Monotone Methods for Equilibrium Selection under Perfect Foresight Dynamics*

DAISUKE OYAMA

Department of Economics, University of Vienna
Hohenstaufengasse 9, A-1010 Vienna, Austria
Daisuke.Oyama@univie.ac.at

SATORU TAKAHASHI

Department of Economics, Harvard University
Littauer Center, Cambridge, MA 02138, USA
stakahas@fas.harvard.edu

AND

JOSEF HOFBAUER

Department of Mathematics, University of Vienna
Strudlhofgasse 4, A-1090 Vienna, Austria
Josef.Hofbauer@univie.ac.at

May 2, 2003; revised September 1, 2003

*We are grateful to Drew Fudenberg, Michihiro Kandori, Akihiko Matsui, Bill Sandholm, and Takashi Ui for helpful comments and discussions. Revised versions will be available at http://mailbox.univie.ac.at/Daisuke.Oyama/papers/pfd_supmod.html.

Abstract

This paper studies equilibrium selection in supermodular games based on perfect foresight dynamics. A normal form game is played repeatedly in a large society of rational agents. There are frictions: opportunities to revise actions follow independent Poisson processes. Each agent forms his belief about the future evolution of action distribution in the society to take an action that maximizes his expected discounted payoff. A *perfect foresight path* is defined to be a feasible path of the action distribution along which every agent with a revision opportunity takes a best response to this path itself. A Nash equilibrium is said to be *absorbing* if there exists no perfect foresight path escaping from a neighborhood of this equilibrium; a Nash equilibrium is said to be *globally accessible* if for each initial distribution, there exists a perfect foresight path converging to this equilibrium. By exploiting the monotone structure of the dynamics, a unique Nash equilibrium that is absorbing and globally accessible for any small degree of friction is identified for certain classes of supermodular games. For games with monotone potentials, the selection of the monotone potential maximizer is obtained. Complete characterizations of absorbing equilibrium and globally accessible equilibrium are given for binary supermodular games. An example demonstrates that unanimity games may have multiple globally accessible equilibria for a small friction. *Journal of Economic Literature* Classification Numbers: C72, C73.

KEYWORDS: equilibrium selection; perfect foresight dynamics; supermodular game; strategic complementarity; stochastic dominance; potential; monotone potential.

Contents

1 Introduction

Supermodular games capture the key concept of strategic complementarity in various economic phenomena. Examples include oligopolistic competition, adoption of new technologies, bank runs, currency crises, and economic development. Strategic complementarity plays an important role in particular in Keynesian macroeconomics (Cooper (1999)). From a theoretical viewpoint, those games have appealing properties due to their monotone structure (Topkis (1979), Milgrom and Roberts (1990), Vives (1990), and Athey (2001)).

A salient feature of supermodular games is that there typically exist multiple Nash equilibria due to strategic complementarities, which raises the question as to which equilibrium is likely to be played. To address the problem of equilibrium selection, game theory has so far proposed two major strands of approaches besides the classic one by Harsanyi and Selten (1988). One is to consider the stability of Nash equilibria in the context of evolutionary dynamics (Kandori, Mailath, and Rob (1993), Young (1993), Kandori and Rob (1995) for stochastic models; Hofbauer (1999) for a deterministic model); the other is to embed the original game in a static incomplete information game and examine the robustness of equilibrium outcomes to a small amount of uncertainty (Carlsson and van Damme (1993), Frankel, Morris, and Pauzner (2003); Kajii and Morris (1997), Morris and Ui (2002)).

In the present paper, we study equilibrium selection in supermodular games based on perfect foresight dynamics, first introduced by Matsui and Matsuyama (1995) for 2×2 games.[1] Our approach is dynamic like that of evolutionary models, but it differs in one crucial aspect. While agents in evolutionary models are myopic and boundedly rational, our model has forward-looking, rational agents. Combined with a dynamic environment with frictions, this allows the possibility for self-fulfilling beliefs to upset strict Nash equilibria, yielding a strong equilibrium selection property. Indeed, Matsui and Matsuyama (1995) demonstrated that in 2×2 coordination games, a self-fulfilling belief enables the society to escape from the risk-dominated equilibrium to the risk-dominant equilibrium but not vice versa, provided that the friction is sufficiently small. The purpose of this paper is to characterize the behavior of the perfect foresight dynamics in the class of supermodular games and to derive equilibrium selection criteria for those games, thereby providing a link between ours and other approaches. In particular, we show that for games with monotone potentials (Morris and Ui (2002)), our selection coincides with that from the incomplete information approach due to Kajii and Morris (1997). On the other hand, there are also disagreements, e.g., in unanimity games with more than two players, the

[1] For studies in economic contexts, see, e.g., Matsuyama (1991, 1992) and Kaneda (1995).

selection criterion based on the Nash product (Harsanyi and Selten (1988)) is not supported.

We consider the following framework. The society consists of N large populations of infinitesimal agents, who are repeatedly and randomly matched to play an N-player normal form game. There are frictions: Each agent must make a commitment to a particular action for a random time interval. Opportunities to revise actions follow Poisson processes which are independent across agents. The dynamics thus exhibits inertia in that the action distribution in the society changes continuously. Unlike in standard evolutionary games, each agent forms his belief about the future path of the action distribution and, when given a revision opportunity, takes an action to maximize his expected discounted payoff. A *perfect foresight path* is defined to be a feasible path of action distribution along which each agent takes a best response against this path itself. While the stationary states of this dynamics correspond to the Nash equilibria of the stage game, there may exist a perfect foresight path that escapes from a strict Nash equilibrium when the degree of friction, defined as the discounted average duration of the commitment, is sufficiently small. We say that a Nash equilibrium state x^* is *absorbing* if for every initial state close enough to x^*, any perfect foresight path must converge to x^*; x^* is *globally accessible* if for any initial state, there exists a perfect foresight path converging to x^*.[2] Our equilibrium selection criterion requires a Nash equilibrium to be uniquely absorbing and globally accessible for any small degree of friction.

Several selection results based on the perfect foresight dynamics have been obtained so far. Matsui and Matsuyama (1995) demonstrate that in 2×2 coordination games, a strict Nash equilibrium is absorbing and globally accessible for any small degree of friction if and only if it is the risk-dominant equilibrium. Beyond 2×2 games, Oyama (2002) appeals to the notion of p-dominance to identify (in a single population setting) a class of games where one can explicitly characterize the set of the perfect foresight paths relevant for stability considerations, showing that a p-dominant equilibrium with $p < 1/2$ is selected. Hofbauer and Sorger (2002) and Kojima (2003) obtain related results based on other risk-dominance concepts in a multiple population setting.[3] Hofbauer and Sorger (1999, 2002) establish the selection of the (unique) potential maximizer for potential games, both in a single population setting (Hofbauer and Sorger (1999)) and in a multi-population setting (Hofbauer and Sorger (2002)). Their results rely on a relationship between the perfect foresight paths and the optimal solutions to an associated optimal control problem, and the Hamiltonian structure

[2] For a given initial state, there may exist multiple perfect foresight paths. Therefore, it is possible that a state is globally accessible but not absorbing. Indeed, we provide an example where there exist multiple globally accessible states when the friction is small; by definition, none of them are absorbing.

[3] Tercieux (2003) considers set-valued stability concepts and obtains a similar result.

that the dynamics has when the stage game is a potential game.

In this paper, we develop methods of analysis based on monotonicity and comparison. An underlying observation is that a perfect foresight path is characterized as a fixed point of the best response correspondence on the set of feasible paths. We show that if the stage game is supermodular, this correspondence is monotone with respect to the partial order over feasible paths induced by the stochastic dominance order on the space of mixed strategies. We then compare perfect foresight paths between different stage games that have a monotone relation in terms of best response and show that an analogue to the comparison theorem from the theory of monotone dynamical systems (Smith (1995)) holds for the perfect foresight dynamics.[4] More precisely, it is shown that if either of the two games is supermodular, the order of best responses between the games is preserved in the perfect foresight dynamics. Due to this fact, we can exploit the stability properties of one game to study those of the other.

We proceed to apply our monotone methods to identify a unique Nash equilibrium that is absorbing and globally accessible for a small friction for some classes of games with monotone properties. First, we study the class of games with *monotone potentials* introduced by Morris and Ui (2002), who show that a monotone potential maximizer (MP-maximizer) is robust to incomplete information (Kajii and Morris (1997)).[5] A normal form game has a monotone potential if it is in a monotone relation (in terms of best response) to a potential game. By invoking the potential game result due to Hofbauer and Sorger (2002), we show that if either the stage game or the monotone potential is supermodular, then an MP-maximizer is globally accessible for any small degree of friction, and a strict MP-maximizer is absorbing for any degree of friction. As a corollary, this implies that a (strict) **p**-dominant equilibrium with $\sum_i p_i < 1$ is selected under the perfect foresight dynamics.

We then analyze the class of binary supermodular games, for which we are able to obtain complete characterizations for absorbing states and for globally accessible states. These characterizations are applied to three classes of binary supermodular games. First, for unanimity games, we show that our selection criterion may not be in agreement with that in terms of Nash product.[6] In fact, the perfect foresight dynamics fails to select a single Nash equilibrium for some unanimity games. Example 5.2.1 in

[4]Hofbauer and Sandholm (2002) show that when the underlying game is supermodular, the perturbed best response dynamics forms a monotone dynamical system. The perfect foresight dynamics, on the other hand, cannot be considered as a dynamical system.

[5]More generally, Morris and Ui (2002) show that a generalized potential maximizer is robust. A monotone potential induces a generalized potential in the case considered here. Frankel, Morris, and Pauzner (2003) show that with an additional condition, an MP-maximizer is selected in global games (Carlsson and van Damme (1993)).

[6]Hofbauer (1999) shows that in unanimity games, the Nash equilibrium with the higher Nash product is selected in his spatio-temporal model.

Subsection 5.2 demonstrates that the two strict Nash equilibria are mutually accessible, actually globally accessible, for a small friction. Second, for games with linear incentives (Selten (1995)), we find a connection to the concept of spatial dominance due to Hofbauer (1999). It is shown that if a strict Nash equilibrium is globally accessible under the perfect foresight dynamics with a small friction, then it is spatially dominant. This implies in particular that for (generic) games with linear incentives, a globally accessible equilibrium is unique if it exists. Third, we introduce the class of games with invariant diagonal, in which all players receive the same payoffs when they all play the same mixed strategies. For this class of games, we obtain the generic uniqueness of absorbing and globally accessible equilibrium for a small friction.

The concept of perfect foresight path requires that agents optimize against their beliefs about the future path of the action distribution and that those beliefs coincide with the actual path. Relaxing the latter requirement, Matsui and Oyama (2002) introduce the model of rationalizable foresight dynamics, where while the rationality of the agents as well as the structure of the society is common knowledge, beliefs about the future path are not necessarily coordinated among the agents. It is instead assumed that the agents form their beliefs in a rationalizable manner: in particular, they may misforecast the future. We show that in supermodular games, an absorbing and globally accessible state under the perfect foresight dynamics is also selected under the weaker assumption of rationalizable foresight.

The paper is organized as follows. Section 2 introduces the perfect foresight dynamics for general finite N-player games and provides a characterization of perfect foresight paths as the fixed points of the best response correspondence on the set of feasible paths. Section 3 studies monotone properties of the perfect foresight dynamics and proves our comparison theorem. It also compares the stability under perfect foresight and that under rationalizable foresight. Section 4 considers games with monotone potentials and establishes the selection of MP-maximizer. Section 5 gives a complete characterization of absorption and global accessibility for the class of binary supermodular games. Detailed analyses are provided for unanimity games, games with linear incentives, and games with invariant diagonal. Section 6 concludes.

2 Perfect Foresight Dynamics

2.1 Stage Game

Let $G = (I, (A_i)_{i\in I}, (u_i)_{i\in I})$ be a normal form game with $N \geq 2$ players, where $I = \{1, 2, \dots, N\}$ is the set of players, $A_i = \{0, 1, \dots, n_i\}$ the finite set of actions for player $i \in I$, and $u_i \colon \prod_{i\in I} A_i \to \mathbb{R}$ the payoff function for player i. We denote $\prod_{i\in I} A_i$ by A and $\prod_{j\neq i} A_j$ by A_{-i}.

Denote by $\mathbb{R}_+$ the set of all nonnegative real numbers and by $\mathbb{R}_{++}$ the set of all positive real numbers. The set of mixed strategies for player i is denoted by

$$\Delta(A_i) = \Big\{ x_i = (x_{i0}, x_{i1}, \dots, x_{in_i}) \in \mathbb{R}_+^{n_i+1} \,\Big|\, \sum_{h \in A_i} x_{ih} = 1 \Big\},$$

which is identified with the n_i-dimensional simplex. We sometimes identify each action in A_i with the element of $\Delta(A_i)$ that assigns one to the corresponding coordinate. The polyhedron $\prod_{i \in I} \Delta(A_i)$ is a subset of the n-dimensional real space endowed with the sup norm $|\cdot|$, where $n = \sum_{i \in I}(n_i + 1)$. For $x \in \prod_i \Delta(A_i)$ and $\varepsilon > 0$, $B_\varepsilon(x)$ denotes the ε-neighborhood of x relative to $\prod_i \Delta(A_i)$, i.e., $B_\varepsilon(x) = \{y \in \prod_i \Delta(A_i) \,|\, |y - x| < \varepsilon\}$.

Payoff functions $u_i(h, \cdot)$ are extended to $\prod_{j \neq i} \Delta(A_j)$, and $u_i(\cdot)$ to $\prod_{j \in I} \Delta(A_j)$, i.e., $u_i(h, x_{-i}) = \sum_{a_{-i} \in A_{-i}} \prod_{j \neq i} x_{ja_j} u_i(h, a_{-i})$ for $x_{-i} \in \prod_{j \neq i} \Delta(A_j)$, and $u_i(x) = \sum_{h \in A_i} x_{ih} u_i(h, x_{-i})$ for $x \in \prod_{j \in I} \Delta(A_j)$. Let $BR^i_{u_i}(x_{-i})$ be the set of best responses to $x_{-i} \in \prod_{j \neq i} \Delta(A_j)$ in pure strategies, i.e.,

$$\begin{aligned} BR^i_{u_i}(x_{-i}) &= \arg\max_{h \in A_i} u_i(h, x_{-i}) \\ &= \{h \in A_i \,|\, u_i(h, x_{-i}) \geq u_i(k, x_{-i}) \text{ for all } k \in A_i\}. \end{aligned}$$

We say that $x^* \in \prod_i \Delta(A_i)$ is a Nash equilibrium if for all $i \in I$ and all $h \in A_i$,

$$x^*_{ih} > 0 \Rightarrow h \in BR^i_{u_i}(x^*_{-i}),$$

and x^* is a strict Nash equilibrium if for all $i \in I$ and all $h \in A_i$,

$$x^*_{ih} > 0 \Rightarrow \{h\} = BR^i_{u_i}(x^*_{-i}).$$

Let $\Delta(A_{-i})$ be the set of probability distributions on A_{-i}. We sometimes extend $u_i(h, \cdot)$ to $\Delta(A_{-i})$. For $\pi_i \in \Delta(A_{-i})$, we write $u_i(h, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) u_i(h, a_{-i})$ and $BR^i_{u_i}(\pi_i) = \arg\max_{h \in A_i} u_i(h, \pi_i)$.

2.2 Perfect Foresight Paths

Given an N-player normal form game, which will be called the stage game, we consider the following dynamic societal game. Society consists of N large populations of infinitesimal agents, one for each role in the stage game. In each population, agents are identical and anonymous. At each point in time, one agent is selected randomly from each population and matched to form an N-tuple and play the stage game. Agents cannot switch actions at every point in time. Instead, every agent must make a commitment to a particular action for a random time interval. Time instants at which each agent can switch actions follow a Poisson process with the arrival rate

$\lambda > 0$. The processes are independent across agents. We choose without loss of generality the unit of time in such a way that $\lambda = 1$.[7]

The action distribution in population $i \in I$ at time $t \in \mathbb{R}_+$ is denoted by

$$\phi_i(t) = (\phi_{i0}(t), \phi_{i1}(t), \dots, \phi_{in_i}(t)) \in \Delta(A_i),$$

where $\phi_{ih}(t)$ is the fraction of agents who are committing to action $h \in A_i$ at time t. Let $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_N(t)) \in \prod_i \Delta(A_i)$. Due to the assumption that the switching times follow independent Poisson processes with arrival rate $\lambda = 1$, $\phi_{ih}(\cdot)$ is Lipschitz continuous with Lipschitz constant 1, which implies in particular that it is differentiable at almost all $t \in \mathbb{R}_+$. Moreover, its speed of adjustment is bounded: $\dot{\phi}_{ih}(t) \geq -\phi_{ih}(t)$, where $\sum_{h \in A_i} \dot{\phi}_{ih}(t) = 0$. We call such a path $\phi(\cdot)$ a feasible path.

