Private Provision of Public Goods
in the Local Interaction Model

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Abstract:

This paper analyses the evolutionary version of the Public Good game (Eshel, Samuelson, and Shaked (1998)) when agents can use imitation and best reply decision rules. I provide a condition, which completely describes agent behavior in the long run, for *any* number of neighbors and *any* total number of agents. Moreover, it is shown that it is enough to have just one decision rule per agent in order to obtain the same long run outcomes. The paper gives an explanation why we might observe irrational cooperation in the rational World. *(JEL* C70, C72, C73)

How should we model individuals' behavior in economic models? Individuals are traditionally assumed to be rational with perfect foresight, now, however boundedly rational behavior is increasingly being studied. In evolutionary game theory, bounded rationality is often modeled by assuming that individuals have finite memories and use a fixed rule to choose a strategy to play. Although the evolutionary nature of such an approach is realistic, a major drawback is that only one decision rule is normally available to a player.

In reality, humans experiment with many different decision rules over time. (See Arthur (1994) for an excellent discussion. He also sites psychology literature.) The literature on experimental economics claims that there is no single rule, which might describe human behavior (see Mookherjee and Sopher (1994, 1997), Cheung and Friedman (1997), Camerer and Ho (1999), Salmon (2001)).

This paper studies an evolutionary model where several decision rules are available to each agent. To the best of my knowledge, it is the first attempt to model agents with sets of decision rules. In every period the agents have to choose a decision rule and then use this rule to select a strategy. The main questions of this paper are "How predictions of this model are different from evolutionary models, where agents have only one decision rule" and "Which rule(s) will survive in the long run?"

There are a few papers in evolutionary game theory where authors model multiple decision rules by assuming that a part of the population for a player **i** position in the game always uses a particular decision rule and another part of the same population always uses another rule. Agents are chosen at random for player positions from the corresponding populations to play the game in every period (see Saez-Marti and Weibull (1999), Matros (2000), Josephson (2002)). The main assumptions of this literature are that any agent plays only one particular rule and any agent from the population **i** has a positive probability to play the player **i** position in the game in every period. The agents in this paper use an optimal rule

from their sets of rules in every period. This optimal rule may be the same decision rule from the set of rules or may vary across periods. Subsection II.2 clarifies the crucial difference between these two approaches. I show that even short run outcomes, or recurrent classes, are different in two cases. It is not surprising after that that these two approaches might lead to different long run predictions. More discussion on this issue can be found in my companion paper, Matros (2002).

Arthur (1994) models agents with several predictions or hypotheses in the form of functions that map the past outcomes into next-period outcomes. However, he offers only computer simulation for a particular "Bar Problem". I study an evolutionary framework similar to Eshel, Samuelson, and Shaked (1998), but my assumptions about agents differ. Whereas the agents in their model imitate others who earn the highest average payoffs in a Public Good game, the agents in my model have a set of two rules: the imitation and the best reply.

In every period, N agents play a Public Good game. There are two strategies in the game. "Production" of a public good is a strictly dominated cooperative strategy. The agents are located on a circle and their utilities are increasing in the number of agents playing the strategy "Production" in their local **2k** neighborhoods. Each agent observes a sample of plays of her **2k** nearest neighbors from the previous period. She then searches for the current optimal decision rule. If the strategy chosen based on her last period decision rule performs best among her **2k** neighbors in the sample, then the optimal decision rule is the last period decision rule. Otherwise, another decision rule is optimal. The boundedly rational agents next use the optimal rule to choose a strategy in the current period, but with some positive probability they make a mistake and instead choose either the best reply rule or the imitation rule at random. As is standard in the literature, the agents also make errors in using the rules with the result that they select a strategy at random with some small probability.

My choice of two decision rules is motivated by the fact that imitation is the first rule that a human being learns as a child. In time, humans become "smarter" and learn the best reply rule. There is a large volume of recent literature concerned with trying to identify the type of learning rules used by subjects in experiments. For example, result 4 in Huck, Normann, and Oechssler (1999) states:

> *If subjects have the necessary information to play best replies, most do so, though adjustment to the best reply is almost always incomplete. If subjects additionally have the necessary information to "imitate the best", at least a few subjects become pure imitators.*

The best reply rule and the imitation rule are typical decision rules in evolutionary models. The best reply rule usually leads to a rational outcome (see Young (1993, 1998), Kandori, Mailath, and Rob (1993), Ellison (1993), Blume (1993), Samuelson (1997)). Bergstrom and Stark (1993), Eshel, Samuelson, and Shaked (1998) show that the imitation rule can lead to a cooperative irrational outcome. Simulation evidence for the imitation behavior can be found in Nowak and May (1992, 1993) and Nowak, Bonhoeffer, and May (1994).

To emphasize the difference in the long run predictions between the imitation and the best reply rules, I consider the Public Good game. On the one hand Eshel, Samuelson, and Shaked (1998) show that the imitation rule leads to the cooperative irrational outcomes in the Public Good game in the long run, even in the presence of mutations that continually introduce the rational strategy into the model. On the other hand, Young (1998) demonstrates that the best reply rule selects the rational uncooperative outcome. Moreover, Matros (2002) shows that the rational uncooperative outcome is a unique long run prediction, if there are

populations of agents for every player position, such that every population contains agents who use either the best reply rule or the imitation rule and every agent in the population **i** has a positive probability to play the player **i** position in the Public Good game in every period. Given these contrasting predictions this paper explores which outcome(s) will arise when agents can use both rules.

First, the short run outcomes are described. Then, I obtain an ergodic Markov process on the finite state space and study the stationary distribution of this process, as the mistake and error probabilities tend to zero. There are two cases: the two neighbors case, **k=1**, and the general case, where each agent has more than two neighbors, **k≥2**. I analyze short and long run outcomes in both cases. It turns out that in the two neighbor case, **k=1**, the short run outcomes depend on the initial conditions, but the only rational uncooperative outcome is a unique long run prediction. The general case is more interesting.

