UCL CENTRE FOR ADVANCED SPATIAL ANALYSIS

UCL

WORKING PAPERS SERIES

Paper 199 - Apr 15

Spatial Interaction, Imperfect Competition and the Evaluation of User Benefit

ISSN 1467-1298

casa

Centre for Advanced Spatial Analysis University College London 1 - 19 Torrington Place Gower St London WC1E 7HB
Tel: +44 (0)20 7679 1782 casa@ucl.ac.uk www.casa.ucl.ac.uk

Spatial Interaction, Imperfect Competition and the Evaluation of User Benefit

Robin Morphet

April 27th 2015

Abstract

The spatial interaction model as represented by the entropy maximising trip distribution model is located in economic theory as a model that represents imperfect competition. In doing so it minimises the consumer deadweight loss associated with imperfect competition although it never eliminates it. The evaluation of benefits under imperfect competition is shown to require the inclusion of changes in land rent but is otherwise similar to the standard cost benefit analysis. A worked example suggests that the inclusion of rents and a correction for double counting results in lower estimates of benefit than obtain when only trip costs are measured.

> *"..regional scientists and geographers have developed several models, ranging from the entropy (Wilson 1967) to the gravity and logit models(Anas 1983), which have proven to be very effective in predicting different types of flows. By ignoring for a long time this body of research, spatial economists have missed a fundamental ingredient of the space-economy."*
>
> Fujita and Thisse 2013

1. Introduction

The reason for examining this area of evaluation is because of its importance in evaluation of transport improvements user conditions of imperfect competition, the understanding of agglomeration in urban areas and the understanding of the spatial interaction model itself.

The practise of cost benefit analysis assumes perfect competition both in relation to the system changes being analysed and in the wider economy. Prest and Turvey (1968) outline four cases where problems may arise. First the public authority concerned may exhibit monopoly behaviour itself, second downstream benefits may arise in monopolistic markets, third the prices of factors of production may include a rental element, and fourth, average costs may exceed marginal costs. All of these imperfections may give rise to errors in evaluation. However, the SACTRA (1998) report identified agglomeration economies as potential benefits which may depend to an extent on the presence of market imperfections. In this analysis we will be primarily concerned with the second case where the impact of monopolistic competition between household and between firms may result in land rent based welfare changes greater than might be expected under perfect competition. Lind (1973) offers three reasons for examining the role of rents in benefit analysis. Firstly, many public interventions have outputs concentrated in specific locations (e.g. flood defences). Secondly, the interrelation between land value changes and total benefits can be difficult to analyse and thirdly, the difficulty of estimating land value changes as a result of intervention is difficult. The analysis of Lind assumed perfect competition which he recognised might be inadequate in the context of the Koopmans and Beckmann (1957) analysis of the quadratic assignment problem in which a given set of interdependencies between firms could preclude a perfectly competitive equilibrium.

The difficulties in assuming perfect competition in a spatially organised economy were shown as being generally applicable by Starret (Starret, 1974; Fujita and Thisse, 2013) who demonstrated that, in a homogeneous space, the competitive equilibrium of perfect competition cannot exist other than under autarky[1] or with zero transport costs. Any spatial interaction must therefore be analysed in terms of imperfect competition. Since, in our derivations of spatial interaction models the origins and destinations are fixed, there can be no autarky unless the origins and destinations for each zone are equal. In this case the nearest we can get to autarky is the filling of only the diagonals of the interaction matrix using zero intrazonal trip costs, but this is unrealistic. We are thus left with the zero transport cost (but non-zero rent) option which in this exercise, we use as a baseline to estimate the consumer deadweight loss, travel and rent expenditure and consumer surplus that result from the spatial interaction under the imperfect competition that arises when travel costs are greater than zero. Starret's economic analysis may be compared to that of S. Evans(1973) in which as β (see equation (1)) tends to infinity and the effect of cost is reduced, so the trip matrix approaches the maximum efficiency given by the solution to the transportation problem of linear programming. In pursuing this analysis we pose a possible response to the above quotation of Fujita and Thisse(2013, p256) and to the statement in Tavasszy (Tavasszy et al, p105) that *"Markets which are notoriously imperfect, such as land and labour have not yet been fully incorporated into the wider economy models".* The basic assumption of the models discussed below, is that transport and land use must be modelled as a joint market and that, in part, our traditional transportation models already do this. In developing the analysis we are conscious of the limitations of the spatial interaction model as an equilibrium analysis which may preclude or require adjustment for some standard economic analyses. The analysis allows the estimation of the double counting between land rent and trip cost savings and it also allows a reinterpretation of that part of consumer surplus gain attributed to new users and not covered by the direct savings in trip costs. The inclusion of land rent within the transport model is shown to be essential to maintain the form of the demand function and hence of the model and also to determine the perfect competition baseline.

2. Imperfect Competition

Diagram 1 below shows the conventional analysis under perfect competition with demand and supply in equilibrium at point B. The inverse demand function is given, in the unconstrained case by

[1] Autarky describes an economy based on self sufficient local subsistence.

$$T_{ij} = N \frac{e^{-\beta c_{ij}}}{\sum_i \sum_j e^{-\beta c_{ij}}} \tag{1}$$

where N is the total number of trips. We use the inverse demand function where quantity is on the y axis and cost (price) on the x axis as this simplifies the integration required in the derivation of consumer surplus and reflects the mathematical view of equation(1) as the definition of a function. The consumer surplus is given by area CB∞ which is the integral $\int_C^\infty T_{ij} dc_{ij}$ for one trip interchange, *ij*. For the system as a whole this becomes $\sum_i \sum_j \int_C^\infty T_{ij} dc_{ij}$ (Neuburger, 1971) although it may also be written as $\int_C^\infty \sum_i \sum_j T_{ij} dc_{ij}$ (Glaister, 1981 eq.2.43) to allow for simultaneous change in prices.

Under imperfect competition the price is set higher (or lower) than that required for perfect competition as shown in diagram 2[2] and , given a new transport development, the price is reduced and the producer surplus may also be reduced. In the model that we use to test our results, the origins and destinations are fixed for both base and design years so as trips are diverted from one *ij* interchange to another they will rise for one interchange and fall for another.

The standard entropy maximising model is given by (Morphet, 2013)

$$p_{ij} = \frac{e^{-\lambda_i - \lambda_j - \beta c_{ij}}}{\sum_i \sum_j e^{-\lambda_i - \lambda_j - \beta c_{ij}}} \tag{2}$$

where the partition function Z is given by

$$Z = \sum_i \sum_j e^{-\lambda_i - \lambda_j - \beta c_{ij}} \tag{3}$$

The demand curve, or rather, a succession of equilibrium points on the demand curve, in figure 1 is determined by equation (2) which may equivalently be written

$$p_{ij} = -\frac{1}{\beta} \frac{\partial \ln(Z)}{\partial t_{ij}} \tag{4}$$

where t_{ij} is given by equation (11) effectively combining the exponents into a single cost of $\lambda_i + \lambda_j + \beta c_{ij}$ (expressed in information terms). We see that once β is fixed by calibration then the demand curve, is determined by Z. Equation (2) is equivalent to an unconstrained model with cost exponent but for the Starret condition, c_{ij} goes to zero and we have

$$p_{ij} = \frac{e^{-\lambda_i - \lambda_j}}{\sum_i \sum_j e^{-\lambda_i - \lambda_j}} = \frac{e^{-\lambda_i - \lambda_j}}{\sum_i e^{-\lambda_j} \sum_j e^{-\lambda_i}} = p_i p_j \tag{5}$$

[2] Appendix 3 shows the various configurations of deadweight loss according to the various rank orders of new cost, old cost and cost at perfect competition.

thus the proportion of trips between *i* and *j* is the product of the independent origin and destination probabilities. When a cost is present these probabilities are in general not independent unless the costs are all equal. However, as we will see, it is the divergence from independence that characterises spatial interaction under imperfect competition.

If we treat the $\frac{\lambda}{\beta}$ terms as land rent per trip (Morphet, 2013) we have

$$\frac{\lambda_i}{\beta} + \frac{\lambda_j}{\beta} = -\frac{1}{\beta}\ln p_i - \frac{1}{\beta}\ln p_j = -\frac{1}{\beta}\ln p_i p_j \qquad (6)$$

as shown in diagram 2 where the $-\frac{1}{\beta}\ln p_i p_j$ term appears as the residual rent (which we term the Starret rent) when c_{ij} goes to zero. It should be noted that this differs from the baseline given by the solution to the transportation problem which is approached as β approaches infinity (Evans, 1973) but the Starret rent baseline preserves the biproportionality of the model. It should be noted that in making this calculation the value of Z has changed which means, from equation (4) that the demand curve has changed. However, from diagram 2 we will see that the equilibrium point of perfect competition should lie on the same demand curve as that given by the general expression for p_{ij}. In Appendix 4 we show how the two demand curves may be brought into alignment.

We may explore equation (2) at a macroscopic level by taking logarithms multiplying by p_{ij}, summing over *i j*, and then dividing by β giving

$$\frac{1}{\beta}\sum_i\sum_j p_{ij}\ln p_{ij} = -\sum_i\sum_j p_{ij}\left(\frac{\lambda_i + \lambda_j}{\beta} + c_{ij}\right) - \frac{1}{\beta}\ln Z \qquad (7)$$

Which on rearrangement gives

$$G = C - \frac{1}{\beta}S + \rho_i A_i + \rho_j A_j \qquad (8)$$

Where $G = -\frac{1}{\beta}\ln Z$ is the Gibbs free energy, S is entropy, $C = \sum_i\sum_j p_{ij}c_{ij}$ the trip cost energy (internal energy) and $\sum_i\sum_j \rho_i A_i + \rho_j A_j = \sum_i\sum_j p_{ij}\frac{\lambda_i + \lambda_j}{\beta}$ where ρ is rent per unit area and A is area. This is the equivalent to the standard expression in statistical mechanics for the Gibbs free energy of

$$G = U - TS + PV \qquad (9)$$

with $1/\beta$ playing the role of temperature and ρA playing in two dimensions, the role of PV, which is pressure times volume, in three dimensions. The Gibbs free energy is equivalent to the measure of consumer surplus in standard economic theory (Williams and Senior 1978) although it should be borne in mind that in the context of spatial interaction it has the same sign as a reduction in trip costs, i.e. its increase will generally register as negative.

