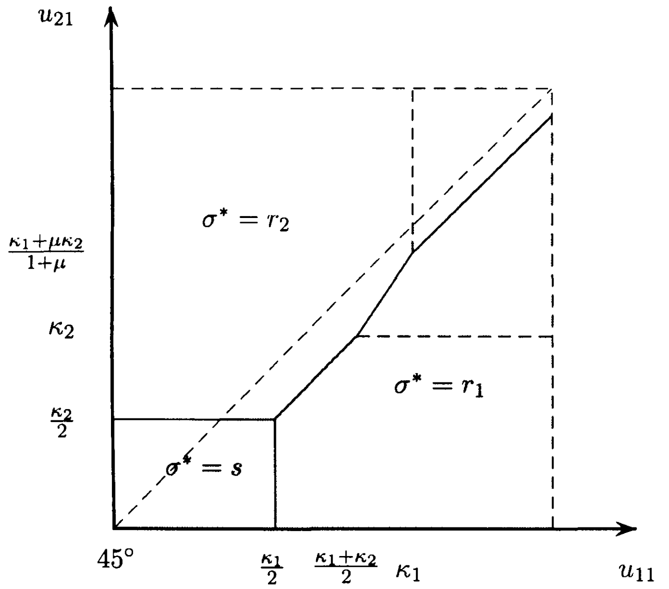


Figure 2.2: Ideal friends in (u_{11}, u_{21})-space. Drawn for $u_{12} < u_{22} < 0$, $u_{11}, u_{21} > 0$, and prior: $p_{11} = p_{21} = \mu$. *Gray*: An ideal friend experiments with r_1 in period 1, and experiments with r_2 in period 2, if r_1 reveals x_{12}. *Transparent*: An ideal friend experiments with r_2 in period 1, and experiments with r_1 in period 2, if r_2 reveals x_{22}.

Choosing when Friends are not Ideal Consider the case where S has no friends who are ideal. To make the problem interesting we assume that S only has friends who are willing to experiment with at most one risky alternative, i.e. we consider friends who uses an optimal strategy $\sigma^* \in \{r_1, r_2\}$, and $\alpha(h) = s$ after any non-trivial history. We distinguish between friends who are *action-homophile* (a-homophile) and *action-heterophile* (a-heterophile). A friend, f, is a-homophile if her action in the first period of her decision problem is the same as the action S would take in the absence of observational learning, that is ex-ante both S and f rank r_i above r_j, $i \neq j$. Otherwise she is a-heterophile.

Interestingly the following proposition establishes conditions under which some decision-makers may prefer to gain more information on a risky alternative which is not preferred ex-ante, that is associate with a-heterophile friends.

Proposition 6 (2^{nd} Best). *Suppose Outcomes and Actions are Observable, and that $\mathcal{S}$ only has friends with optimal strategies $\sigma^* = r_i$, $i = 1, 2$ and $\alpha(h) = s$ after any non-trivial history. Fix an ordinal preference ordering for $\mathcal{S}$, and $\underline{u}_1, \underline{u}_2 < 0$. There exists $\bar{u}_1, \bar{u}_2$ such that $\sigma^*(\mathcal{S}) = r_i$, and $\mathcal{S}$ find it optimal to observe $\sigma^*(f) = r_j$ $i \neq j$.*

Proof. The proof is via example, and is relegated to appendix A.2. □

Two interpretations will be offered. The first relies on the opportunity cost of information, that is what a player could have learned had she observed a player with a different behaviour. The second interpretation relies on the marginal value of information.

The Opportunity Cost of Information To gain some intuition for the result recall that a state is a collection of state realisations for each of the risky arms: $(\omega, \omega') \in \Omega_1 \times \Omega_2$. Let the prior probability of this realisation be denoted $p(\omega, \omega')$.

Let $V_i(\omega, \omega')$, $i = 1, 2$ be the payoff which is obtained if $\mathcal{S}$ learns the i'th element of (ω, ω'), and play is sequentially rational.

For $i, j = 1, 2$, $i \neq j$ define:

$$c_i(\omega, \omega') = \begin{cases} 0 & \text{if } V_i(\omega, \omega') \geq V_j(\omega, \omega') \\ V_j(\omega, \omega') - V_i(\omega, \omega') & \text{otherwise} \end{cases}$$

$c_i(\omega, \omega')$ can be interpreted as the *opportunity cost* of learning arm r_i when the state is (ω, ω'). Note that $c_i \geq 0$. If $c_i(\omega, \omega') > 0$ then $\mathcal{S}$ makes an inference error in state (ω, ω') when she learns the state of arm r_i. The cost depends upon what she could have learned had she learned the state of arm r_j instead.

$\mathcal{S}$ should learn the state of arm r_i iff:

$$C_i \equiv \sum_{(\omega,\omega')} p(\omega, \omega')c_i(\omega, \omega') \leq \sum_{(\omega,\omega')} p(\omega, \omega')c_j(\omega, \omega') \equiv C_j$$

As an example take a decision maker $\mathcal{S}$ such that if $\mathcal{S}$ only has her prior information then $\sigma(\mathcal{S}) = s$, i.e. she plays the safe action in both periods. Now she can choose between learning the state of arm r_1 or r_2. Note that since arms are independent the ordering of arms s and r_j are unaffected by gaining information about arm r_i, $i \neq j$.

Suppose wlog. that $\bar{u}_1 > \bar{u}_2$. Then $C_1 = (1-\mu_1)\mu_2 2\bar{u}_2$, since $\mathcal{S}$ does not take the (constrained) ex-post optimal action (r_2) if she learns that the safe outcome is preferred to the outcome on r_1. $C_2 = \mu_1\mu_2(2\bar{u}_1 - 2\bar{u}_2) + \mu_1(1-\mu_2)2\bar{u}_1$ since if r_2 is learned then in the state where both risky outcomes are preferred to the safe outcome, $\mathcal{S}$ does not take the (constrained) ex-post optimal action, and likewise when she learns that the safe outcome is preferred to the outcome of r_2.

Hence $\mathcal{S}$ prefers to see experimentation with arm r_1 if $\mu_1\bar{u}_1 \geq \mu_2\bar{u}_2$.

The Marginal Value of Information Intuition may also be gained by considering the marginal value of information. Let MV_i denote the *marginal value of information* on arm r_i, $i = 1, 2$. Since the model is Bayesian the value of information is always non-negative.

Consider first the case where $\mathcal{S}$ is not willing to experiment with any of the risky alternatives ex-ante: $\bar{u}_i \leq \frac{\kappa_i}{2}$. $i = 1, 2$. In this case the marginal value of information on arm r_i is:

$$MV_i = \mu_i(2\bar{u}_i),\ i = 1, 2$$

that is learning about an alternative is beneficial because the player learns about outcomes which are preferred to the safe outcome.

Next consider the case where $\mathcal{S}$ experiments with r_2, and chooses action s if the outcome she gets in period 1 is worse than the safe outcome. Furthermore she ranks action s above r_1 in period 1. In terms of parameter restrictions this case is: $\bar{u}_2 \geq \frac{\kappa_2}{2}$, $\bar{u}_1 \leq \frac{\kappa_1}{2}$. We have:

$$\begin{aligned} MV_1 &= \begin{cases} \mu_1\left[2\bar{u}_1 - (\mu_2 2\bar{u}_2 + (1-\mu_2)\underline{u}_2)\right] & if \quad \bar{u}_2 \leq \bar{u}_1\frac{1+\mu_2}{2\mu_2} + \frac{\kappa_2}{2} \\ \mu_1\left[(1-\mu_2)\bar{u}_1\right] & otherwise \end{cases} \\ MV_2 &= -(1-\mu_2)\underline{u}_2 \end{aligned}$$

The value of information on r_1 (the lower ranked risky alternative ex-ante) is either (1) r_1 is revealed to give an outcome which is preferred to the safe outcome, and this is preferred to experimentation on r_2, (2) r_1 is revealed to give an outcome which is preferred to the safe outcome, but experimentation on r_2 is preferred, if r_2 yields an outcome which is worse than the safe outcome, then the player can switch to r_1 in period 2. The value of information on r_1 is thus related to revealing outcomes which are good on r_1. It is related with more frequent play of that risky action. On the other hand, the value of information on r_2 is related to the revelation of outcomes which are worse than the safe outcome. Thus learning more about a risky alternative which is preferred ex-ante is related to less frequent play of that risky alternative.

Consider now the case $\kappa_1 > \bar{u}_1 \geq \frac{\kappa_1}{2}$, $\bar{u}_2 \geq \frac{\kappa_2}{2}$. Then we have:

$$
\begin{aligned}
MV_1 &= \begin{cases} \mu_1 \left[2\bar{u}_1 - (\mu_2 2\bar{u}_2 + (1-\mu_2)\underline{u}_2)\right] & if \quad \bar{u}_2 \leq \bar{u}_1 \frac{1+\mu_2}{2\mu_2} + \frac{\kappa_2}{2} \\ \mu_1 \left[(1-\mu_2)\bar{u}_1\right] & otherwise \end{cases} \\
MV_2 &= (1-\mu_2)\left[(\mu_1 2\bar{u}_1 + (1-\mu_1)\underline{u}_1) - \underline{u}_2)\right]
\end{aligned}
$$

Observe that the value of information on r_1 is the same as the previous case. Learning about r_2 is valuable when it is revealed that the outcome on r_2 is worse than the safe outcome. In this case information has value because it induces experimentation with r_1.

Finally consider $\bar{u}_i \geq \kappa_i$, $i = 1, 2$. Then we have:

$$
\begin{aligned}
MV_1 &= \begin{cases} \mu_1 \left[2\bar{u}_1 - (\mu_2 2\bar{u}_2 + (1-\mu_2)(\underline{u}_2 + \bar{u}_1))\right] & \\ \quad -(1-\mu_1)\left[(1-\mu_2)\underline{u}_1\right] & if \quad \bar{u}_2 \leq \bar{u}_1 \frac{1+\mu_2}{2\mu_2} + \frac{\kappa_2}{2} \\ 0 & otherwise \end{cases} \\
MV_2 &= (1-\mu_2)\left[\mu_1 2\bar{u}_1 + (1-\mu_1)\underline{u}_1 - (\underline{u}_2 + \bar{u}_1)\right]
\end{aligned}
$$

The value of information on r_1 is composed of a benefit when r_1 is revealed to give an outcome which is preferred to the safe outcome. Now there is also an advantage to knowing when r_1 is in a worse state, because $\mathcal{S}$ is willing to experiment with r_1, now she can switch to the safe action instead. The value of information on r_2 is again that it improves decision making when the outcome of r_2 is worse than the safe outcome.

The program:

$$\max_{\sigma} W(\sigma)$$

gives an ex-ante ordering of alternatives to play in period 1, $\succ^a$. Suppose the ordering is $r_i \succ^a s \succ^a r_j$. By sequential rationality learning stops once the player adopts the safe action. What is the value of learning about r_i? Since r_i is the ex-ante preferred action, learning that r_i is indeed better than s, does not add to the marginal value of information, since the player would have learned this even in the absence of learning through observation. But learning that the outcome is worse than the safe outcome is valuable, since the player can switch to the safe action, with the ex-post ordering, $\succ^p$, being: $s \succ^p r_i$, $s \succ^p r_j$, the latter following from independence.

What is the value of learning about r_j? Learning that r_j is worse than s as she expected, ex-ante, does not have value, since she would never have experimented with the alternative, as learning stops in the event that she chooses s. The value then lies in observations which shifts the ex-post ordering of r_j above s. Such an observation implies: $r_j \succ^p s$, and depending on the parameters either $r_i \succ^p r_j$ or $r_j \succ^p r_i$. In the latter case the effect is a first-order effect, since it affects the choice of action in the first period. In the former case the effect is second-order, since the new information is only used if r_i fails.

Now suppose the ex-ante ordering is $r_i \succ^a r_j \succ^a s$. Learning about r_i has the same value as above, that is only observations which ensure: $s \succ^p r_i$ add to the marginal value. New information about r_j now adds to the marginal value regardless of the direction of the information. However information suggesting that $r_j \succ^p r_i$ is a first-order effect, whereas information suggesting $s \succ^p r_j$ is a second-order effect.

More information on an alternative has value only in as much as the information affects the ex-post ordering of alternatives relative to the ex-ante ordering, with positive probability.

The insight that a decision maker may want to learn about a different alternative than that which she favours ex-ante can be related to the literature in information economics on biased contests (Meyer 1991). In this literature when a decision maker has to decide between which of two agents to promote, she performs sequential contests between the agents. In the final contest the decision maker optimally biases the contest in favour of the "current leader", i.e. the one she would promote if there were no more contests. To see the intuition take the simplest example where the are two contests. If there is no bias in the second contest, then if the current leader does not win, then the last contest does not aid the decision maker. However if she biases the tournament in favour of the current leader, and the current loser wins it, then she should optimally promote her, and promote the leader otherwise. Along similar lines in our model a player may want to observe someone quite different from herself, i.e. introduce a bias between her own preferences and the person she is observing because of the information this may reveal. The bias may reveal valuable information about alternatives that she would not learn about otherwise.

In the following box we discuss how the insights developed here, generalises to the case where there are no revealing outcomes.

Non-Revealing Outcomes To fix ideas suppose that a risky alternative may be in one of two states, H and L, and that the outcome distribution generated under H is preferred to the safe outcome, whereas the outcome distribution generated under L is not (for a particular player). Suppose that some regularity condition (such as a MLR-property) is satisfied, such that if the state of an alternative is H then the player is more likely to receive information to that effect.

Suppose that there are no *revealing* outcomes that is even if an arm is in a H state the posterior belief reaches one.

Consider first learning about the alternative which is ranked top ex-ante. Because we will have incomplete learning in general this means that receiving information indicating that the state is H, will make experimentation with that alternative more *persistent*, thus the player is more likely to be staying with the alternative when the state is H. Alternatively information suggesting the state is L will make experimentation less persistent, in expected terms. Thus information suggesting that the state is either H or L now adds to the marginal value of information.

What about information on an alternative which is not preferred ex-ante? If the alternative is ranked below the safe alternative then information which pushes it above the safe action in the ex-post ordering is valuable. In the model examined in this paper only one piece of information is necessary for this to occur. Receiving information supporting that the state is H is valuable in as far as it shifts the ranking of the alternative above the safe action. This ensures that some experimentation takes place with positive probability. On the other hand information suggesting L information is of no value in this case.

If the second ranked alternative is a risky alternative, then H information could lead to the action becoming first ranked (1st order effect). Even if this does not happen if learning on the first ranked alternative stops it will make experimentation more persistent with the alternative (2nd order effect). L information can either shift it below the safe action, or preserve the rank. In either case these effect are of the second order. They will in expected terms diminish the time spent on experimentation with the alternative.

Example 3. *Again consider utility functions of the type* $u_{11}, u_{21} > 0$, $u_{12} < u_{22} < 0$, *and* $p_{11} = p_{21} = \mu$. *Also assume that* $\frac{1-\mu}{2}u_{12} < u_{22} < 0$. *Figure 2.3 illustrates proposition 6.*

Observable Actions

Suppose that only the actions taken by a friend, f, is observable. In this case the action taken in the second period by f is only informative about whether f liked the outcome she received in period 1 of her decision problem, but S receives no new information about the outcome experienced by f in period 2. Hence there are no ideal friends. Furthermore since the period 2 action is only informative about the period 1 outcome, the same can be learned from observing a player who is willing to experiment with at

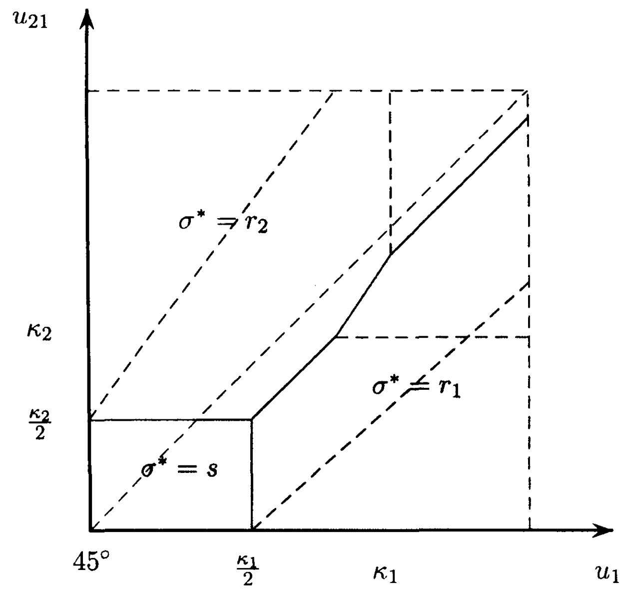


Figure 2.3: Choosing among Friends who use either strategy $\sigma^*(f) = r_1$ or $\sigma^*(f) = r_2$, and $\alpha(h) = s$ after any history h such that the outcome on the arm that was played in period 1 is worse than the safe outcome for f. Drawn for $p_{11} = p_{21} = \mu$, $\frac{1-\mu}{2}u_{12} < u_{22} < 0$ fixed, and $u_{11}, u_{21} > 0$. The dashed lines intersecting axis at $\frac{\kappa_i}{2}$, $i = 1, 2$: if $\bar{u}_i$ lies to the right of (alternatively above) the line then $\mathcal{S}$ prefers to experiment with alternative r_i conditional on knowing that the outcome on arm r_j, $i \neq j$ is preferred to the safe outcome. *Yellow*: It is optimal to observe $\sigma^*(f) = r_1$. *Transparent*: It is optimal to observe $\sigma^*(f) = r_2$.

most one alternative, as can be learned from a player who is willing to experiment with two alternatives.

The characterisation is then equivalent to the characterisation given in the previous section (with observable actions and outcomes), where $\mathcal{S}$ can only choose among friends who are willing to experiment with at most one alternative. We thus present this result as a corollary to proposition 6.

Corollary 1. *Suppose that $\mathcal{S}$ can only observe the actions taken by her friends, and that she has friends with any admissible optimal strategy. Then*

1. *Conditional on observing a friend with $\sigma^* = r_i$, $i = 1.2$ $\mathcal{S}$ can learn the same from observing a friend with $\alpha(h) = s$ as she can from observing a friend with $\alpha(h) = r_j$, $j \neq i$ after any non-trivial history h,*

2. *$\mathcal{S}$ has no ideal friends,*

3. *Under the conditions stated in proposition 6, if it is optimal to observe some $f \in F$, then it is also optimal to observe her when only actions are observable.*

2.4 Concluding Remarks

This paper has investigated a specific reason for why people who live in an uncertain environment can benefit from associating with each other: they might learn from what other people do, which may improve decision making and welfare. There are two main insights of the paper. The first is that when choosing whom to associate with a player who is concerned about wide-ranging experimentation might want to associate with players that are different from herself, in the sense that they are willing to experiment with alternatives that she is not willing to experiment with herself. The other key insight relates to observability. The less a player can infer from observing others, i.e. the less conducive the environment is to learning from others, the more important becomes the consideration that players are similar. This ensures that the player will be able to make good inference from what she learns.

The predictions of the model lend support to some policy interventions. Particularly in many great cities of the United States where ghettoes, with associated bad outcomes of crime and unemployment, predominate, there has been several policy experiments aimed at improving access to economic opportunities.

The *Gatreaux* program in Chicago is one such program (see Rosenbaum (1992, 1995). See also Moffitt in Durlauf and Young (2001, chap. 3)). In these programs residents voluntarily put their names on a list in order to qualify for relocation. The empirical strategy was to make these as close to a natural (randomised) experiment as possible. However since reassignment was voluntary it is doubtful whether the identifying assumption holds. Aside from such methodological problems Rosenbaum finds that residents who ended up (where randomised into) in relatively affluent suburban areas were more successful along a number of economic dimensions. Self-selection into such a programme, and associated good outcomes is indeed what the model presented in this paper delivers. But the model also offers an important caveat to these conclusion. If informational reasons is the main determinant of selection, then people who select into relatively affluent neighbourhoods may not only benefit from from the move, but they may have exerted a positive externality in the neighbourhood they have left. In the context of the model presented here people moving under the program can be interpreted as players that are particularly willing to try out behaviour different from the average behaviour of the population. These players thus benefit more from observing players that are doing something different, giving them a greater incentive to sign up for relocation. Residents remaining now become more constrained in whom they can observe, leading to the conjecture that they experience worse outcomes. If such feedback is present then the parameter of interest is not the average effect of treatment on the treated but rather the average effect of treatment on the population which had the opportunity to participate in the programme.

The insights of the paper may also be related to the literature on biased tournaments

In it's current form the model is not well suited to address structural breaks; the state of the world is drawn at the beginning of play and remains fixed throughout play. Extending the model to allow for structural breaks, may help explain the determinants of social change. As an example of what we have in mind consider the seminal study of an American inner-city ghetto by Anderson (1990, 1999). Anderson explores how a relatively safe and vibrant city neighbourhood transforms as manually unskilled labour jobs gradually disappear. The residents had enjoyed safety and good job-opportunities for many years, but the neighbourhood gradually transformed into a ghetto centered around the drug-economy, as the relatively few who were successful in the conventional economy left the neighbourhood at a faster rate than they were moving in. Adolescents were now emulating behaviour of young and relatively "succesful" participants in the drug-economy. Concurrently they were rejecting the (moral) authority of "old heads", who were still preaching decent (middle-class) values and a strong work ethic in the regular economy.

Such extensions are left for future research.

Chapter 3

Are Gifts-in-Kind Inefficient?

3.1 Introduction

Why do people in a variety of social contexts give gifts-in-kind rather than a monetary gift? A standard microeconomic argument is that gifts-in-kind are inefficient since the gift might not be exactly what the receiver wants, that is the receiver knows her own preferences better than the giver does. By this argument giving a monetary gift is efficient, since a monetary transfer allows the receiver to buy what she likes best.

Following this argument, there has been some interest in estimating the welfare loss of the supposedly inefficient practice of gift giving. Waldfogel (1993) attempts to empirically measure the welfare loss of christmas gift giving. Waldfogel bounds the welfare loss between 10% and 33% of the value of the gifts given[1]. Given the volume of gift giving (according to estimates presented by Waldfogel holiday gift expenditures in the US totalled $38 billion in 1992) this amounts to a significant welfare loss estimated between $4-13 billion[2].

Waldfogel is merely interested in measuring the welfare loss of objects chosen, but leaves the possibility open that receivers may value the act of gift receipt. It is important for Waldfogel's research methodology that these two questions can be separated. The question is whether it is meaningful, or in other words whether this is the parameter of

[1]Along similar lines Waldfogel (2004) presents evidence that own purchases generate between 10% to 18% more value per dollar spent than do gifts.

[2]Waldfogel's work spurred an in the main empirical debate about this purported welfare loss. The debate can be followed in Solnick and Hemenway (1996), List and Shogren (1998), Ruffle and Tykocinski (2000), with replies in Waldfogel (1996, 1998). Using different data and differing methodologies Solnick and Hemenway (1996) and List and Shogren (1998) find that gift giving is perhaps somewhat surprisingly welfare-improving relative to giving money, i.e. the value of an object when given as a gift is valued higher than the monetary equivalent of the object.

interest, in the context of gift giving practices? In the model presented here the value of the gift has two components: an intrinsic value, that is the value of the gift in itself to the receiver, and an informational (extrinsic) value which arises endogeneously from the equilibrium being played, and comes from what the gift can achieve in terms of engaging in social interaction over and above what could be achieved in the absence of gift giving. This is an important correction to the recent empirical economic literature on the practice of gift giving: gift giving takes place in a social context and it cannot meaningfully be studied in isolation from the social context in which it takes place. Simply observing that gift giving is inefficient seen as an isolated social phenomenon does not imply that the institution itself is inefficient: gift giving might be the first link in a longer chain of social exchange. As such this paper provides a theoretical challenge to the observation that giving gifts-in-kind is an inefficient practice.

There is however one sense in which Waldfogel (1993)'s estimates may be useful. If one takes the view that evolutionary pressures select efficient social institutions, then a possible interpretation of the estimated welfare loss in Waldfogel (1993) is that it is a lower bound on the value of social interaction itself. That is people are willing to trade-off inefficient gifts for more efficient social interaction.

Motivations for giving gifts varies. Transfers from parents to their offspring arguably contain an altruistic or perhaps consumption smoothing (i.e. mutual insurance) element. A reason for giving gifts-in-kind in such a context might be paternalistically motivated. There are other contexts where gifts-in-kind with a high generic element are given, such as the widespread practice in middleclass western societies of bringing chocolate, flowers, a bottle of wine, etc. in social encounters. In some contexts however it is important that a gift is kind is not generic, e.g. that the gift embodies that the giver has taken time to think about what the receiver might like. The "wrong" gift might bring the relationship to a premature end, whereas the "right" gift might signal the beginning of a long relationship.

In a simple and stylised model we show that if there is some uncertainty about whom to engage in social exchange with then the practice of gift giving might increase the volume (and quality) of social interaction. Gift giving is seen here to occur at the initial stages of a possibly long-term relationship. Two players from separate populations, G and R, meet and may engage in a long-term relationship/joint project. They bring personal characteristics to the project and the match of these characteristics determine how productive the relationship will be. Personal characteristics (formally a type) are private information and cannot be directly communicated to the other player. Players know the prior distribution of types in the population; ex-ante social interaction is not worthwhile. During their first encounter G receives a signal about R's type. When they meet again she may bring R a gift (at some exogenously fixed cost $c > 0$), which can be either in cash or a gift-in-kind. Next, G and R simultaneously decide whether to engage in the long-term project. The project only goes ahead if both players agree that it should.

The model has multiple equilibria, which we interpret as different social conventions for how social exchange takes place. There are equilibria where gifts-in-kind are used to foster interaction, and equilibria with cash, and an equilibrium where G players in some circumstances give money and give gifts-in-kind in in other circumstances.

In terms of welfare equilibria differ along two dimensions: the *intrinsic* and *informational (extrinsic)* value of gifts. The intrinsic value of the gift is simply the value it has to the receiver. On this dimension equilibria with cash are preferred by the receiver, exactly because a receiver might sometimes get a gifts she does not like. The informational (or extrinsic) value refers to how much social interaction the convention fosters (quantity), and it's ability to sort desirable social interaction from undesirable (quality), that is how much information about the beliefs of the giver about the receivers type the gift is able to convey. In the informational dimension equilibria with gifts-in-kind are generally superior to equilibria with cash. The basic reason is the following. In equilibrium G sends R a gift to induce her to invest in the common project. By giving a gift-in-kind G is also able to communicate to R something about her beliefs of R's type. Thus R is able to condition her decision on whether to engage in the project on more precise beliefs about the beliefs that G had about R's type when she decided to give a gift. With a generic cash gift R only knows that G thinks the project is beneficial, but R is not able to infer which beliefs G had about R's type. On the other hand this is exactly what gifts-in-kind do. R is able to make more precise inference about the beliefs that G had about R when she decided to send a gift. This gives R the choice to refrain from interaction, and benefits both players. By giving gifts-in-kind G allows R relatively more discretion over when to implement the joint project to the benefit of both players. Thus in their ordering of equilibria (according to ex-ante expected payoff) R players face a trade-off between intrinsic and extrensic efficiency.

Noise plays an important role in this argument. G must receive an informative yet noisy signal about the characteristics of R. If G was to learn R's characteristics with probability one, then an equilibrium where cash is given is always at least as good as any equilibrium in which gifts in kind are exchanged.

Multiplicity of equilibria is problematic for the predictive power of the model. Existence of equilibria with gifts-in-kind does not generally rely on what is given. E.g. the model admits equilibria where G players give gifts-in-kind to R players which they believe will not be liked by the receivers. If people have come to expect that they will receive gifts they don't like from people they should continue to interact with, then this is self-confirming, and indeed the best a giver can do is to give such gifts. In a sense different social conventions support equilibrium. Such equilibria are not very intuitive. We address the potential problem of multiplicity by showing that for any upper bound on the quality of the signal that G receives, there is a sufficiently small cost of gift provision such that both players have the same ordering of equilibria w.r.t. expected equilibrium payoffs. In this case an equilibrium with gifts-in-kind is the efficient equilibrium. We appeal to an evolutionary argument to select among equilibria (following Kim and Sobel (1995)) and show that in the long run we will only observe G players who believe that the project is worthwhile giving efficient gifts-in-kind.

Literature Camerer (1988) gave the first contribution to the study of gift giving using game theoretic tools. Camerer studies gift giving in a signalling environment. There are two types of players, types who would like to invest if the other players invest (willing types), and types who free-rides on the investment of others (unwilling types). Types are private information. In this model willing types send costly gifts to distinguish themselves from unwilling types. The crucial modelling assumption is a sorting assumption: the marginal benefit of investment for two willing types, is greater than the marginal benefit to free-riding for unwilling types. Through this mechanism a signalling equilibrium is constructed, where willing types send costly gifts which exactly deter unwilling types to want to pose as willing types. Camerer's model is silent about the nature and desirability of what is exchanged in equilibrium (i.e. whether e.g. cash or gifts in kind are and should be exchanged). Moreover the problem of multiple equilibria is unaddressed by his approach.

Carmichael and MacLeod (1996) study gift giving in an evolutionary model. In their model there are many types, but it suffices to think in terms of a willing (or cooperating) type and an unwilling (or free-riding) type. A player's type is private information. Player's can engage in long-term relationships, the termination/continuation of which the players have some discretion over. Players are matched over time to play a Prisoner's Dilemma Game. The same basic mechanism as in the paper by Camerer is at play in this dynamic set-up: by conditioning continuation of the match by the early exchange of costly gifts, cooperating types can avoid being driven out by evolution, because co-operating players can now exit bad matches prior to incurring large losses by requiring that gifts be exchanged prior to consenting to the relationship. Because defecting types are re-matched relatively more frequently, the pool of unmatched players are relatively abundant with defecting types. Accordingly they will often by matched with other defecting types and will do worse in the population. In their model gifts must be relatively inefficient, since it must be costly to give a gift, but the value cannot be high because that would make defecting types survive by picking up gifts and then exiting. Hence the model gives a possible explanation for why gifts must be relatively worthless. In a sense in the model of Carmichael and MacLeod inefficient gifts is a consequence of the strategic environment: cooperating types cannot survive if gifts are too efficient. The model proposed here may be more suited at explaining gift exchanges in more mature relationships where the stakes are higher for the players.

Finally there is a more recent literature that explains gift giving via non-standard preferences such as altruism and by introducing emotions (Ruffle 1999, Ruffle and Kaplan 2001).

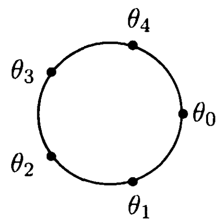
The remaining part of this paper is organised as follows. In section 3.2 the model is introduced. In section 3.3 we characterise equilibria and establish existence. In section 3.4 we look at welfare. In section 3.5 we look at a stochastic evolutionary model, adapting Kim and Sobel (1995)'s model of evolutionary cheap talk to our setting, and show that the process selects a relatively small set of states. In the final section some concluding remarks are provided.

3.2 Model

There are two populations, $\mathcal{G}$ and $\mathcal{R}$. Each member of a population has type $\theta \in \Theta = \{\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\}$. The distribution of types is identical in both populations. At the beginning of play Nature independently draws one individual at random from each population to play the game. The player drawn from $\mathcal{G}$ assumes the role of G(iver) and the player drawn from $\mathcal{R}$ assumes the role of R(eceiver).

Timing In period 1 G receives a signal realisation, τ, about R's type. Then in period 2 she chooses whether or not to give R a gift. If she decides to give R a gift, she can either give a monetary gift, m, or a gift-in-kind, $k \in \Theta$. Note that a gift-in-kind is identified with the set of types Θ. We do this to save on notation. In period 3 G and R simultaneously decide whether to engage in a common project. Then payoffs are realised.

Type Space Each population member in $\mathcal{G}$ and $\mathcal{R}$ is one of 5 types. Types are located on a circle:



Each type is equally likely. At the beginning of play each member of the population knows their own type, but only knows the distribution of types in the other population. This is common knowledge.

Define a distance-measure $d : \Theta^2 \to \mathbb{N}$ which simply counts the minimum number of steps between the types of G and R, e.g. $d(\theta_1, \theta_4) = 2$.

Signal and Interim Types The signal $\tilde{\tau}$ is a random variable with the following distribution, conditional on θ being the true type of player R.

$$\tilde{\tau} = \begin{cases} \theta, & \text{with Pr} = \rho \\ \theta', & \text{with Pr} = \frac{1-\rho}{4}, \ \theta' \neq \theta \end{cases}$$

and $\underline{\rho} \equiv \frac{1}{5} < \rho < 1$.

With probabilty ρ G learns the type of R and with probability $1 - \rho$ the signal is not informative.

Let t_0, t_1 and t_2 denote the interim-type of a G player whose modal belief about the distance between her own type and R's type conditional on receiving τ is on $d(\tau, \theta) = 0$, $d(\tau, \theta) = 1$ and $d(\tau, \theta) = 2$ respectively. $T = \{t_0, t_1, t_2\}$ is the set of interim-types of player G.

Payoffs Players have a vNM-utility function which represents preferences over outcomes of the game.

Gifts If G decides to give a monetary gift, denoted m, then we assume that she gives an amount $c > 0$, c being exogenously fixed, to R. We assume that a monetary gift yields utility c to R. If G decides to give a gift-in-kind then she pays cost c to provide the gift.

G chooses the characteristics of the gift-in-kind. If R is of type θ and she receives a gift-in-kind k with θ characteristics, then her payoff is c, and 0 otherwise. This assumption captures the potential inefficiency of gifts-in-kind. G may also decide not to send R any gift. We denote this action ng.

Project Payoffs If i only engages in the project, then i suffers a payoff loss[3] $l < 0$, and player j, $i \neq j$, gets an exogenous private benefit $b > 0$ independent of d.

If both players decide to engage in the project then the project is realised. Both players receive a project payoff, which is decreasing in distance, d:

$$B(d) = \begin{cases} B & \text{if } d = 0 \\ B - \underline{\delta} & \text{if } d = 1 \\ B - \bar{\delta} & \text{if } d = 2 \end{cases}$$

We assume $\bar{\delta} > \underline{\delta} > 0$ and $B - \underline{\delta} > b > B - \bar{\delta}$. Hence the project is individually rational when $d \leq 1$. Players are *compatible* whenever $d \leq 1$.

If both players choose not to engage in the project then it is not realised and both players get payoff 0.

To make the problem interesting we assume:

$$B - \frac{2(\underline{\delta} + \bar{\delta})}{5} < b \qquad \textbf{(A1)}$$

(A1) implies that types must separate to make investing in the project worthwhile in equilibrium. (A1) together with $B - \underline{\delta} > b$ implies $\underline{\delta} < \frac{2}{3}\bar{\delta}$. Let $\underline{B} = \underline{\delta} + b$ be the minimum level of B, and let $\overline{B} = \frac{2}{5}(\underline{\delta} + \bar{\delta}) + b$ be the maximum level of B. We make the additional assumption that $b < \frac{\underline{\delta}}{2}$.

Our focus will be on $c \leq c_1 = \frac{\underline{\delta}}{5} + \frac{3}{5}b$. c_1 will turn out to be the highest cost level such that for all parameter values there is some equilibrium which implements the project with positive probability[4].

Equilibrium The equilibrium concept is *sequential equilibrium*. We shall restrict attention to a special class of sequential equilibria that we refer to as "symmetric interim-type equilibria". The notion will be defined below.

Let the set of possible actions at the gift giving stage be $\mathbf{G} = \{ng\} \cup \Theta \cup \{m\}$ which includes the action to not give a gift, ng. A strategy for G of type θ is a mapping from

[3]In what follows we shall think of the loss as being sufficiently "small", but strictly positive, so that when we state results we do not explicitly reference this parameter.

