Spatial Voting with Endogenous Timing

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Abstract

We consider a model of (spatial) voting with endogenous timing. In line with what is observed in actual political campaigns, candidates can decide endogenously when and where to locate. More specifically, we analyze endogenous timing in a two-period n-candidate spatial-voting game on the unit interval. We show that this game possesses a pure strategy equilibrium*. The equilibrium* concept is a simplified version of subgame perfection defined by Osborne (1993) for use in games that possess no—or only very complex—subgame perfect equilibria. We demonstrate the latter point by also analyzing the subgame perfect equilibria in *three*-candidate spatial voting with endogenous timing. Our results show that accounting for endogenous timing can eliminate some of the more unappealing equilibrium characteristics of the standard model.

1 Introduction

Among the many simplifying assumptions of the Downsian voting model the, perhaps, most serious is that candidates select their positions simultaneously. In fact, assuming any specific exogenous order in which candidates select their platforms appears in conflict with the evidence. Typically,

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political campaigns last for several weeks, if not months, and candidates decide endogenously on when to take which stance in important matters. With the exception of Osborne's (1993, 2000) pioneering work, which analyzes endogenous timing with three candidates, the literature has remained remarkably quiet on this issue. In particular, we are not aware of general results for more than three candidates.[1] This paper attempts to move a step forward in modeling endogenous timing in voting with an arbitrary number of candidates.

Remarkably, introducing endogenous timing eliminates some of the less attractive properties of the standard model. The standard model predicts, for example, that with $n \geq 4$ parties, both most extreme positions, the extreme left and the extreme right, are taken by two candidates *each*. Given that we are not aware of any countries with two extreme left- and two extreme right-wing parties this appears to be a particularly unappealing feature of the standard model. With endogenous timing we can reconcile theory and empirics in that respect since we find equilibria in which the most extreme positions are *not* taken by two candidates. Rather, candidates are more equally distributed over the political spectrum.

We model endogenous timing in very much the same way as it has been employed in the industrial organization literature (see, e.g., Saloner 1987, Hamilton and Slutsky 1990, or Robson 1990). The general idea is the following. Suppose G is a normal form game with n players whose strategy sets are $S_1, S_2, \ldots, S_n$. Let G' be a two-period game in which all players act simultaneously in the first period. In the first period each player i chooses an action from S_i or chooses to wait. In period 2, all period 1 actions become common knowledge, and then all players who chose to wait in period 1 choose their actions simultaneously. Payoffs are assigned using the payoff matrix of the game G. Then G' is a game with endogenous timing.

In this paper we investigate the equilibrium structure of a spatial voting election game with endogenous timing. More particularly, we start with a standard spatial voting election game in which n candidates simultaneously choose positions on the political spectrum $[0, 1]$ along which voters are uniformly distributed. Each voter supports the nearest candidate. Each candidate's payoff is the length of the set of voters supporting her. Then we introduce endogenous timing as above. We first consider subgame perfect equilibria. Due to the complexity of the problem, we only analyze the special case with $n = 3$. We show that in this case the game has no subgame perfect equilibrium (SPE) in pure strategies. However, there is a rather complex SPE in mixed

[1] One notable exception is Osborne (1993) who also provides a result for $n = 4$ and $n = 5$ (after strengthening some of his assumptions).

strategies. On the equilibrium path of this mixed SPE, two candidates locate in period 1 at 1/4 and 3/4, respectively, whereas the third candidate locates somewhere in between these two positions in period 2.

Due to the difficulties we encounter to find a SPE in this special case, we next consider an alternative equilibrium concept—the equilibrium* notion introduced by Osborne (1993). Unlike a SPE, an equilibrium* is not required to be a Nash equilibrium on, or even to be defined on, every subgame. In particular, an equilibrium* is required to be a Nash equilibrium on any subgame that is reached by a history in the course of which no two players deviate in the same period from the play prescribed by the equilibrium*. It can be argued that an equilibrium* possesses the same stability that is the defining characteristic of subgame perfection. We show that in the general n-player case, our voting model with endogenous timing has an equilibrium* in which any $n-1$ candidates locate at equidistant locations in period 1 whereas the n-th candidate waits until period 2 and then locates somewhere in-between the outermost positions occupied in the first period.