The cases **k=2** and **k>2** are considered separately. The case **k=2** serves as an example. The case **k>2** is quantitatively different from the case **k=2**, but qualitatively the same. All features of the short and long run behavior for the case **k>2** are present in the case **k=2**. It is shown that if agents play the cooperative irrational strategy **P** at all in the long run, then they play this strategy in groups. I prove that both rational and cooperative irrational outcomes might arise if agents can use the best reply rule and the imitation rule. Moreover, the paper presents a condition – Theorem 3 - which completely describes the long run outcomes for *any* number of neighbors. This result also shows that it is enough to give just one decision rule per agent instead of sets of decision rules in order to obtain the same long run outcomes. The main message of the paper is that even though sets of decision rules can better represent human behavior, but the actual long run prediction of the model might coincide with a simpler model, where every agent has just one decision rule. It means that in many cases we do not

need a complicated model, where agents have sets of decision rules, instead one of the simpler models with just one decision rule per agent might be enough.

The rest of the paper is organized as follows. Section I describes the model. Section II analyses the short run prediction of the model. Section III examines the long run prediction of the model, if the agents can make mistakes and errors. Section IV concludes. Proofs are given in the Appendixes.

I. The Model: Public Good Game

We will call the following game the *Public Good Game*. There are **N** players. A player can either produce a public good, which gives one unit of utility to **2k(<N)** of her neighbors and incurs a net cost of **0<c<1** to the player herself, or do nothing at no cost. In other words, every player has two strategies: cooperate with other **2k** players and **P**roduce the public good, or be **S**elfish and not produce the public good. We will call these strategies **P** and **S** respectively. The payoff of agent **i** is then K^P_i**-c** if agent **i** plays the strategy **P** and K^P_i if agent **i** plays the strategy **S**, where $K^P_i \in$ **{0,…,2k}** is the number of **i**'s neighbors, who play the strategy **P**.

The one-shot Public Good game has one strict Nash equilibrium **(S,…,S)**, where all players play the strategy **S**. Moreover, the strategy **P** is strictly dominated by the strategy **S**. If the Public Good game is played repeatedly, then the Folk Theorem can be applied[1] and the play **(P,…,P)**, where all players play the strategy **P**, can be sustained as an equilibrium in the infinitely repeated Public Good Game.

We will consider the following evolutionary version of the Public Good Game. In each discrete time period, **t=1,2**,..., a population of **N** agents plays the Public Good Game. We

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[1] For discussion and applications of Folk Theorem see Fudenberg and Tirole (1991).

assume, as in Ellison (1993); Bergstrom and Stark (1993); and Eshel, Samuelson, and Shaked (1998), that the agents are located around a circle. An agent **i** chooses a strategy $x^t_i \in \{P, S\}$ at time **t** according to a decision rule defined below. The play at time **t** is the vector $x^t = (x^t_1, \ldots, x^t_N)$.

Strategies are chosen as follows. At time **t+1**, each agent **i** inspects a sample $(x^t_{i-k}, \ldots, x^t_{i-1}, x^t_i, x^t_{i+1}, \ldots, x^t_{i+k})$ of size **2k+1** of her **2k** nearest neighbors and herself, taken from the previous play at time **t**. We assume that every agent has two decision rules. They are Best Reply (**BR**) and Imitation (**IM**). The **BR** rule means that an agent plays a strategy in period **t+1**, which is the best reply to her **2k** neighbors strategy distribution $(x^t_{i-k}, \ldots, x^t_{i-1}, x^t_{i+1}, \ldots, x^t_{i+k})$ in the previous period **t**. An agent uses the **IM** rule in period **t+1**, if she plays a strategy, which gives the highest payoff among her **2k** neighbors and herself in the previous period **t**. Suppose that an agent **i** uses the **BR (IM)** rule to choose a strategy in the beginning of the period **t**. The agent **i** uses the **BR (IM)** rule again in the period **t+1**, if this rule is the optimal current decision rule, and she switches to another decision rule **IM (BR)** otherwise. We assume that the last period rule is the *optimal current* decision rule, if an agent **i** observes that the strategy x^t_i, chosen based on the **BR (IM)** rule in the period **t**, gives the highest payoff in the sample $(x^t_{i-k}, \ldots, x^t_{i-1}, x^t_i, x^t_{i+1}, \ldots, x^t_{i+k})$. In other words, agent **i** observes the sample $(x^t_{i-k}, \ldots, x^t_{i-1}, x^t_i, x^t_{i+1}, \ldots, x^t_{i+k})$ and corresponding payoffs of her neighbors. Then she identifies the highest payoff among her neighbors and herself. There exists exactly one strategy, which gives the highest payoff, because of our assumption **0<c<1**. The last period rule is the optimal current decision rule if this strategy is the strategy x^t_i. Note that the agents do *not* observe the decision rules of other agents, just their strategies. The assumption about which rule to use in the current period is very important and we will look at the following examples to see how agents can change the imitation rule into the best reply rule and vice versa. We will call an

agent, whose last period decision rule is **BR**, a *maximizer*, and an agent, whose last period decision rule is **IM**, an *imitator*.

Example 1: Suppose that **k=2** and an imitator observes the following sample

$$P(PS\underline{\textbf{P\{IM\}}}P P)PP,$$

where the strategies in the brackets are the observed sample of the imitator and **P** is her own strategy choice. Only one agent plays the strategy **S** in the sample and the other four agents play the strategy **P**. The second neighbor to the right obtains the highest payoff to the strategy P (*P* marks her), which is **4-c**. However, the agent, who plays the strategy **S**, receives payoff of **4**. Thus the strategy **S** receives the highest payoff in the sample. It means that the imitation is not the optimal rule for the imitator and she will use the best reply rule in the following period.

Suppose now that a maximizer observes the following sample

$$PP(PP\underline{\textbf{S\{BR\}}}SS)SS,$$

where she observes that two of her neighbors play the strategy **S**, the other two play the strategy **P** and **S** is her own strategy choice. The highest payoff to the strategy **S** in the sample is obtained by the maximizer and equals to **2**. The second neighbor to the left obtains the highest payoff to the strategy P (*P* marks her), which is **3-c>2**. Therefore the maximizer finds that the best reply rule is not optimal and will use the imitation rule in the following period.