[4]A characterisation for all cost levels can be found in Appendix B.2.

the space of signal realisations into the space of gifts and an associated action at the investment stage. Beliefs follow from Bayesian updating:

$$s_G(\theta) \;:\; \Theta \rightarrow \mathbf{G} \times \{I, NI\}$$
$$\mu_G(\theta) \;:\; \Theta \rightarrow \Delta(\Theta)$$

A strategy for R of type θ is a mapping from the space of gifts (including no gift) into an action at the investment stage. For each possible gift that R can receive she forms a belief about the type she is facing, using Bayes Rule when it applies:

$$s_R(\theta) \;:\; \mathbf{G} \rightarrow \{I, NI\}$$
$$\mu_R(\theta) \;:\; \mathbf{G} \rightarrow \Delta(\Theta)$$

A strategy profile is a collection of strategies and beliefs of all types of players:

$$\left((s_G(\theta), \mu_G(\theta))_{\theta\in\Theta}, (s_R(\theta), \mu_R(\theta))_{\theta\in\Theta}\right)$$

Symmetric Interim-type Equilibrium We now introduce a restriction on the set of strategies of G that we allow. This entails a within interim-types restriction and an between interim-types restriction on strategies.

Let $h : \Theta \rightarrow \Theta$ be any function which is one-to-one. The interpretation is that h is a convention for how either a signal realisation or a players type is mapped into the set of gifts-in-kind. Thus the mapping $h(\tau) = \tau$, can be interpreted as a convention where G sends gifts that she thinks the receiver likes. Likewise the mapping $h(\theta) = \theta$, can be interpreted as a convention where G sends gifts that she likes herself.

The set of interim-type t_i are G players such that $d(\theta, \tau) = i$, $i = 0, 1, 2$. We impose the following within interim-type restriction. Take a strategy profile. If this profile is a symmetric interim-type strategy profile then all t_i players at the gift-giving stage either do not send a gift (ng), sends a monetary gifts m, send a gift-in-kind which depend on the signal realisation, i.e. $h(\tau)$, or send a gift-in-kind which depend on their own (absolute) type i.e. $h(\theta)$.

We also impose the following between interim-type restriction on strategy-profiles. Take a strategy profile and suppose that t, $t' \in T$ send gifts-in-kind in this profile. Then (1) t and t' both use the same h and (2) t and t' either both send a gift-in-kind which depend on their signal realisation or on their (absolute) type.

We refer to a strategy profile which satisfies the restrictions above as a symmetric interim-type strategy profile. A symmetric interim-type equilibrium is then a sequential equilibrium where the strategy profile is restricted to being a symmetric interim-type strategy profile.

These restrictions are mainly imposed for reasons of tractability. Since gifts serve as a vehicle for communicating the quality of the match between the receiver and the giver, from an efficiency perspective it is desirable e.g. that all G players of a given interim type use the same convention. However ideally one would like this to come out of the model, perhaps via some selection story, rather than assume it. Implicitly we have thus assumed that the process of coordinating on which convention to use has already converged. In this sense within the context of the model considered here, the present analysis provides an upper limit on what can be achieved through gift giving. We do find the restrictions plausible in that they imply that strategies are relatively simple.

Comments A crucial modelling ingredient is that both players face uncertainty about the other player's type. In particular an otherwise identical model, except that the receiver knows the giver's type, but the giver is uncertain about the receiver's type there is no scope for an efficiency gain to giving gifts-in-kind. This follows since R will never invest if she knows that players are not compatible, independent of whether she receives a gift or not. When she knows that players are compatible then a gift from G is sufficient to induce her to invest, because G reveals that she thinks the players are compatible as well, and thus G will also invest. But then equilibrium behaviour is not affected by what is transferred as long as it is sufficiently costly to sort types of G who are willing to invest from types that are not willing to invest. In such a set-up it is indeed efficient to transfer money.

The model presented here makes no claim to generality, we attempt to provide the simplest model to study the question of interest. It can easily be extended to deal with any finite number of types of players, as well as a more general signal distribution.

3.3 Equilibrium Analysis

In this section we characterise symmetric interim-type equilibria and establish existence. First we find necessary conditions for behaviour strategies to constitute an equilibrium of the game.

Then in the following sections we look at sufficient conditions. We first look at strategy profiles where gifts are constrained to monetary gifts. Then we look at strategy profiles where gifts are constrained to gifts-in-kind. Finally we look at hybrid equilibria where gifts-in-kind and monetary gifts are given by different interim types.

3.3.1 Necessary Conditions

The following proposition summarises necessary conditions for type-specific behaviour strategies in equilibrium where the project is implemented with positive probability. At this stage we say nothing about what gifts are used in equilibrium.

Proposition 7. *Suppose $c > b > 0$. The necessary conditions for Equilibrium where the project is implemented with positive probability are:*

Equilibria with no gifts

1. *t_2 play (ng, NI).*
2. *If t_1 play (ng, I) and R invests, then t_0 also play (ng, I).*

Equilibria with gifts

1. *t_2 play (ng, NI).*
2. *If t_1 send gifts and invest, then t_0 also send gifts and invest.*
3. *If type $t \in T \setminus \{t_2\}$ does not invest then she sends no gift.*
4. *R players only invest if they receive a gift.*

The proof is via a series of lemmas, which can be found in Appendix B.1. The intuition is as follows. Interim-types must separate for R to invest in equilibrium. If they do not then by assumption (A1) expected project benefits are sufficiently low that R does not want to invest.

In equilibria with no gifts and where the project is implemented with positive probability (that is some interim type of G invests in the project) R invests unconditionally. Therefore t_2 types optimally respond by not investing in the project. t_0 types invests in the project. Whether t_1 types finds it worthwhile to invest depends on whether the expected project payoff exceed the benefit to free-riding, b.

In equilibria which involves gifts, t_2 types expectations of project payoffs are lower than the benefits to free-riding, but for R to invest they must send gifts. Since the marginal cost of inducing investment from R, c, is larger than the marginal benefit of free-riding, b, t_2 types do not send gifts in any equilibrium. If t_1 types expectations of project payoffs are sufficiently high to make them want to separate from t_2 types and send a gift and investing in the project, then this also applies to t_0 types since t_0 have higher expectations of project benefits than t_1 types.

3.3.2 Equilibria with no Gifts

In this section we characterise all equilibria in which no gifts are sent in equilibrium but where the project is implemented with positive probability. Let the set of strategy profiles where play is constrained to not involving any gifts, but where at least one interim type of G invests, be denoted S_{NG}.

Equilibrium I First we construct an equilibrium where only t_0 types invest in the project whereas other interim types free-ride. In this equilibrium R invests when she does not receive a gift.

Consider the following strategy profile $s^*_{I,NG} \in S_{NG}$:

G-players:

- t_0 play (ng, I)
- t_1, t_2 play (ng, NI)

R-players:

- If ng is received at gift giving stage then play I, otherwise play NI.

We define threshold levels for every B: $\underline{\rho} < \hat{\rho}(B) < \hat{\hat{\rho}}(B) < 1$, with the property that threshold levels are decreasing in B (see appendix B.2.1 for details).

We then have the following characterisation: We have the following characterisation.

Proposition 8. *Suppose play is according to the strategy profile, $s^*_{I,NG}$:*

1. *For any $c > b > 0$ $s^*_{I,NG}$ is an equilibrium profile if and only if $\hat{\rho} \leq \rho \leq \hat{\hat{\rho}}$.*
2. *There is no $b > 0$ such that for all B and ρ $s^*_{I,NG}$ is an equilibrium profile.*

Equilibrium II Next we construct an equilibrium where both t_0 and t_1 types find it worthwhile to invest in the project:

Consider the following strategy profile $s^*_{II,NG} \in S_{NG}$:

G-players:

- t_0 and t_1 play (ng, I)
- t_2 play (ng, NI)

R-players:

- If ng is received at gift giving stage then play I, otherwise play NI.

We have the following characterisation.

Proposition 9. *Suppose play is according to the strategy profile,* $s^*_{II,NG}$*:*

1. *For any* $c > b > 0$ $s^*_{II,NG}$ *is an equilibrium profile if and only if* $\rho \geq \hat{\rho}$*.*

2. *There is no* $b > 0$ *such that for all* B *and* ρ $s^*_{II,NG}$ *is an equilibrium profile.*

By proposition 7 there are no more strategy profiles in S_{NG} which are candidates for equilibria.

It follows directly from propositions 8 and 9 that there is a non-empty set of parameters which does not support an equilibrium with no gifts.

Corollary 2. *There is no* $b > 0$ *such that for all* B *and* ρ *there is some equilibrium with no gifts.*

Figure 3.1 illustrates equilibria with no gifts for fixed b.

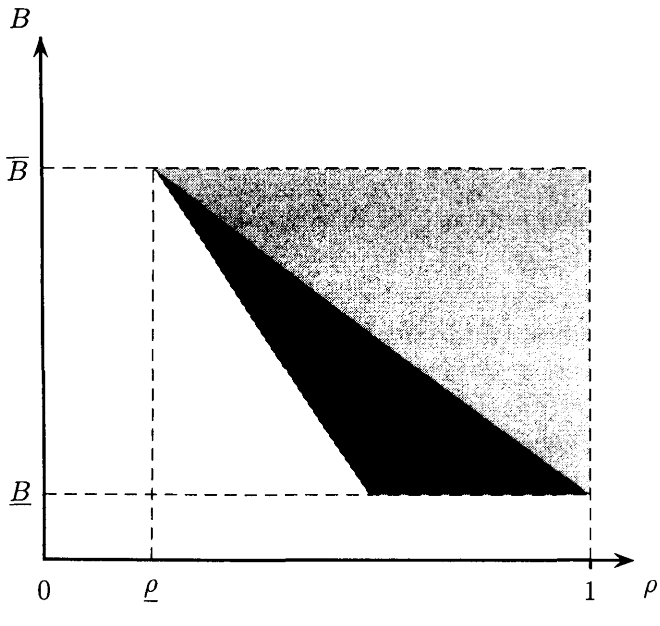


Figure 3.1: The set of Equilibria with no Gifts. Illustrated for the case $4\underline{\delta} = \overline{\delta}$. Darkgrey: Equilibrium $s^*_{I,NG}$ exists. Lightgrey: Equilibrium $s^*_{II,NG}$ exists. Non-Coloured: No investment in equilibrium.

3.3.3 Equilibria with Monetary Gifts

In this section we find and characterise all equilibria where only monetary gifts are used with positive probability in equilibrium. Let the set of strategy profiles where play is constrained to only involving monetary gifts be denoted S_M.

Equilibrium I First we construct an equilibrium where only t_0 types invest and send gifts. Consider the following strategy profile $s^*_{I,M} \in S_M$:

G-players:

- t_0 play (m, I)
- t_1, t_2 play (ng, NI)

R-players:

- If m is received at gift giving stage then play I, otherwise play NI.

In this profile only t_0 types send monetary gifts and invests, while any other type does not invest. R players only invest if they receive a monetary gift.

We define threshold levels for every B: $\underline{\rho} < \hat{\rho}(B) < \hat{\hat{\rho}}(B) < 1$, with the property that threshold levels are decreasing in B (see appendix B.2.2 for details).

We then have the following characterisation:

Proposition 10. *Suppose play is according to the strategy profile,* $s^*_{I,M}$*:*

1. *For fixed* $b < c \leq c_1$ *and for any fixed* B $s^*_{I,M}$ *is an equilibrium profile if and only if* $\rho \in [\hat{\rho}(B), \hat{\hat{\rho}}(B)]$.
2. *There is no cost level* $c > b > 0$ *such that for all* B *and* ρ $s^*_{I,M}$ *is an equilibrium profile.*

The intuition for the result is as follows. For fixed B and c the quality of the signal (parametrised by ρ) can neither be too low nor too high. If it is too low then t_0 types will not find it profitable to try and induce investment, since too much probability is placed on the true state of the world being $d = 2$. If it is too high then t_1 types cannot be deterred from sending a gift as well, since too little probabiltity is placed on the true state of the world being $d = 2$.

Equilibrium II Now we construct an equilibrium where both t_0 and t_1 types invest. Consider the profile $s^*_{II,M} \in S_M$:

G-players:

- t_0, t_1 play (m, I)
- t_2 play (ng, NI)

R-players:

- If m is received at gift giving stage then play I, otherwise play NI.

In this profile t_1 interim types also sends a monetary gift.

For any $b < c \leq c_1$ we define $\underline{B} \leq \tilde{B}(c) < \overline{B}$ (see appendix B.2.2). $\tilde{B}(c)$ has the property that it is increasing in c.

We have the following characterisation:

Proposition 11. *Suppose play is according to the strategy profile, $s^*_{II,M}$:*

1. *For fixed $b < c \leq c_1$ $s^*_{II,M}$ is an equilibrium profile if and only if $B \in [\tilde{B}(c), \bar{B})$ and $\rho \in (\hat{\rho}(B), 1)$.*
2. *There is no cost level $c > b > 0$ such that for all B and ρ $s^*_{II,M}$ is an equilibrium profile.*

This equilibrium corresponds exactly to the case where the quality of the signal is sufficiently high that t_1 types cannot be deterred from sending a gift.

Since we have restricted G-players to only sending monetary gifts, by Proposition 7 there are no other equilibria where only monetary gifts are used in equilibrium.

By combining this observation, with the statements of propositions 10 and 11 we have:

Corollary 3. *For any $b < c \leq c_1$ there is a non-empty set of values of B and ρ such that there is no $s \in S_M$ which is an equilibrium profile.*

Remark 8 (Robustness). *This remark comments on the robustness of our results.*

Consider first the possibility that players can only send a fixed amount of money, but that different denominations are available. In this case equilibria with money will be at least as efficient as any other equilibrium, provided that the cardinality of the space of combinations of money that sums to c is at least equal to the cardinality of the space of types.

Alternatively one could think of an equilibrium construction where t_0 players send m_0 and t_1 players send $m_1 < m_0$. This profile is not an equilbrium since t_0 types will want to pose as t_1 types.

Finally, observe that equilibria with monetary gifts may fail to exist is partially driven by our assumption that the cost of gift provision, c, is not a strategic variable. If we relax this assumption we can have equilibria in where different types send different amounts of money e.g. an interim type t_0 *of type* θ *sends* m_θ, *where* $m_\theta \neq m_{\theta'}$, $\theta \neq \theta'$, *provided that equilibrium transfers between neighbouring types are not too different. In such an equilibrium there will be no inefficency loss for the receiver from receiving gifts that she does not like. This type of equilibrium breaks down in the limit as* ρ, *the quality of the signal, approaches one. Details may be found in Appendix B.3. Such equilibria treat different player types asymmetrically, in events where the expected project payoff are identical. We do not find such equilibria plausible from an ex-ante perspective.*

Figure 3.2 illustrates the equilibria with monetary gifts for cost level $c = c_1$.

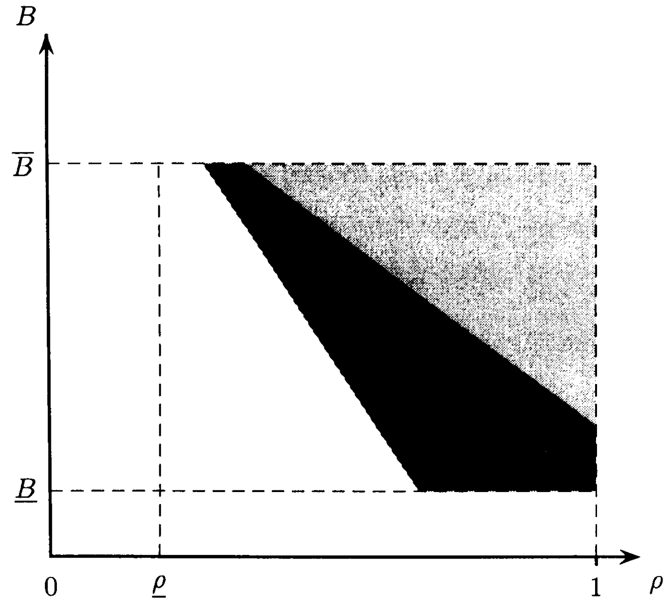


Figure 3.2: The set of Equilibria with Monetary Gifts. Illustrated for the case $b < c = c_1$, and $4\underline{\delta} = \overline{\delta}$. Darkgrey: Equilibrium $s^*_{I,M}$ exists. Lightgrey: Equilibrium $s^*_{II,M}$ exists. Non-Coloured: No monetary gifts are sent in equilibrium.

3.3.4 Equilibria With Gifts-in-Kind

We now turn to equilibria which only makes use of gifts-in-kind. Let the set of profiles which only use gifts-in-kind be denoted S_K. It will turn out that the equilibrium conditions in this section are invariant to the convention used i.e. the h-function. To save on notation we shall therefore be stating results under the restriction that h is the identity mapping. We then state a lemma (lemma 2) which makes the connection clear.

Equilibrium I Consider the following strategy profile, $s^*_{I,K}$, where we initially restrict $h(\cdot)$ to be the identity mapping:

G-players:

- t_0 play (τ, I), where τ is the signal realisation.
- t_1. t_2 play (ng, NI)

R-players:

- If R is of type θ and $d(\tau, \theta) \leq 1$ then play I, otherwise play NI.

That is an R player only invests if the gifts she receives has characteristics that she or one of her neighbours like.

Before characterising equilibrium for every $c \leq c_1$ we define $\underline{B} < \tilde{B}(c) < \overline{B}$ with the property that $\tilde{B}(c)$ is decreasing in c. Also for every B we define $\underline{\rho} \leq \tilde{\rho}(B) < 1$ with the property that $\tilde{\rho}(B)$ is increasing in B (details are in appendix B.2.3).

We have the following characterisation:

Proposition 12. *Suppose play is according to the profile $s^*_{I,K}$:*

1. *For fixed $b < c \leq c_1$ $s^*_{I,K}$ is an equilibrium profile if and only if $B \leq \tilde{B}(c)$ and $\rho \in [\tilde{\rho}(B), 1)$.*

At first sight it may appear slightly surprising that for high quality of the signal realisation it becomes easier to deter t_1 players from sending gifts. The reason is the following. The optimal deviation for a t_1 interim type is for her to send a gift-in-kind which has characteristics that she likes herself. Given the behaviour of R players this has the consequence that in equilibrium the project is never implemented when the true state is $d = 2$, for any ρ. Now suppose the signal quality is close to 1. Thus a t_1 type is almost certain that the project will be implemented but she also knows almost certainly that the true state is $d = 1$. As ρ decreases she knows the true state of the world is now more likely to be $d = 0$ (with an associated higher payoff), while the project is still not implemented when $d = 2$. Therefore for fixed B it becomes harder to deter t_1 as ρ decreases.

We can now comment on equilibrium conditions if players who send gifts do not use the identity mapping.

Lemma 2. *Any one-to-one function h does not change equilibrium conditions.*

Proof. Suppose h is one-to-one. A R player is able to invert the gifts she receives to recover the underlying signal. Thus for any h: $h^{-1}(h(\tau)) = \tau$. □

Given our restriction to equilibria with gifts-in-kind, there are no other equilibria where only t_0 types send gifts-in-kind.

Equilibrium II (Type Revelation) We now construct equilibria where both t_0 and t_1 types send gifts.

Consider the following strategy profile, $s^*_{II,K}$. Again assume that $h(\cdot)$ is the identity mapping:

G-players:

- t_0, t_1 play (θ, I), where θ is G's type.
- t_2 play (ng, NI)

R-players:

- If R is of type θ' and $d(\theta, \theta') \leq 1$ then play I, otherwise play NI. That is an R player only invests if the gifts she receives has characteristics that she or one of her neighbours like.

When h is the identity mapping then we can interpret the equilibrium behaviour of G as dictating that if G believe that she is compatible with R then she gives R something that G likes herself.

We have the following characterisation:

Proposition 13. *Suppose play is according to the profile $s^*_{II,K}$:*

1. *For fixed $b < c < c_1$:*
   (a) *For fixed $B \leq \tilde{B}(c)$: $s^*_{II,K}$ is an equilibrium profile if and only if $\rho \in (\underline{\rho}, \tilde{\rho}(B))$.*
   (b) *For fixed $B > \tilde{B}(c)$: $s^*_{II,K}$ is an equilibrium profile for any ρ.*

Thus whenever there is no equilibrium I, there is an equilibrium II (type revelation) where types who send gifts gives gifts that they like themselves.

Again note that the characterisation holds for any one-to-one function h.

Equilibrium III (Signal Revelation) Finally we construct an equilibrium of the following form:

Consider the following strategy profile, $s^*_{III,K}$. Again assume that $h(\cdot)$ is the identity mapping:

G-players:

- t_0, t_1 play (τ, I), where τ is the signal realisation.
- t_2 play (ng, NI)

R-players:

- If R is of type θ and $d(\tau, \theta) = 0$ then play I, otherwise play NI. That is an R player only invests if the gifts she receives has characteristics that she likes.

We can think of G players in this equilibrium as following the rule: if you think that you are compatible with R, then give her something that you think she will like.

We define for every $B \in (\tilde{B}(c), \frac{1}{3}(\underline{\delta} + \overline{\delta}) + b)$: $\underline{\rho} < \tilde{\tilde{\rho}}(B) < 1$ (see appendix B.2.3 for details). We have the following characterisation:

Proposition 14. *Suppose the interval $(\tilde{B}(c), \frac{1}{3}(\underline{\delta} + \overline{\delta}) + b)$ is non-empty. Also suppose play is according to the profile $s^*_{III,K}$:*

1. *For fixed $b < c \leq c_1$ $s^*_{III,K}$ is an equilibrium profile if and only if $B \in (\tilde{B}(c), \frac{1}{3}(\underline{\delta} + \overline{\delta}) + b)$ and $\rho \in [\tilde{\tilde{\rho}}(B), 1)$.*

By proposition 7 and our restriction interim-type symmetric strategy profiles there are no other equilibrium profiles in the set S_K. Before graphically illustrating equilibria, we note that for $b < c \leq c_1$ there is always some $s \in S_K$ which is an equilibrium profile.

Corollary 4. *For any $b < c \leq c_1$, for any B and ρ there is some $s^* \in S_K$ such that s^* is an equilibrium profile.*

Figure 3.3 illustrates the set of equilibria with gifts-in-kind for $c = c_1$.

Above we established that equilibrium conditions do not depend on h. This means that ex-ante there is no convincing reason why people should either give gifts that they like themselves or gifts that they think the receiver likes. For a given equilibrium with gifts-in-kind receivers (weakly) prefer equilibrium outcomes generated under conventions where h is the identity-mapping. In section 3.5 we analyse whether evolutionary pressures may lead to efficient payoffs.

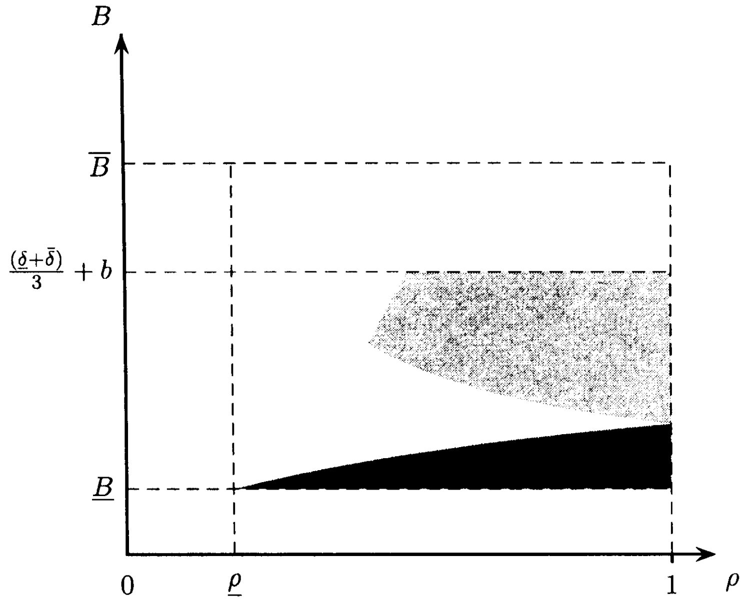


Figure 3.3: The set of Equilibria with Gifts-in-Kind. Illustrated for the case where $\bar{\delta} = 4\underline{\delta}$ and $b < c = c_1$. Darkgrey: Only Equilibrium I exists. Lightgrey: Multiple Equilibria: Equilibrium II (Type Revelation) and Equilibrium III (Signal Revelation) co-exists. Non-Coloured: Equilibrium II (Type Revelation) exists.

3.3.5 Hybrid Equilibria

In this section we construct equilibria where monetary gifts and gifts-in-kind are used by different interim types. Let the set of hybrid strategy profiles be S_H.

Consider the following strategy profile, s_H^*:

G-players:

- t_0 play (m, I).
- t_1 play (τ, I).
- t_2 play (ng, NI)

R-players:

- If R receives m at the gift giving stage then she plays I. If R receives a gift-in-kind and $d(\tau, \theta) = 0$ then play I, otherwise play NI.

In s_H^* t_0 types send money, whereas t_1 types send gifts that they think the receiver likes. R invests whenever she gets a monetary gift, but if she receives a gift-in-kind she only invests if she likes the gift.

For every B we define $\underline{\rho} < \bar{\rho}(B) < 1$, with the property that $\bar{\rho}(B)$ is decreasing in B (see appendix B.2.4 for details).

Proposition 15. *Restrict attention to $s \in S_H$.*

1. *If $B > \frac{\bar{\delta}}{2} + b$ for all B then no $s \in S_H$ is an equilibrium profile.*
2. *For $b < c \leq c_1$ and $B \leq \frac{\bar{\delta}}{2} + b$ for some B:*
   (a) *s_H^* is an equilibrium profile if and only if $B \in [\tilde{B}(c), \frac{\bar{\delta}}{2} + b]$ and $\rho \in [\bar{\rho}(B), 1)$.*
   (b) *The set of hybrid equilibrium profiles $S_H^* \subseteq S_H$ is either empty or $S_H^* = \{s_H^*\}$.*

Note again that this holds for any h which is one-to-one.

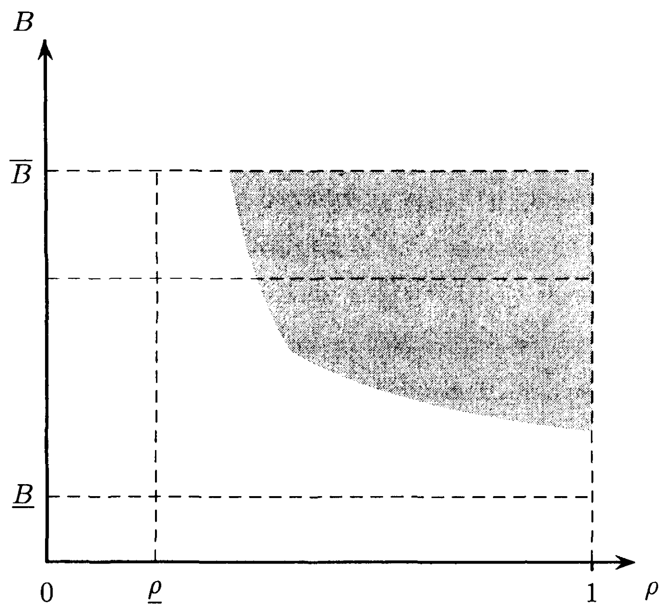


Figure 3.4: Hybrid Equilibria. Illustrated for the case where $b < c = c_1$ and $\bar{\delta} = 4\underline{\delta}$. Lightgrey: Hybrid Equilibrium (Signal Revelation) exists. Non-Coloured: No Hybrid Equilibrium exists.

3.4 Welfare

In this section we establish welfare properties of equilibria. Welfare is evaluated ex-ante, before players are informed of their own type.

One can think of the ordering of equilibria along two dimensions: an *intrinsic* and an *informational (extrinsic)* value of gift giving. The intrinsic value of gift giving refers to the value of the gift in itself, separate from the social interaction that the gift giving practices generate in equilibrium. The extrinsic value refers to how well the gift is at realising the potential of social interaction. The extrinsic value can be separated into two parts: (1) How good is the practice at generating interaction when it is beneficial to do so? (2) How good is the practice at helping players to avoid engaging in the project when it is beneficial to do so?

3.4.1 The Informational Value of Gift Conventions

In this section we highlight two different ways in which to think about the informational value of gift conventions. It is possible to think abstractly as a gift giving convention as an information structure, where the state of the world, which in our context is the distance between the giver and the receiver, is correlated with the gift that the receiver gets. Blackwell's ordering of information structures then say something about which gift conventions a receiver prefers, when considering only the informational value. We first offer a brief review of the Blackwell Ordering of information structures. Then we present the Blackwell ordering in the context of our model. Finally we shall think of gift giving conventions in terms of a standard statistical test subject to type I and type II errors.

Blackwell Ordering Consider two random variables $\tilde{x}_\mu$ and $\tilde{x}_\nu$ which are both correlated with an unobserved state of the world $\tilde{s} \in S = \{1, \ldots, N\}$, N being finite. Let X be the finite sample space of the random variables. Then we can identify the matrices:

$$\mu = \begin{bmatrix} \mu_{11} & ,\ldots, & \mu_{1N} \\ \vdots & \ddots & \vdots \\ \mu_{n1} & ,\ldots, & \mu_{nN} \end{bmatrix} \quad , \quad \nu = \begin{bmatrix} \nu_{11} & ,\ldots, & \nu_{1N} \\ \vdots & \ddots & \vdots \\ \nu_{n1} & ,\ldots, & \nu_{nN} \end{bmatrix}$$

where the entries:

$$\begin{aligned} \mu_{ij} &= \Pr(\tilde{x}_\mu = x_i | \tilde{s} = s_j) \geq 0 \\ \nu_{ij} &= \Pr(\tilde{x}_\nu = x_i | \tilde{s} = s_j) \geq 0 \end{aligned}$$

are the probability of receiving signal x_i conditional on the state of the world being s_j when observing $\tilde{x}_\mu$ and $\tilde{x}_\nu$ respectively. We shall refer to μ and ν as the *information structure* generated by $\tilde{x}_\mu$ and $\tilde{x}_\nu$ respectively.

$\tilde{x}_\mu$ *Blackwell dominates* $\tilde{x}_\nu$ if and only if there exists a stochastic transformation matrix Q:

$$Q = \begin{bmatrix} q_{11} & ,\dots, & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & ,\dots, & q_{nn} \end{bmatrix}$$

where $q_{ij} \geq 0$, $\sum_{i=1}^{n} q_{ij} = 1$, such that:

$$\nu = Q\mu$$

The matrix Q is sometimes called a *garbling* of μ, since the transformation $Q\mu$ garbles the information contained in μ.

Blackwell's Theorem shows that if $\tilde{x}_\mu$ Blackwell dominates $\tilde{x}_\nu$ then independent of the underlying decision problem a decision maker prefers to observe $\tilde{x}_\mu$ rather than $\tilde{x}_\nu$ ex-ante[5]. If $\tilde{x}_\mu$ is Blackwell preferred to $\tilde{x}_\nu$ then we write $\tilde{x}_\mu \succ \tilde{x}_\nu$. Blackwell's result is weak in the sense that the Blackwell ordering is an incomplete ordering. Two information structures $\tilde{x}_\mu$ and $\tilde{x}_\nu$ cannot be ordered if: $\tilde{x}_\mu \not\succ \tilde{x}_\nu$ and $\tilde{x}_\nu \not\succ \tilde{x}_\mu$.

In the context of our model one can think of each equilibrium strategy involving gift giving as inducing a particular information structure which the receiver makes her inference from. In the context of the model the states of the world are the possible types of the gift giver. Since only distance between the types of R and G is payoff relevant, then the possible states of the world are: $d = 0, 1, 2$. The sample space of the signal is the gift space (including no gift). Thus the receipt of a particular gift will depending on which equilibrium being played, be correlated with the state of the world.

We denote an information structure associated with an equilibrium of the game as $\Pi_i(j)$. i refers to the medium of exchange used in the equilibrium, i.e. either gifts in K(ind), M(oney) or a H(ybrid) equilibrium. j refers to the type of equilibrium that is either type *I*, *II* or *III*.

The ordering is displayed in the following table[6].

[5]See e.g. Kihlstrom (1984).

[6]A simple exercise in a system of linear equations. Details is available upon request

	$\Pi_K(II)$	$\Pi_K(III)$	$\Pi_K(I)$	$\Pi_M(II)$	$\Pi_M(I)$	Π_H
$\Pi_K(II)$	–	$\succ$	$N.A.$	$\succ$	$\succ$	$\succ$
$\Pi_K(III)$	$\not\succ$	–	$N.A.$	$\succ$	$\not\succ$	$\not\succ$
$\Pi_K(I)$	$N.A.$	$N.A.$	–	$N.A.$	$\succ$	$N.A.$
$\Pi_M(II)$	$\not\succ$	$\not\succ$	$N.A.$	–	$N.A.$	$\not\succ$
$\Pi_M(I)$	$\not\succ$	$\not\succ$	$\not\succ$	$N.A.$	–	$\not\succ$
Π_H	$\not\succ$	$\not\succ$	$N.A.$	$\succ$	$\succ$	–

An entry $\succ$ indicates that the information structure in the row entry blackwell dominates the column entry. $N.A.$ indicates that the two information structures do not co-exist.

The table shows the Blackwell ordering of information structures for the gift giving game. The receiver prefers Equilibrium II with gifts-in-kind with type revelation ($s^*_{II,K}$), whenever it is available. When the information structure is not available then any decision maker prefers to observe the information structure generated under Equilibrium I with gifts-in-kind ($s^*_{I,K}$).

Note that conditional on sending a gift G and R players preferences over the action of the gift receiver at the investment stage are aligned. In particular if the signal indicates a good match then both players prefer that R invests, whereas if the signal indicates a bad match then it is in the interest of both players that R does not invest (the decision to invest would give both players expected negative payoff, whereas not investing yields a small benefit to R and a small loss to G).

The intuition why equilibria with gifts-in-kind blackwell dominates equilibria with money is roughly that gifts-in-kind is a "richer language" for G to communicate her beliefs about R's type to R. The comparative richness of the language gives R the possibility to condition the investment decision on a richer set of signals, in particular when a gift reveals that teaming up would probably be bad, R has the opportunity to terminate the relationship. Both players prefer this to ending up in a bad match, the small loss that G suffers from the termination being preferable to the larger loss involved in being tied up in a bad match.

Hypothesis Testing and Inference Errors Another useful way to compare information structures comes from looking at inference errors associated with information structures. Take a particular information structure Π, and let α_Π denote the probability that a G player does not send a gift even though the players are compatible (a *type I* error). Let β_{II} be the probability with which the project is implemented even though players are not compatible (a *type II* error).

Since the signal is noisy with positive probability G will be making a *type I* error, that is failing to send a gift even though players are compatible. This error probability α is the same across equilibria with gifts in kind and with money which involves the same intensity of gift giving. On the other hand the equilibrium with gifts-in-kind leaves R some *discretion* over whether to implement the project or not, i.e. depending on the particular gift that she receives. Consequently equilibria with gifts-in-kind involve fewer type II errors, i.e. the probability β that the project is implemented eventhough players are not compatible.

Conditional on gift reception players share the same interest, so leaving some discretion to R benefits both players. This observation relates to the informational value of gift giving, and confirms the Blackwell ordering.

Thus R players ordering of equilibrium outcomes essentially involves a trade-off between the informational and the intrinsic value. Because the intrinsic value of gifts depend on the equilibrium played R players have a preference for gifts conventions which involves money.

3.4.2 Ordering of Equilibrium Outcomes

In this section we first partially characterise G and R's ordering of equilibrium outcomes for any $b < c \leq c_1$. We show that for some parameter values which admit multiple equilibria then G and R's ordering are not identical. Players disagree about which equilibrium they would like to play.

Next we show that disagreement is limited in the following sense. For any fixed level of signal quality $\rho^* < 1$, and when the cost of gift giving is just sufficient to deter freeriding, i.e in the limit as $c = b$, there is a sufficiently small cost c^* such that if $b = c \leq c^*$ then for all levels of B and $\rho \leq \rho^*$ the players ordering of equilibrium outcomes coincide.

Partial Ordering of Equilibria

In this section we partially characterise G and R's ordering of equilibrium outcomes for any $b < c \leq c_1$.

We first present the partial ordering of equilibrium outcomes for G players. We consider two cases. In the first case we restrict attention to welfare in equilibria that involves gift giving. In the second case we consider welfare in all equilibria. The proofs of the statements of this section are all relegated to appendix B.4.