Definition 2.1. A path $\phi \colon \mathbb{R}_+ \to \prod_i \Delta(A_i)$ is said to be *feasible* if it is Lipschitz continuous, and for all $i \in I$ and almost all $t \in \mathbb{R}_+$, there exists $\alpha_i(t) \in \Delta(A_i)$ such that

$$\dot{\phi}_i(t) = \alpha_i(t) - \phi_i(t). \tag{2.1}$$

In equation (2.1), $\alpha_i(t) \in \Delta(A_i)$ denotes the action distribution of the agents in population i who have a revision opportunity during the short time interval $[t, t + dt)$.

Denote by Φ^i the set of feasible paths for population i, and let $\Phi = \prod_i \Phi^i$ and $\Phi^{-i} = \prod_{j \neq i} \Phi^j$. For $x \in \prod_i \Delta(A_i)$, the set of feasible paths starting from x is denoted by $\Phi_x = \prod_i \Phi^i_x$. For each $x \in \prod_i \Delta(A_i)$, Φ_x is convex and compact in the topology of uniform convergence on compact intervals.[8]

An agent in population i anticipates the future evolution of action distribution, and, if given the opportunity to switch actions, commits to an action that maximizes his expected discounted payoff. Since the duration of the commitment has an exponential distribution with mean 1, the expected discounted payoff of committing to action $h \in A_i$ at time t with a given anticipated path $\phi \in \Phi$ is represented by

$$\begin{aligned} V^\theta_{ih}(\phi)(t) &= (1+\theta) \int_0^\infty \int_t^{t+s} e^{-\theta(z-t)} u_i(h, \phi_{-i}(z))\, dz\, e^{-s} ds \\ &= (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} u_i(h, \phi_{-i}(s))\, ds, \end{aligned}$$

where $\theta > 0$ is a common rate of time preference. We view θ as the *degree of friction*. Note that V^θ is well-defined for $\theta > -1$. In particular, V^0 is well-defined.

[7] We can alternatively assume as follows. Each agent exits from his population according to the Poisson process with parameter λ and is replaced by his successor. Agents make once-and-for-all decisions upon entry, i.e., one cannot change his action once it is chosen.

[8] One can instead use the topology induced by the discounted sup norm.

Given a feasible path $\phi \in \Phi$, let $BR^i(\phi)(t)$ be the set of best responses in pure actions to $\phi_{-i} = (\phi_j)_{j \neq i}$ at time t, i.e.,

$$BR^i(\phi)(t) = \arg\max_{h \in A_i} V^\theta_{ih}(\phi)(t).$$

Note that for each $i \in I$, the correspondence $BR^i \colon \Phi \times \mathbb{R}_+ \to A_i$ is upper semi-continuous since V^θ_i is continuous.

A perfect foresight path is a feasible path along which each agent optimizes against that path itself.

Definition 2.2. A feasible path ϕ is said to be a *perfect foresight path* if for all $i \in I$, all $h \in A_i$, and almost all $t \in \mathbb{R}_+$,

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \in BR^i(\phi)(t). \tag{2.2}$$

Note that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$ (i.e., $\alpha_{ih}(t) > 0$ in (2.1)) implies that action h is taken by some positive fraction of the agents in population i having a revision opportunity during the short time interval $[t, t+dt)$. The definition says that such an action must be a best response to the path ϕ itself.

2.3 Best Response Correspondence

For a given initial state $x \in \prod_i \Delta(A_i)$, a best response path for population i to a feasible path $\phi \in \Phi_x$ is a feasible path $\psi_i \in \Phi^i_x$ along which every agent takes an optimal action against ϕ. This defines the *best response correspondence* $\beta^i_x \colon \Phi_x \to \Phi^i_x$ which maps each feasible path $\phi \in \Phi_x$ to the set of best response paths for population i:

$$\beta^i_x(\phi) = \{\psi_i \in \Phi^i_x \mid \dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \in BR^i(\phi)(t) \ \text{ a.e.}\}. \tag{2.3}$$

Let $\beta_x \colon \Phi_x \to \Phi_x$ be defined by $\beta_x(\phi) = \prod_i \beta^i_x(\phi)$. We denote by $\beta \colon \Phi \to \Phi$ the extension of β_x to Φ, i.e., $\beta(\phi) = \beta_{\phi(0)}(\phi)$ for $\phi \in \Phi$.

A perfect foresight path ϕ with $\phi(0) = x$ is a fixed point of $\beta_x \colon \Phi_x \to \Phi_x$, i.e., $\phi \in \beta_x(\phi)$. The existence of perfect foresight paths follows, due to Kakutani's fixed point theorem, from the fact that β_x is a nonempty-, convex-, and compact-valued upper semi-continuous correspondence. This fact can be shown by either of the two characterizations given below.

Remark 2.1. For a given feasible path $\phi \in \Phi_x$, a best response path $\psi \in \beta_x(\phi)$ is a Lipschitz solution to the differential inclusion

$$\dot{\psi}(t) \in F(\phi)(t) - \psi(t) \ \text{ a.e.}, \qquad \psi(0) = x, \tag{2.4}$$

where $F \colon \Phi \times \mathbb{R}_+ \to \prod_i \Delta(A_i)$ is defined by

$$F_i(\phi)(t) = \{\alpha_i \in \Delta(A_i) \mid \alpha_{ih} > 0 \Rightarrow h \in BR^i(\phi)(t)\}, \tag{2.5}$$

which is the convex hull of $BR^i(\phi)(t)$. Since the right hand side of (2.4) is convex- and compact-valued, and upper semi-continuous in t, the existence theorem for differential inclusion (see, e.g., Aubin and Cellina (1984, Theorem 2.1.4)) implies the nonemptiness of the set of solutions, $\beta_x(\phi)$. The convexity of $\beta_x(\phi)$ is obvious. Furthermore, by an argument analogous to that in Matsui and Oyama (2002, Lemma A.2), we can show that $\beta_x(\phi)$ is compact and depends upper semi-continuously on ϕ. For these properties of β_x, we only need the upper semi-continuity of BR^i, which is in turn implied by the continuity of V_i^θ.

Lemma 2.1. *β_x is compact valued and upper semi-continuous.*

Proof. Since the values are contained in the compact set Φ_x, it is sufficient to show that β_x has a closed graph. Let $\{\phi^k\}_{k=1}^\infty$ and $\{\psi^k\}_{k=1}^\infty$ be such that $\psi^k \in \beta_x(\phi^k)$, and assume that $\phi^k \to \phi$ and $\psi^k \to \psi$ as $k \to \infty$. Take any $i \in I$, $h \in A_i$, and $t \in \mathbb{R}_+$ such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$. We want to show that $h \in BR^i(\phi)(t)$.

Observe that for any $\varepsilon > 0$, there exists $\bar{k}$ such that for all $k \geq \bar{k}$,

$$\dot{\psi}_{ih}^k(t^k) > -\psi_{ih}^k(t^k)$$

for some $t^k \in (t-\varepsilon, t+\varepsilon)$. Take a sequence $\{\varepsilon^\ell\}_{\ell=1}^\infty$ such that $\varepsilon^\ell > 0$ and $\varepsilon^\ell \to 0$ as $\ell \to \infty$. Then, we can take a subsequence $\{\psi^{k_\ell}\}_{\ell=1}^\infty$ of $\{\psi^k\}_{k=1}^\infty$ such that $\dot{\psi}_{ih}^{k_\ell}(t^\ell) > -\psi_{ih}^{k_\ell}(t^\ell)$ holds for some $t^\ell \in (t-\varepsilon\ell, t+\varepsilon\ell)$. By assumption, $h \in BR^i(\phi^{k_\ell})(t^\ell)$ for all ℓ. Now let $\ell \to \infty$. Since $BR^i(\cdot)(\cdot)$ is upper semi-continuous, we have $h \in BR^i(\phi)(t)$. ∎

Remark 2.2. The correspondence β_x^i is actually the best response correspondence for an associated differential game, as constructed in Hofbauer and Sorger (2002). With the stage game G, the discount rate $\theta > 0$, and an initial state $x \in \prod_i \Delta(A_i)$ given, the associated differential game is an N-player normal form game in which the set of actions for player $i \in I$ is Φ_x^i and the payoff function for player i is given by

$$J_i(\phi) = \int_0^\infty e^{-\theta t} u_i(\phi(t))\, dt. \tag{2.6}$$

As shown by Hofbauer and Sorger (2002), the perfect foresight paths are precisely the Nash equilibria of this game, due to the following fact.

Lemma 2.2. *For a feasible path $\phi \in \Phi_x$,*

$$\beta_x^i(\phi) = \arg\max_{\psi_i \in \Phi_x^i} J_i(\psi_i, \phi_{-i}).$$

Proof. Follows from Lemma 3.1 in Hofbauer and Sorger (2002). ∎

The continuity of J_i, the quasi-concavity of $J_i(\cdot, \phi_{-i})$, and the compactness of Φ_x^i therefore imply the desired properties of β_x^i.

2.4 Stability Concepts

The path $\bar{\phi}(\cdot)$ such that $\bar{\phi}(t) = x^* \in \prod_i \Delta(A_i)$ for all $t \geq 0$ is a perfect foresight path if and only if x^* is a Nash equilibrium of the stage game. Nevertheless, when the degree of friction $\theta > 0$ is sufficiently small, there may exist another perfect foresight path which converges to another Nash equilibrium, that is to say, self-fulfilling beliefs may enable the society to escape, even from a strict Nash equilibrium. A state x^* is absorbing if no perfect foresight path escapes from a neighborhood of x^*; x^* is the unique absorbing state if it has an additional stability property that we call global accessibility, i.e., for any initial state, there exists a perfect foresight path that converges to x^*.

Definition 2.3. (a) $x^* \in \prod_i \Delta(A_i)$ is *accessible* from $x \in \prod_i \Delta(A_i)$ if there exists a perfect foresight path from x that converges to x^*. $x^* \in \prod_i \Delta(A_i)$ is *globally accessible* if it is accessible from any $x \in \prod_i \Delta(A_i)$.

(b) $x^* \in \prod_i \Delta(A_i)$ is *absorbing* if there exists $\varepsilon > 0$ such that any perfect foresight path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

A globally accessible state is not necessarily absorbing, as there are generally multiple perfect foresight paths from a given initial state. In fact, a (nondegenerate) example in Subsection 5.2 (Example 5.2.1) shows that there may exist multiple globally accessible states when the degree of friction is small; by definition, none of them are absorbing.

Any absorbing or globally accessible state is a Nash equilibrium of the stage game, which follows from the proposition below.

Proposition 2.3. *If $x^* \in \prod_i \Delta(A_i)$ is the limit of a perfect foresight path, then x^* is a Nash equilibrium.*

Proof. Suppose that x^* is the limit of a perfect foresight path ϕ^*. Let $\bar{\phi}$ be the constant path at x^*, i.e., $\bar{\phi}(t) = x^*$ for all $t \geq 0$. Let ϕ^t be the feasible path defined by $\phi^t(s) = \phi^*(s + t)$ for all $s \geq 0$. Then, $\{\phi^t\}_{t \geq 0}$ converges to $\bar{\phi}$ as $t \to \infty$.

Take any $i \in I$ and any $h \in A_i$ with $x^*_{ih} > 0$. For any $T \geq 0$, there exists $t \geq T$ such that $h \in BR^i(\phi^*)(t) = BR^i(\phi^t)(0)$, since ϕ^* is a perfect foresight path that converges to x^*. By the upper semi-continuity of $BR^i(\cdot)(0)$, we have $h \in BR^i(\bar{\phi})(0)$, which implies that h is a best response of player i to x^*_{-i} in the stage game. ▮

3 Supermodularity and Monotonicity

Supermodular games are games in which actions are ordered so that each player's marginal payoff to any increase in his action is nondecreasing in other players' actions. In this section, we first identify monotone properties of the perfect foresight dynamics for supermodular stage games, including the monotonicity of the best response correspondence β with respect to a partial order on Φ, which is induced by the stochastic dominance order over mixed strategies. We then prove a comparison theorem for perfect foresight paths under two different stage games that have a monotonicity relation in terms of best responses. This theorem implies that if either of the two game is supermodular, then one game inherits stability properties from the other. Finally, we show that for supermodular games, stability under perfect foresight is equivalent to that under rationalizable foresight (Matsui and Oyama (2002)).

3.1 Supermodular Games

For $x_i, y_i \in \Delta(A_i)$, we write $x_i \precsim y_i$ if y_i stochastically dominates x_i, i.e.,

$$\sum_{k=h}^{n_i} x_{ik} \leq \sum_{k=h}^{n_i} y_{ik}$$

for all $h \in A_i$. For $x, y \in \prod_i \Delta(A_i)$, we write $x \precsim y$ if $x_i \precsim y_i$ for all $i \in I$ and $x_{-i} \precsim y_{-i}$ if $x_j \precsim y_j$ for all $j \neq i$. Moreover, we define $\phi_i \precsim \psi_i$ for $\phi_i, \psi_i \in \Phi^i$ by $\phi_i(t) \precsim \psi_i(t)$ for all $t \geq 0$; $\phi \precsim \psi$ for $\phi, \psi \in \Phi$ by $\phi_i \precsim \psi_i$ for all $i \in I$; and $\phi_{-i} \precsim \psi_{-i}$ for $\phi_{-i}, \psi_{-i} \in \Phi^{-i}$ by $\phi_j \precsim \psi_j$ for all $j \neq i$. Note that if $\phi(0) \precsim \psi(0)$ and $\dot{\phi}(t) + \phi(t) \precsim \dot{\psi}(t) + \psi(t)$ for all $t \in \mathbb{R}_+$, then $\phi \precsim \psi$.

The game G is said to be *supermodular* if whenever $h < k$, the difference $u_i(k, a_{-i}) - u_i(h, a_{-i})$ is nondecreasing in $a_{-i} \in A_{-i}$, i.e., if $a_{-i} \leq b_{-i}$, then

$$u_i(k, a_{-i}) - u_i(h, a_{-i}) \leq u_i(k, b_{-i}) - u_i(h, b_{-i}).$$

A well-known key property of supermodular games is that if $h < k$ and $x_{-i} \precsim y_{-i}$, then

$$u_i(k, x_{-i}) - u_i(h, x_{-i}) \leq u_i(k, y_{-i}) - u_i(h, y_{-i}).$$

We begin with extending this property to the expected discounted payoff function V_i^θ to show that BR^i is monotone with respect to the partial order on Φ.

Lemma 3.1. *Suppose that the stage game is supermodular. For $\phi, \psi \in \Phi$, if $\phi_{-i} \precsim \psi_{-i}$, then for all $i \in I$ and all $t \in \mathbb{R}_+$,*

$$V_{ik}^\theta(\phi)(t) - V_{ih}^\theta(\phi)(t) \leq V_{ik}^\theta(\psi)(t) - V_{ih}^\theta(\psi)(t)$$

for $h < k$, and

$$\min BR^i(\phi)(t) \leq \min BR^i(\psi)(t),$$
$$\max BR^i(\phi)(t) \leq \max BR^i(\psi)(t).$$

Proof. Suppose $\phi_{-i} \precsim \psi_{-i}$ and fix any t. If $k > h$, then

$$\begin{aligned} V^{\theta}_{ik}(\phi)(t) - V^{\theta}_{ih}(\phi)(t) \\ &= (1+\theta)\int_t^{\infty} e^{-(1+\theta)(s-t)}\big\{u_i(k,\phi_{-i}(s)) - u_i(h,\phi_{-i}(s))\big\}\,ds \\ &\le (1+\theta)\int_t^{\infty} e^{-(1+\theta)(s-t)}\big\{u_i(k,\psi_{-i}(s)) - u_i(h,\psi_{-i}(s))\big\}\,ds \\ &= V^{\theta}_{ik}(\psi)(t) - V^{\theta}_{ih}(\psi)(t). \end{aligned}$$

Next, let $k = \min BR^i(\phi)(t)$. For any $h < k$,

$$V^{\theta}_{ik}(\psi)(t) - V^{\theta}_{ih}(\psi)(t) \ge V^{\theta}_{ik}(\phi)(t) - V^{\theta}_{ih}(\phi)(t) > 0$$

since $h \notin BR^i(\phi)(t)$. Hence, if $\ell \in BR^i(\psi)(t)$, then $\ell \ge k = \min BR^i(\phi)(t)$. We thus have $\min BR^i(\psi)(t) \ge \min BR^i(\phi)(t)$.

The other claim that $\max BR^i(\phi)(t) \le \max BR^i(\psi)(t)$ can be proved similarly. ▌

The next proposition establishes the monotonicity of the best response correspondence β^i over Φ. For $\phi \in \Phi$, a feasible path $\phi^- \in \beta^i(\phi)$ is the smallest element of $\beta^i(\phi)$ if $\phi^- \precsim \phi'_i$ for all $\phi'_i \in \beta^i(\phi)$, and $\phi^+ \in \beta^i(\phi)$ is the largest element of $\beta^i(\phi)$, if $\phi'_i \precsim \phi^+$ for all $\phi'_i \in \beta^i(\phi)$.

