Naive fictitious play in an evolutionary environment.

Michał Ramsza

Department of Mathematical Economics
Warsaw School of Economics
Aleje Niepodleglosci 162
02-554, Warsaw, Poland
michal.ramsza@sgh.waw.pl

Center for Economic Learning and Social Evolution (ELSE)
& Department of Mathematics
University College London
Gower Street
London WC1E 6BT, U.K.
m.ramsza@ucl.ac.uk

Address for correspondence
Dr Michal Ramsza
Department of Mathematics
University College London
Gower Street
London WC1E 6BT, U.K.

May 26, 2006

Abstract

A fictitious play algorithm with a unit memory length within an evolutionary environment is considered. The aggregate average behavior model is proposed and analyzed. The existence, uniqueness and global asymptotic stability of equilibrium is proved for games with a cycling property. Also, a noisy version of the algorithm is considered, which gives rise to a system with a unique, globally asymptotically stable equilibrium for any game.

1 Introduction

A fictitious play algorithm with ever growing memory, denoted FP(∞), was introduce in [Bro51]. Once a memory of FP(∞) is truncated, the class of finite fictitious play algorithms is obtained. The fictitious play algorithm with a memory of length M is denoted FP(M). In this paper we specialize even further and consider a naive fictitious play algorithm, i.e. a finite memory fictitious play algorithm with a unit length, FP(1).

The naive fictitious play algorithm is a well known best-reply algorithm used, among others, in a Cournot's duopoly model, cf. [FL98, MWG95]. It also gave rise to best-reply dynamics[1] of the form $\dot{x} = \beta(x) - x$, where $\beta(\cdot)$ is a best-reply correspondence.

We analyze the FP(1) algorithm in an evolutionary environment with micro-inertia. Therefore, the information available to players is distributed, heterogeneous and delayed. We investigate aggregate average behavior of such a population of players, also with noise.

Section 2 gives details of the underlying multi-agent model. Section 3 provides a model of aggregate average behavior of a population of players using the FP(1) algorithm. Equilibrium is defined and studied in Section 4. Section 5 is concerned with dynamic properties of a system and contains results on uniqueness and asymptotic stability of equilibrium. A noisy version of FP(1) algorithm is studied in Section 6. Finally, a continuous time limit of dynamics is considered in Section 7. We conclude in Section 8.

2 Multi-agent model

We consider only normal form games. A game is a triple $(\mathcal{I}, \mathcal{S}, \mathcal{G})$, where $\mathcal{I}$ is a set of players, $\mathcal{S} = \{1, \dots, S\}$ is a set of pure strategies and $\mathcal{G}$ is a payoff function.

[1] In fact, it is differential inclusion.

These games are played within a certain environment. We are concerned with an evolutionary-like single population environment only. That is, we assume the following properties of the environment. Firstly, there are only bilateral interactions in the environment and these interactions are random, i.e. players are matched into pairs at random. Secondly, the environment is fully anonymous, i.e. players are anonymous to each other and to the environment (in particular no specific information is available to the environment while creating random pairs). Finally, we assume micro-inertia, a feature we define and discuss later on.

All the above assumptions restrict the set of games we are interested in. Namely, we are concerned two-player symmetric normal form games. Therefore, we have $\mathcal{I} = \{1, 2\}$ and a payoff function may be summarized as a matrix $G = \{g_{ij}\}$. We do not make any assumptions on the number of pure strategies except that it is a finite number.

The system comprises $N > 2$ players. At each round $t \in \mathbb{N}$ only K players are chosen to play a game. It is assumed that K is an even number such that $K < N$. The probability that a given player is chosen to play in a round is K/N.

We refer to the assumption $K < N$ as micro-inertia. Even if $K = N$, there may be some inertia at the aggregate level. Although it is still possible, it is not so in general. The assumption that $K < N$ excludes the possibility of extremely rapid shifts of the whole population.

Each player uses the extreme version of finite memory fictitious play, i.e. we assume that a player remembers only the last strategy he has played against (or that a memory has a unit length). This algorithm is the well known best-reply algorithm (as in Cournot duopoly model, cf. [FL98]). Because of the evolutionary nature of the environment, the information available to players is heterogeneous and delayed.