3. Consumer Surplus

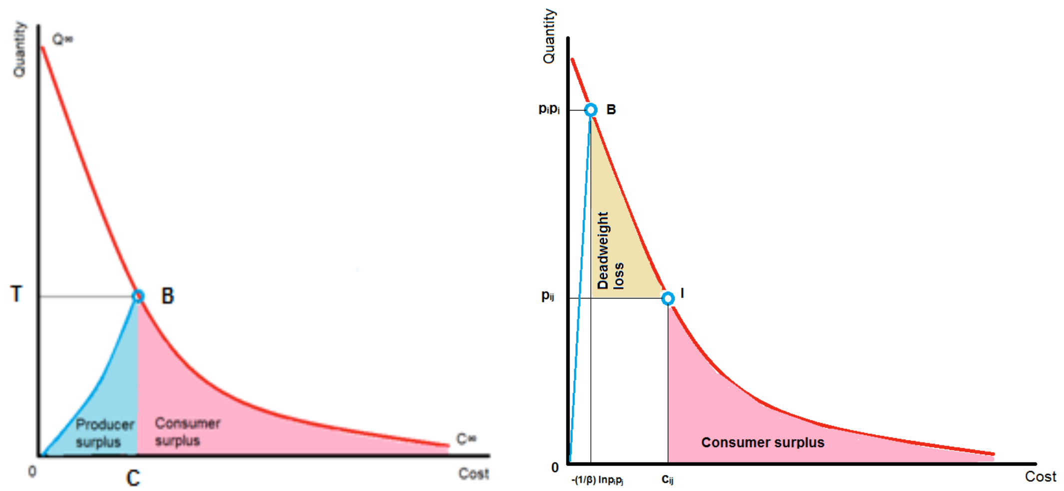
Under the perfect competition conditions of diagram 1 we calculate the consumer surplus, G, as the area $CBC\infty$.

$$
\begin{aligned}
G &= \sum_i \sum_j \int_{c_{ij}}^{\infty} p_{ij}\, dc_{ij} \\
&= \int_{c_{ij}}^{\infty} \sum_i \sum_j p_{ij}\, dc_{ij} \\
&= \int_{c_{ij}}^{\infty} \frac{e^{-\lambda_i - \lambda_j - \beta c_{ij}}}{Z}\, dc_{ij} \qquad \forall i, j \qquad\qquad (10) \\
&= \Big|_{c_{ij}}^{\infty} -\frac{1}{\beta} \ln Z \qquad \forall i, j \\
&= \frac{1}{\beta} \ln Z
\end{aligned}
$$

The resulting integral is a macroscopic state function and hence path independent (see Appendix 2). It should be noted that the integral of Williams and of Neuburger includes the balancing factors (Williams, 1971, Appendix B) although they are not explicitly recognised as von Thunen rents.

4. Deadweight Loss

The case of perfect and imperfect competition is shown in Diagrams 1 a and b. These diagrams represent the position for a single *ij* interchange and it needs to be summed across all *ij* interchanges to get the overall values of trip expenditures and benefits. In such a summation, because trips may rise or fall there is some cancellation in the estimation of the relevant quantities to the extent that some areas may reduce. The deadweight loss or welfare cost (Harberger, 1964) is a measure of the allocative inefficiency of a market under imperfect competition. The consumer element of deadweight loss is shown in diagram 2 as the area *ac1*. A notional producer deadweight loss (assuming a linear function passing through the origin) is given by the area *abc*.



a) Perfect   b) Imperfect

Diagram 1: Perfect and Imperfect Competition

The model says nothing about the supply side so the producer surplus will not figure in this analysis in the same way that it does not form part of the standard methods of benefit analysis. the diagram is slightly unrealistic because it suggests that there is a single demand curve which is continuous but the model only identifies points of equilibrium on a demand curve. In fact there are in diagram 3, two diagrams masquerading as one – a demand curve for the base year on which point 1 lies and a demand curve for the predicted year on which lies point 2. The standard calculation (e.g. the rule of a half) assumes the demand curve is unchanged by the cost changes and for small changes this may be a reasonable assumption. However , it is an unnecessary assumption since knowing the cost changes we can calculate the new value of *Z* which corresponds to a new demand function. The slightly unrealistic diagram 2 for this analysis shows imperfect competition defined by the fact that $c_{ij}^1$and $c_{ij}^2$are both greater than the Starret optimum price. We carry out a similar integration as in section 3 for the area *aeg* which we designate as E, under the inverse demand curve of diagram 2. However, in this case we subsume the generalised trip cost and the rents into a new generalised trip cost, t_{ij}

$$t_{ij} = c_{ij} + \frac{\lambda_i}{\beta} + \frac{\lambda_j}{\beta} \tag{11}$$

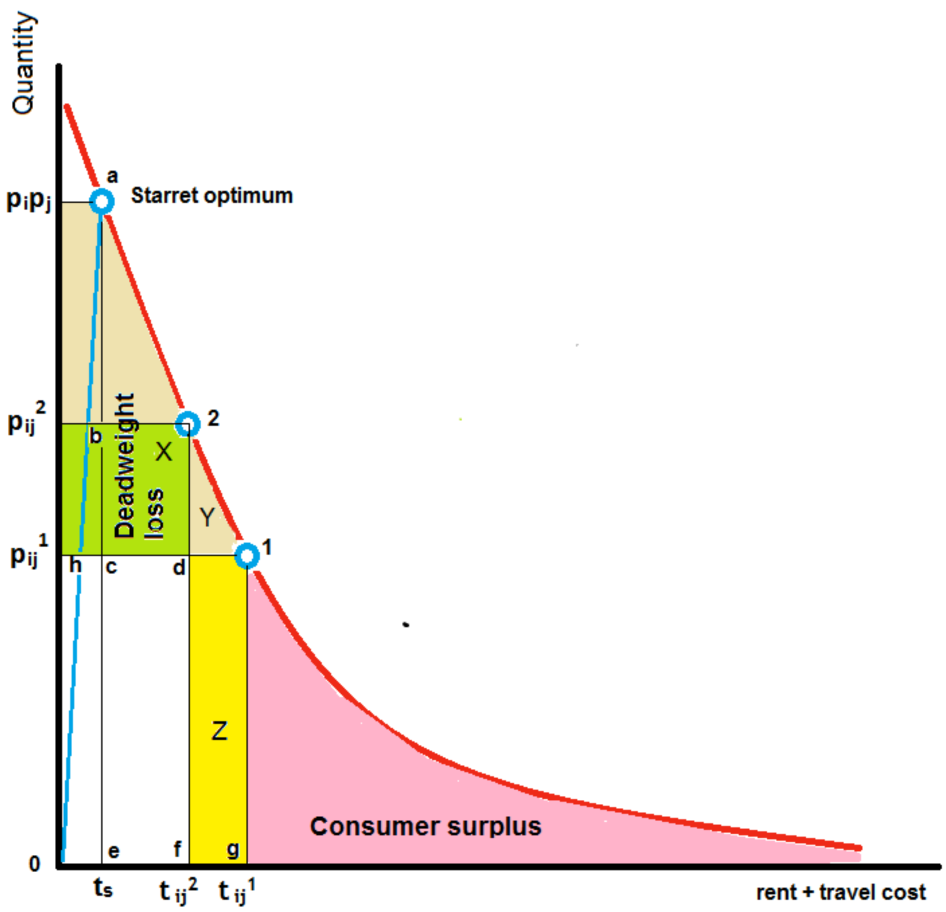


Diagram 2: Imperfect Competition: Travel Cost and Rent

$$E = \int_{-\frac{1}{\beta}\ln p_i p_j}^{t_{ij}} p_{ij} dt_{ij} = \int_{-\frac{1}{\beta}\ln p_i p_j}^{t_{ij}} \frac{e^{-\lambda_i - \lambda_j - \beta t_{ij}}}{Z} dt_{ij} = \left[-\frac{1}{\beta} \ln Z \right]_{-\frac{1}{\beta}\ln p_i p_j}^{t_{ij}}$$

$$= -\frac{1}{\beta}\ln Z + \frac{1}{\beta}\ln \sum_i \sum_j p_i p_j = -\frac{1}{\beta}\ln Z \qquad (12)$$

We may then estimate the area of deadweight loss DWL, (area *ac1* beneath the inverse demand curve and above p_{ij}^{1}) from equation (13) writing p_{ij} for p_{ij}^{1} and as this is a general derivation.

$$\begin{aligned}
DWL &= p_{ij} E - p_{ij}\left(c_{ij} - \left(-\frac{1}{\beta}\ln p_i p_j \right)\right) = E - p_{ij}\left(c_{ij} + \frac{1}{\beta}\ln p_i p_j \right) \\
&= -p_{ij}\frac{1}{\beta}\ln Z - p_{ij}\left(c_{ij} + \frac{1}{\beta}\ln p_i p_j \right) \\
&= -p_{ij}\frac{1}{\beta}\ln Z - p_{ij}c_{ij} - \frac{1}{\beta} p_{ij}\ln p_i p_j \\
&= -p_{ij}\frac{1}{\beta}\ln Z + \frac{1}{\beta} p_{ij}\ln e^{-\beta c_{ij}} - \frac{1}{\beta} p_{ij}\ln p_i p_j
\end{aligned} \qquad (13)$$

This area corresponds to the deadweight loss to consumers taken over one demand curve (i.e. a single value of *Z*) which means we can legitimately estimate its change by subtracting the value based on the base demand curve from that of the predicted demand curve.

