Sampling equilibrium through descriptive simulations.

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Abstract

A definition of sampling equilibrium was introduced in (Osborne and Rubinstein 1998). A dynamic version of the model was introduced in (Sethi 2000). However, a descriptive simulation based on the above idea of procedural rationality (i.e. using the same algorithm of behavior) gave different results, than those achieved in (Osborne and Rubinstein 1998) and (Sethi 2000). The simulation was a starting point for new definitions of both sampling dynamics and sampling equilibrium. *Keywords: bounded rationality, sampling procedure, evolution.*

1 Introduction

A definition of sampling equilibrium was introduced in (Osborne and Rubinstein 1998). It relies on a procedure comprising two distinct phases. In the first one, a player uses in a sequence all available pure strategies. Afterward, it chooses the strategy that has yielded the highest payoff, and then, it switches to the second phase. In the second phase, a player uses the chosen strategy.

A dynamic version of sampling equilibrium was postulated in (Sethi 2000). The proposed dynamics, called sampling dynamics, is typical "evolutionary style" dynamics. A distribution of strategies in a population is sampling equilibrium if and only if it is a stationary point of the sampling dynamics.

The sampling dynamics defined in (Sethi 2000) was postulated. We start with a descriptive simulation (cf. (Chattoe 1996), (Gilbert 1996)), and then, we derive dynamics from the exact underlying model of interactions. Only after we have the appropriate dynamics, we define equilibrium as a stationary point of it, and not the other way around. The new sampling dynamics has different properties as it includes random delays and endogenous noise. As a consequence, a set of equilibria is also different to the original one. We provide basic invariance and existence results.

Section 2 provides a brief summary of the original results on both sampling equilibrium and sampling dynamics. In Section 3 we discuss a simulation and its results. Section 4 contains the main results. Discussion is contained in Section 5. We conclude in Section 6.

2 Sampling equilibrium

A two-player, symmetric, normal form game $\mathcal{G}$ is a triple $\mathcal{G} = (\mathcal{I}, \mathcal{S}, \mathcal{U})$, where $\mathcal{I} = \{1, 2\}$ is a set of players, $\mathcal{S} = \{1, 2, \ldots, S\}$ is a set of pure strategies and $\mathcal{U} = (u_1, u_2)$ is a vector of payoff functions. We will drop the indices referring to players as is usual in the context of symmetric games. A payoff function u is defined through a symmetric $S \times S$ matrix G.

We assume the usual "random-matching" evolutionary scenario. The distribution of players among pure strategies is denoted by $\alpha = (\alpha_1, \ldots, \alpha_S) \in \triangle_S$, where α_j is a fraction of players using a pure strategy $j \in \mathcal{S}$.

The sampling procedure is the following. A player uses each pure strategy once, and then adopts the strategy that has given the highest payoff. The probability of adopting a strategy j given a distribution α is denoted by $w(j, \alpha)$[1]. A distribution α is called sampling equilibrium if and only if it satisfies $\alpha_j = w(j, \alpha)$ for all $j \in \mathcal{S}$ (cf. (Osborne and Rubinstein 1998)). For any normal form game, there exists sampling equilibrium.

The static notion of the sampling equilibrium is accompanied by a notion of sampling dynamics. The dynamics of the form

$$\dot{\alpha}_j = w(j, \alpha) - \alpha_j, \quad \text{for all } \ j \in \mathcal{S}, \tag{1}$$

is called sampling dynamics (cf. (Sethi 2000)). There are two important results. First, a simplex $\triangle_S$ is forward invariant under the sampling dynamics (1). Second, there exists a bijection between the set of sampling equilibria and the set of stationary points of the sampling dynamics (1), i.e. a distribution α is sampling equilibrium if and only if it is a stationary point of (1).

It is useful to see the above definitions at work. Let $\mathcal{G}$ be a two player symmetric game with two pure strategies, $\mathcal{S} = \{1, 2\}$. Let G be a payoff matrix of the game $\mathcal{G}$

$$G = \left[\begin{array}{cc} 5 & 0 \\ 2 & 4 \end{array} \right]. \tag{2}$$

There are two Nash equilibria in pure strategies in the game $\mathcal{G}$. The first one, $(1, 1)$, is payoff dominant. The second one, $(2, 2)$, is risk dominant.

The winning probability of the first strategy equals $w(1, \alpha) = \alpha_1$. Consequently, the sampling dynamics reads $\dot{\alpha}_1 = 0$ and any distribution $\alpha = (\alpha_1, 1 - \alpha_1)$, $\alpha_1 \in [0, 1]$ is (Lapunov) stable sampling equilibrium.