3 Model of average behavior

The model of average behavior consists of a Markov chain describing the evolution of players' histories and an operator providing a distribution of strategies given a distribution of histories. This in turn, defines transition probabilities of histories, and so on. The model of evolution of a distribution of histories $\mu \in \triangle_S$ is the most simple version of a de Bruijn graph, where the set of nodes is just $\mathcal{S}$ and transition probabilities are given by $\mathrm{P}(x)$, where $x \in \triangle_S$ is the distribution of strategies in a population.

Definition 1 *For any $x \in \triangle_S$ the transition matrix $P(x)$ is given by*

$$P(x) = \left(1 - \frac{K}{N}\right) I_S + \frac{K}{N} \begin{bmatrix} x_1 & \cdots & x_S \\ \vdots & & \vdots \\ x_1 & \cdots & x_S \end{bmatrix},$$

where I_S is an $S \times S$ identity matrix.

Once a distribution of histories μ is given, a distribution of strategies x is determined through a best-reply correspondence.

Definition 2 *The matrix $H = [h_{ij}]$ is defined through a best-reply correspondence, i.e. h_{ij} is the probability of playing the j-th pure strategy if the i-th pure strategy was observed.*

The matrix H is a stochastic matrix with a very specific structure. If $\mathrm{BR}(i) = \{j\}$ then $h_{ij} = 1$ and $h_{ij'} = 0$ for all $j' \neq j$. If $\mathrm{BR}(i)$ is not a singleton then we have a uniform distribution on a set $\mathrm{BR}(i)$.

Example 1 Consider a game with a payoff matrix G

$$G = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix},$$

where $a, b > 0$. The best-reply matrix H is given by

$$\mathrm{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

regardless of any particular values of a and b.

Example 2 Consider a game with a payoff matrix G

$$G = \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix},$$

where $a < b$. The best-reply matrix H is given by

$$\mathrm{H} = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix},$$

regardless of any particular values of a and b.

The evolution of the system is described through a system of equations:

$$\mu_{t+1}^T = \mu_t^T \mathrm{P}(x_t), \quad (1)$$

$$x_{t+1}^T = \mu_{t+1}^T \mathrm{H}, \quad (2)$$

where μ_0 is a given initial condition. The first equation provides the definition of evolution, while the second one is an equilibrium condition. It is clear that the system (1)–(2) is well defined, i.e. for any t, $(\mu_t, x_t) \in \triangle_S \times \triangle_S$.

4 Equilibrium

Having described the evolution of the system we define its equilibrium as any fixed point of the dynamics (1)–(2).

Definition 3 *A pair (μ_e, x_e) is an equilibrium if and only if it is a fixed point of the dynamics (1)–(2), i.e. it satisfies the following system of equations*

$$\mu_e^T = \mu_e^T P(x_e), \quad (3)$$

$$x_e^T = \mu_e^T H. \quad (4)$$

The set of all equilibria of a game is denoted $\triangle_{\mathrm{FP}(1)}$.

Proposition 1 *For any two-player symmetric game $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ there exists an* FP(1)*-equilibrium, i.e. $\triangle_{\mathrm{FP}(1)} \neq \emptyset$.*

Proof. The function

$$\triangle_S \times \triangle_S \ni \begin{pmatrix} \mu \\ x \end{pmatrix} \mapsto \begin{pmatrix} \mu^T \mathrm{P}(x) \\ \mu^T \mathrm{H} \end{pmatrix} \in \triangle_S \times \triangle_S$$

is continuous and defined on a compact and convex set $\triangle_S \times \triangle_S$. Hence, by Brouwer's fixed point theorem, there exists an equilibrium. ∎

Proposition 2 *For any two-player symmetric game $(\mathcal{I}, \mathcal{S}, \mathcal{G})$, the set of all equilibrium distributions of strategies is an intersection of a simplex $\triangle_S$ and a kernel of linear operator $H - I$, i.e. it is given by a system of linear equalities*

$$x^T = x^T H \quad \wedge \quad \langle x | \mathbf{1} \rangle = 1, \tag{5}$$

where $\mathbf{1}$ *is a vector of ones.*

Proof. The second equation, $\langle x|\mathbf{1}\rangle = 1$, is obvious. Suppose $(\mu, x) \in \triangle_{\mathrm{FP}(1)}$. Using (3), we have

$$\begin{aligned} \mu^T &= \mu^T \mathrm{P}(x) = \mu^T \left(\left(1 - \frac{K}{N}\right) I_S + \frac{K}{N} \begin{bmatrix} x_1 & \cdots & x_S \\ \vdots & & \vdots \\ x_1 & \cdots & x_S \end{bmatrix} \right) \\ &= \left(1 - \frac{K}{N}\right) \mu^T + \frac{K}{N} x^T. \end{aligned}$$