So summing over *ij* and bearing in mind that *Z* is a constant, we have

$$\sum_i \sum_j DWL = \frac{1}{\beta}\sum_i \sum_j p_{ij} \ln \frac{p_{ij}}{p_i p_j} \qquad (14)$$

Which means that the expected value of the deadweight loss is given by the mutual information times $\frac{1}{\beta}$. The possible configurations of deadweight loss change are shown in Appendix 4. Fortunately the measure of equation (14) applies in all such cases. This means that we can reinterpret the information minimising model (Morphet 1974, Snickars and Weibull 1977, Williams and Senior 1978) as one that minimises the consumer deadweight loss. Such a model minimises a measure of economic inefficiency so far as possible consistent with the constraints. It may also be characterised as that model which, subject to its constraints, minimises the Kullback-Leibler divergence from independence (see equation (5)). We set up the model thus

$$\mathbf{L} = \sum_i \sum_j p_{ij} \ln \frac{p_{ij}}{p_i p_j} - \sum_i \lambda_i \sum_j p_{ij} - p_i - \sum_j \lambda_j \sum_i p_{ij} - p_j - \beta \sum_i \sum_j p_{ij} c_{ij} - C - \lambda_0 \sum_i \sum_j p_{ij} - 1 \qquad (15)$$

Differentiating with respect to p_{ij} and setting the expression to zero gives

$$p_{ij} = p_i p_j \frac{e^{-\lambda_i - \lambda_j - \beta c_{ij}}}{Z} \qquad (16)$$

where

$$Z = \sum_i \sum_j p_i p_j e^{-\lambda_i - \lambda_j - \beta c_{ij}} = e^{\lambda_0} \qquad (17)$$

This is the model posited by Williams and Senior (1978), although not in the context of imperfect information. It is also similar but not identical, to the model suggested by Roy(2004, p32)

Applying the same transformation to equation(16) as in (2) gives

$$G = C - \frac{1}{\beta} S + \rho_i A_i + \rho_j A_j - \frac{1}{\beta} \sum_i \sum_j p_{ij} \ln p_i p_j \qquad (18)$$

The additional term in $p_i p_j$ is the rent expected at the point of perfect competition and zero transport cost. The model gives identical results, in terms of p_{ij}, to the standard entropy maximising model of equation(2) so the Starret rent terms may be combined with the λ terms of equation(16) to give the λ terms of equation(2).

It should be noted that the consumer surplus function for this model is formally similar to that for perfect competition and will , in general, have the same value. We might expect that the impact of imperfect competition would reduce the value of the user benefit compared to that under perfect competition but we will see in section 6 that this is not necessarily the case(see equation (26).

We now repeat the analysis of equation using the model of equation (16) but including the exponents in one term t_{ij}, as in equation (11). This combines generalised trip cost and land rent cost per trip into a new generalised cost, t_{ij} as shown in diagram 2.

We may write for the change in t_{ij} from *1* to *2*

$$\Delta G = \int_1^2 \sum_i \sum_j p_{ij} dt_{ij} = \int_1^2 \sum_i \sum_j \frac{e^{-\beta t_{ij}}}{Z} dt_{ij} = \left[-\frac{1}{\beta} \ln Z \right]_1^2 = \frac{1}{\beta} \ln \frac{Z^1}{Z^2} \qquad (19)$$

It should be emphasised that this change in G is for all interchanges, *ij* and the individual change for interchange *ij*, will be $\frac{1}{\beta}\left(p_{ij}^1 \ln Z^1 - p_{ij}^2 \ln Z^2\right)$ reflecting the proportion or number of trips, depending on whether expected or total values are being used. Of course where we are using probabilities which sum to one then the overall change is simply that shown in equation (19)

It should be noted that in carrying out this integration for deadweight loss (equation(12)) we are assuming that the point of perfect competition will lie on the relevant demand curve. In practice it is unlikely that this will be so and the relevant correction, involving a shift of consumer surplus into rent, is shown in Appendix 4.

5. Maximising the Matching Surplus

The theory of matching (Galichon and Salichie, 2010) in labour markets suggests that the maximisation of the surplus due to matching is consistent with the minimisation of mutual information. Galichon and Salichie construct the expression $E_\pi \Phi(X,Y) - \sigma E_\pi \log \frac{\pi(X,Y)}{P(X)Q(Y)}$ where the first term is a measure of the utility of matching and the second a mutual information measure of deviation from random matching weighted by σ, a measure of the unobserved heterogeneity in the utility of individual pairings not unlike the unobserved random utility of discrete choice theory.

The journey to work matrix is one which shows the attempt to match people with jobs to the benefit of both workers and employers. We may thus argue that minimising the consumer deadweight loss is consistent with maximising a labour market matching surplus.

6. Von Thunen Rents in Appraisal

In the conventional entropy maximising model the von Thunen rents are given by $\frac{\lambda_i}{\beta}$ and $\frac{\lambda_j}{\beta}$ (Morphet, 2013) . In the mutual information model the terms in *i* and *j* including the p_i, p_j terms may be aggregated in the exponent to give rents of $\frac{\lambda_i - \ln p_i}{\beta}$ and $\frac{\lambda_j - \ln p_j}{\beta}$. The two sets of terms for rent should be equivalent up to an additive constant as the model gives rise to the same values of p_{ij} for the same sets of origins and destinations and trip costs.

Under perfect competition we might expect the welfare change as a result of transport cost reductions, to be fully reflected in the rent change (Mishan 1959, SACTRA, 1998). Under imperfect competition we might expect the benefits to be included in both rent change and trip cost change. We can see this more clearly if we modify equation (16) by first taking logarithms, then multiplying all through by p_{ij} and then summing over *i* and *j* and dividing all through by β giving, at equilibrium

$$\frac{1}{\beta}\sum_i \sum_j p_{ij} \ln\left(\frac{p_{ij}}{p_i p_j}\right) = -\left\langle \frac{\lambda_i + \lambda j}{\beta} \right\rangle - \left\langle c_{ij} \right\rangle - \frac{1}{\beta}\ln Z \qquad (20)$$

Which we may write as

$$-G - \frac{1}{\beta} I = \mathrm{C} + \mathrm{R} \qquad (21)$$

Where G is consumer surplus, I is the mutual information modified in this case by $1/\beta$ to convert it from an information to an energy measure, C is trip cost and R is rent. The rents are different from those in equation(8) since the base line Starret rent, $-\frac{1}{\beta}\ln p_i p_j$ is included in the $p_i p_j$ terms. From equation (21) we may write

$$-dG - \frac{1}{\beta} dI = d\mathrm{C} + dR \qquad (22)$$

or

$$\Delta R = \Delta G + \frac{1}{\beta}\Delta I - \Delta \mathrm{C} \tag{23}$$

This is confirmed in Appendix 1, tables A1.1 and 2.

We can see from diagram 2 that the total benefit is given by areas X+Y+Z against which are offset the increase in costs of X . The total benefit is thus Y+Z, which may be familiar as the standard consumer surplus diagram although in this case rents are included. It can however be expressed in a deadweight form as $-\frac{1}{\beta}\Delta I - Z$ bearing in mind that benefits from I and Z are both negative. We have used diagram 2 although we saw its shortcomings earlier in that it subsumes the base year and the planning year demand curves into one. However, if we use the deadweight formulation of benefit it can be written as the difference between the state variables of the two demand curves. We may write using the corrected rents

$$Benefit = \left(\frac{1}{\beta}\sum_i\sum_j T_{ij}^1 \ln\frac{p_{ij}^1}{p_i p_j} + \sum_i\sum_j T_{ij}^1 t_{ij}^1\right) - \left(\frac{1}{\beta}\sum_i\sum_j T_{ij}^2 \ln\frac{p_{ij}^2}{p_i p_j} + \sum_i\sum_j T_{ij}^2 t_{ij}^2\right) \tag{24}$$

but we see from table A1.2 that

$$\begin{array}{lclcl} \Delta(\text{Deadweight Loss}) & + & \Delta(\text{Trip Cost Savings}) & = & \Delta\text{Rent} \\ 10635.41 & + & 5493.987 & = & 16129.4 \end{array} \tag{25}$$

The left hand side of equation(25) equals the right hand side of equation (24) and we see that rental change will only equal trip cost savings when deadweight loss is zero, i.e. at perfect competition. We see that there is double counting between rent and trip costs but the rent exceeds the trip cost savings by the change in deadweight loss.

Using the uncorrected rents we have

$$\begin{array}{lclcl} \Delta(\text{Deadweight Loss}) & + & \Delta(\text{Trip Cost Savings}) & = & \Delta\text{Rent} + \text{Consumer Surplus} \\ 10635.41 & + & 5493.987 & = & 9989.862 + 6139.538 \end{array} \tag{26}$$

This reflects the fact that the corrected rents are the uncorrected rents increased by the consumer surplus. It indicates the fact that the consumer surplus is not an appropriate measure of benefit under imperfect competition. Interestingly, if we calculate the benefits using trip costs only and the rule of a half , we get a value of 16902.61 which is less than 5% greater than the deadweight loss calculation. The rule of a half calculation averages across two demand curves. When these are close the error involved is small as in this case. However the rule of a half could be inappropriate where the demand curves are more separated whereas equation (25) will be correct. It should be noted that the trip cost savings identified under the rule of a half are greater than the trip cost savings in the model (equation(25)). This difference arises because the demand curve of the model is based on costs which include rent (diagram 4) whereas the trip cost only model uses a similar shape of demand curve (because the rents are included as balancing factors) but neglects to include the rents in the cost. If the rule of a half were to be consistent it would use the demands identified in diagram 5, although this would obviously cause other problems.

