Equilibrium Selection and Economic Growth

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1. Introduction

Empirical work on comparative patterns of economic growth and development have suggested the existence of multiple equilibria, i.e. cases where economies with similar sets of economic and social characteristics may end up on long-run growth paths with markedly different levels of per capita income or other important indicators of social welfare. The possibility of such cases was long known in the theoretical literature (e.g. Solow(1956)). Recent theoretical work has revisited this past literature and focussed on the dynamics which would generate such multiplicity of outcomes, together with the appraisal of such concepts as global, conditional and club convergence. The construction of models which have multiple equilibria raises the question of equilibrium selection; and it is on this problem of how equilibrium selection is undertaken that this paper focusses.

At first sight, there may appear little new to be said on the appropriate equilibrium selection methodology. In deterministic models the equilibrium selection process appears reasonably straightforward; the equilibrium growth points will have ”basins of attraction”, any initial set of conditions lying within the ”basin of attraction” of a particular equilibrium point will converge to that point. Different initial conditions lying in different ”basins of attractions” will converge to different equilibria. Equilibrium selection will thus depend on the set of initial conditions. The growth theory literature on selection thus focusses on the nature of the models that will generate multiplicity of outcome, particular attention being paid to aspects of technology, income distribution and savings, or other factors which generate multiple equilibria; little or no consideration is then given to how selection is undertaken; the ”basins of attraction” approach is almost universally taken as embodying the correct methodology.(e.g. Barro and Sala-i-Martin(1992), Galor(1996), Azariades(2001))

Recent work undertaken for the equilibrium selection process in the literature on multiple equilibria in games however suggests an approach that may have some relevance for the study of growth equilibria. (Foster and Young (1990), Kandori,Malaith and Rob(1993), Binmore,Samuelson and Vaughan(1995)). In the case where the development of the economies are not solely determined by initial conditions but may also be influenced by stochastic events, then convergence to the equilibrium determined by presence within a particular ”basin of attraction” may not occur. In general we will find economies wandering in and out of different basins dependent on the influence of the stochastic shocks affecting the economy.

The paper establishes primacy in the determination of equilibrium to what may be termed the ”growth potential function”; it is the maximum value of this function that determines the long run growth path of the economy. The role of ”initial conditions” and therefore the concept of ”basins of attractions”, are absent from determining equilibrium selection.

If the present methodology is accepted there are different implications regarding the role of economic policy in relation to switching between equilibria. In particular, the approach of the ”big push” literature, emphasising the role of capital transfers as leading to switching between equilibria in the long run may be somewhat misplaced.

In order to pursue this alternative methodology, we take as given that the models investigated will generate multiple equilibria; we can thus focus on the alternative solution concept proposed for which standard solutions are well known in the literature. The solution concepts adopted however are of much wider generality, and interested readers may be referred to the relevant game theory literature.(op. cit.)

2. Stochastic Growth Models

In order to differentiate the different equilibrium solution concepts as between solutions arising from the "basins of attraction", and "potential function" approaches, we take as an example the fundamental theory of growth focusing attention on the growth of the per capita capital stock.(amongst the first papers which had a technical formulation of this problem we may note Bourguignon(1974), and Merton(1975)).It should be noted that the majority of these early models concerned the stochastic growth of a single economy, with worked examples focusing on the Cobb-Douglas/ constant savings function model and single equilibria. However we believe the approach adopted here is sufficiently general to cover applications to more modern concerns relating to multiple equilibria, including those of endogenous growth.

The generic equation of stochastic growth will be taken to be of the form,

$$dk = G(k)dt + \sigma(k)dz \tag{1}$$

where k denotes the per capita capital stock; $G(k)$ is the growth function, which is further defined below, and $\sigma^2(k)$ denotes the variance of the change in per capita capital. The properties of these coefficients will depend upon the precise nature of the economic model to be specified. dz is assumed to be a Weiner process of unit variance.

Associated with the s.d.e. (1) there is an equation governing the evolution of the frequency distribution function of k; letting $p(k;t)$ denote this frequency distribution, then we have;

$$\frac{\partial p(k;t)}{\partial t} = -\frac{\partial}{\partial k}(G(k)p(k;t)) + \frac{1}{2}\frac{\partial^2}{\partial k^2}((\sigma^2(k)p(k;t)) \tag{2}$$

the Fokker-Planck or Kolmogorov forward equation (see Appendix 1). The problem then is to solve for $p(k;t)$, $t > 0$; subject to an initial condition, the distribution $p(k;0)$, and the appropriate boundary conditions. Boundary conditions in this case relate to conditions on the solution $p(k;t)$, e.g. if the economy reaches $k = 0$, and transitions to negative values of k are prohibited; or to the value of $p(k;t)$ for very large values of k. If discontinuities in $G(k)$ and $\sigma(k)$ are allowed then additional interface conditions have to be imposed, as the economy traverses from one regime to another; these will be discussed further below.

In specifying boundary conditions it is useful to introduce the notion of probability flow in terms of the distribution $p(k;t)$, defined as,

$$F(k;t) = G(k)p(k;t) - \frac{1}{2}\frac{\partial}{\partial k}(\sigma^2(k)p(k;t)) \tag{3}$$

If the economy cannot cross a particular boundary k^*, then the probability flow at that point must equal zero; thus the boundary conditions that we find it appropriate to impose are,

$$F(0;t) = 0 \tag{4}$$

i.e. the economy cannot have negative values of k; whilst probability is conserved over all possible non-negative values of k, for which we impose the upper boundary condition,

$$\lim_{k \to \infty} F(k;t) = 0 \tag{5}$$

The complete statement of the problem is therefore to find $p(k;t)$ for $t > 0$, which satisfies the FPE (2), the initial condition $p(k;0)$, together with the appropriate boundary conditions, (4),(5), and the normalization condition,

$$\int_{k=0}^{k=\infty} p(k;t) = 1 \tag{6}$$

In Fig.1 we have illustrated the case where three stationary points of $G(k)$ exist; k_1 and k_3 denote the two stable equilibria, whilst k_2 denotes the unstable equilibria.