In the above game, if $\alpha_1(0) = 1$, then $\alpha_1(t) = 1$ for any $t > 0$. At the same

[1] This procedure is called 1-sampling. The probability w is called a winning probability. We consider here only games without tied payoffs, hence we do not assume any tie-breaking rules (cf. (Osborne and Rubinstein 1998)).

time, some fraction of players is using the second pure strategy because of the sampling procedure. This implies that the probability of getting a higher payoff while using the second pure strategy should be positive, which is not the case in the mathematical model.

This awkward behavior of sampling dynamics (1) was the starting point for a development of a descriptive simulation of the population of procedurally rational players using sampling procedure.

3 Simulation

The simulation comprises two classes: an "environment" class and a "player" class. There are n objects of a class player and a single object of a class environment (i.e. there are n players within one environment).

The sole purpose of the "environment" class is to provide a usual "random-matching" evolutionary scenario. It matches $k/2$, $k \leq n$, pairs of player at each time step t, calculates payoffs and sends them back to the chosen players. Players are anonymous to the environment. It neither tracks nor stores any information about players. Any such information is not used in a process of choosing and matching players, i.e. we assume an absolutely anonymous environment.

The behavior of the "player" class consists of two phases: an active one and a passive one. The active phase is the sampling procedure (cf. (Osborne and Rubinstein 1998)). During the passive phase the previously chosen strategy is played until the next sampling procedure starts. The only information a player ever gets is its payoff. The sampling procedure is used to build an association between pure strategies and payoffs. Players are anonymous to each other and to the environment. In particular for a game with two pure strategies, it is possible for a player to start the sampling procedure at time t (i.e. to play the first pure strategy) and continue the procedure at some later time $t + l_1$, $l_1 > 1$ (i.e. to play the second pure strategy). The final decision (i.e. the choice of the best strategy) may be taken at some time $t + l_1 + l_2$, $l_2 > 1$. Any two players may be matched. In particular, two sampling players may play against each other.

We now turn to see the results of the simulation for the game given by the payoff matrix (2). Figures 2 and 3 show results of the simulation for the extremal values of k. The average path of a fraction of passive players using the first pure strategy converges to $\alpha = 1/2$. Although the pace of convergence is heavily influenced by the relation of k and n, the limit itself seems not to depend on it.

The results of the simulation are quite different from the results suggested by the mathematical model. According to the mathematical model, any distribution is Lapunov stable equilibrium. The simulation suggests, that there is single globally asymptotically stable equilibrium.

4 Mathematics behind the simulation

It is necessary to build a mathematical model in order to achieve general results concerning the above simulation. The starting point is a single player. Its behavior comprises two phases: an active one and a passive one. The active phase corresponds to the mechanical part of a procedure, namely sampling subsequent pure strategies. The passive phase consists of playing previously chosen pure strategy. This behavior is summarized in a Markov chain $\mathcal{P}$ depicted on Figure 1.

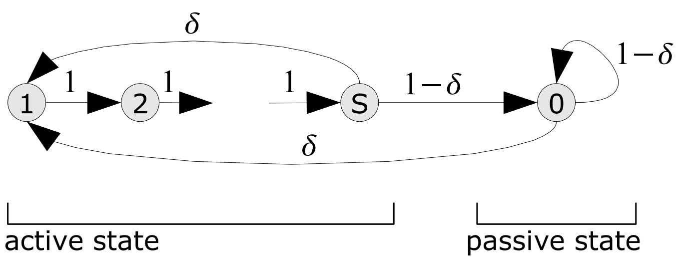

Figure 1: Markov chain corresponding to the behavior of a player.

Proposition 1 *The Markov chain $\mathcal{P}$ corresponding to the behavior of a player* *is aperiodic and irreducible and its invariant measure* π *is given by:*

$$\pi_0 = \frac{1-\delta}{1+\delta(S-1)}, \quad and \quad \pi_j = \frac{\delta}{1+\delta(S-1)}, \; for\ all\ j \in \mathcal{S}. \tag{3}$$

Proof: The first part is obvious. The formulas for the invariant measure π result from simple algebra. ■

The invariant measure π will be denoted by $\pi = (\pi_0, \pi_a)$, where $\pi_a = (\pi_1, \ldots, \pi_S)$. The whole population is a system of n Markov chains $\mathcal{P}$. At any given time t, only k out of n Markov chains are chosen. Therefore, the Markov chain describing the behavior of a single player within such an environment has to be modified, namely its transition matrix is of the form

$$\tilde{\mathcal{P}} = \frac{k}{n}\mathcal{P} + \left(1 - \frac{k}{n}\right) \mathrm{I}_{S+1},$$

where I_{S+1} is $(S+1) \times (S+1)$ identity matrix. The Markov chains $\tilde{\mathcal{P}}$ are correlated. In order to derive a system of equations describing the average behavior of a population on an aggregate level (i.e. the distribution of strategies in a population), it is necessary to know a distribution of players among the nodes $\{0, 1, \ldots, S\}$.

