DYNAMICS FOR INFINITE DIMENSIONAL GAMES[1]

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Introduction

The explosion of interest amongst game-theorists in recent years in the ‘evolutionary’ (learning) dynamics of repeated games, has generally been concerned with 2-player (usually symmetric) games in which each player has available a finite number of pure strategies. The learning dynamics of players chosen from large, usually infinite, populations of such players, are then taken to describe the evolution of the probability with which a randomly chosen player will play a given pure strategy. In a suitable continuous-time limit, in which the game is repeated continuously, these dynamics take the form of a finite-dimensional system of differential equations. Various versions of these dynamics have been extensively studied in various game-theoretic contexts, and up-to-date accounts of much of the recent research in this area can be found in Weibull (1996), Samuelson (1997) and Hofbauer and Sigmund (1999).

[1] An evolutionary dynamics on continuous strategy spaces, which has features in common with the approach presented here, has also recently been developed by Oechssler and Riedel (1998). See also Ponti and Seymour (1998), appendix A, and Oeschler and Riedel (1999) for a somewhat different approaches.

However, in many games the strategy sets open to players are continua, rather than finite sets. For example, in bargaining games over continuously divisible commodities. Little work has been done in exploring evolutionary dynamics in this more general context, not least because of the often formidable technical difficulties involved. Recent exceptions are Hopkins and Seymour (1999), Seymour (1999), Friedman and Yellin (1997), Ponti and Seymour (1998), and Oeschler and Riedel (1998, 1999). However, more adventurous excursions into the evolutionary dynamics of infinite dimensional games are undoubtedly hindered by the lack of an easily accessible and rigorous development of a general theory appropriate to this context. For example, what general forms of dynamics should be associated with 2-player infinite-dimensional games? In particular, what is the infinite dimensional analogue of the Replicator Dynamics? Under what conditions do the usual existence and uniqueness theorems for solutions of systems of ordinary differential equations extend to the infinite dimensional case? Given that solutions exist, when do they exist for all positive times? Under what conditions do infinite dimensional dynamical systems possess equilibria? What is the relation between dynamic equilibria and Nash equilibria of the underlying game? When are dynamic equilibria (locally) stable? When are such equilibria unique? What influence do “mutations” (or “mistakes”) have on the dynamics, in particular in relation to equilibrium selection?

In this paper, I shall give some answers to some of these questions in the context of asymmetric games. In section 1 I develop an abstract formulation of an infinite dimensional, 2-player, asymmetric game, in which mixed strategies are defined as probability measures on the Borel subsets of the space of pure strategies, and illustrate it with the specific example of the simple bargaining game known as the Ultimatum Game. I also discuss the notion of mixed-strategy Nash equilibrium, and prove the infinite dimensional analogue of the key finite-dimensional property that the expected payoffs from any pure strategy in the support of a mixed-strategy equilibrium are equal (Proposition 4). In section 2, I briefly review the relevant parts of the general theory of dynamical systems on a Banach space needed in the sequel, and continue in the spirit of Hofbauer and Sigmund (1990) and Hopkins (1999), to define a very general class of evolutionary dynamics associated with infinite-dimensional games, which, following Hopkins (1999), I shall call *positive definite adaptive* (PDA) dynamics, and which it seems reasonable to suppose will support many natural examples of learning rules likely to be of interest. This class includes the Replicator dynamics, which I consider in detail. A principal feature of PDA dynamics is that a player's non-equilibrium (mixed) strategy will evolve (over a short time period) to a better response to his opponent's recently played strategy. Section 3 is concerned with the relation between Nash equilibria of the underlying game and dynamic equilibria of the evolutionary dynamics. In particular, under suitable assumptions on the class of positive-definite dynamics, asymptotically stable dynamic equilibria are Nash equilibria satisfying certain strictness-type conditions (Proposition 13), and *strongly* strict (pure-strategy) Nash equilibria are asymptotically stable (Proposition 19). This latter notion requires that an equilibrium response should not only be strictly better than any alternative, but that there should in fact be a finite advantage over any alternative stragtegy (precise definitions are given in section 1). In section 4, I consider PDA dynamics augmented by mutations, in which "mistakes" by players are introduced by supposing that a player occasionally uses an exogenously-determined mixed strategy instead of the strategy dictated by the undisturbed dynamic. Alternatively, one can think of a stream of experienced players leaving the game-playing population, to be replaced by naive players who begin by playing the fixed exogenous strategy. In this context, I consider questions concerning the absolute continuity of mixed strategies (*i.e.* whether they are represented by probability *densities*), and discuss the general form of the dynamics when expressed in terms of densities. In particular, I show that the natural home for evolutionary dynamics expressed in these terms is a suitable L_1-space of functions (Proposition 23). Section 6 offers some concluding thoughts.

1. Two-player asymmetric games

Let Ω_1 and Ω_2 be two compact topological spaces. We shall assume for simplicity that Ω_i is a closed and bounded subset of some Euclidean space, $\mathbb{R}^{n_i}$, though most of the arguments which follow could be formulated in a more general context.

The Ω_i are to be construed as the *spaces of pure strategies* in a 2-player, infinite dimensional game; *i.e.* Ω_i is the space of pure strategies available to player-i. To completely specify such a game, we also need *payoff functions*,

$$\pi_1, \pi_2 : \Omega_1 \times \Omega_2 \to I\!R_+, \tag{1}$$

where $I\!R_+$ is the space of non-negative real numbers[2], and $\pi_1(\xi, \eta)$, $\pi_2(\xi, \eta)$ are the payoffs to players-1 and -2, respectively, given that player-1 uses pure strategy $\xi \in \Omega_1$ and player-2 uses pure strategy $\eta \in \Omega_2$.

It would be natural and convenient to be able to assume that the functions π_i are continuous. However, such an assumption is too restrictive, as is shown by the following example.

EXAMPLE 1. *The infinite dimensional Ultimatum Game* Player-1 (Adam) is in possession of a dollar and must make an offer of a split of this dollar with player-2 (Eve). Eve may either accept or reject this offer. If she accepts, she gets what was offered, and Adam keeps the remainder. If she refuses, neither player gets anything.

Take $\Omega_1 = \Omega_2 = [0, 1]$, the unit interval. A pure strategy for Adam is an *offer* to Eve, $\xi \in [0, 1]$. A pure strategy for Eve is an *acceptance level*, $\eta \in [0, 1]$, such that she will accept Adam's offer if and only if it is greater than or equal to η. Thus, the payoff functions,

$$\pi_A, \pi_E : [0, 1] \times [0, 1] \to [0, 1],$$

are given by

$$\pi_A(\xi, \eta) = \begin{cases} 1 - \xi & \text{if } \xi \geq \eta, \\ 0 & \text{if } \xi < \eta. \end{cases} \tag{2a}$$

$$\pi_E(\xi, \eta) = \begin{cases} \xi & \text{if } \xi \geq \eta, \\ 0 & \text{if } \xi < \eta. \end{cases} \tag{2b}$$

But it is clear that neither of these functions is continuous on $[0, 1] \times [0, 1]$.

Note that there is a variation on the Ultimatum Game, which we call the *sub-Utimatum Game*, in which the strict and non-strict inequalities in (2a,b) are reversed. That is, Adam must offer

[2] We shall always assume that payoff functions are bounded, so there is no loss of generality in assuming they are non-negative.

Eve strictly more than η before she will accept. As we shall see later, there are reason for prefering the first version of this game. $\square$

To define a suitable space of functions from which payoff functions can be drawn, we consider the σ-field $\mathcal{B} = \mathcal{B}(\Omega)$ of Borel subsets of the compact space $\Omega \subset I\!R^n$.[3] Let $\mathcal{B}[\Omega]$ denote the linear space whose elements are the uniform limits of real-valued, $\mathcal{B}$-simple functions. Then $\mathcal{B}[\Omega]$ is a Banach space with respect to the norm

$$||f|| = \sup_{\omega \in \Omega} |f(\omega)|, \tag{3}$$

and the set of all real-valued, bounded $\mathcal{B}$-measurable functions is dense in $\mathcal{B}[\Omega]$.[4] The payoff functions we shall consider are non-negative functions in $\mathcal{B}[\Omega_1 \times \Omega_2]$, where $\Omega_1 \times \Omega_2 \subset I\!R^{n_1} \times I\!R^{n_2} = I\!R^{n_1+n_2}$. Such functions have the property that $\pi(\cdot, \eta) \in \mathcal{B}[\Omega_1]$ and $\pi(\xi, \cdot) \in \mathcal{B}[\Omega_2]$ for each $(\xi, \eta) \in \Omega_1 \times \Omega_2$. The Ultimatum game payoffs (2a,b) are clearly of this type.

Now consider mixed strategies for the game. Such strategies should be represented by ‘probability distributions’ over pure strategies. However, in the infinite dimensional context considered here, a probability distribution must be interpreted as a probability *measure*. Thus, for a compact $\Omega \subset I\!R^n$, let $\mathcal{M}[\Omega, \mathcal{B}]$ be the linear space of real-valued, signed, regular, bounded, σ-additive measures on $(\Omega, \mathcal{B})$. Then $X \in \mathcal{M}[\Omega, \mathcal{B}]$ is a probability measure if and only if

$$\begin{array}{ll} a) & X(B) \geq 0 \text{ for all } B \in \mathcal{B}, \\ b) & X(\Omega) = 1. \end{array} \tag{4}$$

We denote the subspace of probability measures by $\mathcal{P}[\Omega, \mathcal{B}]$.

The quantity $X(B)$ is to be interpreted as the probability that a player uses a pure strategy in the subset B. Alternatively, we may interpret $X(B)$ in terms of populations. Thus, suppose

[3] The Borel subsets of Ω are the members of the smallest σ-field of subsets of Ω which contains every closed subset of Ω. Our assumption that $\Omega \subset I\!R^n$ implies that the Borel sets coincide with the Baire sets. If we drop this restriction, then Baire sets and measures must be used throughout. See Yosida (1978), pp 18-19.

[4] A $\mathcal{B}$-simple function is a function which can be represented as a finite linear combination of indicator functions of sets in $\mathcal{B}$. See Dunford and Schwartz (1958), 2.12, p 240.

a player is drawn at random from a large population of size N, each member of which is programmed to play a definite pure strategy in Ω. Then we interpret $X(B)$ as the fraction of this population which plays a pure strategy in B, in the limit as $N \to \infty$. For the asymmetric games considered here, the popuations from which players 1 and 2 are drawn must be distinct.

The space $\mathcal{M}[\Omega, \mathcal{B}]$ is a Banach space with the norm,

$$||X|| = \sup_{||f|| \leq 1} \left| \int_{\Omega} f dX \right|, \tag{5}$$

where f runs over the set of functions $f \in \mathcal{B}[\Omega]$ with $||f|| \leq 1$. The quantity $||X||$ is finite and is known as the *total variation* of X on Ω.[5] The topology on $\mathcal{M}[\Omega, \mathcal{B}]$ defined by this norm is known as the *strong* , or *metric*, topology.

LEMMA 1. Let $\mathcal{P}[\Omega, \mathcal{B}] \subset \mathcal{M}[\Omega, \mathcal{B}]$ be the subset of probability measures on $(\Omega, \mathcal{B})$. Then $\mathcal{P}[\Omega, \mathcal{B}]$ is a closed, convex subset of the unit sphere in $\mathcal{M}[\Omega, \mathcal{B}]$.[6]

Proof. See Appendix. □

There is a family of bilinear pairings,

$$\langle \cdot, \cdot \rangle_B : \mathcal{B}[B] \times \mathcal{M}[\Omega, \mathcal{B}] \to I\!\!R; \qquad \langle g, X \rangle_B = \int_B g dX, \qquad \big(B \in \mathcal{B}\big). \tag{6}$$

We shall often write $\langle g, X \rangle$ for $\langle g, X \rangle_\Omega$. Thus, if $X \in \mathcal{P}[\Omega, \mathcal{B}]$, then $E_X[g] = \langle g, X \rangle$ is the *expected value* of the function g with respect to the probability measure X. More generally, if $X(B) > 0$, then $E_X[g \mid B] = \langle g, X \rangle_B \big/ X(B)$ is the *conditional expectation* of g with respect to X, given that only pure strategies in B are used. In particular, these interpretations apply for payoff functions for the class of games we consider. The main intuitive attraction of the strong topology is contained in the following Lemma.

LEMMA 2. The pairings $\langle \cdot, \cdot \rangle_B$ are continuous.

[5] See Yosida (1978), pp 35-37.

[6] It is fundamental to note that this does *not* imply that $\mathcal{P}[\Omega, \mathcal{B}]$ is compact (at least in the strong topology considered here).

Proof. See Appendix. □

We can now return to mixed strategies. Suppose player-2 uses the mixed strategy $Y \in \mathcal{P}[\Omega_2, \mathcal{B}_2]$. Player-1 then obtains some non-negative *expected* payoff on using a pure strategy $\xi \in \Omega_1$, which we shall denote by $w_Y(\xi)$. More generally, we assume there is a *generalised expected payoff function*

$$w : \mathcal{M}[\Omega_2, \mathcal{B}_2] \to \mathcal{B}_1[\Omega_1]; \qquad Y \to w_Y, \tag{7}$$

which satisfies the following properties:

$$\begin{array}{ll} a) & w \text{ is linear and continuous,} \\ b) & w_Y \geq 0 \text{ on } \Omega_1 \text{ if } Y \geq 0 \text{ on } \mathcal{B}_2, \\ c) & \pi_1(\xi, \eta) = w_{\Delta(\eta)}(\xi), \end{array} \tag{8}$$

where π_1 is the pure strategy payoff function (1), and $\Delta(\eta)$ is the *Dirac* Δ*-measure* at $\eta \in \Omega_2$, defined by,

$$\Delta(\eta)(B) = \begin{cases} 1 & \text{if } \eta \in B, \\ 0 & \text{if } \eta \notin B, \end{cases} \tag{9}$$

for $B \in \mathcal{B}_2$. These measures satisfy the well-known characteristic property,

$$\langle f, \Delta(\xi) \rangle = f(\xi) \quad \text{for } f \in \mathcal{B}[\Omega]. \tag{10}$$

Similarly, we assume there is a generalised payoff function for player-2,

$$v : \mathcal{M}[\Omega_1, \mathcal{B}_1] \to \mathcal{B}_2[\Omega_2]; \qquad X \to v_X, \tag{11}$$

satisfying the obvious properties corresponding to (7).

Why should w be linear in Y? To answer this we begin with the pure strategy payoff functions (1), and construct w and v from them. We wish to define $w_Y(\xi)$ to be the expected payoff to player-1 given that he uses the pure strategy ξ and player-2 uses the mixed strategy Y. Similarly, $v_X(\eta)$ is the expected payoff to player-2 when she uses the pure strategy η and player-1 uses the mixed strategy X. We must therefore define

$$\left.\begin{array}{ll} w_Y(\xi) & = \int_{\Omega_2} \pi_1(\xi, \eta) dY(\eta) = \langle \pi_1(\xi, \cdot), Y \rangle_{\Omega_2}, \\ v_X(\eta) & = \int_{\Omega_1} \pi_2(\eta, \xi) dX(\xi) = \langle \pi_2(\eta, \cdot), X \rangle_{\Omega_1}. \end{array}\right\} \tag{12}$$

Clearly w extends by linearity to a function defined for all $Y \in \mathcal{M}[\Omega_1, \mathcal{B}_1]$, and similarly for v. The formulae (12) therefore explain the linearity assumption (8a). Note that the Fubini-Tonelli Theorem[7] implies that

$$\left.\begin{array}{ll} \langle w_Y, X\rangle_{\Omega_1} & = \langle \pi_1, X \times Y\rangle_{\Omega_1\times\Omega_2} \\ \langle v_X, Y\rangle_{\Omega_2} & = \langle \pi_2, Y \times X\rangle_{\Omega_2\times\Omega_1} \end{array}\right\} \tag{13}$$

We show that (12) and (8) are consistent in the following result.

LEMMA 3. Suppose the pure-strategy payoff functions $\pi_i \in \mathcal{B}[\Omega_1 \times \Omega_2]$ are non-negative. Then, the functions w and v defined by (12) take values in $\mathcal{B}[\Omega_1]$ and $\mathcal{B}[\Omega_2]$, respectively, and satisfy the properties (8).

Proof. The boundedness of w_Y follows from the boundedness of π_1. In fact, since $\pi_1 \geq 0$,

$$\big|w_Y(\xi)\big| = \pi_1^*\big|\langle\frac{\pi_1(\xi,\cdot)}{\pi_1^*}, Y\rangle\big| \leq \pi_1^* \sup_{||f||\leq 1}\big|\langle f, Y\rangle\big| = \pi_1^*||Y||, \tag{14}$$

where $\pi_1^* = sup_{\xi,\eta}\,\pi_1(\xi,\eta) > 0$. That w_Y is $\mathcal{B}_1$-measurable follows from the fact that $\pi_1 \in \mathcal{B}[\Omega_1 \times \Omega_2]$.

To prove (8a), we need only prove continuity in Y. For this, we have, from linearity and (14),

$$\big|w_{Y'}(\xi) - w_Y(\xi)\big| = \big|w_{Y'-Y}(\xi)\big| \leq \pi_1^*||Y' - Y||,$$

so that

$$||w_{Y'} - w_Y|| = \sup_{\xi\in\Omega_1}\big|w_{Y'}(\xi) - w_Y(\xi)\big| \leq \pi_1^*||Y' - Y||.$$

Continuity therefore follows.