Proposition 16. *Consider any fixed c and b such that $c_1 \geq c > b$.*

1. *Consider the set of equilibria in which gifts are sent.*
   (a) *If $s^*_{I,K}$ is an equilibrium profile, then G achieves her highest equilibrium payoff under $s^*_{I,K}$, independent of h.*
   (b) *If $s^*_{II,K}$ is an equilibrium profile, then G achieves her highest equilibrium payoff under $s^*_{II,K}$, independent of h.*
2. *Consider the set of all equilibria:*
   (a) *If $s_{I,K}$ is an equilibrium profile then there is $\tilde{\rho} < 1$ such such that an equilibrium with no gifts is preferred by G to all other equilibrium profiles if $\rho \geq \tilde{\rho}$.*
   (b) *If $s_{II,K}$ then there is $\tilde{\tilde{\rho}} < 1$ such such that an equilibrium with no gifts is preferred by G to all other equilibrium profiles if $\rho \geq \tilde{\tilde{\rho}}$.*

The first part follows from the Blackwell Ordering and the observation that conditional on gift giving R always takes the decision that G would like her to conditional on receiving a particular signal. The fact that conditional on existence each information structure has the same cost, implies that the Blackwell ordering includes all payoff relevant information. Also note that the condition for equilibrium $s^*_{I,K}$ to exist is exactly that $s^*_{II,K}$ does not exist.

To see the intuition for the second part note that as ρ approaches 1, then G's signal reveals R's type almost perfectly. Therefore the probability that the project is implemented when the match is bad approaches 0 as ρ goes to one. But G is sending costly gifts in order to exactly avoid bad matches. Therefore G prefers that R invests uncondítonally rather than having to send a costly gift.

The next proposition shows the partial ordering of equilibrium outcomes for R players. It shows that when parameters are such that only Equilibrium I (either with gifts or money) exists, then R players prefer the equilibrium with gifts-in-kind, where h is the identity mapping, that is G players give gifts that they think R likes (and in fact like themselves). When parameters are such that equilibrium I with gifts-in-kind does not exist, then for every B and $c \leq c_1$ there is some sufficiently large threshold for the signal quality, such that when signal quality is below this threshold then equilibrium II (type revelation), using some h, is the preferred equilibrium outcome. When the signal quality is above the threshold then some other equilibrium is preferred.

Recall that welfare is evaluated ex-ante before players know their own type. We assume that the welfare criterion treats each of the types of R symmetrically. That is among any h which is one-to-one we can restrict attention to h which do not treat different types of R asymmetrically. To this end let h^0 be the identity mapping. Let h^+ be the mapping which for all $\theta \in \Theta$ is mapped to θ', where θ' is located distance one from θ going clockwise around the circle. Likewise define h^- going counter-clockwise.

Proposition 17. *Suppose $c_1 \geq c > b$. Consider the set of all equilibria.*

1. *Suppose $s^*_{I,K}$ is an equilibrium profile. There is a ρ' sufficiently large and a $\underline{B} \leq B' \leq \tilde{B}(c)$, such that for all $B' \leq B \leq \tilde{B}(c)$ and $\rho \geq \rho'$ then R achieves her highest equilibrium payoff under $s^*_{II,NG}$. Otherwise R achieves the highest equilibrium payoff under $s^*_{I,K}$ where $h = h^0$.*

2. *Suppose $s^*_{II,K}$ is an equilibrium profile.*

   (a) *For $B \leq \tilde{B}(c)$: R achieves her highest equilibrium payoff under $s^*_{II,K}$, where h is either h^0, h^+ or h^-.*

   (b) *For $B > \tilde{B}(c)$: There is $\rho(B) < 1$ such that if $\rho \leq \rho(B)$ then R achieves her highest equilibrium payoff under $s^*_{II,K}$ where h is either h^0, h^+ or h^-. If $\rho > \rho(B)$ then some other equilibrium profile achieves a higher payoff.*

The intuition for the result that when $s^*_{I,K}$ is an equilibrium profile, then R may prefer the equilibrium payoff from $s^*_{II,NG}$ is as follows. In $s^*_{I,K}$ only t_0 types invest and send gifts, whereas in $s^*_{II,NG}$ both t_0 and t_1 types invest. Thus it is the relatively high cost of gift provision that deters t_1 types from sending gifts and investing, eventhough expected project benefits are strictly positive. The benefit then of playing according to $s^*_{II,NG}$ is that good matches reach the project stage. The drawback is that more bad matches also reach the project stage, and that R does not receive gifts. As ρ approaches one the probability that a bad match reaches the project stage approaches 0, therefore eventually R will prefer the higher project intensity associated with $s^*_{II,NG}$ rather than receiving gifts.

For the second part note that when $s^*_{II,K}$ is an equilibrium profile then t_1 types also invest and send gifts. Compared to a no gift equilibrium, a gifts in kind equilibrium is preferred on the grounds that gifts help sort bad matches from good matches, and the receiver gets gifts which may give positive utility. Therefore a no gift equilibrium can never be preferred by R to $s^*_{II,K}$.

To gain some intution for the trade-off between Equilibrium II (Type revelation) (restrict attention to the case where h is the identity mapping) and Equilibrium II (Money) note the following. In the type revealing equilibrium, the gift's intrinsic value is less than in the money equilibrium. The probability of an α-error, is identical in the two equilibria. But whereas the probability of a β-error is 0 for the type revealing equilibrium it is positive in the money equilibrium. Now fix B and ρ such that we are at the indifference point between the two equilibria and let B increase. As B increases the cost of making a β-error decreases, whereas the intrinsic value is kept fixed as ρ is fixed. Hence we must move into the region where the equilbrium with money is strictly preferred as B increases.

Consider now the trade-off between Equilibrium II (Type revealing) and Equilibrium III (Signal Revealing) equilibrium. The β-error is 0 in both equilibria. However the α-error in type revealing equilibrium is less than in signal revelation. Again fix B and ρ such that we are on the indifference point between the two equilibrium outcomes. Now increase B. As B increases the cost of making an α-error increases, hence we must move into the region where Equilibrium II is preferred.

In sum, R players face a trade-off between efficient *extrinsic* gift equilibria and efficient *intrinsic* gift equilibria.

Propositions 16 and 17 show that for some parameter values players disagree about which equilibrium they prefer. The intuition is straightforward. In equilibria with gifts, and for a given level of gift giving intensity G players are only concerned about how well the convention is at sorting desirable interaction from undesirable interaction. However they may be willing to trade-off a few bad matches for not having to give gifts. R players however are also concerned about the gift which they receive, thus they will only be willing to trade-off not receiving gifts if the equilibrium leads to more intensive investment. However for a given level of gift giving intensity, the higher the intrinsic value of the gift the more willing they are to trade-off informational efficiency (type revealing equilibrium) for intrinsic efficiency (money, signal or hybrid equilibria).

Partial Ordering - A Limit Case

In this section we ask a different question. Suppose we impose an upper bound on the signal quality $\rho \leq \rho^* < 1$. Is there some sufficiently small $c^* > 0$ such that if the cost of the gift is just sufficient to deter free-riding, i.e. we look at the limit as $c = b$, then for every B and $\rho \leq \rho^*$ among all equilibrium profiles there is some profile such that both G and R achieve their highest equilibrium payoff?

The following proposition show that this is the case.

Proposition 18. *Take any $\rho^* < 1$, and suppose that the cost of gifts is just sufficient to deter free-riding. There exists a $c^*(\rho^*) > 0$ such that if $c = b \leq c^*(\rho^*)$ then for any B and $\rho \leq \rho^*$ both G and R achieves their highest equilibrium payoff, among any equilibrium profile, under $s^*_{II,K}$ and where h is either h^0, h^+ or h^-.*

Proof. First note that in the limit as $c = b$ we cannot sustain $s^*_{I,K}$ as an equilibrium profile. For all parameter values of B and ρ $s^*_{II,K}$ is an equilibrium profile.

Now fix some $\rho^* < 1$. We establish an upper limit on the cost of gift giving $c^*(\rho^*)$ such that if $c \leq c^*(\rho^*)$ then both G and R achieves their highest equilibrium payoff under $s^*_{II,K}$.

It follows from proposition 16 that the relevant comparison for G is between the equilibria $s^*_{II,K}$ and $s^*_{II,NG}$. For G to prefer the equilibrium payoff under $s^*_{II,K}$ to $s^*_{II,NG}$ c must be below the following threshold[7]:

$$c^*_{II,NG} = \frac{3(1-\rho^*)}{13-3\rho^*}\frac{3}{5}\left(\overline{\delta} + \frac{2}{3}\underline{\delta}\right)$$

We now turn to R The threshold values for c^* are as follows: If $c \leq c^*_{III,K}$ then $s^*_{II,K}$, $h \in \{h^0, h^+, h^-\}$ is preferred to $s^*_{III,K}$ where:

$$c^*_{III,K} = \begin{cases} \frac{1-\rho^*}{4(2\rho^*-1)}\underline{\delta}, & \text{if} \quad \rho^* > \frac{1}{2} \\ c_1 & \text{if} \quad \rho^* \leq \frac{1}{2} \end{cases}$$

If $c \leq c^*_H$ then $s^*_{II,K}$, $h \in \{h^0, h^+, h^-\}$ is preferred to s^*_H where:

$$c^*_H = \begin{cases} \frac{1-\rho^*}{2(3\rho^*-1)}(2\underline{\delta} + \overline{\delta}), & \text{if} \quad \rho^* > \frac{1}{3} \\ c_1 & \text{if} \quad \rho^* \leq \frac{1}{3} \end{cases}$$

If $c \leq c^*_{II,M}$ then $s^*_{II,K}$, $h \in \{h^0, h^+, h^-\}$ is preferred to $s^*_{II,M}$ where:

$$c^*_{II,M} = \frac{3}{4}\frac{1-\rho^*}{2-\rho^*}(\overline{\delta} - \underline{\delta})$$

Now let $c^*(\rho^*)$ be equal to the smallest of these thresholds. It then follows that if $c \leq c^*$ then $s^*_{II,K}$, where h is either h^0, h^+ or h^- yields the highest equilibrium payoff to both players. □

[7]Follows from comparing the welfare under the two different equilibria, see proof of proposition 16 in appendix B.4, and setting $b = c$ and noting that the loss from not sending gifts is lowest when $\rho = \rho^*$ and B attains it maximal value $\bar{B}$.

To gain some intuition for the result note that in $s^*_{II,M}$ the probability that R makes a β-error is decreasing in the signal quality ρ. Fix ρ and B such that we are on a point of indifference between the two equilibrium outcomes. Now lower c to $c' < c$. As c' decreases the marginal *intrinsic* value of the money equilibrium falls relative to type revealing equilibrium. Hence at c' we must now be in the region where type revealing equilibrium is preferred.

The argument is similar when R prefers $s^*_{III,K}$ to $s^*_{II,K}$. The probability that R makes an α-error is decreasing in ρ. As the cost falls from c to $c' < c$ the marginal *intrinsic* value of of $s^*_{III,K}$ (signal) falls relative to $s^*_{II,K}$ (type). Hence at c' we must now be in the region where equlibrium II (type) is preferred.

Proposition 18 says that for any level of the signal quality (strictly less than one) there is cost of gift giving sufficiently small, such that if the cost exactly deters the incentive to free-ride then the efficient equilibrium outcome involves the transfer of gifts-in-kind. Irrespective of this result there are still multiple equilibria. In the next section we ask whether an evolutionary process will select strategy profiles that gives rise to efficient payoffs.

3.5 Evolution and Selection

Why might we expect that we are more likely to observe equilibria with gifts in kind rather than equilibria with monetary gifts? In this section we explore the possibility that evolution selects efficient equilibria. In particular we follow Kim and Sobel (1995) and extend the gift-giving game by allowing a round of cheap-talk prior to the gift-giving game. We then look at an evolutionary adaptive process and show that the only equilibrium outcome of the gift giving game which will be observed in the long run is the efficient equilibrium outcome.

Is it reasonable to expect that evolution will have any cutting-power in this set-up? There are several reasons why this might be so. First, the institution of gift-giving has a very long history in human culture (Mauss 1990). This allows evolutionary arguments a sufficient time-scale to operate. Second, we frequently engage in gift-giving, thus we might expect that evolutionary pressures can operate relatively fast. The underlying argument of Waldfogel (1993) is that giving gifts in kind is an inefficient institution and that we should expect in time to see it replaced by the more efficient institution of cash giving. The interpretation that we have in mind is imitation of more succesful strategies rather than reproductive success. Since money in terms of the history of human evolution is a relatively recent development, one might argue that social evolution has not yet had sufficient time to operate on this particular institution. The argument developed here is stronger and shows that even if the institution of cash giving had preceded the institution of giving gifts in kind, then giving gifts in kind will eventually emerge and will be the dominant institution supporting social exchange in the long run.

3.5.1 A Model of Evolution

In this section we adapt the model of Kim and Sobel (1995) to our setting. Kim and Sobel's main result however refer to *games with common interest.* Below we argue that the gift giving game is not a game with common interest, so that we have to extend Kim and Sobel (1995)'s result to our setting.

Suppose that we extend the gift giving game with a round of cheap talk, where only R players are allowed to talk. We refer to this extended game as the communication game. Prior to learning their type a R player may send a message to G from some finite set of words W. W contains at least two words. After R has sent the message the gift giving game is played (which we refer to as the underlying game). A strategy for G maps each statement in W to a strategy in the underlying game. For R a strategy is some statement in W and a strategy in the underlying game.

Suppose there are $5N$ (finite) players in each $\mathcal{G}$ and $\mathcal{R}$ respectively. Each player in a population is matched in a round robin fashion with every player in the other population. The payoffs to the players is the sum of payoffs for each match in the underlying game.

Dynamics Time is discrete: $s = 1, 2, \ldots$ Assume that the population starts in an arbitrary strategy profile. At each time instance each player meets all of the other players in the other population and plays the communication game once.

At the end of each round one member is given the opportunity to revise her strategy. A strategy for player role i, $i = R, G$, z_i, *improves* upon another strategy, z_i', if z_i gives rise to a payoff which is at least as high as the payoff to z_i' against the current population.

We follow Kim and Sobel in assuming that the strategy revision obeys:

1. Exactly one member of the population may change her strategy after each round.

2. A player who has performed worst in her population in a round revises her strategy with positive probability.

3. Any strategy that *improves* upon the strategy being replaced is adopted with positive probability.

4. A strategy that does not improve upon the revising player's strategy is not adopted.

Let a strategy profile be denoted z. Let $A(z)$ be the set of strategy profiles that can be reached when the process is started at z. Extend this notation to sets of population profiles Z, such that $A(Z)$ is the union of $A(z)$ for $z \in Z$.

Following Kim and Sobel (1995) a set Z^* is *stable* if $A(Z^*) = Z^*$.

Games with Common Interests Following Kim and Sobel (1995) a game has *common interests* if in the set of feasible payoffs (of the underlying game) there is a unique point that strongly Pareto-dominates all other feasible payoffs. The canonical example of game with common interest is a two player 2 by 2 coordination game, where one of the equilibria strongly pareto-dominates the other equilibrium

In our context the strategies supporting the efficient equilibrium payoff of the underlying game does not have this property. To see this note that e.g. when $c = b$ and play is according to strategies that lead to efficient equilibrium payoffs then there is another strategy for G players that are preferred by R players, namely that keeping the behaviour of t_0 and t_1 types unchanged, t_2 types instead of not sending gifts and not investing send money gifts and invest. This improves R's expected payoff, since compared to the efficient equilibrium strategy she gets a money transfer and gets to free-ride. Thus R earns a strictly positive payoff from a match with t_2 types, whereas when play is according to the equilibrium profile leading to efficient payoffs she gets payoff 0. Moreover the behaviour of t_2 types do not affect the payoff she gets when faced with either t_0 or t_1 types since these players do not send money.

Comment Note that we only allow R players to talk. The reason is that G players are indifferent between equilibria which differ only by which h is used. However R players are not indifferent. By only allowing R to talk we give her maximum scope for choosing her preferred equilibrium profile. If both players were allowed to talk then the process could visit equilibrium profiles (by *drift*) where $h \notin \{h^0, h^+, h^-\}$. In a model with two sided gift giving allowing both players to talk would allow both players to receive efficient gifts-in-kind.

3.5.2 Selection

The plan of this section is as follows. First we establish that from any starting configuration with positive probability the process will end up in a configuration that leads to payoffs corresponding to an efficient equilibrium payoff in the underlying game. Thus strategies that lead to efficient payoffs are stable. While we are not able to show that once the process hits such a configuration then it will remain there forever, we are able to show that in any element of a stable set G players of type t_0 and t_1 play according to the efficient equilibrium profile for sufficiently low level of the cost of gifts c. The process will also visit states where e.g. t_2 types send gifts and invests. However as this is not an equilibrium of the underlying game G players will eventually update their strategy and the process will return to the efficient equilibrium profile of the underlying game. Thus such a stable set is relatively small.

We focus our analysis on the case where the cost of the gift is just sufficient to deter free-riding that is $c_1 \geq c = b$. This facilitates the analysis without losing the qualitative message. In particular this means that equilibrium $s^*_{II,K}$ exists for all parameter values, and consequently $s^*_{I,K}$ does not exist.

Let Z^* be the set of all strategy profiles where R sends some message and plays according to her part of the strategy profile $s^*_{II,K}$ where $h \in \{h^0, h^+, h^-\}$, in the underlying game, and where G for any message that she receives plays according to her part of $s^*_{II,K}$.

We first show that from any starting strategy profile with positive probability the process hits a strategy profile in Z^*.

Proposition 19. *Suppose* $0 < b = c \leq c^*(\rho^*)$. Z^* *is contained in a stable set.*

Proof. Start from any arbitrary strategy profile where play in the underlying game is not currently according to $s^*_{II,K}$. One by one, while G players strategies remain fixed, let R players revise their strategies such that they respond optimally to the current G population. Suppose this leads them to send message w'. Now let role G players adjust their strategies, one by one, such that they all respond optimally to the strategy of R players. In particular suppose that for any message t_2 types do not send gifts and do not invest (recall that this is a dominant strategy for t_2 types). Further suppose that in response to some unsent message w'' they play according to their part of $s^*_{II,K}$. By proposition 18 $s^*_{II,K}$ leads to efficient equilibrium payoffs. Now let R players revise their strategy such that they send w'' and play according to their part of $s^*_{II,K}$. Finally let G players update their strategy such that in response to any w they play according to $s^*_{II,K}$. □

The next proposition shows that if the process is started in a strategy profile contained in Z^* then via drift the process will leave it again. However for any B and $\rho \leq \rho^* < 1$ there is a sufficiently low cost level $0 < b = c \leq c^{**}(\rho^*)$ such that in any other element of a stable set t_0 and t_1 types play according to $s^*_{II,K}$ in the underlying game.

Proposition 20. *For any B and $\rho \leq \rho^* < 1$ there is a $c^{**}(\rho^*) > 0$ sufficiently small, such that if $b = c \leq c^{**}(\rho^*)$ then a strategy profile z is an element of a stable set if and only if t_0 and t_1 types play according to $s^*_{II,K}$ in the underlying game.*

Proof. The proof is in two steps. In the first step we show that via drift we can transit to a state where t_0 and t_1 play according to $s^*_{II,K}$, while t_2 types do not play according to $s^*_{II,K}$, but play such that inference from t_0 and t_1 types are unaffected (e.g. let them send money and invest) and that we can return to a strategy profile in Z^* again. Then in the second step we show that we cannot transit to other profiles. Details can be found in appendix B.5. □

Taken together the two propositions show that the only states which are stable are states such that t_0 and t_1 types give gifts-in-kind according to an efficient convention.

Corollary 5. *Suppose $b = c \leq c^{**}(\rho^*)$. Then in any stable state G-players of type t_0 and t_1 earn the efficient equilibrium payoff of the underlying game. R-players earn at least a payoff corresponding to the efficient equilibrium payoff of the underlying game.*

Comments Kim and Sobel (1995) show that in games of common interest play will eventually settle down on equilibrium profiles which gives rise to efficient payoffs. The gift giving game presented here is not pure common interest, but shares many of it's features. In particular both players have a common interest in using gifts-in-kind as a vehicle for generating social interaction, at least if the signal quality is not too good. However R players are also interested in extracting gifts and free-riding on t_2 types, even when it is not in t_2 types interest. That is we can find parameter restrictions such that R have common interests with gift giving t_0 and t_1 types but never with t_2. This disagreement is the reason why we are not able to select for profiles which generate the efficient equilibrium payoff.

3.6 Conclusion

This paper has provided a stylised argument supporting the social practice of giving gifts-in-kind as a more efficient convention than a social convention which involves the exchange of money. We identified a trade-off between the extrinsic and intrinsic value of an equilibrium with gifts. We also presented an evolutionary argument which when applied to our model selects an equilibrium with gifts-in-kind. In our model with no gift giving there is no social interaction. Moreover it is important that the gift is given away.

If one could simply signal one's type by wearing a Rolex, then supposedly free-riders could enter a population, buy themselves a Rolex, and then use it repeatedly to free-ride on other players.

An equilibrium with gifts-in-kind is more efficient along the extrinsic dimension relative to equilibria with money, because of how it affects the receiver's beliefs about the giver's beliefs about the receiver's type. When a player receives money she only knows that the giver thinks that the players are compatible, however with positive probability the giver's signal was not correct. When she receives a gift-in-kind e.g. according to the rule: give-her-something-that-you-would-like-yourself then when she receives the gift she is able to distinguish (perfectly) between the event that the two are compatible or not. The personal gift reveals something about the giver's "absolute" type, not just her relative type. This is beneficial to both players because it allows the receiver to avoid interaction in cases where it's in both players interest to do so.

The model presented here does not explain the widespread practice of generic gifts, such as chocolate, flowers etc. But such gifts rarely serve more purpose than to perhaps signal to the receiver that the giver is an interested party and would want to continue interaction. As relationships mature, and the stakes become higher the nature of gift giving changes profoundly into one where what is given becomes important for the decision to continued interaction. According to Caplow (1982, 1984) the wrong gift may put severe strains on a relationship, and may sometimes be the cause of it's termination.

The model may ultimately provide a way of estimating the value of social interaction. E.g. the welfare loss found by Waldfogel (1993) can be interpreted as a lower bound on the value of social interaction.

Chapter 4

Revisiting Schelling's Spatial Proximity Model

4.1 Introduction

Segregation in residential areas is often claimed to be associated with bad economic outcomes. Several empirical studies has shown that neighbourhood effects are important determinants of human capital accumulation, crime-rates, level of unemployment, etc. (Borjas (1995), Topa (2001)). At the same time although the process is clearly dynamic and subject to change, residential segregation is a recurring and remarkable stable social phenomenon in many larger cities in western societies, and is perceived by the polity to strike at the very foundation of social justice.

Schelling's (1969, 1971, 1978)[1] model of local interaction in neighbourhoods is as stunningly simple as the results are striking. Schelling shows that weak incentives for residents to live with people like themselves at the micro-level can lead to remarkable order and relatively high segregation at the macro-level. Segregation emerges in Schelling's model although none of the residents have a strict preference for segregated local neighbourhoods.

In Schelling's model residents live on a line. There are two types of residents A and B. Each resident is concerned about the composition of her local neighbourhood. Her preferences are such that she prefers to have at least one of the two neighbours adjacent to her to be of the same type as herself. Schelling then considers an adaptive procedure through which residents are given the opportunity to revise their current choice of location. The adjustment process ends when no individual can move to a location where she

[1] Schelling (1978) considers interaction in two dimensions and allows for empty locations.

will get higher utility. The process admits many stable neighbourhood configurations ranging from fully integrated to fully segregated configurations. By simulation with pen-and-paper Schelling shows that when the process has converged the composition of local neighbourhoods are often relatively segregated.

In this paper I revisit Schelling's spatial proximity model of interaction in one-dimensional local neighbourhoods[2]. The main question that the paper addresses is: How robust is segregation? In particular how does it depend on the preferences of agents? How does it depend on the behavioural assumptions about how agents find new locations when they decide to leave their current location? Does it depend upon whether location revision is noisy? How long do we have to wait for segregation to obtain, and does it depend on the preferences of the agents? Young (1998, 2001), Pancs and Vriend (2003) have previously revisited Schelling's model using some of the techniques used here[3]. I defer the discussion of these papers and the differences with this paper to a later section.

In a stochastic variant of Schelling's model, where agents occasionally make location mistakes, I show that segregation remains robust under two different assumptions about the preferences of agents: when they have Schelling type preferences and when they have a strict preference for integration. However the time for segregation to obtain depends on preferences. Evolutionary forces tend to push neighbourhood evolution towards segregation, but when agents have a preference for integration waiting times are much longer. The basic reason is the following. With Schelling type preferences in order to transit from a stable state to a (fully) segregated state it is sufficient that agents make relatively costless mistakes moving from one cluster of agents like themselves to a single cluster containing all agents like themselves. When agents have a strict preference for diversity these relatively costless opportunities arise more seldomly, as agents must now move from integrated locations to the border of a large cluster of agents which offers an integrated location. Thus it is the supply of integrated locations which largely determines how fast the segregated state is reached. With Schelling type preferences it is the (much larger) supply of locations which offer (weak) majority neighbourhoods that determines waiting times.

[2]Schelling (1971, 1978) also looked at two-dimensional neighbourhoods but the study of these is outside the scope of this paper.

[3]Möbius (2000) considers a hybrid model of Schelling's neighbourhood tipping model allowing for local interaction. In this model instead of relocating within the residential area unhappy residents leave the area and are replaced with residents from a pool of potential residents. Möbius uses this model to explain rapid ghetto formation in Chicago.

How can the numerical simulation results be reconciled with the empirical observation that we tend to see segregation in a wide variety of settings? In our model the main reason why states that are not fully segregated tend to be observed for long periods of time is that players are boundedly rational in a specific way. In particular they do not attempt to make any predictions about the evolution of neighbourhood structures; but instead optimise myopically. A more sophisticated player who tried to predict the evolution of her local neighbourhood might not only be motivated by living with someone like herself, but also how many others like herself who lives close by. Intuitively since residents sometimes make errors in their location decision, living in clusters with more people like yourself might be more desirable since the probability of ending up living with people different from yourself sometime in the future decreases. I show that if players care about their local neighbourhood composition and the size of the cluster of players like themselves they belong to enters lexicographically, then convergence to the set of segregated states vastly speeds up. One interpretation is that the fear of isolation leads to segregation.

Is the segregation result robust to small pertubations of preferences in the population? In large populations it is reasonable to think that there is some degree of heterogeneity. I assume that a small fraction of residents are "social activists". These residents strictly prefer diversity, but prefer to live in a minority rather than living only with people like themselves (formally I flip their 2nd and 3rd ranked alternative relative to the rest of the population). For the case where all other players have a preference for diversity the presence of just one "social activist" is sufficient to upset the result that segregation obtains in the long run. In fact only integrated states are now stable. This suggests a significant role for "social activists" in creating better outcomes. A single social activist is needed in order for pareto-superior outcomes to be selected in the long run. This suggest that the model where players have a preference for diversity is not robust against a small pertubation to the distribution of preferences in the population. The model with Schelling type preferences is robust to this pertubation.

Related Literature Since this paper is not the first paper to revisit Schelling using the technique of stochastic stability I will give a relatively thourough outline of the main differences to this paper.

Young (1998) presents a formal analysis of a model similar to Schelling's. Again there are two types of residents, and residents have the same preferences as in Schelling. Young considers an adjustment process through which two residents can agree to swap flats, specifically they exchange locations if at least one of the residents are better off and the other is no worse off after the swap. This adjustment process has many stable states, but importantly in every stable state all residents will have at least one neighbour like themselves. Young then introduces a small amount of noise into the system so that unfavourable trades may sometimes take place by mistake. As the noise vanishes there is strong selection among the stable states. Young shows that the only stable states that survive this refinement are precisely the segregated states.

Young (2001) considers a variation of Young (1998) where residents have a strict preference for integration, that is their most preferred outcome is to live in an integrated neighbourhood with one of each type. Young shows that with these preferences segregated states remain the only stochastically stable states. Thus even though residents prefer to live in integrated neighbourhoods the end up living only with their own type. To see how Young reaches his result and to motivate a different formalisation of Schelling's model I look more closely at Young's model. In Young if a resident has only neighbours different from herself then she is *discontent*, if she has only neighbours like herself then she is *moderately content* and if she lives in a mixed neighbourhood then she is *content*. There are no empty locations in the model, instead residents may exchange locations with one another, and they will do so with positive probability only if the exchange is pareto-improving. This assertion is true provided that residents are permitted to give *side payments* to one another. In particular in order to show that the unperturbed dynamic process will converge to a state in which all residents have at least one neighbour like themselves, Young relies on a side payment between two residents of opposite types: one resident who is discontent and one resident who is content (so that after the exchange of locations the content resident is now moderately content, while the discontent resident is content). Schelling's original model does not involve pair-wise flat exchanges, rather discontent residents make unilateral moves (residents simply squeeze in between two other residents). Pair-wise exchanges may be a better "real" world approximation, than Schelling's assumption. However I prefer a formalisation as close to Schelling as possible in order to be able to isolate the separate effects of the building blocks of the model. The formalisation presented here focuses on how *individual* actions lead to segregation more clearly than in Young. In my model individuals make location decisions unilaterally.

Young does not numerically simulate the model. One may wonder how long it takes before the set of stochastically stable state is reached. The simulations presented in this paper suggests that the model a long the lines of Young may display very long waiting times, making the medium run behaviour of the process more interesting.

Pancs and Vriend (2003) also study Schelling's model. Their paper may be viewed as an attempt to isolate the crucial assumptions, responsible for driving the segregation result in the one- and two-dimensional version of Schelling's model. For the purposes of this discussion I focus on the one-dimensional setting. In their variant of Schelling's model Pancs and Vriend show that segregation occur under quite mild assumptions about preferences over neighbourhood composition. In particular segregation obtains even when residents have a strict preference for perfectly diverse neighbourhoods. Although our behavioural assumptions are quite different this paper delivers the same message, with certain qualifications outlined below. Thus our approaches may be viewed as complimentary.

Our papers differ in our behavioural assumptions about how agents find new locations. Pancs and Vriend consider a stochastic non-noisy best-response dynamic[4], and show that the dynamics always converges to the set of segregated states. I consider a better reply dynamics, which I find more appropriate for this low rationality environment[5] and implicitly assume (as did Schelling) that there is some small cost of changing locations. Pancs and Vriend on the other hand not only assume that this is costless, but that players move with positive probability to locations which leaves utility unchanged. This assumption partially drives their segregation result.

Whereas Pancs and Vriend assert that "*A sufficient condition* [for segregation] *on the utility function is that it implies a strict preference for perfect integration*" (Pancs and Vriend 2003, pp. 42-43) this is not the case in our set-up. In our set-up if all players strictly prefer integration to being in a minority to being in a (local) majority, then the only stable outcome is that of perfect integration. This suggest that segregation is non-robust to the behavioural assumptions. Specifically our results differ because Pancs and Vriend consider a best reply dynamics, whereas I assume that strategy revision follows better reply. This is crucial in the one dimensional set-up, since segregation in Pancs and Vriend is driven by the fact that a perfectly integrated location, is strictly preferred, and exists in any configuration (as it does in my set-up). This implies that how players order low ranked neighbourhoods is inconsequential. In contrast when players follow better replies this is no longer the case.

[4]The stochastic elements concern who gets to update and conditional on updating a residents mixes between all of her best responses.

[5]Schelling also seemed to favour this interpretation, stating that a resident moves to the closests location where she is better off.

Perhaps the most important difference between our papers is that the present paper also considers the robustness of segregation when there is some heterogeneity in preferences. In particular I show that slightly perturbing the population composition away from a homogeneous population where players have a strict preference for integration, results in the perfectly integrated state being the only stable configuration. In Pancs and Vriend's set-up such a pertubation would have no effect on their segregation result; effectively a consequence of the best reply dynamics they consider. Supposing that integration is desirable from a social perspective (and thus supposedly that at the individual level integration is desirable), Pancs and Vriend (2003) argue that the forces driving segregation cannot easily be stopped. The current paper has a more positive message: a few "social activists" may indeed help to signficantly overcome segregation.

The remainder of this paper is organised as follows. In section 4.2 I first study analytical properties of Schelling's spatial proximity model when players live on a circle. I numerically quantiyfy how the model performs in terms of time to convergence and properties of equilibria which are selected by the model. I then introduce noise to how residents make location decision and show that the only stochastically stable states are the segregated states. I also present numerical results which show that for the model at hand stochastic stability is misleading about what neighbourhood patterns which will be observed within an economically reasonable time horizon. I then present an alternative model where convergence to the stochastic stable states is relatively fast.

In section 4.3 I assume that residents value diversity per se. Nevertheless I show that in the long run neighbourhoods will be segregated. To test the robustness of this result I allow for heterogeneity in agents neighbourhood preferences. The segregation result is not robust to this pertubation.

In section 4.4 I provide some concluding remarks.

4.2 Local Interaction with Schelling Preferences

In this section I first revisit Schelling's (1969, 1971) model of interaction where residents live on a line. I find conditions for a neighbourhood configuration to constitute a Nash Equilibrium. I also show that the process converges to a Nash Equilibrium starting from any configuration. I complement the analytical results with simulations.

Next I introduce noise to the location decision of residents. I set-up the model in a Markov framework which allows me to use *stochastic stability* (Young 1993) to select among equilibria of the game. I show that in the long run the process will spend almost all of its time in the set of segregated states. Then I show via simulations that the model performs poorly in the sense that the wait until a segregated state is reached, starting from a randomly drawn configuration, is so long as to make it meaningless from an economic perspective. I suggest an alternative model with better properties in terms of the wait until a segregated state is reached. This naturally comes at a cost: I have to make a stronger assumption about the preferences of residents.

4.2.1 Schelling's Model

Schelling (1969, 1971) originally modelled interaction on a line. I follow him closely here except that I connect the line at both ends to form a circle. This allows me to abstract from specifying particular conditions at the two ends of the line, while leaving the qualitative results unchanged. Since I am particularly interested in what elements of the model that drives the selection of equilibria away from diverse neighbourhood structures I limit the analysis to the case where there is an equal (and even) number of residents of each type, such that a (fully) integrated equilibrium exists.

Model

There are two types of players A and B. There are $n > 1$ of each type, so that the total number of players is $2n$. Each player occupies a relative position on the circle, and have preferences over her local neighbourhood composition. Let R be the set of players/residents.

Let L be the set of locations with typical element l_i, $i = 1, \dots, 2n$, that is there are exactly as many locations as there are residents. The *neighbourhood* of location l_i are all locations which are within $r \geq 1$ steps of location l_i:

$$N_i = \{l_j \in L : |i - j| \leq r\}$$

Note that l_i is itself contained in the neighbourhood of l_i. I refer to the players contained in N_i (apart from the player at l_i) as the *neighbours* of l_i.

A configuration σ is an assignment of all players to a position on the circle such that no two players occupy the same position. Given a configuration σ let $\#N_i^t(\sigma)$ be the number of players of type t in the neighbourhood of l_i, $t = A, B$. I sometimes omit σ when there can be no misunderstanding. Let the set of all possible configurations be Σ.

Preferences Each player cares about her local neighbourhood composition. In particular a player of type t currently residing on location l_i gets the following payoff:

$$u_i^t(\sigma) = \begin{cases} 0 & if \quad \frac{\#N_i^t}{2r+1} < \frac{1}{2} \\ 1 & otherwise \end{cases}$$

Note that the player counts herself in the neighbourhood. Let the set of players who have utility 0 in σ be $\underline{R}(\sigma) \subseteq R$.

Dynamics Schelling specifies the following adjustment dynamics. Let time be discrete $\tau = 0, 1, 2, \ldots$. Let the configuration at time τ be $\sigma(\tau)$. Start with an initial (randomly selected) configuration $\sigma(0)$. In each period τ pick one player in $\underline{R}(\sigma(\tau))$ by some exogenously specified procedure e.g. pick the player in $\underline{R}(\sigma(\tau))$ who has the lowest location number. Let her insert herself at a position in which she gets utility 1, such that she has to travel the fewest steps from her current position. If the player vacating l_i inserts herself at position $l_{i'}$ the new configuration $\sigma(\tau + 1)$ is as follows: if the fewest number of steps from l_i and $l_{i'}$ is accomplished by moving counter-clockwise then all players who resided on locations l_j, $j = i', \ldots, i-1$ are moved one position clockwise so that the new location of a player who resided at location l_j is now l_{j+1}. E.g. the player living on l_{i-1} lives on l_i in the new configuration. And equivalently if the fewest number of steps can be accomplished by moving clockwise.

Analysis

In this section I give analytical results for neighbourhood size $r = 1$. When I present simulation result I also examine the case where $r > 1$.