Proposition 3.2. *Suppose that the stage game is supermodular. For $\phi \in \Phi$, $\beta^i(\phi)$ has the smallest element $\min \beta^i(\phi)$ and the largest element $\max \beta^i(\phi)$. If $\phi_i(0) \precsim \psi_i(0)$ and $\phi_{-i} \precsim \psi_{-i}$, then*

$$\begin{aligned} \min \beta^i(\phi) &\precsim \min \beta^i(\psi), \\ \max \beta^i(\phi) &\precsim \max \beta^i(\psi). \end{aligned}$$

Proof. Take ϕ and ψ such that $\phi_i(0) = x_i$, $\psi_i(0) = y_i$, $x_i \precsim y_i$, and $\phi_{-i} \precsim \psi_{-i}$. First, we construct $\phi_i^- = \min \beta^i(\phi)$; the construction of $\max \beta^i(\phi)$ is similar. Define

$$\alpha_i(t) = \min BR^i(\phi)(t),$$

where the right hand side is considered as a mixed strategy. Note that α_i is a lower semi-continuous, and hence, measurable function, since $BR^i(\phi)(\cdot)$ is an upper semi-continuous correspondence. Then, the unique solution ϕ_i^- to

$$\dot{\phi}_i^-(t) = \alpha_i(t) - \phi_i^-(t) \ \text{ a.e.}, \qquad \phi_i^-(0) = x_i,$$

is given by

$$\phi_i^-(t) = e^{-t}x_i + \int_0^t e^{s-t}\alpha_i(s)\,ds.$$

By construction, $\phi_i^- \in \beta^i(\phi)$, and $\phi_i^- \precsim \phi_i'$ for all $\phi_i' \in \beta^i(\phi)$.

On the other hand, any path $\psi_i' \in \beta^i(\psi)$ is given by

$$\psi_i'(t) = e^{-t} y_i + \int_0^t e^{s-t} \alpha_i'(s)\, ds$$

for some $\alpha_i' \colon \mathbb{R}_+ \to \Delta(A_i)$ such that $\alpha_i'(t) \in F_i(\psi)(t)$ for almost all $t \in \mathbb{R}_+$, where $F_i(\psi)$ is defined by (2.5). Since $\phi_{-i} \precsim \psi_{-i}$, it follows from Lemma 3.1 that

$$\min BR^i(\phi)(t) \leq \min BR^i(\psi)(t),$$

and hence, $\alpha_i(t) \precsim \alpha_i'(t)$ for almost all t. Together with the assumption that $x_i \precsim y_i$, this implies that $\phi_i^- \precsim \psi_i'$, thereby completing the proof of $\min \beta^i(\phi) \precsim \min \beta^i(\psi)$. ▮

3.2 Comparison Theorem

Fix I and A. Take two games $G = (I, (A_i)_{i\in I}, (u_i)_{i\in I})$ and $G' = (I, (A_i)_{i\in I}, (v_i)_{i\in I})$ satisfying that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min BR^i_{v_i}(\pi_i) \leq \min BR^i_{u_i}(\pi_i), \tag{3.1}$$

or that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\max BR^i_{v_i}(\pi_i) \leq \max BR^i_{u_i}(\pi_i). \tag{3.2}$$

In this subsection, we consider the relationship between the perfect foresight paths for the stage game G and those for G'. Note that the state space $\prod_i \Delta(A_i)$ is common in both cases. We will show that if G or G' is supermodular, then the perfect foresight dynamics preserves the order of best responses between G and G', and therefore, G inherits stability properties from G'.

We specify the payoff functions to denote by $BR^i_{u_i}(\phi)(t)$ ($BR^i_{v_i}(\phi)(t)$, resp.) the set of best responses for an agent in population i to a feasible path ϕ at time t when the stage game is G (G', resp.). Note that if (3.1) is satisfied, then for any $\phi \in \Phi$ and any $t \in \mathbb{R}_+$,

$$\min BR^i_{v_i}(\phi)(t) \leq \min BR^i_{u_i}(\phi)(t), \tag{3.3}$$

while if (3.2) is satisfied, then for any $\phi \in \Phi$ and any $t \in \mathbb{R}_+$,

$$\max BR^i_{v_i}(\phi)(t) \leq \max BR^i_{u_i}(\phi)(t), \tag{3.4}$$

The following lemma is a key to our comparison theorem. The proof relies on a fixed point argument together with the monotonicity of BR^i.

Lemma 3.3. *Take any $x, y \in \prod_i \Delta(A_i)$ such that $y \precsim x$.*

(a) *Suppose that G and G' satisfy* (3.1) *and that G or G' is supermodular. If a feasible path $\phi \in \Phi_x$ satisfies that for all $i \in I$, all $h \in A_i$, and almost all $t \in \mathbb{R}_+$,*

$$\dot{\phi}_{ih}(t) > -\phi_{ih}(t) \Rightarrow h \geq \min BR^i_{u_i}(\phi)(t), \tag{3.5}$$

then there exists a perfect foresight path $\psi^* \in \Phi_y$ for G' such that $\psi^* \precsim \phi$.

(b) *Suppose that G and G' satisfy* (3.2) *and that G or G' is supermodular. If a feasible path $\psi \in \Phi_y$ satisfies that for all $i \in I$, all $h \in A_i$, and almost all $t \in \mathbb{R}_+$,*

$$\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \Rightarrow h \leq \max BR^i_{v_i}(\psi)(t), \tag{3.6}$$

then there exists a perfect foresight path $\phi^* \in \Phi_x$ for G such that $\psi \precsim \phi^*$.

Proof. We only show (a). For $x, y \in \prod_i \Delta(A_i)$ with $y \precsim x$ and $\phi \in \Phi_x$ satisfying (3.5), define the convex and compact subset $\tilde{\Phi}_y \subset \Phi_y$ to be

$$\tilde{\Phi}_y = \{\psi \in \Phi_y \mid \phi \precsim \psi\}.$$

Let $\beta_{G'}$ be the best response correspondence for the stage game G'. We want to show that $\beta_{G'}(\psi)$ is nonempty for any $\psi \in \tilde{\Phi}_y$. Then we can define a nonempty-valued correspondence $\tilde{\beta}_{G'} \colon \tilde{\Phi}_y \to \tilde{\Phi}_y$ by

$$\tilde{\beta}_{G'}(\psi) = \beta_{G'}(\psi) \cap \tilde{\Phi}_y \qquad (\psi \in \tilde{\Phi}_y),$$

which is also convex- and compact-valued and upper semi-continuous, so that due to Kakutani's fixed point theorem, it has a fixed point $\psi^* \in \tilde{\beta}_{G'}(\psi^*) \subset \tilde{\Phi}_y$, which is a perfect foresight path for G'.

For $\psi \in \tilde{\Phi}_y$, take any $i \in I$, $h \in A_i$, and $t \in \mathbb{R}_+$ such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$. If G is supermodular, then

$$h \geq \min BR^i_{u_i}(\phi)(t) \geq \min BR^i_{u_i}(\psi)(t) \geq \min BR^i_{v_i}(\psi)(t),$$

where the second inequality follows from the supermodularity of G and Lemma 3.1, and the third inequality follows from the assumption of (3.1). If G' is supermodular, then

$$h \geq \min BR^i_{u_i}(\phi)(t) \geq \min BR^i_{v_i}(\phi)(t) \geq \min BR^i_{v_i}(\psi)(t),$$

where the second inequality follows from the assumption of (3.1), and the third inequality follows from the supermodularity of G' and Lemma 3.1. Therefore, in each case, we have

$$h \geq \min BR^i_{v_i}(\psi)(t)$$

for all h such that $\dot{\phi}_{ih}(t) > -\phi_{ih}(t)$.

Now let $\psi' \in \Phi_y$ be given by

$$\dot{\psi'}_i(t) = \min BR^i_{v_i}(\psi)(t) - \psi'_i(t) \text{ a.e.}, \qquad \psi'_i(0) = y_i$$

for all $i \in I$. By construction, we have $\psi' \in \beta_{G'}(\phi)$. Since $\psi'(0) \precsim \psi(0)$ and $\dot{\psi}'(t) + \psi'(t) \precsim \dot{\psi}(t) + \psi(t)$ for almost all t, we also have $\psi' \precsim \psi$, which implies the nonemptiness of $\beta_{G'}(\psi)$. ▮

As a corollary, we have the following result, which is an analogue to the well-known comparison theorem from the theory of monotone (cooperative) dynamical systems (Smith (1995)).

Theorem 3.4. *Take any $x, y \in \prod_i \Delta(A_i)$ such that $y \precsim x$.*

(a) *Suppose that G and G' satisfy* (3.1) *and that G or G' is supermodular. For any perfect foresight path ϕ^* for G with $\phi^*(0) = x$, there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = y$ such that $\psi^* \precsim \phi^*$.*

(b) *Suppose that G and G' satisfy* (3.2) *and that G or G' is supermodular. For any perfect foresight path ψ^* for G' with $\psi^*(0) = y$, there exists a perfect foresight path ϕ^* for G with $\psi^*(0) = x$ such that $\psi^* \precsim \phi^*$.*

Suppose that G or G' is supermodular. This theorem implies that if G is in a monotone relation to G', then G inherits stability properties from G'. First, assume that G and G' satisfy (3.1) and that action profile $\max A = (n_i)_{i \in I}$ is absorbing under the stage game G'. Take any state $x \in B_\varepsilon(\max A)$ for a sufficiently small $\varepsilon > 0$. By Theorem 3.4(a), for any perfect foresight path ϕ^* for G with $\phi^*(0) = x$ there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = x$ such that $\psi^* \precsim \phi^*$. By the assumption that $\max A$ is absorbing under G', ψ^* converges to $\max A$, so that ϕ^* also converges to $\max A$, which implies that $\max A$ is absorbing under G as well.

Second, assume that G and G' satisfy (3.2) and that $\max A = (n_i)_{i \in I}$ is globally accessible under G'. Take any state $x \in \prod_i \Delta(A_i)$. By the assumption that $\max A$ is globally accessible under G', there exists a perfect foresight path ψ^* for G' with $\psi^*(0) = x$ that converges to $\max A$. By Theorem 3.4(b), there exists a perfect foresight path ϕ^* for G with $\phi^*(0) = x$ such that $\psi^* \precsim \phi^*$, which converges to $\max A$. This implies that $\max A$ is absorbing under G as well.

Note that by reversing the orders of actions, the above arguments can be applied to $\min A$. A candidate for the game G' is a potential game, for which the unique potential maximizer is known to be absorbing and globally accessible for any small degree of friction, due to Hofbauer and Sorger (2002). Such a case is considered, with some refinement, in Section 4 considers.

Another corollary to Lemma 3.3 (with $G = G'$) is that in supermodular games, an absorbing state is necessarily a strict Nash equilibrium.

Proposition 3.5. *Suppose that the stage game is supermodular. If $x^* \in \prod_i \Delta(A_i)$ is absorbing, then it is a strict Nash equilibrium.*

Proof. In the light of Proposition 2.3, it is sufficient to show that any Nash equilibrium that is not a strict Nash equilibrium is not absorbing. Suppose that x^* is a non-strict Nash equilibrium. We show the existence of an escaping path from x^*.

Let a'_i (a''_i, resp.) be the smallest (the largest, resp.) among the actions for player i that are best responses to x^*_{-i} in the stage game, and let $a' = (a'_i)_{i \in I}$ and $a'' = (a''_i)_{i \in I}$, which are considered as mixed action profiles. Note that $a' \precsim x^* \precsim a''$ and, by the definition of a non-strict Nash equilibrium, $a' \neq a''$, so that either a' or a'' is different from x^*. Let us assume that $a' \neq x^*$.

Now denote by $\bar{\phi}$ the constant path such that $\bar{\phi}(t) = x^*$ for all t. Note that $BR^i(\bar{\phi})(t)$ coincides with the set of best responses to x^*_{-i} in the stage game, so that $\min BR^i(\bar{\phi})(t) = a'_i$ for all t. Let ψ be the feasible path starting from x^* and converging linearly to a', i.e.,

$$\psi(t) = e^{-t} x^* + (1 - e^{-t}) a'.$$

This path satisfies that $\bar{\phi} \neq \psi$, $\bar{\phi} \precsim \psi$, and $\psi \in \beta(\bar{\phi})$. By Proposition 3.2, there exists a feasible path $\psi' \in \beta(\psi)$ such that $\psi \precsim \psi'$. By Lemma 3.3, there exists a perfect foresight path ψ^* from x^* such that $\psi \precsim \psi^*$, which does not converge to x^*. ∎

A globally accessible state need not be a strict Nash equilibrium in general. Even for the class of strict supermodular games, there are degenerate games where a non-strict, pure-strategy Nash equilibrium is globally accessible. The game given by Figure 1 has a non-strict Nash equilibrium $(0, 1)$, which is globally accessible for any degree of friction. It is an open problem whether every globally accessible state must be a pure Nash equilibrium in generic supermodular games.

	0	1
0	1, 1	1, 1
1	0, 0	1, 1

Figure 1: Globally accessible, non-strict Nash equilibrium

3.3 Stability under Rationalizable Foresight

The concept of perfect foresight path requires that agents optimize their payoffs against their beliefs about the future path of the action distribution and that those beliefs coincide with the actual path. Relaxing the latter requirement, Matsui and Oyama (2002) introduce the model of rationalizable foresight dynamics, where while the rationality of the agents as well as the structure of the society is common knowledge, beliefs about the future path are not necessarily coordinated among the agents. It is instead assumed that the agents form their beliefs in a rationalizable manner: in particular, they may misforecast the future. In this subsection, we consider the stability under the rationalizable foresight dynamics and show that in supermodular games, an absorbing and globally accessible state under the perfect foresight dynamics is uniquely absorbing under the rationalizable foresight dynamics as well.

Following Matsui and Oyama (2002), we define rationalizable foresight paths as follows. First let Ψ^0 be the set of all feasible paths, Φ. Then for each positive integer k, define Ψ^k to be

$$\Psi^k = \big\{\psi \in \Psi^{k-1} \,\big|\, \forall\, i \in I,\ \forall\, h \in A_i,\ \text{a.a.}\, t \in \mathbb{R}_+ : \big[\dot{\psi}_{ih}(t) > -\psi_{ih}(t) \\ \Rightarrow \exists\, \psi' \in \Psi^{k-1} : \psi'(s) = \psi(s)\ \forall\, s \in [0,t] \text{ and } h \in BR^i(\psi')(t)\big]\big\}.$$

Along a path in Ψ^k, an agent having a revision opportunity at time t takes a best response to some path in Ψ^{k-1} while knowing the past history up to time t.[9]

Definition 3.1. Let $\Psi^* = \bigcap_{k=0}^{\infty} \Psi^k$. A path in Ψ^* is a *rationalizable foresight path.*

Our concept of rationalizable foresight paths differs from rationalizability in the associated differential games defined in Remark 2.2. The former incorporates the feature of societal games that different agents in a population can have different beliefs and a single agent can have different beliefs at different revision opportunities, while for the latter, each population acts as a single player, who makes his decision at time zero.

One property of the rationalizable foresight dynamics is that along each rationalizable foresight path, each agent optimizes his payoff against another, possibly different, rationalizable foresight path. We state this without a proof, as it is essentially the same as Proposition 3.2 in Matsui and Oyama (2002).

Proposition 3.6. *A feasible path $\psi \in \Phi$ is in Ψ^* if and only if for all $i \in I$, all $h \in A_i$, and almost all $t \in \mathbb{R}_+$ such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$, there exists $\psi' \in \Psi^*$ such that $\psi'(s) = \psi(s)$ for all $s \leq t$ and $h \in BR^i(\psi')(t)$.*

As in a one-shot game, we have the following relationship between perfect and rationalizable foresight paths. It is verified by observing that every perfect foresight path is contained in each Ψ^k.

[9] Since the environment is stationary and $BR^i(\phi)(t)$ depends only on the behavior of ϕ after time t, in the definition of Ψ^k one can equivalently take ψ' as a path in Ψ^{k-1} that only satisfies $\psi'(t) = \psi(t)$.

Lemma 3.7. *A perfect foresight path is a rationalizable foresight path.*

We define absorption under rationalizable foresight analogously to that under perfect foresight.[10]

Definition 3.2. $x^* \in \prod_i \Delta(A_i)$ is *absorbing* under *rationalizable foresight* if there exists $\varepsilon > 0$ such that any rationalizable foresight path from any $x \in B_\varepsilon(x^*)$ converges to x^*.

An absorbing state under rationalizable foresight is also absorbing under perfect foresight due to Lemma 3.7, but not vice versa in general. For supermodular games, however, we are able to show that the converse is also true.

