Strategic Delay and Efficiency in Global Games

Nathan Larson
Centre for Economic Learning and Social Evolution
University College London
Gower Street, London WC1E 6BT
Email: n.larson@ucl.ac.uk

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Abstract

The global games literature shows that perturbing a complete information coordination game with correlated private information unleashes strategic uncertainty that can act, in domino fashion, to rule out all but a single dominance-solvable equilibrium. We show (in a two-player model) that when players also have the option to delay acting, the domino chain is broken, and multiple equilibria resurface. Furthermore, the combination of strategic uncertainty and the option to delay can have a salubrious effect – inefficiency in all equilibria is lower than without delay and vanishes as the delay cost goes to zero.

1 Introduction

Games in which payoffs are complementary in the actions of several agents often admit multiple Nash equilibria if those actions must be chosen simultaneously. When this is true, each equilibrium is supported by a different set of (correct) beliefs about how other agents will play. Because the Nash concept is silent about which set of beliefs, if any, will arise, considerable effort has been devoted to equilibrium refinements that select unique beliefs in a plausible way. For games with common interests, this has led to a fork in the road: refinements that emphasize uncertainty in agents' beliefs tend to pick the risk dominant equilibrium, while refinements that expand agents' action sets (for example, by allowing pre-play communication) tend to focus beliefs on the Pareto optimal equilibrium. One naturally wonders who will win the arms race when both belief uncertainty and agents' attempts to surmount it are factors. This paper explores this question by relaxing the assumption that agents must act simultaneously in a game that is perturbed away from complete information in the manner of Carlsson and Van Damme (1993). It turns out that the option of delaying one's play, if it is not too costly, helps to prune away inefficient equilibria, even in the presence of incomplete information. When delay is costless, only Pareto optimal equilibria survive. However, the perturbation to information is not innocuous – if delay is costly, then inefficient equilibria are played with positive probability even in the complete information limit.

A simple stylized example may help to convey the intuition behind the results. Suppose Anne and Bob have a tentative plan to go to the theater on Saturday night. After observing different noisy signals about the play's quality (they read the reviews in different papers) on

Friday morning, each must decide independently whether to purchase a ticket or commit to an alternative plan for the evening that does not involve the other one. Suppose that both Anne and Bob prefer seeing the play together rather than alone. Furthermore, suppose that if the quality of the play is good enough (a “triumph”), either agent would prefer to see it regardless of whether the other one goes, and that if the quality is bad enough (a “fiasco”), both would prefer their outside options regardless of what the other one does. The interesting question is what they will do when their signals about the quality of the play are between these extremes, when each would like to go to the play, but only if the other does as well.

The global games literature suggests that this situation can be analyzed *via* iterated dominance. Suppose Anne’s review says that the play is mediocre. She reasons that there is a reasonable chance that Bob has gotten a fiasco review and will take his outside option for sure. Since she would buy a ticket only if she were almost certain that Bob would do the same, she must take her outside option. But then if Bob’s review says that the play is average, he must worry that Anne may have gotten a mediocre review – if the chance of this is high enough, he should take his outside option too. This process also works from the other end – when Anne reads that the play is merely good, she should nonetheless buy a ticket for sure if the chance that Bob has seen a triumphant review is high enough. Applying iterated dominance from both ends may yield a unique threshold signal above which Anne and Bob buy tickets. Although it would be efficient for them to go to the play whenever it is not a fiasco, under the equilibrium threshold they will go to the play only when the quality is substantially higher. However, as long as the noise in their signals is fairly low, Anne and Bob manage to avoid miscoordination most of the time – only for signals close to the threshold do they face a substantial amount of uncertainty about how the other will act.

Now suppose that there is an additional option to purchase tickets at the door on Saturday, at a slightly higher price. To keep things symmetric, suppose that Anne and Bob can delay committing to their outside options until Saturday for the same incremental cost. Furthermore, Anne and Bob will have a chance to talk on Friday night. If the cost of delay is not too high, then when Anne sees a review near her threshold, she may decide to delay her decision until Saturday in order to see what Bob has chosen. If Bob could be sure that she would delay, then he could lower his threshold substantially, buying tickets for all but the fiascos on Friday, with confidence that she would follow his lead on Saturday. In fact, he will not be quite this sure, but even the possibility is enough to encourage him to reduce his threshold a little bit. This in turn will encourage Anne to reduce her threshold a bit as well. After playing out this cycle of threshold reductions, Anne and Bob will be able to coordinate on efficient play-going considerably more often than they could without the option of delay. In fact, we will show that as noise vanishes and as the cost of delay becomes negligible, they always coordinate efficiently.

The next section describes the formal models first without, and then with, the option of delay, and in the latter case derives conditions under which all equilibria exhibit delay. In such an equilibrium, there is a positive probability that both agents delay their decisions. When this happens, it becomes common knowledge that the dominance arguments above do not apply, and so the potential for multiple equilibria re-emerges. Section 3 characterizes two prominent equilibria: an “optimistic” equilibrium in which the agents expect to coordinate on the efficient action in the continuation after both have delayed, and a “pessimistic” equilibrium in which they expect coordination on the inefficient action. In both cases, the gap between equilibrium play and efficient play vanishes as the cost of delay shrinks, but in the "optimistic" equilibrium, convergence to efficiency is at a faster rate. Section 4 provides a similar convergence result for all equilibria. The value-added in the general result is that it places no *a priori* restrictions on the form of strategies; in particular, threshold strategies are not assumed. Section 5 concludes with a discussion of these results in the context of other recent work on strategic timing in games with complementarities.

2 The Model

Before developing the full model, we begin by introducing a simpler version without delay called the *benchmark game* and sketch how iterated dominance can be applied to identify a unique equilibrium of this game. Then we add the delay option to the model and establish a preliminary result showing that this option will be used in equilibrium.

2.1 The Benchmark Model

The benchmark game (BG) that we will look at is a special case of a global game as defined by Carlsson and Van Damme. It is essentially the same as the game that has been used by Morris and Shin (1998) among others to study currency attacks. There are two agents who simultaneously decide whether to invest (I) or not (N). An agent who invests incurs a cost of a and earns a return of v_H if the other agent also invests and v_L if he does not, with $v_H > v_L > 0$. An agent who does not invest always earns 0. Under complete information, this situation is summarized by the following game form.

	I	N
I	$v_H - a, v_H - a$	$v_L - a, 0$
N	$0, v_L - a$	$0, 0$

For moderate investment costs ($a \in (v_L, v_H)$), this game has two strict pure strategy Nash equilibia: the Pareto optimal one in which both agents invest, and another one in which neither does.

Next we introduce incomplete information about the investment cost. Assume that it is common knowledge that the investment cost is a random variable $\tilde{a}$, with realization a, distributed uniformly on $[v_L - \varepsilon, v_H + \varepsilon]$. Each agent receives a private noisy signal of the cost $a_i = a + \varepsilon h_i$, where h_1 and h_2 are independent draws from a symmetric, strictly increasing, atomless distribution H with mean zero and support on $[-1, 1]$. Write $A = [v_L - 2\varepsilon, v_H + 2\varepsilon]$ for the space of possible signals, and observe that a strategy for agent i now must specify an action for each signal he might receive.[1] We call this environment the benchmark game.

We will show that BG has a unique symmetric Nash equilibrium that can be identified by iterated.dominance. The argument goes as follows. Define thresholds $\underline{a}_i$ and $\bar{a}_i$ to be the highest signal below which agent i always chooses I and the smallest signal above which i always chooses N respectively. These thresholds must exist because for small (large) enough a_i, investing (not investing) is a dominant strategy. Now consider agent 1's expected payoff to investing when

[1] To be more precise, a strategy associates a probability distribution over actions with each signal, but in all of the equilibria, agents will play pure strategies almost everywhere.

he receives the signal $a_1 = \bar{a}_2$. Because of the symmetry of the error distributions, he believes it equally likely that agent 2's signal is higher or lower than his. Since agent 2 never invests for $a_2 > \bar{a}_2$, agent 1 expects agent 2 to invest with probability no greater than $\frac{1}{2}$. Thus his expected profit upon investing is

$$\begin{aligned}\pi_1(I \mid a_1 = \bar{a}_2) &= \Pr(2 \text{ invests})v_H + (1 - \Pr(2 \text{ invests}))v_L - \bar{a}_2 \\ &\leq (v_H + v_L)/2 - \bar{a}_2\end{aligned}$$

Furthermore, agent 1's expected profit from investing with signals higher than $\bar{a}_2$ is certainly less than this, as the expected cost will be higher and the probability of co-investment lower. If $(v_H+v_L)/2-\bar{a}_2 < 0$, this means that agent 1's expected profit from investing is strictly negative for all $a_1 \geq \bar{a}_2$. We can conclude then that if $\bar{a}_2 > \bar{v} = (v_H + v_L)/2$, then $\bar{a}_1 < \bar{a}_2$. However, by switching the agents in this argument, it is also true that if $\bar{a}_1 > \bar{v}$, then $\bar{a}_2 < \bar{a}_1$. This leads to a contradiction if $\bar{a}_1$ and $\bar{a}_2$ are both greater than $\bar{v}$, so in a symmetric equilibrium, we must have $\bar{a}_1 = \bar{a}_2 \leq \bar{v}$.

Next consider agent 1's expected payoff to investing when his signal is $a_1 = \underline{a}_2$. In this case, he expects co-investment with probability at least $\frac{1}{2}$ since agent 2 always invests when $a_2 < \underline{a}_2$. Thus his expected payoff to investing is at least $\bar{v} - \underline{a}_2$. Since this is strictly positive when $\bar{v} > \underline{a}_2$, by reasoning parallel to that above, we can conclude that $\bar{v} > \underline{a}_2$ implies $\underline{a}_1 > \underline{a}_2$. Reversing the agents again, we also have $\bar{v} > \underline{a}_1$ implies $\underline{a}_2 > \underline{a}_1$. To avoid a contradiction, we must have $\underline{a}_1 = \underline{a}_2 \geq \bar{v}$ in a symmetric equilibrium. But then because the lower threshold cannot be greater than the upper threshold, we must have $\underline{a}_i = \bar{a}_i = \bar{v}$. Thus the (iterated) dominance regions for N and I completely partition the set of signals, and the equilibrium is unique: invest if and only if the cost signal a_i is less than $\bar{v}$.[2]

Several points are worth noting. First, as ε vanishes, the agents coordinate with probability 1. Moreover, they coordinate on the equilibrium that is risk-dominant given the cost realization, so the effect of strategic uncertainty persists even as that uncertainty vanishes. Moreover, this effect depends crucially on the fact that no matter how small ε is, it is never common knowledge that both equilibria are possible (i.e., that neither strategy is dominant).