Property (8b) follows immediately from (12) and the assumption that π_1 is non-negative. Property (8c) follows from (10) and (12). □

Note that property (8c) identifies the pure strategy $\eta \in \Omega_2$ with the mixed stratgery $\Delta(\eta) \in \mathcal{P}[\Omega_2, \mathcal{B}_2]$.

[7] See Yosida (1978), p. 18.

Given (8) and (12), we can extend w and v to bilinear generalised expected-payoff functions,

$$\Pi_1 : \mathcal{M}[\Omega_1, \mathcal{B}_1] \times \mathcal{M}[\Omega_2, \mathcal{B}_2] \to I\!R \quad \text{and} \quad \Pi_2 : \mathcal{M}[\Omega_2, \mathcal{B}_2] \times \mathcal{M}[\Omega_1, \mathcal{B}_1] \to I\!R,$$

by

$$\left.\begin{aligned} \Pi_1(X, Y) &= \langle w_Y, X \rangle, \\ \Pi_2(Y, X) &= \langle v_X, Y \rangle. \end{aligned}\right\} \tag{15}$$

It then follows that if w and v are given by (12), then the pure strategy payoff functions (1) satisfy

$$\left.\begin{aligned} \pi_1(\xi, \eta) &= \Pi_1\big(\Delta(\xi), \Delta(\eta)\big), \\ \pi_2(\eta, \xi) &= \Pi_2\big(\Delta(\eta), \Delta(\xi)\big). \end{aligned}\right\} \tag{16}$$

We now consider Nash equilibria.

DEFINITION 1. A pair $(\hat{X}, \hat{Y}) \in \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2]$ is a *mixed-strategy Nash equilibrium* (NE) if

$$\langle w_{\hat{Y}}, X \rangle \leq \langle w_{\hat{Y}}, \hat{X} \rangle \quad \text{and} \quad \langle v_{\hat{X}}, Y \rangle \leq \langle v_{\hat{X}}, \hat{Y} \rangle, \tag{17}$$

for all $(X, Y) \in \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2]$. A NE is *strict* (SNE) if the inequalities (17) are strict whenever $X \neq \hat{X}$ and $Y \neq \hat{Y}$.

Important properties of mixed-strategy NE are given in the following Proposition. Recall that, for a measure $X \in \mathcal{M}[\Omega, \mathcal{B}]$, a pure strategy-dependent property holds X*-almost everywhere* (X-a.e.) if it holds everywhere on Ω except on a subset of X-measure zero.

PROPOSITION 4. If $(\hat{X}, \hat{Y})$ is a mixed-strategy Nash equilibrium, then

$$\langle w_{\hat{Y}}, \hat{X} \rangle_{B_1} = \langle w_{\hat{Y}}, \hat{X} \rangle \hat{X}(B_1) \quad \text{and} \quad \langle v_{\hat{X}}, \hat{Y} \rangle_{B_2} = \langle v_{\hat{X}}, \hat{Y} \rangle \hat{Y}(B_2), \tag{18}$$

for all $(B_1, B_2) \in \mathcal{B}_1 \times \mathcal{B}_2$. In addition, $w_{\hat{Y}}(\xi) \leq \langle w_{\hat{Y}}, \hat{X} \rangle$ for each $\xi \in \Omega_1$, with equality $\hat{X}$-a.e., and $v_{\hat{X}}(\eta) \leq \langle v_{\hat{X}}, \hat{Y} \rangle$ for each $\eta \in \Omega_2$, with equality $\hat{Y}$-a.e. That is, $w_{\hat{Y}}$ is constant $\hat{X}$-a.e. and $v_{\hat{X}}$ is constant $\hat{Y}$-a.e.

Proof. Suppose $\hat{X}(B) = 0$ for $B \in \mathcal{B}_1$. Then $\langle w_{\hat{Y}}, \hat{X}\rangle_B = 0$ by absolute continuity[8], so equality holds in (18a). On the other hand, if $\hat{X}(B) > 0$, define a probability measure $X \in \mathcal{P}[\Omega_1, \mathcal{B}_1]$ by

$$X(B') = \frac{\hat{X}(B' \cap B)}{\hat{X}(B)}, \quad (B' \in \mathcal{B}_1).$$

Then, from the Nash equilibrium property (17a), it follows that

$$\frac{\langle w_{\hat{Y}}, \hat{X}\rangle_B}{\hat{X}(B)} = \langle w_{\hat{Y}}, X\rangle \leq \langle w_{\hat{Y}}, \hat{X}\rangle,$$

which gives

$$\langle w_{\hat{Y}}, \hat{X}\rangle_B \;\; \leq \;\; \langle w_{\hat{Y}}, \hat{X}\rangle \hat{X}(B), \tag{19}$$

for all $B \in \mathcal{B}_1$.

Now suppose there exists $B \in \mathcal{B}_1$ for which the inequality (19) is strict. Let $B^c = \Omega - B$. Then (19) also applies to B^c, whence

$$\begin{aligned}\langle w_{\hat{Y}}, \hat{X}\rangle &= \langle w_{\hat{Y}}, \hat{X}\rangle_B + \langle w_{\hat{Y}}, \hat{X}\rangle_{B^c} \\ &< \langle w_{\hat{Y}}, \hat{X}\rangle \hat{X}(B) + \langle w_{\hat{Y}}, \hat{X}\rangle \hat{X}(B^c) \\ &= \langle w_{\hat{Y}}, \hat{X}\rangle \big\{\hat{X}(B) + \hat{X}(B^c)\big\} \\ &= \langle w_{\hat{Y}}, \hat{X}\rangle.\end{aligned}$$

This is a contradiction, and we conclude that equality must hold in (19) for all $B \in \mathcal{B}_1$; *i.e.* (18a) holds. A similar argument establishes (18b).

For the last statement, the NE condition together with (9), gives $w_{\hat{Y}}(\xi) = \langle w_{\hat{Y}}, \Delta(\xi)\rangle \leq \langle w_{\hat{Y}}, \hat{X}\rangle$, for each $\xi \in \Omega_1$. Let $\hat{w}_{\hat{Y}} = w_{\hat{Y}} - \langle w_{\hat{Y}}, \hat{X}\rangle$, and note that (18a) can be written as $\langle \hat{w}_{\hat{Y}}, \hat{X}\rangle_B = 0$ for all $B \in \mathcal{B}_1$. It therefore follows that $\hat{w}_{\hat{Y}} = 0$ $\hat{X}$-a.e. The statements for $v_{\hat{X}}$ are proved similarly. $\square$

Proposition 4 says that, at a NE, the expected payoff to a player from any of the pure strategies which are used with positive probability is the same. In particular, Proposition 4 implies that

[8] See Yosida (1978), Properties of the integral (v), p 17.

a SNE must consist of pure strategies. To see this, let B_0 be the set on which the inequality $w_{\hat{Y}} \leq \langle w_{\hat{Y}}, \hat{X} \rangle$ is strict, and let $B_1 = B_0^c$. Then $\hat{X}(B_0) = 0$ and $\hat{X}(B_1) = 1$ by Proposition 4. Let $\hat{\xi} \in B_1$, and suppose that $\hat{X} \neq \Delta(\hat{\xi})$. Then, by (10) and the strictness condition for SNE,

$$w_{\hat{Y}}(\hat{\xi}) = \langle w_{\hat{Y}}, \Delta(\hat{\xi}) \rangle < \langle w_{\hat{Y}}, \hat{X} \rangle,$$

which contradicts Proposition 4. Hence, $\hat{X} = \Delta(\hat{\xi})$, and $B_1 = \{\hat{\xi}\}$.

At the other extreme, a *fully mixed* NE satisfies $w_{\hat{Y}}(\xi) = \langle w_{\hat{Y}}, \hat{X} \rangle$ and $v_{\hat{X}}(\eta) = \langle v_{\hat{X}}, \hat{Y} \rangle$ for all $(\xi, \eta) \in \Omega_1 \times \Omega_2$. That is, the expected payoffs to each player are the same for every strategy.

As is well known for finite-dimensional asymmetric games, that a SNE is locally asymptotically stable with respect to a large class of evolutionary dynamics. However, this is not the case in infinite dimensions. To see why SNE is not enough to guarantee good properties with respect to evoltionary dynamics, we note that the strictness conditions imply that

$$\pi_1(\hat{\xi}, \hat{\eta}) > \pi_1(\xi, \hat{\eta}) \quad \text{and} \quad \pi_2(\hat{\xi}, \hat{\eta}) > \pi_2(\hat{\xi}, \eta), \tag{20}$$

for all $\xi \neq \hat{\xi}$ and $\eta \neq \hat{\eta}$.[9] However, these strict inequalities are *not* generic properties of payoff functions in $\mathcal{B}[\Omega_1 \times \Omega_2]$. In particular, if $\pi_1(\xi, \hat{\eta})$ is continuous at $\hat{\xi}$, then an arbitrarily small perturbation of π_1 in $\mathcal{B}[\Omega_1 \times \Omega_2]$ will define a payoff function for which the first inequality in (20) fails. For example, define the function $\tilde{\pi}_1(\xi, \eta) \in \mathcal{B}[\Omega_1 \times \Omega_2]$ by

$$\tilde{\pi}_1(\xi, \eta) = \pi_1(\xi, \eta) + \varepsilon I_B(\xi),$$

where I_B denotes the indicator function of a set $B \in \mathcal{B}_1$, and $\varepsilon > 0$. Then from (3), $||\tilde{\pi}_1 - \pi_1|| = \varepsilon$. On the other hand, by continuity of $\pi_1(\xi, \hat{\eta})$ at $\hat{\xi}$, we may choose B with $\hat{\xi} \notin B$ and $0 < \pi_1(\hat{\xi}, \hat{\eta}) - \pi_1(\xi_0, \hat{\eta}) \leq \varepsilon$ for each $\xi_0 \in B$. Then,

$$\tilde{\pi}_1(\hat{\xi}, \hat{\eta}) - \tilde{\pi}_1(\xi_0, \hat{\eta}) = \pi_1(\hat{\xi}, \hat{\eta}) - \pi_1(\xi_0, \hat{\eta}) - \varepsilon \leq 0.$$

[9] The SNE is $(\hat{X}, \hat{Y}) = (\Delta(\hat{\xi}), \Delta(\hat{\eta}))$, and the strictness conditions are $\langle w_{\hat{Y}}, \hat{X} \rangle > \langle w_{\hat{Y}}, X \rangle$ and $\langle v_{\hat{X}}, \hat{Y} \rangle > \langle v_{\hat{X}}, Y \rangle$ for all $X \neq \hat{X}$ and $Y \neq \hat{Y}$. Now take $X = \Delta(\xi)$ and $Y = \Delta(\eta)$ to obtain (20).

Thus, if equality holds for all $\xi_0 \in B$, then each ξ_0 is an alternative best reply to $\hat{\eta}$, and the strictness property for the game with payoff functions $(\tilde{\pi}_1, \pi_2)$ no longer holds (though it is still a NE). On the other hand, if the inequality is strict for some $\xi_0 \in B$, then $(\hat{\xi}, \hat{\eta})$ is no longer even a NE. This underlines the fragility of the notion of SNE in the infinite-dimensional case.

To guarantee good properties, we need a stronger notion than strictness for a NE, which is generic in the appropriate sense.

DEFINITION 2. A SNE $(\hat{X}, \hat{Y}) = (\Delta(\hat{\xi}), \Delta(\hat{\eta}))$ is *super strict* (SSNE) if there exists $\delta_0 > 0$ such that

$$\pi_1(\hat{\xi}, \hat{\eta}) - \pi_1(\xi, \hat{\eta}) \geq \delta_0 \text{ and } \pi_2(\hat{\xi}, \hat{\eta}) - \pi_2(\hat{\xi}, \eta) \geq \delta_0, \tag{22}$$

for all $\xi \neq \hat{\xi}$ and $\eta \neq \hat{\eta}$.

It follows from (22) that it cannot be true that $\pi_1(\xi, \hat{\eta})$ is continuous at $\hat{\xi}$ or that $\pi_2(\hat{\xi}, \eta)$ is continuous at $\hat{\eta}$. That is, $\pi_1(\xi, \hat{\eta})$ has a (discontinuous) 'spike' at $\hat{\xi}$, and $\pi_2(\hat{\xi}, \eta)$ has a spike at $\hat{\eta}$. As we have seen, any pair of payoff functions for which $(\hat{X}, \hat{Y})$ is a SNE can be perturbed by an arbitrarily small perturbation in $\mathcal{B}[\Omega_1 \times \Omega_2]$ to a pair of payoff functions for which $(\hat{X}, \hat{Y})$ is a SSNE. For example, a uniform method to do this is to define a pair of perturbed payoff functions $(\tilde{\pi}_1, \tilde{\pi}_2)$ by:

$$\tilde{\pi}_1(\xi, \eta) = \begin{cases} \pi_1(\xi, \eta) & \text{if } \xi \neq \hat{\xi} \\ (1+p)\pi_1(\xi, \eta) & \text{if } \xi = \hat{\xi} \end{cases}, \qquad \tilde{\pi}_2(\xi, \eta) = \begin{cases} \pi_2(\xi, \eta) & \text{if } \eta \neq \hat{\eta} \\ (1+q)\pi_2(\xi, \eta) & \text{if } \eta = \hat{\eta} \end{cases} \tag{24}$$

with $p, q > 0$ (but otherwise arbitrarily small). Conversely, an SSNE is generic in the sense that all sufficiently small perturbations of the payoff functions retain the property of having an SSNE at $(\hat{X}, \hat{Y})$.

We illustrate the general setup described above in the case of the Ultimatum Game.

EXAMPLE 2. Continuing with the Ultimatum Game described in Example 1, we have that Adam's expected payoff from an offer $\xi \in [0, 1]$, when Eve uses a mixed strategy

$Y \in \mathcal{P}[[0,1],\mathcal{B}]$, is given by (12) with $\pi_1 = \pi_A$ as in (2a):

$$w_Y(\xi) = (1-\xi)\int_0^{\xi} dY = (1-\xi)Y\big([0,\xi]\big).$$

Similarly, if Adam uses a mixed strategy $X \in \mathcal{P}\big[[0,1],\mathcal{B}\big]$, and Eve has acceptance level $\eta \in [0,1]$, then Eve's expected payoff is given by (12) with $\pi_2 = \pi_E$ as in (2b):

$$v_X(\eta) = \int_{\eta}^{1} \xi dX(\xi) = \langle \iota, X\rangle_{[\eta,1]},$$

where ι is the identity map on I.

The Ultimatum Game admits a family of pure-strategy Nash equilibria, $(\hat{X},\hat{Y}) = \big(\Delta(\xi),\Delta(\xi)\big)$ (one for each $\xi \in [0,1]$); *i.e.* Adam offers ξ and Eve accepts any offer greater than or equal to ξ. The equilibrium with $\xi = 0$ is the *subgame perfect* equilibrium. However, none of these equibria are strict, since, if $\xi > 0$,

$$\langle v_{\hat{X}}, \Delta(\eta)\rangle = \pi_E(\eta,\xi) = \xi = \langle v_{\hat{X}}, \hat{Y}\rangle,$$

for all $\eta \leq \xi$ (Eve could equally well have used an acceptance level less than ξ). For the subgame perfect equilibrium, $v_{\hat{X}}(\eta) = \pi_E(\eta, 0) = 0$, so that $\langle v_{\hat{X}}, Y\rangle = \langle v_{\hat{X}}, \hat{Y}\rangle = 0$ for any Y (Eve would receive nothing whatever her strategy).

Now observe the profound difference between the above situation and that pertaining for the sub-Ultimatum game (Example 1). For the latter, it is easy to see that there are no pure strategy Nash equilibria at all. Thus, if Eve accepts Adam's offer ($\xi > \eta$), there is always a strictly better offer Adam could have made which would also have been accepted (*e.g.* $\frac{1}{2}(\xi+\eta)$).

Returning to the Ultimatum game, we can obtain a subgame perfect equilibrium which is a SNE for a modified form of the Ultimatum Game as follows. Restrict Adam and Eve's strategy spaces to $[\alpha, 1]$, where $0 < \alpha < 1$. That is, Adam cannot offer less than α, and this minimum offer is known to Eve, who consequently will not consider an acceptance level less than α. The subgame perfect equilibrium is then $(\hat{X},\hat{Y}) = (\Delta(\alpha),\Delta(\alpha))$, with payoffs $(1-\alpha,\alpha)$. In this case, Eve has no alternative best reply to Adam's offer of α other than to accept, and Adam has no offer against Eve's acceptance level of α which achieves an equal payoff. In fact, Eve has a payoff spike at the acceptance level α, since $\pi_E(\alpha,\alpha) = \alpha > 0$ and $\pi_E(\alpha,\eta) = 0$ for $\eta > \alpha$. On the other hand, there seems to be no intuitively satisfying way of modifying the game which gives Adam a payoff spike at the subgame perfect equilibrium; in fact, $\pi_A(\xi,\alpha) = 1-\xi$ is continuous at $\xi = \alpha$. So, only an artificial (infintessimal) perturbation of Adam's payoff function, as in (21), will yield a subgame perfect equilibrium which is a SSNE. For example, if Adam pays a forfeit (to a third party) measured as a percentage of the deviation of his actual offer from the best offer he could have made. That is, the game now has payoff function for Adam,

$$\pi_A(\xi,\eta) = \begin{cases} 0 & \text{if } \xi < \eta, \\ 1-\xi & \text{if } \xi = \eta, \\ (1-p)(1-\xi) & \text{if } \xi > \eta, \end{cases} \tag{25}$$

where $0 < p \leq 1$ is the percentage penalty. With Eve's payoff function unchanged, (α,α) is now a SSNE for this modified game. □

2. PDA Dynamics

We shall consider dynamical systems on the Banach space $\mathcal{E} = \mathcal{M}[\Omega_1,\mathcal{B}_1] \times \mathcal{M}[\Omega_2,\mathcal{B}_2]$, with the product-space norm. That is, we are given a function,

$$\Xi = (\mathcal{X},\mathcal{Y}) : \mathcal{E} \to \mathcal{E}, \tag{26}$$

which defines the dynamical system,

$$\left.\begin{aligned} \tfrac{dX}{dt} &= \mathcal{X}(X,Y), \\ \tfrac{dY}{dt} &= \mathcal{Y}(X,Y), \end{aligned}\right\} \tag{27}$$

with $X \in \mathcal{M}[\Omega_1,\mathcal{B}_1]$ and $Y \in \mathcal{M}[\Omega_2,\mathcal{B}_2]$.

