Existence of Mixed Strategy Equilibria in a Class of Discontinuous Games with Unbounded Strategy Sets.

Gian Luigi Albano and Alexander Matros*
Department of Economics and ELSE
University College London

February 21, 2002

Abstract

We prove existence of mixed strategy equilibria for a class of discontinuous two-player games with non-compact strategy sets. As a corollary of our main results, we obtain a continuum of mixed strategy equilibria for the first- and second-price two-bidder auctions with toeholds. We also find Klemperer's (2000) result about the existence of mixed strategy equilibria in the classical Bertrand duopoly.

JEL Classification: C72, D43, D44
Keywords: Discontinuous games, Bertrand game, Toehold, First- and Second-Price Auctions

---

*We would like to thank Philippe Jehiel for useful discussions. The support of the Economic and Social Research Council (ESRC) is gratefully acknowledged. The work was part of the programme of the ESRC Research Center for Economic Learning and Social Evolution.

1 Introduction

The aim of this paper is to show the existence of Nash equilibria in mixed strategies for a class of two-player discontinuous games with complete information in which the strategy sets are non-compact. The problem of the existence of equilibria in discontinuous games has been already addressed by Dasgupta and Maskin (1986a, 1986b), Maskin (1986), Simon (1987), Simon and Zame (1990), and, more recently by Reny (1999). Unlike the existing papers, we construct Nash equilibria in mixed strategies when players' strategy sets coincides with the set of real numbers.

Two classes of games fit into our theoretical framework: Two-bidder auctions with toeholds and the standard Bertrand game (with unit demand). In an action with toeholds, two bidders compete for an object. Each of them owns a (strictly positive) share of the object. Their valuations and their shares are common knowledge. Both bidders submit simultaneously sealed bids, the higher bidder gets the object and buys her competitor's share at the selling price. The relevant feature of this game is that each bidder is a buyer and a seller at the same time. A discontinuity arises from the tie breaking rule. If ties are broken through any random device such that a bidder gets the object with probability strictly less than one, players' best responses are not well defined. We prove that, by "opening" the players' strategy space, equilibria in mixed strategies do exist. However, the existence of equilibria is not guaranteed by any fixed point theorem since the strategy set is not compact. We prove existence by construction.

Our approach allows us to derive a continuum of mixed strategy equilibria in the classical Bertrand game. This class of equilibria coincides with what proposed by Klemperer (2000).

The rest of the paper is organized as follows. The next section describes our assumptions and states the main existence results. We provide some examples in Section 3. Section 4 concludes.

2 The Model

We consider two classes of games with complete information and prove existence of mixed equilibria for each of them.

Let $\Gamma_A = (\{i, j\}, \mathsf{R} \times R, (u_i, u_j))$ be a two-player game. Assume that if player i chooses strategy $x \in \mathsf{R}$ and player j plays strategy $y \in \mathsf{R}$, then payoff functions $u_k : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$, $k = i, j$, are

$$(u_i(x,y), u_j(x,y))_A = \begin{cases} (v_i(y), w_j(y)), & \text{if } x > y \\ (w_i(x), v_j(x)), & \text{if } x < y \\ (\alpha v_i(x) + [1-\alpha] w_i(x), [1-\alpha] w_j(x) + \alpha v_j(x)) & \text{if } x = y \end{cases},$$

where $\alpha \in [0,1]$ is a random tie-breaking rule. The model allows any tie breaking rule. We make the following assumptions about functions $v_k(t)$ and $w_k(t)$, $k = i, j$. Since the strategy space is the set of real numbers, players' strategies will be called "numbers".

$$A1.\ w_k(t) \text{ is differentiable, } k = i, j.$$

$$A2.\ w_k'(t) \geq 0,\ k = i, j.$$

Assumption $A2$ simply says that, conditional on player i choosing the "higher" number, player j wants to pick a number as close as possible to the one chosen by player i. This happens, for example in the two-bidder second-price auction with toeholds in which each bidder is a buyer and a seller at the same time. Thus if bidder i submits the higher bid for the object, bidder j wants to sell his share at the highest possible price.