2.2 The Model with Delay

In this section, the benchmark model is augmented by assuming that agents need not always act simultaneously. Now an agent may either pick an action immediately or defer the decision until later. The timing is as follows. First, agents observe their private signals. Then, in period 0, the agents simultaneously decide whether to invest, not invest, or wait (W). If neither agent waits, the game ends immediately, with payoffs as before. Otherwise, each agent choosing to wait incurs a cost of c and the game moves to period 1. In period 1, the agents who have waited observe the actions taken in period 0 and choose either I or N (simultaneously, if both have waited). As soon as both players have chosen an action from $\{I, N\}$ payoffs are realized as before. Call this the asynchronous game (AG).

Two points are worth emphasizing. First, the actions N and W are not equivalent because an agent choosing N does not preserve the option of choosing I later. For this reason, it may be best to think of N as a commitment to invest in an alternative project with constant payoff

[2] Of course equilibrium actions will not be uniquely determined at the threshold signal.

0. When we refer to "not investing" in the sequel, this interpretation should be borne in mind. Second, payoffs are realized once and for all (not period by period) and depend only on the final actions taken by the agents and any relevant waiting costs. To fix ideas, one may think of two firms positioning themselves to enter one of two markets (I or N) that will open in period 1. Committing to one of the markets in period 0 allows a firm to spread out the investment costs associated with entry, saving c, but a firm can also delay its choice until period 1. Profit flows are realized once and for all when the markets open at period 1 and depend only on whether both firms are in market I (but not on when they decided to enter).

It is clear that if the waiting cost is large enough that delay is a dominated strategy, then the equilibria of AG and BG coincide. The next result establishes a rough converse: delay always occurs when it is relatively costless. For brevity, define $\Delta v = v_H - v_L$.

Proposition 1 *If $c < \Delta v/4$, then every sequential equilibrium of AG has a positive probability of delay.*

Proof. Suppose to the contrary that there is an equilibrium in which each player chooses I or N in period 0 for every signal. Then the same argument used to establish the BG equilibrium implies that each agent invests precisely when his signal is greater than $\bar{v}$. Thus, the payoff to agent 1 when his signal is $a_1 = \bar{v}$ is 0. Were he to wait, he would observe 2 choosing either I (if $a_2 < \bar{v}$) or N (if $a_2 > \bar{v}$), events he believes to be equally likely. By waiting and mimicking 2, he earns $\frac{1}{2} \cdot 0 + \frac{1}{2}(v_H - E(\tilde{a} \,|\, a_2 < a_1 = \bar{v})) - c > (v_H - \bar{v})/2 - c = \Delta v/4 - c$, a profitable deviation. ■

This proposition underscores the fact that uncertainty about one's opponent's action has two effects: it encourages the choice of "safe" actions, but it also creates an option value to delaying one's decision until the uncertainty is resolved. The BG equilibrium compresses all strategic uncertainty into a small region around the threshold signal, creating strong incentives for agents with signals in that region to wait. On the other hand, agents with extreme signals are sufficiently sure of how their opponents will act that waiting is not worthwhile. In the sequel, we assume that $c < \Delta v/4$ and show that equilibria exist in strategies with two thresholds for period 0 action: a signal below which the investment is immediately made and a signal above which the decision not to invest is made immediately, with waiting in the intermediate region.

3 Simple Equilibria

One implication of Proposition 1 is that in a symmetric equilibrium, there is a positive chance that both agents wait in period 0. We will refer to the pair of strategies in period 1 after both agents have waited in period 0 as the *continuation game*. A key observation is that it must be common knowledge in the continuation game that neither agent's signal is too extreme. For example, if the equilibrium specifies that each agent always plays I (N) immediately for signals less than $\underline{a}$ (greater than $\bar{a}$), then upon arriving in the continuation game it is common knowledge that both signals lie in $[\underline{a}, \bar{a}]$. If this is a subset of $[v_L, v_H]$, then it is common knowledge that neither agent believes either strategy to be ruled out by dominance. As a result, the continuation game will typically have multiple equilibria. This casts some doubt on whether it will be possible to make sharp predictions about outcomes of AG.

The equilibria constructed below go some way toward answering this question. Both are characterized by simple threshold strategies $(\underline{a}, \bar{a})$ under which an agent invests immediately for cost signals below $\underline{a}$, chooses N immediately for cost signals above $\bar{a}$, and waits for intermediate signals. The first equilibrium assumes an "optimistic" continuation game in which the agents coordinate on investment after both wait, while the second assumes a "pessimistic" continuation in which there is no investment. The equilibria share certain qualitative features. In both cases, the waiting region shrinks as signals grow more precise, with immediate coordination in the noiseless limit. Furthermore, in both cases this coordination is more and more frequently on the efficient investment I as the cost of waiting becomes small.

3.1 An "Optimistic" Equilibrium

We will look for a symmetric equilibrium in threshold strategies as described above. In addition to the thresholds $(\underline{a}_i, \bar{a}_i)$, a full strategy description must specify the action taken in period 1 by an agent with signal a_i who has observed his opponent choose $S \in \{I, N, W\}$ in period 0.

As a first step, observe that an agent who has waited and observed his opponent choose I or N in period 0 will always follow suit. To see why, suppose that in some equilibrium, agent 1 were to choose N after observing 2's choice of I in period 0. This can only happen if 1's posterior belief about $\tilde{a}$, revised to incorporate the fact that $a_2 \leq \underline{a}_2$, makes N a dominant strategy for him. However, 2's choice of I is the most favorable news about I that 1 could possibly receive, so he must expect to play N regardless of what he observes in period 0 after waiting. In this case, he is better off playing N immediately and saving the waiting cost. The same is true for the opposite case.

Now consider the situation in which both agents have waited until period 1. At this point, it is common knowledge that $a_1 \in \overline{W}_1 = [\underline{a}_1, \bar{a}_1]$ and $a_2 \in \overline{W}_2 = [\underline{a}_2, \bar{a}_2]$. Furthermore, it is common knowledge that 1's expectation of $\tilde{a}$ is less than $E(\tilde{a} \,|\, a_1 = \bar{a}_1,\ a_2 \in \overline{W}_2)$, so whenever $\bar{a}_1 \leq v_H$ and $\bar{a}_2 \leq v_H$, it is common knowledge that 1 does not believe N to be a strictly dominant strategy. Extending this argument, when $\overline{W}_i \in [v_L, v_H]$ for $i \in \{1, 2\}$, it is common knowledge that neither player believes either strategy to be strictly dominant.[3] For the time being, we will assume that this condition holds. Consequently, dominance arguments do not restrict the set of equilibrium outcomes in this continuation game. Here we focus on the optimistic continuation in which both agents always choose to invest in period 1 after (W, W) is played in period 0.

Next we turn to identifying optimal period 0 thresholds $(\underline{a}, \bar{a})$ given this continuation. Toward this end, let us suppose that agent 2 uses the strategy $(\underline{a}, \bar{a})$ and consider agent 1's best response. In a symmetric equilibrium, we will need agent 1 to be indifferent between I and W when $a_1 = \underline{a}$ and to be indifferent between W and N when $a_1 = \bar{a}$. This pair of indifference conditions will pin down the equilibrium thresholds. Before jumping into the analysis, it will be helpful to define the distribution of the difference between the errors in the agents' signals G (normalized by ε); that is, $h_1 - h_2 \sim G$. The symmetry of H implies that G is symmetric with mean 0. We let $\phi(k) = 1 - G(k)$ be the probability that the difference in the errors exceeds $k\varepsilon$. Given the agents' uniform prior over $\tilde{a}$, we can then express the probability that agent 1 places

[3] This is not a tight bound - the sets W_i can spill a little bit outside of $[v_L, v_H]$ without upsetting things.

on agent 2 having received a signal substantially higher or lower than his own signal[4]:

$$\begin{aligned}\Pr(a_2 &< a_1 - k\varepsilon \mid a_1) = \phi(k) \\ \Pr(a_2 &> a_1 + k\varepsilon \mid a_1) = \phi(k)\end{aligned}$$

In particular, $\Pr(a_2 < a_1 \mid a_1) = \Pr(a_2 > a_1 \mid a_1) = \phi(0) = 1/2$.

Now we turn to agent 1's best response. Regardless of his signal, he can earn 0 by choosing N. Let us write $\pi_I(a_1)$ and $\pi_W(a_1)$ for the expected payoff he earns by choosing I or W respectively when his signal is a_1. If he chooses I, he will succeed in coordinating with agent 2 whenever agent 2 chooses I or W – that is, whenever $a_2 < \bar{a}$. He will fail if agent 2 chooses N, which happens when $a_2 > \bar{a}$. The probability of this latter event is $\Pr(a_2 > a_1 + (\bar{a} - a_1) \mid a_1) = \phi(k)$, where $k = (\bar{a} - a_1)/\varepsilon$. Thus, the expected payoff to choosing I is

$$\begin{aligned}\pi_I(a_1) &= (1 - \phi(k))(v_H - E(\tilde{a} \mid a_1; a_2 < \bar{a})) + \phi(k)(v_L - E(\tilde{a} \mid a_1; a_2 > \bar{a})) \\ &= v_H - \phi(k)\Delta v - E(\tilde{a} \mid a_1)\end{aligned}$$

Alternatively, if agent 1 were to choose W, he would still end up coordinating on I whenever agent 2 chooses I or W. (In the first case he would follow agent 2's action, and in the second case we have assumed an optimistic continuation.) The difference is that he can avoid miscoordinating when agent 2 chooses N. His expected payoff is

$$\begin{aligned}\pi_W(a_1) &= (1 - \phi(k))(v_H - E(\tilde{a} \mid a_1; a_2 < \bar{a})) + \phi(k)(0) - c \\ &= (1 - \phi(k))(v_H - E(\tilde{a} \mid a_1; a_2 < \bar{a})) - c\end{aligned}$$

where the second term in the first line is written explicitly to emphasize the case in which agent 1 avoids miscoordinating by waiting. Now we would like to show that there exist $(\underline{a}, \bar{a})$ such that agent 1's best response to $(\underline{a}, \bar{a})$ is $(\underline{a}, \bar{a})$. Essentially, this is a matter of showing that Figure 1 accurately represents the relationship between $\pi_I(a_1)$, $\pi_W(a_1)$, and 0. More formally, we need to show that there exist $(\underline{a}, \bar{a})$ such that

1. $\pi_I(a_1)$ is strictly decreasing.
2. $\pi_W(a_1)$ is strictly decreasing and crosses 0 at $\bar{a} < v_H$
3. $\pi_I(a_1) - \pi_W(a_1)$ is strictly decreasing and crosses 0 at $\underline{a}$, with $v_L < \underline{a} < \bar{a}$.