Proposition 2 *Let there be* n *Markov chains* $\mathcal{P}$*, but at each time step* t *only* k *out of* n *are picked. In such a system, the distribution of players among the nodes* $j \in \mathcal{S} \cup \{0\}$ *equals* π*.*

Proof: In fact, the only thing we have to show, is that π is the invariant measure of $\tilde{\mathcal{P}}$. We have

$$\pi^T \, \tilde{\mathcal{P}} = \pi^T \, \left(\frac{k}{n}\mathcal{P} + \left(1 - \frac{k}{n}\right) \mathrm{I}_{S+1}\right) =$$

$$\frac{k}{n} \, \pi^T \, \mathcal{P} + \left(1 - \frac{k}{n}\right) \pi^T \, \mathrm{I}_{S+1} = \frac{k}{n}\pi^T + \left(1 - \frac{k}{n}\right) \pi^T = \pi^T$$

This completes the proof. ■

Let $\alpha(t) = (\alpha_1(t), \ldots, \alpha_S(t))$ denote a distribution of pure strategies among passive players at time t. At any time we have $\alpha(t) \in \triangle_S$, where $\triangle_S = \{\alpha \in$

$\mathrm{R}_+^S : \sum_{j\in\mathcal{S}} \alpha_j = 1\}$. The difference equations, describing average behavior of the system of players, are typical inflow–outflow equations. They read

$$\pi_0 n\alpha_j(t+1) = \pi_0 n\alpha_j(t) + \underbrace{\frac{k}{n}(1-\delta)\pi_S n w(j,\Phi_t)}_{\text{inflow}} - \overbrace{\frac{k}{n}\delta\alpha_j(t)}^{\text{outflow}},$$

where $\Phi_t = (\phi_0, \ldots, \phi_{t-1}, \phi_t)$, $\phi_t = \pi_0\alpha_t + \pi_a$. Dividing both sides by $\pi_0 n$, taking into account formulas (3) and rearranging leads to

$$\alpha_j(t+1) - \alpha_j(t) = \frac{k\delta}{n}\ \left(w(j,\Phi_t) - \alpha_j(t)\right). \tag{4}$$

Proposition 3 *Simplex $\triangle_S$ is forward invariant under the dynamics (4).*

Proof: The proof falls naturally into two parts. First, we can rewrite (4) as

$$\alpha_j(t+1) = \left(1 - \frac{k\delta}{n}\right)\alpha_j(t) + \frac{k\delta}{n} w(j,\Phi_t),$$

that is, $\alpha_j(t+1)$ is a convex combination of $\alpha_j(t)$ and $w(j,\Phi_t)$. Since they both belong to the interval $[0,1]$, it follows that $\alpha_j(t+1) \in [0,1]$ for any $j \in \mathcal{S}$.

Second, assume that $\sum_{j\in\mathcal{S}}\ \alpha_j(t) = 1$. Then

$$\sum_{j\in\mathcal{S}} \alpha_j(t+1) = \sum_{j\in\mathcal{S}} \left(\alpha_j(t) - \frac{k}{n}\delta\alpha_j(t) + \frac{k}{n}\delta w(j,\Phi_t)\right) =$$

$$= \sum_{j\in\mathcal{S}} \alpha_j(t) - \frac{k}{n}\delta\sum_{j\in\mathcal{S}}\alpha_j(t) + \frac{k}{n}\delta\sum_{j\in\mathcal{S}} w(j,\Phi_t) = 1$$

This completes the proof. ■

Equilibrium is defined as a "stationary" point of the dynamics (4). It is a consequence of the dynamics (4) and it comes after it, not before.

Definition 1 *A distribution $\alpha^\diamond \in \triangle_S$ is called sampling equilibrium if and only if it is a "stationary" point of the dynamics (4), i.e. for any t it is $\alpha(t) = \alpha^\diamond$.*

There are static conditions defining sampling equilibrium $\alpha^\diamond$. These conditions are derived from the dynamics (4).