$A3$. There exists $\underline{t} \in \mathsf{R}$, such that $w_k(t) - v_k(t) > 0$, for all $t \geq \underline{t}$ and $k = i, j$.

The explanation for this assumption is as follows. There exists a threshold $\underline{t}$ such that, for all actions greater than $\underline{t}$, each bidder prefers being the "low" strategy player than the "high" strategy player. The example of a two-player auction with toeholds will clarify the role of this assumption. Consider bidder i's valuation v_i. It easy to see that bidder i prefers being the seller (loser) for all prices greater than his own valuation. Assumption $A3$ "guarantees" that the payoff functions are discontinuous if players choose the same number $t \geq \underline{t}$.

We finally assume

$$A4.\ \int_{\underline{t}}^{+\infty} \frac{w_k'(t)\, dt}{w_k(t) - v_k(t)} = +\infty, \text{ for } k = i, j$$

Assumptions A1-A4 guarantee our main result.

Theorem 1 Suppose that assumptions A1-A4 hold. Then the game Γ_A admits a continuum of equilibria in mixed strategies. For any $t^* \geq \underline{t}$, the following probability distribution constitutes a mixed strategy equilibrium:

$$F_j(t) = \begin{cases} 0, & \text{if } t < t^* \\ 1 - \exp\left[-\int_{t^*}^{t} \frac{w_i'(s)ds}{w_i(s)-v_i(s)}\right] & \text{if } t \geq t^* \end{cases}, \tag{1}$$

where $i \neq j$.

Proof. Notice first that the distribution function $F_j(t)$ is a positive, strictly increasing function which satisfies $F_j(t^*) = 0$ and $F_j(+\infty) = 1$, because of assumption A4. We show now that the distribution functions from (1) constitute a mixed strategy equilibrium. Suppose that player j, $i \neq j$, uses the c.d.f. $F_j(t)$ above, then we have to show that (a) player i's expected payoff $\overline{u}_i^A$ is constant if he chooses a number $x \in [t^*, +\infty)$; (b) player i's expected payoff if he chooses a number $x \in (-\infty, t^*)$ is at most $\overline{u}_i^A$.

(a) If player i chooses a strategy $x \in [t^*, +\infty)$, then his expected payoff writes

$$E[u_i(x, F_j(y))] = \int_{t^*}^{x} v_i(s) f_j(s)\, ds + \int_{x}^{+\infty} w_i(x) f_j(s)\, ds, \tag{2}$$

where the first integral in the right-hand side is player i's expected payoff if her number x is greater than the opponent's number y, and the second integral is player i's expected payoff if her number x is smaller than the opponent's number y.

¿From the probability distribution function $F_j(t)$ in (1), it is immediate to get the density function $f_j(t)$:

$$f_j(t) = \begin{cases} 0, & \text{if } t < t^* \\ \frac{w_i'(t)}{w_i(t)-v_i(t)} \exp\left[-\int_{t^*}^{t} \frac{w_i'(s)ds}{w_i(s)-v_i(s)}\right], & \text{if } t \geq t^* \end{cases}.$$

Player i's expected payoff (2) can be rewritten as

$$\begin{aligned} E[u_i(x, F_j(y))] = {} & \int_{t^*}^{x} v_i(y) \frac{w_i'(y)}{w_i(y) - v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\, dz}{w_i(z) - v_i(z)}\right] dy + \\ & + w_i(x) \int_{x}^{+\infty} \frac{w_i'(y)}{w_i(y) - v_i(y)} \exp\left[-\int_{t^*}^{y} \frac{w_i'(z)\, dz}{w_i(z) - v_i(z)}\right] dy. \end{aligned}$$

Using assumption A4, we get

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=w_i\left(x\right)\left(\exp\left[-\int\limits_{t^*}^{x}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]\right)+$$

$$+\int\limits_{t^*}^{x}\left[v_i\left(y\right)-w_i\left(y\right)+w_i\left(y\right)\right]\frac{w_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int\limits_{t^*}^{y}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]dy.$$

Player i's expected payoff becomes:

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=w_i\left(x\right)\left(\exp\left[-\int\limits_{t^*}^{x}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]\right)-$$

$$-\int_{t^*}^{x}w_i'\left(y\right)\exp\left[-\int_{t^*}^{y}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]dy+$$

$$+\int_{t^*}^{x}w_i\left(y\right)\frac{w_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int\limits_{t^*}^{y}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]$$

or

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=w_i\left(x\right)\left(\exp\left[-\int\limits_{t^*}^{x}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]\right)-$$

$$-w_i\left(x\right)\exp\left[-\int_{t^*}^{x}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]+w_i\left(t^*\right)-$$

$$-\int_{t^*}^{x}w_i\left(y\right)\frac{w_i'\left(y\right)dz}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int_{t^*}^{y}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]dy+$$

$$+\int_{t^*}^{x}w_i\left(y\right)\frac{w_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int_{t^*}^{y}\frac{w_i'\left(z\right)dz}{w_i\left(z\right)-v_i\left(z\right)}\right]dy,$$

and we finally obtain

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=w_i\left(t^*\right)=\overline{u}_i^A\text{ for any }x\in\left[t^*,+\infty\right).$$

(b) If player i chooses a strategy $x \in (-\infty, t^*)$, then i's expected payoff is:

$$E\left[u_i\left(x, F_j\left(y\right)\right)\right] = w_i\left(x\right), \tag{3}$$

because $x < y$ in this case. From assumption A2, it follows that $w_i(x) \leq w_i(t^*)$ for any $x \in (-\infty, t^*)$. Hence, any $x \in [t^*, +\infty)$ is a best reply to the probability distribution function $F_j(t)$, from (1).

The same argument applies to player j. Hence the distribution functions from (1) characterizes a mixed equilibrium.¥

We turn to the second possibility now. Consider the game $\Gamma_B = (\{i, j\}, \mathsf{R} \times \mathsf{R}, (u_i, u_j))$. Assume that if player i chooses a number $x \in \mathsf{R}$ and player j chooses $y \in \mathsf{R}$, then the payoff functions $u_k : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$, $k = i, j$, are

$$(u_i(x,y), u_j(x,y))_B = \begin{cases} (v_i(x), w_j(x)), & \text{if } x > y \\ (w_i(y), v_j(y)), & \text{if } x < y \\ (\alpha v_i(x) + [1-\alpha] w_i(x), [1-\alpha] w_j(x) + \alpha v_j(x)) & \text{if } x = y \end{cases},$$

where $\alpha \in (0, 1)$. We make the following assumptions

$B1$. $v_k(t)$ is differentiable, $k = i, j$.

$B2$. $v_k'(t) \leq 0$, $k = i, j$.

Assumption $B2$ captures a feature of the first price auction (with or without toeholds): the higher bidder prefers always to reduce his winning bid (provided that no tie occurs).

$B3$. There exists $\bar{t} \in \mathsf{R}$, such that $v_k(t) - w_k(t) > 0$, for all $t \leq \bar{t}$ and $k = i, j$.

This assumption simply says that there exists a threshold $\bar{t}$ such that, if player i chooses a number x below $\bar{t}$, then player j prefers to choose a number y such that $x < y < \bar{t}$. In a two-bidder first-price auction (with or without toeholds) such a threshold will be the minimum of the two bidders' valuations. Finally we assume

$B4$. $\displaystyle\int_{-\infty}^{\bar{t}} \frac{v_k'(t)\,dt}{w_k(t) - v_k(t)} = +\infty$, for $k = i, j$.

The main result for the class of games Γ_B is the following

Theorem 2 Suppose that assumptions B1-B4 hold. Then the game Γ_B admits a continuum of equilibria in mixed strategies . For any $t^{**} \leq \bar{t}$, the following probability distribution constitutes a mixed strategy equilibrium:

$$F_j(t) = \begin{cases} \exp\left[-\int_t^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right], & \text{if } t \leq t^{**} \\ 1, & \text{if } t \geq t^{**} \end{cases}, \tag{4}$$

where $i \neq j$.