Because we are analysing the difference between two demand curves rather than subsuming them into one, the method is robust even when the changes in the curves are large. Moreover, we are limiting ourselves to the discrete information contained within the model at equilibrium so that no assumptions of a continuous demand curve need be made.

The numerical analysis in Appendix 1 is based on the Arcadia data with the trip cost matrix modified to represent a crossing in East London. The model is not considered particularly realistic as it does not include an assignment stage. However, it brings out some aspects of the model such as the variety of patterns of consumer deadweight loss that may arise in practice and their relative frequency (see Appendix 3).

7. The Necessity for Rent

Without rent we are unable to define a point of perfect competition on the demand curve, which means that we would be unable to analyze the effect of imperfect competition. We know that the city operates in the context of imperfect competition (Fujita and Thisse, 2013) hence effective modelling of transport requires the inclusion of rents. Their analysis under imperfect competition shows them to be a significant contributory factor in benefit evaluation after the effects of double counting have been removed.

The interaction model without explicit recognition of rent is given by equation(2). When the total number of trips is constrained in overall numbers, then a reduction of trip costs in one area will result in increased trips in that area and necessarily, reduced trips elsewhere. The cost, c_{ij}, is fixed so a reduction in p_{ij} would mean that the number of trips between i and j would no longer fall on the demand line thus breaking the consistency of the model. In practice the balancing factors bring the model back into line. Given that conformance with the demand function is required, only a change in cost can bring this about. This means that the λ values, which are a functionally related to the balancing factors, must reflect costs which for dimensional consistency, must be of the form $\frac{\lambda}{\beta}$ and which have been shown to be von Thunen land rents per trip (Morphet 2013). The effect of a reduction in p_{ij} implies an increase in rent per trip which is a welfare increase per trip. Whether this implies an increase in rent overall depends on whether the effect of the increase in rent per trip outweighs the effect of the decrease in trips. However, it is in such areas where should rents per unit area rise, they may be reduced by the densification of the affected zones. Such areas may not be those traditionally associated with the first order effects of transport improvements e.g. around new railway stations but rather the less noticeable second order effects in more distant areas.

The use of land rents in analysis gives a good indication of where the benefits are distributed geographically and they should also inform any post implementation analysis of the proposed improvement and the timing of benefit realisation. Information on land rent changes should be easier to obtain than repeating the survey and modelling process.

A comparison of diagrams 4 and 5 shows how much better behaved the model is when rents are included.

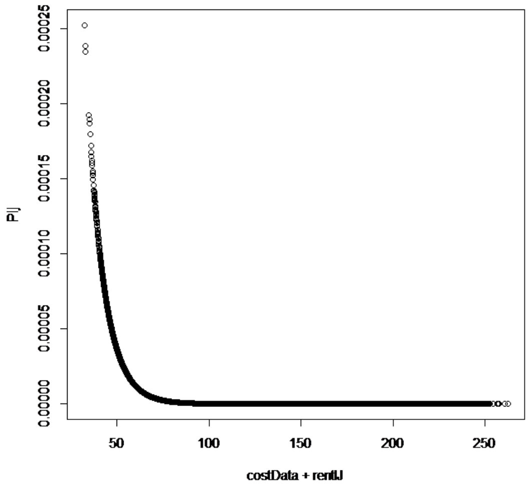


Diagram 4: Demand including rent and travel cost

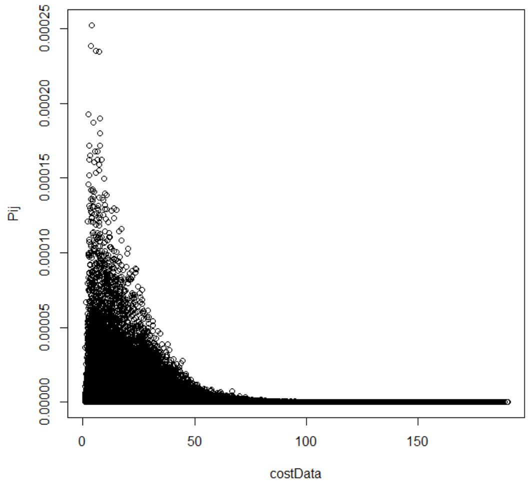


Diagram 5: Demand including only travel cost

8. Conclusions

The model we have used is simple being all modes on a single network with no overall growth in trip ends. Yet it is of a size and complexity which allows some testing of the theoretical conclusions set out above. It is sufficiently testing to expose some of the strengths and shortcomings of the statistical mechanical method of inference and to show the need to use more economic arguments when it comes to issues of double counting. However, we have sought to limit our analysis of benefits to what is permitted in the model. Principally this means that there can be no coalescing of before and after demand curves and so far as possible there can be no moving up and down a demand curve particularly a coalesced curve. It may be argued that where we use integration we are moving along the demand curve but the answers that we get from the integrations (for deadweight loss and consumer surplus) are expressible as simple relations amongst the variables of state.

We have shown how the conventional entropy maximising spatial interaction model, when viewed in the context of Starret's Impossibility Theorem, produces an information minimising model in which the minimand is the mutual information. We have shown that minimising the mutual information is equivalent to minimising the consumer element of the deadweight loss thus placing the derivation firmly in the context of the economics of imperfect competition. Further we can see that such a minimisation is the natural consequence of the underlying labour market seeking to achieve an optimal matching between employers and employed. This latter consideration opens up avenues for further research on the utility transfers between employers and employed and the impact of space and travel cost. On the other hand the reduction in consumer deadweight loss is a benefit only if the unused capacity exists and can be brought into use. Monopolistic competition suggests that such capacity should be available although the lags in adaptation in the housing and public transport markets might delay benefit realisation, as might regulation. However, the trickle down and multiplier effects of the relaxation of such quantity constraints may result in additional welfare effects (Starret, 1988 p159). The concept of deadweight loss is more often seen in the context of taxation which is viewed as a disbenefit which distorts perfect competition. However, deadweight loss is an essential component of imperfect competition and of the spatial economy. It implies increasing returns and hence agglomeration which may be viewed as desirable. The minimisation of consumer deadweight loss may reduce the level of agglomeration on which cities depend and the question of the appropriate size of consumer deadweight loss and its relation to the extent of agglomeration is an area of potential further research.

The increase in land rents and their capitalisation may be considered inequitable as it accrues to land owners (Starret, 1988 p229). For a benefit that is largely created by public investment and by the location choices of the population at large this may seem unfortunate unless a part of it is retrieved through such taxation as tax increment financing (TIF), although this should be taxation on land and not on development and may be contentious in implementation (McInnis, 2015). The analysis set out above may prove useful in the implementation of TIF as proposed (Sandford, 2014) in Battersea (Pickford, 2013) as it will inform the definition of the areas on which the burden of TIF is to fall.

The transfer of consumer surplus to rent involved in the rent correction of Appendix 4 represents the capture of benefits by land owners. If such benefits are required so that they may be taxed then it seems clear that there needs to be an emphasis on benefit realisation which will need to include land use changes that encourage an increase in allocative efficiency and that the projects under consideration should encompass both transport and land use changes and the means used to bring them about. The change in rents may be expected to be larger than that in trip costs since, in considering a trip interchange, ij, the occupiers of i, may only include a minority who travel to destinations that make direct use of the reduced trip cost. However, as the rents of these users rise in relation to reduced trip costs, so do those of their neighbours as they are in the same market. This market, after allowing for differences in size, environment etc., is a market for job potential which takes into account all potential destination zones. The double counting of reduced trip costs and increased rents which, under perfect competition, would be total, is seen under imperfect competition to be partial but decreasingly so as the combined land and transport market moves closer to perfect competition. The ratio of consumer deadweight loss to consumer surplus might be seen as a measure of the efficiency of the land transport market for consumers in the given study area. The question of producer surplus has been neglected both here and in the wider literature. However, on the land use side the rents include those rents accruing to land owners and paid by firms in the destination zones. It may be that, given the levels of subsidy to transport users, there is no real producer surplus. However the treatment of taxation and subsidy, important as it is, is beyond the scope of this note which has concentrated on the estimate of a naive Marshallian consumer surplus in common with much of the literature and current practice. However, there is a close relationship between deadweight loss and the equivalent and compensating variation of Hicks (Prest and Turvey, 1968)

We see that entropy is the dominant part of the deadweight loss and hence the dominant aspect of imperfect competition in the model. Revisiting the derivation of rents from the von Thunen model (Morphet, 2013) we see that we effectively use entropy to relate the von Thunen model of perfect competition to that of imperfect competition. Entropy would seem to be the signature of imperfect competition.

It can be argued (Greenhat et al, 1987, p3) that the spatial analysis of a market is simply a reflection of the wider case of separated markets. Markets may be separated by time, product variety, tariff barriers, product storage time etc. The analysis in space of the interaction model may therefore have wider application where the equivalent of transport cost can be determined.

Appendix 1

Outline Analysis for London and the Outer Metropolitan Area

The study area is shown below (Diagram A1.1) with a red boundary inside which is the blue boundary of the Greater London area inside which a red circle indicates the location of the transport improvement to be evaluated. This site is shown in more detail in diagram AI.2.

Diagram A1.3 a) below shows the plot of trip probabilities against the total of expected trip cost plus expected origin and destination rents for the Arcadia (Batty, 2009) area of London and the South East. The beta value was 0.1095119. Its smoothness suggests a continuous demand curve although it should be borne in mind that it is a composed of a set of equilibrium points. The linearity in diagram b) reflects the exponential structure of the model.