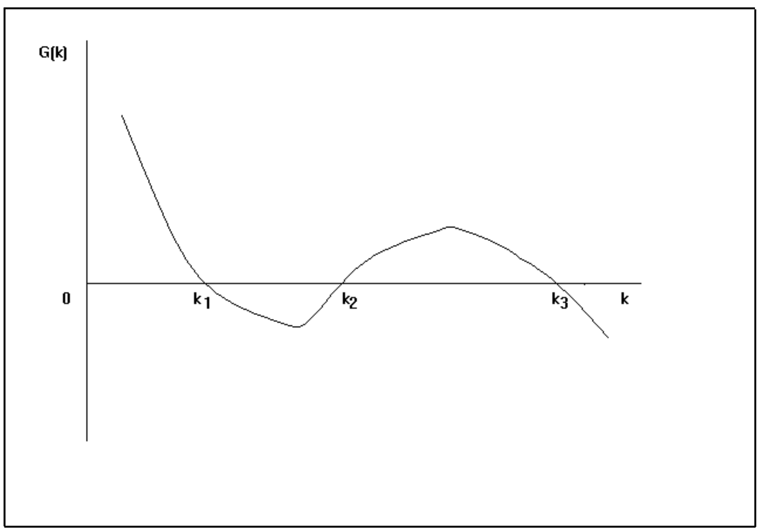

Fig.1

The basin of attraction for k_1 is therefore the set of points of k lying below k_2. whilst the basin of attraction for k_3 is the set of points lying above k_2:

In order to determine the long run growth path of the economy therefore one has to determine the basin of attraction within which the current capital per capita lies. Transference between the two basins of attraction is not possible, unless a sufficient amount of capital were gifted to or removed from the economy. In the deterministic case equilibrium selection, where multiple equilibria exist, is therefore straightforwardly determined by the basin of attraction within which the current per capita capital stock of the economy lies.

In the case where the growth equation is subject to a stochastic perturbation then questions arise as to the analysis of the equilibrium that is selected. A number of definitions of equilibrium selection may be applied in this context; one such concept is the so-called low noise limit selection process under which the stochastic perturbation of the growth equation is taken uniformly to zero. In such cases the low noise limit selects the equilibrium for the economy. Since the economy is subject to random shocks and never settles to a stationary equilibrium, then one can also ask questions as to what is the most likely equilibrium that will be observed.

3. The Ergodic Distribution

In the present paper our main interest will centre on the limiting distribution,

$$\lim_{t \to \infty} p(k;t) = p^*(k) \tag{7}$$

The distribution is deemed to be ergodic if it is independent of the initial distribution $p(k;0)$. An ergodic distribution may not always exist, however assumptions sufficient to guarantee ergodicity standard in the literature are:

(A1) The boundary conditions are reflecting,

(A2) $G(k)$ is bounded on the real line, and $\sigma^2(k) > 0$ on the real line.

Such assumptions are also sufficient to guarantee uniqueness of $p(k;t)$ on the real line, $t > 0$. Existence of $p^*(k)$ is usually shown by construction; the conditions for uniqueness are well known in the literature (see e.g. Friedman(1964), Risken(1984)). For the remainder of this section we shall assume that (A1) and (A2) hold.

The determination of the functional form which satisfies the condition of time invariance is relatively easy to determine from the flow condition (3). If $p^*(k)$ is independent of t, then in equilibrium $p^*(k)$ must satisfy,

$$F(k;t) = G(k)p^*(k) - \frac{1}{2}\frac{\partial}{\partial k}(\sigma^2(k)p^*(k)) = 0 \tag{8}$$

i.e. stationarity of the distribution must imply a zero probability flow not only at the boundaries of $0 \le k < \infty$ but also at every point in its interior.

Solving (8) for $p^*(k)$ we thus have,

$$p^*(k) = \frac{C}{\sigma^2(k)} e^{\phi(k)} \tag{9}$$

where,

$$\phi(k) = 2\int_{k_0}^{k} \frac{G(x)}{\sigma^2(x)} dx \tag{10}$$

$k_0 < k$ is an arbitrary constant, however upon integrating (10), and subst. into (9), this constant can be subsumed in the constant of integration C; which will be determined by the normalization condition (6) i.e.

$$C = \left[\int_{k=0}^{k=\infty} \frac{e^{\phi(k)}}{\sigma^2(k)}\right]^{-1} \tag{11}$$

The function $\phi(k)$ has an important role in linking the stochastic dynamics to the equilibrium distribution eventually generated; in the statistical and physical sciences literature $\varphi(k)$ is called the potential function.

In the case where the variance $\sigma^2(k)$ is assumed to be constant over the interval of k, the relationship between the distribution $p^*(k)$ and the properties of the deterministic dynamical system are particularly straightforward. Letting $\sigma^2(k) = V > 0$, and differentiating (9) successively w.r.t. k we have,

$$\frac{\partial p^*(k)}{\partial k} = p^*(k)\varphi'(k) = \frac{C}{V}e^{\varphi(k)}\varphi'(k) \tag{12}$$

$$\frac{\partial^2 p^*(k)}{\partial k^2} = p^*(k)[\varphi''(k) + (\varphi'(k))^2 \tag{13}$$

Now from (10) we have,

$$\varphi'(k) = 2G(k)/V \tag{14}$$

$$\varphi''(k) = 2G'(k)/V \tag{15}$$

When $G(k) = 0$ we have a rest point of the original deterministic system, and these rest points correspond to stationary points of $\varphi(k)$ and $p^*(k)$. Further, from (13) and (15) we note that at these stationary points of $p^*(k)$,

$$\frac{\partial^2 p^*(k)}{\partial k^2} = p^*(k)\varphi''(k) = (2p^*(k)G'(k))/V \tag{16}$$

and for $p^*(k)$ strictly positive at the stationary points, unstable rest points of $G(k)$; $G'(k) > 0$, correspond to the minima of $p^*(k)$; whilst stable rest points, $G'(k) < 0$ correspond to the maxima of $p^*(k)$.