Assume that the sampling process begins in the period **t=1** from some arbitrary initial play $\mathbf{x^0}$ and some arbitrary initial decision rule distribution $\mathbf{d^0}$. We then obtain a finite Markov chain on the finite state space $(\mathbf{\{P,S\}})^N \times (\mathbf{\{BR,IM\}})^N$ of states of the length **2N** drawn from the strategy space $(\mathbf{\{P,S\}})$ and the decision rule space $(\mathbf{\{BR,IM\}})$, with an arbitrary initial play $\mathbf{x^0}$ and some arbitrary initial decision rule distribution $\mathbf{d^0}$. Given a play $\mathbf{x^t}$ and a decision rule distribution $\mathbf{d^t}$ at time t, the process moves to a state of the form $\mathbf{\{x^{t+1}; d^{t+1}\}}$ in the next period, such a state is called a successor of $\mathbf{\{x^t; d^t\}}$. We will call this process *unperturbed adjusted dynamics* with population size **N** and **2k** neighbors, $\mathbf{AD^{N,k,0,0}}$.

Example 2: Suppose that **k=1**, **N=4**, $\mathbf{x^0}$**=(P,P,S,S)** and $\mathbf{d^0}$**=(IM,IM,BR,BR)**.

In period **1**, each agent **i** inspects a sample $\mathbf{(x^0_{i-1}, x^0_i, x^0_{i+1})}$ of size **3** of her **2** nearest neighbors and herself, taken from the previous play in period **0**. Agents **3** and **4** (**1** and **2**) use the **BR (IM)** rule to choose a strategy in the beginning of period **0**. Agents **3** and **4** use the **BR** rule again in period **1**, because these agents observe that the strategy **S** gives a payoff of **1** unit and the strategy **P** gives a payoff of **1-c** units (there is exactly one neighbor of every agent playing the strategy **P**). Therefore, the strategy **S** gives the highest payoff in the sample.

However, agents **1** and **2** will switch to another decision rule, **BR**, because they also observe that the strategy **S** gives the highest payoff in their samples. It means that $\mathbf{d^1}$**=(BR,BR,BR,BR)**. Hence, all agents will use the **BR** rule in period **1** and play the dominant strategy **S**, $\mathbf{x^1}$**=(S,S,S,S)**. As a result the unperturbed adjusted dynamics process moves to the state $\mathbf{\{x^1; d^1\}}$ in period **1**.

The unperturbed adjusted dynamics process describes the short run behavior in the model, when there are no mistakes and errors in the agents' behavior. Short run prediction is useful, because the predicted outcome(s) arise very fast, due to the local interaction structure

of the model, stay long (until a mistake or an error are made), and depend on the initial condition.

Let us introduce some noise into the model. Humans can often use a specific rule, even though they might know that another rule is better in the current situation. To model that and the situation where a rule may be chosen by a mistake, we suppose that the agents use the optimal rule with probability **1-μ** and use a rule chosen at random with probability **μ≥0**. Moreover, suppose that agents use a rule to choose a strategy with probability **1-ε** and make an error and choose a strategy at random with probability **ε>0**. The resulting *perturbed adjusted dynamics* process $\mathbf{AD^{N,k,\mu,\varepsilon}}$ is an ergodic Markov process on the finite state space $\mathbf{(\{P, S\})^N \times (\{BR, IM\})^N}$. Thus, in the long run, the initial state is irrelevant. We will pay special attention to the analysis of boundedly rational agents which is the case if probabilities of mistakes, **μ**, and errors, **ε**, are small.

II. Short Run: Recurrent Classes

In what follows, we will make use of the following definitions. A *recurrent class* of the process $\mathbf{AD^{N,k,0,0}}$ ($\mathbf{AD^{N,k,\mu,0}}$ for **μ** fixed and **0<μ<1**) is a set of states such that there is zero probability of moving from any state in the class to any state outside, and there is a positive probability of moving from any state in the class to any other state in the class. We call a state **h** *absorbing* if it constitutes a singleton recurrent class. In this section we first analyze the situation when all agents *always* use the optimal rule, **μ=0**, and do *not* make errors. Then, we consider the process $\mathbf{AD^{N,k,\mu,0}}$ for **μ** fixed and **0<μ<1** and find its recurrent classes.

II.1. Case μ=0

Note that if every agent uses the imitation rule and plays the strategy **P**, then each agent obtains the same payoff and therefore the imitation rule is the current optimal decision rule. Moreover, as everyone plays the same strategy **P**, the imitation rule selects strategy **P** in the following period as well. It means that a state of the form:

$$\{(P,\ldots,P);(IM,\ldots,IM)\}=\{P;IM\}$$

is absorbing.

Suppose that all agents play the strategy **S** in the previous period. Either an agent uses the **IM** rule or the **BR** rule, she observes the only strategy **S** in her sample. It means that her decision rule is optimal. Both rules select the strategy **S** again in the following period. Hence, states in the form:

$$\{(S,\ldots,S);(d,\ldots,d)\}=\{S;d\}$$

are absorbing, where **d** may be either **BR** or **IM**. Let all 2^N states of the form **{S;d}** denote the set **{(S;•)}**.

Other recurrent classes depend on the number of neighbors, **k**, and the size of the population, **N**. Let us start from the case **k=1**.

II.1.1. Case k=1

Note that a maximizer can *never* become an imitator in this case, because the maximal possible payoff to the strategy **P** in the maximizer's neighborhood is **1-c**. However, if at least

one neighbor plays the strategy **P**, then the maximizer's payoff associated with her strategy **S** is at least **1**. It means that a maximizer will *always* find her last period decision rule, **BR**, to be optimal and be the maximizer.

