Private Value Perturbations and Informational Advantage in Common Value Auctions

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June 23, 2003

Abstract

We analyze the value of being better informed than one's rival in a two bidder, second price common value auction. In order to do so, we must pare down the continuum of equilibria that typically exists in this setting. We propose selecting an equilibrium that is robust to perturbing the common value of the object with small private value components. Under this selection, we show that having better information about the common value will frequently hurt rather than help a bidder and that the ratio of private value to common value information held by a bidder has a significant effect on the value of information.

1   Introduction

Auction theory is largely silent on what to expect when some bidders are better informed than others about the common value of an object for sale. This is due in part to the fact that second-price common value auctions often present multiple equilibria. When bidders are equally well informed, selecting the symmetric equilibrium appears reasonable, but when some bidders know more than others, there is no obvious or compelling way to choose the "right" equilibrium. This paper resolves the problem by introducing small private value perturbations to the bidders' common value for the object. As these perturbations vanish, bidding converges to a unique equilibrium of the unperturbed model. The value of being better informed than one's rival takes a subtle form in this equilibrium: it is not having more precise information *per se* that is valuable, but rather, having information that is difficult for rivals to free ride on. In certain cases, this reduces to a fairly simple criterion: bidders whose private information contains a higher ratio of private value to common value information do better.

The analysis focuses on two bidder, second price common value auctions with a "sum of independent signals" specification of the common value. In this framework, the variance of the

common value is the sum of the variances of the two bidders' signals, so the noise in Bidder 1's estimate of the common value after observing her signal is related to the variance of Bidder 2's signal, and *vice versa*. In this sense, a bidder with a higher variance signal can be thought of as having relatively more information about the common value.

In an unperturbed second price auction, the value of better information is unclear because of a strong free-riding effect. Because the winning bidder's price is set by the losing bidder's bid, the losing bidder's private information will typically be fully revealed by the auction price. Thus, when a bidder, in formulating a best response to her rival's strategy, considers the event of winning with a particular signal at a particular price, she is effectively conditioning on full information about the common value. Given this, there is no particular reason that a poorly informed bidder needs to bid cautiously. On the other hand, her inference about the common value upon winning at a particular price will be more pessimistic when her rival's strategy is more aggressive, and this *will* lead her to bid more cautiously. By simultaneously making one bidder's strategy more aggressive and the other's more cautious, one can trace out a continuum of equilibria in the unperturbed model, and none of these equilibria depend on the informational asymmetry between the two bidders.

In contrast, when there is a private value component to each bidder's signal, free-riding off a rival's bid becomes more difficult. When a bidder conditions on paying a particular price, she must infer what portion of that price reflects a "premium" for her rival's private value for the object. She should be prepared to win at this price if and only if her own private value for the object exceeds that premium. This leads to a more stringent equilibrium condition: at any price at which the bidders might tie, the expected share of private value reflected in the two bidder's bids must be the same. Under some regularity conditions on the signal distributions, this *robust to private values (RPV)* condition pins down a unique equilibrium.

We focus on the situation in which the bidders' private value components have the same mean, so *ex ante*, neither bidder is expected to have a higher value for the object than the other.[1] We develop intuition for the RPV condition with a few simple examples. The examples illustrate the importance of the slope of the probability density function for a bidder's signal about the common value. When this slope is negative, the bidder has access to the truth about a component of the common value that is commonly thought more likely to be low than to be high. High bids by such a bidder are relatively likely to reflect a high private value rather than a high common value, so this bidder's rival must be particularly cautious. In contrast, a bidder with access to signals drawn from a density function with positive slope is at a disadvantage, as his rival will tend to assume that a high bid confirms the *ex ante* belief that this component of the common value is more likely to be high.

Most of the general results are developed for the case in which both signal density functions are decreasing. Section 5 discusses some of the reasons that this case may be of interest; just

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[1] The case in which one bidder has a small, *ex ante*, private value advantage has been analyzed in Klemperer (1998). Here we want to focus on the effect of differences in information rather than differences in value.

to mention one, we might imagine that prior (unmodeled) stages of competition have selected for bidders in the right tails of otherwise unimodal signal distributions. The first result shows that giving a bidder more information about the common value makes her relatively less likely to win the auction (Proposition 2). Essentially, this is because the common value to private value ratio in her bid goes up, allowing her rival to bid more aggressively. Conversely, giving a bidder more private value information makes her more likely to win. However, there is a sense in which a bidder benefits from better information – if the bidders share the same ratio of private value to common value information, then the bidder with more information wins more often (Corollary 2) and earns a larger surplus (Proposition 3).

We can also say something about the incentives of a seller who has some control over the information released to each bidder. If the seller can control the amount of private value information released to each bidder, it should release information in such a way that the bidder with less information about the common value is more likely to win the auction (Propositions 5 and 6). (Loosely, this is because this bidder claims a smaller information rent.) Alternatively, if the seller can reduce even further the amount of common value information held by this less informed bidder, it can gain by doing so (Proposition 7). Standard results on revenue equivalence allow us to consider whether the seller might do better with a first price auction. It turns out that the answer is linked to whether the less informed bidder wins more or less often than the more informed bidder in the second price auction, and there are situations in which each format does strictly better than the other (Proposition 9).

My paper draws on two strands of the auction literature. The first relates to common value auctions with asymmetrically informed bidders. Here most of the work has focused on first price auctions, often looking at the case in which a single informed bidder competes with several relatively uninformed bidders whose information is in some sense a coarsening of the informed bidder's information (Wilson (1967), Englebrecht-Wiggans, Milgrom, and Weber (1983), Hendricks, Porter, and Wilson (1994), among others). More recently, Laskowski and Slonim (1999) and Kagel and Levin (1999) characterize approximate first price equilibria in models in which better information corresponds to a lower variance signal about the common value. Campbell and Levin (2000) explicitly solve a parametric model and provide examples in which having better information hurts a bidder.

For second price auctions, Milgrom and Weber (1982) and Einy, Haimanko, Orzach, and Sela (2000) show that as in the first price auction, when one bidder has information that encompasses everything known by another bidder, the latter bidder cannot make a profit. Krishna and Morgan (1998) and Mares (2000) deal with asymmetries that arise when *ex ante* symmetric bidders pool their information and bid jointly. Both papers resolve a multiplicity of equilibria by imposing some type of symmetric bidding despite the fact that the bidders are not symmetric. Perhaps the closest paper to mine is Parreiras (2002), who also uses a perturbation approach to select an equilibrium of a second price auction with asymmetrically informed bidders. However, his approach is based on the assumption there is a small first price

component to the auction (rather than a private value component), and his results are quite different.

I also build on past work on auctions with both private and common values. In a seminal paper, Milgrom and Weber (1982) treat the case of symmetric bidders when a bidder's information about its private and common value can be summarized by a one-dimensional signal. Relaxing these assumptions has proved difficult, in large part because the basic toolbox of auction theory relies heavily on finding a natural ordering of bidders' signals. The lack of such an ordering when signals are multi-dimensional can cause problems for the existence and characterization of equilibrium as shown by Jackson (1999). In order to avoid this difficulty, I rely on a relatively special model of private and common values that also been used by Goeree and Offerman (1999) among others. The next section describes the model and partially characterizes equilibria. Section 3 looks at the limit in which private values are small and develops the equilibrium selection result. Section 4 illustrates some implications of the selection result with examples, and Section 5 provides general results. Section 6 concludes with a discussion of possible extensions.

2 The Model

There is a single object to be sold to one of two buyers, 1 and 2, in a second price auction with a reserve price of 0. A buyer's valuation for the object is $v_i = v + \varepsilon z_i$ where εz_i is a private value component and v is common to both bidders. The common value is modeled as the sum of two independent signals: $v = x_1 + x_2$ where x_1 and x_2 are distributed independently with continuously differentiable densities f_1 and f_2 (with common support $X \subset \Re^+)^2$. The private value terms z_1 and z_2 are also random variables drawn independently from continuously differentiable densities g_1 and g_2, while ε is a constant scale factor. We assume that z_i has mean μ_i and variance σ_i^2. Our principal focus will be on the case in which the scale factor ε is small, so that the private component of valuations is small relative to the common value. The only information available to a bidder is a signal $s_i = x_i + \varepsilon z_i = v_i - x_{-i}$. In other words, a bidder receives information about his value that is incomplete because it does not reveal part of the common value v. Furthermore, a bidder does not observe the common and private value elements of his own signal individually, but just the composite signal.[3]

As a prelude to examining equilibrium in the full model, it is helpful to briefly review the set of Nash equilibria of the model with pure common values; we do this in Section 2.1. Then, in Section 2.2, equilibria of the full model are characterized. The following preliminary result will be helpful for both cases.

Lemma 1 *Equilibrium bidding strategies are weakly increasing in signals.*

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[2] Assuming common supports is not strictly necessary, but it will avoid some tedious steps later.

[3] Actually, the equilibria we will identify would be unchanged if a buyer observed x_i and z_i rather than just s_i. Essentially, this is because x_i and z_i provide no information about x_{-i}, so the buyer's belief about his own value v_i does not change.