As an example consider the dynamics of the standard growth model, where $G(k)$ is assumed to have three roots at which the growth rate is zero. Thus with a constant $\sigma^2(p) = V$, such a relationship is represented by Fig 2.

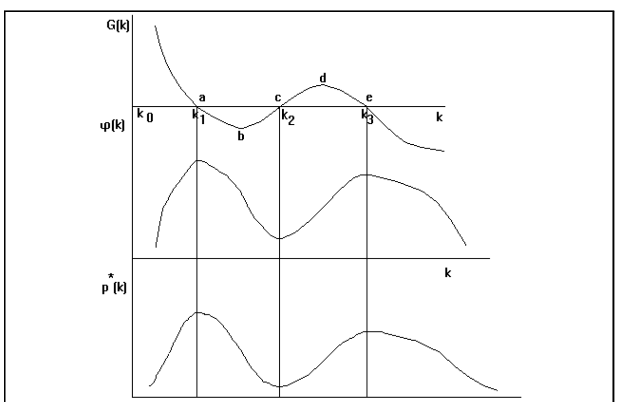

Fig. 2

We may note that: (a) The positions of the stationary points of $\varphi(k)$ and $p^*(k)$ are unaffected by the value of V; this follows from eqs. (14) and (12). (b) The ranking by value of $\varphi(k)$ and hence $p^*(k)$ at these stationary points are unaffected by the value of V. This follows from taking the quotient of (9) defined at any two stationary points with constant variance. (c) The absolute values of $\varphi(k)$ and hence $p^*(k)$ will vary with V. Apparent from (8) and (9).

In the case of a non-constant variance, the equivalence between rest points of the deterministic growth system, and maxima and minima of the equilibrium distribution no longer holds. We assume that $\sigma^2(k) > 0$.over the unit interval for k.

We then have, differentiating equation (9),

$$\frac{\partial p^*(k)}{\partial k} = p^*(k)[\varphi'(k) - \frac{1}{\sigma^2(k)}\frac{\partial \sigma^2(k)}{\partial k}] \tag{17}$$

and,

$$\frac{\partial^2 p^*(k)}{\partial k^2} = p^*(k)[\varphi'(k) - \frac{1}{\sigma^2(k)}\frac{\partial \sigma^2(k)}{\partial k}]^2 + p^*(k)[\varphi''(k) - \frac{\partial}{\partial k}(\frac{1}{\sigma^2(k)}\frac{\partial \sigma^2(k)}{\partial k})] \tag{18}$$

whereas from (18),

$$\varphi'(p) = 2G(p)/\sigma^2(p) \tag{19}$$

$$\varphi''(p) = [2/\sigma^2(p)][\frac{\partial G(p)}{\partial p} - \frac{G(p)}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] \tag{20}$$

Stationary points of $f^*(p)$ thus occur either when $f^*(p) = 0$ or when,

$$\varphi'(p) - \frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p} = 0 \tag{21}$$

i.e. subst. (19) into (21), when,

$$G(p) = (1/2)\frac{\partial \sigma^2(p)}{\partial p} \tag{22}$$

The second order condition for a maximum, $f^*(p) > 0$; requires, from (18) and (20),that,

$$[2/\sigma^2(p)][\frac{\partial G(p)}{\partial p} - \frac{G(p)}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] - \frac{\partial}{\partial p}[\frac{1}{\sigma^2(p)}\frac{\partial \sigma^2(p)}{\partial p}] < 0 \tag{23}$$

Thus the maxima of $p^*(k)$ cannot be determined solely in reference to the deterministic dynamics as reflected in $G(k)$ but also on $\sigma^2(k)$. Indeed, by an appropriate choice of $\sigma^2(k)$ the maxima of $p^*(k)$ can be shifted.

We now however have to consider the low noise limit. Again we assume that $\sigma^2(k)$ can be written in the form,

$$\sigma^2(k) = \beta V^2(k) \tag{24}$$

where letting $\beta \to 0$ we take the variance uniformly to zero over the unit interval. From equation (17) we see the reappearance of the coincidence of the roots of $G(k)$, the deterministic equation of growth with the maxima and minima of the potential function , provided $\beta\frac{\partial V^2(k)}{\partial k} \to 0$ as $\beta \to 0$.

4. The Equilibrium Selection Process

Equilibrium selection would imply that the distribution function $p(k)$ converges to a single point on the interval $[0;\infty)$. Provided the assumptions (A1) and (A2) hold, convergence to such a single point is not possible for the system (2)-(6). However, if we take the limit of the distribution $p^*(k)$ as the stochastic term $\sigma^2(k) \to 0$, then the distribution does collapse, either to a single or multiple points.

We return to the case where $\sigma^2(k)$ is allowed to vary over k, but still retain the assumption that $\sigma^2(k) > 0$: We propose to determine what happens to $p^*(k)$ as $\sigma^2(k)$ uniformly tends to zero. We therefore assume that $\sigma^2(k)$ can be represented in the form,

$$\sigma^2(k) = \beta V^2(k) \tag{25}$$

where $\beta > 0$ is some constant.

We are then interested in,

$$\lim_{\beta \to 0} p^*(k) = \lim_{\beta \to 0} \frac{C}{\beta V^2(k)} e^{\psi(p)/\beta} \tag{26}$$

where,

$$\beta \varphi(k) = \psi(k) = 2\int_0^k \frac{G(x)}{V^2(x)} dx \tag{27}$$

The basic theorem relating to (26) appears to have originated with Pontryagin, Andronov and Witt(1934).