To obtain the standard existence and uniqueness theorems for solutions of (27), we need only the minimal assumption that Ξ be *locally Lipschitz*. That is, for each $(X,Y) \in \mathcal{E}$, there is a neighbourhood, $\mathcal{N}(X,Y) \subset \mathcal{E}$, such that Ξ is bounded on $\mathcal{N}(X,Y)$, and there exists a constant $k \geq 0$ (depending on $\mathcal{N}$) with

$$||\Xi(X',Y') - \Xi(X'',Y'')|| \leq k.\,max\big\{||X'-X''||, ||Y'-Y''||\big\}, \tag{28}$$

for all $(X', Y'), (X'', Y'') \in \mathcal{N}(X, Y)$. Of course, a locally Lipschitz function is continuous.[10]

Recall that, under these assumptions, there is a unique *solution flow* for the system (27), in the form of a continuous map on an open subset $\mathcal{U} \subset I\!R \times \mathcal{E}$,

$$\mathcal{F} : \mathcal{U} \to \mathcal{E}; \qquad (t, e) \to \mathcal{F}_t(e), \tag{29}$$

where $e = (X, Y) \in \mathcal{E}$ is a generic element, having the following properties:

$$\begin{array}{l} a) \;\; \{0\} \times \mathcal{E} \subset \mathcal{U}, \\ b) \;\; \{t \mid (t,e) \in \mathcal{U}\} = \big(-\alpha(e), \beta(e)\big) \subset I\!R, \\ c) \;\; \mathcal{F}_0(e) = e \;\text{ for each }\; e \in \mathcal{E}, \\ d) \;\; \mathcal{F}_{s+t}(e) = \mathcal{F}_s\big(\mathcal{F}_t(e)\big), \\ e) \;\; \mathcal{F}_t(e) = e + \int_0^t \Xi\big(\mathcal{F}_s(e)\big) ds \;\text{ for each }\; (t,e) \in \mathcal{U}. \end{array} \tag{30}$$

In (b), $0 < \alpha(e), \beta(e) \leq \infty$, and $\big(-\alpha(e), \beta(e)\big)$ is the *maximal* interval on which the solution with initial condition e is defined.

DEFINITION 3. A subset $\mathcal{K} \subset \mathcal{E}$ is called *forward invariant* under the flow $\mathcal{F}$ if, for each $e \in \mathcal{K}$ and $0 < \gamma < \beta(e)$, $\mathcal{F}_t(e) \in \mathcal{K}$ for all $t \in [0, \gamma]$. Similarly, $\mathcal{K}$ is *backward invariant* if, for each $e \in \mathcal{K}$ and $0 < \gamma < \alpha(e)$, $\mathcal{F}_t(e) \in \mathcal{K}$ for all $t \in [-\gamma, 0]$.

We have the following important result.

PROPOSITION 5. Let $\mathcal{K} \subset \mathcal{E}$ be a closed subset which is forward invariant under the flow $\mathcal{F}$, and suppose that Ξ is uniformly bounded on $\mathcal{K}$. Then $\beta(e) = \infty$ for each $e \in \mathcal{K}$. Similarly, if $\mathcal{K}$ is backward invariant then $\alpha(e) = \infty$ for each $e \in \mathcal{K}$. If $\mathcal{K}$ is both forward and backward invariant, then, $\mathcal{F}_t : \mathcal{K} \to \mathcal{K}$ is a homeomorphism, with inverse $\mathcal{F}_{-t}$, for each $t \in I\!R$.

Proof. See, for example, Hirsch and Smale (1974), p. 172, Proposition. Their use of compactness in the finite-dimensional case can be replaced by the uniform boundedness assumption given here. □

In applying the general theory outlined above to the case of infinite-dimensional, 2-player games, we shall take $\Xi = (\mathcal{X}, \mathcal{Y})$ to have a particular form. But we shall keep this form

[10] For example Dieudonné (1960), p 281, (10.4.6) and p 299, (10.8.1). Conversely, a continuous function is locally bounded, but need not satisfy (28).

as general as possible, while retaining our focus on the specific application of the theory to evolutionary games. In the subsequent development, we follow, in spirit if not in detail, the ideas of Hopkins (1999) - see also Hopkins and Seymour (1999).

We suppose given a family of locally-Lipschitz functions,

$$Q_\Omega : \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}] \to \mathcal{M}[\Omega, \mathcal{B}], \tag{31}$$

one for each pair (n, Ω), with Ω a compact subset of $I\!R^n$. Associate an *evolutionary game dynamics* to the family of operators (31) using the generalised payoff functions w and v of (7) and (11) defined by the given underlying game with pure strategy space $\Omega_i \subset I\!R^{n_i}$ for player i ($i = 1, 2$), by

$$\left.\begin{array}{l} \frac{dX}{dt} = \mathcal{X}_0(X, Y) = Q_1(w_Y, X), \\ \frac{dY}{dt} = \mathcal{Y}_0(X, Y) = Q_2(v_X, Y). \end{array}\right\} \tag{32}$$

The idea is that these dynamics capture a continuous-time limit, as $\tau \to 0$, of some strategy-learning process, when the given game is repeated indefinitely at intervals of length τ. Thus, we assume the existence of two distinct and large (usually infinite) populations, one from which player-1 is drawn and the other from which player-2 is drawn. The form of the dynamics incorporates the manner in which players are chosen in each round. In this scenario, $X(B)$ is interpreted as the probability that a chosen player-1 will use a pure strategy in B. Observe that the instantaneous rate of change of the mixed strategy X used by player-1 depends both on X and on the payoff he receives when player-2 uses the mixed strategy Y. Similarly, the instantaneous rate of change of player-2's mixed strategy depends both on her current mixed strategy, Y, and on the payoff she receives on player-1's use of the mixed strategy X. An important example of this construction, the Replicator Dynamics, is discussed below. Of course, the dynamics defined by (32) can only be interpreted in game-theoretic terms when X and Y are proper mixed strategies (*i.e.* probability measures). For general X and Y, the dynamics is to be taken as existing only as a formal extension of the interpreted dynamics.

PROPOSITION 6. Suppose Q_Ω is locally Lipschitz on $\mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}]$ for each Ω. Then $\Xi_0 = (\mathcal{X}_0, \mathcal{Y}_0)$ is locally Lipschitz on $\mathcal{E} = \mathcal{M}[\Omega_1, \mathcal{B}_1] \times \mathcal{M}[\Omega_2, \mathcal{B}_2]$.

Proof. See Appendix. □

COROLLARY 7. Suppose that Q_Ω is C^1 (*i.e.* continuously differentiable) on $\mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}]$. Then $\Xi_0 = (\mathcal{X}_0, \mathcal{Y}_0)$ is locally Lipschitz on $\mathcal{E} = \mathcal{M}[\Omega_1, \mathcal{B}_1] \times \mathcal{M}[\Omega_2, \mathcal{B}_2]$.

Proof. A C^1 map is locally Lipschitz. See Hirsch and Smale (1974), Lemma, pp 163-64. □

EXAMPLE 3. *The Replicator Dynamics.* The infinite dimensional Replicator dynamics are defined using (32) and the family of operators,

$$R_\Omega(g, X)(B) = \langle g, X\rangle_B X(\Omega) - \langle g, X\rangle_\Omega X(B), \qquad \big(B \in \mathcal{B}\big). \tag{33}$$

It follows from (33) that, for $f, g \in \mathcal{B}[\Omega]$,

$$\langle f, R_\Omega(g, X)\rangle = \langle fg, X\rangle X(\Omega) - \langle f, X\rangle\langle g, X\rangle. \tag{34}$$

Thus, when X is a probability measure, so that $X(\Omega) = 1$, $\langle f, R_\Omega(g, X)\rangle$ is the *covariance* of f and g with respect to X. Note that R_Ω is in fact linear in g. The dynamics (32) associated to the Replicator operators is called the Replicator dynamics.

The Replicator operators admit an important generalization as follows. Let $a_\Omega : \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}] \to [a_*, a^*]$, with $0 < a_* \leq a^* < \infty$, be a family of locally-Lipschitz functions. Then we define the *Generalized Replicator operators* by: $\hat{R}_\Omega(g, X) = a_\Omega(g, X) R_\Omega(g, X)$. The significance of this generalization lies in the fact that the games we consider are asymmetric, so that the multiplying factors $a_1(w_Y, X)$ and $a_2(v_X, Y)$ which appear in the generalized Replicator dynamics (32), represent different time-scale factors over which the two players' strategies evolve. These factors cannot in general both be absorbed by a single time-scale change, and hence represent important intrinsic features of the dynamics. Also note that, unlike $R_\Omega(g, X)$, the generalization $\hat{R}_\Omega(g, X)$ is not in general linear in g. The relevance of Generalized Replicator dynamics to learning processes which involve sequential sampling and imitation has been emphasised by Schlag (1998) and Hofbauer and Schlag (1998).

From (32), the Replicator dynamics takes the form

$$\left.\begin{aligned} \tfrac{dX}{dt}(B_1) &= \langle w_Y, X\rangle_{B_1} X(\Omega_1) - \langle w_Y, X\rangle_{\Omega_1} X(B_1), \\ \tfrac{dY}{dt}(B_2) &= \langle v_X, Y\rangle_{B_2} Y(\Omega_2) - \langle v_X, Y\rangle_{\Omega_2} Y(B_2), \end{aligned}\right\} \tag{35}$$

for $B_i \in \mathcal{B}_i$. Here, when X and Y are probability measures, the measure $= \langle w_Y, X\rangle_{B_1}$ on $(\Omega_1, \mathcal{B}_1)$ can be interpreted as the "fitness" of the mixed strategy X against Y from the point of view of player-1. Then $\langle w_Y, X\rangle_{\Omega_1}$ is the "mean fitness", and the replicator equation measures the deviation of the fitness from the (probability-weighted) mean fitness. Similarly, the measure $\langle v_X, Y\rangle_{B_2}$ is the "fitness" of the mixed strategy Y against X from the point of view of player-2.

Equations (35) take on a more familiar form when X and Y are defined by *densities*; *i.e.*

$$\left.\begin{array}{l} X(B_1) = \int_{B_1} x(\xi)d\xi, \\ Y(B_2) = \int_{B_2} y(\eta)d\eta, \end{array}\right\} \tag{36}$$

where $d\xi$ and $d\eta$ represent Lebesgue measure on Ω_1 and Ω_2, repectively, and $x(\xi), y(\eta)$ are L_2-functions on Ω_1 and Ω_2, respectively. Then taking $B_1 = d\xi$ and $B_2 = d\eta$ in (36) yields the infintessimal form of these equations,

$$\left.\begin{array}{l} \frac{dx(\xi)}{dt} = x(\xi)\big[w_Y(\xi) - \langle w_Y, X\rangle_{\Omega_1}\big], \\ \frac{dy(\eta)}{dt} = y(\eta)\big[v_X(\eta) - \langle v_X, Y\rangle_{\Omega_2}\big]. \end{array}\right\} \tag{37}$$

Conversely, integration of equations (37) over B_1 and B_2, respectively, recovers equations (35).

The Replicator family of operators (33) is C^1 in the sense of Corollary 7. To see this, recall that the derivative of $Q_\Omega(g, X)$ is (by definition) the continuous linear operator $DQ_\Omega(g, X)$: $\mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}] \to \mathcal{M}[\Omega, \mathcal{B}]$, given by

$$DQ_\Omega(g, X)(\phi, \Phi) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big\{Q_\Omega(g + \varepsilon\phi, X + \varepsilon\Phi) - Q_\Omega(g, X)\big\},$$

For the Replicator operators, the linearity in the first coordinate and the continuity in the second coordinate of R_Ω, implies that this limit is

$$DR_\Omega(g, X)(\phi, \Phi) = R_\Omega(\phi, X) + D_2R_\Omega(g, X)\Phi,$$

where $D_2R_\Omega(g, X) : \mathcal{M}[\Omega, \mathcal{B}] \to \mathcal{M}[\Omega, \mathcal{B}]$ is the partial derivative of $R_\Omega(g, X)$ with respect to X. This latter is easily calculated from (33) to be

$$D_2R_\Omega(g, X)\Phi(B) = \langle g, \Phi\rangle_B X(\Omega) - \langle g, X\rangle\Phi(B) + \langle g, X\rangle_B\Phi(\Omega) - \langle g, \Phi\rangle X(B),$$

$B \in \mathcal{B}$, which is patently continuous (in fact, bilinear) in X and g by Lemma 2. Since R_Ω is continuous, this shows that R_Ω is C^1 and hence that the Replicator dynamics (35) is locally

Lipschitz. Since the multiplying factors $a_\Omega(g, X)$ are assumed to be locally Lipschitz, it follows that the Generalized Replicator operators $\hat{R}_\Omega(g, X)$ are also locally Lipschitz, and in fact C^1 if the $a_\Omega(g, X)$ are C^1. $\square$

DEFINITION 4. An operator Q_Ω is said to be *positive* on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ if, for each $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$,

$$\langle g, Q_\Omega(g, X) \rangle \geq 0, \tag{41}$$

and Q_Ω is said to be *positive definite* on a subset $\mathcal{S} \subseteq \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$, if it is positive and equality holds in (41) for $(g, X) \in \mathcal{S}$ if and only if $\langle g, X \rangle_B = 0$ for all $B \in \mathcal{B}$; *i.e.* if and only if $g = 0$ X-a.e.

The significance of positive-definite conditions has been recognised by Hofbauer and Sigmund (1990), Hopkins (1999), and applied by Hopkins and Seymour (1999).

Now suppose player-1 receives constant payoffs; *i.e.* $\pi_1(\xi, \eta) = \pi_1(\eta)$ for each pure-strategy pair $(\xi, \eta) \in \Omega_1 \times \Omega_2$. Then w_Y = constant. However, if player-1 is receiving strategy-independent payoffs, he has no incentive to update any particular strategy he happens to be using, so in this case we expect $\frac{dX}{dt} = 0$. In view of this, it is reasonable to require our operators to satisfy

$$Q_\Omega(c, X) = 0 \text{ for any constant } c \text{ and each } X \in \mathcal{P}[\Omega, \mathcal{B}]. \tag{42}$$

It follows that equality must hold in (41) when g is constant. However, the only constant function satisfying the positive definite condition is $g = 0$, so that Q_Ω cannot be positive-definite on any set $\mathcal{S}$ containing a pair (g, X) with $X \in \mathcal{P}[\Omega, \mathcal{B}]$ and g a non-zero constant.

On the other hand, given any (g, X), we may represent g uniquely in the form

$$g = g_X + c_X \tag{43}$$

where $\langle g_X, X \rangle = 0$, and c_X is the constant function with value $\langle g, X \rangle$. We may therefore form the derived set $\mathcal{S}' = \{(g_X, X) \mid (g, X) \in \mathcal{S}\}$. Then the only pair $(g, X) \in \mathcal{S}'$ with g constant and $X \in \mathcal{P}[\Omega, \mathcal{B}]$ must have $g = 0$.

A further obvious constraint on the game dynamics (32) arises from the relation $X(\Omega) = 1$ for $X \in \mathcal{P}[\Omega, \mathcal{B}]$. Thus,

$$\frac{dX}{dt}(\Omega) = \frac{d}{dt}\big[X(\Omega)\big] = 0,$$

is the condition for the preservation through time of this normalization condition. We therefore require the operators Q_Ω in (31) to satisfy

$$Q_\Omega(g, X)(\Omega) = \langle 1, Q_\Omega(g, X)\rangle = 0 \text{ for each } (g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]. \tag{44}$$

In view of the above discussion, we consider the class of operators defined as follows.

DEFINITION 5. Let

$$\mathcal{S}_\Omega = \big\{(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}] \mid \langle g, X\rangle = 0\big\}. \tag{45}$$

Then $\mathcal{S}'_\Omega = \mathcal{S}_\Omega$. A family of operators (31) is called *positive definite* if (42) and (44) hold, and Q_Ω is positive on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$, and positive-definite on $\mathcal{S}_\Omega$ in the sense of (41). The dynamics (32) associated with a positive-definite family of operators is called a *Positive-definite-Adaptive*, or *PDA*, dynamics.

The important properties of positive-definite operators are summarized in the following proposition, the proof of which follows easily from the above discussion.