Theorem 3.8. *Suppose that the stage game is supermodular. Then, $x^* \in \prod_i \Delta(A_i)$ is absorbing under rationalizable foresight if and only if it is absorbing under perfect foresight.*

Therefore, in supermodular games, an absorbing and globally accessible state under perfect foresight is the unique state that is absorbing under rationalizable foresight.

The "if" part of this theorem follows from the lemma below. For $x \in \prod_i \Delta(A_i)$, let $\Psi_x^k = \Psi^k \cap \Phi_x$ and $\Psi_x^* = \bigcap_{k=0}^\infty \Psi_x^k$. Note that $\Psi_x^* = \Psi^* \cap \Phi_x$, i.e., Ψ_x^* is the set of rationalizable foresight paths from x.

Lemma 3.9. *Suppose that the stage game is supermodular. Ψ_x^* has the smallest and the largest elements, and these elements are perfect foresight paths.*

Proof. We show that Ψ_x^* has the smallest element and that it is a perfect foresight path. Let ϕ^0 be the smallest feasible path from x (i.e., the linear path from x to $\min A$) and ϕ^k the smallest best response path to ϕ^{k-1}, which is given by

$$\dot{\phi}_i^k(t) = \min BR^i(\phi^{k-1})(t) - \phi_i^k(t) \text{ a.e.}, \qquad \phi_i^k(0) = x_i.$$

Then, $\{\phi^k\}_{k=0}^\infty$ is an increasing sequence in the compact set Φ_x, so that $\{\phi^k\}_{k=0}^\infty$ converges to some $\phi^* \in \Phi_x$. By the upper semi-continuity of β_x, ϕ^* is a perfect foresight path, and hence, an element of Ψ_x^* by Lemma 3.7.

It suffices to show that ϕ^* is a lower bound of Ψ_x^*. Let us show that for all k, ϕ^k is a lower bound of Ψ_x^k. Then, since ϕ^k is a lower bound of Ψ_x^* ($\subset \Psi_x^k$) for all k, the limit ϕ^* is also a lower bound of Ψ_x^*.

First, ϕ^0 is a lower bound of Ψ_x^0. Then suppose that ϕ^{k-1} is a lower bound of Ψ_x^{k-1}. Fix any $\psi \in \Psi_x^k$, and take any i and any t such that ϕ_i^k and ψ_i are differentiable at t. For all h, if $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$, then $h \in BR^i(\psi')(t)$

[10] We can also define global accessibility under rationalizable foresight in a similar manner. Due to Lemma 3.7, it is weaker than that under perfect foresight.

for some $\psi' \in \Psi_x^{k-1}$. Since $\phi^{k-1} \precsim \psi'$ by assumption, it follows from the supermodularity and Lemma 3.1 that $\min BR^i(\phi^{k-1})(t) \leq \min BR^i(\psi')(t) \leq h$ for all h such that $\dot{\psi}_{ih}(t) > -\psi_{ih}(t)$. Therefore, we have $\dot{\phi}_i^k(t) + \phi_i^k(t) \precsim \dot{\psi}_i(t) + \psi_i(t)$ for almost all t, which implies that $\phi^k \precsim \psi$. Hence, ϕ^k is a lower bound of Ψ_x^k. ∎

Proof of Theorem 3.8. "If" part: Take any rationalizable foresight path ψ from x sufficiently close to x^*. By Lemma 3.9, there exist perfect foresight paths ϕ and ϕ' from x such that $\phi \precsim \psi \precsim \phi'$. If x^* is absorbing under perfect foresight, then both ϕ and ϕ' converge to x^*, and therefore, ψ also converges to x^*.

"Only if" part: Follows from Lemma 3.7. ∎

Remark 3.1. All the results in this section hold under more general settings (after appropriate modifications of replacing "$\phi_{-i} \precsim \psi_{-i}$" with "$\phi \precsim \psi$", and (3.1) and (3.2) with (3.3) and (3.4)), where $V_i^\theta(\cdot)(\cdot)\colon \Phi \times \mathbb{R}_+ \to \mathbb{R}^{n_i+1}$ is continuous, and $V_i^\theta(\cdot)(t)\colon \Phi \to \mathbb{R}^{n_i+1}$ is supermodular, i.e., if $\phi \precsim \psi$, then

$$V_{ik}^\theta(\phi)(t) - V_{ih}^\theta(\phi)(t) \leq V_{ik}^\theta(\psi)(t) - V_{ih}^\theta(\psi)(t)$$

for $k > h$. Examples of such functions include the expected discounted payoffs induced by the stage game where the payoff to an agent in population i taking action $h \in A_i$ is given by a continuous function $g_{ih}\colon \prod_i \Delta(A_i) \to \mathbb{R}$. Note here that the payoff function for an agent in population i can depend on the action distribution within population i itself. Such payoff functions describe random matching models within a single population, considered in Matsui and Matsuyama (1995), Hofbauer and Sorger (1999), and Oyama (2002), as well as models with nonlinear payoffs, considered in Matsuyama (1991, 1992) and Kaneda (1995).

4 Games with Monotone Potentials

This section applies the monotonicity argument developed in the previous section to games with monotone potentials introduced by Morris and Ui (2002). Suppose that games G and G' satisfy (3.1) or (3.2). Roughly speaking, G has a monotone potential if G' is a potential game, and action profile $\max A$ is a monotone potential maximizer of G if it is the unique potential maximizer of G'. For potential games, Hofbauer and Sorger (2002) show that the unique potential maximizer is absorbing and globally accessible for any small degree of friction. Therefore, we can conclude from the monotonicity argument in Subsection 3.2 that if G or G' is supermodular, then $\max A$ is absorbing (if (3.1) is satisfied) and globally accessible (if (3.2) is satisfied) for any small degree of friction.

For the precise definition, which is given in the subsection below, two remarks are in order. First, when G' is a potential game, a condition weaker than both (3.1) and (3.2) is sufficient for the global accessibility result. Morris and Ui's (2002) notion of monotone potential employs this weaker condition (see Definition 4.1), while (3.1) corresponds to what we call strict monotone potential (see Definition 4.2). Second, in order to consider action profiles other than $\max A$ and $\min A$, we need some complication.

4.1 Monotone Potential Maximizers

Fix an action profile $a^* \in A$. Let $A_i^- = \{h \in A_i \,|\, h \leq a_i^*\}$ and $A_i^+ = \{h \in A_i \,|\, h \geq a_i^*\}$. For a function $f\colon A \to \mathbb{R}$, a probability distribution $\pi_i \in \Delta(A_{-i})$, and a nonempty set of actions $A_i' \subset A_i$, let

$$BR_f^i(\pi_i|A_i') = \arg\max_{h \in A_i'} f(h, \pi_i),$$

where $f(h, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) f(h, a_{-i})$. We employ the following simplified version of monotone potential.

Definition 4.1. The action profile $a^* \in A$ is a *monotone potential maximizer*, or an *MP-maximizer*, if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min BR_v^i(\pi_i|A_i^-) \leq \max BR_{u_i}^i(\pi_i|A_i^-),$$

and

$$\max BR_v^i(\pi_i|A_i^+) \geq \min BR_{u_i}^i(\pi_i|A_i^+).$$

Such a function v is called a *monotone potential function* for a^*.

In addition, we introduce a slight refinement of MP-maximizer.

Definition 4.2. The action profile $a^* \in A$ is a *strict monotone potential maximizer*, or a *strict MP-maximizer*, if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$,

$$\min BR_v^i(\pi_i|A_i^-) \leq \min BR_{u_i}^i(\pi_i|A_i^-),$$

and

$$\max BR_v^i(\pi_i|A_i^+) \geq \max BR_{u_i}^i(\pi_i|A_i^+).$$

Such a function v is called a *strict monotone potential function* for a^*.

A strict MP-maximizer is always an MP-maximizer, but the converse is not true. In a degenerate game (with at least two action profiles) where payoffs are constant for each player, all the action profiles become MP-maximizers, while none of them is a strict MP-maximizer. For a generic choice of payoffs, an MP-maximizer is a strict MP-maximizer. For supermodular games, a strict MP-maximizer is unique if it exists, due to Theorems 4.1 and 4.2 given below.

MP-maximizer unifies several existing concepts. A (strict) MP-maximizer is a (strict) Nash equilibrium. A unique potential maximizer is a strict MP-maximizer. A (strict) **p**-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is a (strict) MP-maximizer. For games with diminishing returns, MP-maximizer reduces to local potential maximizer (Morris and Ui (2002) and Frankel, Morris, and Pauzner (2003)). See Subsection 4.3 for details.

4.2 Results

For a function $f\colon A \to \mathbb{R}$, a feasible path ϕ, and a nonempty set of actions $A'_i \subset A_i$, let

$$BR^i_f(\phi|A'_i)(t) = \operatorname*{arg\,max}_{h\in A'_i}\ (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} f(h,\phi_{-i}(s))\,ds,$$

where $f(h, x_{-i}) = \sum_{a_{-i}\in A_{-i}}(\prod_{j\neq i} x_j(a_j))f(h,a_{-i})$ for $x_{-i} \in \prod_{j\neq i}\Delta(A_j)$. Note that this can be written as

$$BR^i_f(\phi|A'_i)(t) = BR^i_f(\pi^t_i(\phi)|A'_i)$$

with a probability distribution $\pi^t_i(\phi) \in \Delta(A_{-i})$ which is given by

$$\pi^t_i(\phi)(a_{-i}) = (1+\theta)\int_t^\infty e^{-(1+\theta)(s-t)} \prod_{j\neq i}\phi_{ja_j}(s)\,ds.$$

Let $G_v = (I, (A_i)_{i\in I}, (v)_{i\in I})$ be the potential game in which all players have the common payoff function v. We have the following two theorems. Their proofs are given in Appendix.

Theorem 4.1. *Suppose that the stage game G has an MP-maximizer a^* with a monotone potential function v. If either G or G_v is supermodular, then there exists $\bar{\theta} > 0$ such that a^* is globally accessible for all $\theta \in (0, \bar{\theta})$.*

Theorem 4.2. *Suppose that the stage game G has a strict MP-maximizer a^* with a strict monotone potential function v. If either G or G_v is supermodular, then a^* is absorbing for all $\theta > 0$.*

Given an MP-maximizer a^* and a monotone potential v, observe that the restricted games $G^-_v = (I, (A^-_i)_{i\in I}, (v)_{i\in I})$ and $G^+_v = (I, (A^+_i)_{i\in I}, (v)_{i\in I})$ are potential games with the unique potential maximizer a^*. The proofs of Theorems 4.1 and 4.2 utilize this observation to apply results on potential games by Hofbauer and Sorger (2002).

4.3 Examples

This subsection provides special cases of games with monotone potentials. For games with no monotone potential, see Examples 5.2.1 and 5.4.1.

4.3.1 p-Dominance

Let $\mathbf{p} = (p_1, \ldots, p_N) \in [0, 1)^N$. The notion of **p**-dominance (Kajii and Morris (1997)) is a many-player, many-action generalization of risk-dominance.

Definition 4.3. (a) An action profile $a^* \in A$ is a **p**-*dominant equilibrium* if $a_i^* \in BR^i_{u_i}(\pi_i|A_i)$ for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a^*_{-i}) \geq p_i$.

(b) An action profile a^* is a *strict* **p**-*dominant equilibrium* if $\{a_i^*\} = BR^i_{u_i}(\pi_i|A_i)$ for all $i \in I$ and all $\pi_i \in \Delta(A_{-i})$ with $\pi_i(a^*_{-i}) > p_i$.

A **p**-dominant equilibrium with low enough **p** is an MP-maximizer with a monotone potential function that is supermodular (with appropreate reordering of actions).

Lemma 4.3. *If a^* is a (strict)* **p**-*dominant equilibrium with $\sum_{i\in I} p_i < 1$, then a^* is a (strict) MP-maximizer with the (strict) monotone potential v given by*

$$v(a) = \begin{cases} 1 - \sum_{i\in I} p_i & \text{if } a = a^*, \\ -\sum_{i\in C(a)} p_i & \text{otherwise,} \end{cases}$$

where $C(a) = \{i \in I \mid a_i = a_i^*\}$.

Proof. See Appendix. ▮

By relabeling actions so that $a_i^* = \max A_i$ for all $i \in I$, we can make v supermodular. Therefore, we have the following result as a corollary to Theorems 4.1 and 4.2.

Corollary 4.4. (a) *A* **p**-*dominant equilibrium with $\sum_{i\in I} p_i < 1$ is globally accessible for any small degree of friction.*

(b) *A strict* **p**-*dominant equilibrium with $\sum_{i\in I} p_i < 1$ is absorbing for any degree of friction.*

In particular, a strict **p**-dominant equilibrium with $\sum_{i\in I} p_i < 1$ is uniquely absorbing (and globally accessible) for any small degree of friction.

Remark 4.1. Hofbauer and Sorger's (2002) notion of 1/2-dominance differs from (strict) **p**-dominance (for any **p**) for games with more than two players. An action $a^* \in A$ is said to be 1/2-dominant if $\{a_i^*\} = BR^i_{u_i}(x_{-i}|A_i)$ for all $i \in I$ and all $x_{-i} \in \prod_{j\neq i} \Delta(A_j)$ with $x_{ja_j^*} > 1/2$ for all $j \neq i$. Note the difference between π_i and x_{-i} in the definitions.

4.3.2 Local Potential Maximizers

Frankel, Morris, and Pauzner (2003) introduce the notion of local potential maximizer which we call strict local potential maximizer, while Morris and Ui (2002) give a slightly weaker definition.

Definition 4.4. (a) An action profile $a^* \in A$ is a *local potential maximizer*, or an *LP-maximizer*, if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i\colon A_i \to \mathbb{R}_+$ such that if $h < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\bigl(v(h+1, a_{-i}) - v(h, a_{-i})\bigr) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i}),$$

and if $h > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$\mu_i(h)\bigl(v(h-1, a_{-i}) - v(h, a_{-i})\bigr) \leq u_i(h-1, a_{-i}) - u_i(h, a_{-i}).$$

Such a function v is called a *local potential function* for a^*.

(b) The action profile $a^* \in A$ is a *strict local potential maximizer*, or a *strict LP-maximizer*, if there exists a function $v\colon A \to \mathbb{R}$ with $v(a^*) > v(a)$ for all $a \neq a^*$ such that for all $i \in I$, there exists a function $\mu_i\colon A_i \to \mathbb{R}_{++}$ such that if $h < a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$v(h+1, a_{-i}) - v(h, a_{-i}) \leq \mu_i(h)\bigl(u_i(h+1, a_{-i}) - u_i(h, a_{-i})\bigr),$$

and if $h > a_i^*$, then for all $a_{-i} \in A_{-i}$,

$$v(h-1, a_{-i}) - v(h, a_{-i}) \leq \mu_i(h)\bigl(u_i(h-1, a_{-i}) - u_i(h, a_{-i})\bigr).$$

Such a function v is called a *strict local potential function* for a^*.

The game G is said to have *diminishing marginal returns* if for all $i \in I$, all $h \neq 0, n_i$, and all $a_{-i} \in A_{-i}$,

$$u_i(h, a_{-i}) - u_i(h-1, a_{-i}) \geq u_i(h+1, a_{-i}) - u_i(h, a_{-i}).$$

In games with diminishing marginal returns, the MP-maximizer conditions reduce to the LP-maximizer conditions.

Lemma 4.5. *If the game G has a (strict) LP-maximizer a^* with a (strict) local potential function v and if G or G_v has diminishing marginal returns, then a^* is a (strict) MP-maximizer with a (strict) monotone potential function v.*

Proof. See Appendix. ▮

We have the following result as a corollary to Theorems 4.1 and 4.2.

Corollary 4.6. (a) *Suppose that the stage game G has an LP-maximizer a^* with a local potential function v. If G or G_v has diminishing marginal returns and if G or G_v is supermodular, a^* is globally accessible for any small degree of friction.*

(b) *Suppose that the stage game G has a strict LP-maximizer a^* with a strict local potential function v. If G or G_v has diminishing marginal returns and if G or G_v is supermodular, then a^* is absorbing for any degree of friction.*

In particular, a strict LP-maximizer is uniquely absorbing (and globally accessible) for any small degree of friction, if G or G_v has diminishing marginal returns and G or G_v is supermodular.

4.3.3 Symmetric 3×3 Supermodular Games

Consider symmetric 3×3 games with three Nash equilibria, where $I = \{1, 2\}$, $A_1 = A_2 = \{0, 1, 2\}$, $u_1(h, k) = u_2(k, h)$ for all $h, k \in \{0, 1, 2\}$, and $u_1(h, h) > u_1(k, h)$ for all $k \neq h$. Assume strict supermodularity, i.e., $u_1(h, k) - u_1(h', k) > u_1(h, k') - u_1(h', k')$ if $h > h'$ and $k > k'$. Frankel, Morris, and Pauzner (2003) estabilish that this class of games generically have a strict LP-maximizer. In fact, those games have a strict MP-maximizer although they do not have diminishing marginal returns in general.