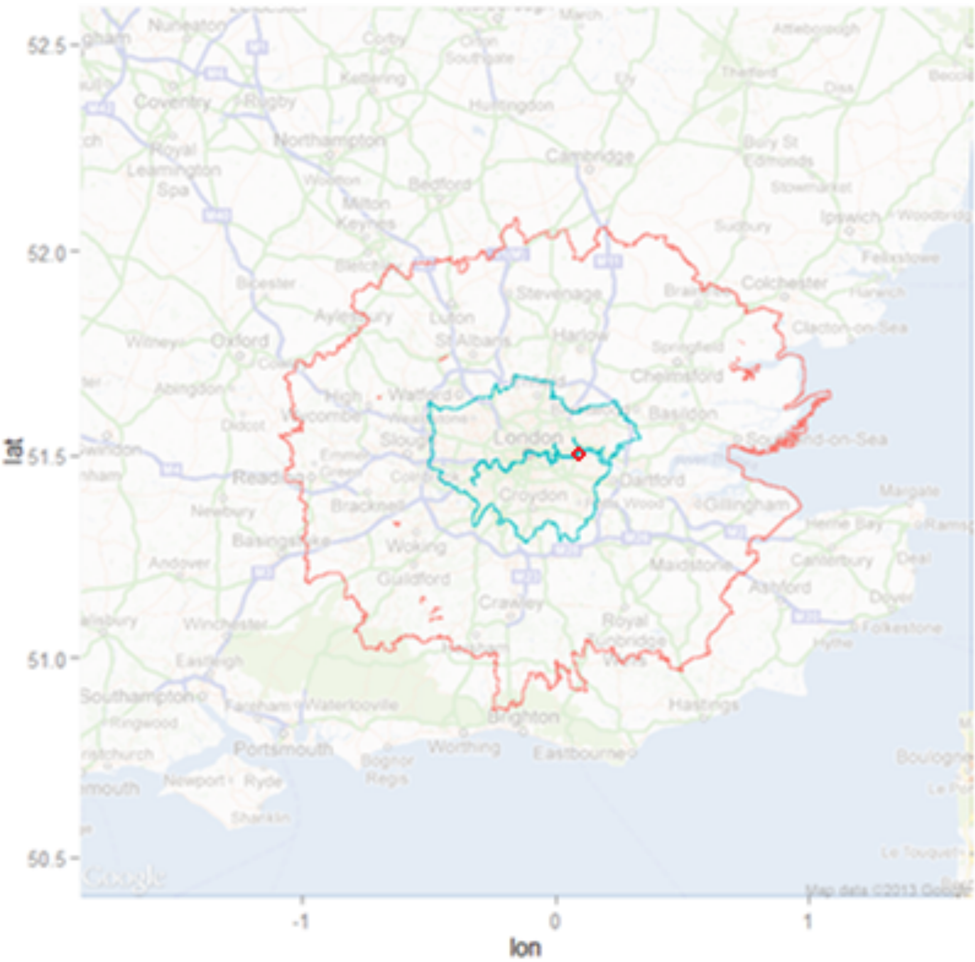


Diagram A1.1

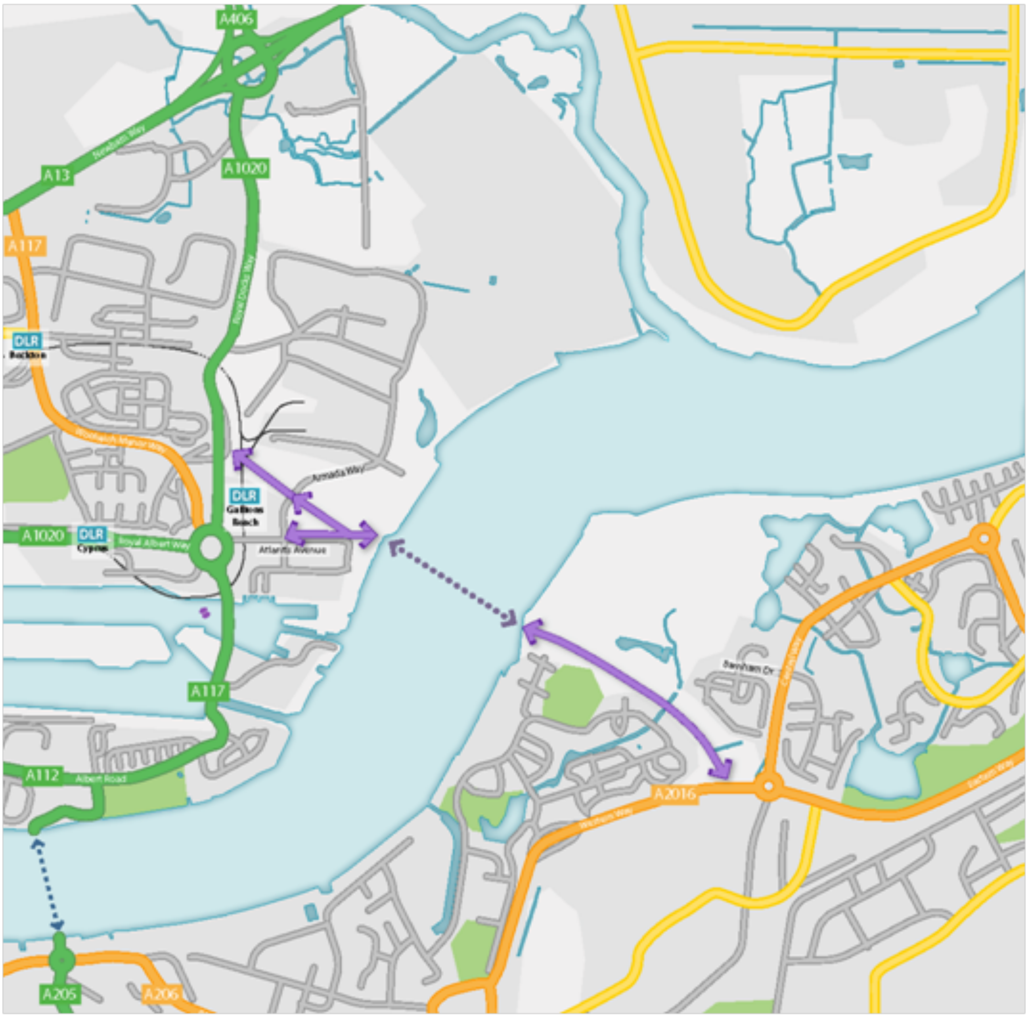
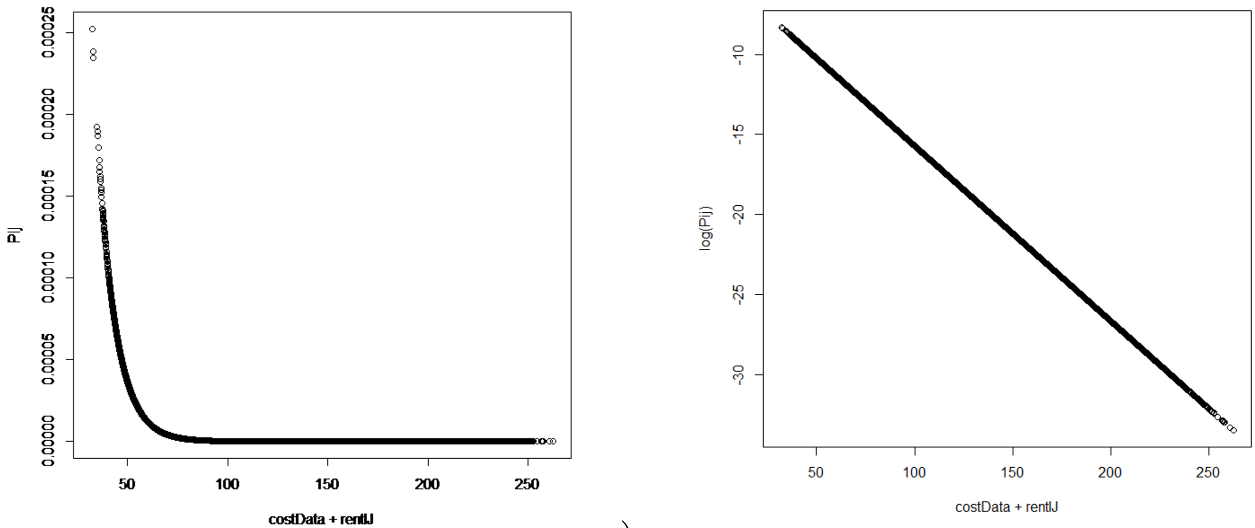
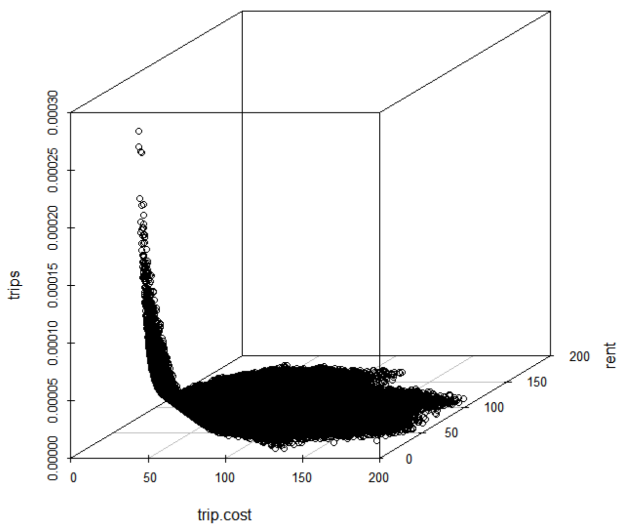