Therefore, at any equilibrium, there is $\mu = x$. Substituting μ with x in (4) yields the result. ∎

The above proposition is quite revealing. The equilibrium does not depend on the size of the system, N, or the micro-inertia K/N. The only element defining an equilibrium is a best-reply matrix. The structure of the set of FP(1)-equilibria is linear, i.e. there is a unique equilibrium or a continuum number of equilibria[2]. This set is always compact and convex.

[2]Even if there is a continuum number of equilibria, it does not mean that any distribution is an equilibrium. This depends on the dimension of a kernel of $\mathrm{H} - I$.

The convexity of the set of FP(1)-equilibria clearly prevents this concept from being anything like the concept of Nash equilibrium. We have the following examples.

Example 3 Consider the game from Example 1. It is trivial to see that $\triangle_{\mathrm{FP}(1)} = \triangle_2 \times \triangle_2$, and so any distribution x is an FP(1)-equilibrium, regardless of values $a, b > 0$.

Example 4 Consider a game with a payoff matrix G

$$G = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},$$

where $a, b > 0$. The best-reply matrix H is given by

$$\mathrm{H} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

regardless of any particular values of a and b. The only distribution x satisfying system (5) is $x_e^T = (1/2, 1/2)$, which is not a Nash equilibrium.

5 Dynamics

We start with a derivation of dynamics in terms of the distribution of histories μ. We have

$$\begin{aligned}
\mu_{t+1}^T &= \mu_t^T \mathrm{P}(x_t) = \left(1 - \frac{K}{N}\right)\mu_t^T + \frac{K}{N} x_t^T \\
&= \left(1 - \frac{K}{N}\right)\mu_t^T + \frac{K}{N}\mu_t^T \mathrm{H} \\
&= \mu_t^T \left(\left(1 - \frac{K}{N}\right) I_S + \frac{K}{N}\mathrm{H}\right) = \mu_t^T \Psi.
\end{aligned}$$

In a similar way, we have

$$\begin{aligned}
x_{t+1}^T &= \mu_{t+1}^T \mathrm{H} = \left(\left(1 - \frac{K}{N}\right)\mu_t^T + \frac{K}{N} x_t^T\right)\mathrm{H} \\
&= \left(1 - \frac{K}{N}\right) x_t^T + \frac{K}{N} x_t^T \mathrm{H} \\
&= x_t^T \left(\left(1 - \frac{K}{N}\right) I_S + \frac{K}{N}\mathrm{H}\right) = x_t^T \Psi.
\end{aligned}$$

The matrix Ψ is a stochastic matrix.

Consider a two-player symmetric game $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ with a payoff matrix G and a set of pure strategies $\mathcal{S} = \{1, \ldots, S\}$. The matrix Ψ defines a directed graph, denoted $\mathcal{H}$, with a set of nodes $\mathcal{S}$ and a set of eges $\mathcal{E}$. A pair $(i, j) \in \mathcal{E}$ if and only if $j \in \mathrm{BR}(i)$. Together with the probabilities defined in Ψ, the graph $\mathcal{H}$ is a homogeneous Markov chain.

Proposition 3 *For any two-player symmetric game $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ the sequence x_t converges to a set $\triangle_{\mathrm{FP}(1)}$ of all* FP(1)*-equilibria.*

Proof. The set of states $\mathcal{S}$ of a Markov chain Ψ can be partitioned, cf. [GS01, Bre99], into a union of a set of transient states and irreducible blocks. Each of the irreducible blocks is ergodic, since it is also aperiodic and positive recurrent. Therefore, the sequence of measures x_t will converge to a measure that satisfies $x^T = x^T \Psi$, hence $x^T = x^T \mathrm{H}$. ■