Proof. Fix a bidding strategy for buyer 2. By bidding b, buyer 1 earns $\pi_1(b \,|\, s_1) = \Pr(b_2 \leq b)E[v_1 - b_2 \,|\, b_2 \leq b] = \Pr(b_2 \leq b)s_1 + \Pr(b_2 \leq b)E[x_2 - b_2 \,|\, b_2 \leq b]$. The cross-partial derivative $\frac{\partial}{\partial b \partial s_1}\pi_1(b \,|\, s_1) = \frac{d}{db}\Pr(b_2 \leq b)$ is weakly positive, so standard results guarantee that buyer 1's optimal bid weakly increases in his signal. ∎

2.1 The Common Value Benchmark

Here we assume that $\varepsilon = 0$, so the setting is one of purely common values, with $v = s_1 + s_2$. For convenience, we assume that the support of the signal distributions is compact: $X = [0, \bar{x}]$. The following lemma and its proof provide a sense of why multiple equilibria arise in this setting. Before proceeding to the lemma, let us introduce some terminology. A bidding function maps signals to bids: $b_i : X \to \Re^+$. We restrict attention throughout to strictly increasing, continuously differentiable bidding functions. The set of bids used by a bidder – that is, the image of b_i – is denoted $B_i \equiv b_i(X)$, and the inverse bidding function is denoted $\phi_i : B_i \to X$. We will focus on equilibria for which $B_1 \cap B_2$ is nonempty so that each bidder has a positive chance of winning the auction. We can also define a bidder's surplus function $Y_i(p \,;\, s_i, \phi_{-i})$. This is the expected surplus earned by a bidder i with signal s_i who wins at a price p, given that her rival's (inverse) bidding strategy is ϕ_{-i}. In the case of pure common values, the surplus function is

$$
\begin{aligned}
Y_i(p \,;\, s_i, \phi_{-i}) &= E(v \,|\, s_i, s_{-i} = \phi_{-i}(p)) - p \\
&= s_i + \phi_{-i}(p) - p
\end{aligned}
$$

In what follows, we will sometimes write $Y_i(p \,;\, s_i)$ or just $Y_i(p)$ when the missing arguments are clear from the context.

Lemma 2 *Any pair of increasing inverse bidding functions (ϕ_1, ϕ_2) that satisfy $\phi_i' < 1$ and $\phi_1(p) + \phi_2(p) = p \;\forall p \in B_1 \cap B_2$ constitute a Nash equilibrium of the auction with pure common values.*

Proof. Suppose that ϕ_1 and ϕ_2 satisfy the conditions of the lemma and consider whether ϕ_1 is a best response to ϕ_2 for an arbitrary signal s_1'. There are two possibilities: either $b' = b_1(s_1') \in B_2$ (Bidder 1 sometimes ties with Bidder 2 given signal s_1'), or $b' \notin B_2$ (Bidder 1 always wins or always loses with signal s_1'). If $b' \in B_2$, then Bidder 1's expected surplus conditional on winning and paying a price equal to her bid is

$$
\begin{aligned}
Y_1(b' \,;\, s_1', \phi_2) &= s_1 + \phi_2(b') - b' \\
&= \phi_1(b') + \phi_2(b') - b' \\
&= 0
\end{aligned}
$$

Furthermore, $Y_1(p; s_1', \phi_2)$ is strictly decreasing in p (because $\phi_2' < 1$), so Bidder 1's surplus function is positive (negative) when she wins at prices less than (greater than) b'. By bidding b', she wins if and only if her expected surplus is positive, so this is a best response.

Alternatively, suppose that $b' < b_2 \ \forall b_2 \in B_2$ (so Bidder 1 never wins with signal s_1). Let $b^* = \inf\{b \in B_1 \cap B_2\} = \inf\{b \in B_2\} > b'$ be the lowest price at which Bidder 1 could win. (b^* exists because we have assumed $B_1 \cap B_2$ nonempty, and the second equality follows from the continuity of b_1.) Of course, $\phi_1(b^*) > s_1$. By assumption, we have

$$
\begin{aligned}
Y_1(b^*; \phi_1(b^*), \phi_2) &= \phi_1(b^*) + \phi_2(b^*) - b^* \\
&= 0
\end{aligned}
$$

Thus Bidder 1's surplus at any price $p \geq b^*$ for which it has a positive chance of winning is strictly negative:

$$
\begin{aligned}
Y_1(p; s_1, \phi_2) &< Y_1(p; \phi_1(b^*), \phi_2) \\
&\leq Y_1(b^*; \phi_1(b^*), \phi_2) \\
&= 0
\end{aligned}
$$

It follows that bidding $b' < b^*$ and never winning is a best response. The argument for the case in which $b' > b_2 \ \forall b_2 \in B_2$ (so that Bidder 1 always wins with signal s_1) is virtually identical and is omitted. Since the choice of s_1 was arbitrary, ϕ_1 is a best response to ϕ_2. An identical argument establishes that ϕ_2 is a best response to ϕ_1. ∎

While the equilibria characterized above are not the only equilibria of the auction, they share certain features are worth highlighting. First, an increase in the price a bidder pays is partially, but not fully, compensated for by an increase in the rival signal that can be inferred ($\phi' < 1$). This is what ensures that when a bidder likes winning at one price, she also likes winning at all lower prices, simplifying the evaluation of optimal strategies. Second, equilibrium requires that Bidder 1's expectation of the common value conditional on tying Bidder 2 at some bid p must be the same as Bidder 2's expectation conditional on that event, and both expectations must be equal to p. This is implied by $\phi_1(p) + \phi_2(p) = p$. However, equilibrium does not put any conditions on the signals $s_1 = \phi_1(p)$ and $s_2 = \phi_2(p)$ for which they tie other than that the signals must sum to p. This leaves substantial scope for one bidder to bid more aggressively than the other.

As an example, consider equilibria in strategies that are linear in signals: $b_i = k_i x_i$. Such equilibria must satisfy the constraint that $\frac{1}{k_1} + \frac{1}{k_2} = 1$. One such equilibrium is the symmetric one, with $b_1 = 2x_1$, $b_2 = 2x_2$. However, there is a continuum of asymmetric equilibria in which one buyer bids more aggressively than under the symmetric strategy ($k_i > 2$) and the other bids less aggressively ($k_{-i} < 2$). In these equilibria, aggressive bidding by the "strong" bidder creates a more severe winner's curse for the weak bidder who knows that if he has won, his

rival's signal must have been quite low. This encourages the weak bidder to bid cautiously. But this in turn alleviates the winner's curse faced by the strong bidder, as winning against a more cautious rival conveys less bad news about the rival's signal. Thus asymmetric bidding can be self-sustaining.

So far, nothing has been said about the distributions of the bidders' signals; this is because the equilibrium set does not depend on those distributions. Of course, this means that the set of equilibrium outcomes does not depend on the relative amount of information about the common value held by each bidder. This is a (perhaps perverse) consequence of the second price format. A less informed buyer need not bid more cautiously than its better informed rival because it never pays more than its rival's bid, which fully reveals its rival's (superior) information. Thus, free-riding renders any informational advantage moot. This will no longer be the case when private values are introduced.

2.2   Private Value Perturbations

Now I turn to the case in which valuations have a private value component ($\varepsilon > 0$). In this case, upon winning, a bidder must worry about how much of the price he is paying reflects his rival's private benefit from winning the object rather than common value to both of them. In what follows, I will assume that the bidders' values are *ex ante* symmetric (that is, $\mu_1 = \mu_2 = 0$) in order to focus on the effects of asymmetries in bidder information (When values are *ex ante* asymmetric, one can demonstrate a strong advantage for the stronger bidder as suggested in Klemperer (1998).) In the rest of this section, we will develop a characterization of equilibrium with private values.

As before, we consider bid functions that are strictly increasing in signals, and hence, invertible. Suppose that Bidder 2's bid function and inverse are given by b_2 and ϕ_2, and once again, consider Bidder 1's best response. His expected value for the object conditional on winning at a price b is given by

$$E(v_1 \mid s_1, s_2 = \phi_2(b)) = s_1 + E(x_2 \mid s_2 = \phi_2(b))$$

and his expected net surplus upon winning and paying b is now equal to

$$Y_1(b\,;s_1) = s_1 + E(x_2 \mid s_2 = \phi_2(b)) - b$$

Notice that because ϕ is increasing, it doesn't matter whether we write these expressions as conditioning on Bidder 2's bid b or her signal $\phi_2(b)$. While in the pure common values case, the middle term in this expression, $E(x_2 \mid s_2 = \phi_2(b))$, is simply equal to Bidder 2's signal $\phi_2(b)$, with private values, Bidder 1 must infer how much to discount Bidder 2's signal to account for her private value component.

As before, we are interested in conditions that ensure that $Y_1(b\,;s_1)$ is decreasing in b, because this in turn will ensure that there is a single b^* that separates the prices at which
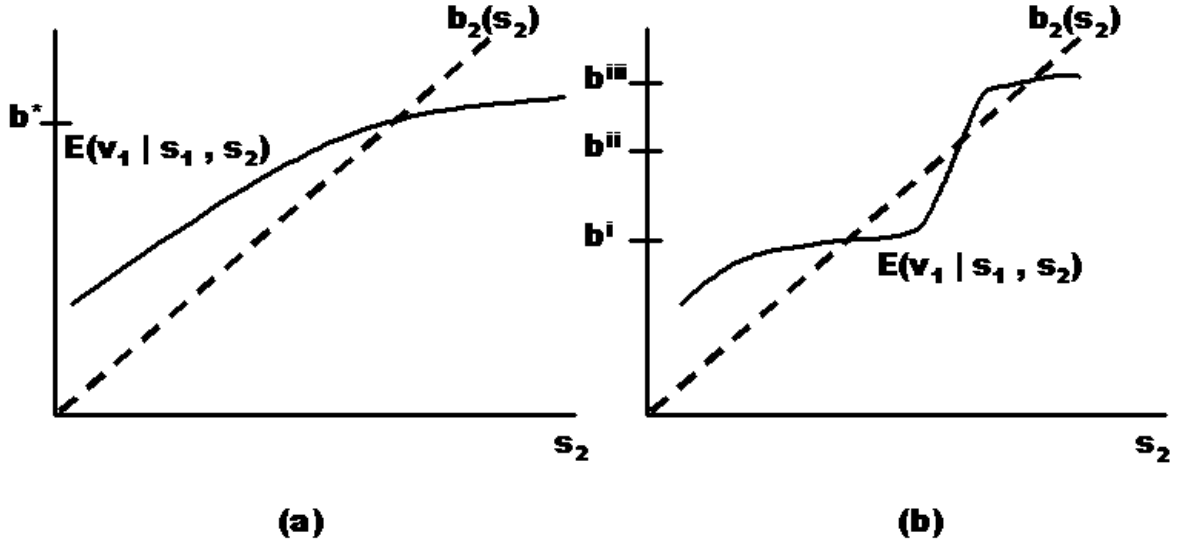
Figure 1: Expected value and price for Bidder 1 as a function of s_2.