Theorem 1. *Distributional Dominance*.

If $p^*(k)$ exists and if $\psi(k)$ attains a unique maximum at k^* in the interval $[0;\infty)$ then,

$$\lim_{\beta \to 0} p^*(k) = p^*(k^*) \tag{28}$$

where $p^*(k^*)$ is the distribution centred on k^*, such that for any $\varepsilon > 0$; $p^*(k^*) = 0$ for $k > k^* + \varepsilon$, and $k < k^* - \varepsilon$.

Proof: Appendix 2.

Corollary 1.

If $p^*(k)$ exists and $\psi(k)$ attains n maxima in the interval $[0;\infty)$ at points k_i^* such that $\psi(k_i^*) = \psi^*$ for $i = 1;\ldots;n$; then,

$$\lim_{\beta \to 0} p^*(k) = p^{**}(k) \tag{29}$$

where $p^{**}(k)$ is a function such that $p^{**}(k_i^*) = p^{**}(k_j^*)$ all $i; j = 1;\ldots;n$; and for any $\varepsilon > 0$; $p^{**}(k) = 0$ for $k > k_i^* + \varepsilon$, and $k < k_i^* - \varepsilon$ for all $i = 1;\ldots;n$.

5. Growth Models with Multiple Equilibria

We now turn to the application of the above methodology to models of growth with multiple equilibria. Let us consider the case where the deterministic growth dynamics are such that two possible stable equilibria exist, at k_1 and k_3 . An economy will therefore converge to that equilibrium depending upon the basin of attraction in which its initial per capita capital stock is positioned. In this instance the basin of attraction for k_1, as shown in Fig.2 is $[0, k_2]$; whilst the basin of attraction for k is the interval $[k_2, \infty]$. The growth process is quite clearly history dependent.

However, when the stochastic formulation is used then movement between the two basins is possible. The question that now occupies us is the nature of the probability distribution for k, that arises; and the form that this distribution takes as the stochastic component tends to disappear. One of the interesting questions that may be asked is the role of the basin of attraction in determining the most probable k that is achieved by the economy in steady growth. One intuitive may run along the lines that it is that equilibrium k which has the largest basin of attraction, and therefore the most likely observation is that an economy is to be found in proximity to this equilibrium.

In fact this observation may be confounded. As should be apparent from the theory outlined in Sect.4 it is not the size of the basin of attraction that is relevant in determining the most probable configuration, but the relative magnitudes of the potential function at the stable equilibria points. Applying theorem 1 to the points k_1 and k_3 , we therefore make the comparison,

$$p^*(k_1) \gtrless p^*(k_3) \tag{30}$$

depends upon whether,

$$\varphi(k_1) \gtrless \varphi(k_3) \tag{31}$$

Remembering that,

$$\varphi(k) = 2\int_{k_0}^{k} \frac{G(x)}{\sigma^2(x)} dx \tag{32}$$

then the criterion may be written as,

$$\int_{k_0}^{k_1} G(x)dx - \int_{k_0}^{k_3} G(x)dx = \int_{k_3}^{k_1} G(x)dx \gtrless 0 \tag{33}$$

Thus the evaluation may either be stated in terms of the potential function or in terms of the integral of the growth function,

Thus in terms of fig.2 the most probable location of the economy is determined by the relative magnitude of the areas abc cde.

Although what appears to be a relatively minor change without regard to the specification of the growth model, quite substantive differences exist with regard to the qualitative implications of certain policy implications.

Assume that Fig.2 applies, that the potential function is highest at k_1. Consider the first a change in endowment which moves the economy from basin of k_1 to the basin of k_2. Such may arise as the result of a sufficiently large capital donation. Now, in the non-stochastic version of the model, such would be sufficient to place the economy on a growth path that leads eventually to the high level steady state k_3 . Thus. the argument that a relatively large quantity of foreign aid might allow an economy to escape from a "poverty trap" at k_1. However, we can now see that the most probable value for this economy does not change, it remains at k_1 . Indeed, the potential function is not altered by the change in the current endowment of the economy; the implication is clear, if policy measures are to permanently affect the most probable configuration of the economy, these should be directed to changing the potential function not the current endowment.

Changing the potential function has two major effects, firstly it changes the position of the growth equilibria, secondly it changes the importance of these two equilibria, in relation to the magnitude of the potential function , and hence the most probable configuration for the economy. The two effects can most clearly be seen by example. Assume that the $G(k)$ function is represented by the simple neoclassical relationship,

$$G(k) = sf(k)/k - (n + \delta) \tag{34}$$

and that the economy's initial value of s, and the other parameters was sufficient to ensure that the potential function was at it's highest at k_3, the high value path; and that the economy was in long run equilibrium at k_3: Now consider a fall in the savings rate, such that the potential function of the equilibrium switch values such that k_1 becomes the higher. Now in the case of the non-stochastic growth model, the economy may expect to find that the value of the upper steady state value of k, k_3 has declined, but that movement from k_3 to a nearby k_3^* is the only consequence. Now we have the additional implication that the most probable steady state value that the economy faces is not at k_3^* but at k_1:

Similar arguments apply to the effects of transitory changes in the savings rate. Let us assume that the economy is initially in the basin of k_1. An increase in the savings rate occurs so that k surpasses k_2 and enters the basin of attraction of k_3: Under the traditional story, even if the savings rate relapses to its old level , the economy will still now stay at the high equilibrium, k_3: However, with a relapse to the old savings level, if the potential value now switches the highest value from k_3 to k_1; then the transitory change in the savings propensity will have no effect on the long-run most probable state if the economy.