While employing this procedure resolves the unintuive features of the standard model, it also demonstrates, on a more technical level, the advantage and usefulness of the equilibrium* concept: An easy to proof equilibrium* may exist for a game that has no SPE and there may be a simple equilibrium* for a game that has only complex or hard-to-find SPE.

Related Literature: The model we study is related to the ones in Palfrey (1984), Osborne (1993) and Osborne (2000). Palfrey studies a three-candidate two-period model with exogenous timing in which two candidates locate in the first period and the third locates in the second period after observing the other two candidates' choices. Candidates are assumed to be vote-maximizers. Palfrey shows that in this setup there exists a limit-equilibrium in which the first-period candidates locate at different positions and the third candidate locates in between the two in the second period. Palfrey's and our model share the features of two location periods and vote-maximizing candidates. However, whereas Palfrey only considers the case of $n = 3$, we consider the n-player setup. Moreover, whereas timing decisions are imposed exogenously in Palfrey's model, we endogenize this decision. Interestingly, restricting our result to the 3-player case, Palfrey's timing and location pattern emerges endogenously in our model.

Osborne (1993) studies two versions of a 3-candidate many-period model. If candidates move in a fixed order, as it is assumed in one version, then there is a unique pure SPE outcome, in which the first candidate enters at the median voter's favorite position, the second candidate stays out, and the third candidate enters at the median. In another version of the model there is an infinite sequence of periods $1, 2, ...$ In each period every player who has not already chosen a position either chooses a position now or chooses to wait until the next period.[2] Given a strategy profile of the candidates, there is a date after which no new entry into the competition occurs. At this date an election is held. The winner is the candidate receiving most votes. Osborne shows that in every SPE of this model only one candidate enters and the other two stay out of the competition.

Osborne (2000) again uses the setup in which each of three candidates may move whenever she wishes. But this time candidates are uncertain about the distribution of the voters' favorite positions. Moreover, each candidate wishes to maximize the probability of winning, has a (symmetric) minimal acceptable probability of winning, p_0, and prefers to stay out of the competition than to enter and win with probability less than p_0. As Osborne shows, if there is sufficient uncertainty, "the game has an equilibrium (essentially a SPE) in which two players enter at distinct positions simultaneously in the first period and the third player either stays out, or, if there is enough uncertainty, enters in the second period at a position between those of the other candidates." (p. 42).

There are two main differences between Osborne's models and our approach. First, we allow for only two periods in which candidates may choose platforms whereas Osborne allows for infinitely many. Second, and more importantly, we are not modeling a winner-take-all election, in which a payoff of, say, 1 for one candidate and 0 for all others would be appropriate, but a proportional representation election in which, for example, our candidates can be thought of as political parties, and each party is awarded a number of seats in parliament proportional to the number of votes it receives.[3]

The paper is organized as follows. In Section 2 we define spatial voting with endogenous timing. In Section 3.1 we consider SPE in the three-player case. Then, in Section 3.2, we discuss the equilibrium* concept and show that the n-candidate spatial-voting model with endogenous timing has a pure strategy equilibrium*. Section 4 offers a short discussion of our findings and concludes with a short list of open problems.

[2]Note that by always choosing to wait, a candidate can stay out of the competition.

[3]In an Hotelling-oligopoly model our assumption of vote-maximizing candidates naturally and realistically translates into demand-maximizing (and thus payoff-maximizing) firms when prices are fixed.

2 The model

For simplicity of exposition[4] we assume voters to be uniformly distributed along $[0, 1]$ with density 1. There are $n(\geq 3)$ candidates labeled $1, 2, ..., n$. The voting game consists of two periods. In period 1 all n candidates act simultaneously. For each i, candidate i chooses a position $x_i \in [0, 1]$ or chooses to wait until period 2. The choices then become common knowledge. In period 2 those candidates who chose to wait in period 1 simultaneously choose positions. After period 2, candidates located at x share all the voters nearer to x than to any other occupied position. Thus, the payoff π_i to candidate $i = 1, 2, ..., n$ is the length of the voter set won by position x_i divided by the number of candidates located at x_i. We assume that a candidate who locates at time $t = 2$ at an already occupied location x may specify x^+ or x^-. Without this standard assumption there is no hope for a Nash equilibrium even for many of the 1-player subgames of the location game with endogenous timing. The following example shows how payoffs are assigned under the assumption:

$$\pi_3(4/5, 2/5, (2/5)^+) = 1/5, \pi_3(4/5, (4/5)^-, (4/5)^-) = 2/5.$$

The assumption says that a candidate can adopt an opponent's political position, yet somehow signal that she is slightly to the left or to the right of that opponent. The assumption provides a nice compromise between the continuous voter distribution of our model and the fact that real elections have only finitely many voters.