However, an imitator can become a maximizer if she plays the strategy **P** and one of the following combinations of strategies occurs in her sample:

(1)   (S,<u>**P**</u>,S),

(2)   (S,<u>**P**</u>,P),S,

(3)   P,(S,<u>**P**</u>,P),

where the strategies in the brackets are the imitator's observed samples, the first line arises if both neighbors of the imitator play the strategy **S**; the second line arises if one neighbor plays the strategy **P** and his neighbor in turn plays the strategy **S**; the third line arises if one neighbor plays the strategy **S** and his neighbor in turn plays the strategy **P**. It is straightforward to see from conditions **(1)-(3)** that an imitator finds her last period rule to be optimal only in the following two cases

(4)   (P,**P**,P),

(5)   S,(S,**P**,P),P,

where the imitator observes only strategy **P** in her sample in the first case and she is in between a cluster of strategy **S** and a cluster of strategy **P** in the second case. Conditions **(1)-(5)** provide a complete description of the absorbing states, if **k=1**. So, the following are the absorbing states:

- The state in which all agents are imitators and play the strategy **P**, **{P;IM}**.

- The states in which all agents play the strategy **S**. The set **{(S;•)}** contains all such states.

- A state in which all agents are imitators and play the strategy **P** except two adjacent agents, who use either the **BR** rule or the **IM** rule and play the strategy **S**:

  …P,…P,S,S,P,…P,…

- A state in which all agents are imitators and play the strategy **P** except three adjacent agents, who use either the **BR** rule or the **IM** rule and play the strategy **S**:

  …P,…P,S,S,S,P,…P,…

- …

- A state in which three adjacent agents are imitators and play the strategy **P**, and other agents use either the **BR** rule or the **IM** rule and play the strategy **S**:

  …S,…,S,P,P,P,S,…S,…

These examples, and combinations constructed from them, include all of the possibilities for the absorbing states. Note that in all absorbing states both strategies must appear in clusters. We then have the following proposition.[2]

PROPOSITION 1: Suppose that **N≥5**. The absorbing states of the unperturbed process **AD^{N,1,0,0}** are (i) the states in which all agents play the strategy **S**, (ii) the state in which all agents are imitators and play the strategy **P**, and (iii) the states in each of which a cluster of imitators, playing the strategy **P**, of the length three or longer are separated by a cluster of imitators or maximizers, playing the strategy **S**, of the length two or longer.

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[2] A formal proof is available from the author on request.

II.1.2. Case k=2

The story is different now. A maximizer, who plays the strategy **S**, can become an imitator if she faces the following situation in her sample:

(6)   (S,S,<u>S</u>,P,P),P,P,

where strategies in the brackets are the observed sample and <u>S</u> is her own strategy choice.

An imitator can become a maximizer if she plays the strategy **P** and one of the following combinations of strategies occurs in her sample:

A) All neighbors play the strategy **S**.

B) Only one neighbor plays the strategy **P**.

C) Two neighbors play the strategy **P** and the maximal payoff to this strategy is less than the maximal payoff to the strategy **S** in the sample.

D) Only one neighbor plays the strategy **S** and her payoff is higher than the maximal payoff to the strategy **P** in the sample.

It is straightforward to see that the state **{P;IM}** and states in the set **{(S;•)}** are absorbing. The following examples show that there are other recurrent classes.

Example 3: Suppose that **N=7** and **k=2**. See Figure 1.

Insert Figure 1 here.

There are five imitators, who play the strategy **P**, and two maximizers, who play the strategy **S**. It is easy to see that every agent finds that her strategy performs best in her sample. For

example, every imitator has the best imitator, marked **IM**, in the sample. The best imitator receives **4-c**, which the highest payoff among all agents. It means that each imitator finds her decision rule to be optimal. Every maximizer does not have the best imitator in the sample, which means that their payoffs, **3**, are the highest in their samples. Therefore both maximizers find that the best reply rule is optimal. It means that every agent finds her last period decision rule to be optimal. These optimal rules select the last period strategies again. Hence, Figure 1 gives an example of an absorbing state.

The following example shows that a recurrent class can contain several states.

Example 4: Suppose that **N=9** and **k=2**. See Figure 2.

Insert Figure 2 here.

There are eight imitators, who play the strategy **P**, and one maximizer, who plays the strategy **S**, in the first circle on Figure 2. The maximizer obtains **4**, which is the highest payoff among all agents. Therefore, all four agents, who have the maximizer's strategy in their samples, find that their decision rules are not optimal. These four agents will use the best reply rule instead of the imitation rule in the following period. The best reply rule selects the strategy **S** in the following period, as it is shown in the second circle on Figure 2. There are five maximizers and four imitators in the second circle. The imitators find that the imitation rule is optimal and they continue to play the strategy **P**. Two maximizers on the boundary between maximizers and imitators observe that two neighbors play the strategy **P** and the other two play the strategy **S**. The highest payoff to the strategy **P** is equal to **3-c**, which is greater that **2**, which is the highest payoff to the strategy **S** in the sample. Therefore, these two maximizers find that

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the best reply rule is not optimal and they use the imitation rule in the following period. We observe this in the third circle on Figure 2. There are three maximizers and six imitators in the third circle. Once again, all imitators find that the imitation is the optimal decision rule and play the strategy **P** in the following period. Two maximizers on the boundary between maximizers and imitators observe that two neighbors play the strategy **P** and the other two play the strategy **S**. The highest payoff to the strategy **P** is equal to **3-c**, which is greater that **2**, which is the highest payoff to the strategy **S** in the sample. Therefore, these two maximizers find that the best reply rule is not optimal and they use the imitation rule in the following period. It explains how the process moves from the third to the first circle on Figure 2. The cycle repeats again. Hence, Figure 2 gives an example of a recurrent class, which contains three states.

With these two examples in mind, it is easy to see how conditions **(6), (A) − (D)** provide a complete description of recurrent classes, if **k=2**. The following are recurrent classes:

- The state in which all agents are imitators and play the strategy **P**.

- The states in which all agents play the strategy **S**. The set **{(S;●)}** contains all such states.

- A state in which all agents are imitators and play the strategy **P** except two adjacent imitators or maximizers, who play the strategy **S**:

    …P,…,P,S,S,P,…P,…

- A set of three states, consisting of:

    …P,…,P,S,P,…P,…

    …P,…,P,S,S,S,S,S,P,…P,…

    …P,…,P,S,S,S,P,…P,….

These examples and combinations constructed from them include all of the possibilities for recurrent classes. As it is in the case **k=1** both strategies must appear in clusters in all recurrent classes.