These claims are verified in the appendix for sufficiently small noise. Observing that $\phi(0) = \frac{1}{2}$, the indifference condition $\pi_W(\bar{a}) = 0$ yields

$$\frac{1}{2}(v_H - E(\tilde{a} \mid a_1 = \bar{a}, a_2 < \bar{a})) = c \tag{1}$$

while the condition that $\pi_I(\underline{a}) = \pi_W(\underline{a})$ gives us

$$\phi(\bar{k})(E(\tilde{a} \mid a_1 = \underline{a}, a_2 > \bar{a}) - v_L) = c \tag{2}$$

[4] To be more precise, these expressions are valid for signals in the range $[v_L, v_H]$. Boundary effects start to become relevant closer to the edges of the support of $\tilde{a}$, but the equilibrium thresholds will never lie above v_H or below v_L.

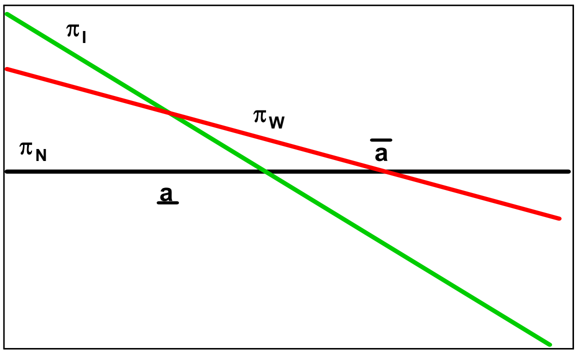

Figure 1: Equilibrium configuration of π_I, π_W, and π_N

where $\bar{k} = (\bar{a} - \underline{a})/\varepsilon$.

This construction allows us to state the following result.

Proposition 2 *When $c < \Delta v/4$ and $\varepsilon < 2c$, a (symmetric) sequential equilibrium of AG exists with threshold strategies defined by (1) and (2) and an optimistic continuation.*

The condition on ε is imposed to ensure that the waiting region $(\underline{a}, \bar{a})$ lies strictly within (v_L, v_H). For positive ε, there is a chance that an agent whose signal lies just outside (v_L, v_H) will have her beliefs about $\tilde{a}$ shifted into (v_L, v_H) by her opponent's period 0 action. Thus, her option value of waiting is positive; if the waiting cost is sufficiently low, she will not act immediately. With a bit more work one can show that the equilibrium we have constructed does not change much in this case, but we will not present this since the emphasis later will be on limiting equilibria with vanishing noise and finite waiting costs. In order to illustrate the intuition behind conditions (1) and (2), we present an example in which the waiting region $(\underline{a}, \bar{a})$ can be explicitly characterized.

Example 1 *Uniform noise*

Suppose that H is a uniform distribution on $[-1, 1]$. Then the distribution of the difference in signals will have a triangular density with support on $[-2\varepsilon, 2\varepsilon]$. One can show that $E(\tilde{a} \,|\, a_1 = \bar{a}, a_2 < \bar{a}) = \bar{a} - \varepsilon/3$, so using (1), the upper threshold is simply $\bar{a} = v_H - 2c + \varepsilon/3$. In other words, the range of signals for which an agent immediately chooses not to invest despite believing that joint investment is Pareto optimal, $[v_H - 2c + \varepsilon/3, v_H]$, is positive and shrinks as the cost of waiting declines. This range also shrinks as uncertainty about the cost of investing grows; this is because greater uncertainty about one's opponent's signal results in greater uncertainty about his action, increasing the option value of waiting. Because the outcomes of I and W coincide except when one's opponent chooses N, the lower threshold depends on the probability of this happening. Here the expressions are more complicated:

$$\begin{aligned} \Pr(a_2 &> \bar{a} \,|\, a_1 = \underline{a}) = \phi(\bar{k} \,|\, \underline{a}) = \frac{1}{8}(2 - \bar{k})^2 \\ E(\tilde{a} \,|\, a_1 &= \underline{a}, a_2 > \bar{a}) = \underline{a} + \frac{4/3 - \bar{k}^2(1 - \bar{k}/3)}{(2 - \bar{k})^2}\varepsilon \end{aligned}$$

and the equilibrium $\underline{a}$ solves a cubic equation. The important point though, is that an agent with signal $\underline{a}$ must expect his opponent to choose N with strictly positive probability, $\phi(\bar{k})$ – otherwise there would be no option value to waiting and he would do strictly better by investing immediately. Thus, $\bar{k} = (\bar{a} - \underline{a})/\varepsilon < 2$, so the width of the waiting region cannot be greater than 2ε.

The Noiseless Limit

Now consider what happens upon approaching the complete information limit, holding the waiting cost fixed. As $\varepsilon \to 0$, (2) goes to

$$\frac{1}{8}(2 - \bar{k})^2(v_L - \underline{a}) = c$$

so in the limit, $\bar{k}$ must be less than 2, and $\bar{a} - \underline{a} = \bar{k}\varepsilon$ must go to 0. In other words, because the width of the waiting region is constrained by the noise in the signals for the reasons discussed above, the waiting region must vanish along with that noise. Then, in the limit, $\underline{a} = \bar{a} = v_H - 2c$, and the agents always coordinate immediately – on (I, I) when $a < v_H - 2c$, and on (N, N) when $a > v_H - 2c$. Even though the option to wait is not exercised in the complete information limit, its influence does not disappear: as the waiting costs shrinks from $\Delta v/4$ to 0, the threshold between coordination regimes grows from $\bar{v}$, its level in the benchmark game, to v_H. If we let both the noise and the delay costs vanish (with the former going to zero first), then the agents coordinate on immediate investment whenever it is efficient to do so.

Furthermore, the limiting behavior of this equilibrium does not depend on the distribution of the noise; for general H we have the same result.

Proposition 3 *In the limit as $\varepsilon \to 0$ of the symmetric equilibrium described above, (I, I) is played immediately if $a < v_H - 2c$ and (N, N) is played immediately otherwise. As $c \to 0$, (I, I) is played whenever it is efficient.*

Proof. Fix a positive c. As in the case where H is uniform, we refer to conditions (1) and (2). As $\varepsilon \to 0$, a_1 and a_2 converge to a, so $E(\tilde{a} \,|\, a_1 = \bar{a}, a_2 < \bar{a}) \to \bar{a}$. Then (1) implies that $\bar{a} \to v_H - 2c$. Next, if $\bar{a}$ and $\underline{a}$ were to differ by more than 2ε, the probability of observing signals more than $\bar{a} - \underline{a}$ apart would be 0, as would be the left-hand side of (2). Since this is inconsistent with (2), $\bar{a} - \underline{a}$ must go to 0 with ε. ■

The form of this particular equilibrium is not entirely robust when limits are taken in the opposite order. To see why, refer to the case with uniform noise. Fixing ε and taking c to 0, the equilibrium conditions give us $\bar{a} = v_H + \varepsilon/3$, or in other words, there are agents who believe N to be a dominant strategy but nonetheless wait. By itself this does not pose a problem; it is simply a statement that the option value of waiting is positive. However, the equilibrium conditions ensure that $E(\tilde{a} \,|\, a_1 = \bar{a}, a_2 < \bar{a}) = v_H$ – that is, with a zero waiting cost, an agent who is just indifferent between N and waiting expects the value of joint investment to be zero conditional on it occurring. The event he conditions on is $a_2 < \bar{a}$ because he expects joint investment to occur whenever 2 does not choose N immediately. However, upon arriving at period 1, he has additional information about 2's action in period 0. His revised expectation of the investment cost is either $E(\tilde{a} \,|\, a_1 = \bar{a}, a_2 < \underline{a}) < v_H$ after observing I or $E(\tilde{a} \,|\, a_1 = \bar{a}, a_2 \in [\underline{a}, \bar{a}]) > v_H$ if 2 waited. Thus he is *ex post* unwilling to play I after observing the (relatively) bad news that his opponent waited also, and the optimistic continuation cannot be sustained.

This equilibrium could be patched up to be robust to the order of limits, but it would still suffer from the criticism that assuming an optimistic continuation after both agents wait is arbitrary. The next section investigates whether any of the flavor of this equilibrium is preserved when the assumption of an optimistic continuation is dropped.

3.2 A "Pessimistic" Equilibrium

The construction is very similar to the previous one with the following exception: now when both agents have waited in period 0, the continuation strategies will specify that they both choose N in period 1. This will change the indifference conditions that determine the waiting region.

First consider the upper threshold. In the optimistic equilibrium, if agent 1 chooses to wait he can expect the final outcome to be (I, I) if $a_2 < \underline{a}$ or $a_2 \in (\underline{a}, \bar{a})$ and (N, N) if $a_2 > \bar{a}$. With a pessimistic continuation, he expects (I, I) only if $a_2 < \underline{a}$ and (N, N) otherwise. This makes waiting less attractive and changes the indifference condition between W and N to

$$\phi(\bar{k})(v_H - E(\tilde{a} \,|\, a_1 = \bar{a}, a_2 < \underline{a})) = c \tag{3}$$

The effect will be to tend to push down $\bar{a}$ relative to the optimistic case.

Next we look at the lower threshold. If agent 1 invests, he expects the outcome to be (I, I) if $a_2 < \bar{a}$ and (I, N) otherwise. Alternatively, if he waits, he expects (I, I) if $a_2 < \underline{a}$ and (N, N) otherwise. Noting that $\Pr(a_2 < \bar{a} \,|\, a_1 = \underline{a}) = 1 - \phi(\bar{k})$ and $\Pr(a_2 < \underline{a} \,|\, a_1 = \underline{a}) = \frac{1}{2}$, the condition for indifference between I and W becomes

$$\begin{aligned} (1 - \phi(\bar{k}))v_H + \phi(\bar{k})v_L - E(\tilde{a} \,|\, a_1 &= \underline{a}) = \frac{1}{2}(v_H - E(\tilde{a} \,|\, a_1 = \underline{a}, a_2 < \underline{a})) - c \\ \phi(\bar{k})\Delta v - \frac{1}{2}(v_H - E(\tilde{a} \,|\, a_1 &= \underline{a}, a_2 > \underline{a})) = c \end{aligned} \tag{4}$$

(The second step applies the law of iterated expectations to $E(\tilde{a} \,|\, a_1 = \underline{a})$.) For the pessimistic equilibrium, an additional regularity condition on the distribution of the noise is needed.