Bidder 1 would like to win $(p < b^*)$ from the prices at which it would prefer to lose $(p > b^*)$. In this case, determining Bidder 1's optimal strategy will be relatively straightforward. With pure common values, it sufficed to constrain the slope of ϕ_2, but with private values, the situation can be more complicated, as illustrated by Figure 1.

If the situation is as in (a), Bidder 1 can win at every price for which his expected surplus is positive by bidding b^*. However, if the situation is as in (b), Bidder 1 would like to win whenever the price is less than b^{iii} but not between b^i and b^{ii}. This is impossible, so he will have to assess the relative likelihood of prices in $[b^i, b^{ii}]$ and $[b^{ii}, b^{iii}]$ in order to determine whether it is better to bid b^i or b^{iii}. Situations like (b) can occur if there are regions for which the slope of $E(x_2 \,|\, s_2)$ is larger than the slope of b_2 with respect to s_2. Such situations can arise naturally under private values; for example, suppose that the common component x_2 is believed to be either low, $U(0, 1)$, or high, $U(2, 3)$, and the private value component is either $-\frac{1}{2}$ or $\frac{1}{2}$ with equal probability. Then for small $\delta > 0$, the expected value of x_2 when $s_2 = 1.5 - \delta$ is 1. (The private value must be high, because the prior rules out $x_2 = 2 - \delta$). However, when $s_2 = 1.5 + \delta$, the expected value of x_2 jumps up discontinuously to $2 + \delta$. (Here, the prior rules out $x_2 = 1 + \delta$, so the private value must be low.) In this example, a higher signal for 2 is particularly good news for 1 because it implies a dramatic shift between the common and private portions of 2's signal. In order to rule out "extreme" swings in beliefs like this one, the following condition is imposed.

A1. $d(E(x_i \,|\, s_i)/ds_i < 1$ (Good news is not *too* good.)

Given the monotonic relationship between signals and bids, we can write Bidder 1's surplus as a function of Bidder 2's signal: $Y_1(s_2 \,; s_1) = s_1 + E(x_2 \,|\, s_2) - b_2(s_2)$. Given A1, this surplus

function $Y_1(b\,;s_1)$ will be decreasing in s_2 if $b_2' > 1$. Existence of equilibrium bidding functions with $b_i' > 1$ is not automatic; we need an additional assumption.

A2. $dE(x_i\,|\,s_i)/ds_i > 0$ (Good news is not bad news.)

The following example helps to illustrate the need for A2. Suppose $x_2 \sim U(0,1)$ and $\varepsilon z_2 = -\varepsilon$ or ε with equal probability. Then $E(x_2\,|\,s_2)$ drops from s_2 to $s_2 - \varepsilon$ as s_2 increases through $1 - \varepsilon$, because a signal higher than $1 - \varepsilon$ means that the possibility that the common value is underestimated (rather than overestimated) can be ruled out. In other words, a higher rival signal can be bad news about the common value. When this is the case, Bidder 1 may find it optimal to raise his bid more slowly than his signal in order to compensate for the bad news implied by (possibly) defeating a higher rival signal. But this would mean that bid functions could have slope less than one, and we would like to rule this out.

Although assumptions A1 and A2 convey some helpful intuition, they make joint demands on the distributions of the common and private values, so it would be useful to have more primitive conditions on the distributions f and g. The next lemma provides these conditions.

Lemma 3 .
    i.   *If x_i and s_i have the monotone likelihood ratio property (MLRP), then A2 is satisfied.*
    ii. *If z_i and s_i have the monotone likelihood ratio property, then A1 is satisfied.*
    iii. *If f_i and g_i are strictly log-concave, then A1 and A2 are satisfied.*

Proof. Appendix ∎

In light of Lemma 3, the following assumption will be used frequently in the sequel.

A3. f_i is strictly log-concave.

If Bidder 1's surplus does decrease monotonically with the price at which it wins, then its optimal bid is given by the unique solution to

$$E(v_1\,|\,s_1, s_2 = \phi_2(b)) = s_1 + E(x_2\,|\,s_2 = \phi_2(b)) = b$$

which can be rewritten as

$$s_1 + \phi_2(b) - \varepsilon E(z_2\,|\,s_2 = \phi_2(b)) = b$$

or equivalently

$$\phi_1(b) + \phi_2(b) - \varepsilon E(z_2\,|\,s_2 = \phi_2(b)) = b$$

That is, he is willing to pay up to the sum of his signal and 2's signal (which he can infer), minus the expected private value component of 2's signal. Of course, the same argument is

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valid for Buyer 2 whose inverse bid function should therefore satisfy

$$\phi_1(b) + \phi_2(b) - \varepsilon E(z_1 \,|\, s_1 = \phi_1(b)) = b$$

Combining these two expressions yields the following equilibrium conditions

$$\lambda(b) \;=\; E(z_1 \,|\, s_1 = \phi_1(b)) = E(z_2 \,|\, s_2 = \phi_2(b)) \qquad (1)$$

$$\phi_1(b) + \phi_2(b) \;=\; b + \varepsilon\lambda(b) \qquad (2)$$

Proposition 1 *Suppose that A1 and A2 are satisfied. Then,*

i) Any pair of increasing inverse bidding functions (ϕ_1, ϕ_2) that satisfy $\phi_i' < 1$ and (1) and (2) for all $b \in B_1 \cap B_2$ constitute a Nash equilibrium.

ii) A Nash equilibrium satisfying $\phi_i' < 1$ exists.

Proof. i) Each bidder's surplus function can now be written as

$$Y_i(p\,;s_i, \phi_{-i}) = s_i + E(x_{-i} \,|\, s_{-i} = \phi_{-i}(p)) - p$$

Differentiating with respect to p indicates that this surplus function is strictly decreasing:

$$
\begin{aligned}
\frac{d}{dp}Y_i(p\,;s_i, \phi_{-i}) \;&=\; \phi_{-i}'(p)\frac{d}{ds_{-i}}E(x_{-i} \,|\, s_{-i} = \phi_{-i}(p)) - 1 \\
&<\; 0
\end{aligned}
$$

The inequality follows from A1 and $\phi_{-i}' < 1$. At this point, the proof proceeds analogously to Lemma 2. ∎

In words, because each buyer bids its expected value conditional on tying with its rival, if there is a bid at which the buyers sometimes tie, then they must have the same expected value for the object at that bid. Subtracting off the common value component, this means that the portion of its rival's bid that each bidder rationally expects to represent private value must be the same for the two bidders. This will restrict the scope for either bidder to be arbitrarily aggressive, as each must assess how likely it is that its rival's willingness to drive up the price reflects a private rather than a common benefit.

3   The Common Value Limit

In this section, we focus on characterizing equilibrium in the limit as private values vanish. This analysis leads to the *robust to private values* condition, which must hold in any limiting equilibrium. Under some regularity conditions, there is a unique equilibrium satisfying the

RPV condition; this equilibrium can be thought of as the most reasonable prediction for the unperturbed common value model.

The strategy will be to develop a more explicit formulation of the equilibrium condition (1), and then take Taylor series approximations of this condition. Taking limits as ε goes to zero leads to the RPV condition. First, the expected private value component of a rival's signal from (1) can be expressed:

$$E(z_1 \mid s_1 = s) = \frac{\int z\, f_1(s - \varepsilon z) g_1(z)\, dz}{\int f_1(s - \varepsilon z) g_1(z)\, dz}$$

We will take Taylor series expansions in the numerator and denominator of this expression as follows. First the numerator:

$$
\begin{aligned}
\int z\, f_1(s - \varepsilon z) g_1(z)\, dz &= f_1(s) \int z\, g_1(z)\, dz \\
&\quad - \varepsilon f_1'(s) \int z^2\, g_1(z)\, dz \\
&\quad + o(\varepsilon) \\
&= -\varepsilon \sigma_1^2 f_1'(s) + o(\varepsilon)
\end{aligned}
$$

As is usual, $o(\varepsilon)$ indicates terms that vanish at a strictly faster rate than ε, and the assumption that $\mu_1 = 0$ is used twice. Next, for the denominator:

$$
\begin{aligned}
\int f_1(s - \varepsilon z) g_1(z)\, dz &= f_1(s) \int g_1(z)\, dz \\
&\quad - \varepsilon f_1'(s) \int z\, g_1(z)\, dz \\
&\quad + o(\varepsilon) \\
&= f_1(s) + o(\varepsilon)
\end{aligned}
$$

Combining these steps, we have

$$E(z_1 \mid s_1 = s) = -\varepsilon \sigma_1^2 \frac{f_1'(s)}{f_1(s)} + o(\varepsilon)$$

and in a similar fashion, for the private component of Buyer 2's signal we have

$$E(z_2 \mid s_2 = s) = -\varepsilon \sigma_2^2 \frac{f_2'(s)}{f_2(s)} + o(\varepsilon)$$

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Therefore, we can rewrite the equilibrium condition (1) as

$$\sigma_1^2 \frac{f_1'(s_1)}{f_1(s_1)} = \sigma_2^2 \frac{f_2'(s_2)}{f_2(s_2)} + o(\varepsilon^0) \quad \text{where}$$
$$s_1 = \phi_1(b)$$
$$s_2 = \phi_2(b)$$

In the limit, as private information becomes negligible, the equilibrium bidding functions must satisfy

$$\sigma_1^2 \frac{f_1'(s_1)}{f_1(s_1)} = \sigma_2^2 \frac{f_2'(s_2)}{f_2(s_2)} \tag{3}$$

for any s_1 and s_2 for which Buyers 1 and 2 submit the same equilibrium bid.