6. Discontinuities in the Growth Function

The above analysis is not restricted to the case where $G(k)$ is a differentiable or continuous function. The model we take to illustrate discontinuities in the growth factor, and consequent multiple stable equilibria, is where a discontinuity in technology occurs at some value of per capita capital. In Fig.3 such a discontinuity occurs at point k_I.

Which is the most probable equilibria that will be observed in long-run equilibrium? Intuitively it will depend on a comparison of areas under the growth function between the lower and upper equilibria ($k_1$and k_3) and the point of disjunction, k_I. Thus if an equilibrium is relatively closer to the growth disjunction it is that equilibria which will be less likely to be observed in practice. It is interesting to note the quite radical difference between this prediction and that associated with the basin of attraction; the basin of attraction of the upper equilibrium is infinite compared to the finite basin of the lower equilibrium however a prediction that the upper equilibria is the most likely to be observed is not necessarily correct; indeed it is the "half-basin" lying between the two equilibria which is important; and even knowledge of this basin may give inaccurate predictions without knowledge of the potential function.(An example of a deterministic model where a disjunction in the growth function is provided by the analysis of Azariadis and Drazen (1990).

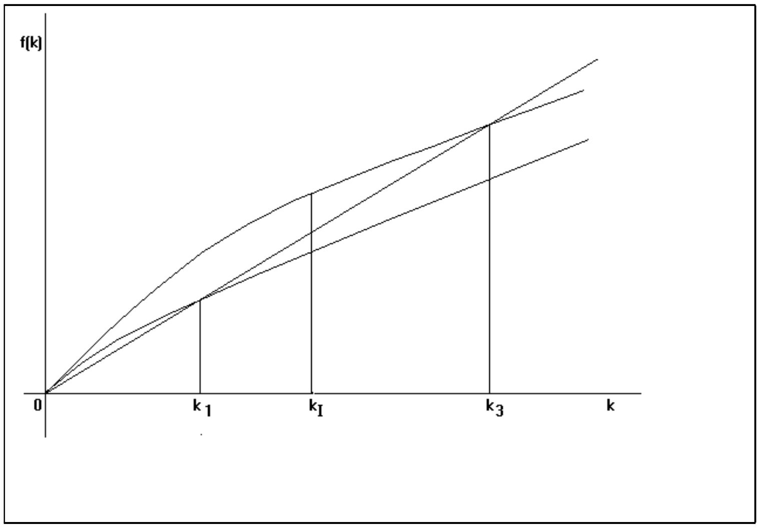

Fig.3

Within each region of k a different production function applies, transition across the interface k_I results in a discrete change in the growth rate and consequent movement towards a new equilibria.

In order to proceed to the solution, let us set up the model The growth process is governed by the following sde's

$$dk = G_1(k)dt + \sigma_1(k)dz \qquad \text{for } 0 \leq k \leq k_I \tag{35}$$

$$dk = G_2(k)dt + \sigma_2(k)dz \qquad \text{for } k_I < k < \infty \tag{36}$$

with corresponding FPE's,

$$\frac{\partial p_1(k;t)}{\partial t} = -\frac{\partial}{\partial k}(F_1(k;t)) \qquad \text{for } 0 \leq k \leq k_I \tag{37}$$

$$\frac{\partial p_2(k;t)}{\partial t} = -\frac{\partial}{\partial k}(F_2(k;t)) \qquad \text{for } k_I < k < \infty \tag{38}$$

where,

$$(F_1(k;t) = -(G_1(k)p_1(k;t)) + \frac{1}{2}\frac{\partial}{\partial k}(\sigma_1^2(k)p(k;t)) \qquad \text{for } 0 \leq k \leq k_I \tag{39}$$

$$(F_2(k;t) = -(G_2(k)p_1(k;t)) + \frac{1}{2}\frac{\partial}{\partial k}(\sigma_2^2(k)p(k;t)) \qquad \text{for } k_I < k < \infty \tag{40}$$

Boundary conditions are such that,

$$F_1(0;t) = \lim_{k \to \infty} F_2(k;t) \tag{41}$$

In addition we require a condition on the interface of the two regimes. We shall assume that the flow from Regime I equals the flow into Regime II; of course this flow may be positive, negative or zero.Thus we require,

$$F_1(k_I;t) = F_2(k_I;t) \tag{42}$$

Further we shall assume that the probability distribution function is continuous on the real line, and so,

$$p_1(k_I;t) = p_2(k_I;t) \tag{43}$$

The solutions for the distributions in each regime are therefore, defined by,

$$p_1^*(k) = \frac{C_1}{\sigma_1^2(k)} e^{\varphi_1(k)} \tag{44}$$

$$p_2^*(k) = \frac{C_2}{\sigma_2^2(k)} e^{\varphi_2(k)} \tag{45}$$

where,

$$\varphi_i(k) = 2\int_{k_{i0}}^{k} \frac{G_i(x)}{\sigma_i^2(x)} dx \qquad i = 1;2 \tag{46}$$

$k_{i0} < k$ are arbitrary constants, however upon integrating as can be seen there now appear to be four arbitrary constants; however the constants k_{i0} can be subsumed within the constants C_1 and C_2. These constants may then be determined by the normalization condition requiring that,

$$\int_0^{k_I} p_1^*(k) + \int_{k_I}^{\infty} p_1^*(k) = 1 \tag{47}$$

together with the interface condition.