Proposition 4 *Let $\alpha^\diamond \in \triangle_S$ be sampling equilibrium. Then it satisfies the following conditions*

$$\alpha_j^\diamond = w(j, \pi_0\alpha^\diamond + \pi_a) \quad \textit{for all } j \in \mathcal{S}. \tag{5}$$

Proof: Let $\alpha^\diamond \in \triangle_S$ be sampling equilibrium. By definition of $\alpha^\diamond$ we know, that for any t, there is $\alpha(t) = \alpha^\diamond$. This implies, that $\phi_t = \pi_0\alpha^\diamond + \pi_a$ for any t. Therefore we have

$$\alpha_j^\diamond = \alpha_j^\diamond - \frac{k}{n}\delta\alpha_j^\diamond + \frac{k}{n}\delta w(j, \pi_0\, \alpha^\diamond + \pi_a) \quad \text{for all} \;\; j \in \mathcal{S}$$

and consequently $\alpha_j^\diamond = w(j, \pi_0\alpha^\diamond + \pi_a)$ for all $j \in \mathcal{S}$. This completes the proof. ■

Proposition 5 *For any symmetric two player game $\mathcal{G}$ there exists sampling equilibrium.*

Proof: Simplex $\triangle_S$ is compact and convex. The mapping $w : \triangle_S \to \triangle_S$ is continuous (polynomial). Hence, by Bouwer's fixed point theorem, there exists $\alpha^\diamond \in \triangle_S$ such that $\alpha_j^\diamond = w(j, \pi_0\alpha^\diamond + \pi_a)$ for all $j \in \mathcal{S}$. This completes the proof. ■

We now turn to study the delays present in the dynamics (4). These delays are obviously random and correlated as far as a single player is concerned. We are interested in an average distribution θ over the set of possible delays D_t at time t, where

$$D_t = \{l \in \mathrm{N}^S \;:\; \sum_{j\in\mathcal{S}} l_j < t\}.$$

Although, it is possible to get exact, closed-form formulas for a distribution θ in simple situations[2], we are, however, interested in its asymptotic properties.

Proposition 6 *Let $k < n$. The asymptotic distribution θ of lags $l_1, l_2, \ldots, l_S$ is of the form*

$$\theta(l_1, \ldots, l_S) = \left(\frac{k}{n-k}\right)^S \prod_{j\in\mathcal{S}} \left(1 - \frac{k}{n}\right)^{l_j}.$$

It is a product of independent geometrical distributions.

[2]For example, in a situation with two pure strategies only, the general formula reads

$$\theta(l_1, l_2) = -\frac{k^2\left(1-\frac{k}{n}\right)^{l_1+l_2}\delta\left(\frac{n-k(\delta+1)}{n}\right)^{-l_1-l_2}\left(\left(\frac{n-k(\delta+1)}{n}\right)^{l_1+l_2} + \delta\left(\frac{n-k(\delta+1)}{n}\right)^t\right)}{(k-n)^2\left(\left(1-\frac{k}{n}\right)^t - \left(\frac{n-k(\delta+1)}{n}\right)^t + \left(\left(1-\frac{k}{n}\right)^t - 1\right)\delta\right)}.$$

It may be used to get the results of the Proposition 6 in the special case. This, however, is quite complicated and it will not be used.

Proof: Let $[\cdot]_j$ denote the jth coordinate of a vector. Let A be a set of all incoming paths up to time $t+1$, i.e. paths of the form $(0,\ldots,2,0)$. Let $B(l)$ be a set of l-incoming paths up to time $t+1$, i.e. paths of the form

$$(0,\ldots,0 \text{ or } S,\overbrace{1,\ldots,1}^{l_1 \text{ times}},\ldots,\underbrace{S,\ldots,S}_{l_S \text{ times}},0),$$

where $l=(l_1,\ldots,l_S)\in D_{t+1}$. Obviously, for any $l\in D_{t+1}$ we have $B(l)\subset A$. The distribution θ, given by the conditional probability $\theta(l)=\mathrm{P}(B(l)|A)=\mathrm{P}(B(l))/\mathrm{P}(A)$ is thus

$$\theta(l)=\frac{\left(\left[\mathrm{e}^1\tilde{\mathcal{P}}^{t-\sum_{j\in\mathcal{S}}l_j}\right]_S\tilde{p}(S,1)+\left[\mathrm{e}^1\tilde{\mathcal{P}}^{t-\sum_{j\in\mathcal{S}}l_j}\right]_0\tilde{p}(0,1)\right)}{\left[\mathrm{e}^1\tilde{\mathcal{P}}^t\right]_S\tilde{p}(S,0)}d(l),$$

where

$$d(l)=\tilde{p}(1,1)^{l_1-1}\tilde{p}(1,2)\ \cdots\ \tilde{p}(S,S)^{l_S-1}\tilde{p}(S,0).$$