As a first step in the analysis I look for patterns that are stable with respect to the adjustment process. Then in the next step I establish that from any initial configuration Schelling's model converges to the an element in the set of stable configurations.

Stable Configurations I will say that a configuration σ is *stable* under Schelling's adjustment procedure if no player will want to change her location, given her current location and the assignment of the other players to locations on the circle.

Definition 3. *A* ***cluster*** *is a contiguous group of at least two players of the same type. A cluster is* minimal *if it is of length 2.*

Definition 4. *A configuration is* ***integrated*** *if all residents belong to a minimal cluster. A configuration is* ***segregated*** *if all A players belong to the same cluster.*

Throughout I will assume that $n \geq 2$ is even. This is a necessary condition for integrated configurations to exist. Note that n even is not required for the characterisation of stable configurations, nor to establish convergence.

I now characterise stable configurations:

Proposition 21. *Suppose $r = 1$ and $n \geq 2$. A configuration is stable with respect to Schelling's dynamics if and only if all players have at least one neighbour of her own type.*

Proof. Suppose a player does not have any neighbours of her own type. Since the player can insert herself between any two players, independent of the configuration, she can move to a location where she has at least one neighbour of her own type.

Now suppose all players have at least one neighbour of her own type. Then she gets utility 1 and has no strict incentive to change her location. □

Let the set of stable configurations be denoted $\Sigma^* \subset \Sigma$.

Remark 9. *Note that if $n \geq 4$ and even the set of stable configurations contain the fully integrated configuration, in which all players live in a diverse neighbourhood:*

$$\cdots AABBAABB \cdots$$

and the segregated configuration:

$$\cdots AAAABBBB \cdots$$

I now establish that Schelling's model with dynamic adjustment converges to some configuration in Σ^* starting from any initial configuration, $\sigma(0) \in \Sigma$.

Definition 5 (Convergence). *The process has **converged** if at any period τ, when players change locations according the dynamic adjustment process, the following holds:*

$$\sigma(\tau + 1) = \sigma(\tau)$$

Given a configuration σ, let σ' be the configuration which is constructed from σ by moving player i from location σ_i to some location l: $\sigma' = \sigma_{(\sigma_i, l)}$.

Definition 6 (Pivotal). *If for some $j \in N_{\sigma_i}(\sigma) \setminus \{i\} : u_j(\sigma) \neq u_j(\sigma')$ then player i is **ex-ante** pivotal, P^-, for player j. If $u_j(\sigma) = u_j(\sigma')$ for all j then she is ex-ante non-pivotal, P_0^-.*

*If for some $j \in N_{\sigma_i}(\sigma) \setminus \{i\} : u_j(\sigma) \neq u_j(\sigma')$ then player i is **ex-post** pivotal, P^+, for player j. If $u_j(\sigma) = u_j(\sigma')$ for all j then she is ex-post non-pivotal, P_0^+.*

In words, if i is *ex-ante* pivotal then the utility of one i's neighbours in σ changes by i's move. If i is *ex-post* pivotal then the utility of one i's neighbours in σ' changes by i's move.

I now show that Schelling's dynamic process indeed converges:

Proposition 22. *Suppose $r = 1$. Starting from any initial configuration $\sigma \in \Sigma$ the process converges to some $\sigma \in \Sigma^*$.*

Proof. Suppose $\sigma(\tau)$ is not stable, otherwise we are done. Let the total number of players who receive utility 0 at time τ be $m(\tau)$. For any configuration we must have $0 \leq m(\tau) \leq 2n$. Let one of these players be i. i lives on location σ_i. i only has neighbours of the other type. Let her move to the location nearest to σ_i such that she has utility 1. Such a position must exist since $n > 1$, denote it l_i. By the rule of movement by inserting herself at l_i she now has one neighbour of each type. Let this new configuration be $\sigma' = \sigma_{(\sigma_i, l_i)}$.

i is either P^+ (for the player of her own type) or P_0^+. If i is P^+ for $j \in N_{l_i}(\sigma') \setminus \{i\}$ then the utility of j is now 1. i is either P^- (for either both or one of her neighbours) or P_0^-. If i is P^- for some $j \in N_{\sigma_i}(\sigma) \setminus \{i\}$ then the utility of j is now 1. Hence the lower and upper bound on the number of players with utility 0 in the next period, $m(\tau + 1)$, is:

$$m(\tau) - 4 \leq m(\tau + 1) \leq m(\tau) - 1$$

The process converges if for any τ: $m(\tau) = 0$. Thus after at most $m(0)$ steps the process has converged. □

Remark 10. *Note that the convergence result does not rely on the order in which players who have utility 0 move.*

Simulations

In this section I present results from numerical simulations of the model. Simulations were programmed in FORTRAN 95 (Compaq Visual Fortran v6.6)[6].

In the actual implementation of the dynamic process I used the following procedure. For each simulation the starting configuration is drawn as follows. I make n draws from the set of locations, without replacement, with each location being equi-probable. Then

[6] The Fortran programs are available on my website: http://homepages.ucl.ac.uk/~uctpv00.

the n players of a particular type are allocated to these locations. The n players of the other type are then allocated to the remaining available locations. The dynamic adjustment process is implemented as follows: in each period I find the first player who does not have any neighbours like herself starting from location 1. This player then moves to the location nearest to her current location where she will have at least one player like herself. This ends the period, and the adjustment process is repeated until all players have at least one neighbour like themselves.

Neighbourhood Radius 1 I now provide some simulation results which shed further light on the analytical results above. In particular I show that the adjustment dynamics described above select a subset of the stable configurations.[7]

Eye Balling I first simulate a few randomly drawn starting configurations, and see where the process ends up. I let $n = 10$. A dot over a player indicates that they player currently has utility 0. Recall that the player at the first position has neighbours at position 2 and 20.

Each succesive line represents one "dotted" player starting from the left who updates her location.

$$\begin{aligned} &\dot{A}\dot{B}\dot{A}BBAAABBAABB\dot{A}\dot{B}AABB \\ \rightsquigarrow\ &BAABBAAABBAABB\dot{A}\dot{B}AABB \\ \rightsquigarrow\ &BAABBAAABBAABBBAAABB \end{aligned}$$

with 4 clusters of each type.

$$\begin{aligned} &B\dot{A}BB\dot{A}\dot{B}\dot{A}\dot{B}AAAA\dot{B}AABB\dot{A}BB \\ \rightsquigarrow\ &BBBAA\dot{B}\dot{A}\dot{B}AAAA\dot{B}AABB\dot{A}BB \\ \rightsquigarrow\ &BBBAAABBAAAA\dot{B}AABB\dot{A}BB \\ \rightsquigarrow\ &BBBAAABBAAAAAABBB\dot{A}BB \\ \rightsquigarrow\ &BBBAAABBAAAAAAABBBBB \end{aligned}$$

with 2 clusters of each type.

[7] I will sometimes refer to a stable configuration as an equilibrium.

As can be seen the adaptive process leads to relatively segregated states, but seldomly states with full segregation.

Table 4.1 gives descriptive statistics about the number of moves to convergence starting from a randomly drawn starting configuration[8]. The standard deviation, and the min and max of the distribution is also reported. The number of residents of each type in the simulations $n = 10, 20, 50, 100$. The number of simulations in each column is 100.000.

Table 4.1: Convergence Time (Schelling)

	Number of Residents of each Type			
	$n = 10$	$n = 20$	$n = 50$	$n = 100$
Mean	2.931	5.767	14.273	28.081
Std.	1.176	1.635	2.588	3.646
Max	8	13	26	43
Min	0	0	4	13

Note: 100.000 Observations per column.

It can be seen from the table that convergence to an equilibrium is fast. Moreover the number of moves to convergence seems to increase linearly in the number of residents.

Table 4.2 shows the mean, standard deviation, the min and max number of clusters for different sizes of the interacting population. Again the number of residents is varied: $n = 10, 20, 50, 100$. Note that an equilibrium with cluster size $k = \frac{n}{2}$ is a (fully) integrated equilibrium where all players have one neighbour of each type, and that an equilibrium with cluster size equal to 1 is a (fully) segregated state. Any state with cluster size between $k = 1$ and $k = \frac{n}{2}$ is an equilibrium.

[8] As pointed out above the starting configurations are chosen in the following way: n out of the $2n$ positions are chosen; all locations having equal probability of being drawn. Each of the n positions are then populated with one type residents, while the remaining n position are populated with the other type. This procedure is adopted throughout the paper.

Table 4.2: Number of Clusters in NE (Schelling)

	Number of Residents of each Type			
	$n = 10$	$n = 20$	$n = 50$	$n = 100$
Mean	2.338	4.491	10.972	21.783
Std.	.591	.831	1.229	1.825
Max	5	8	16	29
Min	1	1	6	14

Note: 100.000 Observations per column.

It can be seen from the table that only a subset of the set of equilibria of the static game is selected.

In the following figure the frequency with which some equilibrium with k clusters of each type is selected under the dynamic adjustment process is graphed for 100.000 randomly drawn starting configurations. It can be seen that the process selects a relative small set of the possible equilibria.

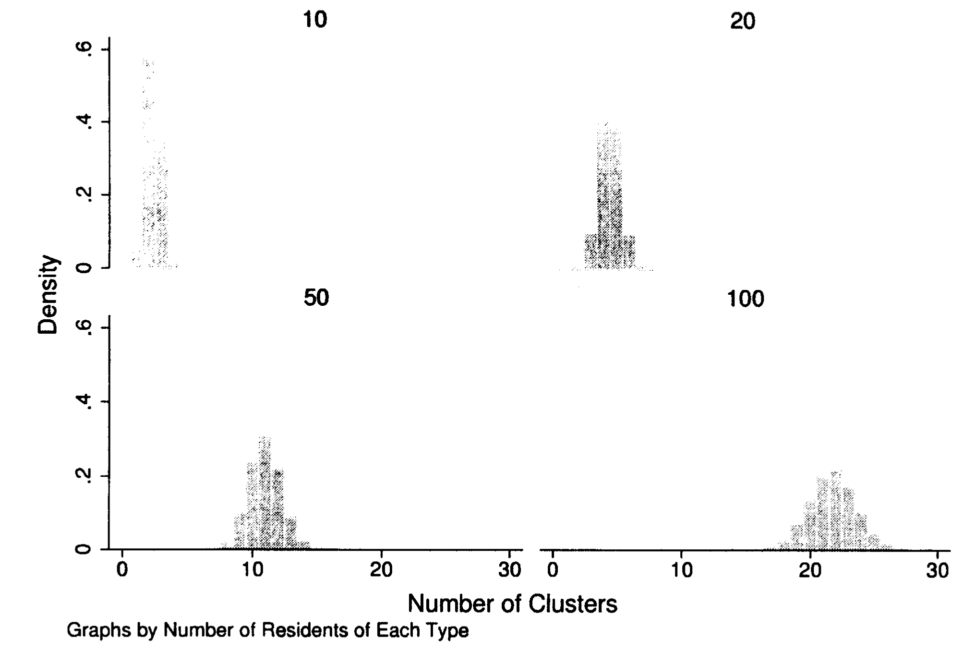


Figure 4.1: Distribution of clusters in NE. Neighbourhood radius = 1. Number of residents of each type: $n = 10, 20, 50$ and 100. 100.000 Observations per graph.

The distribution of clusters in the selected equilibria gives a straightforward proxy for how local neighbourhoods are composed. Even more informative is the distribution of local neighbourhoods. In the next table I report the median fraction of residents who have respectively 1 and 2 neighbours like themselves in their neighbourhood across simulations. Note that in any equilibrium all residents must have at least one neighbour like themselves. Also note that in the fully segregated state the fraction of players with 2 neighbours like themselves is 80%, 90%, 96% and 98% for $n = 10, 20, 50$ and 100 respectively.

Table 4.3: Composition of local neighbourhoods in NE (Schelling)

	Number of Residents of each Type							
	$n = 10$		$n = 20$		$n = 50$		$n = 100$	
Number Like	1	2	1	2	1	2	1	2
Median	.4	.6	.4	.6	.44	.56	.44	.56
Min	.2	0	.1	.2	.24	.36	.28	.42
Max	1	.8	.8	.9	.64	.76	.58	.72

Note: 100.000 Observations per column.

It can be seen from the table that as the number of residents increase the median fraction of players who live in neighbourhoods only with residents like themselves decrease. The distribution also becomes less dispersed around the median as the number of residents increase.

Changing the Neighbourhood Radius I now illustrate how changing the size of how residents conceive of their local neighbourhood affects convergence time and the distribution of the local composition of neighbourhoods. When the neighbourhood radius is greater than one, our clustering measure is no longer reliable as a configuration which is not an equilibrium when $r = 1$ will be an equilibrium when $r > 1$. E.g. when $r = 2$ the configuration where types alternate is now an equilibrium.

The following graph illustrates the convergence time to reach equilibrium. I have fixed the number of residents of each type to $n = 100$, and the neighbourhood radius is varied between $r = 2, 3, 4$ and 5. As can be seen the number of moves to convergence is increasing in the size of neighbourhood radius. This relation is likely to be non-linear, since if individual neighbourhoods equal the whole residential area then all configurations are stable.

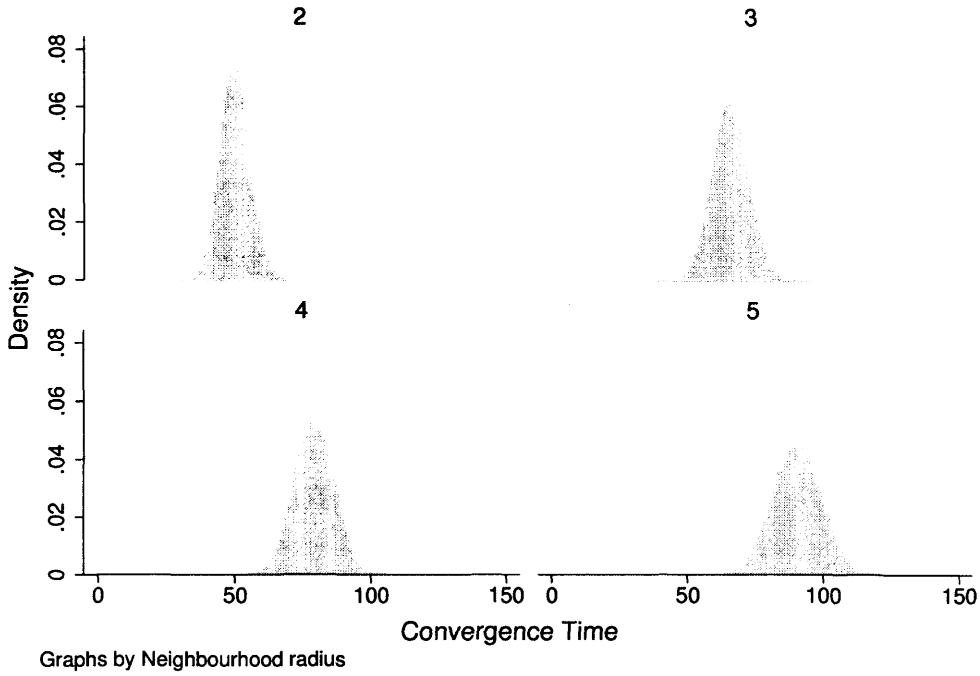


Figure 4.2: Convergence Time to NE. Number of residents of each type = 100. Neighbourhood radius $r = 2, 3, 4$ and 5. 100.000 Observations per graph.

The following table gives summary statistics about the distribution of local neighbourhoods once an equilibrium has been reached.

Table 4.4: Composition of local neighbourhoods in NE (Schelling)

	Neighbourhood Radius											
	$r = 2$			$r = 3$			$r = 4$			$r = 5$		
Number Like	2	3	4	<5	5	6	<7	7	8	<9	9	10
Median	.27	.24	.49	.37	.18	.46	.48	.14	.39	.49	.12	.4
Min	.14	.14	.29	.2	.1	.23	.24	.07	.15	.24	.06	.12
Max	.25	.33	.72	.55	.25	.7	.69	.2	.69	.74	.16	.7

Note: 100.000 Observations per column.

As can be seen from the table there is a significant number of residents who only live with people like themselves. When $r = 2$ only 27% of the residents live in neighbourhoods that are truly diverse, whereas 49% live in segregated neighbourhoods. When $r = 5$ this drops to 40% but nevertheless this is remarkably robust. The extreme distribution when we ask what the maximal fraction of residents who have only neighbours like themselves is also remarkably stable around 70%.

4.2.2 Schelling's Model with Mistakes

Above I established that from any starting configuration the dynamic process converges to some Nash Equilibrium. However the set of Nash Equilibria is "large", leaving the predictions of the model relatively weak. In this section I again look at Schelling's model, but now assume that players sometimes with small probability make location choices that are left unexplained by the model. This allow us to use relatively strong results from Markov Chain Theory. In particular I use the concept of *stochastic stability* (Young 1993) as the criterion for selecting among equilibria of the game. Intuitively a configuration is stochastically stable if it is relatively easy to get to the configuration, and relatively difficult to escape from it once the configuration has been reached, as the probability of making mistakes goes to zero.

I begin this section with a brief overview of the building blocks for the concept of stochastic stability. Then I present a variant of Schelling's model with mistakes, and show that the stochastic stable states are precisely the segregated states. The section ends with some simulation results.

Stochastic Stability

This section gives an overview of the necessary building blocks for the theory of stochastic stability. The review largely follows the exposition in Young (1998, chap. 3.3-3.4).

Elements of Markov Theory Let $P : \Sigma \to \Sigma$ be a finite state (time-homogenous) transition matrix. Specifically for every pair $\sigma, \sigma' \in \Sigma$, the probability of transiting at time t from σ to σ' at time $t+1$ is $P_{\sigma\sigma'}$. $P_{\sigma\sigma'} > 0$ if the process can transit from σ to σ' in one step. Otherwise $P_{\sigma\sigma'} = 0$.

Our interest is in how much time the process spend in various configurations. Suppose the initial state is σ^0. For each $t > 0$, $\mu^t(\sigma|\sigma^0)$ (a random variable) is the relative frequency with which state σ is visited up until period t. As $t \to \infty$ $\mu^t(\sigma|\sigma^0)$ converges almost surely to a probability distribution $\mu^\infty(\sigma|\sigma^0)$, which is the *asymptotic frequency distribution* conditional on starting the process at σ^0. $\mu^\infty(\sigma|\sigma^0)$ can be interpreted as a selection criterion since it tells us which states that have support in the long run frequency distribution. If $\mu^{\infty}(\sigma|\sigma^0)$ is independent of σ^0 then the process is *ergodic.*

A state σ' is *accessible* from σ, $\sigma \to \sigma'$ if

$$P^t_{\sigma\sigma'} > 0, \text{ for some integer } t > 0.$$

where P^t is the t-fold product of P. Two states σ and σ' *communicate* if both are accessible from the other, $\sigma \sim \sigma'$. This relation partitions Σ into a set of equivalence classes, called communication classes. A *recurrent class* of P is a communication class such that no state not in the class is accessible from the class. Let $E_1, \ldots, E_K$ be the K distinct recurrent classes of the chain. A state σ is *recurrent* if it is contained in a recurrent class, otherwise it is *transient.* Another way to understand this partitioning of states, is to see that a state σ is recurrent if conditional on starting the process in σ the probability that the state is visited infinitely often is equal to 1. If the process has only one recurrent class and this class is the entire state space, Σ, then the process is *irreducible.*

Let μ be a solution to:

$$\mu P = \mu, \text{where } \mu \geq 0 \text{ and } \sum \mu(\sigma) = 1$$

It can be shown that the solution is unique if and only if P has a unique recurrent class. μ is then referred to as the *stationary* distribution of P. If P has a unique recurrent class then μ describes the time-average asymptotic behaviour, and is independent of the initial state σ^0, that is:

$$\lim_{t\to\infty} \mu(\sigma|\sigma^0) = \mu^{\infty}(\sigma|\sigma^0) = \mu(\sigma)$$

If the system is also *aperiodic,* then for t large enough the position of the system can be approximated by μ. Let $\nu^t(\sigma|\sigma^0)$ be the probability that the state is σ at time t when it was started in σ^0. Hence we write:

$$\nu^t(\sigma|\sigma^0) = P^t_{\sigma^0\sigma}$$

If the process is irreducible and aperiodic then P^t converges to the matrix P^{∞} in which every row equals the stationary distribution μ:

$$\lim_{t\to\infty} \nu^t(\sigma|\sigma^0) = \mu(\sigma) \text{ for all } \sigma \in \Sigma.$$

that is with probability one $\nu^t(\sigma|\sigma^0)$ and $\mu^t(\sigma|\sigma^0)$ converge to $\mu(\sigma)$.

Perturbed Markov Chains This section looks at a family of perturbations of the Markov Process P, parametrised by the error rate ϵ. P^ϵ is a regular perturbed Markov Process (of P^0) if P^ϵ is irreducible for every $\epsilon \in (0, \epsilon^*]$, and for every $\sigma, \sigma' \in \Sigma$, $P^\epsilon_{\sigma\sigma'}$ approaches $P^0_{\sigma\sigma'}$ at an exponential rate:

$$\lim_{\epsilon \to 0} P^\epsilon_{\sigma\sigma'} = P^0_{\sigma\sigma'},$$

and

$$\text{if } P^\epsilon_{\sigma\sigma'} > 0 \text{ for some } \epsilon > 0, \text{ then } 0 < \lim_{\epsilon \to 0} P^\epsilon_{\sigma\sigma'}/\epsilon^{r(\sigma,\sigma')} < \infty \text{ for some } r(\sigma, \sigma') \geq 0.$$

$r(\sigma, \sigma')$ is the *resistance* of the transition $\sigma \to \sigma'$. Note that $r(\sigma, \sigma') = 0$ if and only if $P^0_{\sigma\sigma'} > 0$.[9]

For any $\epsilon > 0$ the process is now irreducible, and from the results of the previous it has a unique stationary distribution, denote it by μ^ϵ. Following Young (1993) a state σ is *stochastically stable* if:

$$\lim_{\epsilon \to 0} \mu^\epsilon(\sigma) > 0.$$

The main theorem of Young (1993) stated below establishes that for all σ this limit exists and that for each state this limit equals the stationary distribution of the unperturbed Markov process: $\lim_{\epsilon \to 0} \mu^\epsilon(\sigma) = \mu^0(\sigma)$. The theorem also shows how to identify stochastically stable states in terms of the stochastic potential of the recurrent classes of the unperturbed process. This is the final building block that I now introduce.

Suppose the unperturbed process has K distinct recurrent classes, $E_1, \ldots, E_K$. Take pairs of distinct recurrent classes E_i and E_j, $i \neq j$. An ij-path is a sequence of states $\xi = (\sigma_1, \sigma_2, \ldots, \sigma_q)$ that begins in E_i and ends in E_j. The resistance of the ij-path is equal to: $r(\xi) = r(\sigma_1, \sigma_2) + \cdots + r(\sigma_{q-1}, \sigma_q)$. Let $r_{ij} = \min r(\xi)$ be the ij-path that has lowest resistance (note that $r_{ij} > 0$ since E_i and E_j are distinct recurrent classes).

Construct a complete directed graph with K vertices (that is for each vertex k there is exactly one directed edge from k to each of the $K - 1$ remaining vertices). The weight of the directed edge $i \to j$ is r_{ij}. A j-tree is a set of $K - 1$ edges that from every vertex different from j, has a unique directed path in the tree to j. The resistance of a tree is the sum of the resistances along it's edges. The *stochastic potential* of E_j is the minimum resistance among all j-trees. Intuitively the stochastic potential of a recurrent class says something about how "easy" it is to get to the state starting from any of the

[9] If $P^\epsilon_{\sigma\sigma'} = P^0_{\sigma\sigma'} = 0$ for all $\epsilon \in (0, \epsilon^*]$, then $r(\sigma, \sigma') = \infty$.

other recurrent classes[10].

I now state a result from Young (1993) which characterises stochastically stable states in terms of stochastic potentials:

Theorem 1 (Young (1993)). *Let P^ϵ be a regular perturbed Markov process, and let μ^ϵ be the unique stationary distribution of P^ϵ for each $\epsilon > 0$. Then $\lim_{\epsilon \to 0} \mu^\epsilon = \mu^0$, and μ^0 is a stationary distribution of P^0. The stochastically stable states are precisely those states that are contained in the recurrent classes of P^0 having minimum stochastic potential.*

Technically introducing noise allows us to use the powerful results of the previous section for *ergodic* Markov processes. The system's behaviour becomes independent of the starting conditions for sufficiently large t. Moreover the stationary distribution tells us which states are likely to be visited in the long-run.

From an interpretative view one can think of perturbations as testing how robust equilibria are to the continual bombardment of small perturbations. Thus pertubations are a selection device, to select among recurrent states. Bergin and Lipman (1996) show that with state-dependent mutation rates, any recurrent state (of the unperturbed process) can have support in the set of stochastically stable states. Therefore assumptions made about how fast error-rates converge to zero is in no way innocuous.

Schelling's Model with Mistakes

Consider the following modified version of Schelling's model.

In each period $\tau = 0, 1, 2, \ldots$ a player is randomly selected. All players are equally likely of being chosen. Suppose player i at location l_i is chosen. i has the opportunity to move to a randomly selected location l (moving either clockwise or counter clockwise), not identical to her current location, with all locations $l \neq l_i$ being equally likely of being chosen.

The probability that player i moves to l depends on utility difference between the utility at her current location and the utility she would get if she moved to l. Assume that there are numbers $0 < \alpha < \beta < \gamma < \infty$. For $\epsilon \in (0, \epsilon^*]$, the decision to remain or move occur with the following state dependent probabilities:

[10]Ellison (2000) shows how the maximal waiting time can be bounded by using the concepts of radius and co-radius. Intuitively a recurrent state is stochastically stable if it is easy to enter the basin of attraction of the state starting from any of other recurrent state of the unperturbed dynamics, and if it is relatively difficult to exit its basin of attraction when the process is found in this state.

1. If i's utility increases at l then she moves there with probability $1 - \epsilon^{\alpha}$ and remains at her current location with probability ϵ^{α}. The decision to remain is termed a *mistake*.

2. If i is indifferent between her current location and l then she stays at her current location with probability $1 - \epsilon^{\beta}$ and moves with probability ϵ^{β}.

3. If i's utility decreases at l then she stays at her current location with probability $1 - \epsilon^{\gamma}$ and moves with probability ϵ^{γ}.

Mistake probabilities are chosen such that the greater the loss in utility from the mistake the less likely the player is to make it.

Analysis

The first result in this section shows that the only recurrent states are the stable configurations.

Proposition 23. *Suppose $r = 1$. Under the unperturbed dynamics, P^0, a state is recurrent if and only if it is a stable configuration. Moreover any recurrent state is also absorbing.*

Proof. $\Rightarrow$: Suppose $\sigma \in \Sigma^{NE}$. Then all players have at least one neighbour like themselves, and all players have utility 1. Accordingly no player will want to vacate her current location.

$\Leftarrow$: Suppose $\sigma \notin \Sigma^{NE}$. Therefore there must be at least one player who does not have any neighbours like herself. With positive probability this player will be drawn for revision and be matched with a location where she will have at least one neighbour like herself. Hence she will move there. Let this state be σ'. At her former location she must have had two neighbours different from herself. Therefore after she moves these player's utility must weakly increase. Also at her new location the utility of her neighbours must weakly increase: if she has two neighbours like herself then their utility is unchanged. If she has a mixed neighbourhood then the utility of the neighbour different from herself remains unchanged whereas the utility of the neighbour like herself is either unchanged or has increased. Thus in σ' there are at least one player whose utility has increased (the player who has relocated) and no player whose utilty has decreased. Hence either

$\sigma' \in \Sigma^{NE}$ or there is another player who does not have a neighbour like herself. But then we can apply the same procedure again. Since the number of players is finite under P^0 the process arrives at an equilibrium in a finite number of steps.

Finally note that under P^0 if the state is an equilibrium then the process will not leave this state, since no players have a strict incentive to change their location. Thus the state is absorbing. □

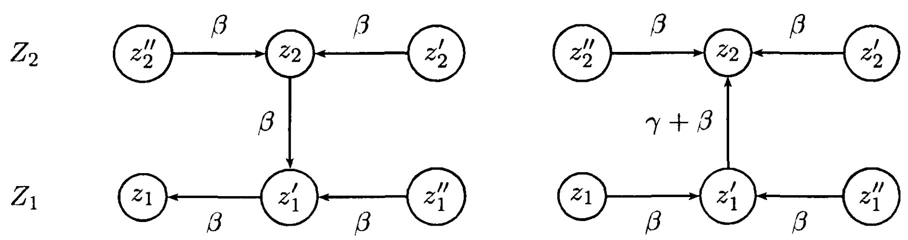
Next I show that the stochastically stable states are precisely the segregated states.

Proposition 24. *Suppose $r = 1$. A state is stochastically stable if and only if it is segregated.*

Proof. The proof can be found in Appendix C.1. □

The intuition for the result follows from the observation in lemma 10 that it is easier (i.e. cheaper) to reduce the number of clusters than it is to increase it. That is from any initial state transitions which tend to reduce the number of clusters are relatively more likely than the reverse. The rest is just accounting.

To get some intuition for how minimal cost trees are constructed using lemmas 10 and 11 consider the simplest case where $K = 2$. The following picture shows two minimal cost trees, the first rooted at $z_1 \in Z_1$, i.e. a segregated state, the other rooted at $z_2 \in Z_2$ i.e. an integrated state.



Tree rooted at z_1 Tree rooted at z_2

A minimal cost tree is constructed in the following way. Horizontal costs are given by lemma 11 and vertical costs are given by lemma 10.

Remark 11. *The claim of proposition 24 would also hold under the weaker (but less plausible) assumption that mutation rates are independent of the magnitude of utility* *loss. Suppose that mistakes which involve moves which leaves the player indifferent or worse off than at her current location both has cost β. Then the claim above still holds since:*

$$\sum_{k<k'} \beta > 0$$

holds for all $k' \geq 2$.

Simulations

In this section I present numerical simulations of the model for various parameter value of the number of residents and the noise level. The section has two main parts. In the first part I turn off the noise, i.e. I set $\epsilon = 0$. This allows us to compare how the stochastic selection procedure affects the convergence time, and clustering in the equilibria selected by the process and compare it to the deterministic version of Schelling. In the second part I examine how the model behaves when there is a positive level of noise present. I contain the simulations to the case $r = 1$.

No Noise Schelling's model contains two deterministic components that are given stochastic counterparts in our model. First, Schelling assumes that residents who enjoy low utility get to update their choice of location in a deterministic way. In Schelling in each round first all players that have no neighbours like themselves are marked out. Then starting from the right end of the line any player that was marked updates her location, until all marked players have either moved to a new location or they have at least one neighbour like themselves. After this a new round begins. Second, Schelling assumes that when a player moves she moves to a location closets to her current location where she has at least one neighbour like herself.

In order to see whether the stochastic counterpart of these rules play any role for convergence time and the clustering in the equilibria that are reached I simulate the model when the noise level is turned off, i.e. $\epsilon = 0$.

The first table shows the convergence time to Nash.

Table 4.5: Convergence Time (Stochastic Schelling)

	Number of Residents of each Type			
	$n = 10$	$n = 20$	$n = 50$	$n = 100$
Mean	44.6	123.9	444.0	1088.9
Std.	36.4	76.0	205.2	417.8
Max	365	666	1977	4112
Min	0	0	61	261

Note: $\epsilon = 0$, 10.000 Observations per column.

Comparing the results to Schelling's original model, it can be seen that convergence is slower for the stochastic version. This is not surprising since players who are never selected under Schelling's procedure is selected for revision in the stochastic version. In particular as the configuration comes close to an equilibrium, with only a few players needing to update their locations, the probability that these players are selected decreases in the stochastic version. Thus convergence rates slow down when the configuration gets close to an equilibrium. Nevertheless convergence is fairly rapid.

I now investigate whether the way players update their location choice has any bearing on what equilibria are eventually reached by the process. The next two tables gives details on this.

Table 4.6: Number of Clusters (Stochastic Schelling)

	Number of Residents of each Type			
	$n = 10$	$n = 20$	$n = 50$	$n = 100$
Mean	2.29	4.34	10.51	20.86
Std.	.70	1.01	1.60	2.25
Max	5	8	17	29
Min	1	1	5	13

Note: $\epsilon = 0$, 10.000 Observations per column.

Comparing with the results from simulating Schelling's original model, it can be seen that the mean number of clusters in the equilibria that are reached under the stochastic version is slightly lower than in Schelling's model. However the difference is not significant. This observation is confirmed by the next table.

Table 4.7: Composition of local neighbourhoods in NE (Stochastic Schelling)

	Number of Residents of each Type							
	$n = 10$		$n = 20$		$n = 50$		$n = 100$	
Number Like	1	2	1	2	1	2	1	2
Median	.4	.6	.4	.6	.44	.56	.42	.58
Min	.2	0	.1	.2	.2	.32	.26	.42
Max	1	.8	.8	.9	.68	.8	.58	.74

Note: $\epsilon = 0$, 10.000 Observations per column.

I conclude that the stochastic updating and selection of potential allocations does not have a significant impact on the distribution of equilibria that are reached. The only significant impact is on the time to convergence.

In the next part I turn on the noise.

Noise I showed analytically that in the long run the process will only visit the segregated states. However the result is silent about how long we have to wait before the long run kicks in. In this case simulations are a useful means of examining whether the selection of stochastic stability is economically meaningful. I am interested in how the model behaves with respect to three time aspects of the model: the short, medium and long run.

Definition 7. *The* ***short run*** *is the time interval:* $\{0, \dots, \tilde{T} - 1\}$, *where* $\tilde{T} \geq 0$ *is the random time where the process hits a recurrent class for the first time. The* ***medium run*** *is the time interval:* $\{\tilde{T}, \dots, \tilde{\tilde{T}}\}$, *where* $\tilde{\tilde{T}} \geq \tilde{T} \geq 0$ *is the random time where the process hits an element in the set of stochastically stable states for the first time. The* ***long run*** *is* $t : t > \tilde{\tilde{T}}$

For the purpose of the simulations I fix: $(\alpha, \beta, \gamma) = (1, 2, 3)$.[11]

The Short Run The following tables show the descriptive statistics of the time of convergence to a recurrent class.

[11]This choice is clearly somewhat arbitrary. The choice affects the rate at which mistake rates go to zero and therefore affects when the prediction of stochastic stability is valid.

Table 4.8: The Short Run (Stochastic Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Mean	42.9	45.2	49.1	123.5	124.8	135.8	454.5	483.3	489.5
Std.	35.8	36.0	37.7	75.8	77.0	81.6	202.4	224.1	238.4
Max	256	228	219	451	507	616	1533	1677	1658
Min	0	0	0	6	12	4	108	110	101

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 500 Observations per column.

The Medium Run The following table shows statistics of the duration of the medium run and the fraction of time spent in states that are visited before the process hits one of the stochastically stable states for the first time.

Table 4.9: Duration of The Medium Run (Stochastic Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
		($\times 10^3$)			($\times 10^4$)			($\times 10^5$)	
Mean	134.5	21.4	4.8	127.9	21.5	5.06	168.8	26.5	7.85
Std.	148.0	24.6	5.1	111.0	18.3	4.19	120.2	18.6	5.75
Max	140.2	153.0	28.9	919.2	116.9	29.6	603.7	111.1	39.4
Min	0	0	0	1.94	0	.047	18.0	2.25	.386

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 500 Observations per column.

From the table above it can be seen that the duration of the medium run increases exponentially in the number of residents.

Table 4.10: Visited States in The Medium Run (Stochastic Schelling)

n	10			20			50		
State	No. Clusters			No. Clusters			No. Clusters		
	2	3-5	OE	2-3	4-10	OE	2-4	5-25	OE
Mean	89.5	9.0	1.6	89.8	9.6	.6	86.7	12.1	1.2
Max	100	99.6	64.6	99.9	79.7	5.1	98.5	68.9	2.7
Min	0	0	0	18.1	0	.06	30.1	.6	.7
Obs.	449			499			500		

Note: $\epsilon = .05$, $(\alpha, \beta, \gamma) = (1, 2, 3)$, OE=Out of Equilibrium. Total number of observations per column: 500. Observations where process transits directly to long run not included.