Condition 1 *G is weakly log-concave.*

Proposition 4 *A (symmetric) sequential equilibrium of AG exists with threshold strategies defined by (3) and (4) and a pessimistic continuation.*

Proof. The proof, which is in the appendix, involves showing first that thresholds satisfying (3) and (4) exist and are unique, and then that using these thresholds is optimal for each agent. ■

As before, the extent of delay is limited by the degree of uncertainty about the investment cost: at each threshold, an agent will only be indifferent between acting immediately and waiting if there is a positive probability that his opponent will take the opposite action to the one he is considering; consequently, the size of the waiting region $\bar{a} - \underline{a}$ is bounded above by the magnitude of the signal noise. (That is, $\phi(\bar{k} \,|\, a) > 0$, and so $\bar{a} - \underline{a} < 2\varepsilon$.) The limit properties of this equilibrium hinge, as earlier, on this fact.

The Noiseless Limit

Because of the bound above, $\bar{a} - \underline{a} \to 0$ as $\varepsilon \to 0$, so in the limit as the noise in signals vanishes, the equilibrium strategies can be characterized by a single threshold $\bar{a}$. To find this limit threshold, we take the limit of conditions (3) and (4) as $\varepsilon \to 0$, yielding

$$\begin{aligned} \phi(\bar{k} \mid \bar{a})(v_H - \bar{a}) &= c \\ \phi(\bar{k} \mid \bar{a})\Delta v - \frac{1}{2}(v_H - \bar{a}) &= c \end{aligned}$$

Solving for the threshold produces

$$v_H - \bar{a} = \sqrt{c^2 + 2c\Delta v} - c \tag{5}$$

It is worth pausing for a moment to compare the limit equilibria generated by optimistic and pessimistic continuations. In both cases, there is immediate coordination with probability 1 as signal noise vanishes. Furthermore, in both cases, the range of costs $(\bar{a}, v_H)$ for which the agents fail to coordinate on investment even though it would be Pareto optimal to do so shrinks as the cost of delay declines. Finally, both cases are outcome equivalent to the benchmark game when $c \geq \Delta v/4$ and produce fully efficient coordination as $c \to 0$.

However, they differ in the rate at which the *rate* at which inefficient coordination vanishes with c. For the optimistic limit equilibrium, this region vanished at rate c, but for the pessimistic limit equilibrium, (5) indicates that it shrinks at the slower rate $c^{1/2}$. In understanding this difference, it may be helpful to think about how these equilibria might arise out of a sequence of deviations from strategies that do not use delay.

Start with the unique equilibrium of BG, in which I (N) is played for signals below (above) $\bar{v}$. For finite ε, these strategies leave agents with signals near $\bar{v}$ uncertain about how their opponents will play (as demonstrated in Proposition 1), giving them an incentive to wait. For the sake of clarity, suppose that only agent 1 is allowed to wait. Then agent 1 should deviate to waiting in some neighborhood of $\bar{v}$. Now consider agent 2's best response to this deviation. Now when he chooses I, joint investment will occur more often than before, because there will be some signals $a_1 > \bar{v}$ for which 1 previously would have played N, but now will wait and follow 2. This makes I more attractive to agent 2, and she should react by shifting her threshold between I and N upward, toward higher costs. But this in turn makes I more attractive to agent 1, and he should respond by shifting his waiting region upward, thus eliciting another response by agent 2, and so on. The upward march of the two agents' thresholds will continue as long as the option value to agent 1, minus c, is greater than zero (the payoff to choosing N). This option value is roughly equal to the probability that agent 2 chooses I ($\tilde{\ }\frac{1}{2}$) times the gain to coordinating on I (roughly $v_H - \bar{a}$). Setting $(v_H - \bar{a})/2 \approx c$ yields $\bar{a} \approx v_H - 2c$.

The logic is similar when both agents can wait. With an optimistic continuation, the option value to waiting at $\bar{a}$ (relative to choosing N) is still roughly $(v_H - \bar{a})/2$ because the chance of an opponent choosing either I or W is $\frac{1}{2}$ and both lead to coordination on I. As with one-sided waiting, this should lead to a threshold of roughly $\bar{a} \approx v_H - 2c$.

When conjectures are pessimistic, the option value to waiting at $\bar{a}$ is lower. While the gain to coordinating on I is still roughly $(v_H - \bar{a})$, this will happen only when the opponent chooses I (rather than I or W), and the probability of this is less than $\frac{1}{2}$. To identify this probability, call it p, more precisely, we can use information about the lower threshold $\underline{a}$. At the lower threshold, choosing I rather than W leads to a gain of about $v_H - \underline{a} \approx v_H - \bar{a}$ if the opponent chooses W and a loss of $\underline{a} - v_L$ if the opponent chooses N (call this latter chance q). As $\underline{a}$ and $\bar{a}$ increase, the gain $v_H - \bar{a}$ decreases. Since $\underline{a} - v_L$ does not decrease, indifference requires that q decreases at roughly the same rate as $(v_H - \bar{a})$. But finally note that p and q are equal – each is the probability that the opponent and own signals differ by more than $\bar{a} - \underline{a}$. Going back to where we started, this means that the option value to waiting at $\bar{a}$ is on the order of $p(v_H - \bar{a}) \approx (v_H - \bar{a})^2$. This leads to a threshold that approaches v_H more slowly ($\bar{a} \approx v_H - \sqrt{c}$) as the cost of delay declines.

4 General Results

As we have seen, by adding delay to the benchmark game, we lose one of the major selling points of the global game approach – the equilibrium prediction is no longer unique. A natural concern is that the equilibrium set may be too large to make any robust predictions about game outcomes. The examples in the last section provide some hope that there may be some properties shared by all equilibria; this section will provide a general result that partially confirms this hope. Before proceeding to the result, we offer two exemplary equilibria that illustrate why general results may be difficult to obtain.

Example 2 *"Non-monotonic" Equilibrium*

Suppose that errors follow the same uniform distribution as in Example 1, and set the model parameters to $v_H = 1, v_L = 0, \varepsilon = .01, c = .02$. Define the following partition of the signal space into seven regions $R_i = (\alpha_{i-1}, \alpha_i)$, with $0 = \alpha_0 < \alpha_1 < ... < \alpha_7 = 1$. The other region boundaries are presented below.

$$\alpha_1 = 0.813\,83 \quad \alpha_2 = 0.824\,47 \quad \alpha_3 = 0.836\,87$$
$$\alpha_4 = 0.852\,52 \quad \alpha_5 = 0.94742 \quad \alpha_6 = 0.96333$$

The following strategy specifying how to act depending on which region one's signal falls in, constitutes a symmetric equilibrium:[5]

R_1	R_2	R_3	R_4	R_5	R_6	R_7
I	W_N	N	W_I	I	W_I	N

In this table, W_I is shorthand for "Wait in period 0. If the opponent waits as well, choose I in period 1, otherwise follow the opponent's action." W_N is defined similarly.

Figures 2 and 3 show the expected payoff to agent 1, as a function of his signal, for each action when agent 2 is playing according to the strategy above. Figure 3 provides a closer look at the "non-monotonic" interval $R_1 - R_5$ in which the best response is to invest for low and high cost signals but choose N for intermediate signals.[6]

One can think of this as a regime-shifting equilibrium in which the agents coordinate on the pessimistic equilibrium of 3.2 for low cost signals and coordinate on the optimistic equilibrium of 3.1 for high cost signals. The two regimes are separated by a buffer region R_4 of signals for

[5] The equilibrium was computed numerically. Details of the computation are available upon request.

[6] To avoid clutter, only the higher of the expected payoffs to W_I and W_N is presented. The kink is the point at which W_I overtakes W_N.

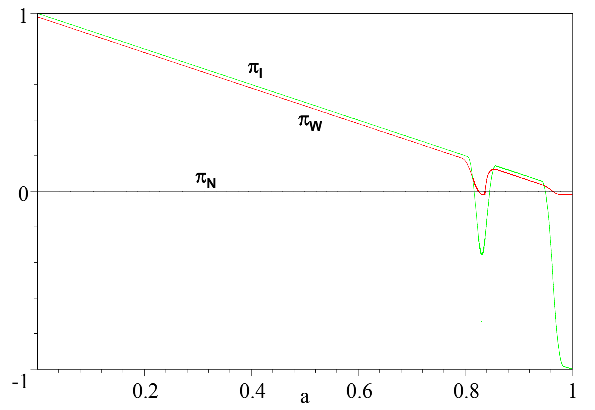

Figure 2: Non-monotonic equilibrium

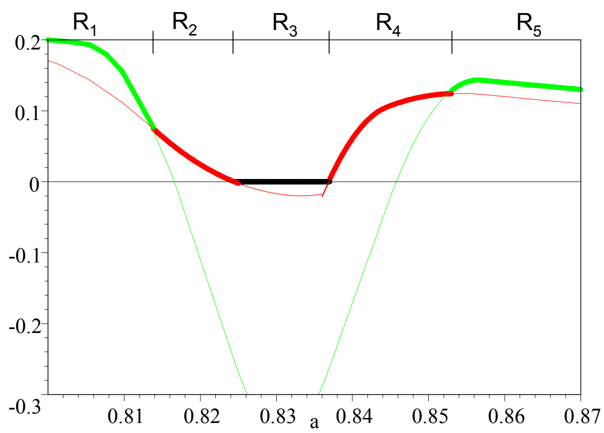

Figure 3: Non-monotonic region

which the agents wait. By way of comparison, the complete information game ($\varepsilon = 0$, $c > 0$) admits as a correlated equilibrium *any* arbitrary partition of the public signal a into sunspot regions, with coordination on I or N depending on the region. Noise in the public signal ($\varepsilon > 0$) disrupts the pinpoint precision required to sustain arbitrary sunspot regions as an equilibrium, but the buffer provided by the waiting regions allows a certain degree of sunspottiness to survive. To highlight one example of how this buffer operates, consider the border signals α_4 and α_5. At both signals, an agent must be indifferent between investing immediately and waiting; however, investing is less attractive at α_5 because the cost is about 10% higher. Compensating for this is the fact that the agent at α_5 faces a smaller risk of being stranded by an opponent playing N than an agent at α_4. (The probabilities are 0.0209 and 0.0237 respectively.) This compensating reduction in risk is made possible by the flexibility of the waiting regions: R_6 provides a wider buffer from the $N-$playing regions than R_4 does.