PROPOSITION 8. Let Q_Ω be a locally-Lipschitz family of operators (31) which are positive definite. Then

a) $\langle g, Q_\Omega(g, X)\rangle \geq 0$ for each $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.

b) $\langle c, Q_\Omega(g, X)\rangle = 0$, if $c =$ constant.

c) Equality holds in (a) if and only if g is constant X-a.e., and then $g = \langle g, X\rangle$ X-a.e.

EXAMPLE 4. *The Replicator Dynamics.* For $X \in \mathcal{P}[\Omega, \mathcal{B}]$, it follows from (34) that

$$\langle g, R_\Omega(g, X)\rangle = \langle g^2, X\rangle - \langle g, X\rangle^2 = \langle \big(g - \langle g, X\rangle\big)^2, X\rangle, \tag{46}$$

is the variance of g with respect to X. Thus, $\langle g, Q_\Omega(g,X)\rangle \geq 0$. Note that equality holds if and only if $X\big(\{\omega \mid g(\omega) \neq \langle g, X\rangle\}\big) = 0$; *i.e.* if and only if g is constant X-a.e. For the generalized Replicator operators (see Example 3), we have $\langle g, \hat{R}_\Omega(g,X)\rangle = a_\Omega(g,X)\langle g, R_\Omega(g,X)\rangle \geq a_*\langle g, R_\Omega(g,X)\rangle \geq 0$, with equality if and only if $\langle g, R_\Omega(g,X)\rangle = 0$, since $a_* > 0$.

From (33), we have, for any X-a.e. constant function c,

$$R_\Omega(c,X)(B) = c\big\{X(B)X(\Omega) - X(\Omega)X(B)\big\} = 0,$$

which verifies (42).[11] Similarly, $\hat{R}_\Omega(c,X) = a_\Omega(c,X)R_\Omega(c,X) = 0$. Again, $\hat{R}_\Omega(g,X)(\Omega) = a_\Omega(g,X)R_\Omega(g,X)(\Omega) = 0$ from (33). It therefore follows that the Replicator and Generalized Replicator families are positive definite in the sense of Definition 3. □

The general significance of a positive definite condition (41) on the operators Q_i in (32) arises from the fact that strategies tend to evolve towards better responses under PDA dynamics. That is, a PDA dynamics is *improving*, asis shown in the following proposition.

PROPOSITION 9. Let $e = (X, Y)$ be a fixed pair a mixed strategies satisfying

$$\langle w_Y, Q_1(w_Y, X)\rangle > 0, \tag{47}$$

and let $\mathcal{F}_t(e) = (X_t(e), Y_t(e))$ be the flow under the PDA dynamics (32) with initial condition e. Then there is a small time interval, $0 < t < \varepsilon$, for which $X_t(e)$ is a strictly better response to Y than X. Similarly, if $\langle v_X, Q_2(v_X, Y)\rangle > 0$, then $Y_t(e)$ is a strictly better response to X than Y.

Proof. Assume that (47) holds. Then w_Y cannot be zero X-a.e., so that $||w_Y|| > 0$. Thus, there is a sufficiently small $\eta > 0$ for which

$$\langle w_Y, Q_1(w_Y, X)\rangle \geq \eta||w_Y|| > 0. \tag{48}$$

[11] Note that $\langle cg, X\rangle = c\langle g, X\rangle$ for any bounded $\mathcal{B}$-measurable function g and X-a.e. constant function c, because the measure $\langle g, X\rangle_B$ is X-absolutely continuous. See footnote 7.

By continuity of Q_1 and of the flow $\mathcal{F}_t(e)$, there is an interval $0 \leq t < \varepsilon$, for which

$$\big|\big|Q_1\big(w_{Y_t(e)}, X_t(e)\big) - Q_1(w_Y, X)\big|\big|$$
$$= \sup_{||g|| \leq 1} \big|\langle g, Q_1\big(w_{Y_t(e)}, X_t(e)\big)\rangle - \langle g, Q_1(w_Y, X)\rangle\big| < \frac{1}{2}\eta.$$

This, together with (48), implies that

$$\langle w_Y, Q_1\big(w_{Y_t(e)}, X_t(e)\big)\rangle \geq \frac{1}{2}\eta||w_Y|| > 0,$$

for each $t \in [0, \varepsilon)$. Now, by the Mean Value Theorem, for each $t \in (0, \varepsilon)$ there is an $s \in (0, t)$ such that,

$$\langle w_Y, X_t(e)\rangle - \langle w_Y, X\rangle = t\langle w_Y, Q_1\big(w_{Y_s(e)}, X_s(e)\big)\rangle \geq \frac{1}{2}\eta||w_Y||t > 0.$$

This shows that, for small values of $t > 0$, $X_t(e)$ is a strictly better response to Y than was X. Thus, if Q_1 satisfies (47), then X evolves in a short time to a better response to Y. $\square$

We have shown that the essential characteristic of positive-definite dynamics is that strategies evolve toward better responses. Clearly, similar arguments apply to Q_2, so that player-2's strategy also evolves toward better responses.

Mutations We can extend the basic form of the dynamics (32) by adding *mutations*. The idea is that matched players who intend (or are programmed) to play a mixed strategy pair, (X, Y), occasionally make 'mistakes' (mutations). These mistakes come with their own fixed, exogenous distributions, $\Theta_i \in \mathcal{P}[\Omega_i, \mathcal{B}_i]$; *i.e.* instead of using the intended mixed strategy X, player-1 sometimes uses the mixed strategy Θ_1, and similarly for player-2. Mistakes for player-i occur with a (usually small) probability $\delta_i \in [0, 1]$. This process has the effect of amending the dynamics (32) to the dynamics defined by

$$\left.\begin{array}{l} \frac{dX}{dt} = \mathcal{X}(X, Y) = (1 - \delta_1)\mathcal{X}_0(X, Y) + \delta_1\big(\Theta_1 - X\big), \\ \frac{dY}{dt} = \mathcal{Y}(X, Y) = (1 - \delta_2)\mathcal{Y}_0(X, Y) + \delta_2\big(\Theta_2 - Y\big). \end{array}\right\} \tag{49}$$

An alternative interpretation of the mutation terms in (49) is that players are leaving their respective populations with probabilities δ_1 and δ_2, and being replaced by naive (inexperienced) players, who bring with them the exogenously determined mixed strategies Θ_i with which they begin playing the game, before any game-specific learning takes place. In this interpretation, the Θ_i can be regarded as distributions over propensities to play pure strategies, conditioned by general cultural influences in the societies from which the players are chosen. One can also think in terms of births and deaths rather than replacement.

We remark that the probabilities δ_i in (49) need not be constants. In fact, we can take them to be endogenously determined by real-valued functions, $\delta_i = \delta_i(X, Y) : \mathcal{E} \to [0, 1]$, which are locally Lipschitz. The mutation terms in (49) are then also locally Lipschitz. Hence, if $\Xi_0 = (\mathcal{X}_0, \mathcal{Y}_0)$ is locally Lipschitz, so is $\Xi = (\mathcal{X}, \mathcal{Y})$. Thus, the theory developed so far applies to the mutation-augmented dynamics defined by (49).

We now show that , provided certain uniformity conditions hold, trajectories of (49) exist for all time $t \geq 0$. We consider two uniformity conditions, as follows.

DEFINITION 6. Let $Q_\Omega : \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}] \to \mathcal{M}[\Omega, \mathcal{B}]$ be a locally Lipschitz family of operators. We say that Q_Ω is *uniformly bounded* on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ if there is a continuous non-decreasing function $\gamma_\Omega : [0, \infty) \to [0, \infty)$, such that $||Q_\Omega(g, X)|| \leq \gamma_\Omega\big(||g||\big)$ for all $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.

DEFINITION 7. Suppose $Q_\Omega : \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}] \to \mathcal{M}[\Omega, \mathcal{B}]$ is C^k for $k \geq 1$. We say that Q_Ω is C^k*-uniformly bounded* on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ if there is a continuous non-decreasing function $\gamma^k_\Omega : [0, \infty) \to [0, \infty)$, such that $max\{||Q_\Omega(g, X)||, ||DQ_\Omega(g, X)||, \ldots, ||D^k Q_\Omega(g, X)||\} \leq \gamma^k_\Omega\big(||g||\big)$ for all $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.

It follows from Corollary 7 that C^k-uniformly bounded implies uniformly bounded.

For example, if $Q_\Omega(g, X)$ is linear in g, then we can take $\gamma_\Omega(x) = Kx$, where the constant $K = sup\{||Q_\Omega(g, X)|| \mid (g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ and $||g|| \leq 1\}$, provided $K < \infty$. Similarly, we can define $\gamma^k_\Omega(x) = K^k x$ with $K^k = sup\{||Q_\Omega(g, X)|| + ||DQ_\Omega(g, X)|| + \ldots +$

$||D^k Q_\Omega(g, X)|| \mid (g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ and $||g|| \leq 1\}$, provided $K^k < \infty$. We shall show later that the Generalized Replicator family is C^1-uniformly bounded on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ (see the proof of Proposition 11 below).

PROPOSITION 10. Let $\mathcal{K} = \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2] \subset \mathcal{E}$. Suppose that

a) Q_Ω is locally Lipschitz on $\mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}]$.

b) $Q_\Omega(g, X)(B) \geq 0$ whenever $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$ and $X(B) = 0$.[12]

c) $Q_\Omega(g, X)(\Omega) = 0$ for each $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.[13]

d) Q_Ω is uniformly bounded on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.

e) $\delta_i : \mathcal{E} \to [0, 1]$ are locally Lipschitz functions $(i = 1, 2)$.[14]

Then $\mathcal{K}$ is forward invariant under the dynamics defined by (49). Hence, trajectories beginning in $\mathcal{K}$ exist for all $t \geq 0$.

Proof. By Lemma 1, $\mathcal{K}$ is closed, bounded and convex. The theorem therefore follows from Propositions 5 and 6, provided we can show that $\mathcal{K}$ is forward invariant and that Ξ is uniformly bounded on $\mathcal{K}$.

Consider an initial condition $e_0 = (X_0, Y_0) \in \mathcal{K}$. Then $\Theta_1(\Omega_1) = X_0(\Omega_1) = 1$. Hence

$$\mathcal{X}(X_0, Y_0)(\Omega_1) = (1 - \delta_1)\mathcal{X}_0(X_0, Y_0)(\Omega_1) = (1 - \delta_1)Q_1(w_{Y_0}, X_0)(\Omega_1) = 0,$$

by condition (c). Similarly, $\mathcal{Y}(X_0, Y_0)(\Omega_2) = 0$.

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[12] This property is weaker than the requirement that $Q_\Omega(g, X)$ be X-absolutely continuous for each g. The latter would require $Q_\Omega(g, X)(B) = 0$ whenever $X(B) = 0$. In fact, if strict inequality holds for some B, then previously unused strategies can be brought into use by the dynamics. This is not possible if equality holds for all B.

[13] That is, property (44) holds.

[14] For example, if δ_i is constant.

Since $e_0 \in \mathcal{K}$, we have $X_0(B) \geq 0$ for all $B \in \mathcal{B}_1$ (*i.e.* X_0 is a probability measure). Suppose that $X_0(B) = 0$ for some B. Then

$$\mathcal{X}(X_0, Y_0)(B) = (1 - \delta_1)Q_1(w_{Y_0}, X_0)(B) + \delta_1\Theta_1(B) \geq \delta_1\Theta_1(B) \geq 0,$$

by condition (b). It follows that $X_t(B)$ can only increase in a short time interval $t \geq 0$. In particular, it cannot become negative. Similarly if $Y_0(B) = 0$ for some $B \in \mathcal{B}_2$.

The above arguments show that $\mathcal{F}_t(e_0) \in \mathcal{K}$ for t in some non-empty interval $[0, \gamma)$. But this clearly implies that $\mathcal{F}_t(e_0) \in \mathcal{K}$ for all $t \geq 0$ for which it is defined, since we have actually shown that, once in $\mathcal{K}$, a forward time trajectory can never leave $\mathcal{K}$.

It remains to show that Ξ is uniformly bounded on $\mathcal{K}$. From Lemma 1 and (e), the function $\delta_1(X, Y)\big(\Theta_1 - X\big)$ is uniformly bounded on $\mathcal{K}$. On the other hand, by (d),

$$||Q_1(w_Y, X)|| \leq \gamma_1\big(||w_Y||\big),$$

where γ_1 is as in Definition 6. But, from (14), $||w_Y|| \leq \pi_1^* < \infty$ for $Y \in \mathcal{P}[\Omega_2, \mathcal{B}_2]$, so that $Q_1(w_Y, X)$ is uniformly bounded on $\mathcal{K}$. The result therefore follows from this and a similar argument for $Q_2(v_X, Y)$. □

We apply Proposition 10 to the mutation-augmented (Generalized) Replicator dynamics defined by the operators (33).

PROPOSITION 11. The Replicator family of operators (33) satisfy conditions (a) to (d) of Proposition 10. Hence, if the δ_i satisfy condition (e), the conclusions of Proposition 10 apply to the mutation-augmented Generalized Replicator dynamics,

$$\left.\begin{aligned} \tfrac{dX}{dt}(B_1) &= (1 - \delta_1)a_1\big[\langle w_Y, X\rangle_{B_1} - \langle w_Y, X\rangle_{\Omega_1}X(B_1)\big] + \delta_1\big[\Theta_1(B_1) - X(B_1)\big], \\ \tfrac{dY}{dt}(B_2) &= (1 - \delta_2)a_2\big[\langle v_X, Y\rangle_{B_2} - \langle v_X, Y\rangle_{\Omega_2}Y(B_2)\big] + \delta_2\big[\Theta_2(B_2) - Y(B_2)\big], \end{aligned}\right\} \tag{50}$$

with $B_i \in \mathcal{B}_i$, $\delta_i = \delta_i(X, Y)$ and $a_1 = a_1(w_Y, X), a_2 = a_2(v_X, Y)$.

Proof. a) is proved in Example 3.

b) In fact $X(B) = 0$ implies $R_\Omega(g, X)(B) = 0$. This follows immediately from the definition (33) and the fact that $\langle g, X\rangle_B$ is X-absolutely continuous.[15]

c) This is immediate from (33).

d) To show that $\hat{R}_\Omega(g, X)$ is uniformly bounded on $B\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$, we have $||\hat{R}_\Omega(g, X)|| \leq a^*||R_\Omega(g, X)||$ where $a_\Omega : \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, X] \to [a_*, a^*]$. It therefore suffices to show that $R_\Omega(g, X)$ is uniformly bounded (Definition 6). In fact, we show that $R_\Omega(g, X)$ is C^1-uniformly bounded (Definition 7).

In Example 3 it is shown that the Replicator operators are C^1. Using the formula in Example 3, we have $||DR_\Omega(g, X)|| \leq ||R_\Omega(g, X)|| + ||D_2R_\Omega(g, X)||$. Hence, $max\{||R_\Omega(g, X)||, ||DR_\Omega(g, X)||\} \leq ||R_\Omega(g, X)|| + ||D_2R_\Omega(g, X)||$. Now,

$$\begin{aligned} ||R_\Omega(g, X)|| &= \sup_{||h||\leq 1} \left|\langle gh, X\rangle - \langle g, X\rangle\langle h, X\rangle\right| \\ &\leq ||g|| \cdot \left\{||X|| + ||X||^2\right\} \\ &= 2||g||. \end{aligned}$$

Similarly,

$$\begin{aligned} ||D_2R_\Omega(g, X)|| &= \sup_{||\Phi||\leq 1} \left|\left|D_2R_\Omega(g, X)\Phi\right|\right| \\ &= \sup_{||\Phi||\leq 1} \sup_{||h||\leq 1} \left|\langle hg, \Phi\rangle - \langle g, X\rangle\langle h, \Phi\rangle + \langle hg, X\rangle - \langle g, \Phi\rangle\langle h, X\rangle\right| \\ &\leq ||g|| \cdot \sup_{||\Phi||=1} \Big\{||\Phi|| + ||X||.||\Phi|| + ||X|| + ||\Phi||.||X||\Big\} \\ &= 4||g||. \end{aligned}$$

Hence, $max\{||R_\Omega(g, X)||, ||DR_\Omega(g, X)||\} \leq 4||g||$ for all $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$, which shows that R_Ω is C^1-uniformly bounded on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.

For the Generalized Replicator dynamics, we have $\hat{R}_\Omega(g, X) = a_\Omega(g, X)R_\Omega(g, X)$. Hence, if a_Ω is C^1,

$$D\hat{R}_\Omega(g, W)(\phi, \Phi) = Da_\Omega(g, X)(\phi, \Phi) \cdot R_\Omega(g, X) + a_\Omega(g, X) \cdot DR_\Omega(g, X)(\phi, \Phi).$$

[15] See footnote 8.

The C^1-uniform boundedness of $\hat{R}_\Omega$ therefore follows by an argument like that above, provided a_Ω is also C^1-uniformly bounded on $\mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$. □

3. Properties of equilibria

We first consider the dynamics (32) defined by a family of continuous operators Q_Ω, without mutations. We shall relate the notion of Nash equilbrium of the underlying game to the notion of an equilibrium (stationary point) of the dynamics (32).

Throughout this section, Q_Ω will be a family of locally Lipschitz operators (31) satisfying the conditions of Proposition 10.

PROPOSITION 12. Suppose Q_Ω satisfies:

a) $Q_\Omega(g, X)$ is X-a.c. for each $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{P}[\Omega, \mathcal{B}]$.

b) $Q_\Omega(g, X) = 0$ whenever $X \in \mathcal{P}[\Omega, \mathcal{B}]$ and $g =$ constant X-a.e.[16]

Then a Nash equilibrium of the underlying game is an equilibrium of the dynamics (32) defined by the Q_Ω.