Lemma 4.7. *Generic symmetric 3×3 supermodular games have a strict MP-maximizer.*

We therefore have the following result as a corollary to Theorems 4.1 and 4.2.

Corollary 4.8. *For generic symmetric 3×3 supermodular games, there exists a unique absorbing and globally accessible state for any small degree of friction.*

This reproduces Theorem 4.3 and resolves their conjecture in Hofbauer and Sorger (2002).

4.3.4 Young's Example

Consider the 3×3 game given in Figure 2(a), taken from Young (1993). Oyama (2002) shows by direct computation that $(2, 2)$ is absorbing and globally accessible for a small degree of friction. In fact, $(2, 2)$ is a strict MP-maximizer with a strict monotone potential function that is supermodular (Figure 2(b)), while the original game is not supermodular (for any ordering of actions). Therefore, our results, Theorems 4.1 and 4.2, also apply to this game.

Note that $(1, 1)$ is stochastially stable (Young (1993)), while it is neither absorbing nor globally accessible when the friction is small.

	0	1	2
0	6, 6	0, 5	0, 0
1	5, 0	7, 7	5, 5
2	0, 0	5, 5	8, 8

(a) Original game

	0	1	2
0	6	5	0
1	5	7	5
2	0	5	8

(b) Monotone potential function

Figure 2: Young's example

5 Binary Supermodular Games

In this section, we restrict our attention to supermodular games with two actions for each player, where $A_i = \{0, 1\}$ for all $i \in I$. Note that in this case, it is not necessary to consider the stochastic dominance order: $\phi \precsim \psi$ if and only if $\phi_{i1}(t) \leq \psi_{i1}(t)$ for all $i \in I$ and all $t \geq 0$. Denoting $p_i = x_{i1}$, we define the *incentive function* $d_i \colon [0,1]^N \to \mathbb{R}$ for player i by

$$d_i(p_1, \ldots, p_N) = u_i(1, x_{-i}) - u_i(0, x_{-i}).$$

We assume that action profiles $\mathbf{0}$, where all players play 0, and $\mathbf{1}$, where all players play 1, are strict Nash equilibria, i.e.,

$$d_i(\mathbf{0}) < 0 < d_i(\mathbf{1})$$

for all i. We further assume that d_i is nondecreasing in each p_j ($j \neq i$) so that the game is a supermodular game. In the next subsection, we first give complete characterizations for the strict Nash equilibrium $\mathbf{1}$ to be globally accessible and to be absorbing, respectively. By reversing the orders of actions, the results can be applied to the other Nash equilibrium $\mathbf{0}$. The subsequent subsections then consider three subclasses of binary supermodular games.

For a feasible path ϕ, denote

$$\begin{aligned} \mathit{\Delta} V_i^\theta(\phi)(t) &= V_{i1}^\theta(\phi)(t) - V_{i0}^\theta(\phi)(t) \\ &= (1+\theta) \int_t^\infty e^{-(1+\theta)(s-t)} d_i(\phi(s))\, ds. \end{aligned}$$

Recall from Lemma 3.1 that if $\phi \precsim \psi$, then $\mathit{\Delta} V_i^\theta(\phi)(t) \leq \mathit{\Delta} V_i^\theta(\psi)(t)$ for all $i \in I$ and all $t \geq 0$ due to the supermodularity.

5.1 General Results

For $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$, let $\phi^{\mathrm{u}}_{\mathbf{T}}$ be the feasible path given by

$$(\phi^{\mathrm{u}}_{\mathbf{T}})_{i1}(t) = \begin{cases} 0 & \text{if } t < T_i \\ 1 - e^{-(t-T_i)} & \text{if } t \geq T_i. \end{cases} \tag{5.1}$$

Denote $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$. For $\mathbf{T} = (T_i)_{i\in I} \in \bar{\mathbb{R}}^N_+$, let $\psi^{\mathrm{d}}_{\mathbf{T}}$ be the feasible path given by

$$(\psi^{\mathrm{d}}_{\mathbf{T}})_{i1}(t) = \begin{cases} 1 & \text{if } t < T_i \\ e^{-(t-T_i)} & \text{if } t \geq T_i \end{cases} \qquad \text{for } i \in S, \tag{5.2}$$

and

$$(\psi^{\mathrm{d}}_{\mathbf{T}})_{i1}(t) = 1 \qquad \text{for } i \notin S, \tag{5.3}$$

where $S = \{i \in I \,|\, T_i \neq \infty\}$.

First, we provide necessary and sufficient conditions for the state $\mathbf{1}$ to be globally accessible for a given degree of friction (Proposition 5.1.1) and for any small degree of friction (Proposition 5.1.2), respectively.

Proposition 5.1.1. *Let $\theta > 0$ be given. The state $\mathbf{1}$ is globally accessible for θ if and only if there exists $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$ such that for all $i \in I$,*

$$\Delta V_i^{\theta}(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0.$$

Proof. See Appendix. ▮

Proposition 5.1.2. *There exists $\bar{\theta} > 0$ such that the state $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if there exists $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$ such that for all $i \in I$,*

$$\Delta V_i^{0}(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0.$$

Proof. See Appendix. ▮

Next, we provide necessary and sufficient conditions for the state $\mathbf{1}$ to be absorbing for a given degree of friction (Proposition 5.1.3) and for any degree of friction (Proposition 5.1.4), respectively. For $S \subset I$, let $\mathbf{0}_S$ be the pure action profile such that i chooses 0 for $i \in S$ and 1 for $i \notin S$.

Proposition 5.1.3. *Let $\theta > 0$ be given. The state $\mathbf{1}$ is absorbing for θ if and only if for any $\mathbf{T} = (T_i)_{i\in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium of the stage game, there exists $i \in S$ such that*

$$\Delta V_i^{\theta}(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) > 0.$$

Proof. See Appendix. ∎

Proposition 5.1.4. *The state* $\mathbf{1}$ *is absorbing for all* $\theta > 0$ *if and only if for any* $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}_+^N$ *such that* $S = \{i \in I \mid T_i \neq \infty\}$ *is nonempty and* $\mathbf{0}_S$ *is a Nash equilibrium of the stage game, there exists* $i \in S$ *such that*

$$\varDelta V_i^0(\psi_{\mathbf{T}}^{\mathrm{d}})(T_i) \geq 0.$$

Proof. See Appendix. ∎

5.2 Unanimity Games

This subsection considers N-player unanimity games. The stage game is given by

$$u_i(a) = \begin{cases} y_i & \text{if } a = \mathbf{0} \\ z_i & \text{if } a = \mathbf{1} \\ 0 & \text{otherwise,} \end{cases} \tag{5.4}$$

where $y_i, z_i > 0$. The incentive function for player i is then given by

$$d_i(p_1, \cdots, p_N) = z_i \prod_{j \neq i} p_j - y_i \prod_{j \neq i} (1 - p_j).$$

Note that this game is supermodular.

For $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, let

$$\begin{aligned} \pi_i(\mathbf{T}; \theta) &= (1 + \theta) \int_{T_i}^{\infty} e^{-(1+\theta)(t - T_i)} \prod_{j \neq i} \left[0 \vee \left\{ 1 - e^{-(t - T_j)} \right\} \right] dt \\ &= (1 + \theta) \int_{\max_j T_j}^{\infty} e^{-(1+\theta)(t - T_i)} \prod_{j \neq i} \left\{ 1 - e^{-(t - T_j)} \right\} dt, \end{aligned} \tag{5.5}$$

and

$$\rho_i(\mathbf{T}; \theta) = (1 + \theta) \int_{T_i}^{\infty} e^{-(1+\theta)(t - T_i)} \prod_{j \neq i} \left\{ 1 \wedge e^{-(t - T_j)} \right\} dt. \tag{5.6}$$

Observe that π_i is continuous and decreasing in θ, while ρ_i is continuous and increasing in θ.

5.2.1 Global Accessibility

For a feasible path $\phi_{\mathbf{T}}^{\mathrm{u}}$ defined by (5.1) with a given $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}_+^N$, the discounted payoff difference is given by

$$\varDelta V_i^\theta(\phi_{\mathbf{T}}^{\mathrm{u}})(T_i) = z_i \pi_i(\mathbf{T}; \theta) - y_i \rho_i(\mathbf{T}; \theta),$$

so that $\mathit{\Delta} V_i^0(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$ if and only if

$$\frac{z_i}{y_i} > \frac{\rho_i(\mathbf{T};0)}{\pi_i(\mathbf{T};0)}.$$

Let

$$U_0 = \left\{ x \in \mathbb{R}^N_{++} \,\middle|\, \exists\, \mathbf{T} \in \mathbb{R}^N_+ \; \forall\, i \in I: \; x_i > \frac{\rho_i(\mathbf{T};0)}{\pi_i(\mathbf{T};0)} \right\}.$$

We immediately have the following from Proposition 5.1.2.

Proposition 5.2.1. *Suppose that the stage game is a unanimity game given by* (5.4)*. Then there exists $\bar{\theta} > 0$ such that $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if $(z_i/y_i)_{i \in I} \in U_0$.*

Symmetrically, there exists $\bar{\theta} > 0$ such that $\mathbf{0}$ is globally accessible for all $\theta \in (0, \bar{\theta})$ if and only if $(y_i/z_i)_{i \in I} \in U_0$.

5.2.2 Absorption

For a feasible path $\psi^{\mathrm{d}}_{\mathbf{T}}$ defined by (5.2) with a given $\mathbf{T} = (T_i)_{i \in I} \in \mathbb{R}^N_+$, the discounted payoff difference is given by

$$\mathit{\Delta} V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) = z_i \rho_i(\mathbf{T};\theta) - y_i \pi_i(\mathbf{T};\theta),$$

so that $\mathit{\Delta} V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0$ if and only if

$$\frac{z_i}{y_i} \geq \frac{\pi_i(\mathbf{T};0)}{\rho_i(\mathbf{T};0)}.$$

Let

$$\begin{aligned} V_0 &= \left\{ x \in \mathbb{R}^N_{++} \,\middle|\, \forall\, \mathbf{T} \in \mathbb{R}^N_+ \; \exists\, i \in I: \; x_i \geq \frac{\pi_i(\mathbf{T};0)}{\rho_i(\mathbf{T};0)} \right\} \\ &= \left\{ x \in \mathbb{R}^N_{++} \,\middle|\, (1/x_i)_{i \in I} \notin U_0 \right\}. \end{aligned}$$

We have the following.

Proposition 5.2.2. *Suppose that the stage game is a unanimity game given by* (5.4)*. Then $\mathbf{1}$ is absorbing for all $\theta > 0$ if and only if $(z_i/y_i)_{i \in I} \in V_0$.*

Symmetrically, $\mathbf{0}$ is absorbing for all $\theta > 0$ if and only if $(y_i/z_i)_{i \in I} \in V_0$.

Proof. "If" part: Suppose that $\mathbf{1}$ is not absorbing for some $\theta > 0$. Then, by Proposition 5.1.4 there exists $\mathbf{T} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty, and for all $i \in S$,

$$\mathit{\Delta} V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) < 0,$$

where $\psi^{\mathrm{d}}_{\mathbf{T}'}$ is given by (5.2) and (5.3). Suppose that $S \neq I$. Then

$$\mathit{\Delta} V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0$$

for all $i \in S$ due to the payoff structure of the unanimity game. Therefore, we have $S = I$, $\mathbf{T} \in \mathbb{R}^N_+$, and

$$\mathit{\Delta} V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}}) < 0,$$

or

$$\frac{z_i}{y_i} < \frac{\pi_i(\mathbf{T}; 0)}{\rho_i(\mathbf{T}; 0)},$$

for all $i \in I$.

"Only if" part: This immediately follows from Proposition 5.1.4. ∎

5.2.3 Two-Player Case

In the case where $N = 2$, the sets U_0 and V_0 reduce to $U_0 = \{(x_1, x_2) | x_1 x_2 > 1\}$ and $V_0 = \{(x_1, x_2) | x_1 x_2 \geq 1\}$, so that $(z_1/y_1, z_2/y_2) \in U_0\, (\subset V_0)$ if and only if $z_1 z_2 > y_1 y_2$. Therefore, by Propositions 5.2.1 and 5.2.2, **1** is absorbing and globally accessible for any small degree of friction if and only if **1** has the higher Nash product over **0**. In the two-player case, this is equivalent to that **1** is risk-dominant.

5.2.4 Three-Player Case

When $N \geq 3$, the complete characterizations given above are rather complex. Here we consider three-player binary games with a symmetry between players 2 and 3. We demonstrate that even for this simple class of games, there may be multiple globally accessible states.

Specifically, we consider the case where

$$(z_1/y_1, z_2/y_2, z_3/y_3) = (r, s, s). \tag{5.7}$$

We can exploit the symmetry due to the following fact. Recall here that if $T_i = T_j$, then $\pi_i(\mathbf{T}; \theta) = \pi_j(\mathbf{T}; \theta)$ and $\rho_i(\mathbf{T}; \theta) = \rho_j(\mathbf{T}; \theta)$.

Proposition 5.2.3. *Suppose that the stage game is given by* (5.4)*. Then* **1** *is globally accessible for any small degree of friction if and only if there exists* $\mathbf{T}$ *such that for all* $i \in I$*,*

$$\frac{z_i}{y_i} \geq \frac{\rho_i(\mathbf{T}; 0)}{\pi_i(\mathbf{T}; 0)}, \tag{5.8}$$

and

$$\frac{z_i}{y_i} \geq \frac{z_j}{y_j} \Rightarrow T_i \leq T_j. \tag{5.9}$$

Proof. It suffices to show that if there exists $\mathbf{T}$ that satisfies (5.8), then there exists $\mathbf{T}'$ that satisfies both (5.8) and (5.9).

Take a $\mathbf{T}$ that satisfies (5.8) and define $\mathbf{T}'$ by

$$T'_i = \min_{j\colon z_j/y_j \le z_i/y_i} T_j$$

for each i. Note that $T'_i \le T_i$ for any i.

Here we fix any i. There exists j such that $T'_i = T_j$ and $z_j/y_j \le z_i/y_i$. Take such a j. Note that $\mathbf{T}_{-j} \ge \mathbf{T}'_{-j}$ and $T_j = T'_j$. Since $\mathbf{T}$ satisfies (5.8) so that

$$\frac{z_j}{y_j} \ge \frac{\rho_j(\mathbf{T}; 0)}{\pi_j(\mathbf{T}; 0)},$$

and π_j is decreasing in $\mathbf{T}_{-j}$ and ρ_j is increasing in $\mathbf{T}_{-j}$, we have

$$\frac{z_j}{y_j} \ge \frac{\rho_j(\mathbf{T}'; 0)}{\pi_j(\mathbf{T}'; 0)}.$$

On the other hand,

$$\frac{\rho_j(\mathbf{T}'; 0)}{\pi_j(\mathbf{T}'; 0)} = \frac{\rho_i(\mathbf{T}'; 0)}{\pi_i(\mathbf{T}'; 0)},$$

since $T'_i = T'_j$. Therefore, it follows from $z_j/y_j \le z_i/y_i$ that

$$\frac{z_i}{y_i} \ge \frac{\rho_i(\mathbf{T}'; 0)}{\pi_i(\mathbf{T}'; 0)},$$

which completes the proof. ∎

A direct computation utilizing Proposition 5.2.3 shows that $\mathbf{1}$ is globally accessible for a small friction if and only if there exists $u \ge 1$ such that

$$r < s, \quad r > \frac{1}{3u^2 - 3u + 1}, \quad s > \frac{3u^2 - 1}{3u - 1},$$

or there exists $v \ge 1$ such that

$$r \ge s, \quad r > 3v - 2, \quad s > \frac{2}{3v - 1}.$$

The above condition is equivalent to that

$$r < s \quad \text{and} \quad r > \frac{2}{(s-1)\sqrt{9s^2 - 12s + 12} + 3s^2 - 5s + 4},$$

or

$$r \ge s \quad \text{and} \quad r > \frac{2}{s} - 1.$$

In the game given by (5.7), $\mathbf{1}$ has the higher Nash product over $\mathbf{0}$ if $rs^2 > 1$. A direct comparison between $r > 1/s^2$ and the above expressions gives the following sufficient condition in terms of Nash product.

Proposition 5.2.4. *In the game given by* (5.7)*, the Nash equilibrium with the higher Nash product is globally accessible for any small degree of friction.*

The converse is not true.