Proposition 4 *For any two-player symmetric game $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ a set of transient states of a Markov chain Ψ is equal to a set of pure strategies removed in a course of iterated elimination of dominated strategies. In particular, if a game is solvable by an iterated elimination of dominated strategies, there is a unique globally asymptotically stable equilibrium which corresponds to a Nash equilibrium.*

Proof. Clearly, if a strategy i is removed in an iterated elimination of dominated strategies, it is a best reply only to strategies that are removed before. Working backwards, we end up with strategies that are removed as the first ones, i.e. they are not a best reply to any strategy (there are no incoming edges to these states). Therefore, the strategy i cannot be recurrent. If a strategy i is transient, it means there is no path from i back to i, that is i is removed by an iterated elimination of dominated strategies.

If a game is solvable by an iterated elimination of dominated strategies, there is a single recurrent state i. Clearly, $x_{t,i} \to 1$. Also, it has to be a Nash equilibrium, since $\{i\} = \mathrm{BR}(i)$. ■

We say that a game $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ has a *cycling property* if the graph $\mathcal{H}$ contains a cycle comprising all nodes.

Proposition 5 *If a two-player symmetric game* $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ *has a cycling property, then there exists the unique globally asymptotically stable* FP(1)-*equilibrium* $x_e \in \mathrm{int}(\triangle_S)$.

Proof. Clearly, if a game has a cycling property, then a best-reply matrix H is irreducible. Therefore, a matrix Ψ is irreducible, aperiodic and positive recurrent, hence ergodic. It follows that there is the unique stationary measure μ_e that charges all states and for any initial measure μ_0, a sequence $\mu_{t+1}^T = \mu_t^T \Psi$ converges to this stationary measure. ■

Proposition 5 says that any game with a cycling property fancies a unique globally asymptotically stable equilibrium. It is the very same property that allows construction of classical non-convergent fictitious play examples in an environment with two players. The most well-known examples are the following.

Example 5 Consider a game with a payoff matrix G, cf. [FK93],

$$G = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix},$$

where $a, b > 0$. This game has a cycling property since $\{2\} = \mathrm{BR}(1)$, and $\{1\} = \mathrm{BR}(2)$. Therefore, there is the unique globally asymptotically stable FP(1)-equilibrium, which is $x_e^T = (1/2, 1/2)$, cf. Example 4.

Example 6 Consider a game with a payoff matrix G, cf. [Sha64, Wei96, FL98],

$$G = \begin{bmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{bmatrix},$$

where $a, b, c > 0$. We have $\mathrm{BR}(1) = \{3\}$, $\mathrm{BR}(3) = \{2\}$ and $\mathrm{BR}(2) = \{1\}$, hence this game has a cycling property. The unique globally asymptotically stable equilibrium, regardless of particular values of parameters a, b and c, is $x_e^T = (1/3, 1/3, 1/3)$.

Example 7 Consider a game with a payoff matrix G, cf. [HS98],

$$G = \begin{bmatrix} 0 & 0 & -1 & a \\ a & 0 & 0 & -1 \\ -1 & a & 0 & 0 \\ 0 & -1 & a & 0 \end{bmatrix},$$

where $a \in \mathbb{R}$. Graphs $\mathcal{H}$ for different values of a are depicted in Figure 1. In all cases this game has a cycling property. The unique globally asymptotically stable equilibrium, regardless of a particular value of parameter a, is $x_e^T = (1/4, 1/4, 1/4, 1/4)$.

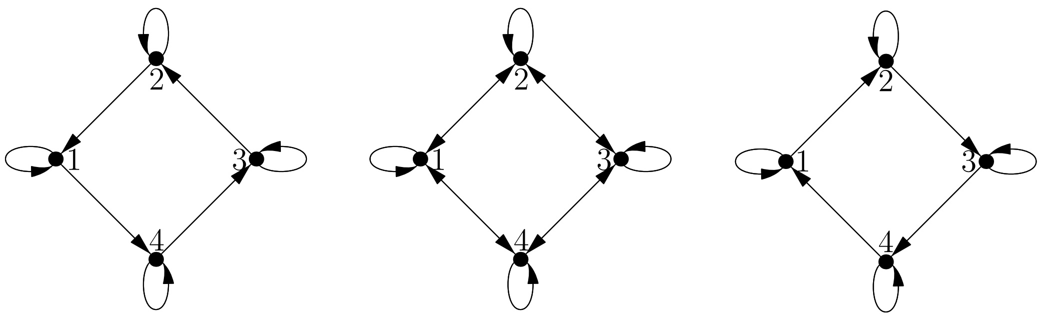

Figure 1: Graphs $\mathcal{H}$ for a game proposed in [HS98] for (from left to right) $a < 0$, $a = 0$ and $a > 0$.