Proof. Let $(\hat{X}, \hat{Y})$ be a Nash equilibrium of the underlying game. It follows from Proposition 4 that $w_{\hat{Y}} = \langle w_{\hat{Y}}, \hat{X} \rangle$, $\hat{X}$-a.e., so that $Q_1(w_{\hat{Y}}, \hat{X}) = 0$ by condition (b). A similar argument shows that $Q_2(v_{\hat{X}}, \hat{Y}) = 0$, and we conclude that $(\hat{X}, \hat{Y})$ is an equilibrium of the dynamics (32). □

Note that condition (a) is not needed in the proof of Proposition 12. It is inserted there for the convenience of future reference.

The converse of Proposition 12 is false; *i.e.* there are dynamic equilibria which are not Nash equilibria. In particular, consider the injective set function (which is not continuous),

[16] If Q_Ω is positive definite, and $g =$ constant X-a.e., then $\langle g, Q_\Omega(g, X) \rangle = 0$, by Proposition 8(c). Condition (b) is stronger than this.

$\Delta \times \Delta : \Omega_1 \times \Omega_2 \to \mathcal{K}$, which identifies the pure-strategy pair (ξ, η) with the mixed-strategy pair $\big(\Delta(\xi), \Delta(\eta)\big)$ (see (10) and (11)). That each of these pure strategy pairs is an equilibrium of the dynamics (32) follows easily from condition (a) of Proposition 12 (in fact, any bounded, $\mathcal{B}$-measurable function is constant $\Delta(\xi)$-a.e.). However, as was shown for the Ultimatum Game in Example 2, not every pair of pure strategies is a Nash equilibrium.

Nevertheless, as we shall show below, it is true that every asymptotically stable dynamic equilibrium is a NE. Before giving our detailed results, we need to discuss what is meant by asymptotic stability in the infinite-dimensional context. Here we shall consider only the most obvious *strong* notion of stability.

DEFINITION 8. Consider $\mathcal{K} = \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2]$ equipped with the (strong) topology induced by the norm metric $||(X, Y) - (X', Y')|| = max\,\{||X - X'||, ||Y - Y'||\}$. A dynamic equilibrium $(\hat{X}, \hat{Y})$ of the PDE dynamic (32) is *locally asymptotically stable* if, given any (strong) neighbourhood U of $(\hat{X}, \hat{Y})$ in $\mathcal{K}$, there exists a (strong) neighbourhood $V \subseteq U$ of $(\hat{X}, \hat{Y})$ such that:

i) $(X_t, Y_t) \in U$ for all $t \geq 0$ and all initial conditions $(X_0, Y_0) \in V$;

ii) $||(X_t, Y_t) - (\hat{X}, \hat{Y})|| \to 0$ as $t \to \infty$ for all initial conditions $(X_0, Y_0) \in V$.

Unfortunaltely this obvious notion of asymptotic stability turns out to be *very* strong, and it is unclear if it is the most suitable definition for use in infinite dimensions. There are weaker notions which also have an intuitive appeal. However, we shall be concerned only with the strong notion defined above in what follows. We begin by generalising to infinite dimensions a result of Samuelson (1997)[17]. First, recall the important property of *pairwise singularity* of measures which we will use often in this section. Thus, $X, X' \in \mathcal{M}[\Omega, \mathcal{B}]$ are mutually singular, written $X \perp X'$, if there exits subsets $B, B' \in \mathcal{B}$ with the following properties:

$$(a)\;\; B \cap B' = \emptyset; \quad (b)\;\; B \cup B' = \Omega; \quad (c)\;\; X'(B) = X(B') = 0. \tag{21}$$

[17] The strictness conditions given in Proposition 13 are the infinite-dimensional analogues of similar conditions given in Samuelson (1997), Proposition 4.3, p 119. See also Samuelson and Zhang (1992).

For example, $\Delta(\xi) \perp \Delta(\xi')$ for $\xi \neq \xi'$.

PROPOSITION 13. Suppose that Q_Ω is positive definite, and that the assumptions of Proposition 12 hold. If $(\hat{X}, \hat{Y})$ is an (locally) *asymptotically stable* equilibrium of the game dynamics (32), then $(\hat{X}, \hat{Y})$ is a Nash equilibrium of the underlying game. Furthermore, $(\hat{X}, \hat{Y})$ satisfies the following strictness conditions.

i) If $\langle w_{\hat{Y}}, X\rangle = \langle w_{\hat{Y}}, \hat{X}\rangle$ for some $X \in \mathcal{P}[\Omega_1, \mathcal{B}_1]$ with $X \neq \hat{X}$, then $v_X \neq \langle v_X, \hat{Y}\rangle$ $\hat{Y}$-a.e.

ii) If $\langle v_{\hat{X}}, Y\rangle = \langle v_{\hat{X}}, \hat{Y}\rangle$ for some $Y \in \mathcal{P}[\Omega_2, \mathcal{B}_2]$ with $Y \neq \hat{Y}$, then $w_Y \neq \langle w_Y, \hat{X}\rangle$ $\hat{X}$-a.e.

Proof. Suppose that $(\hat{X}, \hat{Y})$ is not a Nash equilibrium, and suppose that there exists $X \in \mathcal{P}[\Omega_1, \mathcal{B}_1]$ such that

$$\langle w_{\hat{Y}}, X\rangle > \langle w_{\hat{Y}}, \hat{X}\rangle. \tag{51}$$

Since $Q_1(w_{\hat{Y}}, \hat{X}) = 0$, it follows that $\langle w_{\hat{Y}}, Q_1(w_{\hat{Y}}, \hat{X})\rangle = 0$. This implies that $w_{\hat{Y}} = \langle w_{\hat{Y}}, \hat{X}\rangle$, $\hat{X}$-a.e., by Proposition 8(c).

Let $\hat{w} = w_{\hat{Y}} - \langle w_{\hat{Y}}, \hat{X}\rangle$, so that $\hat{w} = 0$ $\hat{X}$-a.e., and let $B_0 = \{\xi \in \Omega_1 \mid \hat{w} = 0\} \in \mathcal{B}_1$. Setting $B_1 = \Omega_1 - B_0$, we have

$$\Omega_1 = B_0 \cup B_1, \quad B_0 \cap B_1 = \emptyset, \quad \hat{X}(B_0) = 1, \quad \hat{X}(B_1) = 0, \quad \hat{w} = 0 \text{ on } B_0. \tag{52}$$

Thus, (51) can be written

$$\langle \hat{w}, X\rangle = \langle \hat{w}, X\rangle_{B_1} > 0. \tag{53}$$

Now, if $X(B_1) = 0$, it follows that $\langle \hat{w}, X\rangle_{B_1} = 0$ by absolute continuity[18], and we have a contradiction to (53). Hence $X(B_1) > 0$.

Define a measure $X' \in \mathcal{P}[\Omega_1, \mathcal{B}_1]$ by

$$X'(B) = \frac{X(B \cap B_1)}{X(B_1)}, \quad (B \in \mathcal{B}_1).$$

[18] See footnote 8.

It then follows from (52) that $X' \perp \hat{X}$; *i.e.* $X'(B_0) = 0$ and $X'(B_1) = 1$, and from (53) that

$$\langle \hat{w}, X' \rangle = \frac{\langle \hat{w}, X \rangle_{B_1}}{X(B_1)} > 0. \tag{54}$$

Consider the perturbed measure $\tilde{X} = (1-\varepsilon)\hat{X} + \varepsilon X' \in \mathcal{P}[\Omega_1, \mathcal{B}_1]$, with $\varepsilon > 0$ small. Then (54) implies that $\langle \hat{w}, \tilde{X} - \hat{X} \rangle = \langle \hat{w}, \tilde{X} \rangle = \varepsilon \langle \hat{w}, X' \rangle > 0$, which translates to

$$\langle w_{\hat{Y}}, \tilde{X} \rangle > \langle w_{\hat{Y}}, \hat{X} \rangle. \tag{55}$$

Also, at $(\tilde{X}, \hat{Y})$, we have

$$\frac{d}{dt}(\tilde{X} - \hat{X}) = \frac{d\tilde{X}}{dt} = Q_1(w_{\hat{Y}}, \tilde{X}).$$

Thus,

$$\frac{d}{dt}\left[\langle \hat{w}, \tilde{X} - \hat{X} \rangle\right] = \langle \hat{w}, \frac{d}{dt}(\tilde{X} - \hat{X}) \rangle = \langle \hat{w}, Q_1(w_{\hat{Y}}, \tilde{X}) \rangle \geq 0, \tag{56}$$

since Q_Ω is positive definite (Proposition 8(a,b)). Suppose equality holds in (56). Then, by Proposition 8(c), $w_{\hat{Y}} = \langle w_{\hat{Y}}, \tilde{X} \rangle$ $\tilde{X}$-a.e. Hence, by (55), $\hat{w}$ is strictly positive $\tilde{X}$-a.e. But, by construction, $\hat{w} = 0$ on B_0, and $\tilde{X}(B_0) = (1-\varepsilon)\hat{X}(B_0) = (1-\varepsilon) > 0$. This gives a contradiction, and we conclude that strict inequality holds in (56).

It now follows that $\langle \hat{w}, \tilde{X} - \hat{X} \rangle$ is strictly increasing for small times. However, if $(\hat{X}, \hat{Y})$ is locally asymptotically stable, then, in view of (55), $\langle \hat{w}, \tilde{X} - \hat{X} \rangle$ must be non-increasing for sufficiently small ε. We therefore have a contradiction.

A similar argument applies if we assume the existence of a $Y \in \mathcal{P}[\Omega_2, \mathcal{B}_2]$ which violates the Nash equilibrium condition; *i.e.* such that $\langle v_{\hat{X}}, Y \rangle > \langle v_{\hat{X}}, \hat{Y} \rangle$. We have therefore shown that $(\hat{X}, \hat{Y})$ must be a Nash equilibrium.

Now suppose there is an $X \neq \hat{X}$ which is an alternative best reply to $\hat{Y}$; *i.e.* such that $\langle w_{\hat{Y}}, X \rangle = \langle w_{\hat{Y}}, \hat{X} \rangle$. Suppose that $v_X =$ constant $\hat{Y}$-a.e. Form the perturbed strategy $\tilde{X} = (1-\varepsilon)\hat{X} + \varepsilon X$. Then $v_{\tilde{X}} = (1-\varepsilon)v_{\hat{X}} + \varepsilon v_X =$ constant $\hat{Y}$-a.e. By Proposition 12(b), it follws that $Q_2(v_{\tilde{X}}, \hat{Y}) = 0$. On the other hand, $\langle w_{\hat{Y}}, \tilde{X} \rangle = \langle w_{\hat{Y}}, \hat{X} \rangle$, and

$$\frac{d}{dt}\left[\langle w_{\hat{Y}}, \tilde{X} \rangle\right] = \langle w_{\hat{Y}}, Q_1(w_{\hat{Y}}, \tilde{X}) \rangle \geq 0.$$

If the strict inequality holds, then, since Q_Ω is positive definite, $\tilde{X}$ evolves in a short time interval to a strictly better response to $\hat{Y}$ by Proposition 9. But, since $\tilde{X}$ is a best reply to $\hat{Y}$, this is a contradiction. Thus, equality holds, and again by positive definiteness, $w_{\hat{Y}} = \langle w_{\hat{Y}}, \tilde{X}\rangle$ $\tilde{X}$-a.e., and hence $Q_1(w_{\hat{Y}}, \tilde{X}) = 0$. This shows that $(\tilde{X}, \hat{Y})$ is an equilibrium of the dynamics for each $\varepsilon \geq 0$. Hence, $(\hat{X}, \hat{Y})$ cannot be asymptotically stable. We therefore have a contradiction, and the strictness condition (i) is established.

Condition (ii) is proved similarly. □

The strictness condition (i) in Proposition 13, says that, if X is an alternative best reply to $\hat{Y}$, then player-2, using the pure strategies chosen with positive probability under $\hat{Y}$, can distinguish between X and $\hat{X}$. That is, there exists $B_2 \in \mathcal{B}_2$, with $\hat{Y}(B_2) > 0$, such that

$$\langle v_X, \hat{Y}\rangle_{B_2} \neq \langle v_X, \hat{Y}\rangle \hat{Y}(B_2), \quad \text{but} \quad \langle v_{\hat{X}}, \hat{Y}\rangle_{B_2} = \langle v_{\hat{X}}, \hat{Y}\rangle \hat{Y}(B_2).$$

[The latter equality by (18).] A similar interpretation applies to condition (ii). In particular, if $(\hat{X}, \hat{Y}) = \big(\Delta(\hat{\xi}), \Delta(\hat{\eta})\big)$ is a pure-strategy NE which is asymptotically stable, then it follows easily from conditions (i) and (ii) that $(\hat{\xi}, \hat{\eta})$ must be a *strict* NE.

EXAMPLE 6. Consider the Sub-Ultimatum game defined in Example 1. As shown in Example 2, this game has no pure-strategy Nash equilibria at all, and hence cannot have any (pure) asymptotically stable states under any (well-behaved) PDA dynamics. □

We will prove a partial converse of Proposition 13, namely that (with further restrictions) a SSNE is locally asymptotically stable. For this we shall need to make the stronger assumption that Q_Ω is C^1_b, and we first obtain an important property of the derivative in this case.

Suppose Q_Ω is C^1_b, and that $(\hat{X}, \hat{Y}) \in \mathcal{K} = \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2]$ is a dynamic equilibrium of the system (32). We define the operator, $\mathcal{Q}(\hat{X}, \hat{Y}) : \mathcal{E} \to I\!\!R$, by

$$\mathcal{Q}(\hat{X}, \hat{Y})(\Theta, \Phi) = min\left\{\langle w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, \hat{X})\Theta\rangle, \langle v_{\hat{X}}, D_2 Q_2(v_{\hat{X}}, \hat{Y})\Phi\rangle\right\}. \tag{57}$$

The relevant properties of this operator are given in the following Lemma.

LEMMA 14. Let Q_Ω be C_b^1, and $(\hat{X},\hat{Y})$ be a dynamic equilibrium of (32). Then, $\mathcal{Q}(\hat{X},\hat{Y})(X',Y') \geq 0$ for any pair $(X',Y') \in \mathcal{K}$. Further, if X' is $\hat{X}$-a.c or Y' is $\hat{Y}$-a.c, then $\mathcal{Q}(\hat{X},\hat{Y})(X',Y') = 0$.

Proof. For $\varepsilon > 0$, let $L_\varepsilon(\hat{X},\hat{Y})X' = \frac{1}{\varepsilon}\big\{Q_1(w_{\hat{Y}},\hat{X}+\varepsilon X') - Q_1(w_{\hat{Y}},\hat{X})\big\}$, with $X' \in \mathcal{P}[\Omega_1,\mathcal{B}_1]$. Then $D_2Q_\Omega(g,\hat{X})X' = lim_{\varepsilon\to 0}\, L_\varepsilon(\hat{X},\hat{Y})X'$. Further, since $(\hat{X},\hat{Y})$ is a dynamic equilibrium, then $Q_1(w_{\hat{Y}},\hat{X}) = 0$ and $L_\varepsilon(\hat{X},\hat{Y})X' = \frac{1}{\varepsilon}Q_1(w_{\hat{Y}},\hat{X}+\varepsilon X')$. Again, since Q_1 is positive definite, $\langle w_{\hat{Y}}, Q_1(w_{\hat{Y}},\hat{X}+\varepsilon X')\rangle \geq 0$, whence

$$\langle w_{\hat{Y}}, D_2Q_1(w_{\hat{Y}},\hat{X})X'\rangle = \underset{\varepsilon\to 0}{lim}\langle w_{\hat{Y}}, L_\varepsilon(\hat{X},\hat{Y})X'\rangle \geq 0.$$

A similar argument shows that $\langle v_{\hat{X}}, D_2Q_2(v_{\hat{X}},\hat{Y})Y'\rangle \geq 0$ for each $Y' \in \mathcal{P}[\Omega_2,\mathcal{B}_2]$. Hence, $\mathcal{Q}(\hat{X},\hat{Y})(X',Y') \geq 0$.

Now suppose that X' is $\hat{X}$-a.c. Then $\hat{X}+\varepsilon X'$ is $\hat{X}$-a.c for each $\varepsilon > 0$. Since $w_{\hat{Y}} =$ constant $\hat{X}$-a.e., it follows that $w_{\hat{Y}} =$ constant $(\hat{X}+\varepsilon X')$-a.e. Furthermore, $Q_1(w_{\hat{Y}},\hat{X}+\varepsilon X')$ is $(\hat{X}+\varepsilon X')$-a.c. Hence, $\langle w_{\hat{Y}}, Q_1(w_{\hat{Y}},\hat{X}+\varepsilon X')\rangle =$ constant.$\langle 1, Q_1(w_{\hat{Y}},\hat{X}+\varepsilon X')\rangle = 0$. Thus, $\langle w_{\hat{Y}}, L_\varepsilon(\hat{X},\hat{Y})X'\rangle = 0$, and therefore $\langle w_{\hat{Y}}, D_2Q_1(w_{\hat{Y}},\hat{X})X'\rangle = lim_{\varepsilon\to 0}\langle w_{\hat{Y}}, L_\varepsilon(\hat{X},\hat{Y})X'\rangle$ $= 0$. Similarly, $\langle v_{\hat{X}}, D_2Q_2(v_{\hat{X}},\hat{Y})Y'\rangle = 0$ if Y' is $\hat{Y}$-a.c. Hence, $\mathcal{Q}(\hat{X},\hat{Y})(X',Y') = 0$ if either X' is $\hat{X}$-a.c or Y' is $\hat{Y}$-a.c. □

DEFINITION 9. Let Q_Ω be C_b^1, and $(\hat{X},\hat{Y})$ be a dynamic equilibrium of (32). We call $(\hat{X},\hat{Y})$ *strictly non-degenerate for* Q_Ω if there exists $\delta_1 > 0$ such that $\mathcal{Q}(\hat{X},\hat{Y})(X',Y') \geq \delta_1$ for all $(X',Y') \in \mathcal{K}$ with $X' \perp \hat{X}$ and $Y' \perp \hat{Y}$.