In order to determine the relative heights of the probability distribution at the equilibria k_1 and k_3, we consider the ratio,

$$\frac{p_1^*(k_1)}{p_1^*(k_3)} \tag{48}$$

i.e.,

$$\left\{\frac{C_1}{\sigma_1^2(k_1)} e^{\varphi_1(k_1)}\right\} / \left\{\frac{C_2}{\sigma_2^2(k_3)} e^{\varphi_2(k_3)}\right\} = \frac{C_1}{C_2}\frac{\sigma_2^2(k_3)}{\sigma_1^2(k_1)} e^{\varphi_1(k_1) - \varphi_2(k_3)} \tag{49}$$

Now from the interface condition,

$$\frac{C_1}{\sigma_1^2(k_I)} e^{\varphi_1(k_I)} = \frac{C_2}{\sigma_2^2(k_I)} e^{\varphi_2(k_I)} \tag{50}$$

i.e.

$$\frac{C_1}{C_2} = \frac{\sigma_1^2(k_I)}{\sigma_2^2(k_I)} e^{\varphi_2(k_I) - \varphi_1(k_I)} \tag{51}$$

and so the evaluation criteria is,

$$\frac{\sigma_1^2(k_I)}{\sigma_2^2(k_I)}\frac{\sigma_2^2(k_3)}{\sigma_1^2(k_1)} e^{\varphi_1(k_1) - \varphi_2(k_3) + \varphi_2(k_I) - \varphi_1(k_I)} \gtrless 1 \tag{52}$$

If the ratio exceeds unity, then the lower equilibrium at k_1 has the highest potential, and vice versa. The criteria has a simple geometric interpretation when variances are assumed equal and constant over both regimes. In this case the criteria can be written,

$$e^{\varphi_1(k_1) - \varphi_2(k_3) + \varphi_2(k_l) - \varphi_1(k_l)} \gtrless 1 \qquad (53)$$

.

7. Conclusion

The aim of this paper was to suggest a alternative methodological approach to the equilibrium selection problem as found in the economic growth and development literature. The application of the ”basin of attraction” to equilibrium selection may be found to be inappropriate where a substantive stochastic component may be found in equations which model the economy. The appropriate methodology we suggest for such cases is the use of the potential function. The use of the potential function allows the construction of the appropriate probability function which is appropriate for the study if the long term growth path of the economy. The policy implications associated with the use of the potential function are quite profound, insofar as attempting to shift economies onto new growth paths simply by changing the ”initial conditions” from one ”basin to attraction” to another will not result in a shift to a new long run path. In order for this to be accomplished policy measures should be related to those factors which shift the maxima of the potential function. In such a respect the ”potential function” truly reflects the potential for growth.

What role then do the initial conditions and ”basins of attraction” perform. By changing initial conditions, for example, a grant of capital economies can be shifted such that the transition times to one or other of the different equilibria can be affected. This requires a new methodology for the calculation of transition times between equilibria compared to traditional approaches (e.g.). Thus we suggest the following tests to determine which type of policy will lead to the desired outcome for growth which generates maximal wealth. Consider the case of two stable equilibrium paths; equilibrium 2 has a higher per capita welfare level than equilibrium 1; the economy currently lies within the ”basin of attraction” of equilibrium 1. In order to seek a long run equilibrium path mainly in the region of equilibrium 2 then it is first necessary to check the potential value at both these points. If the potential at equilibrium 2 is the greater, then a ”big push” policy structure would succeed in getting the economy onto a path around the highest welfare growth equilibrium. If equilibrium 2 has the lowest potential, then policies should be adopted which change, the relative potential as between the two equilibria in favour of equilibrium 2, then ”big push” policies may be more likely to ensure a satisfactory long run growth path.

Appendix 1. Derivation of the Equation Governing the Evolution of the Probability Distribution Function $f(p,t)$.

Let $p(k,t)$ denote the probability distribution function resulting from the stochastic dynamics implied by (3). In order to derive the equation governing the evolution of $p(k,t)$ we proceed as follows:

Let,

$$H(k,t) = \int_0^k p(x,t)dx \tag{1.1}$$

i.e. the probability mass at or below k at time t; i.e. the normal definition of the cumulative distribution function of k. We assume that within a given interval of time δt, changes in $p(k,t)$ are determined by the transition kernel, $\Gamma(k,k^*,\delta t,t)$; defining the proportion of probability mass at k^* at time t, which at the end of period δt ends up at or below k.

The equation governing the evolution of $H(k,t)$ is then given by,

$$H(k,t+\delta t) = \int_{-\infty}^{+\infty} p(k^*,t)\Gamma(k,k^*,\delta t,t)dk^* = \int_0^{\infty} h(p^*,t)\Gamma(p,p^*,\delta t,t)dp^* \tag{1.2}$$

Note that at this stage of the process k is not restricted to the interval $[0,\infty)$; restrictions on k are specified in terms of the boundary conditions defining the solution. Note further, that $k \gtreqless k^*$, i.e. that the transition kernel may induce zero, positive or negative jumps.

Equation (54) is the well known Chapman-Kolmogorov equation; the Fokker-Planck or diffusion equation may be derived by taking a Taylor series expansion of (54) around (k,t).

Expanding $H(k,t)$ in a Taylor series about the point (k,t), then differentiating with respect to k, we arrive at,

$$\frac{\partial p(k,t)}{\partial t}\delta t = -p(k,t) + \int p(k^*,t)\gamma(k,k^*,\delta t,t)dk^* + o(\delta t) \tag{1.3}$$

where $\gamma(k,k^*,\delta t,t) = \partial\Gamma(k,k^*,\delta t,t)/\partial k$; and $o(\delta t)$ represents a series of terms in δt such that $\lim_{\delta t \to 0} o(\delta t)/\delta t = 0$.