6 Noisy naive fictitious play

We consider two ways of introducing noise into the system. Noise can enter the system through best-reply choices of players, i.e. a player chooses a best reply with high probability and uniform noise over $\mathcal{S}$ with small probability. Alternatively, a player can have a noisy observations, i.e. a player observes a correct history with large probability, or uniform noise over $\mathcal{S}$ with small probability.

If noise is introduced through a best-reply correspondence, it is necessary to modify the best-reply matrix H. We introduce a new noisy best-reply matrix H_ϵ as follows

$$\mathrm{H}_\epsilon = (1-\epsilon)\mathrm{H} + \epsilon\frac{1}{S}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}.$$

The noisy dynamics is the same dynamics as (1)-(2) with H substituted for H_ϵ, i.e. we have

$$\mu_{t+1}^T = \mu_t^T \mathrm{P}(x_t), \tag{6}$$

$$x_{t+1}^T = \mu_{t+1}^T \mathrm{H}_\epsilon. \tag{7}$$

If noise is introduced through observations, it is necessary to modify the transition matrix P. We introduce a new noisy transition matrix P_ϵ as follows

$$\mathrm{P}_\epsilon(x) = \left(1-\frac{K}{N}\right) I_S + \frac{K}{N}\left((1-\epsilon)\begin{bmatrix} x_1 & \cdots & x_S \\ \vdots & & \vdots \\ x_1 & \cdots & x_S \end{bmatrix} + \epsilon\frac{1}{S}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}\right).$$

The noisy dynamics is the same dynamics as (1)-(2) with P substituted for P_ϵ, i.e. we have

$$\mu_{t+1}^T = \mu_t^T \mathrm{P}_\epsilon(x_t), \tag{8}$$

$$x_{t+1}^T = \mu_{t+1}^T \mathrm{H}. \tag{9}$$

We define a noisy FP(1)-equilbrium as any fixed point of either the system (6)-(7) or the system (8)-(9). This does not cause any problems since we have the following.

For the system (6)-(7)

$$\begin{aligned} \mu_{t+1}^T &= \mu_t^T \mathrm{P}(x_t) = \left(1-\frac{K}{N}\right)\mu_t^T + \frac{K}{N}x_t^T \\ &= \left(1-\frac{K}{N}\right)\mu_t^T + \frac{K}{N}\mu_t^T \mathrm{H}_\epsilon \\ &= \mu_t^T\left(\left(1-\frac{K}{N}\right) I_S + \frac{K}{N}\mathrm{H}_\epsilon\right) = \mu_t^T \Psi_\epsilon. \end{aligned}$$

For the system (8)-(9)

$$\begin{aligned}
\mu_{t+1}^T &= \mu_t^T \mathrm{P}_\epsilon(x_t) = \\
&= \left(1-\frac{K}{N}\right)\mu_t^T + \frac{K}{N}\left((1-\epsilon)\,\mu_t^T \mathrm{H} + \epsilon\mu_t^T \frac{1}{S}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}\right) \\
&= \mu_t^T \left(\left(1-\frac{K}{N}\right) I_S + \frac{K}{N}\mathrm{H}_\epsilon\right) = \mu_t^T \Psi_\epsilon .
\end{aligned}$$

In both cases we can repeat the same derivation for x_t and get $x_{t+1}^T = x_t^T \Psi_\epsilon$. The matrix Ψ_ϵ can be rewritten as

$$\Psi_\epsilon = \left(1-\frac{K}{N}\right) I_S + \frac{K}{N}\left((1-\epsilon)\,\mathrm{H} + \epsilon\frac{1}{S}\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix}\right).$$

Clearly, as ϵ converges to 0, the operator Ψ_ϵ converges to Ψ.