Equilibria such as this one may offend our aesthetic taste for simplicity, and they are certainly tedious to compute; categorizing all the possibilities does not appear to be a practical option. Whether we can afford to ignore this sort of equilibrium as a possible description of reality is less clear. For example, financial professionals often claim that market dynamics are characterized by resistance, barriers, and thresholds, terms that sometimes seem to describe non-monotonic behavior and generally only make sense if sunspots are at work. The example is constructed so that by excising the sunspot – that is assigning N to be played on $\{R_4, ..., R_7\}$ without making any changes to the strategy on $\{R_1, R_2, R_3\}$ – we arrive back at the pessimistic equilibrium of Section 3.2. One might be led to conjecture from this that no equilibrium does worse than the simple pessimistic equilibrium in generating investment; however, the next example shows that this is incorrect.

Example 3 *Partial coordination in the continuation game*

All of the equilibria constructed thus far have specified full coordination on either I or N in the continuation game in each waiting region. This example illustrates that partial coordination is also a possibility. The setup is as in the previous example except that we will look at a wider range of values for c and ε. The proposed strategy has the form

$$\begin{array}{cccc} R_1 & R_2 & R_3 & R_4 \\ I & W_I & W_N & N \end{array}$$

where $R_1 = (0, \underline{a})$, $R_2 = (\underline{a}, a^*)$, $R_3 = (a^*, \bar{a})$, $R_4 = (\bar{a}, 1)$. Thus, after both agents have waited, neither can be sure how the other will act. Agents with relatively low cost signals will invest, those with high cost signals will choose N, and sometimes the agents will fail to coordinate. There are now three equilibrium conditions: one at $\underline{a}$, one at $\bar{a}$, and an equation determining the threshold a^* at which the agents switch from W_I to W_N. The condition at $\bar{a}$ is identical to the one for the pessimistic equilibrium (3). However, the condition at $\underline{a}$ differs from (4) because the chance of co-investment when the opponent's signal lies in R_2 makes waiting more attractive than it would be if W_N were played on both R_2 and R_3. If we write $k' = (a^* - \underline{a})/\varepsilon$, so that $\Pr(a_2 > a^* \,|\, a_1 = \underline{a}) = \phi(k')$, then the condition at $\underline{a}$ becomes

$$\phi(\bar{k})E(\tilde{a} \,|\, a_1 = \underline{a}, a_2 > \bar{a}) - \big(\phi(k') - \phi(\bar{k})\big) = c$$

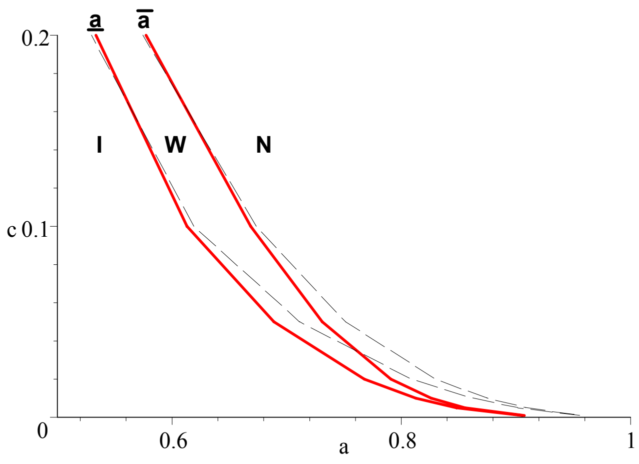

Figure 4: Partial coordination equilibrium

At a^* we must have indifference between W_I and W_N. The two strategies differ only in the event that both agents wait. In this case, we can write $p(a) = \Pr(a_2 \in R_2 \,|\, a_1 = a)$ for the probability that agent 2 invests when $a_1 = a$. Note that $p(a)$ is decreasing in a. Furthermore, we can write $A(a) = E(\tilde{a} \,|\, a_1 = a, a_2 \in \{R_1, R_2\})$ for the expected cost incurred by investing with signal $a_1 = a$ after both agents have waited – this is increasing in a. The threshold a^* between W_I and W_N is defined by the unique solution to

$$\begin{aligned} (1 - p(a))v_L + p(a)v_H - A(a) &= 0, \text{ or} \\ p(a) &= A(a) \end{aligned}$$

Expressions for $p(a)$ and $A(a)$ are derived in the appendix. Figure 4 presents the equilibrium thresholds when the waiting cost is equal to the level of noise, computed for several values of $c = \varepsilon$ between zero and 0.2. The solid lines represent $\underline{a}$ and $\bar{a}$; the horizontal distance between them is the waiting region $R_2 \cup R_3$ for a particular level of c and ε. (The boundary between R_2 and R_3 is omitted.) For comparison, the dashed lines indicate the corresponding waiting region for the pessimistic equilibrium of Section 3.2. The two equilibria are quite close when c and ε are large, but for small c and ε, the range of investment costs for which the partial coordination equilibrium generates investment converges to $[0, 1]$ more sluggishly than for the pessimistic equilibrium. The problem begins with the lower threshold $\underline{a}$, where a switch from W_N to W_I on R_2 tends to make waiting more attractive, pushing $\underline{a}$ down. This switch does not have a first order effect at the upper threshold $\bar{a}$, as agents here never invest after waiting. The second order effect of $\underline{a}$ shifting down is to make waiting less attractive at $\bar{a}$. (The chance of observing the opponent choose I in period 0 declines.) This tends to push $\bar{a}$ down. The overall effect is to shift the entire waiting region down, toward lower investment costs.

In contrast with the earlier examples, the limiting behavior illustrated in Figure 4 applies when c and ε go to zero at the same rate (rather than letting ε vanish first). Nonetheless, the earlier result of convergence to investment for all $a \in (0, 1)$ is still obtained, suggesting that the prior results are not driven by the order of the limits. For technical reasons, the general result below is developed for the ordered double limit (first $\varepsilon \to 0$, then $c \to 0$), but the fundamental logic of the proof appears to survive when c and ε vanish at the same rate.

In all of the equilibria we have presented, as noise vanishes, there is a threshold cost below which investment always occurs, and this threshold approaches the efficient level v_H as the cost of waiting goes to zero. In the rest of this section, we prove that two broad classes of equilibria of AG share this feature. We start with some notation. Let us denote by $\text{AG}(\varepsilon, c)$ the game with level of noise ε and cost of waiting c. We say that a strategy is a *single waiting region (SWR)* strategy if it specifies some $\underline{a}$ and $\bar{a}$ such that I is played below $\underline{a}$, N is played above $\bar{a}$, and some combination of W_I and W_N is played on $(\underline{a}, \bar{a})$. A strategy is *simple* if every contiguous region on which a version of W is played contains only W_I or W_N. The equilibrium strategy of Example 2 is simple but not SWR, while the strategy of Example 3 is SWR but not simple. The equilibrium strategies from Section 3 are both simple and SWR. Let Γ^c_ε (Ψ^c_ε) be the set of symmetric, sequential simple (SWR) equilibria of $\text{AG}(\varepsilon, c)$. Finally, let Γ^c be the set all strategies γ that are limits of strategies in Γ^c_ε as the noise vanishes: $\gamma \in \Gamma^c$ iff $\exists\{\varepsilon_i\}$ and $\{\gamma_{\varepsilon_i}\}$ with $\gamma_{\varepsilon_i} \in \Gamma^c_{\varepsilon_i}$, $\{\varepsilon_i\} \to 0$, and $\{\gamma_{\varepsilon_i}\} \to \gamma$. Define Ψ^c similarly. Notice that Γ^c and Ψ^c are subsets of the set of equilibria of $\text{AG}(0, c)$. We can then prove the following result.

Proposition 5 *There exists a threshold $a^*(c)$ such that*

1. *In every member of Γ^c and Ψ^c, investment occurs for all $a < a^*(c)$.*
2. *$v_H - a^*(c) \to 0$ (at least as fast as $\sqrt{c}$) as $c \to 0$. Thus, as $c \to 0$, every member of Γ^c and Ψ^c involves coordination on investment if and only if it is efficient.*

The intuition of the proof will be sketched here; the formal details are in the appendix. The discussion below should also shed some light on why it is difficult to extend the result to non-simple strategies with multiple waiting regions (or provide a counter-example). The basic idea is to begin with an arbitrary strategy s and "shuffle" its constituent regions to generate a different strategy s' that has the form of either the pessimistic example ($I - W_N - N$) or the partial coordination example ($I - W_I - W_N - N$). Then, indifference relations that apply in an equilibrium best response to s can be mapped into preference inequalities that apply to a best response to s'. The advantage of this mapping is that the derived inequalities will be substantially easier to characterize than the original indifference relations.

This approach is illustrated for a simple strategy in Figure 5. Part (a) depicts the equilibrium strategy s. We draw attention to two points: a_I is the highest signal below which I is *always* played, and a_N is the lowest signal for which N is *ever* played. Since s represents a symmetric equilibrium, each agent must be indifferent between playing I and W_I with signal a_I, and indifferent between N and W_N with signal a_N, if his opponent is playing s. Part (b) shows a tweaked version s' of s in which the entire strategy to the right of a_N is replaced with N. Increasing the chance of facing N tends to make W_I more attractive relative to I, so if $W_I \sim I$ when facing s at a_I, then we have $W_I \succeq I$ when facing s' at a_I. Similarly, the shift from s to s' can only reduce the chance of facing I at a_N, so if $N \sim W_N$ when facing s at a_N, then $N \succeq W_N$ when facing s' at a_N. as well. We can now proceed with these two preference inequalities as for the pessimistic equilibrium of Section 3.