Before transiting to the segregated states the process spends the majority of its time in states which are relatively segregated. Although the duration of the medium run is very long the process spends most of its time in states which are also very segregated.

The Long Run The following table gives statistics about the waiting time until the process hits a stochastically stable state for the first time.

Table 4.11: The Long Run (Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
		$(\times 10^3)$			$(\times 10^4)$			$(\times 10^5)$	
Mean	134.5	21.5	4.8	127.9	21.6	5.08	168.8	26.5	7.86
Std.	148.0	24.6	5.1	111.0	18.3	4.19	120.2	18.6	5.75
Max	140.2	153.0	28.9	919.2	116.9	29.6	603.7	111.1	39.4
Min	.01	.01	.01	1.94	.018	.060	18.0	2.26	.392

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 500 Observations per column.

Since the short run is very short compared to the medium run most of the behaviour of the process is already picked up by the medium run behaviour in table 4.9.

Long Run Behaviour The analytical selection result of stochastic stability is a limit result in the sense that if the noise level goes to zero then the process will spend almost all it's time in the set of stochastically stable states. In many real life situations factors which are unobserved to the modeller affect the decisions of agents. Therefore we might be unwilling to assume that the noise is vanishing. An alternative is to simulate the model starting from a stochastically stable configuration and then observing the time path of the system for a relatively long period. This allows us to numerically quantify the fraction of time that the process spends in the stochastically stable states. It also provides a way to quantify a threshold for the noise level where the prediction of stochastic stability gives a good approximation of where the system is at any time for a sufficiently long time horizon.

The table below shows the fraction of time spent in various configurations, when the process is started in a stochastically stable state. I track the process over 10^7 periods.

Table 4.12: Fraction of Time Spent in Stochastically Stable States (Schelling)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Stochastically Stable States	99.99	99.79	94.9	99.98	89.1	84.7	99.96	55.1	34.3
Other Recurrent States (*)	0	0	3.4	0	10.4	11.4	0	43.6	55.8
Out of Equilibrium	.01	.21	1.8	.02	.5	3.9	.04	1.3	9.9

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, Number of Periods: 10^7. (*) The process only spends time in recurrent states with less than 4 clusters.

It can be seen that the validity of stochastic stability as the prediction of the long run behaviour of the model depends naturally on ϵ but also on the number of residents. In fact for $\epsilon = .1$ and $n = 50$ the process spends a larger fraction of time in recurrent states with cluster size 2 (39.3%) than it does in the stochastically stable states. Our results indicate that given our somewhat arbitrary choice of α, β and γ choosing $\epsilon \leq .02$ yields sufficient predictive power of stochastic stability for $n \leq 50$.

Comments and a Critique The medium run is in fact very long. This suggest that the medium run behavior provides a better approximation of how neighbourhood evolution takes place.

Why does stochastic stability do such a poor job at predicting where we can find the process for reasonable time horizons? The basic intuition is as follows. Suppose the process is started at a state where there are two clusters of each type. The process hits the a stochastically stable state when all players are contained on one of the two locations. Due to our assumption about mistake probabilities we need consider only the events where a player moves from one cluster by mistake to the other cluster. Therefore we can approximate the process by a random walk where the process roughly changes state with probability ϵ^{β}. This base probability has to be modified by how large the cluster is: residents who live in small clusters are less likely to be drawn for revision than residents who live in a larger cluster. Thus as the size of a cluster shrinks the less likely that a player from that cluster is chosen. This introduces a bias towards clusters of equal size.

Take one of the two clusters which is currently of size m_t at time t. The probability with which this cluster grows or shrinks is governed by the following probabilities.

The cluster grows with the following conditional probabilities.

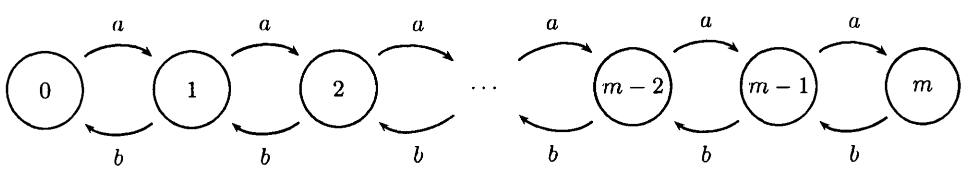
$$Pr(m_{t+1} = m_t + 1 | m_t) = \begin{cases} 0 & \text{for} \quad m_t \in \{0, n\} \\ \frac{(n-m_t)}{n}\epsilon^{\beta} & \text{for} \quad 0 < m_t < n-1 \\ \frac{1}{n} & \text{for} \quad m_t = n-1 \end{cases}$$

and it shrinks with the following probabilities:

$$Pr(m_{t+1} = m_t - 1 | m_t) = \begin{cases} \frac{1}{n} & \text{for} \quad m_t = 2 \\ \frac{m_t}{n}\epsilon^{\beta} & \text{for} \quad 1 < m_t < n \\ 0 & \text{for} \quad m_t \in \{0, n\} \end{cases}$$

The process above is a relatively complex object to analyse. Instead I consider a simplified process.

Consider the following random walk on the set of integers $Z = \{0, \ldots, m\}$, $m > 0$. With probability a the process moves up and with probability b the process moves down, where $a > b > 0$ and $a + b \leq 1$. Note that the process exhibits a positive drift towards m. The following figure illustrates the process:



I am interested in the conditional waiting times w_k, $0 < k \leq m$ until the process hits 0.

These can be determined from the following recursive relation:

$$w_k = a(w_{k+1} + 1) + b(w_{k-1} + 1) + (1 - a - b)(w_k + 1)$$

and the following two boundary conditions:

$$w_0 = 0$$
$$w_m = b(w_{m-1} + 1) + (1 - b)(w_m + 1)$$

This is a second order non-homogenous difference equation with constant coefficients, which can be solved explicitly by standard techniques (see e.g. Sydsaeter and Hammond (1995, p. 750)). Results are summarised in the following lemma.

Lemma 3. *Consider a random walk on the integers between 0 and m. The process moves up with probability a and moves down with probability b, where $a > b > 0$ and $a + b \leq 1$. The conditional waiting time w_k, $0 < k \leq m$, until the process reaches 0 is:*

$$w_k = \frac{1 - \left(\frac{b}{a}\right)^k}{(a-b)(1-\frac{b}{a})\left(\frac{b}{a}\right)^m} - \frac{1}{a-b}k$$

The waiting time until the process reaches 0 is bounded above by $-\frac{1}{a-b}m + o(\left(\frac{a}{b}\right)^m)$.

Proof. A particular solution to the equation follows from the guess: $w_k = Dk$ for some constant D. This leads to the particular solution:

$$w_k^* = -\frac{1}{a-b}k$$

The characteristic equation for the homogenous second order equation is:

$$ar^2 - (a + b)r + b = 0$$

which leads to roots: $r_1 = 1 \wedge r_2 = \frac{b}{a}$. So that the general solution is:

$$w_k = A + B\left(\frac{b}{a}\right)^k - \frac{1}{a-b}k$$

where A and B are constants. The result then follows from the boundary conditions.

The second observation follows from the fact that $w_m \geq w_k$, $0 \leq k < m$. □

This shows that for the random walk with drift the waiting time until the process reaches 0 increases exponentially in m.

The connection to Schelling's model is as follows. Consider a recurrent state of the unperturbed dynamics which contains two clusters of each type. For the purpose of illustrating the connection we need only concern ourselves with one type of players. Also for small ϵ we need only consider the possibility that β mistakes occur (recall $\gamma > \beta$ thus for small ϵ β-mistakes becomes exponentially more likely than γ-mistakes. That is the only movements that we need consider is that one player, by mistake, moves from one cluster to the other cluster. Let the two clusters be denoted c and $\tilde{c}$ respectively A stochastically stable state is reached whenever say all players from cluster c has moved to $\tilde{c}$. All such moves (except the last for $r = 1$) must occur via a β mutation. More importantly the process is biased towards selecting some player from a larger cluster rather than a player from a smaller cluster. To see this note that the probability that the some player from cluster c is selected equals $\frac{n_c}{n}$, since all players are equally likely of being drawn for revision, and with remaining probability some player from cluster $\tilde{c}$ is chosen. Thus as the cluster shrinks the probability that a player from the other cluster is chosen increases. That is the probability that the cluster grows (via a player from cluster $\tilde{c}$ making a location mistake becomes increasingly higher as the cluster shrinks. Thus the process is biased towards clusters of equal size. The main simplification is that for the random walk above I have assumed that the bias does not depend upon the state of the walk, whereas in the Schelling model the closer the state moves to 0 the more unlikely is a downward jump and the more likely an upward jump.

The key observation is that the selection dynamics exhibit a drift towards clusters of equal size. Therefore the maximal waiting time until a stochastically stable state is reached increases exponentially in the number of residents.

This observation leads to a specific criticism of the related work of Young (1998, 2001). The results developed here are equally applicable to Young's variant of Schelling's original model. That is the wait until the process hits a stochastically stable state is so long as to make it uninteresting except for a relative small number of residents.

There is a literature on the speed of convergence to the set of stochastically stable states (Young (1998), Ellison (1993, 2000)). The understanding of that literature is that local interaction greatly speeds up convergence to the set of stochastically stable states. This seems to be at odds with our findings and some comments are in order.

Ellison (2000) introduces the concepts of radius and co-radius to bound waiting times until the process hits the stochastically stable states. Formally he considers a model of evolution with noise by the triple $(Z, P, P(\epsilon))$, where Z is the finite state space, and a P is a Markov transition matrix on Z in discrete time. $P(\epsilon)$ is a family of markov transition families on Z indexed by $\epsilon > 0$, with the property that $P(\epsilon)$ is ergodic, continuous in ϵ

such that $P(0) = P$. Furthermore Ellison defines a cost function: $c : Z \times Z \to \mathbb{R}^+ \bigcup \infty$ such that for all pairs of states $z, z' \in Z$, $\lim_{\epsilon \to 0} P_{zz'}(\epsilon)/\epsilon^{c(z,z')}$ exists and is strictly positive if $c(z, z') < \infty$ (with $P_{zz'}(\epsilon) = 0$ for sufficiently small ϵ if $c(z, z') = \infty$).

The innovation of Ellison (2000) is to characterise the set of stochastically stable states in term of the relation between it's radius and co-radius. This allows him to provide a bound on waiting times. Roughly the radius of a recurrent class (of the unperturbed dynamics) says something about how difficult it is to escape the basin of attraction of the class. The co-radius captures the size of the basin of attraction of the class, and says something about how easy it is to reach the class from other classes. Intuitively if the co-radius is smaller than the radius then we are likely to find the process there for small ϵ. Formally, let Ω be the union of one or more recurrent classes. The basin of attraction of Ω, $D(\Omega)$, is the set of initial states from which the unperturbed process converges to Ω.

The radius of Ω, $R(\Omega)$, is the minimum cost of leaving the basin of attraction of Ω. Let a *path* out of $D(\Omega)$ be a sequence of distinct states $(z_1, z_2, \ldots, z_T)$, where $z_1 \in \Omega$, $z_t \notin Z \setminus D(\Omega)$, $2 \le t \le T-1$, and $z_T \in Z \setminus D(\Omega)$. Let $S(\Omega, Z \setminus D(\Omega)$ be the set of all such paths. Then the radius is defined as:

$$R(\Omega) = \min_{(z_1,\ldots,z_T)\in S(\Omega, Z\setminus D(\Omega))} \sum_{t=1}^{T-1} c(z_t, z_{t+1})$$

Ellison then defines the co-radius:

$$CR(\Omega) = \max_{x\notin\Omega} \min_{(z_1,\ldots,z_T)\in S(x,\Omega)} \sum_{t=1}^{T-1} c(z_t, z_{t+1})$$

With these definitions in place Ellison Theorem 1 can be stated:

Theorem 2 (Theorem 1, Ellison (2000)). *Let $(Z, P, P(\epsilon))$ be a model of evolution with noise, and suppose that some set Ω which is a union of recurrent classes $R(\Omega) > CR(\Omega)$. Then:*

(a) the long-run stochastically stable set of the model is contained in Ω;

(b) for any $y \notin \Omega$, $W(y, \Omega, \epsilon) = O(\epsilon^{-CR(\Omega)})$ as $\epsilon \to 0$.

Unfortunately the theorem has no bite in our context, except for small n. We already know that the set of segregated states are precisely the stochastically stable states. Let this set be denoted Ω. The radius counts the cost of escaping the basin of attraction of the segregated states. Hence $R(\Omega) = \gamma + \beta$ as a γ mutation is required for a player to leave the single cluster of players of her own type, and then a β mistake is required by some other player in the cluster. To find the co-radius we have to find the state x such that among all minimal cost paths from x and into $D(\Omega)$, x is the state which maximises this minimal cost. In our model with n players of each type the co-radius is at least of size $\beta(\frac{n}{2} - 1)$. Since this is the minimum cost of starting from an integrated state and reaching a segregated state. But for $R(\Omega) > CR(\Omega)$ requires that $\gamma > \beta(\frac{n}{2} - 2)$ which fails for large n.

However Ellison also introduces the modified co-radius, which is found by subtracting from the cost of the path the raidus of intermediate limit sets (or recurrent classes). Let $L_1, \ldots, L_r \subset \Omega$ be the sequence of limit sets through which the path passes consecutively, where $L_i \not\subset \Omega$, $i < r$.

$$CR^*(\Omega) = \max_{x \notin \Omega} \min_{(z_1,\ldots,z_T) \in S(x,\Omega)} \sum_{t=1}^{T-1} c(z_t, z_{t+1}) - \sum_{i=2}^{r-1} R(L_i)$$

With this concept Ellison provides a tighter bound on the waiting time:

Theorem 3 (Theorem 2, Ellison (2000)). *Let $(Z, P, P(\epsilon))$ be a model of evolution with noise, and suppose that some set Ω which is a union of recurrent classes $R(\Omega) > CR^*(\Omega)$. Then:*

(a) the long-run stochastically stable set of the model is contained in Ω;

(b) for any $y \notin \Omega$, $W(y, \Omega, \epsilon) = O(\epsilon^{-CR^*(\Omega)})$ as $\epsilon \to 0$.

Using the modified co-radius we can establish that the segregated states are stochastically stable. $CR^*(\Omega) = \beta$ such that $R(\Omega) > CR^*(\Omega)$. The theorem also suggest that the maximal waiting time is of the order $O(\frac{1}{\epsilon^\beta})$, which is independent of n.

Both stochastic stability and the radius-co-radius approach ignores how players are selected for strategy revision, all that matters is that all players get to update with some small probability. A sequence of "right" mutations has to happen for the system to reach the long-run stochastically stable states. For a fixed size of the system, the closer the process gets to a stochastically stable state the more biased is the process against selecting players "required" for the process to transit to the stochastically stable states. As the system grows the bias increases, and the waiting time increases exponentially in the size of the system.

Ellison (1993) and Young (1998, chp. 6) consider local interaction when players play two person coordination game. Both show that the waiting time until the stochastically stable state is reached is independent of the size of the system. The mechanism through which this is established differ between the two papers.

Ellison (1993) considers an adjustment dynamics a la Kandori, Mailath, and Rob (1993). That is in each period *all* players update or adjust their strategy so that their strategy is a best response to play in the previous period. With small probability players tremble. In each period a player plays a two person coordination game with all the other players. Players may weight payoffs differently from different players. In particular Ellison considers the case where players are located on a circle and a player only assign positive (equal) weight to here k nearest neighbours on either side. In the k nearest neighbour model for small enough perturbations the expected waiting time until the process reaches it's long-run steady state distribution, which puts probability mass one on the risk-dominant equilibrium, is independent of the size of the system. The intuition for the result is as follows. It is already well-known that the risk dominant equilibrium has a larger basin of attraction than the other pure equilibrium. This completely determines what will be selected for in the long run. Now suppose we start the process in the equilibrium which is not risk dominant. In order to transit to the risk dominant equilibrium it is sufficient that a suitable "small" group of players (who are connected) mutate to the risk dominant strategy. In the next period this will lead their neighbours to switch to the risk dominant strategy as well. The play of the risk dominant strategy then spreads contagiously. Since the basin of attraction of the risk dominant equilibrium is of larger size, then as ϵ becomes small the risk dominant equilibrium is selected for.

In contrast in my version of Schelling's model there is no contagious element. When play has settled on an equilibrium which is not stochastically stable, all players will have at least one neighbour like themselves. A location mistake (a mutation) will lead at most one other player of the same type to revise her location. This occurs only if the mutating player belong to a cluster of size two (when $r = 1$).

Young (1998, chp. 6) also consider selection in two person coordination games with a risk dominant equilibrium. Young considers a stochastic process in continous time. Players update their strategies according to a Poisson process. When players update their strategy most of the time they play a best reply, and with small probability a non-best reply. The process runs simultaneously for all players. Updates are independently and identically distributed across players. Players interact on a graph, however a given player mainly interacts within relatively small close-knit groups, loosely the group of players that a player interacts with are likely to mainly interact with each other as well. Young asks what the maximum expected wait until a large proportion $1 - p$ of the population the risk dominant equilibrium, this is called the p-intertia of the process. Young then shows that for sufficiently small mistake probabilities and if players interact in close-knit groups then the p-inertia of the process is bounded above, independently of the number of players.

The result relies on the following intuition. First note again that in Young's model the risk dominant equilibrium is the only stochastically stable state. Now since all players live in close-knit groups of a given (small) size, starting from the non-risk dominant equilibrium of the coordination game, the wait until a particular group switches to the risk-dominant equilibrium is bounded above, and crucially does not depend on the total number of players in the population. After this event if the probability that players make mistakes is sufficiently small then this group will continue playing the risk dominant for a long time. Since the process runs simultaneously for all players the waiting time until a large proportion, $1 - p$, of the population is playing the risk-dominant equilibrium is bounded.

It is difficult to see how Young's result can be related to our model. First, Young's result applies to a very small class of games. Second, the result relies on players interacting in small fixed close-knit groups.

How do results depend on the particular distribution of starting configurations that I have chosen for the numerical simulations? In the simulations I have taken an agnostic view, i.e. all possible starting configurations are are equally likely of being chosen. If one has a particular distribution in mind, perhaps for an application, this affects the estimates of convergence time. E.g. one might be interested in the wait starting from configurations that are relatively segregated. Along these lines one could ask the following question: How long is the wait conditional on the percentage of segregated residents in the starting configuration? The argument developed in this section shows that this is unlikely to lower waiting times significantly. This is because the process spends most of its time in configurations that are relatively segregated before eventually transiting to the (fully) segregated states.

One conclusion of the analysis is that the specific assumptions about preferences made in Schelling are not in themselves sufficient to explain why full segregation may occur within an economically interesting time span.[12] I propose a modified version of Schelling

[12] Another response is to ask whether some other concept is better able to capture the behaviour of the process. Möbius (2000) also finds that stochastic stability is bad predictor of how the stochastic process that his model leads to behaves, although it is quite different from the one examined here. Instead he

where players now also have a preference for living in larger clusters rather than small clusters. A motivation is that it might be seen as incorporating into static preferences that players care about more than their current payoffs. If I live in a small cluster then I am more likely to be experiencing that my current co-residents move and thus I end up living in a neighbourhood which only has players different from myself. If I live in a larger cluster then this is less likely to happen and in expectations the wait until it happens is longer. By this argument players actions today are motivated by future payoffs.

4.2.3 An Alternative Version of Schelling

The numerical simulations presented in the last section suggests that even for a relatively small size of residential neighbourhoods the expected wait until the process hits the stochastically stable states are so long as to make them economically uninteresting. Although the stochastic version of Schelling's model is able to explain why some segregation occur relatively fast, it is not able to explain convincingly why neighbourhoods segregate completely. The aim of this section is to present a model where complete segregation occurs relatively fast, while still being the only stable outcome that will be observed in the long run.

In this section I build indirectly into the preferences of the players some concern for future payoffs. More sophisticated players might try to make some prediction about how their neighbourhood evolves over time. This might motivate some players to leave their current location for a new location where they in the short run receive the same payoff. Specifically a player who lives in a relatively small cluster of players that are similar to herself might want to leave her current neighbourhood in order to live in a larger cluster since this cluster is less likely to break up in the short to medium run. Thus the player is likely to enjoy higher payoffs in the future. I model this concern via introducing cluster sizes lexicographically into the payoffs of players, that is whenever two locations gives identical local neighbourhoods then the player evaluates her preference for the locations by comparing the cluster sizes and then she moves to the location which has the highest local concentration of players that are like herself.

I keep other details of the model unchanged from that analysed in the previous section. I present numerical simulations which show that this variant of Schelling leads to complete and rapid segregation.

relies on the concept of *clustering*, see Möbius (2000) for details.

A Concern for Future Payoffs

Given a configuration $\sigma \in \Sigma$ let the cluster size at location l be the number of contin-guously connected players of the same type as the resident on location l and such that location l is included in the cluster. Let n_l be the size of the cluster which l is a part of.

I endow a player of type $t = A, B$ with the following preferences:

$$l \succ l' \quad \Leftrightarrow \quad \begin{cases} \frac{\#N_l^t}{2r+1} \geq \frac{1}{2} & and \quad \frac{\#N_{l'}^t}{2r+1} < \frac{1}{2} \quad \textbf{or} \\ \frac{\#N_l^t}{2r+1}, \frac{\#N_{l'}^t}{2r+1} \geq \frac{1}{2} & and \quad n_l > n_{l'} \end{cases}$$

The dynamic process is identical to the one used in the previous section. That is in every period a resident and a potential location is drawn randomly. The resident evaluates utility at her current location and the potential location, and this determines the probabilities with which she remains or vacates her current location.

Suppose player i residing at l_i is drawn for revision, and that she has the opportunity to move to $l \neq l_i$.

1. If i's utility increases at l then she moves there with probability $1 - \epsilon^{\alpha}$ and remains at her current location with probability ϵ^{α}. The decision to remain is termed a *mistake.*

2. If i is indifferent between her current location and l then she stays at her current location with probability $1 - \epsilon^{\beta}$ and moves with probability ϵ^{β}.

3. If i's utility decreases if she vacates l_i for l then she stays with probability $1 - \epsilon^{\gamma}$ and moves with probability ϵ^{γ}.

I assume $0 < \alpha < \beta < \gamma < \infty$.

Analysis

I begin by characterising the set of recurrent states, which I state without proof.

Proposition 25 (Modified Schelling). *Suppose $r = 1$. A state is recurrent under P^0 if and only if:*

1. *All players have at least one neighbour like themselves.*

2. *All type t clusters are of the same size, $t = A, B$.*

Moreover any recurrent state is also absorbing.

Remark 12. *The set of recurrent states under the modified stochastic version of Schelling is a subset of the set of recurrent states of the stochastic version of Schelling. A recurrent state with $k > 1$ clusters of each type exists if* $\mod(n, k) = 0$.

I now characterise stochastically stable states. Let $\{k_s^*\}$ be the sequence of natural numbers which satisfy:

$$\begin{aligned} \mod(n, k^*) &= 0 \\ \frac{n}{k^*} &\geq 2 \end{aligned}$$

Since n is even note that the largest k^* in the sequence, K^*, equals K. Let S denote the length of the sequence (S naturally depends on n).

Proposition 26 (Modified Schelling). *Suppose $r = 1$. A state is stochastically stable if and only if it is segregated.*

Proof. The proof may be found in Appendix C.2. □

Simulations

In this section I present numerical simulation results for the variant of Schelling's model that I analysed above.

To stay as close to the stochastic model presented in the previous section I again fix: $(\alpha, \beta, \gamma) = (1, 2, 3)$.

The Short Run The following tables gives descriptive statistics about the time to convergence to a recurrent class. I also report the median, since for small n the distribution is skewed.

Table 4.13: The Short Run (Modified Schelling)

n		10			20			50			100	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1	.02	.05	.1
		($\times 10^3$)			($\times 10^3$)			($\times 10^3$)			($\times 10^3$)	
Median	.11	.11	.12	.31	.32	.34	1.09	1.12	1.21	2.66	2.77	2.99
Mean	11.1	.94	.23	1.2	.41	.36	1.12	1.16	1.25	2.74	2.85	3.10
Std.	71.2	5.38	.65	20.0	1.47	.18	.28	.29	.32	.56	.59	.68
Max	1321	126.8	16.1	1185	65.9	8.43	3.37	3.28	3.94	6.75	6.92	9.10
Min	0	.01	.01	.08	.07	.07	.44	.47	.46	1.24	1.36	1.46

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 10.000 Observations per column.

Compared to the stochastic version of Schelling's model the short run has a longer duration. This is not surprising since the equilibria of the modified version of Schelling are a subset of the equilibria of the stochastic version of Schelling.

The Medium Run The following table shows the statistics about the duration of the medium run and the fraction of time spent in states that are visited before the process hits one of the stochastically stable states for the first time.

Table 4.14: Duration of The Medium Run (Modified Schelling)

n		10			20			50			100	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1	.02	.05	.1
Median	0	0	0	0	0	0	0	0	0	0	0	0
Mean	288	43.1	4.03	0	0	.035	0	.046	.13	0	0	.39
Std.	8220	967	93.0	0	0	2.04	0	4.6	7.6	0	0	20.9
Max	482348	36305	4129	0	0	162	0	461	606	0	0	1895
Min	0	0	0	0	0	0	0	0	0	0	0	0

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 10.000 Observations per column.

The duration of the medium run is generally short. In most of the simulations once the process has converged to an equilibrium of the game, it has converged to an element in the set of stochastically stable states.

The Long Run The following table gives statistics about the wait until the process hits a stochastically stable state for the first time.

Table 4.15: The Long Run (Modified Schelling)

n	10			20			50			100		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1	.02	.05	.1
		$(\times 10^3)$			$(\times 10^3)$			$(\times 10^3)$			$(\times 10^3)$	
Median	.11	.11	.12	.31	.32	.34	1.09	1.12	1.21	2.66	2.77	2.99
Mean	11.3	.98	.23	1.2	.41	.36	1.12	1.16	1.25	2.74	2.85	3.10
Std.	71.7	5.46	4.33	20.0	1.47	.18	.28	.29	.32	.56	.59	.68
Max	1321	126.8	16.1	1185	65.9	8.43	3.37	3.28	3.95	6.75	6.92	9.10
Min	.01	.01	.01	.08	.07	.07	.44	.47	.46	1.24	1.36	1.46

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, 10.000 Observations per column.

The fact that the medium run is on average very short is reflected in the wait until the process hits an element in the set of stochastically stable states. The wait is very close to the duration of the short run. The simulations suggest that the wait is linearly increasing in n, the number of residents.

The table below shows the fraction of time spent in various configurations, when the process is started in a stochastically stable state. I track the process over 10^7 periods.

Table 4.16: Fraction of Time Spent in Stochastically Stable States (Schelling)

n	10			20			50			100		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1	.02	.05	.1
SSS	99.99	99.93	99.35	99.99	99.92	99.20	99.99	99.89	98.68	99.99	99.79	97.06
ORS (*)	0	0	.0017	0	.0003	.031	0	.0047	.3477	0	.0273	1.643
OE	.01	.07	.65	.0028	.0830	.76	.0012	.1055	.9772	.0002	.1798	1.297

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, Number of Periods: 10^7. SSS=Stochastically Stable States, ORS=Other Recurrent States, OE=Out of Equilibrium. (*) The process only spends time in recurrent states with less than 3 clusters.

As can be seen from the table compared to the stochastic version of Schelling, stochastic stability is a good predictor of long run behavior even for levels of $\epsilon = .1$. Also the wait until the process enters an element in the set of the stochastically stable states is relatively short. This is a consequence two features of the dynamic process: (1) how easy it is to leave the basin of attraction of the set of stochastically stable states once it has been reached, (2) how easy it is to enter the basin of attraction of the set of stochastically stable states from any of the other recurrent states of the unperturbed dynamics. The analysis revealed that a single β mutation is sufficient to push the unperturbed process into the basin of attraction of the segregated states, thus it is relatively easy to enter the basin of attraction of the stochastically stable states. This accounts for the relatively fast transition to the stochastically stable states. On the other hand once the process has reached the set of stochastically stable states it is relatively difficult to leave it again. In particular more than one γ mutation is required in order for the process to leave the set of segregated states again. In the stochastic version of Schelling only one γ and a β mutation is needed. This accounts for the segregated states being relatively stable once they have been reached.

4.3 Local Interaction with Preferences for Diversity

In the models analysed so far residents are indifferent between living in integrated and segregated local neighbourhooods. In this section I assume that players have a preference for diversity. I am interested in whether individual incentives to avoid living in a local minority are sufficiently strong to have welfare consequences, that is whether a dynamic process will select equilibria that are relatively or fully segregated. I am also interested in how the model behaves before the process reaches it's long run distribution over visited states. Therefore this section also contains numerical simulation results for the short and medium run.

4.3.1 A Model with Preference for Diversity

To stay as closely to Schelling's original model as possible I only modify the utility function[13].

Preferences Given a configuration $\sigma \in \Sigma$ a player of type t who resides on location l_i

[13] I have also investigated a model where players cannot squeeze in between other players. Instead the circle has more locations than residents, such that there are empty locations. A moving player can only move to a location which is empty. In this model I show that only the fully segregated states are stochastically stable.

has the following utility function:

$$u_i^t(\sigma) = \begin{cases} 0 & if \quad \frac{\#N_i^t}{2r+1} \leq \frac{1}{3} \\ 1 & if \quad \frac{1}{3} < \frac{\#N_i^t}{2r+1} \leq \frac{2}{3} \\ x & if \quad \frac{\#N_i^t}{2r+1} > \frac{2}{3} \end{cases}$$

where $\frac{1}{2} < x < 1$.

That is players value diverse local neighbourhoods, but they prefer to live in a ghetto of players like themselves, to living in a ghetto of players who are not like themselves.

Dynamics In each period $\tau = 0, 1, 2, \ldots$ one player and a location is chosen at random, with all players and all locations having positive probability of being chosen. The probability that she moves to the location depends on the utility difference between her current location and the new location.

Specifically let player i, currently living on l_i, be drawn for location revision and let l be the location she has the opportunity to move to. Assume that there are numbers: $0 < \alpha < \beta < \gamma < \delta < \psi < \infty$. For $\epsilon \in (0, \epsilon^*]$ I assume that the decision to stay at or vacate l_i for l is determined by the following procedure:

1. If i's utility increases at l then she moves there with probability $1 - \epsilon^\alpha$ and remains at l_i with probability ϵ^α. The decision to remain is termed a *mistake*.

2. If i is indifferent between her current location and l then she stays at her current location with probability $1 - \epsilon^\beta$ and moves with probability ϵ^β.

3. If i currently currently has utility 1 and l gives utility x then she stays with probability $1 - \epsilon^\gamma$ and moves with probability ϵ^γ.

4. If i currently has utility x and l gives utility 0, then she stays with probability $1 - \epsilon^\delta$ and moves with probability ϵ^δ.

5. If i currently has utility 1 and l gives utility 0 then she stays with probability $1 - \epsilon^\psi$ and moves with probabilty ϵ^ψ.

Analysis

I begin by characterising the the set of recurrent classes under the unperturbed dynamics, P^0.

Proposition 27. *Suppose $r = 1$ and that players have a players have a preference for diversity. Under P^0 a configuration is recurrent if and only if each player belongs to a cluster. Moreover:*

1. *For any $\sigma, \sigma' \in \Sigma$, $\sigma \neq \sigma'$ such that both have $k < K$ clusters and all players belong to a cluster then σ and σ' are contained in the same recurrent class.*

2. *If σ is integrated, i.e. $k = K$, then it is recurrent and absorbing.*

Proof. I first show that the integrated state is recurrent and absorbing. Then I show that under the unperturbed dynamcis any state with $k < K$ clusters of players is recurrent and any other state which also has k clusters belong to the same class. Finally I show that any other state is transient.

Step 1a Suppose σ is integrated. Then σ consist of $K \equiv \frac{n}{2}$ clusters players of each type, and each cluster is minimal, i.e. of length 2. Thus all players receive utility 1, and no player has a strict incentive to change her location.

Step 1b Take any state σ with $k < K$ clusters. Let the size of clusters in σ be $n_{1(\sigma)}, \ldots, n_{k(\sigma)}$ for some t. Suppose that $k(\sigma)$ is not minimal. I show that the process can transit to any σ' such that that $n_{k(\sigma')} = n_{k(\sigma)} - 1$ and and $n_{k'(\sigma)} = n_{k'(\sigma)} + 1$, for some $1 \leq k' < k$. Since $k(\sigma)$ is not minimal there must be a player who gets utility x. She can move to the edge of $k'(\sigma)$ where she gets utility 1, thus the move occurs with positive probability, and the process has transited to a state σ' with k clusters of size $n_{1(\sigma)}, \ldots, n_{k'(\sigma)} + 1, \ldots, n_{k(\sigma)} - 1$. Also note that the process can transit back to σ since in σ' there must be a player in $k'(\sigma')$ who gets utility x.

Notice that the process cannot transit to states with a different number of clusters of each type. For the segregated states this is obvious, so assume that σ has $1 < k < K$ clusters. By the transition dynamics the process can transit to a state where we have $K - 1$ minimal clusters, and one cluster containing the remaining players of type t. However any of the $K - 1$ minimal clusters cannot be broken up by the dynamics since all players live in integrated neighbourhoods, and thus receive highest possible utility.

Step 2 Finally assume that in σ there is at least one player who does not belong to a cluster. This player has utility 0. Thus with positive probability under P^0 she will be drawn for revision and move to a location where she has at least one neighbour like herself. The new configuration is $\sigma' \neq \sigma$. If there are more players who do not belong to a cluster then let them update the location choice. With positive probability the process will arrive at a state σ'' in which all players belong to a cluster. Once the process has hit

σ'' it cannot go back to σ, since by step 1b above a cluster will never vanish. Moreover no player has an individual incentive to move to a location where she only has neighbours which are not like herself. This shows that σ is transient. □

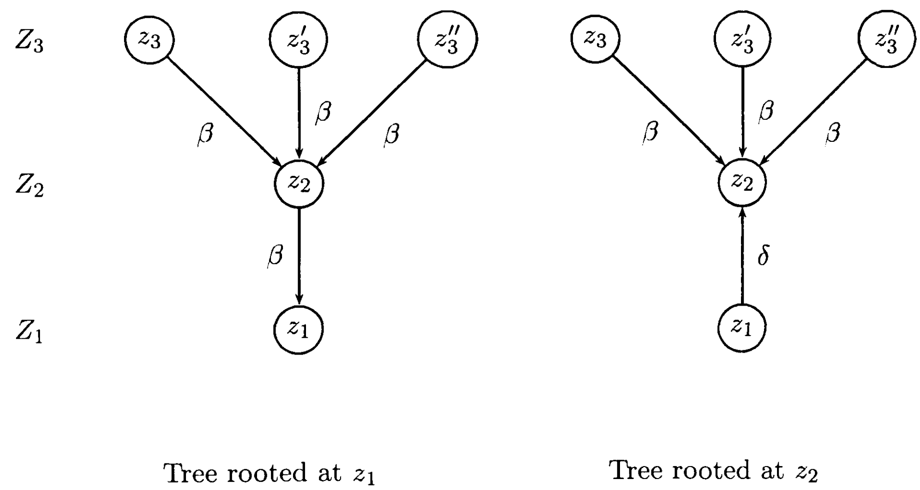
Remark 13. *Notice that the only equilibrium of the game is the integrated state ($k = K$). Nevertheless configurations with $k < K$ clusters, and where all players belong to a cluster, are stable in the sense that the process will visit them infinitely often if started in that configuration. That is although players have individual micro-incentives to relocate the macro structure is stable.*

I now establish that set of segregated states are the only stochastically stable states.

Proposition 28. *A state is stochastically stable if and only if it is segregated.*

Proof. The proof can be found in Appendix C.3. □

The construction of minimal cost trees is shown in the following graph for the case $K = 3$. I show the trees for the segregated state and a state with 2 clusters.



There are two effects which drives the selection result. First players of one type do not internalise the positive externality which they have on players of another type. Second myopic optimization means that players do not anticipate that if they move to a location which is currently unattractive other players of their own type may follow and a higher degree of integration may follow in its wake. However the last effect is less robust, since if players are able to anticipate the evolution of neighbourhoods then for a sufficiently long time horizon they will realise that with high probability their local neighbourhood will evolve from a diverse neighbourhood into a segregated neighbourhood. This might deter players from making the initial unattractive relocation move.