3 Analysis

As mentioned in the Introduction, our analysis comes in two parts. We will first consider SPE. However, in doing so, we will confine ourselves to the special case of $n = 3$. The reason is that the general case appears to be intractable. As it turns out, in the special case of $n = 3$ there is no *pure* SPE and finding a SPE in *mixed* strategies is already extremely cumbersome even for this special case. It thus seems elusive to find a SPE for the general case of n players. In the second part of the analysis we therefore consider the notion of an equilibrium* as defined in Osborne (1993). Using this concept we will prove much more easily that there are equilibria* in pure strategies in the general n-player case.

[4]It is easy to see that our main results (Propositions 1 and 3) can be proved in the very much the same way for the case of any continuous, atomless distribution of voters.

3.1 Subgame perfect equilibrium

Note that a strategy in an extensive-form game is a complete plan of behavior in the sense that it assigns an action to a player at each of his information sets. Thus, in the above voting game a strategy of a candidate is a tuple $(x_i^1, f_i(x_{-i}^1))$ where x_i^1 either specifies a position for period 1 or indicates that the candidate waits, i.e. $x_i^1 \in [0,1] \cup \{w\}$ with w indicating the decision to wait. The function $f_i(x_{-i}^1)$ is a mapping $\times_{j \neq i}([0,1] \cup \{w\}) \to [0,1]$ specifying the candidates's position choice in period 2 in response to what the other candidates did in the first period in case she has decided to wait. So, a strategy in the voting game is a rather complicated mathematical object. But to fully specify a player's strategy is necessary if one wants to apply the notion of a SPE in order to solve the game.

3.1.1 A Mixed Strategy Subgame Perfect Equilibrium for the 3-player game

In this subsection we will consider the case $n = 3$. We will begin by studying a game called *Two Entrants and an Incumbent*. *Two Entrants and an Incumbent* is a 2-person game. Voters are uniformly distributed on $[0,1]$ with density 1 and an incumbent is located at $a \in [0,1]$. Then two entering candidates (the two players) locate simultaneously on $[0, a^-] \cup [a^+, 1]$.

Proposition 1 *Two Entrants and an Incumbent has a pure strategy Nash equilibrium if and only if* $a \in [0, \frac{1}{4}] \cup [\frac{3}{4}, 1] \cup \{\frac{1}{2}\}$.

Proof Suppose $a \in [\frac{3}{4}, 1]$. Let both entrants locate at $a/3$. Suppose candidate 1 relocates to x. If $0 \leq x < a/3$, then $\pi_1(x, a/3) < a/3 = \pi_1(a/3, a/3)$. If $a/3 < x \leq a^-$, then $\pi_1(x, a/3) = a/3 = \pi_1(a/3, a/3)$. If $a^+ \leq x \leq 1$, then $\pi_1(x, a/3) \leq 1 - a \leq 1 - 3/4 = 1/4 \leq a/3 = \pi_1(a/3, a/3)$. If $x = a$, then $\pi_1(x, a/3) = 1/2 - a/3 \leq 1/4 \leq \pi_1(a/3, a/3)$. Therefore, $(a/3, a/3)$ is a Nash equilibrium if $a \in [3/4, 1]$.

Similarly, if $a \in [0, 1/4]$ the strategy combination $(1 - (1-a)/3, 1 - (1-a)/3)$ is a Nash equilibrium and if $a = 1/2$, then (a^-, a^+) is a Nash equilibrium.