Definition 1 *A pair of Nash equilibrium bidding strategies* (b_1, b_2), *with inverses* (ϕ_1, ϕ_2) *is said to be* **robust to private values (RPV)** *if for every pair of signals such that* $b_1(s_1) = b_2(s_2)$, *(3) is satisfied.*

Figure 2 illustrates the intuition underlying (3) for an example in which the private value component is either $\varepsilon\Delta$ or $-\varepsilon\Delta$ with equal probability. A bidder who learns its rival's signal to be s can infer that the common component of its rival's signal is either x^* or x^{**}. Given the equal likelihood of a high or low private value, the posterior likelihood placed on x^* relative to x^{**} is the same as the prior: $f(x^*)/f(x^{**})$. The posterior probability of x^* is $f(x^*)/(f(x^*) + f(x^{**}))$ and the expected deviation of x from s, given s, is

$$
\begin{aligned}
E(x \mid s) - s &= \frac{f(x^*)}{f(x^*) + f(x^{**})}(s - \varepsilon\Delta) + \frac{f(x^{**})}{f(x^*) + f(x^{**})}(s + \varepsilon\Delta) - s \\
&= \frac{f(x^{**}) - f(x^*)}{f(x^*) + f(x^{**})}\varepsilon\Delta \\
&\approx \frac{2\varepsilon\Delta f'(s)}{2f(s)}\varepsilon\Delta \\
&\propto \frac{f'(s)}{f(s)}\sigma_z^2
\end{aligned}
$$

In this case, $f'(s)$ is negative, so the common component is likely to be lower than s. The effect of a larger variance for z is to spread out the possible values of x. This in turn has two effects: it increases the disparity in likelihood between low and high values of x, and it increases the standard deviation of the difference between the common value and the signal. Each of these effects is of order σ_z.

Although (3) restricts possible equilibria, it may not pin down a unique equilibrium. For some distributions f_1 and f_2 there may be multiple solutions to (3). Furthermore, the RPV condition has no impact on bids that always lose or always win. The former issue disappears if we are willing to assume that f_1 and f_2 are sufficiently regular. In particular, if f_i is strictly
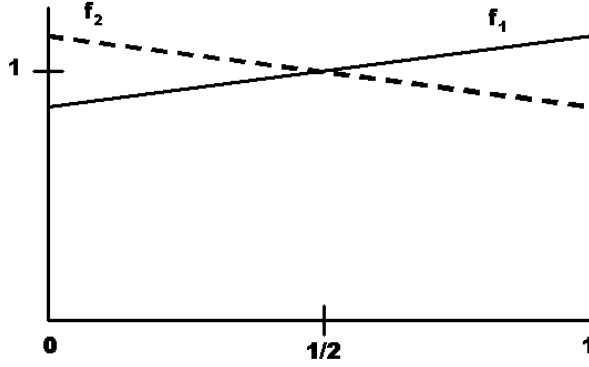
Figure 2:

log-concave, then $\sigma_i^2 f_i'(s)/f_i(s)$ is strictly decreasing. In this case, (3) uniquely identifies the equilibrium (except for bids that win with probability 0 or 1).

4   Examples

In order to develop intuition for the way in which the RPV condition operates, we present three examples.

4.1   "Almost Uniform" Priors

Suppose that the distribution of the common value components is given by

$$f_1(x) = 1 + (\tfrac{1}{2} - x)m$$
$$f_2(x) = 1 + (x - \tfrac{1}{2})m , \quad x \in [0, 1]$$

for m small, as illustrated in Figure 3.

Then (3) can never be satisfied for any s_1 and s_2 in $(0, 1)$ because $f_1' = -m < 0 < m = f_2'$ on this range. In words, Bidder 2 always expects its rival's signal to underestimate x_1, while the opposite is true for Bidder 1, regardless of their signals. Consequently, 2 is always prepared to outbid 1 whenever they are tied. The only case in which this is not true is at the maximum and minimum signals, 1 and 0, where f_1' and f_2' are not defined. Thus the only equilibrium that survives private value perturbation is the one in which the bidders tie only at the highest

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Figure 3:

(lowest) signal for Bidder 1 (2) and Bidder 2 wins the auction with probability 1:

$$b_1(s) = s$$
$$b_2(s) = s + 1$$

This is true regardless of the distributions of private values and for all $m > 0$, so even as m approaches zero and both prior distributions converge to identical uniform distributions, the auction outcome remains completely asymmetric.

4.2 Truncated Normal Priors (1)

Here we assume the common value components to be distributed according to truncated normal distributions. That is,

$$f_i(x) = \begin{cases} \tilde{f}_i(x)/\left(1 - \tilde{F}_i(0)\right) & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\tilde{F}_i$ and $\tilde{f}_i$ are the c.d.f. and p.d.f. of a normal random variable with distribution $N(\bar{x}_i, \alpha_i^2)$, where $\bar{x}_i > 0$. We suppose that $\bar{x}_i$ is large relative to α_i so that the effect of the truncation at zero is minimal. For this distribution, $f_i'(s)/f_i(s)$ takes the relatively simple form $(s - \bar{x}_i)/\alpha_i^2$, so (3) reduces to

$$(s_1 - \bar{x}_1)\left(\frac{\sigma_1}{\alpha_1}\right)^2 = (s_2 - \bar{x}_2)\left(\frac{\sigma_2}{\alpha_2}\right)^2$$

for all signals s_1 and s_2 for which the bidders tie. If we write $\gamma_1 = \left(\frac{\sigma_1}{\alpha_1}\right)^2$ Using the fact that if the bidders tie at s_1 and s_2 then $b_1(s_1) = b_2(s_2) = s_1 + s_2$, we can restate this as follows: for

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any bid b at which the bidders might tie, we must have

$$\phi_1(b) = \frac{\gamma_2 b + D}{\gamma_1 + \gamma_2}$$

$$\phi_2(b) = \frac{\gamma_1 b - D}{\gamma_1 + \gamma_2}$$

where

$$D = \gamma_1 \bar{x}_1 - \gamma_2 \bar{x}_2,$$

$$\gamma_i = \left(\frac{\sigma_i}{\alpha_i}\right)^2$$

Suppose $\underline{b}$ is the smallest bid at which the two bidders might tie.[4] Then one of the bidders must be at her minimum signal – either $\phi_1(\underline{b}) = 0$ or $\phi_2(\underline{b}) = 0$. (If this were not true – if, for example, $b_1(0) \leq b_2(0) < \underline{b}$ – then continuity of b_1 and b_2 would guarantee that there would be some s_1 such that $b_1(s_1) = b_2(0)$, contradicting the minimality of $\underline{b}$.) The identity of the bidder at her minimum signal depends on the sign of D: if $D > 0$, then $s_2 = 0$ ties with $s_1 = D/\gamma_1$ at $\underline{b} = D/\gamma_1$. On the other hand, if $D < 0$, then $s_1 = 0$ ties with $s_2 = -D/\gamma_2$ at $\underline{b} = -D/\gamma_2$. To sum up, the RPV bidding functions are given by

$$b_1(s) = \begin{cases} \bar{x}_1 + \bar{x}_2 + (s - \bar{x}_1)(1 + \kappa_1) & \text{for } s \geq \max\{0, D/\gamma_2\} \\ s & \text{otherwise} \end{cases}$$

$$b_2(s) = \begin{cases} \bar{x}_1 + \bar{x}_2 + (s - \bar{x}_2)(1 + \kappa_2) & \text{for } s \geq \max\{0, -D/\gamma_1\} \\ s & \text{otherwise} \end{cases}$$

where $\kappa_1 = 1/\kappa_2 = \gamma_1/\gamma_2$. That is, each bidder bids $\bar{x}_1 + \bar{x}_2$ (for large $\bar{x}_i$ this is approximately the *ex ante* expected value of the good) plus a "correction" for having a higher or lower than expected signal. This correction comprises the full value of the deviation of the own signal plus an estimate, $(s - \bar{x}_i)\kappa_i$, of the deviation of the rival's signal, conditional on tying. The overall responsiveness of each bid function to the signal depends on the parameter κ_i. A bidder's κ_i is larger if the ratio of publicly relevant (α_i) to privately relevant information (σ_i) in his signal is smaller than for his rival. The underlying logic is that a bidder raises its bid for two reasons as its signal increases – first to account for the increase in value indicated by its own signal, and second to account for the increase in value implied by tying with a higher rival signal. If the relative importance of private value – which is effectively noise to the first bidder – in the rival's bid is large, then the bidder will tend to discount this secondary effect. Conversely, if the ratio of private information in its rival's signal is low, the secondary effect will be large, and the bidder will be justified in raising its bid more as its signal rises.