Simulations

In this section I present simulation results for the model analysed above. For the purpose of the simulations I have fixed: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$.

I begin by looking at the short run behaviour of the model.

The Short Run Table 4.17 shows the time until the process hits a recurrent class of the unperturbed dynamics for the first time.

Table 4.17: Time to Convergence - The Short Run (Diversity)

n		10			20			50	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Mean	31.3	35.8	34.5	88.3	86.4	98.9	303.5	307.4	336.8
Std.	23.1	25.9	28.0	50.0	45.2	54.0	117.9	117.7	150.6
Max	157	129	161	316	327	285	827	753	1166
Min	0	0	0	16	16	20	89	89	94

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 200 Observations per column.

The next table details the number of clusters in the configuration when the process hits a recurrent state of the unperturbed dynamics for the first time.

Table 4.18: Number of Clusters (Diversity)

n		10			20			50	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
Mean	2.87	2.93	2.95	5.62	5.55	5.58	14.1	13.87	13.84
Std.	.73	.72	.70	1.06	1.04	.98	1.57	1.58	1.64
Max	4	4	4	8	8	8	18	19	18
Min	1	2	1	3	3	3	10	10	10

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 200 Observations per column.

I now turn to the medium run.

The Medium Run In the simulations I track the process for a maximum of 10^7 periods. The expected wait until the process hits the set of stochastically stable states for the first time is rapidly increasing in the number of residents in the neighbourhood. In particular for $n \geq 20$ the process does not hit the stochastically stable states during the period in which I track the process. Therefore the medium run behaviour of the process, i.e. the states that are visited in the medium run become the most economically interesting time period. Since the long run is reached within 10^7 periods for very few observations I report the fraction of time the process spends in the recurrent classes of the unperturbed dynamics, which are not stochastically stable.

Table 4.19: Visited States in The Medium Run (Diversity)

n		10			20				50		
State	2	3-5	OE	2-3	4-5	6-10	OE	2-6	7-12	13-25	OE
Mean	75.40	24.49	.113	10.55	85.64	3.616	.188	0	95.01	4.54	.453
Max	99.96	80.14	.721	18.93	93.67	7.413	.223	0	98.99	10.76	.374
Min	19.69	0	.030	4.030	77.81	.888	.146	0	88.76	.602	.544

Note: $\epsilon = .05$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, OE=Out of Equilibrium. For $n > 10$ the process does not reach the set of stochastically stable states in 10^7 periods. Observations per column: 200.

The Long Run In the table below I start the process in the stochastically stable states and track it for 10^7 periods. I record the fraction of time which the process spends in the the different states.

Table 4.20: Fraction of Time Spent in Stochastically Stable States (Diversity)

n		10			20			50	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
SSS	22.62	1.65	1.82	5.03	.491	.034	.648	.643	.107
2-3 Clusters	77.38	98.05	96.72	46.89	12.41	5.54	27.81	.815	.059
> 3 Clusters	0	.207	.706	48.07	86.94	92.99	71.53	98.06	96.38
OE	.003	.091	.748	.010	.163	1.44	.013	.480	3.46

Note: $(\alpha, \beta, \gamma) = (1, 2, 3)$, Number of Periods: 10^7. SSS=Stochastically Stable States, OE= Out of Equilibrium (transient state).

For reasonable levels of noise stochastic stability is not a valid predictor of where the process will spend most of its time, even for relatively small sizes of the residential neighbourhood. Compared to the stochastic version of Schelling presented in section 4.2 the process spends much more time in states which are not stochastically stable. This is due to the change in the preferences of residents over local neighbourhood composition. In particular in the stochastic version of Schelling if the process is started in a segregated state and a resident by mistake moves to a location with players different from herself then no player like herself has a strict incentive to follow her. On the other hand if residents have a preference for diversity then a mistake opens up a "beachhead" from which a new cluster can form at least temporarily. Since the new cluster is small players from the old cluster is more likely to be drawn for revision and might get the opportunity to move to a diverse neighbourhood. Thus the process will spend a significant amount of time outside the segregated state.

Comments From the simulations it can be seen that expected wait until the process hits the set of stochastically stable is indeed very long, even for a small number of residents. Thus stochastic stability is quite uninformative about the behaviour of the system for large time periods. Comparing the model where players have a preference for diversity to the stochastic version of Schelling's model it is also clear that the expected wait until a segregated state is reached is much longer when players have a preference for diversity[14]. Thus it appears that although for both models evolutionary pressures pushes the process towards segregated states, in the medium run individual preferences over outcomes play a significant role. Also for the prediction of stochastic stability to be valid the level of noise must be significantly smaller than in the stochastic version of Schelling. This is because starting from a segregated state a location mistake now leads the process directly out of the segregated states, since the location mistake has opened up an opportunity for a player who only lived with players like herself to move to a diverse neighbourhood. Thus for a given level of noise it is relatively easier to leave the stochastically stable states.

4.3.2 Some Alternative Models

The simulations in the previous section showed that although the segregated states are the only stochastic stable states the wait until the process hits the set of segregated states increases rapidly in the number of residents. In fact the process spends most of

[14]The intuition follows the line of lemma 3.

it's time in relatively segregated states with around two clusters of each type. This is due to the fact that once the process hits a state with two clusters of each type of player it can spend a very long time in states with relatively few clusters, because people who live only with players like themselves are motivated by the edges where they can seek out diverse neighbourhoods. Therefore under the unperturbed dynamics a cluster will spend a long time oscillating between almost disappearing or almost being the single cluster of that type, before a mutation at the "right" time will eventually lead it to either vanish or become the only cluster which contains residents of a particular.

If one is willing to make the additional assumption that residents care about the size of the cluster in which they live (a similar set up to the modified version of Schelling's model analysed in an earlier section) then this is a sufficient condition for segregated states to be the uniquely stochastically stable states, and moreover convergence to the set of stochastically stable states is rapid. This is one route I take in this section.

A second question which comes out the model with preferences for diversity is that although the stochastically stable and segregated states take a very long time to arrive to, nevertheless in the medium run the process visits relatively segregated states. Evolutionary pressures still favor relatively segregated states, even though players have a preference for diversity. This seems to be related to the fact that players are purely motivated by current utility. If a player of type t was willing to move to a ghetto of type t' players, then surely others like himself would eventually follow and diverse neighbourhoods might form. One potential and realistic way forward is to assume that there is some heterogeneity in preferences. I introduce a small portion of players into the population who have different preferences from the majority of the population. I assume that these players most preferred neighbourhood is a diverse one, but that they prefer living with people different from themselves to only living with people like themselves. One interpretation of this type preferences is simply that they are "perverse" in some sense (relative to the majority of the population). However another interpretation is that these players are "sophisticated" in the sense that are willing to sacrifice short term utility for achieving better outcomes in the longer run - they are more patient than the rest of the population.

I continue to formulate the model in a perturbed Markov framework. However in the unperturbed dynamics only the integrated states are recurrent, so in a sense there is no equilibrium selection problem. It might however take a very long time to reach an integrated state, and the presence of a majority of players with "normal" preferences means that evolutionary pressures tend to work against integration. I therefore ask the question whether the presence of a small fraction of "sophisticated" players is able to push the population towards more integrated outcomes.

A Model where players have a Concern for Future Payoffs

In this section I formulate a model where the motivation is identical to the one given in section 4.2.3. That is whenever a player is drawn for location revision and the neighbourhood composition is identical at her current and her potential location, then she compares the size of the clusters and chooses the location which offers the bigger cluster size.

Analysis In this section I first characterise the recurrent states of the unperturbed process. Then I show that the stochastically stable states are precisely the segregated states.

Proposition 29. *Suppose $r = 1$ and that players have a preference for diversity and a concern for future payoffs. Under P^0 a state is recurrent if and only if it is segregated or it is integrated. Moreover all segregated states are in the same recurrent class, and all integrated states are absorbing.*

Proof. $\Leftarrow$: Suppose σ is integrated. Then all clusters have the same size and all players live in diverse neighbourhoods. Thus no player has a strict individual incentive to deviate. Suppose σ is segregated. Then $2r$ players of each type have diverse neighbourhoods while the remaining players live only with people like themselves. Players who live with people like themselves will have individual incentive to move to the edge of the cluster they reside in. The new state is also segregated.

$\Rightarrow$: I now show that all other states are transient. Suppose σ is neither segregated nor integrated. Under P^0 we can transit to a state, σ', in which all players who does not have at least one neighbour like themselves live with at least one like themselves. Thus in σ' all players belong to a cluster of at least minimal size. If σ' is neither segregated nor integrated then σ' has $1 < k < K$ clusters. By the unperturbed dynamics we can then transit to a state in there are $k - 1$ type t minimal clusters, and one cluster containing the remaining residents, which is at least of length 4, denote it k'. Now let one player from each of the $k - 1$ minimal clusters one by one be drawn for revision. Since the other clusters are minimal and k' is at least of length 4, when a player is drawn for revision and she has the opportunity to move to the edge of k' (such that she gets a diverse neighbourhood) then she strictly prefers this since at least 2 more residents of type t live in k' than in her current cluster. Thus by the unperturbed dynamics this player moves. Next let the remaining players from the $k-1$ minimal clusters be drawn for revision. These players only have neighbours different from themselves. Thus they strictly prefer any location which puts them in the k' cluster. Thus by the unperturbed dynamics we have transited to a segregated state. □

Proposition 30. *Suppose $r = 1$, that players have a preference for diversity and a concern for future payoff. A state is stochastically stable if and only if it is segregated.*

Proof. Let Z_1 denote the recurrent class containing all segregated states and let Z_K denote the set of integrated states, with cardinality $|Z_K|$. First I establish the stochastic potential of a segregated state. Take any integrated state. In an integrated state all players live in clusters of minimal size. Let a player from cluster k be drawn for revision and suppose that she has the opportunity to move to an edge of cluster k', $k' \neq k$, with players like herself. She will have the same local neighbourhood at both locations and the cluster size are identical. Thus she is indifferent between the two locations. Thus the move has cost β. Now the process has transited to a state in which there is one cluster which is greater than all other clusters of that type. Hence by the unperturbed dynamics, with positive probability the process transits to a segregated state. That is the stochastic potential of the set of segregated states is:

$$\beta|Z_K|$$

I now establish that the stochastic potential of an integrated state is strictly larger. Take any segregated state. To transit to an integrated state has at least cost δ since all players either live with people like themselves or live in a diverse neighbourhood. Likewise the cost of transit from one integrated state to another distinct integrated state is at least β. Hence the stochastic potential of an integrated state is at least:

$$\beta(|Z_K| - 1) + \delta$$

Since $\delta > \beta$ it follows the only segregated states are stochastically stable. □

In now simulate the model.

Simulations In this section I simulate the model.

I again fix: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, so as to make a direct comparison with the model presented in the previous section more straightforward.

The Short Run The following tables show descriptive statistics about the time of convergence to a recurrent class.

Table 4.21: The Short Run (Modified Diversity)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
					($\times 10^3$)			($\times 10^3$)	
Median	860	837	980	5.577	5.976	7.464	47.28	62.33	395.9
Mean	982.9	991.8	1110	6.339	6.849	9.306	52.76	81.28	558.3
Std.	642.3	675.9	759.6	3.596	4.345	7.007	31.33	62.08	565.6
Max	13	20	7	31.78	28.43	41.70	293.2	431.4	4081
Min	3442	4112	4858	.672	.704	.669	8.28	11.14	9.37

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 500 Observations per column.

The Medium Run The following table gives various statistics about the duration of the medium run and the fraction of time spent in states that are visited before the process hits one of the stochastically stable states for the first time.

Table 4.22: Duration of The Medium Run (Modified Diversity)

n	10			20			50		
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
		($\times 10^3$)			($\times 10^3$)				
Median	0	0	0	0	0	0	0	0	0
Mean	.885	.052	.054	0	0	0	0	0	.026
Std.	15.6	.257	.265	0	0	0	0	0	.581
Max	340.8	2.89	2.79	0	0	0	0	0	13
Min	0	0	0	0	0	0	0	0	0

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 500 Observations per column.

The duration of the medium run is generally short. In most of the simulations once the process has converged to an equilibrium of the game, it has converged to an element in the set of stochastically stable states.

The Long Run The following table gives statistics about the waiting time until the process hits a stochastically stable state for the first time.

Table 4.23: The Long Run (Modified Diversity)

n		10			20			50	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
		$(\times 10^3)$			$(\times 10^3)$			$(\times 10^3)$	
Median	.940	.897	1.012	5.557	5.976	7.464	47.28	62.33	395.9
Mean	1.868	1.044	1.164	6.339	6.849	9.306	52.76	81.28	558.3
Std.	15.61	.678	.755	3.596	4.345	7.007	31.33	62.08	565.6
Max	340.8	4.112	4.558	31.78	28.43	41.70	293.2	431.4	4081
Min	.028	.020	.022	.672	.704	.669	8.28	11.14	9.37

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, 500 Observations per column.

The fact that the medium run is on average very short is reflected in the wait until the process hits an element in the set of stochastically stable states. The wait is very close to the duration of the short run.

The table below shows the fraction of time spent in various configurations, when the process is started in a stochastically stable state. I track the process over 10^7 periods.

Table 4.24: Fraction of Time Spent in Stochastically Stable States (Schelling)

n		10			20			50	
ϵ	.02	.05	.1	.02	.05	.1	.02	.05	.1
SSS	99.94	98.88	90.04	99.67	89.16	49.02	95.81	41.33	2.85
SNI	.054	.933	8.40	.312	10.39	47.44	4.14	57.52	88.77
SI	.008	.182	1.52	.017	.45	3.53	.045	1.15	8.38

Note: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$, Number of Periods: 10^7. SSS=Stochastically Stable States, SNI=States where no player is isolated, SI=States where players are isolated.

The model performs reasonably well for $\epsilon \leq .02$.

A Model with Sophisticated Residents

I assume that a small fraction of players have "sophisticated" preferences in the sense that while their most preferred neighbourhood is a diverse one, they prefer to live in isolated neighbourhoods rather than living in ghettoes with people like themselves.

This leads to the following formulation of "sophisticated" preferences. Given a configuration $\sigma \in \Sigma$ a player of type t with "sophisticated" preferences residing on location l_i, has the following utility function:

$$v_i^t(\sigma) = \begin{cases} x & if \quad \frac{\#N_i^t}{2r+1} \leq \frac{1}{3} \\ 1 & if \quad \frac{1}{3} < \frac{\#N_i^t}{2r+1} \leq \frac{2}{3} \\ 0 & if \quad \frac{\#N_i^t}{2r+1} > \frac{2}{3} \end{cases}$$

where $\frac{1}{2} < x < 1$.

I make the additional assumption that all players only care about the type of the players that reside in their local neighbourhood, and not whether they are "sophisticated" or not.

Apart from this modification the stochastic process is left unchanged.

Analysis Let $\lceil y \rceil$ denote the smallest integer greater than or equal to y. I have the following result.

Proposition 31. *Suppose $r = 1$ and that $\lceil \rho n \rceil \geq 1$ have "sophisticated" preferences, while $n - \lceil \rho n \rceil$ have a preference diversity, where $0 < \rho < 1$. Under P^0 a state is recurrent if and only if it is integrated.*

Proof. $\Leftarrow$: It is immediate that an integrated state is recurrent, since in an integrated state all players have utility 1, thus no player has an individual strict incentive to move.

$\Rightarrow$: I now show that the integrated states are the only recurrent states. First let a player of type t with "sophisticated" preferences be denoted t^s while players who have "normal" preferences are simply denoted by their type, t. Take any state σ which is not integrated. By the unperturbed dynamics the process can transit to a state in which all players of type t, $t = A, B$ has at least one neighbour like themselves. Also by the unperturbed dynamics we can transit to a state where all type t^s players live in a diverse neighbourhood. Let this state be σ'. Note that σ' must have at least

$$\left\lceil \frac{\lceil \rho n \rceil}{2r} \right\rceil$$

clusters of each type to ensure that all "sophisticated" players live in a diverse neighbourhood. Suppose that σ' has $k < \frac{n}{2}$ clusters, otherwise we are done.

I now show that we can transit to a state with $k+1$ clusters of each type. Since n is even and $k < \frac{n}{2}$ there is at least two players of each type who do not live in diverse neighbourhoods. Thus by the unperturbed dynamics we can transit to a state with k clusters and such that there is a cluster of each type which is at least of length 4, and this cluster contains at least one "sophisticated" player. Let the cluster of at least length 4 containing type t players be denoted k_t'. Suppose that in k_t' the "sophisticated" player no longer lives in a diverse neighbourhood. If she does then let one of the other players in k_t' who only have neighbours like themselves be drawn for revision and given the opportunity to move to the edge of the cluster taking up the location that the "sophisticated" player had. Now let the t^s type player be drawn for revision and suppose that she draws a location contained in $k_{t'}'$, $t' \neq t$, such that two type t' players live on either side (this is possible since $k_{t'}'$ is of at least length 4). Since the t^s player prefers to live only with people different from herself to living only with people like herself she will move under the unperturbed dynamics. Now there is a second player in k_t who only live with people like herself. Let her be drawn for revision and let her move next to the "sophisticated" player who just moved. Both these players now live in a minimal cluster and have a diverse neighbourhood. Also by their moves a new cluster of type t' players have formed. Thus we have transited to a state with $k+1$ clusters. Since the moves did not rely on the size of k, if $k+1 < \frac{n}{2}$ then we can repeat the procedure until we have reached an integrated state. □

Remark 14. *Note that the proof only requires that there is at least one "sophisticated" player. Thus the results holds even for the case where only one type of players has some player who is "sophisticated".*

Also note that the Schelling's model in which players do not have a preference for diversity is unaffected by the introduction of a small fraction of "sophisticated" players.

Simulations Since the unperturbed process has a unique recurrent class the process will spend most of it's time in the integrated states for a sufficiently long time horizon, if ϵ is sufficiently small. However there might be other reason for considering positive and non-vanishing levels of noise: it might capture elements of reality which the model does not pick up, or the modeller do not wish to model. Therefore in this section I both simulate the model without noise and with noise. Simulating the model without noise gives us an estimate of the waiting time until the process hits an integrated state. Simulating the model with noise allows me to see to what extent evolutionary pressures which tend to work against integration (for the majority of players) may be countered by the introduction of some heterogeneity in preferences over local neighbourhood composition.

No Noise I first simulate the model without noise. I fix the proportion of "sophisticated" players $\rho = .1$ throughout.

In the first table I record the time when the process first hits a state in which all players who has a preference for diversity has at least one neighbour like themselves and all "sophisticated" players live in a diverse neighbourhood. Such states mimick recurrent states in a model where all players have a preference for diversity. I refer to them as "pseudo-recurrent".

Table 4.25: Time of Convergence to Pseudo-Recurrent States (Heterogeneous Players)

n	10	20	50
Mean	63.3	262.7	2145.9
Std.	51.0	207.0	1517.8
Max	348	1695	10437
Min	0	5	137

Note: $\epsilon = 0$, $\rho = .1$, 500 obs. per column.

In the next table I report the number of clusters when the process first hits a pseudo-recurrent state.

Table 4.26: Clustering (Heterogeneous Players)

n	10	20	50
Mean	3.14	6.40	17.10
Std.	.628	.870	1.51
Max	5	9	21
Min	1	4	11

Note: $\epsilon = 0$, $\rho = .1$, 500 Observations per column.

A straight forward comparison with the clustering in the diversity model with homogeneous agents shows that even in the short run the presence of actively diversity seeking players leads to significantly higher levels of integration.

In the following table I start the process at a randomly drawn configuration and then track it for 10^7 periods or until the integrated state is reached.

Table 4.27: Time of Convergence to Integrated State (Heterogeneous Players)

n	10 ($\times 10^3$)	20 ($\times 10^4$)	50
Mean	12.8	99.31	$> 10^7$
Std.	13.3	97.66	-
Max	114.7	570.3	-
Min	.014	.248	-

Note: $\epsilon = 0$, $\rho = .1$, 500 Observations per column.

The wait until the process reaches an integrated state increases rapidly as n increases.

Noise The simulations without noise showed that for reasonable time horizons the process will be out of equilibrium most of the time, at least for large n. It therefore becomes interesting to see what disequilibrium states are visited by the process. Here noise is not used for selection purposes but is introduced to capture elements of the location decision process which are not captured by the specification of preferences. For the simulations in this section I fix $\epsilon = .02$. As in the previous section I have fixed: $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$ for all simulations. Since the majority of players have diversity seeking preferences, evolutionary pressure tends to push neighbourhoods configurations towards segregated states. At the same time a small fraction of "sophisticated" players are constantly opening up new opportunities for the formation of diverse local neighbourhoods, which tends to push the evolution of the system towards more integrated states.

In the following tables I show the mean fraction of time with which the process visits different states for $n = 10, 20$ and 50 respectively. I track the process for 10^7 periods. Since I am mainly interested in how the presence of a few "sophisticated" players affect the welfare of the majority of players who have a preference for diversity the tables indicate the fraction of time that the process spends in a state with a particular distribution of local neighbourhoods for players who have a preference for diversity. E.g. in table 4.28 I simulate the model with $n = 10$. Since the fraction of "sophisticated" players is fixed to .1 this leaves a total of 18 players who have a preference for diversity.

The table is then read as follows. The time average fraction that the process spends in a state where all 18 players live in a diverse neighbourhood is 33.83. That is the process spends about a third of the total time in states in which all players live in a diverse neighbourhood. It spends 36.67% of the time in states in which 14 out of the 18 players have a diverse neighbourhood, the remaining 4 players live in ghettoes only with player like themselves.

Table 4.28: Visited States ($n = 10$) (Heterogeneous Players)

	No. Integrated					
No. Isolated	0-9	10-13	14	15-17	18	Sum
0	.22	12.05	36.67	12.25	33.83	95.02
1	.07	1.87	.21	2.11	0	4.26
2-18	.04	.53	.13	.01	0	.72
Sum	.34	14.45	37.01	14.37	33.83	100

Note: $\rho = .1$, $\epsilon = .02$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$. Number of periods: 10^7

For $n = 10$ the process spends the majority of the time close to or at an integrated state. 85.2% of the time is spent in states where at least 77.8% of the players live in diverse neighbourhoods. Hardly any players live in isoilated neighbourhoods.

I now increase the total number of residents to 40 players. That is 36 of the players have a preference for diversity.

Table 4.29: Visited States ($n = 20$) (Heterogeneous Players)

	No. Integrated				
No. Isolated	0-19	20-25	26-29	30-36	Sum
0	.24	26.83	42.41	16.62	86.10
1	.10	3.59	5.71	1.90	11.31
2-36	.03	.92	1.34	.30	2.60
Sum	.38	31.34	49.46	18.82	100

Note: $\rho = .1$, $\epsilon = .02$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$. Number of periods: 10^7

States with full integration is now hardly reached, however the process still spends about 68.3% of its time in states where at least 72.2% of the players live in diverse neighbourhoods.

When $n = 50$ there are now 90 players who have a preference for diversity.

Table 4.30: Visited States ($n = 50$) (Heterogeneous Players)

	No. Integrated				
No. Isolated	0-59	60-69	70-75	76-90	Sum
0	4.94	42.60	17.98	1.79	67.32
1	1.85	16.33	4.88	.42	23.49
2	.70	5.11	1.46	.09	7.36
3-90	.22	1.30	.30	.01	1.83
Sum	7.71	65.35	24.62	2.31	100

Note: $p = .1$, $\epsilon = .02$, $(\alpha, \beta, \gamma, \delta, \psi) = (1, 2, \frac{5}{2}, 3, 4)$. Number of periods: 10^7

The process spends about 92.3% of its time in states where at least 66.7% of the players live in diverse neighbourhoods.

4.4 Conclusion

Residential areas may become segregated for a variety of reasons e.g. if everybody prefer to live with rich people then rich people can segregate themselves by outbidding their poor counterparts for residential locations.

In this paper people have preferences over whom they would like to have in their local neighbourhood. If they do not like their current neighbourhood then they can move to one which they like better. This is the basic model suggested by Schelling (1969, 1971).

In the basic formalisation of Schelling's model of this paper, the fear of isolation eventually leads to segregation. This is true even if all residents prefer to live in mixed neighbourhoods. This is a rather bleak message. On the other hand I also showed that if residents prefer to live in mixed neighbourhoods then the presence of just a few "social activists" can have a significant impact on the overall composition of local neighbourhoods.

Attempts to understand the underlying process through which ghettoes form and develop can be found across the social sciences. Anderson (1990, 1999) gives a brilliant and lucid account of the dynamics of ghetto formation in a US metropolis. A general change in the productive environment (the gradual disappearance of skilled labour jobs vs. the appearance of non-productive predatory behaviour of the underground drugs-economy) led to the the loss of authority of the older generation over the reproduction of social norms among youngsters. This process is then exacerbated by the black professional middle-class leaving the residential area in order to avoid the side- effects of the drugs-economy. In Anderson's analysis this is crucial to the formation of the ghetto. If these people remained in the community, they would become role models for how hard work and education can pay off in the new productive environment in contrast to the risky get-rich-fast schemes of the drugs-economy. Crucially his analysis reveals that ethnic homogeneity is not equivalent to the negatively charged ghetto. Ethnic homogeneity can be associated with a vibrant, healthy and dynamic community.

Schelling (1971) also considered neighbour interaction in a two-dimensional space. Investigating how more general geometries affect the evolution of neighbourhoods is left for future research.

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Appendix A

Appendix for Chapter 2

This appendix deals with the case where the observing player can only learn about one of the risky alternatives. In the first section we establish general conditions under which an observing player prefers to learn about alternative r_1 or r_2. Then in second section we establish conditions such that any observing player who ex-ante prefers r_1 will want to observe experimentation with r_1, and any observing player who ex-ante prefers r_2 either prefers to observe experimentation with r_1 or r_2.

A.1 The General Case

In this section we establish general conditions for given preferences of the observing player to prefer to learn about alternative r_1 or alternative r_2.

A.1.1 $\mu_i \bar{u}_i \leq \frac{\mu_i \kappa_i}{2}$, $i = 1, 2$

Note in this case $\sigma(\mathcal{S}) = s$. We have the following:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_1$	$2\bar{u}_1$	0	0
r_2	$2\bar{u}_2$	0	$2\bar{u}_2$	0

Thus and assuming wlog that $\bar{u}_1 \geq \bar{u}_2$,

$$\begin{aligned} C_1 &= (1-\mu_1)\mu_2 2\bar{u}_2 \\ C_2 &= \mu_1\mu_2(2\bar{u}_1 - 2\bar{u}_2) + \mu_1(1-\mu_2)2\bar{u}_1 \end{aligned}$$

Such that it is optimal to observe $\sigma(f) = r_1$ iff:

$$\mu_2\bar{u}_2 \leq \mu_1\bar{u}_1$$

and $\sigma(f) = r_2$ otherwise.

A.1.2 $\mu_1\bar{u}_1 \leq \frac{\mu_1\kappa_1}{2}$ and $\mu_2\bar{u}_2 \geq \frac{\mu_2\kappa_2}{2}$

Case I: $\mu_2\bar{u}_2 \leq \frac{\bar{u}_1(1+\mu_2)}{2} + \frac{\mu_2\kappa_2}{2}$

In this case we have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_1$	$2\bar{u}_1$	$2\bar{u}_2$	$\underline{u}_2$
r_2	$2\bar{u}_2$	0	$2\bar{u}_2$	0

We have:

$$\begin{aligned} C_1 &= (1-\mu_1)(1-\mu_2)(0-\underline{u}_2) \\ C_2 &= \mu_1\mu_2(2\bar{u}_1 - 2\bar{u}_2) + \mu_1(1-\mu_2)2\bar{u}_1 \end{aligned}$$

Hence $\sigma(f) = r_1$ is optimal iff:

$$\mu_2\bar{u}_2 \leq \bar{u}_1 - \frac{\mu_2\kappa_2}{2}\frac{1-\mu_1}{\mu_1}$$

and $\sigma(f) = r_2$ otherwise.

Case II: $\mu_2\bar{u}_2 \geq \frac{\bar{u}_1(1+\mu_2)}{2} + \frac{\mu_2\kappa_2}{2}$

In this case we have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_2$	$\underline{u}_2 + \bar{u}_1$	$2\bar{u}_2$	$\underline{u}_2$
r_2	$2\bar{u}_2$	0	$2\bar{u}_2$	0

The only relevant case is when $\bar{u}_1 + \underline{u}_2 \geq 0$. Then $\sigma(f) = r_1$ is optimal iff:

$$\mu_1\bar{u}_1 \geq -\underline{u}_2 = \frac{\mu_2\kappa_2}{1-\mu_2}$$

and $\sigma(f) = r_2$ otherwise.

A.1.3 $\frac{\mu_1\kappa_1}{2} \leq \mu_1\bar{u}_1$ and $\mu_2\bar{u}_2 \leq \frac{\mu_2\kappa_2}{2}$

Case I: $\mu_1\bar{u}_1 \leq \frac{1+\mu_1}{2}\bar{u}_2 + \frac{\mu_1\kappa_1}{2}$

In this case we have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_1$	$2\bar{u}_1$	0	0
r_2	$2\bar{u}_2$	$2\bar{u}_1$	$2\bar{u}_2$	$\underline{u}_1$

We have:

$$C_1 = (1-\mu_1)\mu_2 2\bar{u}_2$$
$$C_2 = \mu_1\mu_2(2\bar{u}_1 - 2\bar{u}_2) - (1-\mu_1)(1-\mu_2)\underline{u}_1$$

Hence $\sigma(f) = r_1$ is optimal iff:

$$\mu_1\bar{u}_1 \geq \bar{u}_2 - \frac{\mu_1\kappa_1}{2}\frac{1-\mu_2}{\mu_2}$$

and $\sigma(f) = r_2$ otherwise.

Case II: $\mu_1\bar{u}_1 \geq \frac{1+\mu_1}{2}\bar{u}_2 + \frac{\mu_1\kappa_1}{2}$

In this case we have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_1$	$2\bar{u}_1$	0	0
r_2	$2\bar{u}_1$	$2\bar{u}_1$	$\underline{u}_1 + \bar{u}_2$	$\underline{u}_1$

Only relevant case is $\underline{u}_1 + \bar{u}_2 \geq 0$, we have:

$$C_1 = (1-\mu_1)\mu_2(\underline{u}_1 + \bar{u}_2)$$
$$C_2 = -(1-\mu_1)(1-\mu_2)\underline{u}_1$$

Hence $\sigma(f) = r_1$ is optimal iff:

$$\mu_2\bar{u}_2 \leq -\underline{u}_1 = \frac{\kappa_1\mu_1}{1-\mu_1}$$

and $\sigma(f) = r_2$ otherwise.

A.1.4 $\mu_i\bar{u}_i \geq \frac{\mu_i\kappa_i}{2}$, $i = 1, 2$

Case I: $\mu_1\bar{u}_1 \leq \frac{1+\mu_2}{2}\bar{u}_2 + \frac{\mu_1\kappa_1}{2}$ **and** $\mu_2\bar{u}_2 \leq \frac{1+\mu_1}{2}\bar{u}_1 + \frac{\mu_2\kappa_2}{2}$

We have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_1$	$2\bar{u}_1$	$2\bar{u}_2$	$\underline{u}_2$
r_2	$2\bar{u}_2$	$2\bar{u}_1$	$2\bar{u}_2$	$\underline{u}_1$

If $\bar{u}_1 \geq \bar{u}_2$ and $\underline{u}_2 \geq \underline{u}_1$ then $\sigma(f) = r_1$ is always preferred. The only relevant case is $\bar{u}_1 \geq \bar{u}_2$ and $0 \geq \underline{u}_1 \geq \underline{u}_2$ (alternatively $\bar{u}_2 \geq \bar{u}_1$ and $\underline{u}_2 \geq \underline{u}_1$). Here we have:

$$C_1 = (1-\mu_1)(1-\mu_2)(\underline{u}_1 - \underline{u}_2)$$
$$C_2 = \mu_1\mu_2(2\bar{u}_1 - 2\bar{u}_2)$$

Thus if

$$\mu_2\bar{u}_2 \leq \mu_2\bar{u}_1 + \frac{\kappa_1}{2}(1-\mu_2) - \frac{\kappa_2\mu_2}{2}\frac{1-\mu_1}{\mu_1}$$

then $\sigma(f) = r_1$ is preferred. And $\sigma(f) = r_2$ otherwise.

Case II: $\mu_1\bar{u}_1 \geq \frac{1+\mu_2}{2}\bar{u}_2 + \frac{\mu_1\kappa_1}{2}$

In this case we have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_1$	$2\bar{u}_1$	$2\bar{u}_2$	$\underline{u}_2$
r_2	$2\bar{u}_1$	$2\bar{u}_1$	$\underline{u}_1 + \bar{u}_2$	$\underline{u}_1$

Suppose again that $\bar{u}_1 \geq \bar{u}_2$ and $0 \geq \underline{u}_1 \geq \underline{u}_2$, then:

$$\begin{aligned} C_1 &= (1-\mu_1)(1-\mu_2)(\underline{u}_1 - \underline{u}_2) \\ C_2 &= (1-\mu_1)\mu_2(2\bar{u}_2 - (\underline{u}_1 + \bar{u}_2)) \end{aligned}$$

Hence $\sigma(f) = r_1$ is optimal iff:

$$\mu_2\bar{u}_2 \geq \mu_2\kappa_2 - \frac{\mu_1}{1-\mu_1}\kappa_1$$

and $\sigma(f) = r_2$ otherwise.

Case III: $\mu_2\bar{u}_2 \geq \frac{1+\mu_1}{2}\bar{u}_1 + \frac{\mu_2\kappa_2}{2}$

We have:

$\mathbb{P}$(Event)	$\mu_1\mu_2$	$\mu_1(1-\mu_2)$	$(1-\mu_1)\mu_2$	$(1-\mu_1)(1-\mu_2)$
r_1	$2\bar{u}_2$	$\underline{u}_2 + \bar{u}_1$	$2\bar{u}_2$	$\underline{u}_2$
r_2	$2\bar{u}_2$	$2\bar{u}_1$	$2\bar{u}_2$	$\underline{u}_1$

Thus unless $0 \geq \underline{u}_2 \geq \underline{u}_1$ then $\sigma(f) = r_2$ is always preferred. Hence suppose $0 \geq \underline{u}_2 \geq \underline{u}_1$, then:

$$\begin{aligned} C_1 &= \mu_1(1-\mu_2)(2\bar{u}_1 - (\underline{u}_2 + \bar{u}_1)) \\ C_2 &= (1-\mu_1)(1-\mu_2)(\underline{u}_2 - \underline{u}_1) \end{aligned}$$

Thus $\sigma(f) = r_1$ is optimal iff:

$$\mu_1\bar{u}_1 \leq \mu_1\kappa_1 - \frac{\mu_2}{1-\mu_2}\kappa_2$$

and $\sigma(f) = r_2$ otherwise.