Next suppose $a \in (1/2, 3/4)$ and candidates 1 and 2 locate at x_1 and x_2 respectively. (Case 1.) $x_1 \in [a^+, 1] \cup \{a\}$ and $x_2 \neq a^-$. Then $\pi_2(x_1, a^-) = a > \pi_2(x_1, x_2)$. (Case 2.) $x_1 \in [a^+, 1] \cup \{a\}$ and $x_2 = a^-$. Then $\pi_1(1-a, x_2) = (a + 1 - a)/2 = 1/2 > 1 - a \geq \pi_1(x_1, x_2)$. (Case 3.) $x_2 \in [a^+, 1] \cup \{a\}$. Proceed as in cases 1 and 2. (Case 4.) $x_1, x_2 \in [0, a^-]$ and $x_1 \neq x_2$. Without loss of generality, $x_1 < x_2$. Then $\pi_1((x_1 + x_2)/2, x_2) > \pi_1(x_1, x_2)$. (Case 5.) $0 \leq x_1 = x_2 \leq a/3$.

Then $\pi_1(a^+, x_2) = 1 - a > 1/4 > a/3 \geq \pi_1(x_1, x_2)$. (Case 6.) $a/3 < x_1 = x_2 \leq a^-$. Then $0 < (x_1 + a)/4 < x_1 = x_2$ so that $\pi_1((x_1 + a)/4, x_2) > (x_1 + a)/4 = \pi_1(x_1, x_2)$.

Therefore, there is no pure strategy Nash equilibrium if $a \in (1/2, 3/4)$ or symmetrically, if $a \in (1/4, 1/2)$.□

Proposition 2 *Two Entrants and an Incumbent has a symmetric, continuous mixed strategy Nash equilibrium when $a \in (1/4, 3/4)$.*

Proof Fix $a \in (1/4, 3/4)$. Let

$$F(x) = \begin{cases} 0 & \text{for} \quad 0 \leq x \leq b \\ 1 + c(a - 3x)^{-1/3} & \text{for} \quad b \leq x \leq a^- \\ k(2 + a - 3x)^{-1/3} & \text{for} \quad a^+ \leq x \leq B \\ 1 & \text{for} \quad B \leq x \leq 1 \end{cases} \tag{1}$$

where $0 < c < (2a)^{1/3}$, $b = (a + c^3)/3$, $0 < k < (2 - 2a)^{1/3}$ and $B = (2 + a - k^3)/3$. Note that F, b and B depend on c and k. We begin to show that for the proper choice of c and k (F, F) is a mixed strategy Nash equilibrium by establishing some facts about F. Using the definitions of c, b, k, and B,

$$a/3 < b < a < B < 1 - (1 - a)/3 \tag{2}$$

By (2), F is well defined. By (1), F is increasing on $[b, a^-] \cup [a^+, B]$ and continuous on $[0, a^-] \cup [a^+, 1]$. To insure that $F(a^-) = F(a^+)$, set $1 + c(-2a)^{-1/3} = k(2 - 2a)^{-1/3}$. Solving,

$$k = (2 - 2a)^{1/3} + c(1 - 1/a)^{1/3} \tag{3}$$

As c varies between its bounds 0 and $(2a)^{1/3}$, k varies between its bounds $(2 - 2a)^{1/3}$ and 0.

Next, for c fixed, $\pi_1(y, F)$ is a constant function of y on (b, a^-) since

$$\frac{d\pi_1(y, F)}{dy} = 0 \text{ for } y \in (b, a^-),$$

as will now be shown:

$$\begin{aligned} \frac{d\pi_1(y, F)}{dy} &= d\left(\int_b^y ((a - x)/2)F'(x)dx + \int_y^{a^-} ((x + y)/2)F'(x)dx + (1 - F(a^-))(a + y)/2\right)/dy \\ &= ((a - y/2)F'(y) - (y/2)F'(y) + d((y/2)(F(a^-) - F(y)))/dy + (1 - F(a^-))/2 \\ &= F'(y)(a/2 - 3y/2) - F(y)/2 + 1/2 \end{aligned}$$

By (1), $F(y) = 1 + c(a - 3y)^{-1/3}$ for $y \in (b, a^-)$ so that

$$\frac{d\pi_1(y,F)}{dy} = c(a-3y)^{-4/3}(a/2 - 3y/2) - (1 + c(a-3y)^{-1/3})/2 + 1/2 = 0.$$

This completes the proof that for c fixed, $\pi_1(y, F)$ is a constant function of y on (b, a^-). In fact, the definition (1) was derived by solving the differential equation $\frac{d\pi_1(y,F)}{dy} = 0$.