Diagram A1.2

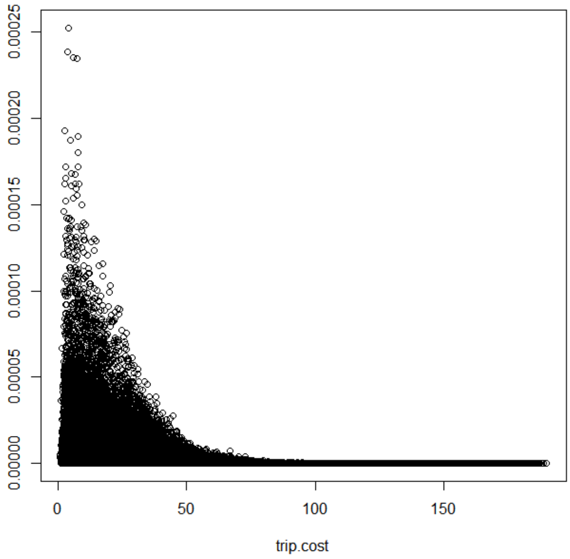


a) b)

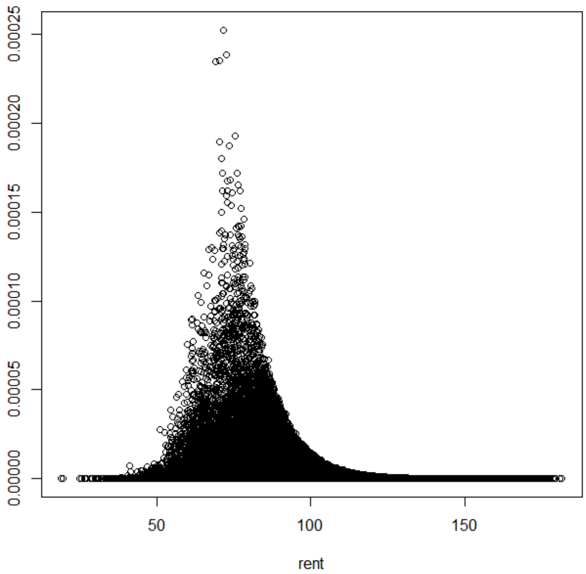
Diagram A1.3



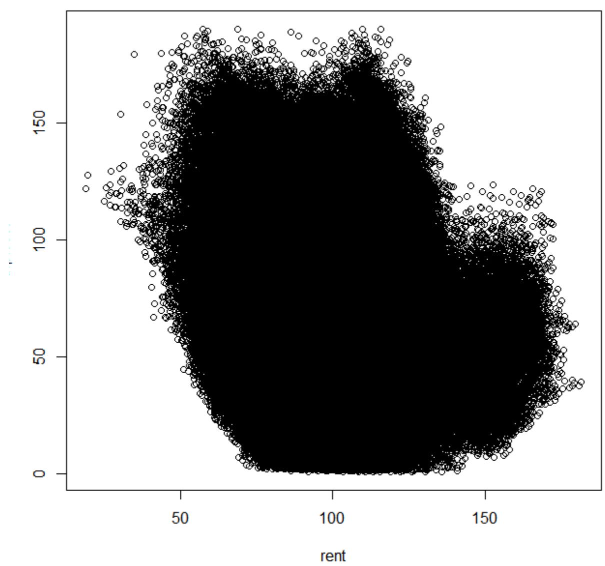
a) 3D plot of trips rent and trip cost



b) trip cost v trips



c) rent v trips



d) rent v trip cost

Diagram A1.4

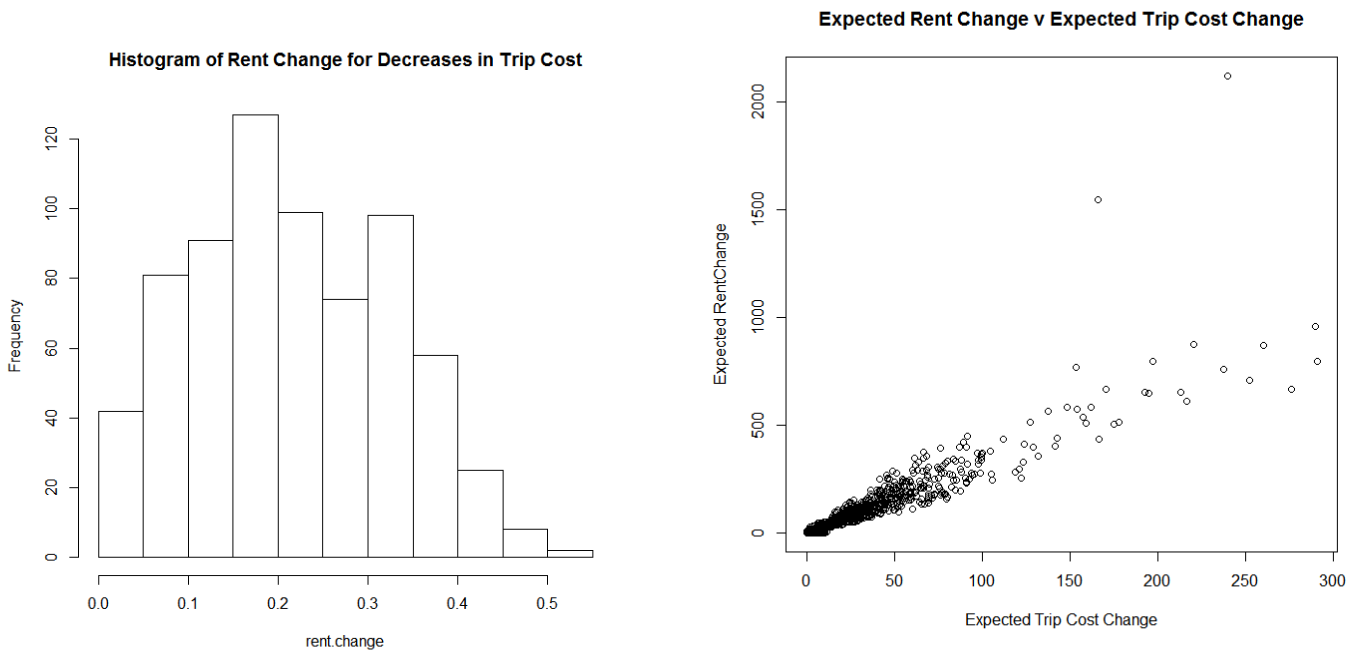
Diagram A1.4 above, shows in *a)* the plot of raw values of trips(p_{ij}) against rent$\left(\dfrac{\lambda_i+\lambda_j}{\beta}\right)$ and trip cost, (c_{ij}). The relationship is simple to discern in three dimensions but difficult to see in the diagrams *b)* to *c)* where the variables are plotted as pairs. Diagrams A1.3 & 4 show the importance of considering trip cost and rent simultaneously when analysing the workings of the model in terms of an analytic relationship between trips and costs.

The model was run for the base year costs and for a new set of costs modified to reflect the impact of the crossing. The origins and destinations are the same for both runs of the model. In table A1.1 and A1.2, below statistics are shown for the model before and after the introduction of the crossing with the final column showing the difference between the two states. The first two output columns are expectations whilst the difference is a total (i.e. N x expected difference). Table A1.1 shows the uncorrected rents and table A1.2 the rents corrected as set out in Appendix 4. Both tables have been constructed using the Arcadia trip data on which the diagrams A1.3 a) and b) are based. *DWL* is the consumer deadweight loss, *G* is the consumer surplus, *U* the combined rent and travel cost and *TS* the entropy times $1/\beta$.

We see, from equation (21) and (22) that, bearing in mind the need to include the $-\frac{1}{\beta}\ln p_i p_j$ term in the rents (although in this example its change is negligible)

$$\Delta G - \Delta DWL = \Delta U - \Delta\left\langle \frac{1}{\beta}\ln p_i p_j \right\rangle \qquad \text{(A0.27)}$$

In considering the relation between rents and trip costs it is important to know whether a decrease in trip cost consistently produces an increase in rent cost per trip. Diagram A1.5a shows a histogram of rent changes, $\Delta\left(\frac{\lambda_i + \lambda_j}{\beta}\right)$ for all those *ij* interchanges where cost has decreased (Δc_{ij})- all the changes are positive, confirming the consistency of sign. This is not always true when expected rents, $\Delta p_{ij}\left(\frac{\lambda_i + \lambda_j}{\beta}\right)$ and expected trip costs ($\Delta p_{ij}\, c_{ij}$), are compared. Diagram A1.5b shows the countervailing effect, at least at relatively high expected trip cost changes, of a redistribution of trips towards lower trip cost interchanges which results in higher expected (total) trip costs and higher expected (total) rents.



a) b)

Diagram A1.5

Tables A1.2 and 3 compare base year and predicted variables for uncorrected and corrected rents as used in equations (25) and (26). The final column in each table shows the difference between the first two columns times N and is thus a column of total costs and benefits enabling comparison with the rule of a half.