A.2 Proof of Proposition 6

The proof is by example. We take an observing player who in the absence of observational learning is willing to experiment with r_2 only. We fix $\underline{u}_1 \leq \underline{u}_2 < 0$, and then proceed to show that we can find pairs $(\bar{u}_1, \bar{u}_2)$ such that the observing player prefers to observe experimentation with r_1

Proof of Proposition 6. Fix $\underline{u}_1 \leq \underline{u}_2 < 0$. Take the case where $\bar{u}_1 \leq \frac{\kappa_1}{2}$ and $\bar{u}_2 \leq \frac{\kappa_2}{2}$, which is the case that in the absence of observational learning the player is willing to experiment with r_2 only. Further specialise to the case where $\mu_2\bar{u}_2 \leq \frac{\bar{u}_1(1+\mu_2)+\mu_2\kappa_2}{2}$ (case I in appendix A.1.2). It then follows from A.1.2 that it is optimal to observe a player experimenting with r_1 iff condition (*) is satisfied:

$$\mu_2\bar{u}_2 \leq \bar{u}_1 - \frac{\mu_2\kappa_2}{2}\frac{1-\mu_1}{\mu_1}$$

We now proceed to show that there are non-empty pairs $(\bar{u}_1, \bar{u}_2)$ such that this condition is satisfied. Fix $\bar{u}_1$ at it's upper boundary $\bar{u}_1 = \frac{\kappa_1}{2}$. Inserting this in the equation above leads to the following condition on $\bar{u}_2$:

$$\bar{u}_2 \leq \frac{\kappa_1}{2\mu_2} - \frac{\kappa_2}{2}\frac{1-\mu_1}{\mu_1}$$

The lower bound on $\bar{u}_2$ is $\frac{\kappa_2}{2}$. The condition:

$$\frac{\kappa_1}{2\mu_2} - \frac{\kappa_2}{2}\frac{1-\mu_1}{\mu_1} \geq \frac{\kappa_2}{2}$$

implies $\kappa_1\mu_1 \geq \kappa_2\mu_2$. Now note that (*) is increasing in $\bar{u}_1$. Thus if $\kappa_1\mu_1 \geq \kappa_2\mu_2$ then there are pairs $(\bar{u}_1, \bar{u}_2)$ such that the player prefers to see experimentation with r_1. Conversely if $\kappa_1\mu_1 \leq \kappa_2\mu_2$ then by a symmetric argument we can find observing players who would in the absence of observational learning be willing to experiment with r_1 only, but prefers to see experimentation with r_2. □

Appendix B

Appendix for Chapter 3

B.1 Necessary Conditions

B.1.1 Preliminaries

We require some notation before proceeding. Let a strategy for a G player who is interim type $t \in T$, be denoted $s_t \in \mathbf{G} \times \{I, NI\}$, where we for ease of notation do not include beliefs, and have suppressed the fact that in general the strategy may depend on the signal realisation or the givers (absolute) type. Also let a strategy profile where the strategy of interim type t is removed be denoted s^{-t}.

We first establish that for all interim types the strategy: send gift and do not invest is strictly dominated by the strategy: do not send a gift and do not invest.

Lemma 4. *For any interim type $t \in T$ the strategy $s_t = (g, NI)$, $g \in \mathbf{G} \setminus \{ng\}$ is strictly dominated by the strategy $s'_t = (ng, NI)$.*

Proof. For any s^{-t} the *maximal* payoff from strategy s_t is:

$$b - c < 0$$

This payoff accrues if the gift that the player sends leads R to invest with probability one.

But for any s^{-t} the *minimal* payoff from s'_t is:

$$0$$

which is the payoff to t if R does not invest after not receiving a gift. □

Remark 15. *It follows from the lemma above that we need only consider strategies for a type t player which takes the form: $s_t \in \{(ng, NI), (ng, I), (g, I)$ for some $g \in \mathbf{G} \setminus \{ng\}\}$.*

Lemma 5. *For t_2 types the strategy $s_{t_2} = (g, I)$ for some $g \in \mathbf{G} \setminus \{ng\}$ is strictly dominated by $s'_{t_2} = (ng, NI)$.*

Proof. For any s^{-t_2} the *maximal* payoff from strategy s_{t_2} is:

$$-c < 0$$

since t_2 interim types have modal beliefs on $d = 2$ and thus expected project benefits are strictly negative.

But the *minimal* payoff from s'_{t_2} is:

$$0$$

which accrues if R does not invest with positive probability conditional on not recieving a gift. □

Remark 16. *It follows from the two lemmas above that in any equilibrium of the game t_2 interim types play either (ng, NI) or $(ng. I)$.*

B.1.2 Equilibria with no gifts

We now turn to establishing necessary conditions for existence of equilibria where no interim types send gifts.

The following lemma establishes that in any equilibrium of the game where no interim types send gifts then t_2 types do not invest.

Lemma 6. *If there is an equilibrium where no gifts are sent, then t_2 play (ng, NI).*

Proof. The proof is by contradiction. Suppose there is an equilibrium where no interim types send gifts but t_2 types invest. In this equilibrium R either invests or she does not invest conditional on ng.

Suppose first that she invests. But then t_2 recieves negative expected payoff from (ng, I) whereas she receives payoff $b > 0$ from (ng, NI). Contradiction.

Suppose next that R does not invest. Then (ng, I) yields a small payoff loss l whereas (ng, NI) yields payoff 0. Contradiction. □

Lemma 7. *If there is an equilibrium with no gifts where R and t_1 types invests, then t_0 types also invest.*

Proof. The proof is by contradiction. Suppose there is an equilibrium with no gifts where t_1 types invests but t_0 types do not. Since t_1 types invests in equilibrium it must be that their expected payoff from the project exceed the benefit of free-riding b, because R invests. In the proposed equilibrium t_0 earn payoff b. But then t_0 have a profitable deviation, since expected project payoffs are strictly higher for t_0 types than for t_1 types. Contradiction. □

B.1.3 Equilibria with Gifts

We now turn to equilibria in which some types send gifts.

Lemma 8. *If there is an equilibrium where gifts are sent by at least one type $t \in T \setminus \{t_2\}$ then t_2 types play (ng, NI).*

Proof. The proof is by contradiction. From remark 16 we need only consider the possibility that t_2 types play (ng, I) in the proposed equilibrium. Consider first the case where R invests conditional on ng. In this case t_2 earns negative expected payoffs since she has beliefs modal on $d = 2$. Contradiction since by playing (ng, NI) she can earn $b > 0$. Next consider the case where R does not invest conditional on ng. In this case t_2 has a small payoff loss l, but if she deviates to (ng, NI) she gets payoff 0. Contradiction. □

Next we note that if there is an equilibrium where t_1 types sends gifts and invests then t_0 types also sends gifts and invests. This statement is the equivalent of the monotonicity property of equilibria with no gifts (lemma 7).

Lemma 9. *If there is an equilibrium where t_1 types send gifts and invests, then t_0 types also sends gifts and invests.*

Proof. Proof is by contradiction. Suppose there is an equilibrium where t_1 types sends gifts and invests, but t_0 types do not. From lemma 8 it follows that t_2 types do not send gifts, thus R does not invest conditional on ng. t_0 types do not send gifts and earn 0 payoff. Note that since t_1 types send gifts and invests, it must be that R invests with positive probability. Also expected project payoffs for t_1 are at least c, the cost of sending a gift. But then t_0 types have a profitable deviation, which is to mimick t_1 types. This follows since t_0 types have strictly greater expected project payoffs than t_1 types. Contradiction. □

We now complete the proof of proposition 7.

Proof of Proposition 7. The proof is divided into two parts. The first part establishes the necessary conditions for an equilibria with no gifts, and the second part establishes condtions for equilibria with gifts.

Equilibrium with no Gifts Claim 1 is simply lemma 6. Claim 2 is lemma 7.

Equilibrium with Gifts Claim 1 is lemma 8. Claim 2 is lemma 9. Claim 3 is lemma 4.Claim 4 follows from lemma 8 which says that t_2 types do not send gifts in equilibrium. From assumption (A1) it then follows that expected project payoffs from investing conditonal on not receiving a gift is strictly negative. □

B.2 Equilibrium Analysis

To ease the number of cases to consider we make the stronger assumption that $\bar{\delta} \geq 2\underline{\delta}$. This assumption only affects the characterisation for cost-levels greater than c_1.

First define the following cost levels: $c_1 \equiv \frac{\delta}{5} + \frac{3}{5}b$, $c_2 \equiv b + \frac{\bar{\delta}}{3} - \frac{2}{3}\underline{\delta}$, $c_3 \equiv b + \frac{2}{5}\bar{\delta} - \frac{3}{5}\underline{\delta} > c_2$, $c_3 > c_1$, and $c_4 \equiv \frac{2}{5}(\underline{\delta} + \bar{\delta}) + b > c_3$.

B.2.1 Equilibria with no gifts

In this section we show how the existence of equilibrium is invariant to the benefit to free-riding b, and only depends on the parameters $\underline{\delta}$ and $\bar{\delta}$. For ease of exposition we shall assume that the loss of being free-ridden on is neglible.

Definition 8. *For $b > 0$ and $B \in (\underline{B}, \overline{B})$ define:*

$$\hat{\rho}(B, b) = \left\{\rho \mid B - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) = b\right\}$$

and:

$$\hat{\hat{\rho}}(B, b) = \left\{\rho | B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} = b\right\}$$

Note that $\hat{\rho} < \hat{\hat{\rho}}$, and that the threshold values are decreasing in B. The first threshold value ensures that t_0 players find it worthwhile to invest, whereas the second ensure that t_1 players find it worthwhile to invest.

Equilibrium I

Interim Incentives of G players The incentive constraints of t_0 are:

$$B - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) \geq b$$

The incentive constraints for t_1 are:

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \leq b$$

If t_1 types are deterred from investing then so are t_2 since project benefits are decreasing in type subscript.

Interim Incentives of R players The incentive constraint of a R player who receives ng is:

$$B - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) \geq b$$

We now conclude that if:

$$B - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) \geq b \tag{B.1}$$

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \leq b \tag{B.2}$$

are satisfied then Equilibrium I with no gifts exists. The first constraint ensure that the expected project benefits are sufficiently high for t_0 types to invest. The second constraint says that project benefits cannot be too high, otherwise t_1 types will want to invest.

Proposition 32. *Suppose play is according to the strategy profile, $s^*_{I,NG}$:*

1. *For any $c > b > 0$ $s^*_{I,NG}$ is an equilibrium profile if and only if $\hat{\rho} \leq \rho \leq \hat{\hat{\rho}}$.*

2. *There is no $b > 0$ such that $s^*_{I,NG}$ is an equilibrium profile for all B and ρ.*

We shall show the last statement. Fix B at the lower bound, $B = \underline{\delta} + b$. We now show that the intersection is for some $\rho > \underline{\rho}$.

$$\underline{\delta} + b - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) = b$$

Solving for ρ gives:

$$\rho = \frac{\bar{\delta} - \underline{\delta}}{\bar{\delta} + \underline{\delta}} > \underline{\rho}$$

which implies that $\frac{2}{3}\bar{\delta} > \underline{\delta}$ which follows by assumption.

Equilibrium II

Interim Incentives of G players Since the expected value of project benefits decreases monotonically in type subscript the binding incentive constraint is for t_1 types.

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \geq b$$

Interim Incentives of R players Since R players receive a convex combination of project payoffs from t_0 and t_1 types in the proposed equilibrium, whenever a t_1 type find it worthwhile to invest, then R also finds it wortwhile.

We conclude that if

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \geq b$$

then an Equilibrium II exists.

Proposition 33. *Suppose play is according to the strategy profile,* $s^*_{II,NG}$*:*

1. *For any* $c > b > 0$ $s^*_{II,NG}$ *is an equilibrium profile if and only if* $\rho \geq \hat{\rho}$.
2. *There is no* $b > 0$ *such that* $s^*_{II,NG}$ *is an equilibrium profile for all* B *and* ρ.

B.2.2 Equilibria with Money

In this section we show that there are two types of equilibria where money is used as a medium to induce investment from R players. In the first equilibrium only G players who receive a signal realisation that R is of the same type as G sends gifts, in the other equilibrium G players who receives a signal realisation suggesting that the players are *compatible* send gifts.

Define:

Definition 9. *For* $b \leq c \leq c_4$ *define:*

$$\bar{\rho}(B,c) = \min\left\{\left\{\rho \mid B - \frac{1-\rho}{2}(\underline{\delta}+\bar{\delta}) = c\right\}, 1\right\}$$

For $b \leq c \leq c_3$ *define:*

$$\hat{\rho}(B,c) = \min\left\{\left\{\rho | B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} = c\right\}, 1\right\}$$

Note that in their common domain $\bar{\rho} < \hat{\rho}$, and that the threshold values are decreasing in B for fixed c. The first threshold value ensures that t_0 players find it worthwhile to invest, whereas the second ensure that t_1 players find it worthwhile to invest.

Equilibrium I

Interim Incentives of G players The incentive constraints of t_0 are:

$$\begin{aligned} B - \frac{1-\rho}{2}(\underline{\delta}+\bar{\delta}) &\geq c \\ B - \frac{1-\rho}{2}(\underline{\delta}+\bar{\delta}) &\geq b \end{aligned}$$

from which it follows that a neccesary condition is $c \geq b$.

The incentive constraints for t_1 are:

$$\begin{aligned} 0 &\geq b - c \\ B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} &\leq c \end{aligned}$$

If t_1 types are deterred from sending gifts then so are t_2, since the first constraint is type-independent, and project benefits are decreasing in type subscript.

Interim Incentives of R players The incentive constraint of a R player who receives m is:

$$B - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) \geq b$$

Note that in equilibrium an R player who does not receive a gift expects G to play NI, therefore her incentive constraint is non-binding.

We now conclude that if $c \geq b$ and:

$$B - \frac{1-\rho}{2}(\underline{\delta} + \bar{\delta}) \geq c \tag{B.3}$$

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \leq c \tag{B.4}$$

then Equilibrium I with money exists. The first constraint ensure that the expected project benefits are sufficiently high for t_0 types to invest. The second constraint says that project benefits cannot be too high, otherwise t_1 types will want to send gifts (and induce investment).

Proposition 34. *Suppose play is according to the strategy profile, $s^*_{I,M}$:*

1. *If $c > c_4$ then there is no B and ρ such that $s^*_{I,M}$ is an equilibrium profile.*

2. *If $c_3 \leq c \leq c_4$ then $s^*_{I,M}$ is an equilibrium profile if and only if $B \in (c, \bar{B})$ and $\rho \geq \hat{\rho}$.*

3. *If $b \leq c \leq c_3$ then $s^*_{I,m}$ is an equilibrium profile if and only if $B \in (\max(\underline{B}, c), \bar{B})$ and $\hat{\rho} \leq \rho \leq \hat{\hat{\rho}}$.*

4. *There is no cost level $c \geq b > 0$ such that $s^*_{I,m}$ is an equilibrium profile for all B and ρ.*

Equilibrium II

Interim Incentives of G players Continue to assume that $c \geq b$. Since the expected value of project benefits decreases monotonically in type subscript the binding incentive constraint is for t_1 types.

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \geq c$$

A t_2 player should not send gifts in equilibrium:

$$B - \frac{1-\rho}{2}\underline{\delta} - \frac{1+3\rho}{4}\bar{\delta} \leq c$$

Which is not binding under assumption (A1).

Interim Incentives of R players When the incentive constraint of t_1 types are satisfied then R's incentive constraint is automatically satisfied since conditional on investing when she receives m she gets a convex combination of the project payoffs from t_0 and t_1 types and project payoffs are decreasing in type subscript.

Thus we find that when $c \geq b$ then:

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\bar{\delta} \geq c$$

is a necessary condition for Equilibrium II. The left hand side is the marginal benefit of the project, conditional on receiving a signal realisation suggesting players are compatible. The right hand side is the marginal cost of the project, c.

Define $\tilde{B}(c) = \underline{\delta} + c$.

Proposition 35. *Suppose play is according to the strategy profile, $s^*_{II,M}$:*

1. *If $c > c_3$ then there is no B and ρ such that $s^*_{II,M}$ is an equilibrium profile.*
2. *If $b \leq c \leq c_3$ then $s^*_{II,M}$ is an equilibrium profile if and only if $B \in (\tilde{B}(c), \bar{B})$ and $\rho \geq \hat{\hat{\rho}}$.*
3. *There is no cost level $c \geq b > 0$ such that $s^*_{II,M}$ is an equilibrium profile for all B and ρ.*

B.2.3 Equilibrium with Gifts-in-Kind

Equilibrium I

Here we construct an equilibrium where only t_0 types send gifts, whereas the two other types t_1, t_2 do not send gifts.

Interim Incentives of G players Assume $c \geq b$. The incentive constraints of t_0 is:

$$\frac{1+\rho}{2}B - \frac{1-\rho}{2}\underline{\delta} \;\geq\; c$$

The constraint says that the marginal benefit from sending the gift should be greater than the marginal cost of sending the gift.

A t_1 type should not send gifts in equilibrium. She might consider two possible deviations: (1) Send $k = \tau$ and I(nvest), (2) Send $k = \theta$, i.e. distort the signal, and I(nvest). It turns out that we only need to deter the second deviation. This is so because the prior probability of the project being implemented is the same for both deviations, but with deviation (1), with positive probability the project will be implemented even when $d = 2$. Thus we have:

$$\frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta} \leq c$$

and to deter her from sending a gift but not investing:

$$\frac{1+\rho}{2}b \leq c$$

If a t_1 player is deterred then so is t_2.

Interim Incentives of R players The incentive constraint of a R player of type θ who receives $d(\tau,\theta) = 1$ is:

$$B - \underline{\delta} \geq b$$

which is satisfied by assumption. We can then sum up the equilibrium conditions:

$$\frac{1+\rho}{2}B - \frac{1-\rho}{2}\underline{\delta} \;\geq\; c \qquad \text{(B.5)}$$

$$\frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta} \;\leq\; c \qquad \text{(B.6)}$$

$$\frac{1+\rho}{2}b \;\leq\; c \qquad \text{(B.7)}$$

Define:

Definition 10. *The problem is* regular *at c^* if*

$$\{(B,\rho)|\frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta} \leq c\}$$

is convex. Let $\underline{c}$ be the largest c such that the problem is regular.

Note that if the problem is regular at c^* then it is regular for all $c \leq c^*$. With our assumptions the problem is regular for all $c \leq c_1$.

Let:

$$\begin{aligned}\tilde{\rho}(B) &= \min\left(\left\{\rho | \frac{1+\rho}{2}B - \frac{1-\rho}{2}\underline{\delta} = c\right\}, 1\right) \\ \tilde{\rho}(B) &= \min\left(\left\{\rho | \frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta} = c\right\}, 1\right)\end{aligned}$$

Proposition 36. *Suppose the problem is not regular at $c = c_3$. Also suppose play is according to the profile $s^*_{I,K}$:*

1. *For $c > c_4$ there is no B and ρ such that $s^*_{I,K}$ is an equilibrium profile.*
2. *For $c_3 \leq c \leq c_4$ then $s^*_{I,K}$ is an equilibrium profile if and only if $B \in (max(\underline{B}, c), \bar{B})$ and $\rho \geq \tilde{\rho}(B)$.*
3. *For $c_1 < c < c_3$:*
   (a) *For any $c > \underline{c}$: $s^*_{I,K}$ is an equilibrium profile if and only if $B \in (c, \bar{B})$ and $\rho \in [\tilde{\rho}(B), \bar{\tilde{\rho}}(B)]$.*
   (b) *For any $c \leq \underline{c}$: $s^*_{I,K}$ is an equilibrium profile if and only if $B \in (\underline{B}, \tilde{B}(c))$ and $\rho \in [\tilde{\rho}(B), 1)$.*
4. *For $c \leq c_1$ $s^*_{I,K}$ is an equilibrium profile if and only if $B \leq \tilde{B}(c)$ and $\rho \in [\tilde{\rho}(B), 1)$.*

Equilibrium II (Type Revelation)

In this equilibrium G players follow the rule: if you think you are compatible with R, then give her something you would like yourself.

Interim Incentives of G players Given the behaviour of R players t_1 should want to send gifts:

$$\frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta} \geq c$$

and to induce her to invest conditional upon sending a gift:

$$\frac{1+\rho}{2}b \leq c$$

Interim Incentives of R players Since

$$B - \underline{\delta} \geq b$$

holds by assumption, we can now sum up the equilibrium conditions:

$$\frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta} \geq c \tag{B.8}$$

$$\frac{1+\rho}{2}b \leq c \tag{B.9}$$

Proposition 37. *Suppose the problem is not regular at $c = c_3$. Also suppose play is according to the profile $s^*_{II,K}$:*

1. *For $c > c_3$ then there is no B and ρ such that $s^*_{II,K}$ is an equilibrium profile.*
2. *For $c < c_3$:*
   (a) *For $c > \underline{c}$: $s^*_{II,K}$ is an equilibrium profile if and only if $B \in (\tilde{B}(c), \bar{B})$ and $\rho \in [\tilde{\rho}(B), 1)$.*
   (b) *For $c \leq \underline{c}$ and $B \leq \tilde{B}(c)$ $s^*_{II,K}$ is an equilibrium profile if and only if $\rho \in [\tilde{\bar{\rho}}(B), 1)$. For $B > \bar{B}(c)$ $s^*_{II,K}$ is an equilibrium profile for any ρ.*

Equilibrium II (Signal Revelation)

Interim Incentives of G players As we shall see in equilibrium R players cannot invest after learning that $d(g^{-1}(k), \theta) = 1$. Otherwise t_1 types will be tempted to distort their signal. The incentive constraint of a t_1 player is:

$$\rho(B - \underline{\delta}) \geq c$$

to induce her to send a gift. To induce her to send the gift expected in equilibrium, and not $k = \theta$ (θ is G's own type), we must have:

$$\frac{1-\rho}{4}B \leq c$$

and to induce her to invest, conditional upon sending the gift:

$$\rho b \leq c$$

Interim Incentives of R players Now consider the incentives of a R player who receives $d(g^{-1}(\tau), \theta) = 0$. She will invest if:

$$B - \frac{2}{3}\underline{\delta} \geq b$$

which holds by assumption. A R player who receives $d(g^{-1}(\tau), \theta) = 1$ should not invest in equilibrium:

$$B - \frac{1}{3}(\underline{\delta} + \bar{\delta}) \le b$$

We can now sum up the necessary conditions for equilibrium:

$$\rho(B - \underline{\delta}) \ge c \quad \text{(B.10)}$$

$$\frac{1-\rho}{4}B \le c \quad \text{(B.11)}$$

$$\rho b \le c \quad \text{(B.12)}$$

$$B - \frac{1}{3}(\underline{\delta} + \bar{\delta}) \le b \quad \text{(B.13)}$$

Or alternatively if (B.11) does not hold:

$$\frac{1-\rho}{4}B \le \rho(B - \underline{\delta})$$

provided that (B.10) holds.

Define for fixed B and c:

$$\tilde{\bar{\rho}}(B,c) = \max\left(\{\rho|\frac{1-\rho}{4}B = c\}, \{\rho|\rho(B - \underline{\delta}) = c\}\right)$$

Proposition 38. *Suppose the problem is not regular at $c = c_3$. Also suppose play is according to the profile $s^*_{III,K}$:*

1. *For $c > c_2$ then there is no B and ρ such that $s^*_{III,K}$ is an equilibrium profile.*
2. *For $c < c_2$ and $\tilde{B}(c) \le \frac{1}{3}(\underline{\delta} + \bar{\delta}) + b$:*
   (a) *For $\tilde{B}(c) \le B \le \frac{1}{3}(\underline{\delta} + \bar{\delta}) + b$ $s^*_{III,K}$ is an equilibrium profile if and only if $\rho \in [\tilde{\bar{\rho}}(B), 1)$*

B.2.4 Hybrid Equilibrium

Profile 1 Consider first an equilibrium construction where G sends a gift-in-kind if she receives a signal which indicates that both players are of the same type, and where she sends money if she believes that players are of different, but compatible types. We show that such an equilibrium does not exist because G has a profitable deviation, in the event that she is supposed to send money.

t_1 should send money. In equilibrium R's beliefs about the true distance between the players, conditional on receiving money, are:

$$Pr(d = 0|m) = \frac{\frac{1}{5}2\frac{1-\rho}{4}}{\frac{2}{5}} = \frac{1-\rho}{4}, \; Pr(d = 1|m) = \frac{1+3\rho}{4}, \; Pr(d = 2|m) = \frac{1-\rho}{2}$$

Supposing these beliefs are sufficient to induce investment from R we now check that t_1 has no incentive to deviate. Her expected payoff is:

$$B - \frac{1+3\rho}{4}\underline{\delta} - \frac{1-\rho}{2}\overline{\delta}$$

But then t_1 has a profitable deviation which is to send a gift-in-kind which reveals her own type. This induces all R players such that $d \leq 1$ to invest, yielding the following payoff:

$$\rho(B - \underline{\delta}) + \frac{1-\rho}{4}(B - \underline{\delta}) + \frac{1-\rho}{4}B$$

which is profitable as long as $B \leq \overline{\delta}$. Hence there is no hybrid equilibrium of this type.

Profile 2 We now check the two remaining possible hybrid equilibria. Consider first the case where t_1 sends a gift-in-kind which reveals her type. This induces investment from types such that $d \leq 1$. Now return to t_0 who sends money. This leads to the following conditional beliefs of R.

$$Pr(d = 0|m) = \rho,\ Pr(d = 1|m) = \frac{1-\rho}{2},\ Pr(d = 2|m) = \frac{1-\rho}{2}$$

Investment yields the following expected payoff:

$$B - \frac{1-\rho}{2}(\underline{\delta} + \overline{\delta})$$

But then t_0 has a profitable deviation which is to reveal her type via a gift-in-kind, yielding the following payoff:

$$\rho B - \frac{1-\rho}{2}(B - \underline{\delta})$$

Hence we cannot sustain this equilibrium.

Profile s_H^* Finally, consider the case where in the event t_1 sends a gift-in-kind which reveals her signal.

First, let us establish conditional beliefs of R. Let k_d denote the gift in kind which is distance d away from R's type.

Now, if R receives k_0 she will know that it $d = 1$, since if $d = 0$ and she receives a gift-in-kind then it cannot come from $d = 0$ since then G should send money in equilibrium. It cannot come from $d = 2$ either since G with this signal would not send gifts.

If R receives k_1 then with equal probability it must come from either $d = 0$ or $d = 2$. Thus she should invest if:

$$B - \frac{\overline{\delta}}{2} \geq b$$

Thus whether investment occurs depend on the parameters of the model.

Finally conditional on k_2 with equal probability the state is either $d = 1$ or $d = 2$, so there will be no investment after this event.

To induce G to invest under this equilibrium profile we have:

$$\rho(B - \underline{\delta}) \geq c \tag{B.14}$$

if $B - \frac{\bar{\delta}}{2} < b$ and

$$\frac{1+\rho}{2}B - (\rho\underline{\delta} + \frac{1-\rho}{4}\bar{\delta})$$

otherwise.

Take first the case where $B - \frac{\bar{\delta}}{2} \geq b$. Then G has a profitable deviation, which is to send a gift matching her own type. This yields a payoff:

$$\frac{1+\rho}{2}B - \frac{1+3\rho}{4}\underline{\delta}$$

which is a profitable deviation.

Now suppose $B - \frac{\bar{\delta}}{2} < b$. In this case there is no profitable deviation if:

$$\rho(B - \underline{\delta}) \geq \frac{1-\rho}{4}B \tag{B.15}$$

so in this case there is an equilibrium.

Define for any B and c:

$$\bar{\rho}(B, c) \quad = \quad \max\left(\{\rho|\rho(B - \underline{\delta}) = c\}, \{\rho|\rho(B - \underline{\delta}) = \frac{1-\rho}{4}B\}\right)$$

We summarise our findings.

Proposition 39. *Suppose the problem is not regular at c_3. Assume $B \leq \frac{\bar{\delta}}{2} + b$ for some B. Restrict attention to $s \in S_H$.*

1. *For $c > c_3$ there is no equilibrium $s^* \in S_H$.*
2. *For $c < c_3$ and $\tilde{B}(c) \leq B \leq \frac{\bar{\delta}}{2} + b$:*
   (a) *s_H^* is an equilibrium profile if and only if $\rho \in [\bar{\rho}(B), 1)$.*
   (b) *The set of hybrid equilibrium profiles $S_H^* \subseteq S_H$ is either empty or $S_H^* = \{s_H^*\}$.*

B.3 Robustness

This section looks at a variation of the model where the amount of money that givers spend is a strategic variable.

We first consider the case where only t_0 interim types invest. Then we extended the result to the case where t_1 types also invest.

To this end let $t_0(\theta)$ denote a t_0 interim type of G which is of type $\theta \in \Theta$.

We shall look for an equilibrium in which each $t_0(\theta)$ sends a money gift of size m_θ, and where $m_\theta \neq m_{\theta'}$, $\theta \neq \theta'$.

Let $f : \Theta \to \mathbb{R}_+$, be a one-to-one mapping that maps types of G into sums of money given.

Consider the strategy profile:

G-players:

- $t_0(\theta)$ play (m_θ, I)
- t_1, t_2 play (ng, NI)

R-players:

- If m is received and $d(f^{-1}(m), \theta_R) \leq 1$ then play I, otherwise play NI.

First observe that the behaviour of t_0 players reveal their type to R. Therefore whenever R receives a monetary payment that comes from a type like herself, or like one of her neighbours then it is sequentially rational to invest.

Next, note that if $\min_\theta f(\theta)$ is sufficiently large to deter t_2 and t_1 types from investing then we equilibrium is established if: (1) All t_0 interim types find it worthwhile to invest given their equilibrium transfer, and (2) $t_0(\theta)$ does not find it worthwhile to pose as $t_0(\theta')$, $\theta \neq \theta'$.

Given the behaviour of R it can only be beneficial for $t_0(\theta)$ to try and pose as a neighbour, whose equilibrium transfer is lower.

Thus we must have:

$$2\frac{1-\rho}{4}(B-\underline{\delta}) + \rho B - m_\theta \quad \geq \quad \frac{1-\rho}{4}(B-\underline{\delta}) + \frac{1-\rho}{4}(B-\overline{\delta})\rho B - m_{\theta'}$$

which leads to the condition:

$$\frac{1-\rho}{4}(\overline{\delta}-\underline{\delta}) \equiv \Delta(\rho) \quad \geq \quad m_\theta - m_{\theta'} \tag{B.16}$$

where θ' is a neighbour of θ.

Let $\underline{m}^i > 0$ be the smallest amount of money that deters interim types t_j, $j > i$ from sending gifts in the proposed equilibrium. Note that $\underline{m}_i$ is (weakly) decreasing in i since expected project payoffs is increasing in i. Also let $\overline{m}^i$ be the highest monetary transfer which is incentive compatible with types t_j, $j \leq i$ sending money. Note that $\overline{m}^i$ is also (weakly) decreasing in i, and furthermore $\overline{m}^i > \underline{m}^i$, whenever t_i earns a strictly positive payoff in equilibrium. If $\overline{m}^i = \underline{m}^i$ then equilibrium cannot be sustained.

First note that when $\rho = 1$, $\Delta(\rho) = 0$ and the equilibrium breaks down. However when $\rho < 1$ it is possible to construct equilibria of this form.

We can now state:

Proposition 40. *Consider a variation of the Gift game where the amount of money that G can give is a strategic variable. For $\rho < 1$, and for any two neighbours: θ and θ' let f satisfy: (i) $|f(\theta) - f(\theta')| < \Delta(\rho)$, (ii) $\max_\theta f(\theta) < \overline{m}^0$, (iii) $\min_\theta f(\theta) > \underline{m}^0$. Then there exists an equilibrium where t_0 types gives monetary gifts according to f, t_1 and t_2 do not send gifts. R invests iff $d(f^{-1}(m_\theta), \theta_R) \leq 1$.*

We now consider an equilibrium where also t_1 types find it worthwhile to invest. It can easily be verified that condition (B.16) must also hold for t_1.

We have:

Corollary 6. *Consider a variation of the Gift game where the amount of money that G can give is a strategic variable. For $\rho < 1$, and for any two neighbours: θ and θ' let f satisfy: (i) $|f(\theta) - f(\theta')| < \Delta(\rho)$, (ii) $\max_\theta f(\theta) < \overline{m}^1$, (iii) $\min_\theta f(\theta) > \underline{m}^1$. Then there exists an equilibrium where t_0, t_1 types gives monetary gifts according to f, t_2 do not send gifts. R invests iff $d(f^{-1}(m_\theta), \theta_R) \leq 1$.*

B.4 Welfare

B.4.1 Information Structures

This section list the information structures generated under equilibria with gifts.

$$\Pi(I,K) = \begin{array}{c} \\ \tau_m \\ \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_\emptyset \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & d_1 & d_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \rho & 0 & 0 \\ 0 & \frac{1-\rho}{4} & 0 \\ 0 & 0 & \frac{1-\rho}{4} \\ 1-\rho & \frac{3+\rho}{4} & \frac{3+\rho}{4} \end{pmatrix} \end{array}$$

$$\Pi(II,K) = \begin{array}{c} \\ \tau_m \\ \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_\emptyset \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & d_1 & d_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \frac{1+\rho}{2} & 0 & 0 \\ 0 & \frac{1+\rho}{2} & 0 \\ 0 & 0 & \frac{3(1-\rho)}{4} \\ \frac{1-\rho}{2} & \frac{1-\rho}{2} & \frac{1+3\rho}{4} \end{pmatrix} \end{array}$$

$$\Pi(III,K) = \begin{array}{c} \\ \tau_m \\ \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_\emptyset \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & d_1 & d_2 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ \rho & \rho & 0 \\ \frac{1-\rho}{2} & \frac{1-\rho}{4} & \frac{1-\rho}{4} \\ 0 & \frac{1-\rho}{4} & \frac{1-\rho}{2} \\ \frac{1-\rho}{2} & \frac{1-\rho}{2} & \frac{1+3\rho}{4} \end{pmatrix} \end{array}$$

$$\Pi(I,M) = \begin{array}{c} \\ \tau_m \\ \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_\emptyset \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & d_1 & d_2 \end{array} \\ \begin{pmatrix} \rho & \frac{1-\rho}{4} & \frac{1-\rho}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1-\rho & \frac{3+\rho}{4} & \frac{3+\rho}{4} \end{pmatrix} \end{array}$$

$$\Pi(II,M) = \begin{array}{c} \\ \tau_m \\ \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_\emptyset \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & d_1 & d_2 \end{array} \\ \begin{pmatrix} \frac{1+\rho}{2} & \frac{1+\rho}{2} & \frac{3(1-\rho)}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{1-\rho}{2} & \frac{1-\rho}{2} & \frac{1+3\rho}{4} \end{pmatrix} \end{array}$$

$$\Pi(H) = \begin{array}{c} \\ \tau_m \\ \tau_0 \\ \tau_1 \\ \tau_2 \\ \tau_\emptyset \end{array} \begin{array}{c} \begin{array}{ccc} d_0 & d_1 & d_2 \end{array} \\ \begin{pmatrix} \rho & \frac{1-\rho}{4} & \frac{1-\rho}{4} \\ 0 & \rho & 0 \\ \frac{1-\rho}{2} & 0 & \frac{1-\rho}{4} \\ 0 & \frac{1-\rho}{4} & \frac{1-\rho}{4} \\ \frac{1-\rho}{2} & \frac{1-\rho}{2} & \frac{1+3\rho}{4} \end{pmatrix} \end{array}$$

B.4.2 Blackwell Ordering

In this section we show how the Blackwell ordering is constructed.

We are looking for a square matrix, Q, such that: all elements are weakly positive, and the sum of elements of each column sums to one. An information structure μ blackwell dominates ν iff $Q\mu = \nu$.

We now compare the various information structures.

$\Pi(II, M)$ does not blackwell dominate $\Pi(II, K)$ since:

$$\begin{aligned} q_{21}\frac{1+\rho}{2} + q_{25}\frac{1-\rho}{2} &= \frac{1+\rho}{2} \\ q_{21}\frac{1+\rho}{2} + q_{25}\frac{1-\rho}{2} &= 0 \end{aligned}$$

leads to a contradiction.

However the following Q matrix shows that $\Pi(II, K)$ blackwell dominates $\Pi(II, M)$:

$$\begin{array}{ccccc} q_{11} & 1 & 1 & 0 & 1 \\ q_{21} & 0 & 0 & 0 & 0 \\ q_{31} & 0 & 0 & 0 & 0 \\ q_{41} & 0 & 0 & 0 & 0 \\ q_{51} & 0 & 0 & 1 & 0 \end{array}$$

$\Pi(II, K)$ blackwell dominates $\Pi(III, K)$ by the following Q-matrix:

$$\begin{array}{ccccc} q_{11} & \frac{2\rho}{1+\rho} & \frac{2\rho}{1+\rho} & 0 & 0 \\ q_{21} & \frac{1-\rho}{(1+\rho)} & \frac{1-\rho}{2(1+\rho)} & \frac{1}{3} & 0 \\ q_{31} & 0 & \frac{1-\rho}{2(1+\rho)} & \frac{2}{3} & 0 \\ q_{41} & 0 & 0 & 0 & 0 \\ q_{51} & 0 & 0 & 0 & 1 \end{array}$$

However:

$$\begin{aligned} q_{12}\rho + q_{13}\frac{1-\rho}{2} + q_{15}\frac{1-\rho}{2} &= \frac{1+\rho}{2} \\ q_{12}\rho + q_{13}\frac{1-\rho}{4} + q_{14}\frac{1-\rho}{4} + q_{15}\frac{1-\rho}{2} &= 0 \end{aligned}$$

leads to a contradiction. Hence $\Pi(II, K)$ is not blackwell dominated by $\Pi(III, K)$.