Before stating our main result on asymptotic stability, we first give a more convenient characterisation of local neighbourhoods in the strong (metric) topology. This is achieved in the next two lemmas.

LEMMA 15. Let $X, X' \in \mathcal{P}[\Omega,\mathcal{B}]$ with $X' \perp X$. Then $||X'-X|| = 2$.

Proof. Choose $B, B' \in \mathcal{B}$ with $B \cup B' = \Omega$, $B \cap B' = \emptyset$, and $X(B') = X'(B) = 0$. Let $f = I_{B'} - I_B$. Then $||f|| = 1$, and $|\langle f, X' - X\rangle| = 2$. Hence, $||X' - X|| \geq 2$. On the other hand, the triangle inequality gives $||X' - X|| \leq ||X'|| + ||X|| = 2$, since $\mathcal{P}[\Omega, \mathcal{B}]$ is a subset of the unit sphere in $\mathcal{M}[\Omega, \mathcal{B}]$. □

LEMMA 16. Let $\hat{X} = \Delta(\hat{\xi}) \in \mathcal{P}[\Omega, \mathcal{B}]$. Then any $X \in \mathcal{P}[\Omega, \mathcal{B}]$ with $X \neq \hat{X}$, may be uniquely decomposed in the form $X = (1-\alpha)\hat{X} + \alpha X'$, where $0 < \alpha \leq 1$, $X' \in \mathcal{P}[\Omega, \mathcal{B}]$ and $X' \perp \hat{X}$. Further, $\alpha = \frac{1}{2}||\hat{X} - X||$.

Proof. Write $X_0(B) = X(B \cap B_0)$ and $X_1(B) = X(B \cap B_1)$. Then $X = X_0 + X_1$, and both X_0 and $X_1$are non-negative measures with $X_0 \perp X_1$. Further, $\hat{X}(B) = 0$ if and only if $B \cap B_0 = \emptyset$, and in this case $X_0(B) = 0$. This shows that X_0 is $\hat{X}$-absolutely continuous, and in fact, $X_0 = a\hat{X}$, where $a = X(B_0) \geq 0$. By the Lebesgue decomposition theorem[19] in $\mathcal{M}[\Omega, \mathcal{B}]$, this decomposition of X is unique. Also, $a = X(B_0) \leq X(\Omega) = 1$, so that $0 \leq a \leq 1$. Setting $\alpha = 1 - a$, then $X(B_0) + X(B_1) = X(\Omega) = 1$ implies that $\alpha = X(B_1)$. If $\alpha = 0$, then $0 \leq X_1(B) = X(B \cap B_1) \leq X(B_1) = 0$, so that $X_1 = 0$, and $X = \hat{X}$. If $\alpha > 0$, write $X' = \alpha^{-1}X_1$. Then $X' \geq 0$ and $X'(\Omega) = \alpha^{-1}X(B_1) = \alpha^{-1}\alpha = 1$. Hence, $X = (1-\alpha)\hat{X} + \alpha X'$ with $X' \in \mathcal{P}[\Omega, \mathcal{B}]$ and $X' \perp \hat{X}$.

For the last part, we have $\hat{X} - X = \alpha(\hat{X} - X')$, and hence $||\hat{X} - X|| = 2\alpha$ by Lemma 15.
□

DEFINITION 10. Let $\hat{X}^{\perp} = \{X' \in \mathcal{P}[\Omega, \mathcal{B}] : X' \perp \hat{X}\}$. We define the *strong* ε-*neighbourhood* of $\hat{X}$ to be

$$N_\varepsilon(\hat{X}) = \left\{(1-\alpha)\hat{X} + \alpha X' : X' \in \hat{X}^{\perp} \text{ and } 0 \leq \alpha < \frac{1}{2}\varepsilon\right\}. \tag{58}$$

It then follows from Lemma 16 that, for $X \in N_\varepsilon(\hat{X})$, $||X - \hat{X}|| = 2\alpha < \varepsilon$. Thus, $N_\varepsilon(\hat{X}) = \left\{X \in \mathcal{P}[\Omega, \mathcal{B}] : ||X - \hat{X}|| < \varepsilon\right\}$. That is, $N_\varepsilon(\hat{X})$ coincides with the usual notion of the open ε-ball in the norm-induced topology on $\mathcal{P}[\Omega, \mathcal{B}]$.

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[19] See Dunford and Schwartz, 1958, Theorem 14, p 132.

LEMMA 17. If $(\hat{X}, \hat{Y}) = (\Delta(\hat{\xi}, \hat{\eta})$ is a SSNE (satisfying (22)), then

$$min\left\{\langle w_{\hat{Y}}, \hat{X} - X'\rangle, \langle v_{\hat{X}}, \hat{Y} - Y'\rangle\right\} \geq \delta_0, \tag{59}$$

for all $(X', Y') \in \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2]$ with $X' \perp \hat{X}$ and $Y' \perp \hat{Y}$.

Proof. The first condition (22) may be written as, $\langle w_{\hat{Y}}, \hat{X}\rangle - w_{\hat{Y}}(\xi) \geq \delta_0 \left(1 - I_{\{\hat{\xi}\}}(\xi)\right)$ for all $\xi \in \Omega_1$. Now apply $\langle \cdot, X'\rangle$ to this inequality, and note that $\langle I_{\{\hat{\xi}\}}, X'\rangle = 0$ because $\hat{X} \perp X'$ implies $X'(\{\hat{\xi}\}) = 0$. We then obtain $\langle w_{\hat{Y}}, \hat{X} - X'\rangle \geq \delta_0$. A similar argument shows that $\langle v_{\hat{X}}, \hat{Y} - Y'\rangle \geq \delta_0$. □

LEMMA 18. If $(\hat{X}, \hat{Y})$ is a SNE, then $\langle w_{\hat{Y}}, Q_1(f, \hat{X})\rangle = \langle v_{\hat{X}}, Q_2(g, \hat{Y})\rangle = 0$ for any $(f, g) \in \mathcal{B}[\Omega_1] \times \mathcal{B}[\Omega_2]$.

Proof. By Proposition 4, $w_{\hat{Y}} = \langle w_{\hat{Y}}, \hat{X}\rangle =$ constant $\hat{X}$-a.e. By Proposition 12(a), $Q_1(f, \hat{X})$ is $\hat{X}$-a.c. Hence, if $\omega_{\hat{Y}}$ is non-constant on a set $B \in \mathcal{B}$, then $\hat{X}(B) = 0$, and hence $Q_1(f, \hat{X})(B) = 0$. It follows that

$$\langle w_{\hat{Y}}, Q_1(f, \hat{X})\rangle = \langle w_{\hat{Y}}, \hat{X}\rangle\langle 1, Q_1(f, \hat{X})\rangle = 0$$

by condition (42). A similar argument shows that $\langle v_{\hat{X}}, Q_2(g, \hat{Y})\rangle = 0$. □

We are now is a position to give our main stability result.

PROPOSITION 19. Let Q_Ω be C_b^2, and $(\hat{X}, \hat{Y})$ be a SSNE which is strictly non-degenerate for Q_Ω. Then $(\hat{X}, \hat{Y})$ is a strongly locally-asymptotically-stable stationary point of the PDA dynamic (32).

Proof. Consider the positive-definite Lyapunov function:

$$L(X, Y) = \langle w_{\hat{Y}}, \hat{X} - X\rangle + \langle v_{\hat{X}}, \hat{Y} - Y\rangle. \tag{60}$$

Then,

$$\frac{dL}{dt}(X, Y) = -\langle w_{\hat{Y}}, Q_1(w_Y, X)\rangle - \langle v_{\hat{X}}, Q_2(v_X, Y)\rangle. \tag{61}$$

Now take $(X,Y) = \Big((1-\alpha_1)\hat{X} + \alpha_1 X', (1-\alpha_2)\hat{Y} + \alpha_2 Y'\Big)$, where $0 \le \alpha_1, \alpha_2 \le 1$, and $(X',Y') \in \hat{X}^\perp \times \hat{Y}^\perp$. Then, $||X - \hat{X}|| = 2\alpha_1$ and $||Y - \hat{Y}|| = 2\alpha_2$, and hence from (60),

$$L(X,Y) = \alpha_1 \langle w_{\hat{Y}}, \hat{X} - X' \rangle + \alpha_2 \langle v_{\hat{X}}, \hat{Y} - Y' \rangle \le 2(\alpha_1 + \alpha_2) \le 2\alpha, \tag{62}$$

where $\alpha = max\{2\alpha_1, 2\alpha_2\} = ||(X,Y) - (\hat{X},\hat{Y})||$. On the other hand, if $(\hat{X},\hat{Y})$ is an SSNE, then from Lemma 17 there exists $\delta_0 > 0$ such that (59) holds. Thus,

$$L(X,Y) \ge (\alpha_1 + \alpha_2)\delta_0 \ge \frac{1}{2}\alpha\delta_0. \tag{63}$$

Now consider

$$\begin{aligned}
&\langle w_{\hat{Y}}, Q_1(w_Y, X) \rangle = \langle w_{\hat{Y}}, Q_1(w_Y, \hat{X}) \rangle \\
&+ \alpha_1 \langle w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, \hat{X})(X' - \hat{X}) \rangle + \alpha_2 \langle w_{\hat{Y}}, D_1 Q_1(w_{\hat{Y}}, \hat{X})(Y' - \hat{Y}) \rangle + \mathcal{O}\left[\alpha^2\right] \\
&= \alpha_1 \langle w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, \hat{X}) X' \rangle + \alpha_2 \langle w_{\hat{Y}}, D_1 Q_1(w_{\hat{Y}}, \hat{X}) Y' \rangle + \mathcal{O}\left[\alpha^2\right]
\end{aligned}$$

by Taylor expansion and Lemma 18. On the other hand,

$$\langle w_{\hat{Y}}, D_1 Q_1(w_{\hat{Y}}, \hat{X}) Y' \rangle = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \langle w_{\hat{Y}}, Q_1(w_{\hat{Y} + \varepsilon Y'}, \hat{X}) \rangle = 0,$$

again by Lemma 18. Thus,

$$\langle w_{\hat{Y}}, Q_1(w_Y, X) \rangle = \alpha_1 \langle w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, \hat{X}) X' \rangle + \mathcal{O}\left[\alpha^2\right].$$

Using a similar calculation for $\langle v_{\hat{X}}, Q_2(v_X, Y) \rangle$, we obtain from (61)

$$\begin{aligned}
&\frac{dL}{dt}(X,Y) = \\
&- \alpha_1 \langle w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, D_2 Q_1(w_{\hat{Y}}, \hat{X}) X' \rangle - \alpha_2 \langle v_{\hat{X}}, D_2 Q_2(v_{\hat{X}}, D_2 Q_2(v_{\hat{X}}, \hat{Y}) Y' \rangle + \mathcal{O}\left[\alpha^2\right] \\
&\le -(\alpha_1 + \alpha_2) \mathcal{Q}(\hat{X}, \hat{Y})(X', Y') + \mathcal{O}\left[\alpha^2\right] \\
&\le -\frac{1}{2}\alpha \mathcal{Q}(\hat{X}, \hat{Y})(X', Y') + \mathcal{O}\left[\alpha^2\right].
\end{aligned}$$

Thus, by the strict non-degeneracy of $(\hat{X},\hat{Y})$, there exists a $\delta_1 > 0$ such that

$$\frac{dL}{dt}(X,Y) \le -\delta_1 \alpha + \mathcal{O}\left[\alpha^2\right].$$

Since the remainder term is bounded, it follows that there is an $\varepsilon > 0$ such that

$$\frac{dL}{dt}(X,Y) \leq -\frac{1}{2}\delta_1 \alpha \tag{64}$$

for all $\alpha = ||(X,Y) - (\hat{X},\hat{Y})|| < \varepsilon$; in particular, for all $(X,Y) \in N_\varepsilon(\hat{X}) \times N_\varepsilon(\hat{Y})$. Hence, from (62) and (64),

$$L(X_t,Y_t) \leq L(X_0,Y_0)\, exp\left\{-\delta_2 t\right\}, \tag{65}$$

for all $t \geq 0$ for which $\alpha_t = ||(X_t,Y_t) - (\hat{X},\hat{Y})|| < \varepsilon$, where $\delta_2 = \frac{1}{4}\delta_1$.

Let $U = \{(X,Y) : ||(X,Y) - (\hat{X},\hat{Y})|| < \varepsilon \text{ and } L(X,Y) < \frac{1}{2}\delta_0 \varepsilon\}$. Since L is continuous in the strong topology with $L(\hat{X},\hat{Y}) = 0$, U is an open neighbourhood of $(\hat{X},\hat{Y})$. Furthermore, it follows from (63) and (65) that for $(X_0,Y_0) \in U$

$$\frac{1}{2}\delta_0 \alpha_t \leq L(X_t,Y_t) \leq L(X_0,Y_0)\, exp\left\{-\delta_2 t\right\} < \frac{1}{2}\delta_0 \varepsilon\, exp\left\{-\delta_2 t\right\},$$

as long as $\alpha_t < \varepsilon$. But these inequalities imply that $L(X_t,Y_t) < \frac{1}{2}\delta_0\varepsilon$ and $\alpha_t = ||(X_t,Y_t) - (\hat{X},\hat{Y})|| < \varepsilon$, and hence that $(X_t,Y_t) \in U$ for all $t \geq 0$. Thus, $\alpha_t = ||(X_t,Y_t) - (\hat{X},\hat{Y})|| \to 0$ as $t \to \infty$, and we conclude that $(\hat{X},\hat{Y})$ is strongly-locally-asymptotically stable. $\square$

Proposition 19 may be applied to the Replicator dynamics as follows.

COROLLARY 20. An SSNE equilibrium is a locally asymptotically stable dynamic equilibrium of the generalized Replicator dynamics.

Proof. We use the formula in the proof of Proposition 11,

$$D_2\hat{R}_\Omega(g,W)\Phi = D_2 a_\Omega(g,X)\Phi \cdot R_\Omega(g,X) + a_\Omega(g,X) \cdot D_2 R_\Omega(g,X)\Phi.$$

Let $X' \in \hat{X}^\perp$. Then

$$\langle w_{\hat{Y}}, D_2\hat{R}_1(w_{\hat{Y}},\hat{X})X'\rangle = a_1(w_{\hat{Y}},\hat{X})\langle w_{\hat{Y}}, D_2 R_1(w_{\hat{Y}},\hat{X})X'\rangle,$$

since $R_1(w_{\hat{Y}},\hat{X}) = 0$. Now $a_\Omega(g,X) \geq a_* > 0$, and

$$\langle w_{\hat{Y}}, D_2 R_1(w_{\hat{Y}},\hat{X})X'\rangle = \langle w_{\hat{Y}}^2, X'\rangle - 2\langle w_{\hat{Y}}, X'\rangle\langle w_{\hat{Y}},\hat{X}\rangle + \langle w_{\hat{Y}}^2,\hat{X}\rangle$$

$$= \text{Var}_{X'}(w_{\hat{Y}}) + \text{Var}_{\hat{X}}(w_{\hat{Y}}) + \big(\langle w_{\hat{Y}}, \hat{X}\rangle - \langle w_{\hat{Y}}, X'\rangle\big)^2$$
$$= \text{Var}_{X'}(w_{\hat{Y}}) + \big(\langle w_{\hat{Y}}, \hat{X}\rangle - \langle w_{\hat{Y}}, X'\rangle\big)^2$$
$$\geq \langle w_{\hat{Y}}, \hat{X} - X'\rangle^2,$$

since $\text{Var}_{\hat{X}}(g) = 0$ for any g, because $\hat{X} = \Delta(\hat{\xi})$. Since $(\hat{X}, \hat{Y})$ is a SSNE and $X' \perp \hat{X}$, we have $\langle w_{\hat{Y}}, \hat{X} - X'\rangle \geq \delta_0 > 0$, and hence $\langle w_{\hat{Y}}, D_2\hat{R}_1(w_{\hat{Y}}, \hat{X})X'\rangle \geq a_*\delta_0^2 > 0$. A similar argument shows that $\langle v_{\hat{X}}, D_2R_2(v_{\hat{X}}, \hat{Y})Y'\rangle \geq a_*\delta_0^2$ for $Y' \perp \hat{Y}$. Thus, $\mathcal{Q}(\hat{X}, \hat{Y})(X', Y') \geq a_*\delta_0^2$ whenever either $X' \perp \hat{X}$ or $Y' \perp \hat{Y}$. This shows that $(\hat{X}, \hat{Y})$ is strictly non-degenerate for the generalized Replicator operators $\hat{R}_\Omega$, and the result therefore follows from Proposition 18. □