Uncorrected Rents Analysis			
	New	Old	Difference
DWL	15.075700	15.077400	-10635.413
G	-43.479950	-43.478960	-6139.538
U	71.127880	71.127160	4495.875
av.rent	48.675370	48.673770	9989.862
trip.cost	22.452510	22.453390	-5493.987
rent-trip.cost	26.222860	26.220380	15483.849
rent+trip.cost	71.127880	71.127160	4495.875
$\frac{1}{\beta}$ S	114.607800	114.606100	10635.413
$\frac{1}{\beta}$ I	15.075700	15.077400	-10635.413
rule of half (trip costs only)			16902.610
N	6245233		
Beta	0.10951190		

Table A1.1

Corrected Rents Analysis			
	New	Old	Difference
DWL	15.075700	15.077400	-10635.410
G	0.000000	0.000000	0.000
U	114.607800	114.606100	10635.410
av.rent	92.155320	92.152740	16129.400
trip.cost	22.452510	22.453390	-5493.987
rent-trip.cost	69.702810	69.699350	21623.390
rent+trip.cost	114.607800	114.606100	10635.410
$\frac{1}{\beta}$ S	114.607800	114.606100	10635.410
$\frac{1}{\beta}$ I	15.075700	15.077400	-10635.410
rule of half (trip costs only)			16902.610
N	6245233		
Beta	0.10951190		

Table A1.2

$$\text{DWL} = \frac{1}{\beta}\mathrm{I} = \frac{1}{\beta}\sum_i \sum_j p_{ij} \ln p_{ij}$$

$$\text{G} = \text{U} - \frac{1}{\beta}\text{S}$$

$$\text{U} = \sum_i \sum_j p_{ij} t_{ij}$$

av.rent$=\sum_i \sum_j p_{ij} \frac{\lambda_i + \lambda_j}{\beta}$

trip.cost$=\sum_i \sum_j p_{ij} c_{ij}$

Appendix 2

Integrability of the Consumer Surplus

The consumer surplus measure of equation (10)is open to some criticism as if it cannot be expressed as a perfect differential it may require evaluation as a path integral. However, the result of the path integral is not necessarily of interest in evaluation. Rather, we need the summation of such results for the evaluation of transport investment. This means that we can operate at the macro level of equations (9) and (21) and examine the integrability of the macro quantities. The integrability conditions were described by Hotelling(1932) and quoted in Williams(1976). However, at the macro level these conditions simply reflect the Maxwell relations of thermodynamics (Morphet, 2013, Callen, 1985 and Ott et al, 2000) which were derived in 1871.

If we take equation (10)

$$G = \frac{1}{\beta} \ln \sum_i \sum_j e^{-\beta t_{ij}} = \frac{1}{\beta} \ln Z \tag{A0.28}$$

differentiating with respect to t_{ij}

$$\frac{\partial G}{\partial t_{ij}} = \frac{1}{\beta} . \frac{-\beta e^{-\beta t_{ij}}}{Z} = -p_{ij} \tag{A0.29}$$

and similarly differentiating with respect to t_{ab}

$$\frac{\partial G}{t_{ab}} = -p_{ab} \tag{A0.30}$$

Differentiating equation(A0.29) with respect to t_{ab} we get

$$\frac{\partial^2 G}{\partial t_{ab} \partial t_{ij}} = -\frac{\partial p_{ij}}{\partial t_{ab}} \tag{A0.31}$$

and doing the same for equation(A0.30) with respect to t_{ij} we get

$$\frac{\partial^2 G}{\partial t_{ij} \partial t_{ab}} = -\frac{\partial p_{ab}}{\partial t_{ij}} \tag{A0.32}$$

but, since the evaluation of a partial differential equation is independent of the order of differentiation we have

$$\frac{\partial^2 G}{\partial t_{ab}\partial t_{ij}} = \frac{\partial^2 G}{\partial t_{ij}\partial t_{ab}} \tag{A0.33}$$

so from equations (A0.31)and (A0.32) we may write

$$\frac{\partial p_{ij}}{\partial t_{ab}} = \frac{\partial p_{ab}}{\partial t_{ij}} \tag{A0.34}$$

These are the Hotelling conditions for integrability.

Including rent means that the generalised cost formulation of equation (11) is continuous which obviates some but not all, of the concerns over Hotelling's conditions for integration under the demand curve (Kozlik, 1942) . That the demand curve as a whole can be treated continuously may be seen from the Appendix 1 diagrams (a) – (b), provided rents are included. With rents not included we get the rather less tractable Appendix 1 diagrams (c) – (d). However the path independence of the aggregate measures may be determined more directly by their role as state variables justifying the evaluation of benefit in terms of their differences in value at equilibrium states 1 and 2 (Callen 1985). The aggregate differentials give rise to the Maxwell relations (Morphet,2013, Callen 1985, s7.1)

Appendix 3

Estimating the Deadweight Loss

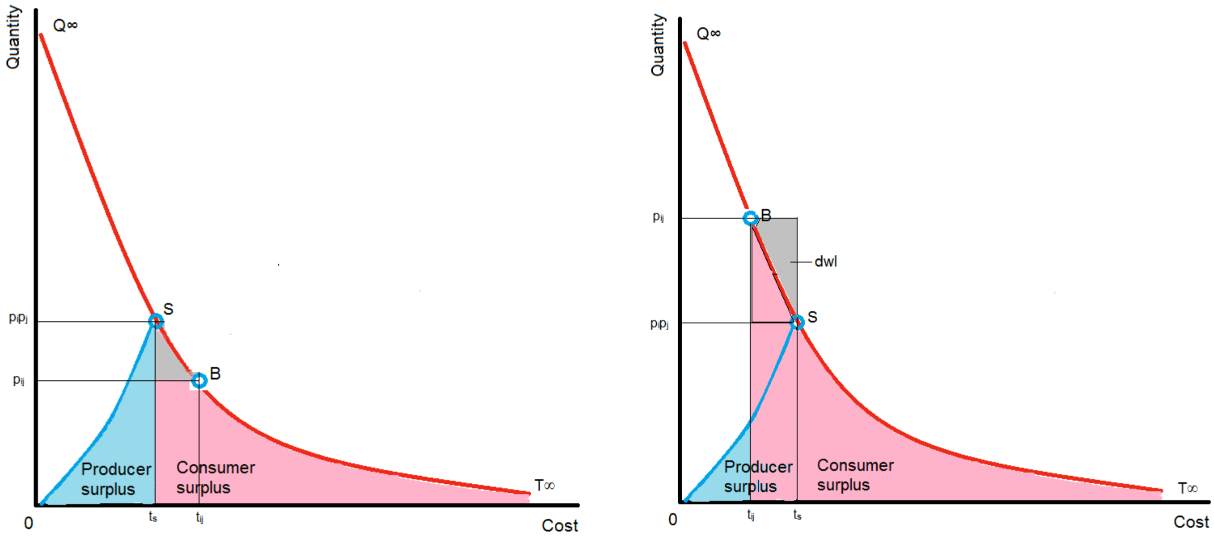


DiaGram A3.1

In diagram A3.1 we show the two basic cases of deadweight loss. In the first the cost, t_{ij} is greater than t_s, the perfect competition optimum. This is the model analysed in equations (12) to (14). It remains to derive the equivalent formulation for the second case where t_{ij} is less than t_s. To estimate the deadweight loss shown in grey we designate as B, the area of the rectangle with corners $Bt_{ij}t_s$ including the grey area. We designate as A, the area $S\ Bt_{ij}t_s$ under the demand curve. We may then write

$$\sum_i \sum_j B = p_{ij}\left(t_s - t_{ij}\right) \qquad \text{(A0.35)}$$

and

$$\begin{aligned}\sum_i \sum_j A &= \int_{t_{ij}}^{t_s} \sum_i \sum_j p_{ij}\,dt_{ij} = \int_{t_{ij}}^{t_s} \sum_i \sum_j \frac{e^{-\beta t_{ij}}}{Z}\,dt_{ij} = \left[-\frac{1}{\beta}\ln Z\right]_{t_{ij}}^{-1/\beta \ln p_i p_j} \\ &= +\frac{1}{\beta}\ln Z_{t_{ij}} - \frac{1}{\beta}\ln \sum_i \sum_j p_i p_j = \frac{1}{\beta}\ln Z_{t_{ij}} \qquad \forall i, j\end{aligned} \qquad \text{(A0.36)}$$

so considering all interchanges, ij

$$\begin{aligned}B - A &= \sum_i \sum_j \left(p_{ij}\left(t_s - t_{ij}\right) - \frac{1}{\beta} p_{ij} \ln Z_{t_{ij}} \right) \\ &= \sum_i \sum_j -\frac{1}{\beta} p_{ij}\left(\ln e^{\ln p_i p_j} - \ln e^{-\beta t_{ij}}\right) - \frac{1}{\beta} p_{ij} \ln Z_{t_{ij}} \\ &= \frac{1}{\beta} \sum_i \sum_j p_{ij} \ln \frac{p_{ij}}{p_i p_j}\end{aligned} \qquad \text{(A0.37)}$$

which is the mutual information as derived in equations (12) to (14) for the first diagram.