Figure 6 shows the procedure for a SWR strategy. In this case, s' involves shuffling the interval (a_I, a_N) to shift all play of I (W_N) to the left (right). Strategy I does equally well as W_I against either I or W_I but elicits full investment against W_N, while W_I does not. At a_I, an agent faces W_N less often under s' than under s, so if he is indifferent between I and

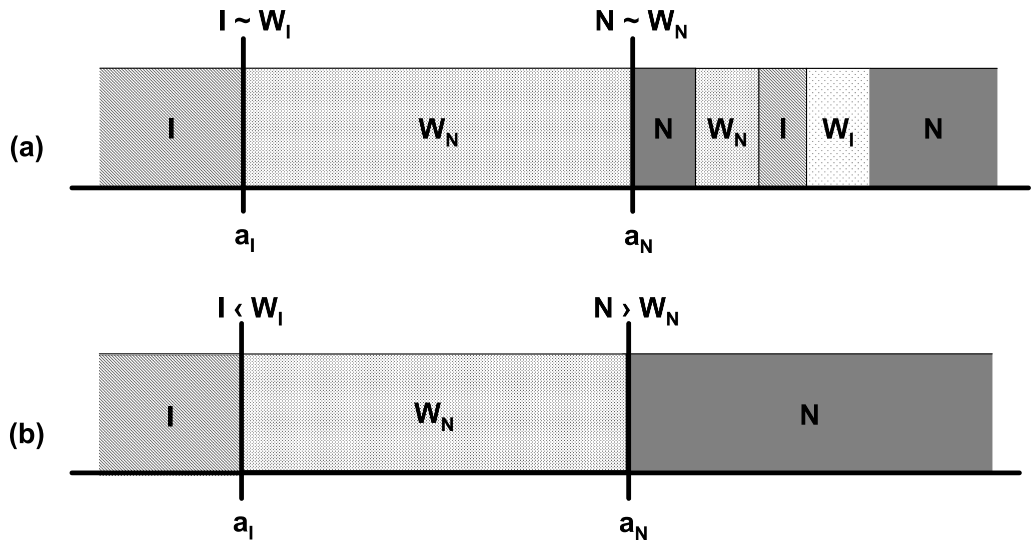

Figure 5: Simple strategy

W_I against s, he must prefer W_I at a_I when facing s'.[7] Now consider a_N. W_N outperforms N only when the opponent chooses I; otherwise both earn 0. An agent with signal a_N faces I less often under s' than under s, so in the best response to s', $N \succeq W_N$ at a_N. We are left in a situation similar to the partial coordination equilibrium above. The first two constraints will imply that if $v_H - a_I$ fails to converge to 0 as $c \to 0$, then the W_I region must expand to fill the entire interval (a_I, a_N). But this would mean that the chance of facing N after both agents have waited must go to 0. By appealing to a third constraint, that W_N is preferred to W_I at a_N. we arrive at a contradiction. This is also where the difficulty with extending the proof to non-simple, multiple waiting region strategies lies. If W_N were played on a region slightly above a_N, then the probability of facing N after both agents wait need not go to zero, even if W_I expands to fill (a_I, a_N). Thus, the contradiction needed to rule out the possibility that $v_H - a_I$ is positive in the limit cannot be established.

5 Discussion

We have shown that for a large class of equilibria, strategic uncertainty interacts with the possibility of delay to restrict outcomes in a sensible way: investment occurs whenever it is efficient *and* worth waiting for. Both strategic uncertainty and delay are necessary for this result – the former provides an incentive for costly delay, and the latter provides a way to break up inefficient equilibria. Furthermore, the form of the strategic uncertainty is crucial as well; the correlation in the agents' information and the "boundary conditions" imposed by the dominance of each action in an extreme region mean that even as ε goes to zero, there is yet some agent in some state of the world whose uncertainty about his opponent's action is large. It is the unraveling started by this agent's decision to wait that generates the result.

[7] The fact that shifting regions toward a_I means they are more likely to be faced by an agent with signal a_I follows from the fact that the error distributions are single-peaked.

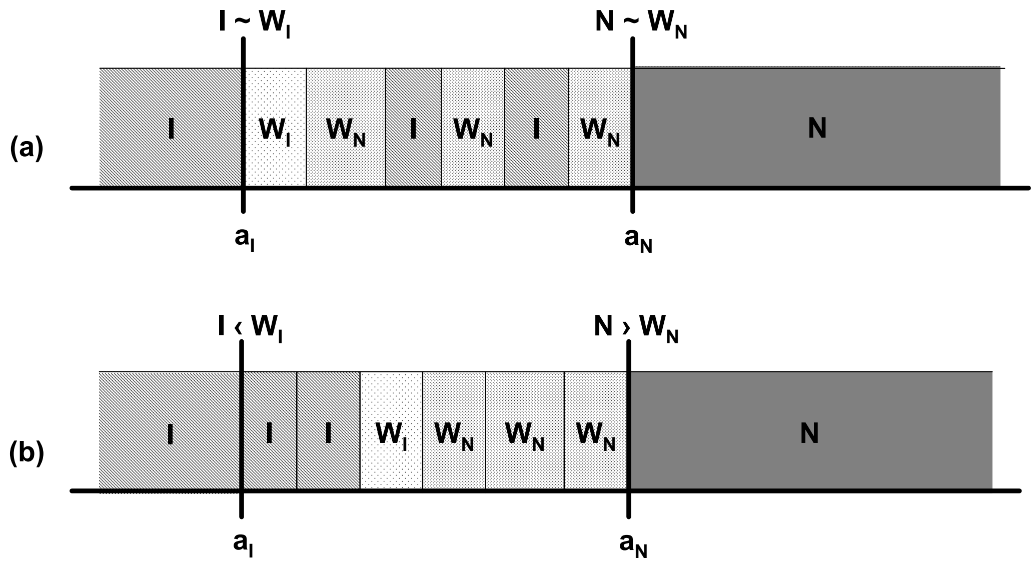

Figure 6: SWR strategy

Endogenous timing in coordination games has been well explored by Chamley and Gale (1994) and Gale (1995) among others, but the study of how endogenous timing and global game-style strategic uncertainty is less developed. In two recent papers, Chamley (2003a, 2003b) develops dynamic models in which a continuum of investors must coordinate on a currency attack. Under a different information structure than mine, he also concludes that multiple equilibria can survive and explores policy interventions by a central bank that could rule out "attack" equilibria. The closest paper to mine is Dasgupta (2001). It is worth taking some time to outline the difference between the two approaches, as they are largely complementary.

Like Chamley, Dasgupta looks at a continuum of small agents, while I focus on the two agent case. Both papers confront the possibility that sufficiently agents will delay that there is no dominant strategy in the continuation game. Dasgupta solves this problem by introducing uncertainty about how many agents have acted and how many have delayed, permitting a second application of iterated dominance arguments to be applied. With a small number of agents, as in my model, this assumption is less justified; instead I focus on characteristics of the set of continuation equilibria. Finally, Dasgupta specializes to normally distributed noise and focuses on monotonic equilibria, allowing him to derive extensive comparative statics results, while I look for weaker results on a larger class of equilibria and for arbitrary specifications of noise. *Inter alia*, he identifies a free-riding effect – if delay costs are too low, not enough agents may be willing to step forward and lead with the efficient action. As a result, efficiency may be highest with intermediate delay costs. With two agents, leadership involves less of a risk – as long as the other agent has waited, he will follow suit – and consequently, lower delay costs always improve welfare.

Coordination games have been widely used because they capture situations with complementarities in a simple way, but that simplicity is also their Achilles heel. It is well understood that the outcomes of coordination games are not always robust when the games are enriched in natural ways, as by introducing noise. We have argued that it is equally mistaken to ignore second-order refinements – in this case, when agents can respond to the introduction of noise through delay, the outcomes can be quite different from those predicted by the original model and the first-order refinement. This paper provides a taste for the variety spanned by these possible outcomes and shows that some general predictions can still be made. However, much more work remains to broaden these generalizations further and extend them to more sophisticated models of timing.

6 Appendix

Verification of the claims leading to Proposition 1
Claim 1: $\pi_I(a_1)$ is strictly decreasing.

Observe that $\pi_I(a_1) = v_H - \phi((\bar{a} - a_1)/\varepsilon)\Delta v - E(\tilde{a} \,|\, a_1)$. Both the second and third terms are strictly decreasing in a_1.

Claim 2: There exists $\bar{a} < v_H$ such that $\pi_W(a_1)$ is strictly decreasing and crosses 0 at $\bar{a}$.

For the first part, fix an arbitrary $\bar{a} < v_H$. Then $\pi_W(a_1) = (1 - \phi((\bar{a} - a_1)/\varepsilon))(v_H - E(\tilde{a} \,|\, a_1; a_2 < \bar{a})) - c$, where both terms in parentheses are positive and strictly decreasing in a_1 for $a_1 < v_H$. For the second part, substitute $\pi_W(\bar{a}) = \frac{1}{2}(v_H - E(\tilde{a} \,|\, a_1 = \bar{a}; a_2 < \bar{a})) - c$. At $\bar{a} = v_H$, this expression is less than $\frac{1}{2}(v_H - (v_H - \varepsilon)) - c = \varepsilon/2 - c < 0$. At $\bar{a} = v_L$, $\pi_W(\bar{a})$ is greater than $\frac{1}{2}(v_H - v_L) - c > 0$. By continuity, $\pi_W(\bar{a}) = 0$ for some $\bar{a}$ in (v_L, v_H).

Claim 3: $\pi_I(a_1) - \pi_W(a_1)$ is strictly decreasing and crosses 0 at $\underline{a}$, with $v_L < \underline{a} < \bar{a}$.

$\pi_I(a_1) - \pi_W(a_1) = c - \phi((\bar{a} - a_1)/\varepsilon)(E(\tilde{a} \,|\, a_1; a_2 > \bar{a}) - v_L)$. The components of the second term are both positive (the latter for $a_1 > v_L$) and increasing, so $\pi_I(a_1) - \pi_W(a_1)$ is strictly decreasing for $a_1 > v_L$. Furthermore,

$$\pi_I(v_L) - \pi_W(v_L) > c - \phi((\bar{a} - v_L)/\varepsilon)((v_L + \varepsilon - v_L) > c - \varepsilon/2 > 0$$

If we can show $\pi_I(\bar{a}) - \pi_W(\bar{a}) < 0$, then we are done. Toward this end, note that

$$\begin{aligned} \pi_I(\bar{a}) - \pi_W(\bar{a}) &= c - (E(\tilde{a} \,|\, a_1 = \bar{a}; a_2 > \bar{a}) - v_L)/2 \\ &= (v_H + v_L)/2 - (E(\tilde{a} \,|\, a_1 = \bar{a}; a_2 > \bar{a}) + E(\tilde{a} \,|\, a_1 = \bar{a}; a_2 < \bar{a}))/2 \\ &= \bar{v} - \bar{a} \end{aligned}$$

The second line follows by adding the identity $\pi_W(\bar{a}) = 0$. Furthermore, $\pi_W(\bar{a}) = 0 > (v_H - \bar{a})/2 - c = (\frac{\Delta v}{2} + \bar{v} - \bar{a})/2 - c = (\Delta v/4 - c) + (\bar{v} - \bar{a})$. By assumption, the first term is positive, so we must have $\bar{a} > \bar{v}$. But then $\pi_I(\bar{a}) - \pi_W(\bar{a}) < 0$ as claimed, so there must be some $\underline{a}$ such that $\pi_I(\underline{a}) - \pi_W(\underline{a}) = 0$ and $v_L < \underline{a} < \bar{a}$.

Proof of Proposition 4
Step 1: (3) and (4) have a unique solution.