PROPOSITION 2: Suppose that **N≥6**. The recurrent classes of the unperturbed process **AD^{N,2,0,0}** are (i) the states in which all agents play the strategy **S**, (ii) the state in which all agents are imitators and play the strategy **P**, and (iii) the sets containing states in each of which a cluster of imitators, playing the strategy **P**, of the length four or longer are separated by clusters of maximizers or imitators, playing the strategy **S**, of the length five or less. These sets are either singletons (in which case all clusters of the strategy **S** are of the length two) or contain three states [in which case any cluster of the strategy **S** of the length one (three or five) in one of the states moves to a cluster of the strategy **S** of the length five (one or three) in the other].

PROOF: See the Appendix A.

II.1.3. General Case k>2

After we have found out the recurrent classes in the case **k=2**, it is easy to see that the following are recurrent classes if **k>2**:

- The state in which all agents are imitators and play the strategy **P**.
- The states in which all agents play the strategy **S**. The set **{(S;●)}** contains all such states.

- A state in which all agents are imitators and play the strategy **P** except two adjacent imitators or maximizers, who play the strategy **S**:

  …P,…,P,S,S,P,…P,…

- A set of three states, consisting of:

  …P,…P,S,P,…P,…

  …P,…P,S,…,S,P,…P,…

  …P,…,P,S,S,S,P,…P,…,

where there is one agent, who plays the strategy **S** in period **t**; there are **2k+1** agents, who play the strategy **S** in period **t+1**; there are three agents, who play the strategy **S** in period **t+2**. The cycle is repeated after that.

These examples and combinations constructed from them include all of the possibilities for recurrent classes. Again, both strategies must appear in clusters in all recurrent classes.

THEOREM 1: Suppose that **k>2** and **N≥k+4**. The recurrent classes of the unperturbed process $\mathbf{AD^{N,k,0,0}}$ are (i) the states in which all agents play the strategy **S**, (ii) the state in which all agents are imitators and play the strategy **P**, and (iii) sets containing states in which there is a cluster of imitators, playing the strategy **P**, of the length **k+2** or longer are separated by clusters of maximizers or imitators, playing the strategy **S**, of the length **2k+1** or less. These sets are either singletons (in which case all clusters of the strategy **S** are of the length two) or contain three states [in which case any cluster of the strategy **S** of the length one (three or **2k+1**) in one of the states moves to a cluster of the strategy **S** of the length **2k+1** (one or three) in the other].

PROOF: See the Appendix A.

The following subsection clarifies the main difference between the approach of this paper and other papers in the field. Typically authors model multiple decision rules by assuming that a part of the population for a player **i** position in the game always uses a particular decision rule and another part of the same population always uses another rule. Agents are chosen at random for player positions from the corresponding populations to play the game in every period (see Saez-Marti and Weibull (1999), Matros (2000), Josephson (2002)). I will fix **0<μ<1**, such that the agents can use the best reply rule and the imitation rule in every period. We can think that there is a population of agents for every player position in the Public Good game and that **μ** share of each population contains agents, who always use the best reply rule, and the other part of the populations, *1-μ* share, always use the imitation rule.

II.2. 0<μ<1

If agents can use both decision rules in every period with positive probability, then only rational outcomes will survive as the short run outcomes. The intuition is simple: if occasionally all agents use the best reply rule in one period, then only the strategy **S** will be present thereafter. The following proposition is a corollary of Matros (Theorem 1, 2002).

PROPOSITION 3: If **μ** is fixed and such that **0<μ<1**, then only states from the set {(•,S)} are absorbing states of the unperturbed process **AD**^{N,k,μ,0}.

PROOF: See the Appendix A.

This finishes the description of the short run outcomes.

III. Long Run: Mutations

So far we have considered only the short run behavior. The long run behavior of the adjusted dynamics process $\mathbf{AD^{N,k,\mu,\epsilon}}$ will be analyzed in this section by letting agents make mistakes or/and errors. Firstly, we consider the situation, where both mistakes and errors tend to zero, $\boldsymbol{\mu \to 0}$, $\boldsymbol{\epsilon \to 0}$. Secondly, we analyze the situation, where $\boldsymbol{\mu}$ is positive and fixed (the agents can always use both decision rules) and the probabilities of errors tend to zero, $\boldsymbol{\epsilon \to 0}$.

III.1. $\boldsymbol{\mu \to 0}$, $\boldsymbol{\epsilon \to 0}$

From Propositions 1 and 2 and Theorem 1, it follows that there are four "types" of the recurrent classes. In most cases, mistakes in each "type" of the recurrent classes, which can happen with probability $\boldsymbol{\mu}$, lead to another recurrent class of the same type. Agents have to make errors, which can happen with probability $\boldsymbol{\epsilon}$, in order to move from a recurrent class of one type to a recurrent class of another type. Movements between the recurrent classes of different types rather than movements inside of the recurrent classes are of our interest. To make the exposition as simple as possible we suppose that $\boldsymbol{\mu = \epsilon}$ in this section. This assumption affects the long run outcomes. Although the results in general case $\boldsymbol{\mu \to 0}$, $\boldsymbol{\epsilon \to 0}$ are quantitatively different from the case $\boldsymbol{\mu = \epsilon \to 0}$, but the case $\boldsymbol{\mu = \epsilon \to 0}$ contains all qualitative features of the general case. Moreover, we can avoid all complications with the order of limits with our assumption. We are now in a position to state the main results on the long run behavior.

III.1.1. Case k=1

Proposition 1 claims that there are three types of candidates for the long run outcomes. The following result specifies a unique prediction.

PROPOSITION 4: The limiting distribution of the adjusted dynamics process $\textbf{AD}^{\textbf{N,k,}\varepsilon\textbf{,}\varepsilon}$ puts positive probability only on states from the set **{(S;•)}**, where all agents play the strategy **S.**

PROOF: See the Appendix B.

We shall consider more interesting cases now. The case **k=2** is the first to be analyzed. The general case **k>2** will follow it.

III.1.2. Case k=2

Proposition 2 states that there are four types of short run outcomes. The following result specifies when the irrational cooperative behavior will be observed in the long run.