Since F is continuous on $[0, a^-]$, $\pi_1(y, F)$ is a continuous function of y for $y \in [b, a^-]$ so that $\pi_1(y, F)$ is a constant function of y for $y \in [b, a^-]$.

Also, it is clear that for $y \in [0, b)$, $\pi_1(y, F) < \pi_1(b, F)$ since with probability 1 candidate 2 locates in $[b, a^-] \cup [a^+, B]$ so that candidate 1 increases his voter set by moving from y to b. In summary, if candidate 1 is limited to playing $y \in [0, a^-]$, candidate 1 maximizes his payoff against F by choosing any $y \in [b, a^-]$.

To show that candidate 1 limited to playing $y \in [a^+, 1]$ maximizes his payoff against F by choosing any $y \in [a^+, B]$, it is necessary to show $\frac{d\pi_1(y,F)}{dy} = 0$ for $y \in (a^+, B)$.

$$\begin{aligned}
\frac{d\pi_1(y,F)}{dy} &= d\left(F(a^-)(1-(a+y)/2) + \int_{a^+}^{y}(1-(x+y)/2)F'(x)dx + \int_{y}^{B}((x-a)/2)F'(x)dx\right)/dy \\
&= -F(a^-)/2 + (1-y/2)F'(y) - d((y/2)(F(y) - F(a^+)))/dy - ((y-a)/2)F'(y) \\
&= F'(y)(2+a-3y)/2 - F(y)/2
\end{aligned}$$

By (1), $F(y) = k(2 + a - 3y)^{-1/3}$ for $y \in (a^+, B)$ so that

$$\frac{d\pi_1(y,F)}{dy} = k(2+a-3y)^{-4/3}(2+a-3y)/2 - k(2+a-3y)^{-1/3}/2 = 0$$

For $y \in (B, 1]$. $\pi_1(y, F) < \pi_1(B, F)$, since with probability 1 candidate 1 increases his market share by moving from y to B. If candidate 1 is limited to $y \in [a^+, 1]$, then candidate 1 maximizes his payoff by choosing any $y \in [a^+, B]$.

To show that (F, F) is a Nash equilibrium for some choice of c, it is sufficient to show that for the proper choice of c, candidate 1 maximizes his payoff against F by locating in $[b, a^-] \cup [a^+, B]$. By our work above it is enough to show that for the proper choice of c, $\pi_1(a^-, F) = \pi_1(a^+, F)$.

It is now convenient to introduce c into our notation and write $F_c(x)$ in place of $F(x)$. Fix $x_0 \in (a/3, a^-]$. As $c \to 0$, $b \to a/3$. Therefore $\lim_{c\to 0} F_c(x_0) = \lim_{c\to 0}(1 + c(a - 3x_0)^{-1/3}) = 1$. In other words, for small c player 2 playing F_c almost certainly locates near $a/3$. Then for small c

$$\pi_1(a^-, F_c) \approx a/3 < 1 - a \approx \pi_1(a^+, F_c) \tag{4}$$

Next fix $x_0 \in [a^+, 1-(1-a)/3)$. As $c \to (2a)^{1/3}$, $k \to 0$ and $B \to (2+a)/3 = 1-(1-a)/3$. Therefore $\lim_{c\to(2a)^{1/3}} F_c(x_0) = \lim_{c\to(2a)^{1/3}} k(2+a-3x_0) = 0$. For c near $(2a)^{1/3}$, player 2 playing F_c almost certainly locates near $1-(1-a)/3$. For c near $(2a)^{1/3}$

$$\pi_1(a^-, F_c) \approx a > (1-a)/3 \approx \pi_1(a^+, F_c) \tag{5}$$

Since $\pi_1(a^-, F) - \pi_1(a^+, F_c)$ is a continuous function of c, by (4) and (5) and the Intermediate Value Theorem there must be a $\overline{c} \in (0, (2a)^{1/3})$ such that $\pi_1(a^-, F_{\overline{c}}) = \pi_1(a^+, F_{\overline{c}})$. For this $\overline{c}$, $(F_{\overline{c}}, F_{\overline{c}})$ is a Nash equilibrium.□

We can now establish the claim made at the beginning of this section.