It is worth elaborating on what is *not* implied by this equilibrium. It is not true that the bidder with relatively more private information bids higher, on average, than his rival – both bidders bid approximately $\bar{x}_1 + \bar{x}_2$ on average. It is also not true that the bidder with

---

[4]Technically, this should be the infimum, but the more rigorous argument would be essentially the same.

more private information bids higher relative to his signal. Rather, he bids relatively more aggressively when he has a high signal, and relatively more weakly when his signal is low.

As an illustration, consider the case in which Bidder 2 has a small private value component but Bidder 1's signal is purely common value. Formally, we look at the limit as σ_1^2 goes to zero, holding σ_2^2 fixed. Then κ_1 goes to zero and κ_2 goes to infinity. Bidder 1 simply bids his signal plus $\bar{x}_2$. Bidder 2 bids her signal when it is less than $\bar{x}_2$ and bids unboundedly high when her signal exceeds $\bar{x}_2$. Thus, the outcome of the auction is determined entirely by Bidder 2. When Bidder 2's signal is low, Bidder 1 wins the auction and pays s_2. When Bidder 2's signal is high, she wins and pays $s_1 + \bar{x}_2$. For large $\bar{x}_1$ and $\bar{x}_2$, the chance that s_2 exceeds $\bar{x}_2$ is approximately $\frac{1}{2}$, and expected auction revenues are approximately $\frac{1}{2}(E(s_1) + \bar{x}_2) + \frac{1}{2}E(s_2 \,|\, s_2 < \bar{x}_2) \lesssim \bar{x}_1/2 + \bar{x}_2$.

Contrast this with the standard symmetric equilibrium when $\alpha_1 = \alpha_2 = 1$ and $\bar{x}_1 = \bar{x}_2 = 10$. The average common value is 20 but in the RPV equilibrium, revenues are approximately 14.60. In the symmetric equilibrium, each bidder bids twice its signal and so the bidder with the higher signal wins. The average revenue is at least the expected second highest bid which is roughly equal to 18.87.[5] Notice that the bidders do not share at all equally in the extra surplus generated by the small private value asymmetry. In the symmetric equilibrium, each earns a surplus of 0.56 on average. In the RPV equilibrium, Bidder 2 nets $s_2 - 10$ whenever $s_2 > 10$ and zero otherwise, yielding an expected surplus of approximately 0.40. On the other hand, Bidder 1 nets s_1 whenever $s_2 < 10$, for an expected surplus of approximately 5. Bidder 2 suffers because the private value perturbation induces Bidder 1 to bid more aggressively with low signals, and all else equal, Bidder 2 tends to win when Bidder 1 has a low signal. Bidder 1 gains dramatically because Bidder 2 bids much less aggressively when his signal is low, which is when Bidder 1 tends to win. This is intended in part as a cautionary example – even when bidders are symmetrically informed about the common value, small differences in private value information can tip the auction outcome far away from the symmetric equilibrium.

4.3   Truncated Normal Priors (2)

In this example, each common value component is assumed to be distributed as the right tail of a normal distribution. Formally, $x_i = |y_i|$ where $y \sim N(0, \alpha_i^2)$. The analysis of bids is as above, and we have

$$
\begin{aligned}
b_1(s) &= (1 + \kappa_1)s \\
b_2(s) &= (1 + \kappa_2)s \quad \text{for all } s \geq 0
\end{aligned}
$$

In contrast to the previous example, the priors here always place greater weight on low common values than on high common values, so when uncertainty about the private value component of a rival's bid induces greater reliance on priors, the result is always a shift toward less aggressive bidding. To make the example concrete, take $\alpha_1 = 1$ and $\alpha_2 = 2$, so 2's signal explains four

[5]At least, because we assume that no sale takes place if both bids are negative.
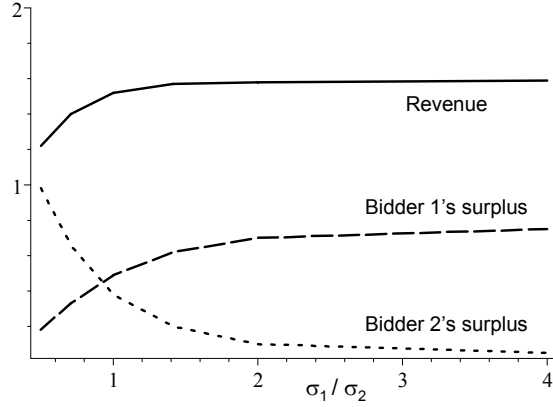
Figure 4:

times as much of the variation in the common value as 1's does. Table 1 and Figure 4 show the expected surplus for each bidder and the expected auction revenue for selected ratios of the private information held by the two bidders.

Table 1

σ_1/σ_2	1's Surplus	2's Surplus	Revenue	Prob. 1 wins
$\frac{1}{2}$	0.18	0.98	1.22	0.30
$\frac{1}{\sqrt{2}}$	0.33	0.66	1.40	0.50
1	0.49	0.38	1.52	0.71
$\sqrt{2}$	0.62	0.20	1.57	0.85
2	0.70	0.10	1.58	0.92
4	0.75	0.05	1.59	0.96

The first row, $\sigma_1/\sigma_2 = \frac{1}{2}$, is the case in which each bidder has the same amount of private information relative to public information ($\kappa_1 = \kappa_2$), but Bidder 2 is more completely informed than Bidder 1. The two bidders bid symmetrically as a function of their signals, but 2 wins more often because its signal is on average higher. In the second row, the distribution of bids is the same for the both bidders which means that Bidder 1 is bidding more aggressively; it is induced to do so because it has relatively more private value information than Bidder 2. Nonetheless, Bidder 1 still earns less surplus than Bidder 2. It is not until Bidder 1 catches up in terms of the *absolute* amount of private information ($\sigma_1/\sigma_2 = 1$).that he begins to earn more than Bidder 2. As Bidder 1's private value information advantage grows, both his surplus and auction revenue continue to grow as well.

Note that in contrast to the previous example, holding more private information is unambiguously good for Bidder 1. This is true because in the right tails, uncertainty about a rival's

bid unambiguously induces the other bidder to shade down its expectation of the common value, whereas in Example 4.2, bidders with low signals tended to find such uncertainty encouraging rather than discouraging. Note also that the seller does better when there is relatively more uncertainty *ex ante* about the private value of the bidder who is more poorly informed about the common value (that is, higher σ_1/σ_2). That is because (with small private value components) this is the bidder who commands the smaller information rent, so it is in the seller's interest for him to bid more aggressively and win more often. This has implications for optimal information release by the seller, which we explore in the next section.

5 General Results

This section is devoted to formalizing some of the results that were illustrated in the previous section. We begin by defining one particular notion of what it means for one bidder to to have better information about the common value than another. The definition will allow us to compare signals when one signal distribution is essentially a compressed or dispersed version of the other.

Definition 2 *We say that distributions F_1 and F_2 are comparable if a random variable $y \sim F_2$ has the same distribution as $ax + c$, where $x \sim F_1$ and a and c are real numbers. If $a \gtrless 1$ we say that F_2 contains more (less) information than F_1.*

An alternative formulation of this definition is that $F_2(s) = F_1((s - c)/a)$ for some a and c. This may seem to run counter to the standard intuition that a more concentrated signal is more informative, but one must recall that in the sum of independent signals model, the residual uncertainty in Bidder 1's assessment of the common value is reflected by F_2, so when F_2 is relatively more dispersed, 1 is relatively worse informed, and *vice versa*.

In most of what follows, we impose the following additional assumption.

A4. f_i is strictly decreasing.

A4 ensures that the sign of the inference effect is negative – the presence of private value in a rival's signal means that the other bidder must be more be more cautious. The main reason for focusing on this case is practical; with an unambiguous sign on the inference effect, it is possible to say a great deal about the comparative statics of revenue and the bidder surpluses. However, there are other reasons that A4 may represent a relevant class of distributions. One reason is prior competition. As we alluded to in the introduction, one might think of the two-bidder auction as the final stage of a competitive process in which agents with low signals about the common value have already dropped out. In this case, it may be a reasonable shortcut to suppose that the two surviving agents have "right tail" signals for which A4 applies, even if the underlying signal distribution is more general. Alternatively, suppose that agents have some control over their information sources. As illustrated in 4.1, agents have little incentive

to acquire precise information about aspects of the common value for which prior beliefs are optimistic (i.e., increasing density), as this will tend to handicap them in the bidding. Instead, agents should seek information on features for which the common priors ascribe more weight to low versus high values.

Proposition 2 *Suppose the bidders have comparable common value signals that satisfy A3 and A4. Fix F_1 and the private value perturbation parameters σ_1^2 and σ_2^2. Then in an RPV equilibrium,*

i. As Bidder 2's common value signal grows more informative, its probability of winning declines.

ii. If $\sigma_1^2 = \sigma_2^2$, then the bidder with the less informative common value signal is more likely to win the auction.

iii. Fixing the common value signals, Bidder 2's probability of winning increases with its private value information.

iv. If the bidders have identical common value signals, then the bidder with more private value information is more likely to win.