PROPOSITION 5: If **N>48**, then the limiting distribution of the adjusted dynamics process $\textbf{AD}^{\textbf{N,k,}\varepsilon\textbf{,}\varepsilon}$ puts positive probability on all recurrent classes except states in which all agents play the strategy **S.**

PROOF: See the Appendix B.

III.1.3. Case k>2

The general case is similar to the case **k=2**. Although in Proposition 5 we only describe a condition, which guarantees the irrational cooperative behavior in the long run, the following result completely describes all possibilities in the general case.

THEOREM 3: If $N>4(k+1)(k+2)$, then the limiting distribution of the adjusted dynamics process $\mathbf{AD}^{N,k,\varepsilon,\varepsilon}$ puts positive probability on all recurrent classes except absorbing states in which all agents play the strategy **S**. If $N<4(k+1)(k+2)$, then the limiting distribution of the adjusted dynamics process $\mathbf{AD}^{N,k,\varepsilon,\varepsilon}$ contains only the absorbing states, where all agents play the strategy **S**. If $N=4(k+1)(k+2)$, then the limiting distribution of the adjusted dynamics process $\mathbf{AD}^{N,k,\varepsilon,\varepsilon}$ puts positive probability on all recurrent classes.

PROOF: See the Appendix B.

Theorem 3 describes for which parameters **N** and **k** the long run prediction of the model contains cooperative irrational outcomes. If we fix **k**, such that **2k+1=N**, then the unique prediction of the model is the outcome where all agents play the strategy **S**. Now, if we keep fixed **k** and start to increase the number of agents on the circle, **N**, then the rational outcome – all agents play the strategy **S** – will be the unique long run prediction until **N=4(k+1)(k+2)**. If we increase the number of agents on the circle, **N**, further, then the long run prediction changes. The irrational cooperative outcomes will appear if the number of agents on the circle is much bigger that the number of neighbors, or formally, **N>4(k+1)(k+2)**. Similar effects arise if the number of agents on the circle, **N**, is fixed and we start to decrease the number of neighbors, **k**.

Theorem 3 shows also that for any given pair of parameters **N** and **k**, there exists one rule per agent, instead of the set of two rules, such that if all agents use this particular rule, then the long run prediction is the same as if the agents had the sets of rules. In particular, if **N<4(k+1)(k+2)**, then such a rule is the best reply rule, if **N≥4(k+1)(k+2)**, then such a rule is the imitation rule.

III.2. Fixed μ, 0<μ<1

From proposition 3 and theorem 2, we conclude that all agents must play the strategy **S** in the long run, because all agents play the strategy **S** in every absorbing state. This result is consistent with similar results in Saez-Marti and Weibull (1999), Matros (2000) and Josephson (2002).

IV. Conclusion

This paper analyzes the evolutionary version of the Public Good game where agents can use two decision rules. I model this framework as a Markov chain on a finite state space. It is shown that the short run behavior of the system can have four types: (i) *all agents play the cooperative production strategy*, (ii) *all agents play the rational selfish non-productive strategy*, (iii) there are agents playing both strategies so that *between two clusters of agents playing the cooperative strategy there is a cluster of two agents playing the rational strategy* or (iv) there are agents playing both strategies so that *between two clusters of agents playing the cooperative strategy there is a cycle, when the number of agents playing the rational strategy in the cluster varies* from one to **2k+1**, then to three and then back to one.

This result is consistent with the results in Bergstrom and Stark (1993) and Eshel, Samuelson, and Shaked (1998), where agents use only the imitation rule and have two, **k=1**, or four, **k=2**, neighbors. One of the contributions of this paper is to show that this result is robust to the introduction of the best reply rule into the system.

Another contribution is a condition - Theorem 3 - between the number of neighbors, **k**, and the total number of the agents, **N**, which completely describes the long run behavior of the perturbed process. So far in the literature (see Bergstrom and Stark (1993) and Eshel, Samuelson, and Shaked (1998)) only the two, **k=1**, and four neighbor, **k=2**, cases have been considered. I demonstrate that even with the possibility of using the best reply rule, the cooperative irrational behavior might be sustained not only in the short run, but also in the long run. At the same time, my condition shows the limit of Eshel, Samuelson, and Shaked (1998) results for an arbitrary number of neighbors. They fix the number of the neighbors, two or four, and find the total number of agents, **N**, such that the irrational cooperative behavior will survive in the long run. Theorem 3 describes the long run outcomes for any number of the neighbors, **k**, and for any total number of agents, **N**.

Theorem 3 also reveals that it is enough to have just one decision rule per agent in order to obtain the same long run outcomes as we obtain in the case of the sets of two decision rules. Which rule to use for this purpose depends on the parameters **N** and **k** of the model. This result is very important, because it means that we can use simple models with just one decision rule to analyze human behavior: the long run prediction is the same with more complicated models. It is an open and interesting question for the future research: whether it is always the case that for *any* sets of decision rules and *any* game there always exists a unique decision rule per agent, such that the long run prediction in both cases are the same.

In concluding it should be noted that my model could be extended to the case where agents can use any sets of rules. For discussion of this see my companion paper, Matros

(2002), where I analyze a bigger set of rules, but assume that agents have a positive probability to use any rule from their sets of rules in every period. It is possible that the different choices of the set of rules may change the results of this paper, but the exploration of this possibility is left for future research.

Appendix A

PROOF OF PROPOSITION 2: It is straightforward that the states in which all agents play the strategy **S** or all agents are imitators playing the strategy **P** are absorbing.