Proof. Notice first that distribution function $F_j(t)$ is a positive, strictly increasing function which satisfies $F_j(-\infty) = 0$ and $F_j(t^{**}) = 1$, because of assumption B4. We show now that the distribution functions from (4) constitute a mixed equilibrium. Suppose that player j, $i \neq j$, uses the c.d.f. $F_j(t)$ above, as in the proof for Theorem 1, we have to show that (a) player i's expected payoff $\overline{v}_i^B$ is constant if he chooses a number $x \in (-\infty, t^{**}]$; (b) player i's expected payoff if he chooses a number $x \in (t^{**}, +\infty)$ is at most $\overline{v}_i^B$.

(a) If player i chooses a number $x \in (-\infty, t^{**}]$, then i's expected payoff is:

$$E\left[u_i(x, F_j(y))\right] = \int_{-\infty}^{x} v_i(x)\, f_j(y)\, dy + \int_x^{t^{**}} w_i(y)\, f_j(y)\, dy. \tag{5}$$

¿From the probability distribution function $F_j(t)$ in (4), it is immediate to get the density function $f_j(t)$:

$$f_j(t) = \begin{cases} \frac{v_i'(t)}{w_i(t) - v_i(t)} \exp\left[-\int_t^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right], & \text{if } t \leq t^{**} \\ 0, & \text{if } t \geq t^{**} \end{cases}.$$

The expected i's payoff (5) can be rewritten as

$$E\left[u_i(x, F_j(y))\right] = v_i(x)\, F_j(x) +$$

$$+ \int_x^{t^{**}} \left[v_i(y) + w_i(y) - v_i(y)\right] \frac{v_i'(y)}{w_i(y) - v_i(y)} \exp\left[-\int_y^{t^{**}} \frac{v_i'(s)\,ds}{w_i(s) - v_i(s)}\right] dy.$$

This is equivalent to

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(x\right)F_j\left(x\right)+\int_x^{t^{**}}v_i'\left(y\right)\exp\left[-\int_y^{t^{**}}\frac{v_i'\left(s\right)ds}{w_i\left(s\right)-v_i\left(s\right)}\right]dy+$$

$$+\int_x^{t^{**}}v_i\left(y\right)\frac{v_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int_y^{t^{**}}\frac{v_i'\left(s\right)ds}{w_i\left(s\right)-v_i\left(s\right)}\right]dy,$$

or

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(x\right)\exp\left[-\int_x^{t^{**}}\frac{v_i'\left(s\right)ds}{w_i\left(s\right)-v_i\left(s\right)}\right]-v_i\left(x\right)\exp\left[-\int_x^{t^{**}}\frac{v_i'\left(s\right)ds}{w_i\left(s\right)-v_i\left(s\right)}\right]+$$

$$+v_i\left(t^{**}\right)-\int_x^{t^{**}}v_i\left(y\right)\frac{v_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int_y^{t^{**}}\frac{v_i'\left(s\right)ds}{v_i\left(s\right)-w_i\left(s\right)}\right]dy+$$

$$+\int_x^{t^{**}}v_i\left(y\right)\frac{v_i'\left(y\right)}{w_i\left(y\right)-v_i\left(y\right)}\exp\left[-\int_y^{t^{**}}\frac{v_i'\left(s\right)ds}{v_i\left(s\right)-w_i\left(s\right)}\right]dy.$$

Finally we obtain

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(t^{**}\right)=\overline{v}_i^B \text{ for any } x\in\left(-\infty,t^{**}\right].$$

(b) If player i chooses a number $x\in\left(t^{**},+\infty\right)$, then i's expected payoff is:

$$E\left[u_i\left(x,F_j\left(y\right)\right)\right]=v_i\left(x\right),$$

because $x>y$ in this case. From assumption B2, it follows that $v_i\left(x\right)\leq v_i\left(t^{**}\right)$ for any $x\in\left(t^{**},+\infty\right)$. Hence any number $x\in\left(-\infty,t^{**}\right]$ is a best reply to the probability distribution function $F_j\left(t\right)$ in (4). The same reasoning applies to player j. Hence the distribution function in (4) constitutes a mixed strategy equilibrium. ¥

3 Applications

In this section we provide three examples that fit into the classes of games Γ_A and Γ_B.