Proof. For simplicity, we assume that $c = 0$ so $F_2(s) = F_1(s/a)$. (The proof would not change with a non-zero c.) The RPV equilibrium condition is given by

$$\frac{1}{a}\frac{f_1'(s_2/a)}{f_1(s_2/a)}\sigma_2^2 = \frac{f_1'(s_1)}{f_1(s_1)}\sigma_1^2$$

Define the function $h(u) = f_1'(F_1^{-1}(u))/f_1(F_1^{-1}(u))$ which is the value of f_1'/f_1 at the signal corresponding to percentile u of F_1. The assumptions ensure that h is negative and strictly decreasing. Then we can write the equilibrium condition as

$$h(u_2) = a\frac{\sigma_1^2}{\sigma_2^2}h(u_1)$$

where u_1 and u_2 are the percentiles (from F_1 and F_2 respectively) of the signals for which the bidders tie. We will write $u_1 = \gamma(u_2)$ for the mapping implied by this condition. To show i., observe that as a increases, each value of u_2 must tie with a strictly smaller value u_1; that is, $\gamma(u_2)$ decreases for every u_2. But 2's probability of winning the auction is just $\int_0^1 \Pr(\text{win} \mid u_2)\, du_2 = \int_0^1 \gamma(u_2)\, du_2$, so we are done. For step ii., we have

$$\frac{h(u_2)}{h(u_1)} = a$$

so $\gamma(u_2) \gtrless u_2$ iff. $a \lessgtr 1$. If $\gamma(u_2) > u_2$ then 2's probability of winning is $\int_0^1 \gamma(u_2)\, du_2 > \int_0^1 u_2\, du_2 = \frac{1}{2}$, and conversely if $\gamma(u_2) < u_2$. Steps iii. and iv. are essentially identical to steps i. and ii. and are omitted. ∎

The intuition is just as in the example of 4.3. When higher levels of the common value are believed to be relatively less likely, the prospect that an opponent's high bid could reflect his high private value is unambiguously bad news. The news gets worse as the relative influence of that private value on the opponent's bid grows. However, the bad news is mitigated if the opponent's signal resolves relatively more uncertainty about the common value. If we define α_i^2 to be the variance of s_i, relative bidding can be characterized as follows.

Corollary 1 *Under the conditions of Proposition 2, the bid distributions of the two bidders are identical (and so each bidder is equally likely to win) when $\alpha_2/\alpha_1 = \sigma_2^2/\sigma_1^2$. Otherwise, the bidder with relatively more private value information (less common value information) is more likely to win.*

Proof. Note that $\alpha_2/\alpha_1 = a$ and when $a\sigma_1^2/\sigma_2^2 = 1$, the equilibrium condition is $h(u_2) = h(u_1)$, so $\gamma(u_2) = u_2$. This means that each bidder uses the same mapping from the percentile of his signal to his bid, so the distribution of bids is identical. If $a\sigma_1^2/\sigma_2^2 > 1$ then we have $\gamma(u_2) < u_2$ as in the previous proof, and so 1 is more likely to win. When this inequality is reversed, 2 is more likely to win. ∎

This corollary indicates that as we give one bidder more information, we must give him private value information at at least half the rate of common value information in order to prevent his chances of winning from eroding.

Corollary 2 *Assume the conditions of Proposition 2. If the bidders have the same relative variance of private value to common value information, then the better informed bidder is more likely to win.*

Proof. This follows directly from Corollary 1. ∎

In this narrow sense, bidders with better information outperform those with worse information.

It is more difficult to say anything conclusive about relative payoffs, but the following proposition provides a partial result.

Proposition 3 *Suppose A3 and A4 hold and in the RPV equilibrium the better informed bidder wins at least half of the time. Then the better informed bidder has the higher average payoff.*

Proof. Suppose that 2 is better informed than 1 and that the bidders are equally likely to win the auction. Consider an arbitrary bid b. The bidders tie at b when s_1 and s_2 share the same percentile in F_1 and F_2 and $s_1 + s_2 = b$, i.e., when $s_1 = s_1^b = (b-c)/(1+a)$ and $s_2 = s_2^b = as_1 + c$. Conditional on the winning price being b, it is equally likely that $s_1 = s_1^b$ and $s_2 \geq s_2^b$ (2 has won) or $s_2 = s_2^b$ and $s_1 \geq s_1^b$ (1 has won). In the latter case, 1 earns an expected surplus of $E_{F_1}(s_1 \mid s_1 \geq s_1^b) + s_2^b - b = E_{F_1}(s_1 \mid s_1 \geq s_1^b) - s_1^b$. In the former case, 2

earns

$$
\begin{aligned}
E_{F_2}(s_2 \,|\, s_2 \geq s_2^b) + s_1^b - b &= E_{F_2}(s_2 \,|\, s_2 \geq s_2^b) - s_2^b \\
&= a(E_{F_1}(s_1 \,|\, s_1 \geq s_1^b) - s_1^b) \\
&> E_{F_1}(s_1 \,|\, s_1 \geq s_1^b) - s_1^b
\end{aligned}
$$

where the second step follows from the comparability of the two distributions. Since this inequality holds at every possible winning bid, we conclude that 2's overall expected surplus from the auction is higher.

Now suppose that 2 wins more than half of the time. Then it will be true that the bidders tie at b for some $s_1^{b\prime}$ and $s_2^{b\prime}$ such that $s_1^{b\prime} + s_2^{b\prime} = b$ and $s_2^{b\prime} < \hat{s} = as_1^{b\prime} + c$. Then, conditional on a winning price of b, 2 is more likely to be the winner. By the same arguments as above, 1's surplus conditional on being the winner is $E_{F_1}(s_1 \,|\, s_1 \geq s_1^{b\prime}) - s_1^{b\prime}$, 2's surplus conditional on being the winner is $E_{F_2}(s_2 \,|\, s_2 \geq s_2^{b\prime}) - s_2^{b\prime}$, and $E_{F_1}(s_1 \,|\, s_1 \geq s_1^{b\prime}) - s_1^{b\prime} < E_{F_2}(s_2 \,|\, s_2 \geq \hat{s}) - \hat{s}$. Then because $E_{F_2}(s_2 \,|\, s_2 \geq k) - k$ is decreasing in k for log-concave distributions, we have $E_{F_2}(s_2 \,|\, s_2 \geq s_2^{b\prime}) - s_2^{b\prime} > E_{F_2}(s_2 \,|\, s_2 \geq \hat{s}) - \hat{s}$, and hence $E_{F_2}(s_2 \,|\, s_2 \geq s_2^{b\prime}) - s_2^{b\prime} > E_{F_1}(s_1 \,|\, s_1 \geq s_1^{b\prime}) - s_1^{b\prime}$. Because 2 is more likely to be the winner, we have *a fortiori* that 2's expected surplus conditional on the price being b is greater than 1's, and so we are done. ∎

Furthermore, we can state conditions under which more private value information improves a bidder's payoff.

Proposition 4 *Suppose A3 and A4 hold and the bidders' signals are comparable. Then a bidder's expected payoff is increasing in its level of private value information.*

Proof. Consider an increase in Bidder 1's private information level from σ_1^2 to σ_1^{2*}. This implies a decrease in $\gamma(u_2)$ to some $\gamma^*(u_2) < \gamma(u_2)$; that is, 1 ties 2 with lower percentile bids than previously. Consider 1's payoff conditional on an arbitrary signal percentile u_2 for 2. Under σ_1^2 and σ_1^{2*} 1 earns

$$
\int_{\gamma(u_2)} F_1^{-1}(u_1) + F_2^{-1}(u_2) - (F_1^{-1}(\gamma(u_2)) + F_2^{-1}(u_2)) \, du_1 = \int_{\gamma(u_2)} F_1^{-1}(u_1) - F_1^{-1}(\gamma(u_2)) \, du_1
$$

and

$$
\int_{\gamma^*(u_2)} F_1^{-1}(u_1) + F_2^{-1}(u_2) - (F_1^{-1}(\gamma^*(u_2)) + F_2^{-1}(u_2)) \, du_1 = \int_{\gamma^*(u_2)} F_1^{-1}(u_1) - F_1^{-1}(\gamma^*(u_2)) \, du_1
$$

respectively, where $F_1^{-1}(\gamma(u_2)) + F_2^{-1}(u_2)$ is 2's bid when its signal percentile is u_2 and the tying function is γ. The change in 1's payoff under σ_1^{2*} is

$$
\int_{\gamma(u_2)} F_1^{-1}(\gamma(u_2)) - F_1^{-1}(\gamma^*(u_2)) \, du_1 + \int_{\gamma^*(u_2)}^{\gamma(u_2)} F_1^{-1}(u_1) - F_1^{-1}(\gamma^*(u_2)) \, du_1
$$

Observe that $\gamma^*(u_2) < \gamma(u_2)$ guarantees that both integrals are positive, so 1's payoff increases under σ_1^{2*}, conditional on u_2. Since u_2 was chosen arbitrarily, 1's unconditional payoff must increase as well. ∎

Next, we turn to the incentives of the seller. Suppose the seller can release information to each bidder that is relevant to its private value for the object; I will take the reduced-form interpretation that this means the seller can affect σ_1^2/σ_2^2. Who would the seller prefer to inform? The next two propositions indicate a sense in which it is better to release information to a more poorly informed bidder.

Proposition 5 *Assume A3 and A4. Consider the seller's choice between manipulating σ_1^2/σ_2^2 so that the bidders are equally likely to win and manipulating σ_1^2/σ_2^2 so that the bidder with less information about the common value always wins. If information about the common value is sufficiently asymmetric, the seller will prefer to have the less informed bidder always win.*

Proof. As before, assume $a > 0$ so that 2 is better informed about the common value. Write $t_2 = s_2/a$ and note that t_2 and s_1 have the same distribution. When σ_1^2/σ_2^2 is such that the bidders are equally likely to win, the condition for which signals s_1 and s_2 tie is $s_1 = s_2/a = t_2$. The corresponding bid functions are $b_1 = (1+a)s_1$ and $b_2 = (1+\frac{1}{a})s_2 = (1+a)t_2$, and so the seller's revenue is just $(1+a)E(\min(s_1,t_2))$.