Example 5.2.1. Let $y_1 = 6 + c > 0$, $y_2 = y_3 = 1$, and $z_1 = z_2 = z_3 = 2$ (see Figure 3). This game is a modified version of an example in Morris and Ui (2002, Example 1).[11] If $c > 0$, then $\mathbf{0}$ is globally accessible for a small friction, while if $c < 2\sqrt{6}$, then $\mathbf{1}$ is globally accessible for a small friction. Therefore, if $0 < c < 2\sqrt{6}$, the game has two globally accessible states simultaneously when the friction is small. Note that $\mathbf{0}$ ($\mathbf{1}$, resp.) has the higher Nash product if $c > 2$ ($c < 2$, resp.).

On the other hand, one can show that if $c \leq 0$, then $\mathbf{1}$ is absorbing for any degree of friction, while if $c \geq 2\sqrt{6}$, then $\mathbf{0}$ is absorbing for any degree of friction.

	0	1
0	$6+c$, 1, 1	0, 0, 0
1	0, 0, 0	0, 0, 0

0

	0	1
0	0, 0, 0	0, 0, 0
1	0, 0, 0	2, 2, 2

1

Figure 3: Multiple globally accessible states

5.3 Binary Games with Linear Incentives

This subsection considers N-player binary supermodular games with linear incentives (Selten (1995)). A binary game is said to have linear incentives if the incentive function for player i takes the form (with $p_j = x_{j1}$)

$$d_i(p_1, \dots, p_N) = \sum_{j=1}^{N} \alpha_{ij} p_j - s_i$$

with $\alpha_{ii} = 0$. If $\sum_{j=1}^{N} \alpha_{ij} - s_i > 0$ and $s_i > 0$ for all i, then both $\mathbf{0}$ and $\mathbf{1}$ are strict Nash equilibria. We assume that $\alpha_{ij} \geq 0$ for $i \neq j$ so that the game is supermodular. (The special case $\alpha_{ij} = \alpha_{ji}$ leads to a potential game and has been considered in Hofbauer and Sorger (2002).)

[11] One can verify that $\mathbf{0}$ is not an MP-maximizer for any c, while $\mathbf{1}$ is an MP-maximizer if and only if $c < -2$.

We restate the characterization for global accessibility given in Proposition 5.1.2 in the following form.

Lemma 5.3.1. *In a binary supermodular game,* **1** *is globally accessible for any small* $\theta > 0$ *if and only if there exists* $(r_i)_{i\in I} \in \mathbb{R}^N$ *such that*

$$\int_0^\infty e^{-t} d_i(\Psi(r_1 - r_i + t), \ldots, \Psi(r_N - r_i + t))\, dt > 0 \tag{5.10}$$

for all $i \in I$*, where* Ψ *is given by*

$$\Psi(z) = \begin{cases} 0 & \text{for } z \leq 0, \\ 1 - e^{-z} & \text{for } z > 0. \end{cases}$$

Proof. Given $(T_i)_{i\in I}$ in Proposition 5.1.2, set, for example, $r_i = -T_i$. ∎

It is an open problem even in the class of games with linear incentives whether condition (5.10) implies (or even is equivalent to) that **1** is absorbing.

There is a relation to the concept of *spatial dominance* due to Hofbauer (1999). He considers a spatial model with populations of agents each of which is distributed along the real line, where agents move randomly on it and interact locally across populations. This can be modeled mathematically by a system of reaction-diffusion equations for the spatial distributions of actions. The local interaction is assumed to be governed by the myopic best response dynamics introduced by Gilboa and Matsui (1991). Each Nash equilibrium corresponds to a spatially homogeneous stationary action distribution. A Nash equilibrium $p^* \in [0, 1]^N$ is called *spatially dominant* if its basin of attraction contains an open set in the compact-open topology. If initially the population is close to p^* on a sufficiently large (but finite) interval, then it will converge to p^* everywhere. This implies that every game has at most one spatially dominant equilibrium. Hence this model provides a way of selecting a unique equilibrium for many important games; e.g., in 2×2 coordination games the risk-dominant equilibrium is spatially dominant. However, many games have no spatially dominant equilibrium at all.

The connection with the perfect foresight dynamics follows from the following fact, which holds for general binary supermodular games.

Lemma 5.3.2 (Hofbauer (1999), Lemma 1). *In a binary supermodular game,* **1** *is spatially dominant if there exists* $(r_i)_{i\in N} \in \mathbb{R}^N$ *such that*

$$d_i(\Phi(r_1 - r_i), \ldots, \Phi(r_N - r_i)) > 0 \tag{5.11}$$

for all $i \in I$*, where* Φ *is given by*

$$\Phi(z) = \begin{cases} e^z/2, & \text{for } z \leq 0, \\ 1 - e^{-z}/2 & \text{for } z > 0. \end{cases}$$

Lemma 5.3.3. $\int_0^\infty e^{-t}\Psi(z+t)dt = \Phi(z)$.

Proof. If $z \leq 0$,

$$\int_0^\infty e^{-t}\Psi(z+t)dt = \int_{-z}^\infty e^{-t}\{1 - e^{-(z+t)}\}dt = e^z/2,$$

and if $z > 0$,

$$\int_0^\infty e^{-t}\Psi(z+t)dt = \int_0^\infty e^{-t}\{1 - e^{-(z+t)}\}dt = 1 - e^{-z}/2,$$

as claimed. ∎

Lemma 5.3.4. *If d_i is linear, then the two conditions* (5.10) *and* (5.11) *are equivalent.*

Proof. By Lemma 5.3.3 and the linearity of d_i,

$$\begin{aligned}
&\int_0^\infty e^{-t} d_i(\Psi(r_1 - r_i + t), \dots, \Psi(r_N - r_i + t))dt \\
&= d_i\left(\int_0^\infty e^{-t}\Psi(r_1 - r_i + t)dt, \dots, \int_0^\infty e^{-t}\Psi(r_N - r_i + t)dt\right) \\
&= d_i(\Phi(r_1 - r_i), \dots, \Phi(r_N - r_i)),
\end{aligned}$$

which implies the claim. ∎

Combining Lemmas 5.3.1, 5.3.2, and 5.3.4 establishes the following implication.

Proposition 5.3.5. *In a binary supermodular game with linear incentives, if the strict Nash equilibrium* $\mathbf{1}$ *(or* $\mathbf{0}$*) is globally accessible for any small θ, then it is spatially dominant.*

Since a game has at most one spatially dominant equilibrium, this proposition implies in particular that in binary supermodular games with linear incentives, $\mathbf{0}$ and $\mathbf{1}$ cannot be simultaneously globally accessible (in contrast to the example of unanimity games in Subsection 5.2).

It is open whether the converse of Lemma 5.3.2 holds (no counter-example has been found so far). If it is true, it implies the equivalence between global accessibility under the perfect foresight dynamics with a small friction and spatial dominance of a strict Nash equilibrium for the class of binary supermodular game with linear incentives.

The linearity of the incentive functions d_i is crucial in the proof of Lemma 5.3.4. Indeed, the agreement between the selected equilibrium by spatial dominance and the one by the perfect foresight dynamics fails for nonlinear incentives. One class of examples are unanimity games in Subsection 5.2 for which the equilibrium with the higher Nash product is spatially dominant (see Hofbauer (1999)). Another example will be given in Subsection 5.4.

5.4 Binary Games with Invariant Diagonal

This subsection considers N-player binary supermodular games with invariant diagonal. A binary game is said to have an *invariant diagonal* if the incentive functions satisfy

$$d_1(p, \dots, p) = \cdots = d_N(p, \dots, p).$$

This class of games includes games with "equistable biforms" introduced in Selten (1995). We assume that d_i is nondecreasing in each p_j ($j \neq i$) so that the game is supermodular.

Denote by $D(p)$ the restriction of any d_i to the diagonal $p = p_1 = \cdots = p_N$. Observe that $D(p)$ is nondecreasing in p. This game has a potential function along the diagonal, which is defined by

$$v(p) = \int_0^p D(q)\, dq. \tag{5.12}$$

Proposition 5.4.1. *Suppose that the stage game is a binary supermodular game with an invariant diagonal. Let v be the potential function along the diagonal given by* (5.12)*. If $v(1) > v(0)$, then*

(a) *there exists $\bar{\theta} > 0$ such that $\mathbf{1}$ is globally accessible for all $\theta \in (0, \bar{\theta})$;*

(b) *$\mathbf{1}$ is absorbing for all $\theta > 0$.*

Proof. (a) Along the linear path ϕ from $\mathbf{0}$ to $\mathbf{1}$, which is given by $\phi_{i1}(t) = 1 - e^{-t}$ for all $i \in I$,

$$\begin{aligned} \Delta V_i^0(\phi)(0) &= \int_0^\infty e^{-s} D(1 - e^{-s})\, ds \\ &= \int_0^1 D(p)\, dp = v(1). \end{aligned}$$

Hence, if $v(1) > v(0) = 0$, then $\Delta V_i^0(\phi)(0) > 0$, implying that $\mathbf{1}$ is globally accessible for any small $\theta > 0$ by Proposition 5.1.2.

(b) If $v(1) > v(0) = 0$, then there exists $p < 1$ such that $v(p) > 0$. Take such a p and any perfect foresight path ϕ with $\phi_{i1}(0) \geq p$ for all $i \in I$. Note that $\phi_{i1}(t) \geq pe^{-t}$. Then,

$$\begin{aligned} \Delta V_i^\theta(\phi)(0) &= (1+\theta) \int_0^\infty e^{-(1+\theta)s} d_i((\phi_{i1}(s))_{i \in I})\, ds \\ &\geq (1+\theta) \int_0^\infty e^{-(1+\theta)s} D(pe^{-s})\, ds \\ &\geq \int_0^\infty e^{-s} D(pe^{-s})\, ds \\ &= \frac{1}{p} \int_0^p D(q)\, dq = \frac{v(p)}{p} > 0, \end{aligned}$$

where the first inequality follows from the monotonicity of D, and the second inequality follows from the stochastic dominance relation between the distributions on $[0, \infty)$ with the density functions $(1+\theta)e^{-(1+\theta)s}$ and e^{-s}. Hence, we have $\phi_{i1}(t) = 1 - (1 - \phi_{i1}(0))e^{-t}$ for all $t \geq 0$, and therefore, ϕ converges to $\mathbf{1}$, implying that $\mathbf{1}$ is absorbing (independently of $\theta > 0$). ▌

Similarly, if $v(0) > v(1)$, then $\mathbf{0}$ is globally accessible for any small $\theta > 0$ and absorbing for any $\theta > 0$. Therefore, for generic binary supermodular games with invariant diagonal, either $\mathbf{0}$ or $\mathbf{1}$ is a unique absorbing and globally accessible state for any small degree of friction (even though there may be other strict equilibria).

Remark 5.4.1. A state $x^* \in \prod_i \Delta(A_i)$ is *linearly stable* if for any $x \in \prod_i \Delta(A_i)$, the linear path from x to x^* is a perfect foresight path. One can verify that for binary supermodular games with invariant diagonal, if $v(1) > v(0)$, then $\mathbf{1}$ is linearly stable for any small degree of friction.

Remark 5.4.2. The above result extends to the class of games with "monotone diagonal". Let $D_i(p) = d_i(p, \dots, p)$ and $v_i(p) = \int_0^p D_i(q)\, dq$. It can be shown precisely in the same way that if $v_i(1) > v_i(0)$ for all $i \in I$, then $\mathbf{1}$ is globally accessible for any small $\theta > 0$ and absorbing for any $\theta > 0$.

Example 5.4.1. Consider the following three player game (see Figure 4): If all three players match their actions, then their payoffs are given by $u_i(\mathbf{0}) = a > 0$ and $u_i(\mathbf{1}) = d > 0$. For other action profiles, if i matches $i+1$ with action 0, then i's payoff is $b > 0$; if i matches $i+1$ with action 1, then i's payoff is $c > 0$; otherwise, all players receive payoff 0. Suppose here that $a > b$ and $d > c$. Note that this game is supermodular and has an invariant diagonal.[12] Proposition 5.4.1 implies that if $2a + b > c + 2d$, then $\mathbf{0}$ is absorbing and globally accessible for a small friction, while if $2a+b < c+2d$, then $\mathbf{1}$ is absorbing and globally accessible for a small friction.

The selection criterion based on MP-maximization, on the other hand, yields a limited prediction: One can verify that $\mathbf{0}$ is an MP-maximizer if and only if $a > c + d$, while $\mathbf{1}$ is an MP-maximizer if and only if $a + b < d$. For this game, the notion of u-dominance introduced by Kojima (2003) gives the same condition: $\mathbf{0}$ is u-dominant if and only if $a > c + d$, while $\mathbf{1}$ is u-dominant if and only if $a + b < d$.[13]

Spatial dominance selects a different equilibrium for this game as the equilibrium with the larger best response region on the diagonal is spatially dominant, i.e., $\mathbf{0}$ is spatially dominant if and only if $a + b > c + d$, while $\mathbf{1}$ is spatially dominant if and only if $a + b < c + d$.

[12]This game is not a (weighted) potential game, since it has a better reply cycle.

[13]In general, MP-maximization and u-dominance give different conditions.

	0	1
0	a, a, a	0, 0, b
1	0, b, 0	c, 0, 0

0

	0	1
0	b, 0, 0	0, c, 0
1	0, 0, c	d, d, d

1

Figure 4: Game with invariant diagonal

6 Conclusion

In this paper, we have considered the problem of equilibrium selection for the class of supermodular games by embedding the normal form game into the perfect foresight dynamics. Different strict Nash equilibria may have different stability properties when the degree of friction is small, based on which we can select a particular equilibrium. By exploiting the monotone structure of the dynamics, we have established equilibrium selection results for some classes of supermodular games. On the other hand, we have found that in certain unanimity games, no Nash equilibria have the required stability property.

For games with monotone potentials, a monotone potential maximizer is uniquely absorbing and globally accessible for any small degree of friction. This implies that our perfect foresight approach leads to the same prediction as that via the incomplete information approach, where Morris and Ui (2002) show that a monotone potential maximizer is robust to incomplete information (Kajii and Morris (1997)). For binary supermodular games with linear incentives, global accessibility under the perfect foresight dynamics implies spatial dominance of a strict Nash equilibrium (Hofbauer (1999)).

Appendix

A.1 Proof of Theorem 4.1

Suppose that a^* is an MP-maximizer with a monotone potential function v. In this subsection, we show that a^* is globally accessible for any small degree of friction. Let $A'_i \subset A_i$ denote a set of actions for player i that contains a^*_i. This set will be taken as $A^-_i = \{h \in A_i \,|\, h \leq a^*_i\}$ or $A^+_i = \{h \in A_i \,|\, h \geq a^*_i\}$. For the potential game $G'_v = (I, (A'_i)_{i \in I}, (v)_{i \in I})$ with the unique potential maximizer $a^* \in A'$, consider the following optimal control problem with a given initial state $z \in \prod_i \Delta(A'_i)$:

$$\text{maximize} \quad J(\phi) = \int_0^\infty e^{-\theta t} v(\phi(t))\, dt \tag{A.1a}$$

$$\text{subject to} \quad \phi \in \Phi'_z, \tag{A.1b}$$

where Φ'_z is the set of feasible paths defined on $\prod_i \Delta(A'_i)$ with the initial state z. The state z will be taken as $\min A = (0, \ldots, 0)$ or $\max A = (n_1, \ldots, n_N)$.

Lemma A.1.1. *There exists $\bar{\theta} > 0$ such that for any $\theta \in (0, \bar{\theta})$ and any $z \in \prod_i \Delta(A'_i)$, any optimal solution to the optimal control problem* (A.1) *converges to a^*.*

Proof. Follows from Lemma 1 in Hofbauer and Sorger (1999) and Lemmas 4.2 and 4.3 in Hofbauer and Sorger (2002) applied to the restricted potential game G'_v. ▮

Lemma A.1.2. *Let X be a nonempty compact set endowed with a partial order $\precsim$. Suppose that for all $x \in X$, the set $L_x = \{y \in X \mid y \precsim x\}$ is closed. Then X has a minimal element.*

Proof. Take any totally ordered subset X, and denote it by X'. Since $\{L_x\}_{x \in X'}$ consists of nonempty closed subsets of a compact set and has the finite intersection property, $L^* = \bigcap_{x \in X'} L_x$ is nonempty. Any element $x^* \in L^*$ is a lower bound of X' in X. Therefore, it follows from Zorn's lemma that X has a minimal element. ▮

Lemma A.1.3. *For any $z \in \prod_i \Delta(A'_i)$, there exist optimal solutions ϕ^- and ϕ^+ to the optimal control problem* (A.1) *such that*

$$\dot{\phi}_i^-(t) = \min BR_v^i(\phi^- | A'_i)(t) - \phi_i^-(t),$$
$$\dot{\phi}_i^+(t) = \max BR_v^i(\phi^+ | A'_i)(t) - \phi_i^+(t)$$

for all $i \in I$ and almost all $t \geq 0$.