3.1 The Bertrand Model

The standard two-firm Bertrand model with unit demand and zero marginal costs belongs to the class Γ_A. The firms' payoff functions write

$$\left(u_i\left(x,y\right),u_j\left(x,y\right)\right)_A=\begin{cases}\left(0,y\right), & \text{if } x>y\\ \left(x,0\right), & \text{if } x<y\\ \left(0.5x,0.5x\right) & \text{if } x=y\end{cases},$$

or $v_i\left(t\right)=v_j\left(t\right)\equiv 0$, $w_i\left(t\right)=w_j\left(t\right)=t$, $\underline{t}=0$, and $\alpha=0.5$. It is easy to check that all assumptions A1-A4 are fulfilled. As a corollary of Theorem 1, we obtain an identical result to the one proposed by Klemperer (2000):

Proposition 3 There exists continuum of equilibria in mixed strategies in the Bertrand model. For any $\underline{p}>0$, the following probability distribution constitutes a mixed equilibrium:

$$F\left(p\right)=\begin{cases}0, & \text{if } p<\underline{p}\\ 1-\frac{\underline{p}}{p} & \text{if } p\geq\underline{p}\end{cases}.$$

Note that the expected profit of each firm in the mixed equilibrium is $\underline{p}>0$. The appealing feature of this class of equilibria is that firms use a very simple Cauchy distribution function which is parametrized by the lower bound of the support $\underline{p}$. The main drawback is that we can always make firms' expected profits arbitrarily high. This is due to the fact that consumers' valuation for the good can be infinitely high.

3.2 Auctions with toeholds

Another class of games that fits into the theoretical model analyzed in Section 2 is a two-bidder auction with toeholds. Two risk-neutral bidders are interested in acquiring an object. Bidder $i\left(j\right)$ has a valuation $v_i\left(v_j\right)$ and owns a share $\theta_i\left(\theta_j=1-\theta_i\right)>0$ of the object. Bidders' values and shares are common knowledge. Bidders submit bids simultaneously. The higher bidder gets the object and pays either her bid in the first price auction, or the opponent's bid in the second price auction. If bids are equal, then the object is allocated to bidder i with probability $\alpha\in[0,1]$ and to bidder j with probability $(1-\alpha)\in[0,1]$. Thus, bidder i's payoff is $v_i-(1-\theta_i)p$, if he wins, and $\theta_i p$, if he loses, where p is the selling price. We consider below two possible mechanisms: first- and second-price auctions.

3.2.1 The Second-Price Auction

Suppose that the two bidders compete in a second-price auction. It is easy to see that the auction game with toeholds is exactly game Γ_A, where

$$(u_i(x,y), u_j(x,y))_A =$$

$$= \begin{cases} (v_i - (1-\theta_i)y, (1-\theta_i)y), & \text{if } x > y \\ (\theta_i x, v_j - \theta_i x), & \text{if } x < y \\ (\alpha[v_i - (1-\theta_i)x] + (1-\alpha)\theta_i x, \alpha(1-\theta_i)x + (1-\alpha)(v_j - \theta_i x)) & \text{if } x = y \end{cases},$$

or $v_i(t) = v_i - (1-\theta_i)t$, $v_j(t) = v_j - \theta_i t$, $w_i(t) = \theta_i t$, $w_j(t) = (1-\theta_i)t$, and $\underline{t} > \max\{v_i, v_j\}$. It is easy to check that all assumptions A1-A4 are fulfilled. As a corollary of Theorem 1, we have

Proposition 4 There exists a continuum of mixed strategy equilibria in the sealed bid second-price auction in which player i randomizes over bids in the interval $[\underline{b}, +\infty)$ according to the distribution function

$$F_i(b) = 1 - \left(\frac{\underline{b} - v_j}{b - v_j}\right)^{1-\theta_i}, \; i \neq j,$$

where $\underline{b}$ is any number greater than $\max\{v_i, v_j\}$.