EXAMPLE 7. *Modified Ultimatum Game.* Consider the modified form of the Ultimatum Game discussed in Example 2, in which there is a positive lower bound $\alpha > 0$ for offers from Adam, whose payoff function is given by (25) with $p > 0$. Then the subgame perfect equilibrium is the unique SSNE. By Proposition 18, this equilibrium is therefore locally asymptotically stable for suitable PDA dynamics. □

EXAMPLE 8. As discussed in section 2, any game with a SNE, $(\hat{X}, \hat{Y})$, may be perturbed by an arbitrarily small perturbation to a game for which $(\hat{X}, \hat{Y})$ is a SSNE, and which is therefore locally asymptotically stable for suitable PDA dynamics. For example, the perturbation (24) with $p, q > 0$ arbitrarily small. □

Symmetric games. The theory in this paper has been developed for asymmetric games. However, there is no difficulty in adapting it to symmetric games. For a symmetric game, we have $\Omega_1 = \Omega_2 = \Omega$. Only one payoff function $w = v$ in (7) or (11) is required, and the payoffs to players 1 and 2 from a mixed strategy pair, (X, Y), are given by:

$$\left.\begin{aligned} \Pi_1(X,Y) &= \langle w_Y, X\rangle, \\ \Pi_2(Y,X) &= \langle w_X, Y\rangle. \end{aligned}\right\} \tag{66}$$

[*cf.* (15).] The general form of the (mutation-augmented) game dynamics (49), defined by a family of operators (31) is then,

$$\left.\begin{aligned} \tfrac{dX}{dt} &= Q(w_Y, X) + \delta\big(\Theta - X\big), \\ \tfrac{dY}{dt} &= Q(w_X, Y) + \delta\big(\Theta - Y\big). \end{aligned}\right\} \tag{67}$$

with $Q = Q_\Omega$ and $\delta \geq 0$. Here, we think of both players as chosen from the same population, which is subject to a single mutation probability, δ, associated with a single mutation distribution, $\Theta \in \mathcal{P}[\Omega, \mathcal{B}]$. However, in this interpretation, we should think of $X(B)$ as the probability that *any* player chosen from the single population will play a pure strategy in B. In other words, as far as her propensity to play a particular pure strategy is concerned, a chosen player does not care whether she occupies the role of player-1 or player-2. Thus, the same mixed strategy should apply to both players, so that we must take $X = Y$ in (67), and therefore (67) reduces to the single equation,

$$\frac{dX}{dt} = Q(w_X, X) + \delta\big(\Theta - X\big). \tag{68}$$

EXAMPLE 9. *The Nash Demand Game*. There are two players who must bid, independently and simultaneously, for a share of an infinitely divisible utility pie of unit size. A strategy for each player is a bid $\xi \in [0, 1]$, representing the share demanded by the player. If ξ_1 and ξ_2 are the bids of the two players, the rules of the game stipulate that the players get their demands if $\xi_1 + \xi_2 \leq 1$, but get nothing if $\xi_1 + \xi_2 > 1$. However, in the case $\xi_1 + \xi_2 < 1$, the outcome is inefficient since there is a positive surplus, $1 - \xi_1 - \xi_2$, which is discarded.

The strategy space is $\Omega = [0, 1]$, and the payoff function is

$$\pi(\xi, \eta) = \begin{cases} \xi & \text{if } \xi \leq 1 - \eta, \\ 0 & \text{if } \xi > 1 - \eta. \end{cases} \tag{69}$$

This is easily seen to be $\mathcal{B}_{12} = \mathcal{B}_{21}$-measurable. The associated payoff function, $w : \mathcal{M}[\Omega, \mathcal{B}] \to \mathcal{B}[\Omega]$, is given by

$$w_Y(\xi) = \xi \int_0^{1-\xi} dY(\eta) = \xi Y\big([0, 1 - \xi]\big), \tag{70}$$

and satisfies conditions (8).

The strategy pairs $(\xi, 1 - \xi)$ form a continuous family of pure-strategy Nash equilibria. Each of these equilibria is strict, but none of them is a SSNE. However, we can modify the game slightly as follows. Suppose each player has to pay a penalty, expressed as a percentage of the his share of the pie, for an inefficient outcome. Thus, if the players bid ξ_1 and ξ_2 with $\xi_1 + \xi_2 < 1$

(an inefficent outcome), then player i forfeits a fixed percentage $p\xi_i$, with $0 < p \leq 1$. This modified game has payoff function

$$\pi(\xi_1, \xi_2) = \begin{cases} 0 & \text{if } \xi_1 + \xi_2 > 1, \\ \xi_1 & \text{if } \xi_1 + \xi_2 = 1, \\ (1-p)\xi_1 & \text{if } \xi_1 + \xi_2 < 1. \end{cases} \quad (71)$$

For this game, each pair $(\xi, 1-\xi)$ with $\xi \in (0,1)$ is a SSNE, and hence is locally asymptotically stable for any well-behaved PDA dynamics by Proposition 18. At first sight this seems counterintuitive because the pairs $(\xi, 1-\xi)$ form a continuum in pure-strategy space $\Omega \times \Omega$. However, it must be remembered that, however close ξ and ξ' are (with $\xi' \neq \xi$) in $\Omega = [0,1]$, the corresponding mixed strategies $\Delta(\xi)$ and $\Delta(\xi')$ are far appart in the strong topology. In fact $||\Delta(\xi) - \Delta(\xi')|| = 2$ by Lemma 15. The basin of attraction of $(\xi, 1-\xi)$ contains a strong neighbourhood of the form (58), and therefore only measures of the form $X = (1-\varepsilon)\Delta(\xi) + \varepsilon X'$ with $X' \perp \Delta(\xi)$ and $\varepsilon > 0$ small, are asymptotically attracted to this equilibrium; in particular, $X = (1-\varepsilon)\Delta(\xi) + \varepsilon\Delta(\xi')$.[26] Thus, a very large proportion of the probability weight remains concentrated at ξ for relevant small perturbations away from equilibrium. □

4. Absolute continuity and probability densities

Let Q_Ω be a family of operators satisfying the hypotheses of Propositions 10 and 12. Fix a mixed strategy $\Theta \in \mathcal{P}[\Omega, \mathcal{B}]$. We have in mind that Θ should be one or other of the mutation distributions in the mutation-augmented dynamical system (49), but for the present we do not need to specify this. Denote by $\mathcal{D}[\Omega, \Theta] \subset \mathcal{B}[\Omega]$, the subset of *density functions* for Θ; *i.e.* functions θ satisfying

$$\theta \geq 0 \quad \text{and} \quad \langle \theta, \Theta \rangle = 1. \quad (72)$$

The set $\mathcal{D}[\Omega, \Theta]$ is closed and convex.

Let $\mathcal{M}[\Omega, \mathcal{B}, \Theta] \subset \mathcal{M}[\Omega, \mathcal{B}]$ be the closed subspace of measures which are Θ-a.c., and $\mathcal{P}[\Omega, \mathcal{B}, \Theta] = \mathcal{M}[\Omega, \mathcal{B}, \Theta] \cap \mathcal{P}[\Omega, \mathcal{B}]$. For $\Theta \geq 0$ (not necessarily normalized), there is a linear

[26] With similar considerations at $1 - \xi$ for the other player.

map, $J_\Theta : \mathcal{B}[\Omega] \to \mathcal{M}[\Omega, \mathcal{B}, \Theta]$, given by

$$J_\Theta f(B) = \langle f, \Theta \rangle_B \qquad (B \in \mathcal{B}), \tag{73}$$

and if $\Theta \in \mathcal{P}[\Omega, \mathcal{B}]$, J_Θ maps $\mathcal{D}[\Omega, \Theta]$ into $\mathcal{P}[\Omega, \mathcal{B}, \Theta]$. It is easy to see that $||J_\Theta f|| \leq ||f||.||\Theta||$, from which it follows that J_Θ is continuous. The kernel of J_Θ is the null space of Θ; *i.e.* $\text{Ker}J_\Theta = \big\{ f \in \mathcal{B}[\Omega] \mid f = 0 \;\; \Theta\text{-a.e.} \big\}$, and the Radon-Nikodým theorem[20] implies that J_Θ induces an isometric isomorphism,

$$J_\Theta : L_1[\Omega, \mathcal{B}, \Theta] \to \mathcal{M}[\Omega, \mathcal{B}, \Theta]. \tag{74}$$

Recall our assumption that $Q_\Omega(g, X)$ is X-a.c. for each g. If, in addition, X is Θ-a.c., then so is $Q_\Omega(g, X)$. It therefore follows from the above discussion that there is an operator

$$q_\Omega : \mathcal{B}[\Omega] \times \mathcal{D}[\Omega, \Theta] \to L_1[\Omega, \mathcal{B}, \Theta], \tag{75}$$

satisfying

$$J_\Theta x = X \quad \text{and} \quad J_X \big[q_\Omega(g, x) \big] = Q_\Omega(g, X). \tag{76}$$

Hence, from (73)[21],

$$J_\Theta \big[q_\Omega(g, x) x \big] = Q_\Omega(g, X). \tag{77}$$

The above discussion may be generalized if we assume the stronger condition that $Q_\Omega(g, X)$ is X-a.c. for all $(g, X) \in \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}]$. Recall that this means that $Q_\Omega(g, X)(B) = 0$ whenever the total variation of X on $B \in \mathcal{B}$ is zero; *i.e.* $|X|(B) = ||X||_B = 0$.[22] Since $|X|$ is a positive measure, there is a unique function $\hat{q}_\Omega(g, X) \in L_1[\Omega, \mathcal{B}, \Theta]$ such that $J_{|X|} \left(\hat{q}_\Omega(g, X) \right) = Q_\Omega(g, X)$. On the other hand, the function x in (74) is no longer necessarily

[20] See Dunford and Schwartz (1958), Theorem 10.2, p 176.

[21] See Dunford and Schwartz (1958), Theorem 10.4, p 179, for the fact that absolutely continuous measures can be combined in this way.

[22] See Dunford and Schwartz (1958), Lemma 4.13, p 131. Note that this condition holds for the family of operators (33) defining the Replicator dynamics.

a density, but exists as an element of $L_1[\Omega, \mathcal{B}, \Theta]$. By the Jordan decomposition Theorem[23] we may write $X = X^+ - X^-$, where $X^\pm$ are positive measures, and $|X| = X^+ + X^-$. Hence, $x = x^+ - x^-$, with $x^\pm$ non-negative L_1-functions. Then $J_\Theta(|x|) = J_\Theta(x^+ + x^-) = |X|$. Now let $B_\pm \in \mathcal{B}$ be the support of $x^\pm$, and define $I_x \in \mathcal{B}[\Omega]$ by $I_x(\xi) = \pm 1$ if $\xi \in B_\pm$, and zero otherwise. If we set $q_\Omega(g, x) = I_x \hat{q}_\Omega(g, X)$, then (77) holds for any X.

The properties of q_Ω are summarized in the following proposition.

PROPOSITION 21 If Q_Ω satisfies the hypotheses of Propositions 10 and 12, then q_Ω given by (75) to (77) satisfies the following.

a) $xq_\Omega(g, x)$ is locally Lipschitz on $\mathcal{B}[\Omega] \times L_1[\Omega, \mathcal{B}, \Theta]$.

b) Either $x(\xi) = 0$ or $q_\Omega(1, x)(\xi) = 0$.

c) $\langle xq_\Omega(g, x), \Theta\rangle_\Omega = 0$.

d) $g = 0$ Θ-a.e. implies $xq_\Omega(g, x) = 0$ Θ-a.e.

If, in addition, Q_Ω is positive definite, then:

e) $\langle gxq_\Omega(g, x), \Theta\rangle \geq 0$.

f) $\langle gxq_\Omega(g, x), \Theta\rangle = \langle xg_x q_\Omega(g_x, x), \Theta\rangle$, where $g_x = g - \langle gx, \Theta\rangle$.

g) Equality holds in (e) if and only if $xg_x = 0$ Θ-a.e.

Proof. See Appendix. □

Now let $\Theta = (\Theta_1, \Theta_2) \in \mathcal{P}[\Omega_1, \mathcal{B}_1] \times \mathcal{P}[\Omega_2, \mathcal{B}_2]$ be a pair of probability measures, and let $\mathcal{E}_\Theta = \mathcal{M}[\Omega_1, \mathcal{B}_1, \Theta_1] \times \mathcal{M}[\Omega_2, \mathcal{B}_2, \Theta_2] \subset \mathcal{M}[\Omega_1, \mathcal{B}_1] \times \mathcal{M}[\Omega_2, \mathcal{B}_2] = \mathcal{E}$. We wish to show that the subspace $\mathcal{E}_\Theta$ is invariant under the mutation augmented dynamics (49) defined by the mutation strategies Θ.

23 See Yosida (1978), Theorem 2, p 36, or Dunford and Schwartz (1958), Theorem 1.8, p 98.

LEMMA 22 If $Q_\Omega(g,X)$ is X-a.c. for any $(g,X) \in \mathcal{B}[\Omega] \times \mathcal{M}[\Omega,\mathcal{B}]$, then $\mathcal{E}_\Theta$ is invariant under the dynamics (49); *i.e.* $\Xi\big(\mathcal{E}_\Theta\big) \subset \mathcal{E}_\Theta$.

Proof. From (77), if X is Θ_1-a.c., then so is $Q_1(w_Y,X)$ for any $Y \in \mathcal{M}[\Omega_2,\mathcal{B}_2]$. The right hand side of the first equation of (49) is just a linear combination of terms in $\mathcal{M}[\Omega_1,\mathcal{B}_1,\Theta_1]$, and hence is also in $\mathcal{M}[\Omega_1,\mathcal{B}_1,\Theta_1]$. This is true for each $Y \in \mathcal{M}[\Omega_2,\mathcal{B}_2]$. Similar remarks apply to the second equation of (49), and so the result follows. $\square$

PROPOSITION 23 Suppose $Q_\Omega(g,X)$ is X-a.c. for any $(g,X) \in \mathcal{B}[\Omega] \times \mathcal{M}[\Omega,\mathcal{B}]$. Then the mutation augmented dynamics (49) with mutation strategies $\Theta = (\Theta_1,\Theta_2)$, when restricted to $\mathcal{E}_\Theta$, is equivalent to the dynamics defined on $L_1[\Omega_1,\mathcal{B}_1,\Theta_1] \times L_1[\Omega_2,\mathcal{B}_2,\Theta_2]$ given by

$$\frac{dx}{dt} = (1-\delta_1)xq_1(w_y,x) + \delta_1(1-x), \tag{78a}$$

$$\frac{dy}{dt} = (1-\delta_2)yq_2(v_x,y) + \delta_2(1-y), \tag{78b}$$

where $q_i(g,z) = q_{\Omega_i}(g,z)$ is defined by (76), and we have written w_y, v_x for w_Y, v_X, with $X = J_1x, Y = J_2y$ and $J_i = J_{\Theta_i}$. Similarly, $\delta_i = \delta_i(X,Y) = \delta_i\big(J_1x, J_2y\big)$. Further, the closed convex set $\mathcal{D}[\Omega_1,\Theta_1] \times \mathcal{D}[\Omega_2,\Theta_2]$ is invariant under the flow defined by (78).

Proof. The subspace $\mathcal{E}_\Theta \subset \mathcal{E}$ is closed, and hence is a Banach space. Thus, by Lemma 22, $\mathcal{E}_\Theta$ is invariant under the flow $\mathcal{F}_t$ of the dynamics (49). Write $\mathcal{F}_t(X,Y) = \big(X(t),Y(t)\big)$. Then $(X,Y) \in \mathcal{E}_\Theta$ implies $X(t) \in \mathcal{M}[\Omega_1,\mathcal{B}_1,\Theta_1]$ for each t.[24] Hence, there exists a Θ_1-essentially unique function $x(t) \in \mathcal{B}_1[\Omega_1]$ with $J_1x(t) = X(t)$. Further, by Proposition 10 and the properties of J, $x(t) \in \mathcal{D}[\Omega_1,\Theta_1]$ if $X \in \mathcal{P}[\Omega_1,\mathcal{B}_1,\Theta_1]$. Similarly, there exists a unique $y(t) \in L_1[\Omega_2,\mathcal{B}_2,\Theta_2]$ with $J_2y(t) = Y(t)$. Writing $f_t(x,y) = \big(x(t),y(t)\big)$, we have that $(J_1 \times J_2)(f_t(x,y)) = \mathcal{F}_t(X,Y)$. Further, the uniqueness of f_t implies that the conditions (30c,d) for f follow from those for $\mathcal{F}$, so that f defines a flow on $L_1[\Omega_1,\mathcal{B}_1,\Theta_1] \times L_1[\Omega_2,\mathcal{B}_2,\Theta_2]$.

Since $lim_{t\to 0}\big\{\frac{1}{t}\big(X(t) - X\big)\big\}$ exists and is equal to $\frac{dX}{dt} \in \mathcal{M}[\Omega_1,\mathcal{B}_1]$, the sequence $\big\{\frac{1}{t_n}\big(X(t_n) - X\big)\big\}$ is a Cauchy sequence in $\mathcal{M}[\Omega_1,\mathcal{B}_1,\Theta_1]$ for any sequence $\{t_n\}$ with

[24] In fact, the proof of Lemma 21 shows that this is true even if Y is not Θ_2-absolutely continuous.