$\Pi(II, K)$ blackwell dominates $\Pi(I, M)$ by the following Q-matrix:

$$\begin{array}{ccccc} q_{11} & \frac{2\rho}{1+\rho} & \frac{1-\rho}{2(1+\rho)} & \frac{1}{3} & 0 \\ q_{21} & 0 & 0 & 0 & 0 \\ q_{31} & 0 & 0 & 0 & 0 \\ q_{41} & 0 & 0 & 0 & 0 \\ q_{51} & \frac{1-\rho}{1+\rho} & \frac{1+3\rho}{2(1+\rho)} & \frac{2}{3} & 1 \end{array}$$

Since,

$$\begin{aligned} q_{21}\rho + q_{25}(1-\rho) &= \frac{1+\rho}{2} \\ q_{21}\frac{1-\rho}{4} + q_{25}\frac{3+\rho}{4} &= 0 \end{aligned}$$

$\Pi(II, K)$ is not blackwell dominated by $\Pi(I, M)$.

$\Pi(II, K)$ blackwell dominates $\Pi(H)$ by the following Q-matrix:

$$\begin{matrix} q_{11} & \frac{2\rho}{1+\rho} & \frac{1-\rho}{2(1+\rho)} & \frac{1}{3} & 0 \\ q_{21} & 0 & \frac{2\rho}{1+\rho} & 0 & 0 \\ q_{31} & \frac{1-\rho}{1+\rho} & 0 & \frac{1}{3} & 0 \\ q_{41} & 0 & \frac{1-\rho}{2(1+\rho)} & \frac{1}{3} & 0 \\ q_{51} & 0 & 0 & 0 & 1 \end{matrix}$$

Since,

$$\begin{aligned} q_{31}\rho + q_{33}\frac{1-\rho}{2} + q_{35}\frac{1-\rho}{2} &= 0 \\ q_{31}\frac{1-\rho}{4} + q_{33}\frac{1-\rho}{4} + q_{34}\frac{1-\rho}{4} + q_{35}\frac{1+3\rho}{4} &= 0 \end{aligned}$$

implies $q_{31} = q_{33} = q_{34} = q_{35} = 0$, then:

$$q_{31}\frac{1-\rho}{4} + q_{32}\rho + q_{34}\frac{1-\rho}{4} + q_{35}\frac{1-\rho}{2} = 0$$

we must have: $q_{32} = \frac{1+\rho}{2\rho} > 1$ which is a contradiction. $\Pi(II, K)$ is not blackwell dominated by $\Pi(H)$.

$\Pi(II, K)$ blackwell dominates $\Pi(II, M)$ by the following Q-matrix:

$$\begin{matrix} q_{11} & 1 & 1 & 1 & 0 \\ q_{21} & 0 & 0 & 0 & 0 \\ q_{31} & 0 & 0 & 0 & 0 \\ q_{41} & 0 & 0 & 0 & 0 \\ q_{51} & 0 & 0 & 0 & 1 \end{matrix}$$

Since,

$$\begin{aligned} q_{21}\frac{1+\rho}{2} + q_{25}\frac{1-\rho}{2} &= \rho \\ q_{21}\frac{3(1-\rho)}{4} + q_{25}\frac{1+3\rho}{4} &= 0 \end{aligned}$$

$\Pi(II, K)$ is not blackwell dominated by $\Pi(II, M)$.

Since,

$$\begin{aligned} q_{12}\rho + q_{13}\frac{1-\rho}{2} + q_{15}\frac{1-\rho}{2} &= \rho \\ q_{12}\rho + q_{13}\frac{1-\rho}{4} + q_{14}\frac{1-\rho}{4} + q_{15}\frac{1-\rho}{2} &= \frac{1-\rho}{4} \end{aligned}$$

we have: $q_{13} - q_{14} = \frac{5\rho-1}{1-\rho}$, so that as $\rho \to 1$ the difference becomes unbounded, a contradiction. Hence $\Pi(I, M)$ is not blackwell dominated by $\Pi(III, K)$.

Since,

$$\begin{aligned} q_{21}\rho + q_{25}(1-\rho) &= \rho \\ q_{21}\frac{1-\rho}{4} + q_{25}\frac{3+\rho}{4} &= 0 \end{aligned}$$

$\Pi(III, K)$ is not blackwell dominated by $\Pi(I, M)$.

Since.

$$\begin{aligned} q_{21}\rho + q_{23}\frac{1-\rho}{2} + q_{25}\frac{1-\rho}{2} &= \rho \\ q_{21}\frac{1-\rho}{4} + q_{23}\frac{1-\rho}{4} + q_{24}\frac{1-\rho}{4} + q_{15}\frac{1+3\rho}{4} &= 0 \end{aligned}$$

$\Pi(III, K)$ is not blackwell dominated by $\Pi(H)$.

Since,

$$\begin{aligned} q_{22}\rho + q_{23}\frac{1-\rho}{2} + q_{25}\frac{1-\rho}{2} &= 0 \\ q_{23}\frac{1-\rho}{4} + q_{24}\frac{1-\rho}{2} + q_{25}\frac{1+3\rho}{4} &= 0 \end{aligned}$$

$q_{22} = q_{23} = q_{24} = q_{25} = 0$. But,

$$q_{22}\rho + q_{23}\frac{1-\rho}{4} + q_{24}\frac{1-\rho}{4} + q_{25}\frac{1-\rho}{2} = \rho$$

a contradiction. $\Pi(H)$ is not blackwell dominated by $\Pi(III, K)$.

By,

q_{11}	1	1	1	0
q_{21}	0	0	0	0
q_{31}	0	0	0	0
q_{41}	0	0	0	0
q_{51}	0	0	0	1

$\Pi(I, K)$ blackwell dominates by $\Pi(I, M)$.

Since,

$$\begin{aligned} q_{21}\rho + q_{25}(1-\rho) &= \rho \\ q_{21}\frac{1-\rho}{4} + q_{25}\frac{3+\rho}{4} &= 0 \end{aligned}$$

$\Pi(I.K)$ is not blackwell dominated by $\Pi(I,M)$.

From,

$$\begin{aligned} q_{21}\rho q_{25}(1-\rho) &= 0 \\ q_{21}\frac{1-\rho}{4} + q_{25}\frac{3+\rho}{4} &= \rho \end{aligned}$$

$\Pi(H)$ is not blackwell dominated by $\Pi(I,M)$.

By,

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{matrix}$$

$\Pi(H)$ blackwell dominates $\Pi(I,M)$.

Since,

$$\begin{aligned} q_{21}\frac{1+\rho}{2} + q_{25}\frac{1-\rho}{2} &= 0 \\ q_{21}\frac{1+\rho}{2} + q_{25}\frac{1-\rho}{2} &= \rho \end{aligned}$$

$\Pi(H)$ is not blackwell dominated by $\Pi(II,M)$.

$\Pi(H)$ blackwell dominates $\Pi(II,M)$, by:

$$\begin{matrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

B.4.3 Proof of Propostion 16

It follows from the Blackwell ordering that among equilibria with gifts G prefers either $s^*_{I.K}$ or $s^*_{II.K}$ depending upon which of these is an equilibrium profile.

For the second part we note for equilibria with no gifts the probability with which type invests:

$$\Pi(I,NG) = \begin{matrix} \\ I \\ NI \end{matrix} \begin{matrix} d_0 & d_1 & d_2 \\ \left(\begin{matrix} \rho & \frac{1-\rho}{4} & \frac{1-\rho}{4} \\ 1-\rho & \frac{3+\rho}{4} & \frac{3+\rho}{4} \end{matrix}\right) \end{matrix}$$

$$\Pi(II, NG) = \begin{matrix} & d_0 & d_1 & d_2 \\ I & \frac{1+\rho}{2} & \frac{1+\rho}{2} & \frac{3(1-\rho)}{4} \\ NI & \frac{1-\rho}{2} & \frac{1-\rho}{2} & \frac{1+3\rho}{4} \end{matrix}$$

First we calculate welfare:

$$\begin{aligned}
W^G_{I,K} &= \frac{1}{5}(\rho(B-c)) + \frac{2}{5}(\frac{1-\rho}{4}(B-\underline{\delta}-c)) + \frac{2}{5}(-\frac{1-\rho}{4}c) \\
W^G_{II,K} &= \frac{1}{5}(\frac{1+\rho}{2}(B-c)) + \frac{2}{5}(\frac{1+\rho}{2}(B-\underline{\delta}-c)) + \frac{2}{5}(-\frac{3(1-\rho)}{4}c) \\
W^G_{I,NG} &= \frac{1}{5}(\rho B + (1-\rho)b) + \frac{2}{5}(\frac{1-\rho}{4}(B-\underline{\delta}) + \frac{3+\rho}{4}b) + \frac{2}{5}(\frac{1-\rho}{4}(B-\bar{\delta}) + \frac{3+\rho}{4}b) \\
W^G_{II,NG} &= \frac{1}{5}(\frac{1+\rho}{2}B + \frac{1-\rho}{2}b) + \frac{2}{5}(\frac{1+\rho}{2}(B-\underline{\delta}) + \frac{1-\rho}{2}b) + \frac{2}{5}(\frac{3(1-\rho)}{4}(B-\bar{\delta} + \frac{1+3\rho}{4}b)
\end{aligned}$$

For fixed $c \leq c_1$ and $b < c$ we now calculate the treshold value for given B $\rho(B)$ such that when $\rho \geq \rho(B)$ then some no gifts equilibrium is preferred.

$$\rho_{I,NG}(B) = 1 - \frac{2(c+4b)}{\bar{\delta} - B}$$

Note that when B attains it's lower bound $B = \underline{B}$ then $\rho_{I,NG} < 1$, and that threshold values are decreasing in B. We can also bound $\rho_{I,NG}$ from below since it follows that for $B = \underline{B}$ welfare is higher in $s^*_{I,K}$ when $s^*_{I,NG}$ does not exist. The condition for existence of $s^*_{I,NG}$ is: $B \geq \frac{1+\rho}{2}(\underline{\delta} + \bar{\delta}) + b$, setting $B = \underline{B}$ yields the lower bound: $\frac{\bar{\delta}-\underline{\delta}}{\bar{\delta}+\underline{\delta}}$, such that $\frac{\bar{\delta}-\underline{\delta}}{\bar{\delta}+\underline{\delta}} < \rho(\underline{B}) < 1$. It then follows that when $s^*_{I,K}$ exists then for any $\rho \geq \rho(\underline{B})$ then some equilibrium with no gifts is preferred by G.

We now turn to the case where $s^*_{II,K}$ is an equilibrium profile. We shall show that for $B = \tilde{B}$ there is a $\rho\tilde{B} < 1$ such that $s_{II,NG}$ is preferred for $\rho > \rho\tilde{B}$. We shall then use a continuity argument to extend the result until we reach the border with $s^*_{I,K}$.

The threshold value for $s_{II,NG}$ to be preferred to $s^*_{II,K}$ is: $\rho(B) = 1 - \frac{6c+2b}{3(\bar{\delta}-B)}$ which is decreasing in B.

Now note that $B = \tilde{B}(c)$ the threshold value for ρ is strictly less than one. Note that this also holds at the lower bound of $B = \underline{B}$. Hence since the threshold value is continuous in B when B is decreased below $B = \tilde{B}(c)$ towards $\underline{B}$ we must eventually cross into the set of parameters for which $s^*_{I,K}$ is now an equilibrium and we are done.

B.4.4 Proof of Proposition 17

It is useful to look at the information structures:

In general equilibria differ along two dimensions: (i) the probability with which state dependent projects are implemented, (ii) the (intrinsic) utility of gifts. Gifts in kind are in general superior along the first dimension, whereas gifts with cash are more desirable along the second dimension.

Let the event that G sends a gift be denoted E_1 and let E_2 be the event that R invests conditional on E_1, and E_3 the event that she free-rides. It is clear that $E_2 \bigcap E_3 = \emptyset$, and let v denote the intrinsic value of the gift to the receiver.

Information structure Π is preferred to Π' if:

$$\sum_{i=0}^{2} \mathbb{P}(d_i) \left[\mathbb{P}_i(E_1|\Pi)\mathbb{P}_i(E_2|\Pi) - \mathbb{P}_i(E_1|\Pi')\mathbb{P}_i(E_2|\Pi')\right) \right] B(d_i) \geq$$

$$\sum_{i=0}^{2} \mathbb{P}(d_i) \left[\mathbb{P}_i(E_1|\Pi')(v(d_i,\Pi') + \mathbb{P}_i(E_3|\Pi')b) - \mathbb{P}_i(E_1|\Pi)(v(d_i,\Pi) + \mathbb{P}_i(E_3|\Pi)b)\right]$$

The left hand side of the equation is the differential informational value between Π and Π'. The right hand side is the differential intrinsic value between the information structures.

We first restrict attention to equilibria which involves gifts. We have the following comparison:

$s_{II,K}$ is preferred to s_H if:

$$(1-\rho)(B - \underline{\delta} + \overline{\delta}) \quad \geq \quad c(1+3\rho)$$

From which we can see that as $\rho \longrightarrow 1$ then $LHS \longrightarrow 0$ whereas $RHS \longrightarrow 4c$, so for sufficiently large ρ the inequality fails.

We can also see that for fixed ρ there is a c sufficiently small (but) positive so that the inequality is satisfied.

$s_{II,K}$ is preferred to $s_{I,M}$ if:

$$B(1+3\rho) \quad \geq \quad c(1-\rho) + (1+3\rho)\underline{\delta} - (1-\rho)\overline{\delta}$$

as $\rho \longrightarrow 1$ we get $B \geq \underline{\delta}$ which holds by assumption. As $\rho \longrightarrow \frac{1}{5}$ we get $B \geq \frac{c-\overline{\delta}}{2} + \underline{\delta}$. Hence if $c \geq \overline{\delta}$ we are done. Since $c_1 = \frac{\delta}{5} + \frac{3}{5}b$ and $b < \frac{\delta}{2}$ we get $\overline{\delta} > \frac{\delta}{2}$, which is satisfied by assumption. Hence $s_{II,K}$ is preferred to $s_{I,M}$ for any parametervalue.

$s_{II,K}$ is preferred to $s_{II,M}$ if:

$$-3(1-\rho)(B - \overline{\delta}) \quad \geq \quad (5-\rho)c$$

It thus follows that as $\rho \rightarrow 1$ $s_{II,M}$ must eventually be preferred. For fixed $\rho < 1$ there is a sufficiently small $c > 0$ such that $s_{II,K}$ is preffered.

s_H is preferred to $s_{II,M}$ if:

$$B \leq \frac{\underline{\delta}}{4} + \frac{\overline{\delta}}{2} - c$$

letting c and b go to 0, s_H is preferred to $s_{II,M}$ if:

$$\frac{2}{5}(\underline{\delta} + \overline{\delta}) \leq \frac{\underline{\delta}}{4} + \frac{\overline{\delta}}{2}$$

which is satisfied by assumption. However we can say nothing general.

$s_{II,K}$ is preferred to $s_{III,K}$ if:

$$3(1-\rho)(B - \frac{2}{3}\underline{\delta}) \geq (5\rho - 1)c$$

which is not satisfied for ρ sufficiently close to 1. Again for fixed ρ there is a c sufficiently small so that $s_{II,K}$ is preferred to $s_{III,K}$.

s_H is preferred to $s_{III,K}$ if:

$$B - \frac{\underline{\delta} + \overline{\delta}}{2} \geq -c$$

Since the upper bound for existence of $K_{II}S$ is $\frac{\underline{\delta}+\overline{\delta}}{3} + b$ if we can show that:

$$\frac{\underline{\delta} + \overline{\delta}}{3} + b \leq \frac{\underline{\delta} + \overline{\delta}}{2} - c$$

then we have shown that $s_{II,K}$ is always preffered. Using $c = c_1$ and $b = \frac{\underline{\delta}}{2}$ we get the condition: $\overline{\delta} \geq 5\underline{\delta}$ which does not hold in general.

$s_{II,K}$ is preferred to $s_{I,M}$ if:

$$(6\rho - 2)B - (5\rho - 1)\underline{\delta} + (1-\rho)\overline{\delta} \geq (2 - 6\rho)c$$

As $\rho \rightarrow 1$ we get $B \geq \underline{\delta} - c$ which is satisfied by assumption. As $\rho = \frac{1}{3}$ we get $\overline{\delta} \geq \underline{\delta}$ which is also satisfied. When $\rho \rightarrow \frac{1}{5}$ which in general is not a valid boundary we get $B \geq \overline{\delta} - c$ which is not satisfied in general.

$s_{III,K}$ is preferred to $s_{II,M}$ if:

$$B \geq \frac{1}{3}\underline{\delta} + \frac{1}{2}\overline{\delta} - c$$

we need to show that:

$$\frac{1}{3}\underline{\delta} + \frac{1}{2}\overline{\delta} - c \geq \frac{1}{3}(\underline{\delta} + \overline{\delta}) + b$$

this boils down to the condition $\frac{\overline{\delta}}{6} \geq c + b$, so that as $c, b \to 0$ we get the desired result.

$s_{I,K}$ is preferred to $s_{I,M}$ if:

$$B \leq \overline{\delta} - 2c$$

In the relevant range $B \leq \underline{\delta} + c$. So that for $c = c_1$ and $b = \frac{\delta}{2}$ we get $\overline{\delta} \geq \frac{3}{2}\underline{\delta}$ which is satisfied by assumption. Hence $s_{I,K}$ is preferred to $s_{I,M}$.

Finally consider the comparison with equilibria with no gifts. Note first that in the region where $s_{II,K}$ exists this equilibrium is always preferred by R to an equilibrium with no gifts, since sorting is better, and R recieves gifts she likes with positive probability. Hence the only case we need to consider is the region where $s_{I,K}$ is an equilibrium profile. First note that $s_{I,NG}$ can never be preferred since the same intensity of investment occur in both equilibria, but $s_{I,K}$ has better sorting, and R receives gifts that she likes with positive probability.

Thus we need only compare $s_{I,K}$ to $s_{II,NG}$. We get:

$$c \geq \frac{5-\rho}{2\rho}B - \frac{1+3\rho}{2\rho}\underline{\delta} - \frac{3(1-\rho)}{2\rho}\overline{\delta} - \frac{1-\rho}{2\rho}b$$

Letting $\rho \to 1$ we get $c \geq 2(B - \underline{\delta})$. When $B = \underline{B}$ this gives the condition $c \geq 2b$ which is not ruled out. Letting $B = \underline{\delta} + c$ we get a contradiction. Thus there must be some $B' < \tilde{B}(c)$ and ρ' such that if $B \geq B'$ and $\rho \geq \rho'$ then $s_{II,NG}$ is preferred.

B.5 Evolution

Proof of proposition 20:

Proof. **Step 1** For any B and $\rho \leq \rho^* < 1$ suppose $b = c \leq c^*(\rho^*)$. Also suppose that in the current profile R players send message w and play according to $s^*_{II,K}$ and that in response to any message G players play according to $s^*_{II,K}$. Now let G players strategy drift one by one such that in response to some unsent message w' t_0 and t_1 types play according to $s^*_{II,K}$, while t_2 types do not play according to $s^*_{II,K}$, but play such that inference from t_0 and t_1 types are unaffected (e.g. let them send money and invest).

Now let R players one-by-one update their strategy. In particular let them send message w' and in the underlying play according to $s^*_{II,K}$, that is in particular they do not invest after receiving money. This transition occurs with positive probability since when faced with t_0 and t_1 types R players earn identical payoffs, but when she faces t_2 types she earns a strictly positive payoff, whereas when play in the underlying game was according to $s^*_{II,K}$ she earned 0. Let this profile be denoted z'. Clearly $z' \notin Z^*$.

When G players are given the opportunity to revise their strategy t_2 types will again play according to $s^*_{II,K}$ in response to any message. Thus we can return to $z \in Z^*$ with positive probability.

Step 2 Take any $\rho^* < 1$. We show that for any B and $\rho \leq \rho^* < 1$ there is a $0 < b = c \leq c^{**}(\rho^*)$ where $c^{**}(\rho^*) \leq c^*(\rho^*)$ such that starting from a profile where $s^*_{II,K}$ is played in the underlying game R will never want to deviate to a strategy where t_0 and t_1 types do not play according to $s^*_{II,K}$. We already know that $s^*_{II,K}$ is R's preferred equilibrium outcome of the underlying game. Now consider the following non-equilibrium strategies which G players could drift to in response to some unsent message w':

- s_1 t_0 types play according to $s^*_{II,K}$ and $h = h^0$, t_1 and t_2 types send money and invests.
- s_2 t_0 types play according to $s^*_{II,K}$ and $h = h^0$, t_1 types send money and invests, and t_2 types do not send money but invests.
- s_3 t_0 types send money and invests, t_1 types play according to $s^*_{II,K}$ and $h \in \{h^+, h^-\}$, t_2 types send no money but invests.

If $c \leq c_1^{**}(\rho^*) = \frac{1-\rho^*}{13-\rho^*}\underline{\delta}$ then R players do not find it profitable to deviate when G players have drifted to s_1 in response to w'. If $c \leq c_2^{**}(\rho^*) = \frac{4}{5}\frac{1-\rho^*}{9-\rho^*}(\frac{3}{2}\overline{\delta} - \underline{\delta})$ then R players do not find it profitable to deviate when G players have drifted to s_2 in response to w'. Finally when G players have drifted to s_3 R players do not find it profitable to deviate if: $c \leq c_3^{**}(\rho^*) = \frac{4}{5}\frac{1-\rho^*}{13-\rho^*}(\frac{3}{2}\overline{\delta} - \underline{\delta})$. Note that $c_2^{**}(\rho^*) < c_3^{**}(\rho^*)$ so that $c_3^{**}(\rho^*)$ is never binding. Let $c^{**}(\rho^*) = \min(c_1^{**}(\rho^*), c_2^{**}(\rho^*))$. □

Appendix C

Appendix for Chapter 4

C.1 Proof of Proposition 24

Proposition 24. *Suppose $r = 1$. A state is stochastically stable if and only if it is segregated.*

I prove the proposition via a series of lemmas. Let Z_k, $1 \leq k \leq \frac{n}{2} \equiv K$ be the set of states with k clusters of each type, and $|Z_k|$ is the number of elements in Z_k. I first find the minimum resistance of a segregated state.

Lemma 10. *For any $1 \leq k \leq K - 1$ there is a state $z \in Z_k$ such that the minimum cost of transiting from z to some $z' \in Z_{k+1}$ is $\gamma + \beta$. For any $k \geq 2$ there is a state $z \in Z_k$ such that the minimum cost of transiting from z to some $z' \in Z_{k-1}$ is β.*

Proof. The proof is in two main steps. In the first step we show that the minimum cost of transiting from some $z \in Z_k$ to a suitable state $z' \in Z_{k+1}$, $1 \leq k \leq K - 1$ has minimum cost $\gamma + \beta$. In the second step we show that the minimum cost of transiting from some $z \in Z_k$ to a suitable state $z' \in Z_{k-1}$, $k \geq 2$ has cost β.

Step 1 Start from some $z \in Z_k$ with the property that for both types of residents there is a cluster which is not *minimal* and has at least 4 residents. Since $1 \leq k \leq K - 1$ there must be at least one cluster which is not minimal, for all $z \in Z_k$. Furthermore since n is even either there is a cluster which contains at least 4 residents or there are at least two clusters which are not minimal. So suppose that z has a cluster of size at least 4, and let them be denoted k_A and k_B respectively.

Now let one of the residents in k_A make a mistake such that she moves to k_B and inserts herself such that she has at least two B neighbours on either side. This mistake has cost γ. Now let another resident in k_A make a location mistake such that she insert herself next to the first A who moved. This mistake has cost β. We have now arrived at some $z' \in Z_{k+1}$ at cost $\gamma + \beta$.

It remains to be shown that this cost is minimal. First observe that given a state with $1 \leq k < K$ clusters all players have utility one. Thus a new cluster can only form if a player makes a mistake. Moreover the only type of mistake which leads a new cluster to form with positive probability is that a player moves from say cluster k_A to a location where she has only players of opposite type. If she moves somewhere else then cluster k_A contains one less of type A but she is added to another pre-existing cluster of type A players. Therefore the player must make a location mistake which has cost γ (since she gets utility zero). In the ensuing state all players but one (the player who made the mistake) has utility one. If a player from cluster k_A moves next to the player who previously moved both players will have utility one. But the player had utility one before, therefore a β mistake is required for a new cluster to form.

Step 2 Start from some $z \in Z_k$ where $2 \leq k \leq K$, with the property that z has a minimal cluster for at least one of the types. There is such a state $z \in Z_k$ since $k > 1$.

Let one of the residents in the minimal cluster, call it $\tilde{k}$, move to another cluster of residents of her own type (recall $k > 1$ so there is another cluster). This has cost β. Now let the remaining player in $\tilde{k}$ be drawn for location revision and suppose she draws a new location which gives her at least one neighbour like herself. This occurs with positive probability since there are $k - 1$ other clusters in z. Thus she will move at cost 0, and we have transited to a state $z' \in Z_{k-1}$. To see that this cost is minimal notice that since we started out from a state in which all players have utility one, the lower bound on the cost of transition is exactly β. □

Recall that the aim is to construct minimal cost trees for all recurrent states. Also recall that the lower bound on the minimal cost of transition is β. That is if I can prove that a transition has cost β then it is minimal. In the next lemma I show that for any $z \in Z_k$ I can construct a *path* which includes all $z' \in Z_k \setminus \{z\}$ and that this path has resistance $\beta(|Z_k| - 1)$ that is the path has minimum resistance.

Lemma 11. *For any $1 \leq k \leq K$ there is a minimum resistance path ξ, which ends in $z \in Z_k$ and such that for all $z' \in Z_k$: $z' \in \xi$, and $z'' \notin Z_k$: $z' \notin \xi$ with cost $\beta(|Z_k| - 1)$.*

Proof. We only show the proof for the segregated states. The proof is analogous for all other classes.

Suppose that in $z \in Z_1$ the A cluster begins at position l. Thus it ends at position

$l + n - 1$. We now show that the minimum cost of transiting to a state $z' \in Z_1$ such that the A cluster begins at position $l + 1$ is β.

Let the player at position l be drawn for revision and suppose she has the opportunity to move to position $l + n$. Let her move counter clockwise around the circle to this location. Since her utility is the same at both locations the cost of this move is β. The A cluster now starts at position $l + 1$.

Now we construct the path ξ. $\xi = (z^1, \ldots, z^{|Z_k|})$ where $z^i \in \xi : z^i \in Z_k$, $i = 1, \ldots, |Z_k|$ and $z = z^{|Z_k|}$. Two elements z^i and z^{i+1}, $i = 1, \ldots, |Z_k| - 1$ has the property that in z^i the A cluster begins at position l and in z^{i+1} it begins at postion $l + 1$. The cost of this path is the sum of the individual transitions. By the argument above the cost of ξ is $\beta(|Z_k| - 1)$. □

I can now prove proposition 24:

Proof of Proposition 24. It follows from lemmas 10 and 11 that the stochastic potential of a state $z \in Z_1$ is:

$$\beta(|Z_1| - 1) + \sum_{k \geq 2} \beta(|Z_k| - 1) + \sum_{k \geq 2} \beta$$

where the two first parts follow from lemma 11 and the third part follows from lemma 10.

For any state $z' \in Z_{k'}$, $k' \neq 1$ the stochastic potential is:

$$\sum_{k<k'} (\beta|Z_k| + \gamma) + \beta\,(|Z_{k'}| - 1) + \sum_{k>k'} \beta|Z_{k'}|$$

In then follows that $z \in Z_1$ has minimum stochastic potential if:

$$\sum_{k<k'} \gamma \ > \ 0$$

for all $k' \geq 2$. The claim then follows immediately since $\gamma > 0$. □

C.2 Proof of Proposition 26

Proposition 26 (Modified Schelling). *Suppose $r = 1$. A state is stochastically stable if and only if it is segregated.*

Again the proof proceeds via a number of intermediate steps.

Lemma 12. *For any $k_s^* \in \{k_s^*\}_{s=1}^{S-1}$ there is a state $z \in Z_{k_s^*}$ such that the minimum cost of transiting from z to some $z' \in Z_{k_{s+1}^*}$ is:*

$$\gamma(k_{s+1}^* - k_s^*)\frac{n}{k_{s+1}^*}$$

For any $k_s^* \in \{k_s^*\}_{s=2}^{S}$ from any state $z \in Z_{k_s^*}$ the minimum cost of transiting from z to some $z' \in Z_1$ is β.

Proof. **Step 1** Take some $z \in Z_{k_s^*}$. To transit to $z' \in Z_{k_{s+1}^*}$ $k_{s+1}^* - k_s^*$ new clusters must form. Thus at least $k_{s+1}^* - k_s^*$ γ mutations are needed. All players who belong to any of the original clusters is strict worse off at any of the seeds of a new cluster, since they are smaller in size than all of the original clusters. The size of each cluster is $\frac{n}{k_{s+1}^*}$. Thus each of the seeds must be complemented by $\frac{n}{k_{s+1}^*} - 1$ further γ mutations.

Step 2 Take any $z \in Z_{k_s^*}$, where $s > 1$. Since all clusters have the same size players are indifferent between which cluster they belong to. Now let one of the players leave their current cluster and join another cluster. This has cost β. This cluster now attracts demand from all other cluster and so no further mutations are needed to reach a segregated state. □

The second step is a corollary to lemma 11 and is stated without proof.

Corollary 7. *For any $k^* \in \{k_s^*\}$ starting from $z \in Z_{k^*}$ the minimum resistance path ξ such that for all $z' \in Z_{k^*} \setminus \{z\} : z' \in \xi$ is $\beta(|Z_{k^*}| - 1)$.*

I can now conclude the proof.

Proof of proposition 26. The idea of the proof is first to argue that the stochastic potential of a segregated state provides a lower bound. Then I argue that any state which is not segregated does not attain this lower bound.

Step 1 By lemma 12 for any k_s^*, $s > 1$, the cheapest way to go from a state in $Z_{k_s^*}$ to a a state in $Z_{k_1^*}$ is via a single β mutation. Thus the sum of cost is $(S-1)\beta$. Moreover by lemma 7 for any k_s^* the minimum resistance path has cost $\beta(|Z_{k_s^*}| - 1)$. Thus the stochastic potential of a segregated state is:

$$\beta(|Z_{k_1^*}| - 1) + \sum_{s=2}^{S} \beta|Z_{k_s^*}|$$

Step 2 The proof is then completed by observing that leaving the set of segregated states has minimum cost at least:

$$\gamma(k_2^* - k_1^*)\frac{n}{k_2^*}$$

This completes the proof since $\gamma > \beta$. □

C.3 Proof of Proposition 28

Proposition 28. *A state is stochastically stable if and only if it is segregated.*

In order to prove the proposition let $|Z_k|$ be the number of states with k clusters, $1 \leq k \leq K$, and recall that any two states $z, z' \in Z_k$ $z \neq z'$ and where $k < K$ z and z' are in the same recurrent class.

Lemma 13. *For $1 < k \leq K$ the minimum cost of transiting from $z \in Z_k$ to $z' \in Z_{k-1}$ is β.*

Proof. Take some state $z \in Z_k$, $1 < k \leq K$. z either contains a minimal cluster or via the unperturbed dynamics with positive probability the process can transit to a state with k clusters and at least one minimal cluster. Hence assume that z contains a minimal cluster. Let one of the residents in the minimal cluster be drawn for revision and let the location she can move to be on the edge of another cluster containing residents of her own type. Since the resident lives in a minimal cluster she currently has utility 1, and if she moves to her new location she will also get utility 1. Thus a mistake at cost β is required. Now let the remaining resident in the minimal cluster in z be drawn for revision. Since she lived in a minimal cluster she now only has neighbours different from herself. Therefore by the unperturbed dynamics the process can transit to a state z' which contains only $k - 1$ clusters. □

Lemma 14. *For $1 \leq k < K$ the minimum cost of transiting from $z \in Z_k$ to $z' \in Z_{k+1}$ is δ.*

Proof. Take some state $z \in Z_k$, $1 \leq k < K$. Since $k < K$ there is at least two players of type t who only has neighbours like themselves. Let one of these player be i and assume that she currently lives on location l. i has utility x. Moreover either z contains a cluster of type t', $t' \neq t$ players of at least length 4 or via the unperturbed dynamics the process can transit to a state with a cluster of at least length 4 of type t' players. Hence assume that z contains such a cluster and that it begins at location $l' > l$. Now let i be drawn for revision and suppose that she has the opportunity to move clockwise around the circle to $l' + 2$. By construction $l' + 2$ gives i a neighbourhood which only contains neighbours different from herself. Thus a mistake at cost δ is required for her to move. Since z had $k < K$ clusters there is another player j of type t who only had neighbours like herself z. After i has moved this player still only has neighbours like herself. Thus via the unperturbed dynamics, with positive probability this player will be drawn for revision, and suppose she has the opportunity to move next to i. At this location she will have a diverse neighbourhood, thus at no further cost she will move next to i. The cluster containing i and j is minimal thus both players enjoy their highest utility. Moreover going counter clockwise next to this cluster there is now a minimal cluster of type t' players, and going clockwise there is a cluster at least of length 2. Thus we have transited to a state $z' \in Z_{k+1}$.

It remains to be shown that this cost is minimal. That is it we need to show that any sequence of β and γ mistakes is not sufficient to transit to some $z' \in Z_{k+1}$.

First we rule out that a sequence of β mistakes. Assume that $s \geq 1$ β mistakes are sufficient for the process to transfer to some $z' \in Z_{k+1}$. Any single β cannot lead a player to insert herself in a neighbourhood where she gets utility 0 unless she had utility 0 before. Players who made a β mistake and had utility 1 and x respectively must have moved to a pre-existing cluster. If a player has utility 0 it must be because all other players in the cluster she belonged to in z have moved. All of the players must have moved to a cluster that existed in z. But then no new clusters can form.

Suppose $s \geq 1$ β mistakes is the minimal cost to transfer to a state $z' \in Z_{k+1}$. Thus after $s - 1$ mistakes a single β mistake is required. Let the state which is arrived at after $s - 1$ mistakes be denoted z^{s-1}. Players who made a β mistake must have had either utility 1 or utility x. That is the player who moves by mistake must move from a cluster which existed in z_{s-p} next to a player who formed part of a cluster in z. That is up until $s - 1$ no new clusters have been formed. Since precisely s β mistakes are required, by the hypothesis there is a β mistake which will lead to the formation of a new cluster. In z^{s-1} all players have either utility 1, x or 0. Clearly a β mistake by players with utility 1 or x does not lead a new cluster to form since they will move to a position where there is at least one player like themselves. A player with utility 0 who makes a β mistake will lead to the dissolution of previously existing cluster, and will with positive probability lead to the formation of a new cluster. However the total number of clusters cannot increase.

Now we rule out that a sequence of γ mistakes lead to some $z' \in Z_{k+1}$. After $s - 1$ γ mistakes, exactly one γ mistake is required by the hypothesis. As in the argument above, γ mistakes cannot lead to the formation of new clusters only the dissolution of old clusters with positive probability. To see this note that a γ mistake leads players to vacate in integrated location for a location where they only have neighbours like themselves. Since this move must lead players to abandon a pre-existing cluster for another pre-existing cluster, no new clusters are formed with pos. prob. after $s - 1$ mistakes. But after $s - 1$ mistakes a player who makes a γ mistake moves to a cluster which existed in the previous state.

Finally we have to rule out that any sequence of γ and β mistakes will lead to some $z' \in Z_{k+1}$. By the argument above after any β or γ mistake no new cluster is formed without the dissolution of an old cluster. □

I can now complete the proof of proposition 28:

Proof of proposition 28. By lemma 13 the stochastic potential of $z \in Z_1$ is:

$$\beta|Z_K| + \beta(K - 2)$$

The stochastic potential of $z \in Z_k$, $1 < k < K$ is:

$$\beta|Z_K| + \beta(K - (k + 1)) + \delta(k - 1)$$

and the stochastic potential of $z \in Z_K$ is at least:

$$\beta(|Z_K| - 1) + \delta(K - 1)$$

since players in z enjoy the highest possible utility.

Since $\delta > \beta$ only the states in Z_1 has minimum stochastic potential. □