The interesting feature of this class of mixed strategy equilibria is that the expected payoff to each bidder is $\underline{b} > \max\{v_i, v_j\}$ and is independent from each bidder's valuation!

3.2.2 The First-Price Auction

Suppose that the selling mechanism is the first-price auction. Then, the auction game with toeholds belongs to the class Γ_B with payoff functions

$$(u_i(x,y), u_j(x,y))_B =$$

$$= \begin{cases} (v_i - (1-\theta_i)x, (1-\theta_i)x), & \text{if } x > y \\ (\theta_i y, v_j - \theta_i y), & \text{if } x < y \\ (\alpha[v_i - (1-\theta_i)x] + (1-\alpha)\theta_i x, \alpha(1-\theta_i)x + (1-\alpha)(v_j - \theta_i x)) & \text{if } x = y, \end{cases}$$

that is, $v_i(t) = v_i - (1-\theta_i)t$, $v_j(t) = v_j - \theta_i t$, $w_i(t) = \theta_i t$, $w_j(t) = (1-\theta_i)t$, and $\bar{t} < \min\{v_i, v_j\}$. It is easy to check that all assumptions B1-B4 are fulfilled. As a corollary of Theorem 2, we have

Proposition 5 There exists a continuum of mixed strategy equilibria in the sealed bid first-price auction in which player i randomizes over bids in the interval $(-\infty, \bar{b}]$ according to the distribution function

$$F_i(b) = \left(\frac{v_j - \bar{b}}{v_j - b}\right)^{\theta_i},\ i \neq j,$$

where $\bar{b}$ is any number lower than $\min\{v_i, v_j\}$.

The expected payoff of bidder i in the mixed strategy equilibrium is $v_i - (1-\theta_i)\bar{b}$, where $\bar{b} < \min\{v_i, v_j\}$. Note that this payoff does depend upon bidder i's valuation, which was not the case in the second price auction.

4 Conclusion

The main feature of the class of games studied in this paper is the presence of externalities between players. We have pointed out that the use of a random tie-breaking rule makes this game discontinuous. We have shown that, if the players' strategy space coincides with the set of real numbers, a continuum of Nash equilibria in mixed strategies do exist.

One might wonder what would happen if we modified the game in such a way to allow for a deterministic tie-breaking rule. In our toeholds example, since valuations are common knowledge, one could think of breaking a tie in favor of the bidder with the higher valuation for the object. This formulation has been analyzed by Ettinger (2001). The author shows that the first- and second-price auctions admit a unique equilibrium in undominated strategies. It is easy to prove that, if bidders can play weakly dominated strategies, the set of equilibrium outcomes both in first- and second-price auctions will coincide with the interval between bidders' valuations[1]. However, whenever a random tie-breaking rule is introduced, the set of equilibrium outcomes will be found outside the interval between bidders' valuations.

[1] A proof of this statement can be provided by the authors upon request.

References

[1] Ettinger, D. (2001): “Bidding among friends and enemies”, CERAS, *mimeo*.

[2] Dasgupta P. and E. Maskin (1986): ”The Existence of Equilibrium in Discontinuous Economic Games, I: Theory”, *Review of Economic Studies*, 1-26.

[3] Dasgupta P. and E. Maskin (1986): ”The Existence of Equilibrium in Discontinuous Economic Games, II: Applications”, *Review of Economic Studies*, 27-41.

[4] Klemperer P (2000): ”Why Every Economist Should Learn Some Auction Theory”, Invited Lecture to 8th World Congress of the Econometric Society.

[5] Maskin E. (1986): ”The Existence of Equilibrium with Price-Setting Firms”, *American Economic Review*, 76, 382-386.

[6] Simon L. (1987): ”Games with Discontinuous Payoffs”, *Review of Economic Studies*, 569-597.

[7] Simon L. and W. Zame (1990): “Discontinuous Games and Endogenous Sharing Rules,” *Econometrica*, 58, 861-872.

[8] Reny P. (1999): ”On the Existence of Pure and Mixed Strategy Nash Equilibria in Discontinuous Games,” *Econometrica*, 67, 1029-1056.