$t_1 > t_2 > \ldots > t_n \to 0$. Hence, from (74), $\left\{\frac{1}{t_n}\big(x(t_n) - x\big)\right\}$ is a Cauchy sequence in $L_1[\Omega_1, \mathcal{B}_1, \Theta_1]$. Since this latter space is complete[25], it follows that $\frac{1}{t_n}\big(x(t_n) - x\big) \to \frac{dx}{dt} \in L_1[\Omega_1, \mathcal{B}_1, \Theta_1]$ as $n \to \infty$. Further, by the continuity of J_1, we have $J_1\left(\frac{dx}{dt}\right) = \frac{dX}{dt}$, which also shows that $\frac{dx}{dt}$ is independent of the sequence $\{t_n\}$. Thus, $lim_{t\to 0}\frac{1}{t}\big(x(t) - x\big)$ exists in $L_1[\Omega_1, \mathcal{B}_1, \Theta_1]$ and is equal to $\frac{dx}{dt}$. Now use (77) to obtain

$$J_1\left(\frac{dx}{dt}\right) = \frac{dX}{dt} = J_1\big[(1-\delta_1)xq_1(w_Y, x) + \delta_1(1-x)\big]$$

It therefore follows from (74) that equation (78a) holds in $L_1[\Omega_1, \mathcal{B}_1, \Theta_1]$. A similar argument serves to establish equation (78b). $\square$

EXAMPLE 9. *The Replicator dynamics.* For the family of operators (33) defining the Replicator dynamics, and a fixed measure $\Theta \in \mathcal{P}[\Omega, \mathcal{B}]$, the corresponding family of operators (75) (when $X \geq 0$) is easily seen to be

$$r_\Omega(g, x) = \langle x, \Theta\rangle g - \langle gx, \Theta\rangle. \tag{79}$$

In particular, the mutation-augmented Replicator dynamics (50) take the form

$$\left.\begin{array}{rcl} \frac{dx}{dt} &=& (1-\delta_1)x\big[\langle x, \Theta_1\rangle w_y - \langle w_y x, \Theta_1\rangle\big] + \delta_1(1-x), \\ \frac{dy}{dt} &=& (1-\delta_2)y\big[\langle y, \Theta_2\rangle v_x - \langle v_x y, \Theta_2\rangle\big] + \delta_2(1-y), \end{array}\right\} \tag{80}$$

for $x, y \geq 0$. Clearly, the equations (80) extend formally to the whole of $L_1[\Omega_1, \mathcal{B}_1, \Theta_1] \times L_1[\Omega_2, \mathcal{B}_2, \Theta_2]$. Also, if x is a Θ_1-density and y is a Θ_2-density, then $\langle x, \Theta_1\rangle = \langle y, \Theta_2\rangle = 1$. When the Θ_i are Lebesgue measures, and $\delta_i = 0$, this gives the form (37) of the Replicator dynamics.

Observe that if $\delta_1 > 0$ on $\mathcal{D}[\Omega_1, \Theta_1] \times \mathcal{D}[\Omega_2, \Theta_2]$, then any solution of (78) must have $x_t(\xi) > 0$ for all $\xi \in \Omega_1$, and $t > 0$. However, this does not necessarily mean that all pure strategies get used with positive probability. For example, if there exists an open set $B_1 \in \mathcal{B}_1$ with $\Theta_1(B_1) = 0$, we have $X_t(B_1) = \langle x_t, \Theta\rangle_{B_1} = 0$, for all t, so that pure strategies in B_1 never get used.

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[25] See Dunford and Schwartz (1958), Theorem 6.6, p 146.

For densities with full support, some of the properties of q_Ω listed in Proposition 21 simplify. Thus, if $x > 0$ on Ω, then Proposition 21(b) implies that $q_\Omega(1, x) = 0$ on Ω_1. Proposition 21(d) reads: $g = 0$ Θ-a.e. implies $q_\Omega(g, x) = 0$ Θ-a.e. Finally, Proposition 21(g) implies that equality holds in (e) if and only if $g_x = 0$ Θ-a.e.

6. Final Remarks

The theory presented in this paper leaves many questions unanswered. Perhaps the most pressing need is to fill the gap between the necessary and sufficient conditions of Propositions 13 and 19 for a Nash equilibrium to be asymptotically stable. Of course, the same gap exists also in the finite-dimensional case[27]. This is a question of the appropriate refinement of Nash equilibrium. The requirement for a SSNE in Proposition 19 is very strong indeed. However, in the infinite-dimensional case, it is false that a strict NE is necessarily locally asymptotically stable. This is perhaps unsurprising in view of the fact that an arbitrarily small perturbation of the payoff function can convert a SNE (which is not a SSNE) into a strictly dominated strategy. Nevertheless, it seems likely that if less restrictive conditions could be obtained for the Replicator dynamics, then the same conditions would work more generally for an enlarged class of positive-definite dynamics, perhaps with additional restrictions, such as the non-degeneracy assumption of definition 5.

Of course, the notion of an SSNE is very strong, and probably few games of interest have such equilibria. It may therefore be prefereable to seek out weaker concepts of local asymptotic stability than that discussed here. In effect this means considering other possible topologies besides the strong topology; for example, the so-called weak*-topology on the space of measures $\mathcal{M}[\Omega, \mathcal{B}]$[28]. This is a matter for further research.

Some of the most difficult technical questions probably centre around the effect of mutations on the underlying positive-definite game dynamics. In particular, we have seen for the Ultimatum

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[27] See, for example, Samuelson (1997) , chapter 4.

[28] Dunford ansd Schwartz (1958), Chapter V, section 3; Also Yosida (1978), p 111. Topologies weaker than the strong topology ave been considered by Oeschler and Riedel (1999)

Game and the Nash Demand game that Nash equilibria may occur in extended components (Examples 2 and 6), rather than as isolated points. In this situation, we have no useful stability theorems. In fact, one of the motivations for introducing (low probability) mutations is the hope that their effect will be to 'condense' a component of Nash equilibria down to a single isolated equilibrium of the augmented dynamics, which it might then be possible to show is stable. This is one possible approach to 'equilibrium selection'. However, there is no general theory (as far as I know) which covers this situation, even in the finite-dimensional case, so that each example must be treated *ab initio*, simulation often being the only available option.[29] A useful first step would be to understand when an extended component of Nash equilibria is asymptotically attracting for mutation-free dynamics. The complex equilibrium situation which can arise when mutations are introduced is evident in the work of Seymour (1999) on equilibria in the infinite-dimensional Ultimatum Game.

Finally, examples of positive-definite game dynamics other than the Replicator dynamics, such as those discussed in Hopkins (1999) in the finite-dimensional case, would be interesting to examine in their infinite-dimensional form.

[29] For example Gale, Binmore and Samuelson (1995). See also Samuelson (1997), chapter 5.

Appendix

Proof of Lemma 1. For convexity, note that any convex combination of probability measures is also a probability measure.

To show that $\mathcal{P}[\Omega,\mathcal{B}]$ is a subset of the unit sphere, note that, for any $\mathcal{B}$-measurable function f, with $||f|| \leq 1$, we have

$$\int_\Omega f dX \leq \sup_{\omega\in\Omega}\{f(\omega)\}\int_\Omega dX \leq \sup_{\omega\in\Omega}\big|f(\omega)\big|X(\Omega) \leq 1,$$

and

$$\int_\Omega f dX \geq \inf_{\omega\in\Omega}\{f(\omega)\}\int_\Omega dX \geq -\sup_{\omega\in\Omega}\big|f(\omega)\big|X(\Omega) \geq -1.$$

Thus,

$$||X|| = \sup_{||f||\leq 1}\Big|\int_\Omega f dX\Big| = 1.$$

It remains to show that $\mathcal{P}[\Omega,\mathcal{B}]$ is a closed subset of $\mathcal{M}[\Omega,\mathcal{B}]$. To do this, we shall show that the complement of $\mathcal{P}[\Omega,\mathcal{B}]$ is open. Let $X \in \mathcal{P}[\Omega,\mathcal{B}]^c$, the complement of $\mathcal{P}[\Omega,\mathcal{B}]$. Then either $X(\Omega) > 1$, or there exists $B \in \mathcal{B}$ such that $X(B) < 0$. In the latter case, let $\theta \in \mathcal{M}[\Omega,\mathcal{B}]$ satisfy $||\theta|| < \varepsilon$ for some $0 < \varepsilon \leq -\frac{1}{2}X(B)$. Then, if I_B is the indicator function of B,

$$\begin{aligned}
(X+\theta)(B) &= X(B) + \int_\Omega I_B d\theta \\
&\leq X(B) + \Big|\int_\Omega I_B d\theta\Big| \\
&\leq X(B) + \sup_{||f||\leq 1}\Big|\int_\Omega f d\theta\Big| \\
&= X(B) + ||\theta|| \\
&< X(B) + \varepsilon \\
&< 0.
\end{aligned}$$

Thus, the open ball of radius ε, centre X is contained in $\mathcal{P}[\Omega,\mathcal{B}]^c$. On the other hand, if $X(\Omega) > 1$, choose $\varepsilon \leq \frac{1}{2}\big(X(\Omega) - 1\big)$. Then,

$$(X+\theta)(\Omega) = X(\Omega) + \int_\Omega d\theta$$

$$
\begin{aligned}
&\geq X(\Omega) - \Big|\int_\Omega d\theta\Big| \\
&\geq X(\Omega) - \sup_{||f||\leq 1}\Big|\int_\Omega f d\theta\Big| \\
&= X(\Omega) - ||\theta|| \\
&> X(\Omega) - \varepsilon \\
&> 1.
\end{aligned}
$$

Again, the open ball of radius ε, centre X is contained in $\mathcal{P}[\Omega, \mathcal{B}]^c$. This shows that $\mathcal{P}[\Omega, \mathcal{B}]^c$ is an open subset of $\mathcal{M}[\Omega, \mathcal{B}]$, and therefore completes the proof. □

Proof of Lemma 2. Fix $B \in B$. If g is a bounded $\mathcal{B}$-measurable function defined on B, then g may be extended to a bounded $\mathcal{B}$-measurable function $\hat{g}$ defined on Ω, by setting $\hat{g} = 0$ on $\Omega - B$. Clearly $||\hat{g}|| = ||g||_B = sup_{\omega \in B}|g(\omega)|$. Thus, by (5), for $X \in \mathcal{M}[\Omega, \mathcal{B}]$,

$$
\begin{aligned}
||X||_B &= \sup_{||g||_B \leq 1} \big|\langle g, X\rangle_B\big| \\
&= \sup_{||\hat{g}|| \leq 1} \big|\langle \hat{g}, X\rangle\big| \\
&\leq \sup_{||f|| \leq 1} \big|\langle f, X\rangle\big| \\
&= ||X||,
\end{aligned}
$$

where $f \in \mathcal{B}[\Omega]$. Hence,

$$
\begin{aligned}
|\langle g', X'\rangle_B - \langle g, X\rangle_B| &= |\langle g', X'\rangle_B - \langle g, X'\rangle_B + \langle g, X'\rangle_B - \langle g, X\rangle_B| \\
&\leq |\langle g' - g, X'\rangle_B| + |\langle g, X' - X\rangle_B| \\
&\leq ||g' - g||_B||X'||_B + ||g||_B||X' - X||_B \\
&\leq ||g' - g||_B||X'|| + ||g||_B||X' - X||. \qquad (A1)
\end{aligned}
$$

It therefore follows that $|\langle g', X'\rangle_B - \langle g, X\rangle_B| \to 0$ as $max\big\{||g' - g||_B, ||X' - X||\big\} \to 0$, which proves the result. □

Proof of Proposition 6. Fix $(X, Y) \in \mathcal{M}[\Omega_1, \mathcal{B}_1] \times \mathcal{M}[\Omega_2, \mathcal{B}_2]$. From (32),

$$
||\mathcal{X}_0(X'', Y'') - \mathcal{X}_0(X', Y')|| = ||Q_1(w_{Y''}, X'') - Q_1(w_{Y'}, X')||.
$$

By hypothesis, there is a neighbourhood, $\mathcal{U}_1 \times \mathcal{N}_1 \subset \mathcal{B}_1[\Omega_1] \times \mathcal{M}[\Omega_1, \mathcal{B}_1]$, of (w_Y, X), and a positive constant l_1, such that

$$||Q_1(w_{Y''}, X'') - Q_1(w_{Y'}, X')|| \leq l_1. \, max\{||X'' - X'||, ||w_{Y''} - w_{Y'}||\},$$

whenever $(w_{Y''}, X''), (w_{Y'}, X') \in \mathcal{U}_1 \times \mathcal{N}_1$. By property (8a), there is a constant l_2 such that $||w_{Y''} - w_{Y'}|| = ||w_{Y''-Y'}|| \leq l_2||Y'' - Y'||$.[30] Setting $\mathcal{N}_2 = w^{-1}(\mathcal{U}_1)$, the continuity of w implies that $\mathcal{N}_2$ is a neighbourhood of Y in $\mathcal{M}[\Omega_2, \mathcal{B}_2]$. Then, for $(X', Y'), (X'', Y'') \in \mathcal{N}_1 \times \mathcal{N}_2$, we have

$$||\mathcal{X}_0(X'', Y'') - \mathcal{X}_0(X', Y')|| \leq k_1. \, max\{||X'' - X'||, ||Y'' - Y'||\},$$

where $k_1 = max\{l_1, l_1 l_2\}$. A similar argument applies to $\mathcal{Y}_0$, and the proposition therefore follows. $\square$

Proof of Proposition 21. (a) Fix $(g, x) \in \mathcal{B}[\Omega] \times L_1[\Omega, \mathcal{B}, \Theta]$, and let $X = J_\Theta[x]$. Since Q_Ω is locally Lipschitz, we may choose an open neighbourhood $\mathcal{N} \subset \mathcal{B}[\Omega] \times \mathcal{M}[\Omega, \mathcal{B}]$ of (g, X) in which Q_Ω is bounded and Lipschitz, with constant k. Let $U = J_\Theta^{-1}[\mathcal{N}]$. Then, by the continuity of J_Θ, U is an open neighbournood of (g, x) in $L_1[\Omega, \mathcal{B}, \Theta] \times L_1[\Omega, \mathcal{B}, \Theta]$. For $(g', x'), (g'', x'') \in U$, and $X' = J_\Theta[x'], X'' = J_\Theta[x'']$, it follows from (68) and (71) that

$$\begin{aligned}
\left|x' q_\Omega(g', x') - x'' q_\Omega(g'', x'')\right|_1 &= ||J_\Theta\left[x' q_\Omega(g', x') - x'' q_\Omega(g'', x'')\right]|| \\
&= ||Q_\Omega(g', X') - Q_\Omega(g'', X'')|| \\
&\leq k. \, max\{||g' - g''||, ||X' - X''||\} \\
&= k. \, max\{||g' - g''||, |x - x'|_1\}.
\end{aligned}$$

This proves (a).

(b) Suppose $X \geq 0$. Let $B_0 \in \mathcal{B}$ be the set on which $x(\xi) = 0$, so that $x(\xi) > 0$ on $B_1 = B_0^c$. Then $X(B_0) = \langle x, \Theta \rangle_{B_0} = 0$ and $X(B_1) = \langle x, \Theta \rangle_{B_1} > 0$, unless $\Theta(B_1) = 0$. But, this latter eventuality can occur only if $X = 0$, in which case we can take $x = 0$ everywhere. Otherwise,

30 See Yosida (1978), p 43, Corollary 2.

write $Q_\Omega(g,X) = Q^+_\Omega(g,X) - Q^-_\Omega(g,X)$. Then $Q_\Omega(1,X) = 0$ implies $Q^\pm_\Omega(1,X) = 0$. We therefore have $\langle xq^\pm_\Omega(1,x),\Theta\rangle_{B_1} = Q^\pm_\Omega(1,X)(B_1) = 0$, from which we conclude that $q^\pm_\Omega(1,x) = 0$ Θ-a.e. on B_1, and hence that $q_\Omega(1,x) = 0$ Θ-a.e. on B_1. However, since $q_\Omega(1,x)$ is only defined Θ-a.e., we can take $q_\Omega(1,x) = 0$ everywhere on B_1. The extension of the arguement to arbitrary X is straightforward - see the discussion after (71). This proves (b).

(c) From (67) and (71), $\langle xq_\Omega(g,x),\Theta\rangle = J_\Theta[xq_\Omega(g,X)](\Omega) = Q_\Omega(g,X)(\Omega) = 0$ (see (44)). This proves (c).

(d) If $g = 0$ Θ-a.e. and X is Θ-absolutely continuous, then $g = 0$ X-a.e. Hence, $Q_\Omega(g,X) = 0$ by Proposition 12(a). Thus, from (70), $xq_\Omega(g,x) = 0$ Θ-a.e. This proves (d).

(e) It follows from (67) that if $f \in \mathcal{B}[\Omega]$, then $\langle f, J_\Theta\theta\rangle = \langle f\theta,\Theta\rangle$. Hence, from (71),

$$\langle gq_\Omega(g,x)x,\Theta\rangle = \langle g, J_\Theta[q_\Omega(g,x)x]\rangle = \langle g, Q_\Omega(g,X)\rangle \geq 0, \tag{A2}$$

since Q_Ω is positive definite. This proves (e).

(f) Note that $g_x = g - \langle g,X\rangle = g_X$. The result then follows from Proposition 8 and (A2).

(g) By Proposition 8, equality holds in (A2) if and only if $g_x = g_X = 0$ X-a.e. Hence, if and only if $xg_x = 0$ Θ-a.e. $\square$

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