To find the remaining recurrent classes, consider what happens to a cluster of imitators playing the strategy **P**. From conditions **(A) – (D)**, it follows that any cluster of the strategy **P** of the length one, two or three will immediately disappear. So, imitators can play the strategy **P** in groups of the length four or longer. Consider what happens to a cluster of agents playing the strategy **S**. From the condition **(6)**, it follows that any such a cluster of the strategy **S** of the length three or longer will shrink in the following period.[3] It will shrink until the cluster becomes the length of two or one. The cluster of agents playing the strategy **S** of the length two will not change. However, if only one agent plays the strategy **S** among four neighbors, each of whom plays the strategy **P**, then the whole neighborhood – all five agents - will play the strategy **S** in the following period. Note that four neighbors will change their decision rules first in the following period. Then this cluster of agents playing the strategy **S** shrinks to the cluster of the length three (two agents change their decision rules from the best reply to the imitation), or to the cluster of the length four (one agent changes his decision rules from

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[3] Note that the "shrinking" agents will change their decision rules. In general it might be the case that some agents use the imitation rule to select the strategy **S** in the current period. These agents will switch to the best reply rule and play the strategy **S** in the following period. However, the strategy choices of imitators, who play the strategy **P**, do not change in these circumstances and after finite number of periods the cluster of agents playing the strategy **S** of length one or two will arise.

the best reply to the imitation and the other switches from the imitation to the best reply). Then the cluster of the strategy **S** shrinks to the length one or two and the cycle is repeated again.

PROOF OF THEOREM 1: It is obvious that the states in which all agents play the strategy **S** or all agents are imitators playing the strategy **P** are absorbing.

To find the remaining recurrent classes, consider what happens to a cluster of imitators playing the strategy **P**. Note that any cluster of the strategy **P** of the length **1**, **2**,…, **k+1** will immediately disappear. So, imitators can play the strategy **P** in groups of the length **k+2** or longer. Consider what happens to a cluster of agents playing the strategy **S**. Any such a cluster of the length three or longer will shrink in the following period.[4] It will shrink until the cluster of the strategy **S** of the length of two or one. The cluster of agents playing the strategy **S** of the length two will not change. However, if there is only one agent who plays the strategy **S** among her **2k** neighbors, each of whom plays the strategy **P**, then the whole neighborhood – all **2k+1** agents - will become maximizers and play the strategy **S** in the following period. Then this cluster of agents playing the strategy **S** shrinks to the cluster of the length three, then one and the cycle is repeated again.

PROOF OF PPOPOSITION 3: It is immediate that any state from the set **{(●,S)}** is absorbing. We will show that there are no other absorbing states. Suppose that the unperturbed process $\mathbf{AD^{N,k,\mu,0}}$ is in an arbitrary state in period **t**. There is a positive probability that every agent uses the best reply rule in period **t+1**, because **μ>0** and fixed. It means that all agents will play the dominant strategy **S** in period **t+1**.

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[4] It might stay the same size one more period if some agents use the imitation rule to select the strategy **S** in the current period. These imitators will switch their decision rule to the best reply and will play the strategy **S** again in the following period. Imitators, who play the strategy **P** in the current period, will use the same rule in the following period and play the strategy **P**.

Appendix B

We will need the following definitions.

Definition 1: $\mathbf{AD^{N,k,\epsilon,\epsilon}}$ is a **regular perturbed Markov process** if $\mathbf{AD^{N,k,\epsilon,\epsilon}}$ is irreducible for every $\boldsymbol{\epsilon}$, such that $\boldsymbol{\epsilon} \in (\mathbf{0}, \boldsymbol{\epsilon}^*]$, and for every state $\mathbf{h}$, $\mathbf{h'} \in (\{\mathbf{P}, \mathbf{S}\})^{\mathbf{N}} \times (\{\mathbf{BR}, \mathbf{IM}\})^{\mathbf{N}}$, $\mathbf{AD^{N,k,\epsilon,\epsilon}_{hh'}}$ approaches $\mathbf{AD^{N,k,0,0}_{hh'}}$ at an exponential rate, i.e. $\mathbf{\lim_{\epsilon \to 0} AD^{N,k,\epsilon,\epsilon}_{hh'} = AD^{N,k,0,0}_{hh'}}$ and if $\mathbf{AD^{N,k,\epsilon,\epsilon}_{hh'} > 0}$ for some $\boldsymbol{\epsilon} > \mathbf{0}$, then $\mathbf{0 < \lim_{\epsilon \to 0} AD^{N,k,\epsilon,\epsilon}_{hh'} / (\epsilon^r_{h \to h'})) < \infty}$ for some $\mathbf{r_{h \to h'} \geq 0}$. The real number $\mathbf{r_{h \to h'}}$ is the **resistance** of the transition $\mathbf{h \to h'}$.

LEMMA 1: The adjusted dynamics process $\mathbf{AD^{N,k,\epsilon,\epsilon}}$ is a regular perturbed Markov process.

PROOF: $\mathbf{AD^{N,k,\epsilon,\epsilon}}$ is a regular perturbed Markov process for the same reason as shown by Young (1998) when he considers adaptive play.

Definition 2: (Young, 1993) Let $\boldsymbol{\rho(\epsilon)}$ be the unique stationary distribution of an irreducible process $\mathbf{AD^{N,k,\epsilon,\epsilon}}$. A state $\mathbf{h}$ is **stochastically stable** if $\mathbf{\lim_{\epsilon \to 0} \rho_h(\epsilon) > 0}$.

Let the process $\mathbf{AD^{N,k,\epsilon,\epsilon}}$ have recurrent classes $\mathbf{E_1, ..., E_M}$. For each pair of distinct recurrent classes, a **pq**-path is a sequence of states $\zeta = (\mathbf{h_p, ..., h_q})$ beginning in $\mathbf{E_p}$ and ending in $\mathbf{E_q}$. The *resistance* of this path is the sum of the resistances on the edges composing it. Let $\mathbf{r_{pq}}$ be the least resistance over all **pq**-paths. Construct a complete directed graph with $\mathbf{M}$ vertices, one for each recurrent class. The weights on the directed edge $\mathbf{E_p \to E_q}$ is $\mathbf{r_{pq}}$. A tree rooted at $\mathbf{E_l}$ is a set of $\mathbf{M\text{-}1}$ directed edges such that, from every vertex different from $\mathbf{E_l}$, there is a

unique directed path in the tree to E_l. The resistance of such a rooted tree $\Psi(E_l)$ is the sum of resistances r_{pq} on its $M-1$ edges. The *stochastic potential* of a recurrent class E_l is the minimum resistance over all trees rooted at E_l. The following theorem is analogous to results of Freidlin and Wentzell (1984) on Wiener processes. Foster and Young (1990) introduced the theorem to economics for continuous state spaces. Young (1993, 1998) contains a discrete version of the theorem.