Rewrite the equations in terms of $\bar{k}$ and $\underline{a}$, so (3) becomes

$$\phi(\bar{k})(v_H - E(\tilde{a} \,|\, a_1 = \underline{a} + \bar{k}\varepsilon, a_2 < \underline{a})) = c$$

Note that the left-hand side of (4) is increasing in $\underline{a}$, and for a fixed $\bar{k}$ can be made larger than c (for $\phi(\bar{k}) > c/\Delta v$) by taking $\underline{a}$ close to v_H and smaller than c by taking $\underline{a}$ close to v_L. Write $a_m(\bar{k})$ for the unique $\underline{a}$ that satisfies (4), given $\bar{k}$. Because the left-hand side of (4) is decreasing in $\bar{k}$, $a_m(\bar{k})$ is increasing.

Next, note that the left-hand side of (3) is decreasing in $\underline{a}$. For a fixed $\bar{k}$, it will be larger than c (again, as long as $\phi(\bar{k}) > c/\Delta v$) for $\underline{a}$ near v_L and smaller than c for $\underline{a}$ near v_H. Write $a_n(\bar{k})$ for the $\underline{a}$ that solves (3), given $\bar{k}$. LHS(3) decreasing in $\bar{k}$ implies $a_n(\bar{k})$ decreasing. Furthermore, $a_m(0) \approx v_L + 2c < a_n(0) \approx v_H - 2c$, and $a_m(\phi^{-1}(c/\Delta v)) \approx v_H > a_n(\phi^{-1}(c/\Delta v)) \approx v_L$, so there is a unique $\bar{k}$ such that $a_m(\bar{k}) = a_n(\bar{k})$.

Step 2: $(\underline{a}, \bar{a})$ is a best response to $(\underline{a}, \bar{a})$

Suppose that 2 plays $(\underline{a}, \bar{a})$. Clearly I is a best response for 1 for any signals that rule out facing N, i.e., for $a_1 \le \bar{a} - 2\varepsilon$. For $a_1 = \underline{a} - x > \bar{a} - 2\varepsilon$, the expected payoff difference between I and W is

$$\begin{aligned}
\Pr(W)(v_H - E(\tilde{a}\,|\,\underline{a} - x, a_2 &\in [\underline{a}, \bar{a}])) - \Pr(N)(E(\tilde{a}\,|\,\underline{a} - x, a_2 > \bar{a}) - v_L) + c \\
(\phi(x) - \phi(x + \bar{k}))(v_H - E(\tilde{a}\,|\,\underline{a} - x, a_2 &\in [\underline{a}, \bar{a}])) - \phi(x + \bar{k})(E(\tilde{a}\,|\,\underline{a} - x, a_2 > \bar{a}) - v_L) + c \\
&\phi(x)(v_H - a') - \phi(x + \bar{k})(\Delta v + \Delta a) + c
\end{aligned}$$

where $a' = E(\tilde{a}\,|\,\underline{a} - x, a_2 \in [\underline{a}, \bar{a}])$ and $\Delta a = E(\tilde{a}\,|\,\underline{a} - x, a_2 > \bar{a}) - E(\tilde{a}\,|\,\underline{a} - x, a_2 \in [\underline{a}, \bar{a}])$. Both a' and Δa are decreasing in x; for Δa, this is a consequence of logconcavity of G. Furthermore, logconcavity of ϕ implies that $\phi(x)/\phi(x + \bar{k})$ is increasing in x. Suppose that 1 were indifferent between I and W for some positive x^*. Then $\phi(x)(v_H - a') - \phi(x + \bar{k})(\Delta v + \Delta a) < 0$ on a neighborhood of x^*. But then the payoff difference can be written

$$\phi(x + \bar{k})\left(\frac{\phi(x)}{\phi(x + \bar{k})}(v_H - a') - (\Delta v + \Delta a)\right) + c$$

The term in parentheses is negative near x^* and increasing (becoming less negative) in x. It is multiplied by a term that is positive and decreasing in x, so the entire expression is increasing in x. That is, I (W) is strictly preferred for signals in a neighborhood below (above) $\underline{a} - x^*$. This is true for any indifference point $x^* \ge 0$, including $x^* = 0$, so I must be strictly preferred for all signals below $\underline{a}$. (Otherwise the expected payoff difference would have to cross 0 with positive slope for some signal below $\underline{a}$, a contradiction.) This argument extends directly to show that W is strictly preferred to I for signals above $\underline{a}$.

As for the upper threshold, N is a clear best response for signals large enough that there is no hope (or fear) of facing I. The relative payoff to W (vs. N) is increasing in the probability of facing I, which in turn increases as a_1 decreases, so $\bar{a}$ is unique, and N (W) is strictly preferred for higher (lower) signals.

Proof of Proposition 5

Let a_I be the boundary below which I is always played and let a_N be the boundary below which N is never played, that is, $a_I = \sup\{a_{I'} : I \text{ is played at } a\ \forall a \le a_{I'}\}$ and $a_N = \sup\{a_{N'} : N \text{ is never played at } a\ \forall a \le a_{N'}\}$. Suppose that $a_I = a_N$. Then continuity of payoffs requires that I and N both have an expected payoff of 0 at a_I. A deviation to waiting at a_I has payoff at least $\frac{1}{2}(v_H - E(\tilde{a}\,|\,a_1 = a_I, a_2 < a_I)) - c > \frac{1}{2}(v_H - a_I) - c$ (because with probability 1/2 one's opponent has a lower signal for which I is always played). If $a_I < v_H - 2c$, then this last term is positive, and waiting is strictly preferred at a_I, and by continuity, in a neighborhood of a_I, contradicting the definitions of a_I and a_N. Thus it must be that $a_I \geq v_H - 2c \geq v_H - (\sqrt{c^2 + 2c\Delta v} - c)$ (for all $c < \Delta v/4$), and so the proposition is satisfied.

Suppose instead that $a_I < a_N$. We will establish a bound for equilibria with pessimistic conjectures about play after (W, W) and then show that the bound can only be tighter for other conjectures. We will write $\Pr(R \,|\, a_{R'})$ for the probability that 2 plays R conditional on $a_1 = a_{R'}$, $R, R' \in \{I, N, W\}$. There are two cases to consider, depending on whether I is ever played on the interior of $[a_I, a_N]$.

Case 1: Only W is played on $[a_I, a_N]$

Then 1 must be indifferent between I and W at a_I, so we have

$$\Pr(N \,|\, a_I)(E(\tilde{a} \,|\, a_I, 2 \text{ plays } N) - v_L) - \Pr(W \,|\, a_I)(v_H - E(\tilde{a} \,|\, a_I,\ 2 \text{ plays } W)) = c$$

The expectations of $\tilde{a}$ must lie within ε of a_I, so

$$\begin{aligned}
\Pr(N \,|\, a_I)(a_I + \varepsilon - v_L) &> \Pr(W \,|\, a_I)(v_H - (a_I + \varepsilon)) + c \\
\Pr(N \,|\, a_I)\Delta v &> (1 - \Pr(I \,|\, a_I))(v_H - (a_I + \varepsilon)) + c \\
&> (1 - \Pr(I \,|\, a_I))(v_H - a_I) + c - \varepsilon \\
\Pr(N \,|\, a_I)\Delta v + \Pr(I \text{ and } a_2 > a_N \,|\, a_I)(v_H - a_I) &> (1 - \Pr(I \text{ and } a_2 < a_N \,|\, a_I))(v_H - a_I) + c - \varepsilon \\
(\Pr(N \,|\, a_I)\Delta v + \Pr(I \text{ and } a_2 > a_N \,|\, a_I))\Delta v &> (1 - \Pr(I \text{ and } a_2 < a_N \,|\, a_I))(v_H - a_I) + c - \varepsilon \\
\Pr(a_2 > a_N \,|\, a_I)\Delta v &> (1 - \Pr(I \text{ and } a_2 < a_N \,|\, a_I))(v_H - a_I) + c - \varepsilon \\
\Pr(a_2 > a_N \,|\, a_I)\Delta v &> (1 - \Pr(a_2 < a_I \,|\, a_I))(v_H - a_I) + c - \varepsilon \\
\Pr(a_2 > a_N \,|\, a_I)\Delta v &> \frac{1}{2}(v_H - a_I) + c - \varepsilon \\
\phi(\bar{k}) &> \frac{(v_H - a_I)/2 + c - \varepsilon}{\Delta v}
\end{aligned}$$

where $\bar{k} = (a_N - a_I)/\varepsilon$. The fact that $\Pr(a_2 > a_N \,|\, a_I) > 0$ establishes that $\bar{k} < 2$.

Next, 1 must be indifferent between N and W at a_N, yielding

$$\begin{aligned}
\Pr(I \,|\, a_N)(v_H - E(\tilde{a} \,|\, a_N, 2 \text{ plays } I)) &= c \\
\Pr(I \,|\, a_N)(v_H - a_N - \varepsilon) &< c \\
\Pr(I \text{ and } a_2 < a_I \,|\, a_N)(v_H - a_N - \varepsilon) &< c \\
\phi(\bar{k})(v_H - a_N - \varepsilon) &< c \\
\phi(\bar{k})(v_H - a_I - 3\varepsilon) &< c \\
\phi(\bar{k}) &< \frac{c}{v_H - a_I - 3\varepsilon}
\end{aligned}$$

Together, these imply

$$\frac{x/2 + c - \varepsilon}{\Delta v} < \frac{c}{x - 3\varepsilon}$$

letting $x = v_H - a_I$. We will return to this inequality after addressing Case 2.

Case 2: I is played somewhere on the interior of $[a_I, a_N]$

The logic of Case 1 hinged on two inequalities. In the latter, an agent who plays N at a_N must expect to see I sufficiently rarely. When I is never played on $[a_I, a_N]$ but always played below a_I, this bounds $\bar{k}$ below. If I is also sometimes played on $[a_I, a_N]$, then in principle, a tighter lower bound on $\bar{k}$ could be applied. On the other hand, in the former inequality, an agent who plays W at a_I must expect to see N sufficiently often *relative to* W, putting an upper bound on $\bar{k}$ relative to how often W is seen. When I is played somewhere on $[a_I, a_N]$ it displaces W, so this upper bound must become looser. The difficulty in Case 2 is in showing that the tightening upper bound matters more than the loosening lower bound.

Suppose that σ is the measure of signals in the interval $[a_I, a_N]$ for which I is played. Consider an alternative profile in which I is played everywhere on $[a_I, a_I + \sigma]$ and W is played everywhere on $[a_I + \sigma, a_N]$. That is, all play of I is squeezed to the left end of the interval. The strategy of the proof will be to show that the inequalities that must hold for the true profile imply a set of inequalities for this alternative profile that can be treated in a way similar to Case 1.