On the other hand, when $\sigma_1^2/\sigma_2^2 \to \infty$ so that 1 wins with arbitrarily low signals, the price is set with probability $\to 1$ by 2's bid. This is simply $s_2 + E(s_1 \mid s_1\text{ties with }s_2) \to s_2$, so the seller's revenue approaches $E(s_2) = aE(t_2)$. This is better for the seller if

$$aE(t_2) - (1+a)E(\min(s_1,t_2)) > 0$$

or

$$(1+a)E(t_2 - \min(s_1,t_2)) - E(t_2) > 0$$

But $E(t_2 - \min(s_1,t_2))$ and $E(t_2)$ are positive constants, so the inequality holds for a large enough. ∎

This echoes the example in 4.3. Again, the idea is that the better informed bidder has a larger information rent when he wins, so the seller does better by encouraging his rival to win more often. For some examples, the policy of allowing the less informed bidder to always win appears to be optimal, but in other cases there is an interior optimum in which the less informed bidder wins with probability greater than 1/2 but less than 1.

Proposition 6 *Assume A3 and A4 and consider the situation in which the two bidders are equally likely to win. The seller can always do better (worse) by providing slightly more (less) private value information to the bidder with the less informative common value signal.*

Proof. The strategy is to sign the slope of expected revenues with respect to the ratio σ_1^2/σ_2^2 at the point where the bidders are equally likely to win. The details are in the appendix. ∎

Next I ask what can be said about the seller's preferences over the common value information held by the bidders. Suppose that the seller can release more or less information about the common value to one of the two bidders exclusively (say Bidder 2); I model this by assuming that the seller can manipulate a. Here I must be somewhat careful, as changing a will generally change the mean of Bidder 2's common value signal, but I do not mean to assume that the seller can change the average common value of the object. The solution is somewhat crude: I assume that there is a third component of the common value about which the two bidders are both uninformed, although they know its distribution and mean. As the seller increases (decreases) a, the changes in the average signal observed by 2 are absorbed by opposite changes in the mean of the third component. The idea is that the additional information supplied to Bidder 2 is incremental information about this third component.[6] In this environment, we have a companion result to the previous proposition.

Proposition 7 *Assume A3 and A4 and consider the situation in which the two bidders are equally likely to win. The seller can always do better (worse) by providing slightly less (more) common value information to the bidder with the less informative common value signal.*

Proof. Appendix. ∎

There are two factors at play in this result. The seller has a bias against releasing common value information exclusively to one of the bidders because it increases that bidder's information rent without eroding the exclusivity of the information held by the other bidder. This is a feature of the additive independent signals model, and presumably if the seller could release information to 2 that would induce more overlap between his information and that of Bidder 1, this effect might be reversed. The second factor is the now familiar effect in which giving a bidder more common value information relative to private value information makes him a weaker bidder.

The multiplicity of equilibria in the unrefined second price auction contrasts with the corresponding first price auction, for which there is a unique equilibrium. With the goal of comparing the seller's revenue across the two auction formats, I introduce a result adapted from Parreiras (2002).

Proposition 8 *If in the second price auction, the buyers' bids are distributed identically in equilibrium, then the expected revenue from the first and second price auctions is the same.*

Given Corollary 1, this means that the comparison of first and second price auctions can be reduced to a comparison of second price auction revenues across different relative levels of private values. As indicated by the example in 3.2.3, these revenues may be higher or lower than those of the first price auction. More formally, we have the following.

---

[6]I sidestep substantial technical issues in assuming this information structure to be feasible. The main difficulty is in preserving the comparability of F_1 and F_2 while adding information to F_2; this seems likely to impose very strong restrictions on the form of F_1 and F_2.

Proposition 9 *Assume A3 and A4. If one common value signal is sufficiently more informative than the other, then there exist levels of private information such that*

i. The bidder with the less informative common value signal wins more often, and expected revenue is greater than in a first price auction.

ii. The bidder with the more informative common value signal wins more often, and expected revenue is lower than in a first price auction.

Proof. For i., apply Propositions 5 and 8. For ii., we can adapt the proof of Proposition 5 to show that when the bidder with the less informative common value signal also has sufficiently little private value information relative to its rival, revenue is lower than it would be if the bidders were equally likely to win. Then we apply Proposition 8 again. ∎

While I suspect that a stronger result holds, I have been unable to prove it.

Conjecture 1 *Under A3 and A4, second price expected revenue exceeds first price expected revenue if and only if the bidder with the less informative common value signal is more likely to win in the second price auction.*

6 Concluding Remarks

The model has potentially interesting implications for information acquisition in auctions. For example, one implication of Proposition 2 is that there may be relatively general conditions under which the marginal benefit to a bidder of acquiring more information about the common value is negative. Conversely, there are fairly general conditions under which acquiring private value information is beneficial. To be more precise, these effects arise when a bidder is *known* to have acquired more information, as in each case it is the strategic reaction of rivals that is important. One may conjecture that bidders will attempt to adopt a strong posture by putting disproportionate research into aspects of the object that other bidders are unlikely to care about, even if these aspects are relatively insignificant relative to the overall value of the object. Meanwhile, research about the common value aspects may be relatively minimal. This need not have serious consequences for efficiency in the current framework, but if there were a chance that the object could fail to sell (due to a reserve price, perhaps), or if private values were small but not negligible, then underinvestment in common value information could be more of a problem.

There are also implications for the kind of information sources that bidders will seek out. The analysis shows that a bidder is better off drawing a signal for which f' is negative because this discourages her rivals from tagging along on her high bids. As a practical matter, we may expect firms to skew their research toward learning about "lottery ticket" features of the common value – that is, features that are expected to reveal modestly bad news about the common value most of the time and very good news occasionally. (Or if they don't skew their research in this way, they should at least try to give the appearance of doing so publicly.)

One possible complaint against the model is that it does not provide a true selection result because the set of equilibria that are robust to *some* private value perturbation σ_1/σ_2 is typically still large. In order to get a precise equilibrium prediction, one must know *which* private value perturbation best represents the situation. There are two responses to this criticism. First, even if the appropriate ratio σ_1/σ_2 is not known, the RPV condition still puts relatively stringent restrictions on the form of equilibrium bidding functions. For example, "sunspot" equilibria in which the bidders switch from cautious to aggressive strategies (and *vice versa*) at arbitrary prices and arbitrarily often will not survive. This may be valuable for empirical work, as the number of free parameters to estimate is reduced from a continuum to just one. Second, a selection that does not depend on the relative amounts of private value information held by the bidders runs the risk of throwing out information that is actually important in determining auction outcomes.

Finally, a note of caution. A theme that emerges from this paper as well as several others in the theoretical literature (e.g., Klemperer (1998), Parreiras (2002)) is that second price auction outcomes can be very sensitive to small asymmetries among bidders when values are "almost" common. Here it is small asymmetries in the amount of private value information that modulate the effect of much larger asymmetries in information about the common value. However, empirical and experimental work consistently reveals that bidders account for the information held by others, but not as fully as they should. One hopes that the effects of differential information identified in the theory would show up qualitatively, if perhaps less powerfully, in the data; experimental work on auctions with differential information would be valuable in determining whether this is true.

Appendix: Proofs

Lemma 3

i. We will drop unnecessary subscripts. Choose arbitrary s and $s' > s$. There exists some x^* such that $f(x^* \mid s) = f(x^* \mid s')$. Then because the MLRP holds for x and s, we have

$$\frac{f(x \mid s')}{f(x^* \mid s')} > \frac{f(x \mid s)}{f(x^* \mid s)}$$

for all $x > x^*$, and hence $f(x \mid s') > f(x \mid s)$ for all $x > x^*$. Similarly, we have $f(x \mid s') < f(x \mid s)$ for all $x < x^*$. But this suffices to show that $F(x \mid s') < F(x \mid s)$ from which it follows that $E(x \mid s') > E(x \mid s)$.

ii. Because $E(x \mid s) = s - \varepsilon E(z \mid s)$, it will be enough to show that $E(z \mid s)$ is increasing in s, which can be proved just as for i.

iii. We will show that g log-concave implies that $f(x' \mid s)/f(x \mid s)$ is increasing in s whenever $x' > x$. First, note that

$$\frac{f(x' \mid s)}{f(x \mid s)} = \frac{f(x')}{f(x)} \frac{g(\frac{s-x'}{\varepsilon})}{g(\frac{s-x}{\varepsilon})}$$

so if $s' > s$,

$$\frac{f(x'\,|\,s')}{f(x\,|\,s')} > \frac{f(x'\,|\,s)}{f(x\,|\,s)} \Leftrightarrow g(\frac{s'-x'}{\varepsilon})g(\frac{s-x}{\varepsilon}) > g(\frac{s-x'}{\varepsilon})g(\frac{s'-x}{\varepsilon})$$

But $\frac{s'-x'}{\varepsilon} + \frac{s-x}{\varepsilon} = \frac{s-x'}{\varepsilon} + \frac{s'-x}{\varepsilon}$ and $s - x'$ and $s' - x$ bracket $s - x$ and $s' - x'$, so the inequality holds by log-concavity of g. Finally, we can apply i. to show A2. The proof that log-concavity of f implies A1 is very similar and is omitted.