In the limit, as $t\to\infty$, the above formula yields

$$\lim_{t\to\infty}\theta(l)=\frac{(\pi_S\tilde{p}(S,1)+\pi_0\tilde{p}(0,1))}{\pi_S\tilde{p}(S,0)}d(l)=\left(\frac{k}{n-k}\right)^S\prod_{j\in\mathcal{S}}\left(1-\frac{k}{n}\right)^{l_j}.$$

That is, the distribution of lags is a product of asymptotically independent geometrical distributions. This completes the proof. ∎

We now turn to see if the results of the mathematical model, given by the dynamics (4), are consistent with the results of the simulation. Let $\mathcal{G}$ be the game (2). We have the following.

Proposition 7 *Let $\mathcal{G}$ be the game (2) and let $k\le n$, $\delta\in(0,1)$. In the game $\mathcal{G}$, there is single sampling equilibrium $\alpha^\diamond=(1/2,1/2)$. The equilibrium is globally asymptotically stable.*

Proof: First, we have (cf. Proposition 4)

$$\alpha_1^\diamond=w(1,\Phi)\ \Rightarrow\ \alpha_1^\diamond=\pi_1+\pi_0\alpha_1^\diamond\ \Rightarrow\ \alpha_1^\diamond=\frac{1}{2}.$$

This asserts the first claim.

Second, the difference equation (4) reads

$$\alpha_1(t+1) = \left(1 - \frac{k\delta}{n}\right)\alpha_1(t) + \frac{k\delta}{n}\sum_{l\in D_t}\theta(l)\phi_{t+1-l_1-l_2}.$$

It can be rewritten as

$$\alpha_1(t+1) = \left(1 - \frac{k\delta}{n}\right)\alpha_1(t) + \frac{k\delta}{n}\left(\pi_1 + \pi_0\sum_{l\in D_t}\theta(l)\alpha_1(t+1-l_1-l_2)\right). \quad (6)$$

Let consider any path $(\alpha(0),\ldots,\alpha(t))$ satisfying the dynamics (6). Let $\epsilon_t = \alpha_1(t) - 1/2$ and let $\gamma_t = \max_{l\leq t}|\epsilon_l|$. Substituting $\alpha_1(t) = 1/2 + \epsilon_t$ and rearranging leads to

$$\epsilon_{t+1} = \left(1 - \frac{k\delta}{n}\right)\epsilon_t + \frac{k\delta}{n}\pi_0\sum_{l\in D_t}\theta(l)\epsilon_{t+1-l_1-l_2}.$$

Obviously, we have

$$|\epsilon_t| \leq \gamma_t \quad \text{and} \quad \left|\sum_{l\in D_t}\theta(l)\epsilon_{t+1-l_1-l_2}\right| \leq \gamma_t.$$

Using the above inequalities, we get the following

$$|\epsilon_{t+1}| \leq \left(1 - \frac{k\delta}{n}\right)\gamma_t + \frac{k\delta}{n}\pi_0\gamma_t = \underbrace{\left(1 - \frac{k\delta}{n}(1-\pi_0)\right)}_{\Gamma}\gamma_t,$$

where the constant Γ belongs to the interval $(0,1)$. This completes the proof. ■

The constant Γ shows how the particular parameters of the model influence the convergence. The increase of k/n will result in decrease of Γ, thus resulting in faster convergence. The increase of the probability δ of switching to the active phase will also increase the rate of convergence. Concluding, the results of the mathematical model are consistent with the results of the simulation.

5 Discussion

The dynamics (4) is different from the original formulation in two aspects. First, the players at time t are faced with a different distribution of strategies, i.e. with a distribution ϕ_t instead of just α_t. Second, the process of sampling involves delays, hence the winning probability has to be based on a history of distributions Φ_t rather than on a single distribution.

The sampling equilibrium as defined in (Osborne and Rubinstein 1998) will be called simply “sampling equilibrium” (SE). The sampling equilibrium as defined here will be reffered to as “anonymous sampling equilibrium” (ASE). The (in)stability results developed in (Sethi 2000), (Ramsza 2005) concern SE.