Proposition 6 *For any two-player symmetric game* $(\mathcal{I}, \mathcal{S}, \mathcal{G})$ *and any* $\epsilon \in (0, 1]$ *there exists a unique globally asymptotically stable noisy* FP(1)*-equilibrium.*

Proof. For any game and any $\epsilon \in (0, 1]$, the matrix Ψ_ϵ is irreducible, aperiodic and positive recurrent. Therefore, there exists a unique stationary measure x_e, and a sequence x_t converges to x_e. ■

Proposition 2 holds for a noisy version of the FP(1)-equilibrium, i.e. any noisy FP(1)-equilibrium has to satisfies a system of linear equations

$$x^T = x^T \mathrm{H}_\epsilon \quad \wedge \quad \langle x|\mathbf{1}\rangle = 1,$$

where the first equation can be further rewritten as

$$x^T = (1-\epsilon)x^T \mathrm{H} + \epsilon\frac{1}{S}\mathbf{1}.$$

A noisy version of FP(1)-equilibrium is a mixture of a simple FP(1)-equilibrium and uniform noise.

Since as ϵ converges to 0 the noisy equilibrium converges to a simple equilibrium, it can be thought of as a selecting device between different FP(1)-equilibria.

Example 8 Consider a game from Example 1. The system of equations defining a noisy FP(1)-equilibrium reads

$$x_1 - x_2 = 0 \quad \wedge \quad x_1 + x_2 = 1,$$

regardless of values of $a, b > 0$ and ϵ. The unique globally asymptotically stable equilibrium is $x_e = 1/2, 1/2$.

7 Continuous time limit

Suppose that during a time interval $\triangle t$ the probability of a player being active $p(\triangle t)$ is proportional to the time interval $\triangle t$, i.e. $p(\triangle t) = \lambda \triangle t + o(\triangle t)$, where λ is intensity of play. We then have

$$\mathrm{P}(x, \triangle t) = (1 - p(\triangle t))\,\mathrm{I}_S + p(\triangle t) \begin{bmatrix} x_1 & \cdots & x_S \\ \vdots & & \vdots \\ x_1 & \cdots & x_S \end{bmatrix},$$

which leads to

$$x^T_{t+\triangle t} - x^T_t = p(\triangle t) x^T \left(\mathrm{H} - \mathrm{I}_S\right).$$

Dividing by $\triangle t$ and taking the limit as $\triangle t \to 0$ we obtain a system of linear first order differential equations

$$\dot{x}^T = \lambda\, x^T \left(\mathrm{H} - \mathrm{I}_S\right). \qquad (10)$$

The dynamics (10) is well defined, i.e. a simplex $\triangle_S$ is forward invariant under the dynamics (10). First, we have

$$\langle \dot{x} | \mathbf{1} \rangle = \left\langle \lambda\, x^T \left(\mathrm{H} - \mathrm{I}_S\right) | \mathbf{1} \right\rangle = \lambda \left\langle\, x^T \mathrm{H} | \mathbf{1} \right\rangle - \lambda \left\langle\, x^T \mathrm{I}_S | \mathbf{1} \right\rangle = \lambda - \lambda = 0.$$

Second, suppose some $x_l = 0$. But then the $\dot{x}_l \geq 0$ because the only negative elements in a matrix $\mathrm{H} - \mathrm{I}_S$ are on a diagonal. Also, if some $x_l = 1$ then $\dot{x}_l \leq 0$.

In a similar fashion we can derive a continuous time version of noisy FP(1) dynamics

$$
\begin{aligned}
\dot{x}^T &= \lambda\, x^T\, (\mathrm{H}_\epsilon - \mathrm{I}_S) \\
&= \lambda\, x^T\, (\mathrm{H} - \mathrm{I}_S) + \epsilon\lambda\, x^T \left(\frac{1}{S} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \end{bmatrix} - \mathrm{H} \right).
\end{aligned}
$$

Clearly, as $\epsilon \to 0$ the noisy dynamics converges to the dynamics (10).

Example 9 Consider a game from the Example 1. The noisy continuous time version of FP(1) dynamics reads

$$\dot{x}_1 = \epsilon \frac{K}{2N} (1 - 2x_1) \,.$$

Regardless of values $a, b > 0$, $N > K > 2$ or a noise level $\epsilon \in (0, 1)$ the unique globally asymptotically stable equilibrium is $x_e = (1/2, 1/2)$.