Proposition 1(ii) (Existence)

The proof is by construction. A1 ensures that $\lambda_i(s_i) = E(z_i\,|\,s_i)$ is strictly increasing in s_i. Let Z_i be the image of λ_i and let Z be the closure of $Z_1 \cap Z_2$. For signals s_1 and s_2 such that $\lambda_1(s_1) \in Z$ and $\lambda_2(s_2) \in Z$ we implicitly define the maps Q_1 and Q_2 :

$$\lambda_2(Q_2(s_1)) = \lambda_1(s_1)$$
$$\lambda_1(Q_1(s_2)) = \lambda_2(s_2)$$

That is, $Q_2 = Q_1^{-1}$ links pairs of signals that map to the same value of $E(z_i\,|\,s_i)$; λ_i increasing ensures that Q_i exists and is increasing. The domain of Q_{-i} is denoted $S_i = \lambda_i^{-1}(Z)$. Let $U_i = \lambda_i^{-1}(Z_i \setminus Z)$ be the set of signals for i that cannot be paired with a signal for $-i$ mapping to the same value of z. In our construction, a signal in U_i will either be non-competitive (if $\lambda_i(s_i) < \inf\{z \in Z\}$) or will always win (if $\lambda_i(s_i) > \sup\{z \in Z\}$). Continuity ensures that if U_1 includes non-competitive signals, then U_2 does not (and similarly for signals that always win). Let the bidding functions be defined by

$$b_i(s) = \begin{cases} \tilde{b}_i(s) & s \in U_i \\ s + Q_{-i}(s) - \varepsilon\lambda_{-i}(Q_{-i}(s)) & s \in S_i \end{cases}$$

where the only constraint on $\tilde{b}_i$ is that it has slope greater than one and satisfies smooth pasting conditions at the boundaries of U_i and S_i. Notice that for $s \in S_i$ the bidding function can be written $b_i(s) = s + E(x_{-i}\,|\,s_{-i} = Q_{-i}(s))$. Therefore, A2 and Q_{-i} increasing ensure that $b_i'(s) > 1$. By construction, the bid functions overlap precisely for $B_1 \cap B_2 = b_1(S_1) = b_2(S_2)$. Note that $b_1(s_1) = b_2(Q_2(s_1))$. Furthermore, because the bid functions are strictly increasing, $b_1(s_1) = b_2(s_2)$ only if $s_2 = Q_2(s_1)$. Thus for any s_1 and s_2 that tie, we have $\lambda_1(s_1) = \lambda_2(Q_2(s_1)) = \lambda_2(s_2)$, so (1) holds. For tying signals s_1 and s_2 with $b_1(s_1) = b_2(s_2) = b$ and $\lambda_1(s_1) = \lambda_2(s_2) = z_b$, we have

$$\begin{aligned} b &= s_1 + s_2 - \varepsilon z_b \\ &= \phi_1(b) + \phi_2(b) - \varepsilon\lambda(b) \end{aligned}$$

so (2) holds. Therefore the constructed bidding functions satisfy the conditions in i) for a Nash equilibrium.

Proposition 6

We can write expected revenue as the sum of expected revenue from the two bidders:

$$\int \Pr(u_2 > \gamma^{-1}(u_1))(F_1^{-1}(u_1)+F_2^{-1}(\gamma^{-1}(u_1))\,du_1 + \int \Pr(u_1 > \gamma(u_2))(F_1^{-1}(\gamma(u_2))+F_2^{-1}(u_2))\,du_2$$

The first integrand is just the bid made by 1 when his signal falls in percentile u_1 times the probability that this bid determines the price. The second integrand is the equivalent expression for 2. We can rewrite this as

$$\int (1 - \gamma^{-1}(u_1))(F_1^{-1}(u_1) + F_2^{-1}(\gamma^{-1}(u_1))\,du_1 + \int (1 - \gamma(u_2))(F_1^{-1}(\gamma(u_2)) + F_2^{-1}(u_2))\,du_2$$

Differentiating with respect to the ratio of private value information, we have

$$\frac{d(\text{Expected Revenue})}{d(\sigma_1^2/\sigma_2^2)} = \tag{4}$$

$$- \left(\int Z(u_1)(F_1^{-1}(u_1) + F_2^{-1}(\gamma^{-1}(u_1))\,du_1 + \int Y(u_2)(F_1^{-1}(\gamma(u_2)) + F_2^{-1}(u_2))\,du_2 \right)$$

$$+ \left(\int (1 - \gamma^{-1}(u_1)) \frac{Z(u_1)}{f_2(F_2^{-1}(\gamma^{-1}(u_1)))}\,du_1 + \int (1 - \gamma(u_2)) \frac{Y(u_2)}{f_1(F_1^{-1}(\gamma(u_2)))}\,du_2 \right)$$

where $Y(u_2) = d(\gamma(u_2)/d(\sigma_1^2/\sigma_2^2)$ and $Z(u_1) = d(\gamma^{-1}(u_1)/d(\sigma_1^2/\sigma_2^2)$. Using the implicit definition of γ, $h(u_2) = kh(\gamma(u_2))$, $k = a\frac{\sigma_1^2}{\sigma_2^2}$, we have

$$Y(u_2) = -\frac{1}{k^2} \frac{h(u_2)}{h'(h^{-1}(h(u_2)/k))} \qquad\qquad Z(u_1) = \frac{h(u_1)}{h'(h^{-1}(kh(u_1)))}$$

We intend to evaluate this derivative for the value of σ_1^2/σ_2^2 at which the bid distributions are symmetric, i.e., for $\sigma_1^2/\sigma_2^2 = 1/a$. In this case, $k = 1$ and $\gamma(u_2) = u_2$. Substituting in above, we have $Y(u) = -Z(u) = -h(u)/h'(u)$. Using this, we relabel the integration variables in (4) to combine the integrals, giving us

$$\left. \frac{d(\text{Expected Revenue})}{d(\sigma_1^2/\sigma_2^2)} \right|_{\sigma_1^2/\sigma_2^2=1/a} =$$

$$- \left(\int Z(u)(F_1^{-1}(u) + F_2^{-1}(u)\,du + \int Y(u)(F_1^{-1}(u) + F_2^{-1}(u))\,du \right)$$

$$+ \left(\int (1 - u) \frac{Z(u)}{f_2(F_2^{-1}(u))}\,du + \int (1 - u) \frac{Y(u)}{f_1(F_1^{-1}(u))}\,du \right)$$

$$= \int (1 - u)Z(u) \left(\frac{1}{f_2(F_2^{-1}(u))} - \frac{1}{f_1(F_1^{-1}(u))} \right) du$$

The first two terms in the integrand are positive ($Z(u) > 0$ because h and h' are negative by

A3 and A4).   The third term is positive whenever $a > 1$:   because $f_2(x) = \frac{1}{a} f_1(x/a)$ and $F_2^{-1}(u) = a F_1^{-1}(u)$, we have $f_2(F_2^{-1}(u)) = \frac{1}{a} f_1(F_1^{-1}(u))$.   (This is just a consequence of f_2 being more spread out than f_1 at every percentile.)   Thus it is always to the seller's benefit to skew the ratio of private value information in favor of the bidder with less common value information when the bidders would otherwise be equally likely to win.

Proposition 7

As above, we look at the derivative of expected revenue with respect to the proposed change at the point $a \sigma_1^2 / \sigma_2^2 = 1$.   It should be clear that the derivative with respect to a is almost identical to the derivative with respect to σ_1^2 / σ_2^2 calculated above.   The only difference is an additional term that arises because of the change in the distribution F_2 and the corresponding change in the mean of the third common value component.   Using $F_2^{-1} = a F_1^{-1}$, this term is given by

$$
\frac{\partial}{\partial a} \left(
\begin{array}{c}
\int (1 - \gamma^{-1}(u_1))(F_1^{-1}(u_1) + a F_1^{-1}(\gamma^{-1}(u_1)))\, du_1 \\
+ \int (1 - \gamma(u_2))(F_1^{-1}(\gamma(u_2)) + a F_1^{-1}(u_2))\, du_2
\end{array}
\right) - \bar{s}_1
$$
$$
= \int (1 - \gamma^{-1}(u_1)) F_1^{-1}(\gamma^{-1}(u_1))\, du_1 + \int (1 - \gamma(u_2)) F_1^{-1}(u_2)\, du_2 - \bar{s}_1
$$

where $\bar{s}_1 = E(s_1) = dE(s_2)/da$ is the amount by which the mean of the third component needs to be adjusted.   Combining the integrals and evaluating at $a \sigma_1^2 / \sigma_2^2$, this term becomes

$$
2 \int (1 - u) F_1^{-1}(u)\, du - \bar{s}_1
$$

or by making the change of variables $s = F_1^{-1}(u)$

$$
\int 2 f_1(s)(1 - F_1(s))\, s\, ds - \bar{s}_1
$$
$$
= \quad E(\min(x, y) \mid x, y \sim F_1) - \bar{s}_1
$$

which is negative.   This simply reflects the fact that if there is a component of the value about which both bidders are symmetrically uninformed, both bidders will bid up to the full expected value for that component.   As the seller transfers information about this component to Bidder 2, he creates an information rent for 2 that *ceteris paribus* reduces revenues.   To get the total effect of a change in a, we must add this to the derivative calculated in the previous proof (which reflects the changes in revenue due to changes in bidding behavior).   For $a < 1$, this derivative is negative as well.   Thus, the overall effect on revenue of releasing more common value information to Bidder 2 is negative when Bidder 2 has worse common value information than Bidder 1.

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