First consider indifference at a_N. Just as before, we have

$$\begin{aligned} \Pr(I \mid a_N)(v_H - a_N - \varepsilon) &< c \\ (\Pr(I \text{ and } a_2 \in [a_I, a_N] \mid a_N) + \Pr(I \text{ and } a_2 < a_I \mid a_N))(v_H - a_N - \varepsilon) &< c \end{aligned}$$

Now imagine squeezing all play of I to the left. Because the conditional distribution of a_2 is log-concave, the probability that a_2 lies in $[a_I, a_I + \sigma]$ conditional on $a_1 = a_N$ is smaller than the probability that a_2 lies in any other measure σ subset of $[a_I, a_N]$ conditional on $a_1 = a_N$. Thus we have $\Pr(a_2 \in [a_I, a_I + \sigma] \mid a_N) < \Pr(I \text{ and } a_2 \in [a_I, a_N] \mid a_N)$, giving us

$$\begin{aligned} \Pr(a_2 < a_I + \sigma \mid a_N)(v_H - a_N - \varepsilon) &< c \\ \phi(\bar{k} - \sigma/\varepsilon)(v_H - a_N - \varepsilon) &< c \\ \phi(\bar{k} - \sigma/\varepsilon) &< \frac{c}{v_H - a_I - 3\varepsilon} \end{aligned} \tag{6}$$

Now consider a_I. Picking up from the sixth line above,

$$\begin{aligned} \Pr(a_2 &> a_N \mid a_I)\Delta v > (1 - \Pr(I \text{ and } a_2 < a_N \mid a_I))(v_H - a_I) + c - \varepsilon \\ \Pr(a_2 &> a_N \mid a_I)\Delta v > (\frac{1}{2} - \Pr(I \text{ and } a_2 \in [a_I, a_N] \mid a_I))(v_H - a_I) + c - \varepsilon \end{aligned}$$

Once again, imagine that all play of I occurs in $[a_I, a_I + \sigma]$. Again by log-concavity of G, the probability that a_2 lies in $[a_I, a_I + \sigma]$, conditional this time on $a_1 = a_I$, is greater than the probability that a_2 lies in any other measure σ subset of $[a_I, a_N]$. Consequently, $\Pr(I \text{ and } a_2 \in [a_I, a_N] \mid a_I) < \Pr(a_2 \in [a_I, a_I + \sigma] \mid a_I)$, so

$$\begin{aligned} \Pr(a_2 > a_N \mid a_I)\Delta v &> (\frac{1}{2} - \Pr(a_2 \in [a_I, a_I + \sigma] \mid a_I))(v_H - a_I) + c - \varepsilon \\ \phi(\bar{k})\Delta v &> \phi(\sigma/\varepsilon)(v_H - a_I) + c - \varepsilon \\ \frac{\phi(\bar{k})}{\phi(\sigma/\varepsilon)} &> \frac{v_H - a_I}{\Delta v} + \frac{c - \varepsilon}{\phi(\sigma/\varepsilon)\Delta v} \\ \frac{\phi(\bar{k})}{\phi(\sigma/\varepsilon)} &> \frac{v_H - a_I}{\Delta v} + \frac{2(c - \varepsilon)}{\Delta v} \end{aligned} \tag{7}$$

where the last step follows because $\phi(\sigma/\varepsilon) < 1/2$ for $\sigma > 0$. As in Case 1, we would like to compare lines (6) and (6); logconcavity of G permits this.

Lemma 1 *For $u, v > 0$, $\phi(u)\phi(v) \geq \phi(u+v)/2$.*

Proof. Write $\psi(x) = \ln \phi(x)$. Because ψ is concave (by logconcavity of G, which implies logconcavity of ϕ), we have $\psi(0) + \psi(u+v) \leq \psi(0+u) + \psi((u+v)-u) = \psi(u) + \psi(v)$. Substituting $\phi(0) = 1/2$ into this equation yields the result. ■

Applying this lemma to (6) and (6), we arrive at

$$\frac{c}{v_H - a_I - 3\varepsilon} > \frac{v_H - a_I}{2\Delta v} + \frac{c - \varepsilon}{\Delta v}$$

just as in Case 1. Solving the resulting quadratic equation for $x = v_H - a_I$ yields

$$x < -c + \frac{5}{2}\varepsilon + \sqrt{c^2 + c\varepsilon + \frac{1}{4}\varepsilon^2 + 2c\Delta v}$$

Taking limits as ε goes to 0 establishes the result that was claimed.

Finally, consider an equilibrium in which an arbitrary combination of I, W_I, and W_N are played on $[a_I, a_N]$, where W_s refers to waiting and playing s if the other agent also waits. At a_N, N is at least weakly preferred to the more profitable of W_I and W_N, so we certainly have $N \succsim W_N$. At a_I, the better of W_I and W_N is weakly preferred to I. If this is W_N, we have $W_N \succsim I$ at a_I and $N \succsim W_N$ at a_N, and we can proceed just as in Case 2 above. Suppose instead that $I \sim W_I \succsim W_N$ at a_I. Then we proceed with the two conditions $W_I \succsim I$ at a_I and $N \succsim W_N$ at a_N. First note that, as before, if N is a better response than W_N to the true equilibrium strategy at a_N, it will also be a better response to the strategy in which N is always played to the right of a_N (but which otherwise corresponds with the equilibrium strategy). This is also true for $W_I \succsim I$ at a_I. Suppose that on the interval $[a_I, a_N]$, I is played for a measure $\sigma\varepsilon$ of signals and W_I is played for a measure $\gamma\varepsilon$ of signals. Similarly to Case 2, we can construct an alternative profile in which all play of I is shifted to the left end of $[a_I, a_N]$ and all play of W_N is shifted to the right end of the interval. W_I and I have different outcomes only when N or W_N are faced. At a_I, this shift doesn't affect the chance of facing N and reduces the chance of facing W_N (the situation in which I does better than W_I), so the reply preference $W_I \succsim I$ at a_I continues to hold. W_N and N have different outcomes only when I is faced (in which case W_N is preferred to N). This shift reduces the chance of facing I at a_N, so the reply preference $N \succsim W_N$ at a_N also continues to hold. To summarize, we have two conditions on best replies to the adjusted profile in which I is played below $a_I + \sigma$, N is played above a_N, W_I is played on $[a_I + \sigma\varepsilon, a_I + \sigma\varepsilon + \gamma\varepsilon]$, and W_N is played on $[a_I + \sigma\varepsilon + \gamma\varepsilon, a_N]$. As above, the condition that $N \succsim W_N$ at a_N leads to the inequality

$$\phi(\bar{k} - \sigma) < \frac{c}{v_H - a_I - 3\varepsilon}$$

The condition $W_I \succsim I$ at a_I leads to the inequality

$$\phi(\bar{k})(a_I - v_L + \Delta v + \varepsilon) > \phi(\sigma + \gamma)\Delta v + c$$

Thus, as $\varepsilon \to 0$, we have

$$\begin{aligned} \phi(\bar{k}-\sigma) &< \frac{c}{v_H - a_I} \quad (8) \\ \phi(\bar{k})(a_I - v_L + \Delta v) &> \phi(\sigma+\gamma)\Delta v + c \quad (9) \end{aligned}$$

There are two subcases to consider. Suppose first that in the limit of the ε equilibria, W_N is preferred to W_I at a_N. Then W_N will still be preferred under the shifted profile (as it makes facing W_I less likely). But this means that the probability of facing W_N conditional on a_N and both agents waiting must be greater than p^*, where p^* is defined by

$$\begin{aligned} (1-p^*)(v_H - a_I) + p^*(v_L - a_I) &= 0 \\ p^* &= \frac{v_H - a_I}{\Delta v} \end{aligned}$$

Under the shifted profile, this probability is just

$$p' = \frac{\phi(0) - \phi(\bar{k}-\sigma-\gamma)}{\phi(0) - \phi(\bar{k}-\sigma)}$$

Now we combine the first two $\varepsilon \to 0$ conditions with $p' \geq p^*$, and suppose toward a contradiction that $v_H - a_I$ were to converge to 0 more slowly that at rate $\sqrt{c}$ as $c \to 0$. From (8), $\phi(\bar{k}-\sigma)$ would have to go to 0 at a rate faster than $\sqrt{c}$. But then because $\phi(\bar{k}) < \phi(\bar{k}-\sigma)$, $\phi(\bar{k}) \to 0$ faster than $\sqrt{c}$ as well. Then (9) means that $\phi(\sigma+\gamma) \to 0$ faster than $\sqrt{c}$. Together these mean that $\bar{k} \to 2$ and $\sigma+\gamma \to 2$, so $\phi(\bar{k}-\sigma-\gamma) \to \phi(0) = 1/2$. How quick is this convergence? For c small, so $\bar{k}$ and $\sigma+\gamma$ near 2, $\phi(\bar{k}-\sigma-\gamma) \to \phi(0)$ faster than $\sqrt{c}\frac{\phi'(\bar{k}-\sigma-\gamma)}{\phi'(\sigma+\gamma)}$ which is greater than $\sqrt{c}$ because ϕ is steeper near 0 than near its tails (because it is single-peaked). Altogether, this means that $p' \to 0$ faster than $\sqrt{c}$, but then $p' \geq p^*$ means that $v_H - a_I \to 0$ faster than $\sqrt{c}$, a contradiction.

Now consider the second subcase, in which W_I is preferred to W_N in the limit of the ε equilibria. Then the probability of facing W_I, conditional on a_N and both agents waiting must be greater than $1-p^*$. Then the condition that $N \succsim W_I$ at a_N gives us

$$\phi(\bar{k}-\sigma) + \Pr(\text{face } W_I) < \frac{c}{v_H - a_I}$$

and $\Pr(\text{face } W_I) > \Pr(\text{face } W)(1-p^*)$. Furthermore, $\Pr(\text{face } W) < 1/2 - \phi(\bar{k}-\sigma)$, so we have

$$\frac{1}{2} - \frac{v_H - a_I}{\Delta v}\left(\frac{1}{2} - \phi(\bar{k}-\sigma)\right) < \frac{c}{v_H - a_I}$$

In this case, $v_H - a_I$ must go to 0 at least at rate c. If it did not, the right hand side would go to 0, so the left hand side would have to do so as well. This would only be possible with $\phi(\bar{k}-\sigma) \to 0$ and $a_I \to v_L$. But if a_I (and hence a_N) goes to v_L, then choosing I would be strictly better (with a payoff at least $\Delta v/2$) at a_N than N, another contradiction. Thus, in every case, the limiting a_I (as $\varepsilon \to 0$) goes to v_H as the waiting cost vanishes at rate $\sqrt{c}$ or faster.

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