Equivalently, (54) may be written in terms of the size of the jump in k; letting $q = k - k^*$, we have,

$$\frac{\partial p(k,t)}{\partial t}\delta t = -p(k,t) + \int p(k-q,t)\gamma^*(k,k-q,\delta t,t)dq + o(\delta t) \tag{1.4}$$

Expanding $p(k-q,t)$ and $\gamma^*(k,k-q,\delta t,t)$ in Taylor series about the points (k,t) and (q,k) respectively, and letting,

$$\mu_j = \int_{-\infty}^{+\infty} q^j\gamma^*(q,k,\delta t,t)dq, \quad j = 1,2,\ldots \tag{1.5}$$

i.e. the jth moment of the jump function, we may show,

$$\int_{-\infty}^{+\infty} f(k-q,t)\gamma^*(q,k-q,\delta t,t)dq = p(k,t) + \sum_{j=1}^{\infty}\left[\frac{(-1)^j}{j!}\frac{\partial^j}{\partial k^j}(\mu_j p(k,t))\right] \tag{1.6}$$

Hence substituting (54) in (54) we have,

$$\frac{\partial p(k;t)}{\partial t}\delta t = \sum_{j=1}^{\infty}\left[\frac{(-1)^j}{j!}\frac{\partial^j}{\partial k^j}(\mu^j p(k;t))\right] + o(\delta t) \tag{1.7}$$

The particular case of (54) we consider is where we assume $\gamma^*(q;k;\delta t;t)$ to be a normal distribution in the size of the jump, q, with mean and second moment defined respectively as,

$$\mu^1 = u(k;t)\delta t\ ;\ \mu^2 = \sigma^2(k;t)\delta t \tag{1.8,1.9}$$

Since $\gamma^*(q;k;\delta t;t)$ is assumed normal, all odd moments higher than the first are zero; whilst for the even moments we have,

$$\mu^{2j} = \frac{(2j)!}{j!2^j}(\mu^2)^j; \tag{1.10}$$

$j = 1; 2; \ldots\ldots$

Hence all moments of $\gamma^*(q;k;\delta t;t)$ higher than the second are either identically zero, or of order $o(\delta t)$. Dividing through both sides of (1.7) by δt, and letting $\delta t \to 0$; we arrive at the Fokker-Planck equation (5) with appropriate definitions for the mean and variance.

APPENDIX 2

Theorem 1. *Distributional Dominance* .

If $p^*(k)$ as defined by (17) exists and if $\phi(k)$, as defined by (18) attains a unique maximum at k^* in the interval $[0, \infty)$ then,

$$\lim_{\sigma^2(k) \to 0} p^*(k) = \delta(k - k^*) \tag{2.1}$$

where δ is the Dirac delta function, i.e. $\delta(k - k^*) = 1$ if $k = k^*$, and $\delta(k - k^*) = 0$ if $k \neq k^*$.

Proof

We have in equilibrium,

$$G(k)p(k,t) = \frac{1}{2}\frac{\partial}{\partial k}(\sigma^2(k)p(k,t)) \tag{2.2}$$

and thus the equilibrium distribution,

$$p^*(k) = \frac{C}{\sigma^2(k)}e^{\phi(k)} \tag{2.3}$$

where C is the normalization constant, and,

$$\phi(k) = 2\int_0^k \frac{G(x)}{\sigma^2(x)}dx \tag{2.4}$$

What happens to $p^*(k)$ as $\sigma^2(k)$ decreases uniformly ? Let $\sigma^2(k) = \beta V^2(k)$ where β is a parameter which we shall allow to tend to zero. Then,

$$p^*(k) = \frac{C}{\beta V^2(k)}e^{\psi(k)/\beta} \tag{2.5}$$

where,

$$\psi(k) = 2\int_0^k \frac{G(x)}{V^2(x)}dx \tag{2.6}$$

We also have the normalization condition,

$$\int_{-\infty}^{+\infty} p^*(x)dx = 1 \tag{2.7}$$

Thus subst. (54) in (54) we have,

$$\frac{\beta}{C} = \int_{-\infty}^{+\infty} \frac{1}{V^2(x)}e^{\psi(x)/\beta}dx \tag{2.8}$$

and so,

$$p^*(k) = e^{\psi(p)/\beta}/V^2(k)\int_{-\infty}^{+\infty} \frac{1}{V^2(x)}e^{\psi(x)/\beta}dx \tag{2.9}$$

Let $\psi(k)$ attain a unique maximum at k^*, and let $\psi(k^*) = J$; then we can define $\psi^*(k) = \psi(k) - J$, and note that,

$$p^*(k) = e^{\psi(k)/\beta}/V^2(k)\int_{-\infty}^{+\infty} \frac{1}{V^2(x)}e^{\psi(x)/\beta}dx$$

$$= e^{(\psi(k)-J)/\beta}/V^2(k)\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{(\psi(x)-J)/\beta}dx$$

$$= e^{\psi^*(k)/\beta}/V^2(k)\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{(\psi^*(x))/\beta}dx \tag{2.10}$$

and we thus ensure that at $k^*; \psi^*(k^*) = 0$: In order to continue we require *Lemma 1* ;

Lemma 1

$$\lim_{\beta\to 0}\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{(\psi^*(x))/\beta}dx = B(\beta) \tag{2.11}$$

where $\beta^{1/n}A(k''/M^2 \le B(\beta) \le \beta^{1/n}A(k')/M^2; M$ = Min $V^2(p)$ and $A(k''); A(k')$ are constants independent of β.