Proposition 3 *The 3-player location game with endogenous timing played on* $[0,1]$ *with consumers distributed uniformly has a mixed strategy subgame perfect equilibrium, but no pure strategy subgame perfect equilibrium.*

Proof By Proposition 1, if two players wait in period 1 and the third player locates at $a \in (1/4, 1/2) \cup (1/2, 3/4)$ in period 1, the resulting subgame has no pure strategy Nash equilibrium. Therefore the 3-player location game with endogenous timing has no pure strategy SPE.

Since the 3-player location game with endogenous timing possesses an equilibrium* (s^* of Proposition 4) to show that it possesses a mixed strategy SPE, it is enough to show that every proper subgame possesses a SPE.

Shaked (1982) showed that the subgame that results when all three players wait in period one has a mixed strategy Nash equilibrium. Each subgame that results when exactly two players wait in period 1 has a pure or mixed strategy Nash equilibrium by Propositions 1 and 2. It is quite easy to see that every subgame that results when exactly one player waits in period 1 has a pure strategy Nash equilibrium: if players 1 and 2 locate at x_1 and x_2 respectively in period 1, then x_1^-, x_1^+, x_2^- or x_2^+ is a Nash equilibrium of the 1-player period 2 game. Since all the subgames in this paragraph are simultaneous play games, all these Nash equilibria are trivially subgame perfect.□

Remark 1 *Note that on the equilibrium path of the mixed strategy subgame perfect equilibrium in Proposition 3 two candidates locate at 1/4 and 3/4, respectively in period 1 whereas the third candidate locates at $1/4^+$ in period 2.*

3.2 Equilibrium*

To circumvent the difficulties of fully specifying strategies for each of the players and of showing that these strategies constitute a (mixed) Nash equilibrium in each subgame, we proceed as in Osborne (1993, 2000) by applying the notion of an *equilibrium**. This notion requires only a partial specification of the players' strategies. The general idea of an *equilibrium** is "that no player should be able to increase her payoff by changing her action in any period, given that the behavior of the players in the subgame to which the deviation leads is optimal" (Osborne 1993, p. 142). As Osborne (1993, p. 142) argues, "[t]o check that a strategy profile σ meets this condition, no information is needed about the behavior that σ prescribes in subgames that are reached when more than one player deviates from σ in some period." More precisely, an *equilibrium** is defined as follows (adapted from Osborne 1993, p. 142):

A substrategy σ_i of player i for a game G is a subset $H(\sigma_i)$ of the set of histories of G together with a function that assigns an action of player i to every history in $H(\sigma_i)$. More accurately, the function must assign an action of player i to $h \in H(\sigma_i)$ unless player i is not required to act in the period following h (in other words, unless player i has no information set in the game tree in the period following h). A profile σ of substrategies is an *equilibrium** if (1) for every player i, $H(\sigma_i)$ includes all histories that result when at most one player deviates from σ in any given period and (2) after any such history, no player can increase her payoff by a unilateral change of strategy, given that the other players continue to adhere to σ.

Clearly, including condition (1) in the definition of an *equilibrium** makes sure that the substrategies contain enough information to determine whether condition (2) is satisfied. Note that it is much easier to work with this notion of equilibrium rather than SPE since one does not have to worry about the existence of an equilibrium in subgames that can only be reached by a deviation of two or more players in a given period.

We now use two examples to illustrate the definition of equilibrium*. In Figure 1 subgame K begins at node c. Suppose K is a simultaneous play game with no mixed strategy Nash equilibrium (for example, suppose players 1 and 2 simultaneously choose a positive integer and the player who chooses the larger integer wins 10 while the other player wins 4. In case of a tie each wins 7. Then

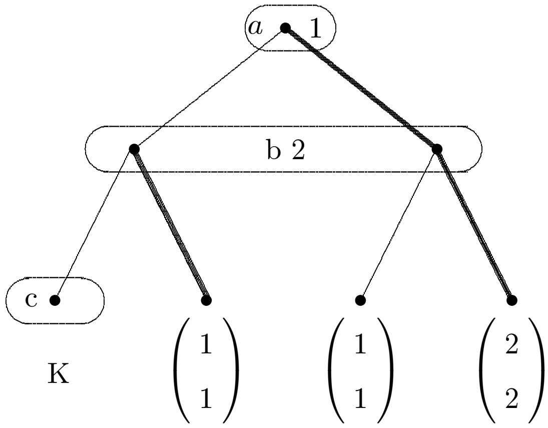

Figure 1: An example

G possesses no mixed strategy SPE and consequently no pure strategy SPE; however, s^* defined by the bold lines in Figure 1 is a pure strategy equilibrium*. Notice that s^* satisfies condition (1) of the definition of equilibrium* since the history of G leading to K requires that *two* players deviate from s^* in the first period of the game.