THEOREM 2: (Young 1998) Let $AD^{N,k,\varepsilon,\varepsilon}$ be a regular perturbed Markov process and let $\rho(\varepsilon)$ be the unique stationary distribution of $AD^{N,k,\varepsilon,\varepsilon}$, for $\varepsilon > 0$. Then, $\lim_{\varepsilon \to 0} \rho(\varepsilon) = \rho(0)$ exists and is a stationary distribution of $AD^{N,k,0,0}$. The stochastically stable states are precisely the states contained in the recurrent classes of $AD^{N,k,0,0}$, having minimum stochastic potential.

PROOF OF PROPOSITION 4: From Proposition 1, it follows that any absorbing state can contain N, $N-2$, $N-3$, $N-4$,…, 4, 3 or 0 imitators, who are playing the strategy P. Note that it is enough to make just one error to move from an absorbing state with m imitators, playing the strategy P, to an absorbing state with $m-1$ imitators, playing the strategy P, where $m \in \{4,5,…,N-2\}$. Similarly, one error is enough to move from the absorbing state, where N imitators play the strategy P, to the state, where $N-3$ imitators play the strategy P; and from the absorbing state, where 3 adjacent imitators play the strategy P and other agents play the strategy S, to the state, where all agents play the strategy S. Note that there must be at least three errors to leave the state, where all agents play the strategy S. It means that the states where all agents play the strategy S have minimal *stochastic potential*.

PROOF OF PROPOSITION 5: From Proposition 2, it follows that any recurrent class can contain N, $N-2$, $N-3$, $N-4$,…, 5, 4, or 0 imitators, who are playing the strategy P. Note that it

is enough to make just one error to move between two recurrent classes, which have a cluster of the strategy **P** of the length **m** and a cluster of the strategy **P** of the length **m+1**, where **m**∈ **{3,4,…,N-3}**. One error is enough to move between the absorbing state **{P;IM}** and recurrent classes, where **N-2** imitators play the strategy **P**. It means that all recurrent classes different from the absorbing states in which all agents play the strategy **S** have the same stochastic potential.

Four errors are required to move from the absorbing state in which all imitators play the strategy **S**, to states in which only **2** agents play the strategy **S** (for **N** even), or to sets of blinkers in which **1**, **5**, or **3** agents play the strategy **S**. Note that all absorbing states in the set **{(S;•)}** have the same stochastic potential, because one error is enough to move between absorbing states **{S;BR}** and **{(S,…,S);(IM,BR,…,BR)}**, **{(S,…,S);(IM,IM,BR,…,BR)}** and **{(S,…,S);(IM,BR,…,BR)}**, …, **{S;IM}**, and **{(S,…,S);(IM,…,IM,BR)}**.

What is the smallest number of errors, which it is necessary to make in order to move from a recurrent class with at least four imitators, who play the strategy **P**, to a recurrent class, where all agents play the strategy **S**? Proposition 2 tells us that the string of imitators playing the strategy **P** is at least of the length four and the string of the strategy **S** is at most of the length five in any recurrent class. There must be at least one error per string of the strategy **P** in order to move to an absorbing state, where all agents play the strategy **S**. After such an error every string of the strategy **P** must be at most of the length three in order to disappear in the following period. It is possible for a string of the maximal length of seven. That string must be between two strings of the strategy **S**, each of those has the maximal length of five. Hence, at least **N/12** errors are necessary to move from any recurrent class to an absorbing state in which all agents play the strategy **S**. It means that all absorbing sets except the states, where all agents play the strategy **S**, have minimal *stochastic potential,* if **N>48**.

PROOF OF THEOREM 3: From Theorem 1, it follows that any recurrent class can contain **N**, **N-2**, **N-3**, **N-4**,…, **k+3**, **k+2**, or **0** imitators, who are playing the strategy **P**. Note that it is enough to make just one error to move between two recurrent classes, which have a cluster of the strategy **P** of the length **m** and a cluster of the strategy **P** of the length **m+1**, where **m**∈{**k+2,k+3,…,N-3**}. One error is enough to move between an absorbing state **{P;IM}** and an absorbing state, where **N-2** imitators play the strategy **P**. It means that all recurrent classes different from the absorbing states in which all agents play the strategy **S** have the same stochastic potential.

(**k+2**) errors are required to move from the absorbing state in which all imitators play the strategy **S** to a recurrent class in which only **2** agents play the strategy **S** (for **N** even) or to sets of blinkers in which **1**, **2k+1**, or **3** agents play the strategy **S**. These (**k+2**) errors must create a cluster of the strategy **P** of the length **k+2**. Note that all absorbing states in the set **{(S;•)}** have the same stochastic potential, because one error is enough to move between absorbing states **{S;BR}** and **{(S,…,S);(IM,BR,…,BR)}**, **{(S,…,S);(IM,IM,BR,…,BR)}** and **{(S,…,S);(IM,BR,…,BR)}**, …, **{S;IM}**, and **{(S,…,S);(IM,…,IM,BR)}**.

What is the smallest number of errors, which it is necessary to make in order to move from a recurrent class with at least (**k+2**) imitators, who play the strategy **P**, to a recurrent class, where all agents play the strategy **S**? Theorem 1 tells us that the string of imitators playing the strategy **P** is at least of the length (**k+2**) and the string of the strategy **S** is at most of the length (**2k+1**) in any recurrent class. There must be at least one error per string of the strategy **P** in order to move to an absorbing state, where all agents play the strategy **S**. After such an error every string of the strategy **P** must be at most of the length (**k+1**) in order to disappear in the following period. It is possible for a string of the maximal length of (**2k+3**). That string must be between two strings of the strategy **S**, each of those has the maximal length of (**2k+1**). Hence, at least **N/[(2k+3)+(2k+1)]=N/4(k+1)** errors are necessary to move

from any recurrent class to an absorbing state in which all agents play the strategy **S**. It means that all recurrent classes except the absorbing states, where all agents play the strategy **S**, have minimal *stochastic potential*, if **(k+2)<N/4(k+1)**, or **4(k+1)(k+2)<N**. The statement of the theorem follows immediately.

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