Because of the above differences it is instructive to review how the original dynamics (1) was derived (we follow (Sethi 2000)). There is some unspecified (possibly large) population. In each of the small time intervals h the proportion of individuals leaving the population is $1-\mathrm{e}^{-\lambda h}$. The proportions of new entrants is therefore the same to keep the size of the populations constant. It is assumed that, according to the sampling algorithm, new entrants adopt the j-th strategy with probability $w(j, \alpha(t))$. The inflow-outflow equation reads

$$\alpha_j(t+h) = \mathrm{e}^{-\lambda h}\alpha_j(t) + (1-\mathrm{e}^{-\lambda h})w(j, \alpha(t)).$$

Rearranging to

$$\frac{\alpha_j(t+h) - \alpha_j(t)}{h} = \frac{1-\mathrm{e}^{-\lambda h}}{h}\left(w(j, \alpha(t)) - x_j(t)\right),$$

taking the limit $h \to 0$, and setting $\lambda = 1$ leads to the dynamics (1).

The environment proposed in (Sethi 2000) is in general the same as the one considered here. The individuals leaving the environment and those coming into the environment may be considered the same, hence making the system closed. So, at any time there is a proportion of the population running the sampling algorithm in order to choose the new strategy, which is basically the same situation as the one considered here. However, there is one crucial difference. In the system considered in (Sethi 2000), the sampling fraction of the population is matched exclusively against not sampling individuals. This may be achieved only if the sampling players have information about the “state” of other players. Whether it is implemented in an “environment” class or a “player“ class is not important. This difference removes the endogenous noise π. Also, it is assumed that the whole sampling procedure fits into the time interval h, and so there are no delays.

It is interesting to see whether the (in)stability results, concerning certain pure-strategy SE (cf. (Sethi 2000), (Ramsza 2005)), carry on to the fully anonymous situation considered here (ASE).

The instability conditions based on the notion of inferiority[3] concern border equilibria. Also, the examples, where dominated strategies survive along convergent trajectories of the system, contain border equilibria, which correspond to strictly dominant strategies. Therefore, it is important to characterize these equilibria.

Definition 2 *Let $\mathcal{G}$ be any 2-player symmetric game. A strategy $j \in \mathcal{S}$ is called certainly worse if there exists a strategy $m \in \mathcal{S}$, $m \neq j$, such, that*

$$\min_{v \in \mathcal{S}} u(m, v) > \max_{v \in \mathcal{S}} u(j, v).$$

Proposition 8 *Let $\mathcal{G}$ be any symmetric 2-player game, and let $\alpha^{\diamond}$ be its sampling equilibrium (ASE). Then, $\alpha^{\diamond}_j = 0$ if and only if $j \in \mathcal{S}$ is certainly worse.*

Proof: If $j \in \mathcal{S}$ is certainly worse, then at any sampling equilibrium (ASE) $\alpha^{\diamond}_j = 0$. Suppose that $\alpha^{\diamond}_j = w(j, \pi_0\alpha^{\diamond} + \pi_a) = 0$. Since $\pi_a \neq 0$, then all sequences of sampling have positive probabilities regardless of α. Because $\alpha^{\diamond}_j = 0$, the payoff yielded by the strategy j has to be smaller than payoff for some other strategy on any sampling sequence. This implies that j is certainly worse. ■

Suppose, that a profile $(1, 0, \ldots, 0)$ is sampling equilibrium (ASE). This implies that all strategies, except the first one, are certainly worse. This however implies further, that such a profile cannot be (twice) inferior[4]. In fact, if such a profile is sampling equilibrium (ASE), it is the only sampling equilibrium of the game, as well as the only Nash equilibrium of the game. Also, it is globally asymptotically stable (cf. (Ramsza 1999)). The instability conditions formulated in (Sethi 2000) consider profiles that either are not sampling equilibria (ASE), or are not inferior.

[3]The twice inferior profiles are considered in 2-player games, cf. (Sethi 2000).
[4]Since these conditions concern only the payoff function, and not the environment.

Similarly, the stability conditions (superiority) formulated in (Ramsza 2005), based on the notion of stochastic dominance, are too weak to guarantee asymptotic stability, since they are weaker than strict dominance. Any pure strategy sampling equilibrium (ASE) has to correspond to the unique superior strategy of a game, but it is possible for a strategy to be superior and not be equilibrium (ASE).

As $\delta \to 0$ the set of the sampling equilibria (ASE) defined in Proposition 4 will converge to a certain subset of the set of sampling equilibria (SE) defined in (Osborne and Rubinstein 1998). Finally, as $\delta = 0$, the equation (5) will define exactly the same set of distributions as the one defined originally (cf. (Osborne and Rubinstein 1998)). It is tempting to think of the model developed here as a selecting device among the equilibria (SE) defined in (Osborne and Rubinstein 1998). However, the dynamics for $\delta = 0$ reads $x(t+1) = x(t)$, which is obvious since nobody is sampling. So, the environment modeled in (Sethi 2000) cannot be considered a limit of the fully anonymous environment considered here, even though the static conditions of Propostion 4 will converge to the original definition.