On the other hand, consider the game of Figure 1 altered slightly by making the information set labeled b into two information sets. Then s^* is not an equilibrium* since K is the result of a history in which player 1 deviates in period 1 and player 2 then deviates in period 2 (note that by turning b into two information sets, we no longer have both players playing in period 1, but we now have player 1 playing in period 1 and player 2 playing in period 2). In fact the new game possesses no equilibrium*.

Concerning the relation between an equilibrium* and a SPE, note that a SPE is an equilibrium*, and that if every proper subgame has a SPE then any equilibrium* can be used as a starting point in the construction of a SPE.

As was stated above, the strengths of the equilibrium* concept are first that an equilibrium* strategy profile possesses the same stability that is the defining characteristic of a SPE; and second that a simple equilibrium* may exist for a game with no SPE or for a game which possesses only complicated, or hard-to-find SPE. We think this will become clear when comparing the results in the preceding subsection with the ones that will be derived in the following subsection.

3.2.1 A pure equilibrium* for the n-player game

Consider the following substrategy profile s^*:[5]

(i) Candidates $1, 2, ..., n-1$ locate at $k, 3k, 5k, ..., 1-k$ with $k = \frac{1}{2(n-1)}$, respectively, in the first period. After a history in which only candidate $i \in \{1, 2, \ldots, n-1\}$ deviates by choosing to wait until period 2, candidate i locates at the point he deviated from, $(2i-1)k$.

(ii) Candidate n chooses to wait in the first period and

- locates at k after a history in which the other candidates chose to locate as described in the first sentence of (i);
- locates at $x+$ after a history in which the set of occupied positions is $\{x, 3k, 5k, ..., 1-k\}$ with $x < k$;
- locates at $x-$ after a history in which the set of occupied positions is $\{x, 3k, 5k, ..., 1-k\}$ with $k < x \leq 3k$;
- locates at $3k-$ after a history in which the set of occupied positions is $\{x, 3k, 5k, ..., 1-k\}$ with $x > 3k$;
- similarly for a deviation in which locations $\{k, 3k, 5k, ..., 1-3k\}$ were chosen in the first period but location $1-k$ was not.
- locates at $(2i-1)k$ after a history in which the set of occupied positions is $\{x, k, ..., (2i-3)k, (2i+1)k, ..., 1-k\}$ with $x \neq (2i-1)k$, $i \in \{2, 3, ..., n-2\}$.
- locates at $(2i-1)k$ after a history in which only candidate $i \in \{1, 2, \ldots, n-1\}$ deviated by choosing to wait.

The next proposition will show that s^* is an equilibrium* of the voting game. The importance of this result is that it describes behavior that is much more in line with what we can observe in the field than the equilibria of the standard model. The most extreme positions, k and $1-k$, are only occupied by one candidate each, i.e., there is just one candidate on the extreme left and one on the extreme right. The standard model predicts that these positions are occupied by two

[5] Of course, there is no loss of generality letting candidates $1, 2, ..., n-1$ be the ones who commit in period 1 according to s^* and letting candidate n be the one who waits until period 2.

parties each. The difference is particularly striking for $n = 4$, a case not uncommon in European elections. The standard model with simultaneous moves predicts two left-wing parties at $1/4$ and two right-wing parties at $3/4$ while the center of the political spectrum is unoccupied. In contrast, s^* predicts one left-wing candidate who locates himself early at $1/6$, one right-wing candidate who chooses at $5/6$, a centristic candidate at $1/2$ and a fourth candidate who waits initially. In the case of s^* the fourth candidate also chooses an extreme left position; but, as the proof of the next proposition will reveal, it is easy to construct other equilibria where he chooses either a right-wing or a centristic platform. In any case, s^* predicts a wider variety of political platforms than one would expect in the standard model. And, perhaps most importantly, it predicts that there will be centristic parties with more than two candidates.