Example 10 Consider a game from the Example 7 with $a > 0$. The noisy continuous time version of FP(1) dynamics reads

$$
\begin{aligned}
\dot{x}_1 &= \frac{K(4\epsilon x_3 - 4x_3 + 4x_1(\epsilon - 2) + 4x_2(\epsilon - 1) - 3\epsilon + 4)}{4N}, \\
\dot{x}_2 &= \frac{K(-4x_2 - 4x_1(\epsilon - 1) + \epsilon)}{4N}, \\
\dot{x}_3 &= \frac{K(-4x_3 - 4x_2(\epsilon - 1) + \epsilon)}{4N}.
\end{aligned}
$$

Regardless of values $a > 0$, $N > K > 2$ or a noise level $\epsilon \in (0, 1)$ the unique globally asymptotically stable equilibrium is $x_e = (1/4, 1/4, 1/4, 1/4)$.

8 Conclusions

We derived a model of aggregate behavior of a population of players using naive fictitious play in an evolutionary environment. The derived model is different from the best-reply dynamics, cf. [HS98]. Rather, it is reminiscent of the concept of a sampling equilibrium, cf. [OR98].

The FP(1)-equilibria, defined as fixed points of the derived dynamics, exist for any two-player symmetric game[3]. The set $\triangle_{\mathrm{FP}(1)}$ of all FP(1)-equilibria has a simple linear structure, i.e. $\triangle_{\mathrm{FP}(1)} = \ker(\mathrm{H} - I) \cap \triangle_S$. Therefore, it is convex and compact.

The dynamics always converges to the set $\triangle_{\mathrm{FP}(1)}$. If a game has the cycling property, there is a unique globally asymptotically stable FP(1)-equilibrium. It is interesting that it is exactly this property that allowed to build classical non-convergence examples, cf. [FK93, Sha64, HS98, FY98].

The noisy version of the model provides the uniqueness and global asymptotic stability of an equilibrium, regardless of the considered game. Since, as the noise level disappears, the noisy dynamics converges to the simple FP(1) dynamics, the noisy dynamics may be considered a selecting device among many FP(1)-equilibria.

We provide also a continuous time versions of FP(1) dynamics with and without noise. The continuous time versions are simple systems of linear first order differential equations. This is different from replicator dynamics, which is quadratic, and best-reply dynamics, which in fact is differential inclusion.

Further research is required to study and understand fictitious play algorithms with longer but still finite memories within the evolutionary environment. However, the future research should proceed along the paths developed herein.

References

[Bre99] Pierre Bremaud. *Markov chains, Gibbs fields, Monte Carlo simulations and queues.* New York, Berlin, Heidelberg: Springer-Verlag, 1999.

[Bro51] G. Brown. Iterative solutions of games by fictitous play. In *Activity Analysis of Production and Allocation.* New York: Wiley, 1951.

[3] In fact, it can be easily shown that they exist for any finite game.

[FK93] Drew Fudenberg and David Kreps. Learning mixed equilibria. *Games and Economic Behavior*, 5:320–367, 1993.

[FL98] Drew Fudenberg and David Levine. *The theory of learning in games.* Cambridge, Massachustets, London: MIT Press, 1998.

[FY98] Dean P. Foster and H. Peyton Young. On the nonconvergence of fictitious play in coordination games. *Games and Economic Behavior*, 25:79–96, 1998.

[GS01] Geoffrey R. Grimmet and David R. Strizaker. *Probability and random procesess.* Oxford University Press, 2001.

[HS98] Josef Hofbauer and Karl Sigmund. *Evolutionary games and population dynamics.* Cambridge, New York: Cambridge University Press, 1998.

[MWG95] Andreu Mas-Colell, Michael Whinston, and Jerry Green. *Microeconomic Theory.* New York, Oxford: Oxford Iniversity Press, 1995.

[OR98] Martin Osborne and Ariel Rubinstein. Games with procedurally rational players. *American Economic Review*, 88:834–847, 1998.

[Sha64] Lloyd Shapley. Some topics in two-person games. In *Advances in Game Theory.* Princeton University Press, 1964.

[Wei96] Jorgen Weibull. *Evolutionary game theory.* Cambridge, Massachustets, London: MIT Press, 1996.