It is also interesting to characterize equilibria in scenarios with large amount of noise, i.e. where δ is close to 1. We have the following.

Proposition 9 *Let $\mathcal{G}$ be any symmetric 2-player game with S pure strategies. There exists $\underline{\delta}$ such that for any $\delta > \underline{\delta}$ there is unique sampling equilibrium $\alpha^\diamond$. The unique sampling equilibrium $\alpha^\diamond$ is approximately $\alpha_j^\diamond \approx v_j(\mathcal{G})(1/S)^S$, where $v_j(\mathcal{G})$ is a number of winning sequences for the j-th strategy in a game $\mathcal{G}$.*

Proof: The sampling equilibrium is defined through a system of algebraic equations $\alpha = w(j, \pi_0\alpha + \pi_a)$, $j \in \mathcal{S}$. It is easy to see, that as $\delta \to 1$, the polynomials $w(j, \pi_0\alpha + \pi_a)$ converge to constants $v_j(\mathcal{G})\pi_j^S = v_j(\mathcal{G})(1/S)^S$. Hence, for large enough values of δ, there is unique sampling equilibrium $\alpha^\diamond$, and its value is approximately $\alpha_j^\diamond \approx v_j(\mathcal{G})(1/S)^S$. ■

It is important to note, that as $\delta \to 1$ the passive fraction of population π_0

decreases to 0. Finally, as $\delta = 1$ there is no interpretation for a solution of the system (5). In the game (2) for δ close to 1 the unique sampling equilibrium (ASE) is still $(1/2, 1/2)$. In the "voluntary exchange" game, cf. (Osborne and Rubinstein 1998), given by a matrix

$$G = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 0 & 3 & 6 \end{bmatrix},$$

for δ close to 1, the unique sampling equilibrium (ASE) is around $(14/27, 8/27, 5/27)$. In the "three-action coordination game", cf. (Sethi 2000), given by a matrix

$$G = \begin{bmatrix} 2 & 6 & 6 \\ 0 & 7 & 3 \\ 1 & 4 & 8 \end{bmatrix},$$

for δ close to 1, the unique sampling equilibrium (ASE) is around $(9/27, 7/27, 11/27)$. It is interesting that the largest probability is attached to the strategy constituting the payoff dominant Nash equilibrium. Also in the former game, the largest probability is attached to the dominant strategy.

6 Conclusions

We built a descriptive simulation of an evolutionary system of players using sampling procedure to choose between pure strategies. The results of the simulation were qualitatively different to the results of the original mathematical model. As a consequence, a new mathematical model of average dynamics was proposed. Equilibrium was defined as a "stationary" point of the dynamics. The results of the new mathematical model (c.f. Proposition 7) were checked against the results of the simulation. The results of both were consistent.

We proved some results concerning the new mathematical model of the sampling dynamics. In particular, we showed the existence of the new sampling equilibrium (c.f. Proposition 5). We also provided further details concerning delays present in the dynamics (4) (c.f. Proposition 6).

The mathematical model developed here may be extended in various directions. One can change the sampling procedure to include many samplings or even a random number of samples. The order of sampling may also be different or random. But as long as the procedure used is composed of two distinct phases, active and passive, and the matching is anonymous, there are two effects present: random delays and endogenous noise. The results presented in (Osborne and Rubinstein 1998), (Sethi 2000), (Ramsza 2005) may be restored only by removing anonymity of players, e.g. through removing the endogenous noise. This can be done in a variety of settings.

The approach presented here is to derive the aggregate average dynamics from the underlying exact model of interactions. This method has been previously taken up by various authors for different procedures in different settings (cf. (Björnerstedt and Weibull 1996), (Börgers and Sarin 1997), (Brenner 1992), (Schlag 1998)). It seems, however, that it has been rarely based so closely on descriptive simulations but rather on "mental experiments", and as such prone to errors resulting from neglecting the algorithmic issues.