Proof

$$\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx = \int_{p^*-h}^{p^*+h}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx + \int_{-\infty}^{p^*-h}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx$$

$$+\int_{p^*+h}^{+\infty}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx \tag{2.12}$$

where h is some small positive constant. The maximum of $\psi^*(k)$ occurs at k^* when $\psi^*(k^*) = 0$; thus the first integral on the R.H.S. of (54) contains a term which is independent of β. Outside the interval $[k^*+h; k^*-h]$ we have $\psi^*(k) < 0$; and thus each term in the second and third integrals on the R.H.S. of (54) tend to zero as β tends to zero; thus,

$$\lim_{\beta\to 0}\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx = \int_{p^*-h}^{p^*+h}\frac{1}{V^2(x)}e^{\psi^*(x)/\beta}dx \tag{2.13}$$

Now consider the function $\psi^*(k)$ in the interval $[k^*-h; k^*+h]$; the function is such that we can choose positive integers $K^1; K^2$, and an even integer n, such that,

$$-K^1(k-k^*)^n \ge \psi(k) \ge K^2(k-k^*)^n \tag{2.14}$$

where $K^2 > K^1 > 0$: Thus,

$$\frac{1}{V^2(k)}e^{-K^1(k-k^*)^n/\beta} \ge \frac{1}{V^2(k)}e^{\psi^*(k)/\beta} \ge \frac{1}{V^2(k)}e^{-K^2(k-k^*)^n/\beta} \tag{2.15}$$

Now,

$$\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{-K^1(k-k^*)^n/\beta}dx = \int_{k^*-h}^{k^*+h}\frac{1}{V^2(x)}e^{-K^1(k-k^*)^n/\beta}dx$$

$$+\int_{-\infty}^{k^*-h}\frac{1}{V^2(x)}e^{-K^1(k-k^*)^n/\beta}dx + \int_{k^*+h}^{+\infty}\frac{1}{V^2(x)}e^{-K^1(k-k^*)^n/\beta}dx \tag{2.16}$$

The last two integrals on the R.H.S. of (54) tend to zero as β tends to zero, and therefore,

$$\lim_{\beta\to 0}\int_{-\infty}^{+\infty}\frac{1}{V^2(x)}e^{-K^1(k-k^*)^n/\beta}dx = \int_{k^*-h}^{k^*+h}\frac{1}{V^2(x)}e^{-K^1(k-k^*)^n/\beta}dx \tag{2.17}$$

Let M^2 = Min $V^2(k) > 0$; then,

$$\int_{-\infty}^{+\infty} \frac{1}{V^2(x)} e^{-K^1(k-k^*)^n/\beta} dx \leq \frac{1}{M^2} \int_{-\infty}^{+\infty} e^{-K^1(k-k^*)^n/\beta} dx \tag{2.18}$$

Letting $z = (x - k^*)/\beta^{1/n}$ then,

$$\int_{-\infty}^{+\infty} e^{-K^1(x-k^*)^n/\beta} dx = \beta^{1/n} \int_{-\infty}^{+\infty} e^{-K^1 z^n} dz = \beta^{1/n} A(K^1) \tag{2.19}$$

and so,

$$\int_{k^*-h}^{k^*+h} \frac{1}{V^2(x)} e^{-K^1(k-k^*)^n/\beta} dx = \int_{-\infty}^{+\infty} \frac{1}{V^2(x)} e^{-K^1(k-k^*)^n/\beta} dx \leq \beta^{1/n} A(K^1)/M^2 \tag{2.20}$$

We may similarly show that,

$$\int_{k^*-h}^{k^*+h} \frac{1}{V^2(x)} e^{-K^1(k-k^*)^n/\beta} dx \geq \beta^{1/n} A(K^2)/M^2 \tag{2.21}$$

and so,

$$\lim_{\beta \to 0} \int_{-\infty}^{+\infty} \frac{1}{V^2(x)} e^{\psi^*(x)/\beta} dx = B(\beta) \tag{2.22}$$

where $\beta^{1/n} A(K^2)/M^2 \leq B(\beta) \leq \beta^{1/n} A(K^1)/M^2$; $M = \text{Min}\, V^2(p)$ and $A(K^2)$; $A(K^1)$ are constants independent of β. Thus,

$$p^*(k) = e^{\psi^*(k)/\beta} / V^2(k) B(\beta) \tag{2.23}$$

will tend to zero as $\beta \to 0$ for every value of k except $k = k^*$, when it will become infinitely large. From the normalization condition $\int p^*(k) = 1$; therefore we characterise,

$$\lim_{\sigma^2(k) \to 0} p^*(k) = \delta(k - k^*) \tag{2.24}$$

REFERENCES

Azariadis,C. (2001) ”The Theory of Poverty Traps: What have we learned?”(mimeo,UCLA)

Azariadis,C. and A. Drazen, (1990), ”Threshold Externalities and Economic Development”, *Quarterly Journal of Economics*, 105, 501-26

Barro,R.J. and Sala-i-Martin, X (1992), ”Convergence”, *Journal of Political Economy*, vol.46, 385-406.

Binmore, K.,L.Samuelson and R.Vaughan (1995), ”Musical Chairs: Modelling Noisy Evolution”, *Games and Economic Behavior*, 11, 1-35.

Bourguinon,F.(1974), ”A Particular Class of Continuous Time Stochastic Models”, *Journal of Economic Theory* (9), 141-158.

Foster,D. and P.Young (1990), ”Stochastic Evolutionary Game Dynamics”, *Journal of Theoretical Biology*, 38, 219-232.

Galor,O. (1996), ”Convergence: Inferences from Theoretical Models”, *Economic Journal*, 106, 1056-69.

Friedman,A. (1964), *Partial Differential Equations of Parabolic Type*, Prentice Hall, Englewood Cliffs.

Kandori,M., G.Malaith and R.Rob(1993), ”Learning, Mutation and Long-Run Equilibria in Games”, *Econometrica*, 61, 29-56.

Merton, R.C. (1975), ”An Asymptotic Theory of Growth Under Uncertainty”,*Review of Economic Studies*, 42(3),375-393.

Pontryagin L., A.Andronov and A.Vitt (1933), ”On the Statistical Treatment of Dynamical Systems, translation in Noise in Nonlinear Dynamical Systems: Vol.1, Theory of Continuous Fokker-Planck Systems (eds. F.Moss and P.V.E.McClintock) (1989), Cambridge University Press,Cambridge.

Risken, H. (1984), *The Fokker-Planck Equation*, Springer Verlag, Berlin.

Solow, R. (1956), ”A Contribution to the Theory of Economic Growth”, *Quarterly Journal of Economics*, 70, 65-94.