Proof. Fix $z \in \prod_i \Delta(A'_i)$. We only show the existence of ϕ^-; the existence of ϕ^+ is shown similarly. Since the functional J is continuous on Φ'_z, the set of optimizers is a nonempty, closed, and hence compact subset of Φ'_z. Hence a minimal optimal solution (with respect to the order $\phi \precsim \psi$, defined by $\phi(t) \precsim \psi(t)$ for all $t \geq 0$) exists by Lemma A.1.2. Let ϕ^- be such a minimal solution.

Take any $i \in I$ and consider the feasible path ϕ_i given by $\phi_i(0) = z_i$ and

$$\dot{\phi}_i(t) = \min BR_v^i(\phi^- | A'_i)(t) - \phi_i(t)$$

for almost all $t \geq 0$. Since by Lemma 2.2, for almost all $t \geq 0$ there exists $\alpha_i(t)$ in the convex hull of $BR_v^i(\phi^- | A'_i)(t)$ such that

$$\dot{\phi}_i^-(t) = \alpha_i(t) - \phi_i^-(t),$$

we have $\phi_i \precsim \phi_i^-$. On the other hand, since ϕ_i is a best response to ϕ_{-i}^- for the game G_v' by construction, we have

$$J(\phi_i, \phi_{-i}^-) \geq J(\phi^-) = \max_{\psi \in \Phi_z'} J(\psi)$$

by Lemma 2.2, meaning that the path (ϕ_i, ϕ_{-i}^-) is also optimal. Hence, the minimality of ϕ^- implies $\phi_i^-(t) = \phi_i(t)$ for all $t \geq 0$. Therefore, we have

$$\dot{\phi}_i^-(t) = \min BR_v^i(\phi^- | A_i')(t) - \phi_i^-(t)$$

for almost all $t \geq 0$, as claimed. ▌

Lemma A.1.4. *There exists $\bar{\theta} > 0$ such that the following holds for all $\theta \in (0, \bar{\theta})$: there exists a feasible path ϕ^- such that $\phi^-(0) = \min A$, $\lim_{t\to\infty} \phi^-(t) = a^*$, and*

$$\dot{\phi}_i^-(t) = \min BR_v^i(\phi^- | A_i^-)(t) - \phi_i^-(t)$$

for all $i \in I$ and almost all $t \geq 0$; and there exists a feasible path ϕ^+ such that $\phi^+(0) = \max A$, $\lim_{t\to\infty} \phi^+(t) = a^*$, and

$$\dot{\phi}_i^+(t) = \max BR_v^i(\phi^+ | A_i^+)(t) - \phi_i^+(t)$$

for all $i \in I$ and almost all $t \geq 0$.

Proof. Follows from Lemmas A.1.1 and A.1.3. ▌

Proof of Theorem 4.1. Suppose that v is a monotone potential function for a^*. Take ϕ^- and ϕ^+ as in Lemma A.1.4. In what follows, we fix a sufficiently small $\theta > 0$ so that both ϕ^- and ϕ^+ converge to a^*.

Now fix any $x \in \prod_i \Delta(A_i)$. We want to construct a set of feasible paths from x to a^* such that the restriction of the best response correspondence β to this set has a fixed point. Define ψ^-, $\psi^+ \in \Phi_x$ by

$$\dot{\psi}_i^-(t) = \min BR_v^i(\phi^- | A_i^-)(t) - \psi_i^-(t),$$
$$\dot{\psi}_i^+(t) = \max BR_v^i(\phi^+ | A_i^+)(t) - \psi_i^+(t)$$

for all $i \in I$ and almost all $t \geq 0$. Note that $\phi^- \precsim \psi^- \precsim \psi^+ \precsim \phi^+$ as $\phi^-(0) \precsim x \precsim \phi^+(0)$, and thus $\lim_{t\to\infty} \psi^-(t) = \lim_{t\to\infty} \psi^+(t) = a^*$. Let $[\psi^-, \psi^+] = \{\phi \in \Phi_x \,|\, \psi^- \precsim \phi \precsim \psi^+\}$. We will show, as in the proof of Lemma 3.3, that $\tilde{\beta}(\phi) = \beta(\phi) \cap [\psi^-, \psi^+]$ is nonempty for any $\phi \in [\psi^-, \psi^+]$. Then, since $[\psi^-, \psi^+]$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\phi^* \in \tilde{\beta}(\phi^*) \subset [\psi^-, \psi^+]$. Since both ψ^- and ψ^+ converge to a^*, ϕ^* also converges to a^*.

Take any $\phi \in [\psi^-, \psi^+]$. Suppose first that the original game G is supermodular. Then, we have

$$\min BR_v^i(\phi^- | A_i^-)(t) \leq \max BR_{u_i}^i(\phi^- | A_i^-)(t) \leq \max BR_{u_i}^i(\phi | A_i^-)(t),$$

where the first inequality follows from the assumption that v is a monotone potential, and the second inequality follows from the supermodularity of u_i and Lemma 3.1. Similarly, we have

$$\max BR^i_v(\phi^+|A^+_i)(t) \geq \min BR^i_{u_i}(\phi^+|A^+_i)(t) \geq \min BR^i_{u_i}(\phi|A^+_i)(t).$$

Suppose next that the potential game G_v is supermodular. Then, we have

$$\min BR^i_v(\phi^-|A^-_i)(t) \leq \min BR^i_v(\phi|A^-_i)(t) \leq \max BR^i_{u_i}(\phi|A^-_i)(t),$$

where the first inequality follows from the supermodularity of v and Lemma 3.1, and the second inequality follows from the assumption that v is a monotone potential. Similarly, we have

$$\max BR^i_v(\phi^+|A^+_i) \geq \max BR^i_{u_i}(\phi|A^+_i)(t) \geq \min BR^i_{u_i}(\phi|A^+_i)(t).$$

Therefore, in each case, we have

$$\max BR^i_{u_i}(\phi|A^-_i)(t) \geq \min BR^i_v(\phi^-|A^-_i)(t),$$
$$\min BR^i_{u_i}(\phi|A^+_i)(t) \leq \max BR^i_v(\phi^+|A^+_i)(t)$$

for all $i \in I$ and all $t \geq 0$, so that there exists $h \in BR^i(\phi)(t)$ such that

$$\min BR^i_v(\phi^-|A^-_i)(t) \leq h \leq \max BR^i_v(\phi^+|A^+_i)(t).$$

Define

$$\tilde{F}_i(\phi)(t) = F_i(\phi)(t) \cap \left[\min BR^i_v(\phi^-|A^-_i)(t), \max BR^i_v(\phi^+|A^+_i)(t)\right],$$

where $F_i(\phi)(t)$ is defined in (2.5), and $[\alpha_i, \alpha'_i] = \{\alpha''_i \in \Delta(A_i) | \alpha_i \precsim \alpha''_i \precsim \alpha'_i\}$ denotes the order interval. Then the differential inclusion

$$\dot{\psi}(t) \in \tilde{F}(\phi)(t) - \psi(t), \qquad \psi(0) = x$$

has a solution ψ as in Remark 2.1. Since $\tilde{F}_i(\phi)(t) \subset F_i(\phi)(t)$, we have $\psi \in \beta(\phi)$. By the construction of ψ^-, ψ^+, and ψ, we have $\psi^- \precsim \psi \precsim \psi^+$. Thus, we have $\psi \in \tilde{\beta}(\phi) = \beta(\phi) \cap [\psi^-, \psi^+]$, implying the nonemptiness of $\tilde{\beta}(\phi)$. ∎

A.2 Proof of Theorem 4.2

Suppose that a^* is a strict MP-maximizer with a strict monotone potential function v. In this subsection, we show that a^* is absorbing for any degree of friction. For a nonempty set of actions $A'_i \subset A_i$ that contains a^*_i, consider the potential game $G'_v = (I, (A'_i)_{i \in I}, (v)_{i \in I})$.

Lemma A.2.1 (Hofbauer and Sorger (2002)). *Suppose that G'_v is a potential game with a unique potential maximizer $a^* \in A'$. Then, a^* is absorbing for all $\theta > 0$.*

Proof of Theorem 4.2. Suppose that v is a strict monotone potential function with the strict MP-maximizer a^*, and let $A_i^- = \{h \in A_i \,|\, h \le a_i^*\}$ and $A_i^+ = \{h \in A_i \,|\, h \ge a_i^*\}$. By Lemma A.2.1, a^* is absorbing in each of the restricted potential games $G_v^- = (I, (A_i^-)_{i\in I}, (v)_{i\in I})$ and $G_v^+ = (I, (A_i^+)_{i\in I}, (v)_{i\in I})$. Let

$$
\begin{aligned}
x_\varepsilon^- &= \varepsilon \min A + (1-\varepsilon) a^*, \\
x_\varepsilon^+ &= \varepsilon \max A + (1-\varepsilon) a^*
\end{aligned}
$$

for $\varepsilon \in [0, 1]$.

Choose a small $\varepsilon > 0$ so that any perfect foresight path from x_ε^- and x_ε^+ converges to a^* in G_v^- and G_v^+, respectively. Fix any state $x \in \prod_i \Delta(A_i)$ close to a^* satisfying

$$
x_\varepsilon^- \precsim x \precsim x_\varepsilon^+,
$$

and let ϕ^* be any perfect foresight path from x in the original game G. We want to show that ϕ^* converges to a^*.

Let ϕ^- and ϕ^+ be the feasible paths such that $\phi^-(0) = x_\varepsilon^-$, $\phi^+(0) = x_\varepsilon^+$, and

$$
\begin{aligned}
\dot{\phi}_i^-(t) &= \min BR_{u_i}^i(\phi^* | A_i^-)(t) - \phi_i^-(t), \\
\dot{\phi}_i^+(t) &= \max BR_{u_i}^i(\phi^* | A_i^+)(t) - \phi_i^+(t)
\end{aligned}
$$

for all $i \in I$ and almost all $t \ge 0$. By the construction of ϕ^- and ϕ^+, we have $\phi^- \precsim \phi^* \precsim \phi^+$. Note that ϕ^- and ϕ^+ are feasible paths on $\prod_i \Delta(A_i^-)$ and $\prod_i \Delta(A_i^+)$, respectively.

In the following, we find perfect foresight paths $\psi^{-,*}$ and $\psi^{+,*}$ from x_ε^- and x_ε^+ in G_v^- and G_v^+, respectively, satisfying $\psi^{-,*} \precsim \phi^-$ and $\phi^+ \precsim \psi^{+,*}$. Then, since a^* is absorbing both in G_v^- and in G_v^+, $\psi^{-,*}$ and $\psi^{+,*}$ converge to a^*, and thus, ϕ^* also converges to a^* as ϕ^* satisfies $\psi^{-,*} \precsim \phi^* \precsim \psi^{+,*}$. We only show the existence of $\psi^{-,*}$; the existence of $\psi^{+,*}$ is proved similarly.

Let $\tilde{\Phi}_{x_\varepsilon^-} = \{\psi \in \Phi_{x_\varepsilon^-} \,|\, \psi \precsim \phi^-\}$. Consider the dynamics with the stage game G_v^-. We will show that $\tilde{\beta}(\psi) = \beta(\psi) \cap \tilde{\Phi}_{x_\varepsilon^-}$ is nonempty for any $\psi \in \tilde{\Phi}_{x_\varepsilon^-}$. Then, since $\tilde{\Phi}_{x_\varepsilon^-}$ is convex and compact, it follows from Kakutani's fixed point theorem that there exists a fixed point $\psi^{-,*} \in \tilde{\beta}(\psi^{-,*}) \subset \tilde{\Phi}_{x_\varepsilon^-}$, as desired.

Take any $\psi \in \tilde{\Phi}_{x_\varepsilon^-}$. If G is supermodular, then

$$
\min BR_v^i(\psi | A_i^-)(t) \le \min BR_{u_i}^i(\psi | A_i^-)(t) \le \min BR_{u_i}^i(\phi^* | A_i^-)(t),
$$

where the first inequality follows from the assumption that v is a strict monotone potential, and the second inequality follows from the supermodularity of u_i and Lemma 3.1.

If G_v is supermodular, then

$$\min BR_v^i(\psi|A_i^-)(t) \leq \min BR_v^i(\phi^*|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

where the first inequality follows from the supermodularity of v and Lemma 3.1, and the second inequality follows from the assumption that v is a strict monotone potential.

Therefore, in each case, we have

$$\min BR_v^i(\psi|A_i^-)(t) \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t),$$

so that there exists $h \in BR_v^i(\psi|A_i^-)(t)$ such that

$$h \leq \min BR_{u_i}^i(\phi^*|A_i^-)(t).$$

Then, there exists a best response ψ' to ψ in the game G_v^- such that $\psi'(0) = x_\varepsilon^-$ and $\psi' \precsim \phi^-$, which can be constructed as in the proof of Theorem 4.1. ∎

A.3 Proofs for Subsection 4.3

Proof of Lemma 4.3. Let v be given as in the lemma. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. Observe that $v(h, \pi_i) = \sum_{a_{-i} \in A_{-i}} \pi_i(a_{-i}) v(h, a_{-i})$ is constant for all $h < a_i^*$, so that $\min BR_v^i(\pi_i|A_i^-)$ is either 0 or a_i^*. It is sufficient to consider the case where $a_i^* = \min BR_v^i(\pi_i|A_i^-)$.

Since

$$\begin{aligned} v(a_i^*, \pi_i) - v(0, \pi_i) &= \pi_i(a_{-i}^*) \cdot (1 - p_i) + \sum_{a_{-i} \neq a_{-i}^*} \pi_i(a_{-i}) \cdot (-p_i) \\ &= \pi_i(a_{-i}^*) - p_i, \end{aligned}$$

it follows that $a_i^* = \min BR_v^i(\pi_i|A_i^-)$ if and only if $\pi_i(a_{-i}^*) > p_i$.

Therefore, if a^* is a $\mathbf{p}$-dominant equilibrium, then $a_i^* \in BR_{u_i}^i(\pi_i|A_i^-)$, i.e., $a_i^* = \max BR_{u_i}^i(\pi_i|A_i^-)$; if a^* is a strict $\mathbf{p}$-dominant equilibrium, then $\{a_i^*\} = BR_{u_i}^i(\pi_i|A_i^-)$, i.e., $a_i^* = \min BR_{u_i}^i(\pi_i|A_i^-)$. ∎

Proof of Lemma 4.5. (a) Suppose that a^* is an LP-maximizer with a local potential function v. We show that if G or G_v has diminishing marginal returns, then a^* is an MP-maximizer with the same function v. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. We only show that $\max BR_v^i(\pi_i|A_i^-) \leq \max BR_{u_i}^i(\pi_i|A_i^-)$. Let $\overline{a}_i = \max BR_v^i(\pi_i|A_i^-)$. If $\overline{a}_i = \min A_i$, then $\overline{a}_i \leq \max BR_{u_i}^i(\pi_i|A_i^-)$ is satisfied. Then consider the case of $\overline{a}_i > \min A_i$.

Since a^* is an LP-maximizer, we have

$$\mu_i(h)\big(v(h+1, a_{-i}) - v(h, a_{-i})\big) \leq u_i(h+1, a_{-i}) - u_i(h, a_{-i})$$

for all $a_{-i} \in A_{-i}$, and therefore,

$$\mu_i(h)\big(v(h+1,\pi_i) - v(h,\pi_i)\big) \leq u_i(h+1,\pi_i) - u_i(h,\pi_i)$$

for all $h < \overline{a}_i$. On the other hand, we have

$$v(\overline{a}_i,\pi_i) - v(\overline{a}_i - 1,\pi_i) \geq 0$$

by the definition of $\overline{a}_i$.

In the case where G has diminishing marginal returns, since

$$u_i(\overline{a}_i,\pi_i) - u_i(\overline{a}_i - 1,\pi_i) \geq 0,$$

we have

$$u_i(\overline{a}_i,\pi_i) - u_i(h,\pi_i) \geq 0$$

for all $h < \overline{a}_i$, which implies that $\overline{a}_i \leq \max BR^i_{u_i}(\pi_i|A^-_i)$.

In the case where G_v has diminishing marginal returns, we have

$$v(\overline{a}_i - m + 1,\pi_i) - v(\overline{a}_i - m,\pi_i) \geq 0$$

for all $m = 1, \dots, \overline{a}_i$, so that

$$u_i(\overline{a}_i,\pi_i) - u_i(h,\pi_i) \geq 0$$

for all $h < \overline{a}_i$, which implies that $\overline{a}_i \leq \max BR^i_{u_i}(\pi_i|A^-_i)$.

(b) Suppose that a^* is a strict LP-maximizer with a strict local potential function v. We show that if G or G_v has diminishing marginal returns, then a^* is a strict MP-maximizer with the same function v. Fix any $i \in I$ and $\pi_i \in \Delta(A_{-i})$. We only show that $\min BR^i_v(\pi_i|A^-_i) \leq \min BR_{u_i}(\pi_i|A^-_i)$. Let $\underline{a}_i = \min BR^i_v(\pi_i|A^-_i)$. If $\underline{a}_i = \min A_i$, then $\underline{a}_i \leq \min BR^i_{u_i}(\pi_i|A^-_i)$ is satisfied. Then consider the case of $\underline{a}_i > \min A_i$.