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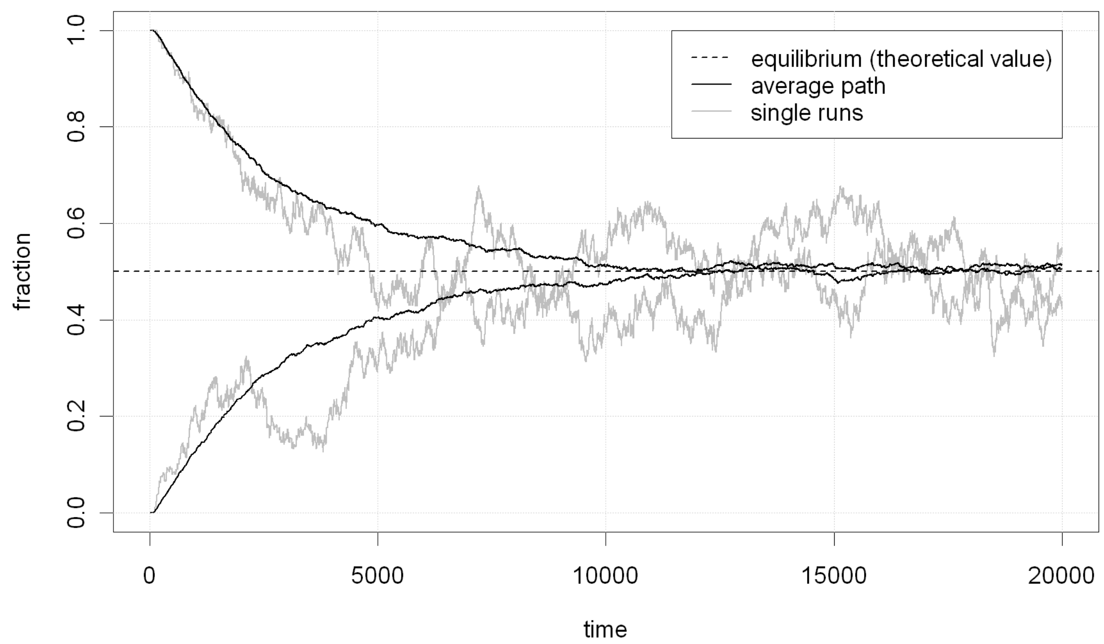

Figure 2: Results of the simulation for the coordination game (cf. Section 2) with $\delta = 1/10$, $n = 100$ and $k = 1$. Gray lines are single realizations. Black lines are average realizations.

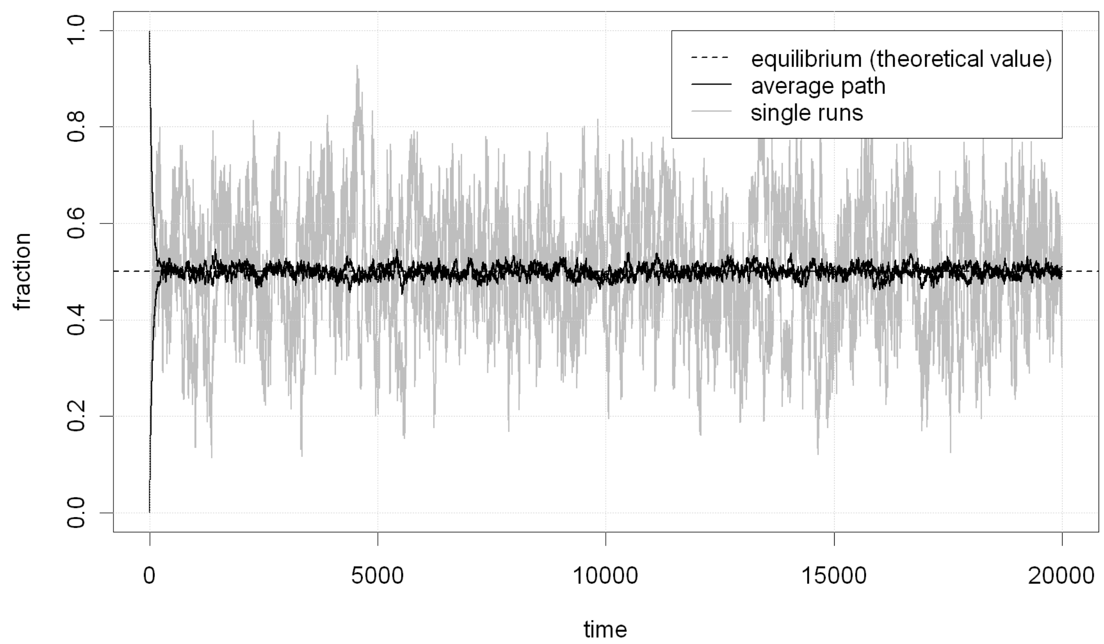

Figure 3: Results of the simulation for the coordination game (cf. Section 2) with $\delta = 1/10$, $n = 100$ and $k = 50$. Gray lines are single realizations. Black lines are average realizations.