Proposition 4 *The substrategy profile s^* is an equilibrium* of the voting game with endogenous timing.*

Proof To see that s^* is an equilibrium* we have to show that none of the candidates has an incentive to deviate from the behavior prescribed by s^*. Let us first consider candidate n. It is easy to see that s^* prescribes optimal behavior for candidate n in each of the classes of subgames treated in (ii). For example, if no one deviates, candidate n locates at k, earns payoff k, and can do no better; if candidate $i \in \{1, 2, \ldots, n-1\}$ deviates by waiting, then both n and i locate at $(2i-1)k$, n earns payoff k and can do no better; if candidate $n-1$ deviates by locating at $1-2k$ in period 1, then n locates at $(1-2k)^+$, earns payoff $2k$ and can do no better; etc.

Next consider candidate 1: If no one deviates both 1 and n locate at k and 1 earns payoff k. Candidate 1 can do no better as is seen by checking candidates n's reaction to all of 1's possible deviations. For example, if 1 waits, n locates at k and 1 earns at most k. If 1 locates at $2k$ in period 1, n locates at $(2k)^-$ in period 2 and 1 earns $k/2$.

Concerning candidate $n-1$, if no one deviates from s^*, $n-1$ earns $2k$ and can do no better. For example, if $n-1$ waits, n locates at $1-k$ and $n-1$ can earn at most k. If $n-1$ locates at $1-2k$ in period 1, n locates at $(1-2k)^+$ and $n-1$ earns $k/2$.

Finally consider candidate $i \in \{1, 2, \ldots, n-1\}$. If no one deviates i wins $2k$ and can do no better which can be seen without even considering n.□

3.2.2 Comparing other equilibria*

As indicated above, there are other equilibria* where one candidates waits and positions himself at a different location in period 2. Moreover, it is easy to see that there are also equilibria* where all candidates move in the first period—choosing the equilibrium locations of the standard simultaneous-move game. The reader may, thus, ask what we have really gained by introducing endogenous timing. We would like to offer several points: The model as such is more plausible. Its equilibrium structure is richer and there exist some equilibria$*$ that appear much more in line with evidence than the equilibria of the standard model. Moreover, comparing the different equilibria* we find that the ones where one candidate waits in the first period do not only yield a more realistic distribution of platforms but are also more stable. In particular, in an equilibrium* where all candidates move simultaneously, all candidates have alternative best responses (like choosing a location somewhere between other candidates or waiting and then choosing a location). In contrast, in the equilibrium* identified above only players 1 and n have alternative best replies while all candidates would be strictly worse off by changing their action. For $n = 3$ there is a pure equilibrium* where one candidates waits but no pure equilibrium* where all candidates move simultaneously which lets the equilibrium* identified above appear more parsimonious. Finally, let us reinterpret our model as one of firm behavior in an oligopoly market with product differentiation. When $n > 3$, we see that the equilibrium* in which one firm waits to locate is more efficient than the equilibria in the one-period model as firms are spread out more evenly.

4 Conclusion

We have investigated a two-period spatial-voting game with endogenous timing. Our main result, is that in the general n-player case, this game has an equilibrium* in which any $n-1$ candidates evenly spread out in period 1 whereas the n-th candidate waits until period 2 and then chooses a platform.

On a purely game-theoretic level we have demonstrated the usefulness of the equilibrium* concept for sequential games as put forward by Osborne (1993). Unlike a subgame perfect equilibrium, an equilibrium* only requires a strategy vector to be a Nash equilibrium on any subgame that is reached by a history in the course of which no two players deviate in the same period from the play prescribed by the equilibrium*. This makes it much more easy to work with. In fact, according to our analysis in Section 3.1, there is little hope of finding a (mixed) subgame perfect equilibrium in the general n-player case of our spatial-voting model, whereas we showed with comparative ease that the general model does possess an equilibrium* in pure strategies.

On a substantive level, we consider our analysis another step toward more realistic voting models that allow for political campaigning during which candidates decide when and where to locate on the political spectrum. As we have shown here, making the assumptions more realistic can also make the equilibrium predictions more appealing. In particular, we have seen that with endogenous timing there are equilibria with more political variety and fewer extremist parties than the standard model predicts.

Obvious avenues for future work are to allow for more periods or the analysis of continuous-time models. Also, it might be interesting to relax the assumption of perfect commitment by allowing candidates to relocate (at some cost) or analyzing reputation formation in repeated voting games.

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