Since a^* is a strict LP-maximizer, we have

$$v(h+1,a_{-i}) - v(h,a_{-i}) \leq \mu_i(h)\big(u_i(h+1,a_{-i}) - u_i(h,a_{-i})\big)$$

for all $a_{-i} \in A_{-i}$, and therefore,

$$v(h+1,\pi_i) - v(h,\pi_i) \leq \mu_i(h)\big(u_i(h+1,\pi_i) - u_i(h,\pi_i)\big)$$

for all $h < \underline{a}_i$. On the other hand, we have

$$v(\underline{a}_i,\pi_i) - v(\underline{a}_i - 1,\pi_i) > 0$$

by the definition of $\underline{a}_i$.

In the case where G has diminishing marginal returns, since

$$u_i(\underline{a}_i,\pi_i) - u_i(\underline{a}_i - 1,\pi_i) > 0,$$

we have

$$u_i(\underline{a}_i, \pi_i) - u_i(h, \pi_i) > 0$$

for all $h < \underline{a}_i$, which implies that $\underline{a}_i \leq \max BR^i_{u_i}(\pi_i | A_i^-)$.

In the case where G_v has diminishing marginal returns, we have

$$v(\underline{a}_i - m + 1, \pi_i) - v(\underline{a}_i - m, \pi_i) > 0$$

for all $m = 1, \ldots, \underline{a}_i$, so that

$$u_i(\underline{a}_i, \pi_i) - u_i(h, \pi_i) > 0$$

for all $h < \underline{a}_i$, which implies that $\underline{a}_i \leq \max BR^i_{u_i}(\pi_i | A_i^-)$. ∎

A.4 Proofs for Section 5

We will need the following lemma.

Lemma A.4.1. *For all $i \in I$ and all $t \geq 0$,*

(a) *for any $\mathbf{T} \in \mathbb{R}^N_+$, $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$ is decreasing in $\theta \geq 0$,*

(b) *for any $\mathbf{T} \in \bar{\mathbb{R}}^N_+$ with $S = \{i \in I \,|\, T_i \neq \infty\}$, $\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(t)$ is nondecreasing in $\theta \geq 0$, and is increasing in $\theta \geq 0$ if $d_i(\mathbf{1}) > d_i(\mathbf{p})$ for $\mathbf{p} = (p_j)_{j \in I}$ such that $p_j = 0$ for $j \in S$ and $p_j = 1$ for $j \notin S$.*

This lemma is a consequence of the stochastic dominance relation among distributions on $[t, \infty)$ induced by discount factors: the distribution on $[t, \infty)$ with density function $(1 + \theta)e^{-(1+\theta)(s-t)}$ stochastically dominates the one with density function $(1 + \theta')e^{-(1+\theta')(s-t)}$ for $0 \leq \theta < \theta'$. The statements follow from the facts that $d_i((\phi^{\mathrm{u}}_{\mathbf{T}})_1(s))$ is nondecreasing in $s \geq 0$ and increasing in $s \geq \max_{j \in I} T_j$, and that $d_i((\psi^{\mathrm{d}}_{\mathbf{T}})_1(s))$ is nonincreasing in $s \geq 0$, and decreasing in $s \geq \max_{j \in S} T_j$ if $d_i(\mathbf{1}) > d_i(\mathbf{p})$.

We first prove the global accessibility results.

Proof of Proposition 5.1.1. "If" part: Suppose that there exists $\mathbf{T} = (T_i)_{i \in I}$ such that for all i,

$$\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0.$$

Since $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$ is increasing in t by Lemma A.4.1, $\Delta V_i^\theta(\phi^{\mathrm{u}}_{\mathbf{T}})(t) \geq 0$ holds for all $i \in I$ and all $t \geq T_i$. This implies that $\phi^{\mathrm{u}}_{\mathbf{T}}$ satisfies

$$(\dot{\phi}^{\mathrm{u}}_{\mathbf{T}})_{i1}(t) > -(\phi^{\mathrm{u}}_{\mathbf{T}})_{i1}(t) \Rightarrow 1 = \max BR^i(\phi^{\mathrm{u}}_{\mathbf{T}})(t)$$

for almost all $t \in \mathbb{R}_+$. Therefore, by Lemma 3.3 (with $G = G'$), for any $x \in \prod_i \Delta(A_i)$ there exists a perfect foresight path ϕ^* from x such that $\phi^{\mathrm{u}}_{\mathbf{T}} \precsim \phi^*$. Since $\phi^{\mathrm{u}}_{\mathbf{T}}$ converges to $\mathbf{1}$, ϕ^* also converges to $\mathbf{1}$. This implies that $\mathbf{1}$ is globally accessible.

"Only if" part: Suppose that **1** is globally accessible, so that there exists a perfect foresight path ϕ such that $\phi(0) = \mathbf{0}$ and $\lim_{t\to\infty} \phi(t) = \mathbf{1}$. Take such a perfect foresight path ϕ and let

$$T_i = \inf\{t \geq 0 \,|\, \dot{\phi}_{i1}(t) > -\phi_{i1}(t)\}$$

for each $i \in I$. Note that $T_i < \infty$ for all $i \in I$.

For $\mathbf{T} = (T_i)_{i\in I}$ defined above, define $\phi^{\mathrm{u}}_{\mathbf{T}}$ as in (5.1). Since $\phi \precsim \phi^{\mathrm{u}}_{\mathbf{T}}$, we have

$$\mathit{\Delta} V^{\theta}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq \mathit{\Delta} V^{\theta}_i(\phi)(T_i) \geq 0$$

due to the supermodularity. ▮

Proof of Proposition 5.1.2. "If" part: Take a $\mathbf{T} = (T_i)_{i\in I} \in \mathbb{R}^N_+$ such that

$$\mathit{\Delta} V^{0}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$. Since $\mathit{\Delta} V^{\theta}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i)$ is continuous in θ, there exists $\bar{\theta} > 0$ such that for all $\theta \in (0, \bar{\theta})$,

$$\mathit{\Delta} V^{\theta}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$, implying that **1** is globally accessible for all $\theta \in (0, \bar{\theta})$ by Proposition 5.1.1.

"Only if" part: Suppose that **1** is globally accessible for a small $\theta > 0$. Then, by Proposition 5.1.1 there exists $\mathbf{T}$ such that

$$\mathit{\Delta} V^{\theta}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) \geq 0$$

for all $i \in I$. Since $\mathit{\Delta} V^{\theta}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i)$ is strictly decreasing in θ by Lemma A.4.1, it follows that

$$\mathit{\Delta} V^{0}_i(\phi^{\mathrm{u}}_{\mathbf{T}})(T_i) > 0$$

for all $i \in I$. ▮

Next we prove the absorption results. For Proposition 5.1.3, we show the following.

Lemma A.4.2. *Let $\theta > 0$ be given. The state* **1** *is absorbing for θ if and only if for any $\mathbf{T} = (T_i)_{i\in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty, there exists $i \in S$ such that*

$$\mathit{\Delta} V^{\theta}_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) > 0.$$

Proof. "If" part: Note first that by the uniform continuity of d_i, for each positive integer m, there exists $\varepsilon^m > 0$ such that for any $\mathbf{p} = (p_j)_{j\in I}$, $\mathbf{q} = (q_j)_{j\in I} \in [0,1]^N$ with $p_j \geq q_j - \varepsilon^m$ for all $j \in I$, we have

$$d_i(\mathbf{p}) \geq d_i(\mathbf{q}) - \frac{1}{m}$$

for all $i \in I$. Then, for any feasible paths ϕ and ψ such that $\phi_{j1}(t) \geq \psi_{j1}(t) - \varepsilon^m$ for all $j \in I$ and $t \geq 0$, we have

$$\mathit{\Delta} V_i^{\theta}(\phi)(t) \geq \mathit{\Delta} V_i^{\theta}(\psi)(t) - \frac{1}{m}$$

for all $i \in I$ and $t \geq 0$.

Suppose that $\mathbf{1}$ is not absorbing. Take any positive integer m and the corresponding ε^m given above. There exist $x \in \prod_i \Delta(A_i)$ with $x_{1i} > 1 - \varepsilon^m$ and a perfect foresight path ϕ^m with $\phi^m(0) = x$ that does not converge to $\mathbf{1}$. Take any such perfect foresight path ϕ^m.

Define

$$T_i^m = \inf\{t \geq 0 \,|\, \dot{\phi}_{i1}^m(t) < 1 - \phi_{i1}^m(t)\},$$

and $S^m = \{i \in I \,|\, T_i^m \neq \infty\}$. Note that S^m is nonempty as ϕ^m does not converge to $\mathbf{1}$. Since ϕ^m is a perfect foresight path and $\mathit{\Delta} V_i^{\theta}(\phi^m)(t)$ is continuous in t, we must have

$$\mathit{\Delta} V_i^{\theta}(\phi^m)(T_i^m) \leq 0 \tag{A.2}$$

for $i \in S^m$.

Define $\tilde{\mathbf{T}}^m = (\tilde{T}_i^m)_{i\in I}$ by $\tilde{T}_i^m = T_i^m - \min_j T_j^m$. Take feasible paths $\psi_{\mathbf{T}^m}^{\mathrm{d}}$ and $\psi_{\tilde{\mathbf{T}}^m}^{\mathrm{d}}$ as in (5.2) and (5.3).

Observe that

$$\phi_{i1}^m(t) \geq (\psi_{\mathbf{T}^m}^{\mathrm{d}})_{i1}(t) - \varepsilon^m$$

for all $i \in I$ and $t \geq 0$. It follows from the definition of ε^m that

$$\begin{aligned}\mathit{\Delta} V_i^{\theta}(\phi^m)(T_i^m) &\geq \mathit{\Delta} V_i^{\theta}(\psi_{\mathbf{T}^m}^{\mathrm{d}})(T_i^m) - \frac{1}{m}\\ &= \mathit{\Delta} V_i^{\theta}(\psi_{\tilde{\mathbf{T}}^m}^{\mathrm{d}})(\tilde{T}_i^m) - \frac{1}{m},\end{aligned}$$

so that

$$\mathit{\Delta} V_i^{\theta}(\psi_{\tilde{\mathbf{T}}^m}^{\mathrm{d}})(\tilde{T}_i^m) - \frac{1}{m} \leq 0 \tag{A.3}$$

for any $i \in S^m$ by (A.2).

Now let $m \to \infty$. Since the set of feasible paths Φ is compact, $\{\psi_{\tilde{\mathbf{T}}^m}\}_{m=1}^{\infty}$ has a convergent subsequence $\{\psi_{\tilde{\mathbf{T}}^{m(k)}}^{\mathrm{d}}\}_{k=1}^{\infty}$ with a limit, which is written as

$\psi^{\mathrm{d}}_{\mathbf{T}}$ for some $\mathbf{T} \in \bar{\mathbb{R}}^N_+$. Note that $\lim_{k\to\infty} \tilde{\mathbf{T}}^{m(k)} = \mathbf{T}$. Since $\min_{i\in I} \tilde{T}^m_i = 0$ for all m, $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty due to the finiteness of I. Moreover, since ΔV^θ_i is continuous on $\Phi \times \mathbb{R}_+$, we have

$$\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$$

for any $i \in S$ by (A.3).

"Only if" part: Suppose that there exists $\mathbf{T} = (T_i)_{i\in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty, and

$$\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$$

for any $i \in S$. Since $\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(t)$ is decreasing in t, $\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(t) \leq 0$ holds for all i and all $t \geq T_i$. This implies that $\psi^{\mathrm{d}}_{\mathbf{T}}$ satisfies

$$(\dot{\psi}^{\mathrm{d}}_{\mathbf{T}})_{i0}(t) > -(\psi^{\mathrm{d}}_{\mathbf{T}})_{i0}(t) \Rightarrow 0 = \min BR^i(\psi^{\mathrm{d}}_{\mathbf{T}})(t)$$

for almost all $t \in \mathbb{R}_+$. Therefore, by Lemma 3.3 (with $G = G'$), there exists a perfect foresight path ϕ^* from $\mathbf{1}$ such that $\phi^* \precsim \psi^{\mathrm{d}}_{\mathbf{T}}$. Since $\psi^{\mathrm{d}}_{\mathbf{T}}$ is such that $(\psi^{\mathrm{d}}_{\mathbf{T}})_{i1}(t) \to 0$ as $t \to \infty$ for $i \in S$, it follows that ϕ^* does not converge to $\mathbf{1}$. Therefore, $\mathbf{1}$ is not absorbing. ▌

Proof of Proposition 5.1.3. By Lemma A.4.2, we only need to show that if for any $\mathbf{T}$ such that $S = \{i \in I | T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, there exists $i \in S$ such that $\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) > 0$, then the same condition holds for any $\mathbf{T}$ such that $\mathbf{0}_S$ is not necessarily a Nash equilibrium. Suppose not, and choose $\mathbf{T}$ and S such that S is maximal among all subsets that violate the condition. Then $\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$ for any $i \in S$. Since $\mathbf{0}_S$ is not a Nash equilibrium, (i) there exists $j \in S$ such that $d_j(\mathbf{p}_S) > 0$, or (ii) there exists $j \notin S$ such that $d_j(\mathbf{p}_S) < 0$, where $\mathbf{p}_S = (p_1, \ldots, p_N)$ such that $p_i = 0$ for $i \in S$ and $p_i = 1$ for $i \notin S$. In case (i), however, by the supermodularity,

$$d_j(\mathbf{p}_S) \leq \Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0,$$

which is a contradiction. Therefore, case (ii) holds. Choose such a j.

Define $\mathbf{T}' = (T'_1, \ldots, T'_N)$ by $T'_i = T_i$ for $i \neq j$ and T'_j as a sufficiently large but finite number. Then $\psi^{\mathrm{d}}_{\mathbf{T}'} \precsim \psi^{\mathrm{d}}_{\mathbf{T}}$, so that

$$\Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_i) \leq \Delta V^\theta_i(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \leq 0$$

for $i \in S$ by the supermodularity. Moreover, since $\Delta V^\theta_j(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_j)$ converges to $d_j(p_1, \ldots, p_N) < 0$ as $T'_j \to \infty$, we have

$$\Delta V^\theta_j(\psi^{\mathrm{d}}_{\mathbf{T}'})(T'_j) < 0.$$

This contradicts the maximality of S. ▌

Proposition 5.1.4 follows immediately from the following.

Lemma A.4.3. *The following conditions are equivalent:*

(a) $\mathbf{1}$ *is absorbing for all* $\theta > 0$*;*

(b) *there exists* $\bar{\theta}$ *such that* $\mathbf{1}$ *is absorbing for all* $\theta \in (0, \bar{\theta})$*;*

(c) *for any* $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}^N_+$ *such that* $S = \{i \in I \,|\, T_i \neq \infty\}$ *is nonempty and* $\mathbf{0}_S$ *is a Nash equilibrium of the stage game, there exists* $i \in S$ *such that*

$$\Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0.$$

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (c): Suppose that there exists $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty, $\mathbf{0}_S$ is a Nash equilibrium, and $\Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) < 0$ for all $i \in S$. Fix such a $\mathbf{T}$. Since $\Delta V^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i)$ is continuous in θ, there exists $\hat{\theta} > 0$ such that for all $\theta \in (0, \hat{\theta})$,

$$\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) < 0$$

for all $i \in S$, implying that $\mathbf{1}$ is not absorbing for any $\theta \in (0, \hat{\theta})$ by Proposition 5.1.3.

(c) $\Rightarrow$ (a): Suppose (c). For each $\mathbf{T} = (T_i)_{i \in I} \in \bar{\mathbb{R}}^N_+$ such that $S = \{i \in I \,|\, T_i \neq \infty\}$ is nonempty and $\mathbf{0}_S$ is a Nash equilibrium, take $i \in S$ as in (c). Let $\mathbf{p} = (p_j)_{j \in I} \in [0, 1]^N$ be such that $p_j = 0$ if $j \in S$ and $p_j = 1$ if $j \notin S$.

By the monotonicity of d_i, we have $d_i(\mathbf{1}) \geq d_i(\mathbf{p})$. If $d_i(\mathbf{1}) = d_i(\mathbf{p})$, then for any $\theta > 0$,

$$\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) = d_i(\mathbf{1}) > 0$$

by the monotonicity of d_i. If $d_i(\mathbf{1}) > d_i(\mathbf{p})$, then $\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i)$ is increasing in θ by Lemma A.4.1, so that for any $\theta > 0$,

$$\Delta V_i^\theta(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) > \Delta V_i^0(\psi^{\mathrm{d}}_{\mathbf{T}})(T_i) \geq 0.$$

It follows that $\mathbf{1}$ is absorbing for all $\theta > 0$ by Proposition 5.1.3. ∎

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