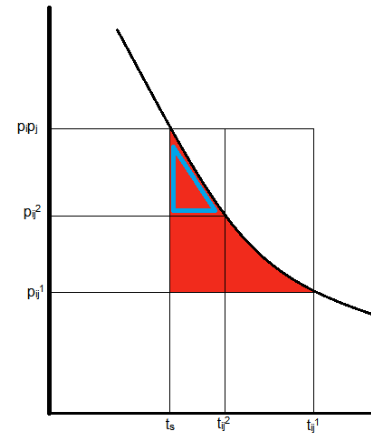
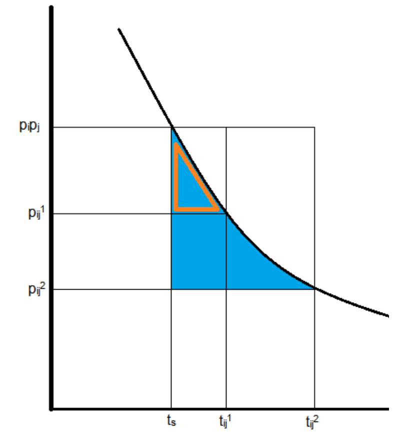
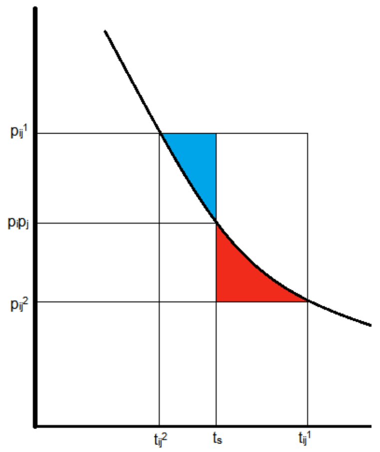
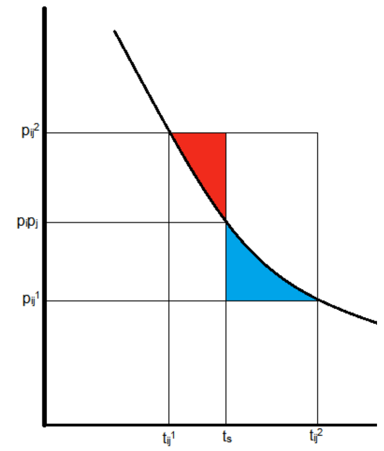
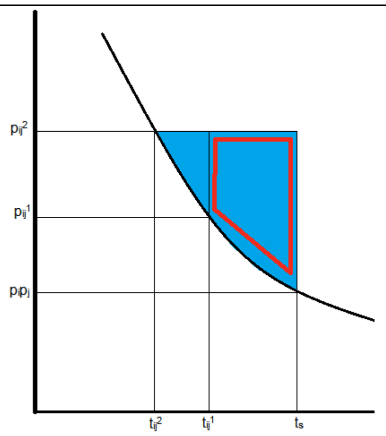
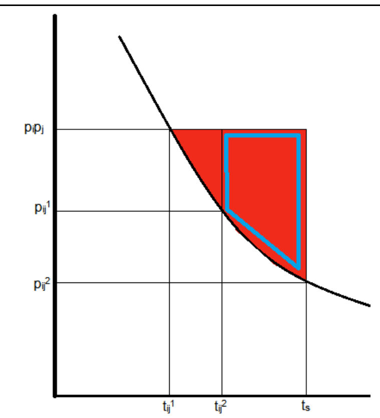
	$t_{ij}^1 > t_{ij}^2$	$t_{ij}^2 > t_{ij}^1$
$\{t_{ij}^1 , t_{ij}^2 \} > t_s$	 *n=1330803*	 *n=1298463*
t_s between $\{t_{ij}^1 , t_{ij}^2 \}$	 *n=181*	 *n=164*
$\{t_{ij}^1 , t_{ij}^2 \} < t_s$	 *n=280111*	 *n=212567*

Diagram A3.2

Diagram A3.2 above, shows the various configurations of deadweight loss when two equilibrium positions are considered. The red areas refer to the base year position and the blue to the predicted year. The numbers show the number of cases of each pattern observed in using the Arcadia model to analyse the Thames crossing (Appendix 1). They indicate that anything that can happen, will happen. It will be seen that the ordering of the costs $\{t_{ij}^1,t_{ij}^2,t_s\}$and of the probabilities $\{p_{ij}^1,p_{ij}^2,p_ip_j\}$ are inversely related, as might be expected from the downward slope of the demand curve. However, this relationship is not consistent unless the rents are corrected as shown in Appendix 4.

Appendix 4

Correcting the balancing factors

Introduction

Equation (A0.38) shows the basic doubly constrained transportation model, first with

$$p_{ij} = \frac{e^{-\lambda_i - \lambda_j - \beta c_{ij}}}{\sum_i \sum_j e^{-\lambda_i - \lambda_j - \beta c_{ij}}} = \frac{e^{-\beta t_{ij}}}{\sum_i \sum_j e^{-\beta t_{ij}}} = \frac{e^{-\beta t_{ij}}}{Z} \qquad \text{(A0.38)}$$

rents and trip cost separated and second with them combined into a new overall cost, *t*. The denominator *Z,* the partition function is given by

$$Z = \sum_i \sum_j e^{-\lambda_i - \lambda_j - \beta c_{ij}} = \sum_i \sum_j e^{-\beta t_{ij}} \qquad \text{(A0.39)}$$

and we may write equation (A0.38) as

$$p_{ij} = -\frac{1}{\beta} \frac{\partial \ln Z}{\partial t_{ij}} \qquad \text{(A0.40)}$$

from which we see that, given β, the demand curve is entirely determined by *Z*.

The factors λ_i and λ_j are unique only up to a constant which makes little difference when the before and after flows and values of *Z* are not too different . However fixing them would be useful in making comparisons with measured values of land rent and in making comparisons where *Z* and the flows change substantially. However, fixing them will require additional information which we derive from the model under perfect competition.

Perfect Competition

From Starret's impossibility theorem we find perfect competition arises only when the costs of transport are zero. This gives us

$$p_{ij} = p_i p_j = \frac{e^{-\lambda_i - \lambda_j}}{\sum_i \sum_j e^{-\lambda_i - \lambda_j}} = \frac{e^{-\lambda_i}}{\sum_i e^{-\lambda_i}} \frac{e^{-\lambda_j}}{\sum_j e^{-\lambda_j}} \qquad \text{(A0.41)}$$

We see that for this expression of the model the value of *Z* is *1* since we can write

$$p_{ij} = \frac{p_i p_j}{\sum_i \sum_j p_i p_j} = \frac{p_i p_j}{1} \qquad \text{(A0.42)}$$

From this we see that, as written, the perfect competition value of *Z* differs from that of the general model of equations (A0.38) and (A0.39). However, the point of perfect competition should lie on the demand curve in the same way as shown in equation (A0.38). We need to ensure the two values of *Z* coincide so that equation (A0.40) still holds.

Fixing the Balancing Factors

There are two ways of ensuring the value of Z is consistent for the cases of both perfect and imperfect competition. We can match the demand curve of equation (A0.38) to that of equation (A0.41) by adjusting the exponent of equation(A0.38) or vice versa. The two methods are shown in diagrams 1 and 2. In diagram 1 the curve of perfect competition (blue) is shifted left to a new position (green) that coincides with the demand curve (red) of equation (A0.38). Conversely, in diagram 2, the demand curve (red) is moved rightwards to a new position (green) that coincides with the perfect competition curve (blue). The latter process is regarded as more fundamental as the value of Z is fixed at 1 by the inherent structure of the model. Moving the other way gives the modelled value of Z which is to some extent, arbitrary. However, the approaches are equivalent when we are considering *changes* in benefit.

The graphs have been constructed using Arcadia data with a doubly constrained model calibrated and run in R. The correction applied for shifting the demand curve onto the perfect competition curve (diagram 2) is calculated thus:

$$p_{ij} = \frac{e^{-\lambda_i-\lambda_j-\beta c_{ij}}}{\sum_i \sum_j e^{-\lambda_i-\lambda_j-\beta c_{ij}}} = \frac{e^{-\lambda_i-\lambda_j-\beta c_{ij}}}{Z} = \frac{e^{-\lambda_i-\lambda_j-\beta c_{ij}} . {}^{1}\!/_{Z}}{Z . {}^{1}\!/_{Z}} = e^{-\lambda_i-\lambda_j-\beta c_{ij}-\ln Z} \qquad \text{(A0.43)}$$

which can be tidied into

$$p_{ij} = e^{-\beta\left(\frac{\lambda_i}{\beta}+\frac{1}{2\beta}\ln Z+\frac{\lambda_j}{\beta}+\frac{1}{2\beta}\ln Z+c_{ij}\right)} \qquad \text{(A0.44)}$$

The lnZ factor has been split equally between the row and column factors to preserve biproportionality and because in the model, the trip costs are fixed. It will be seen that in this formulation the consumer surplus is zero since lnZ is zero when Z is 1. However, all its value has been incorporated into the rents which are a proper measure of benefit. It should be noted that equation(A0.44) is identical to equation (2) thus emphasizing that the system is indifferent to rearrangement of consumer surplus into rent. The adjustment also ensures consistency between the ordinal relations of p_{ij} and t_{ij} so that $p_{ij}^1 > p_{ij}^2$ implies $t_{ij}^1 < t_{ij}^2$ etc. The correction is also attractive in that it seems to overcome the problem of negative rents which can arise in the uncorrected model. For von Thunen the rents could never be less than zero.

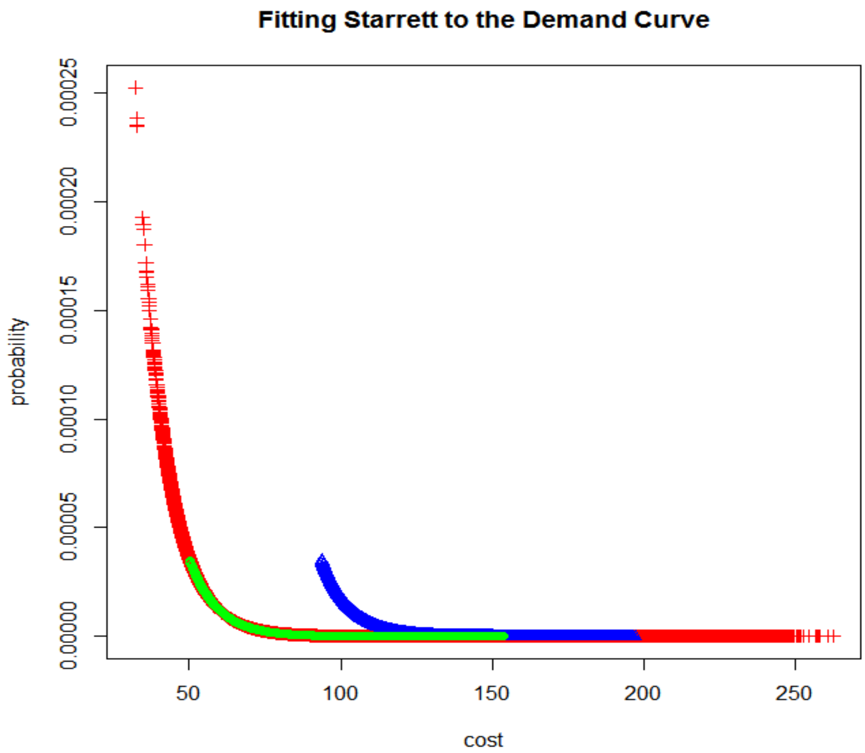


Diagram 1

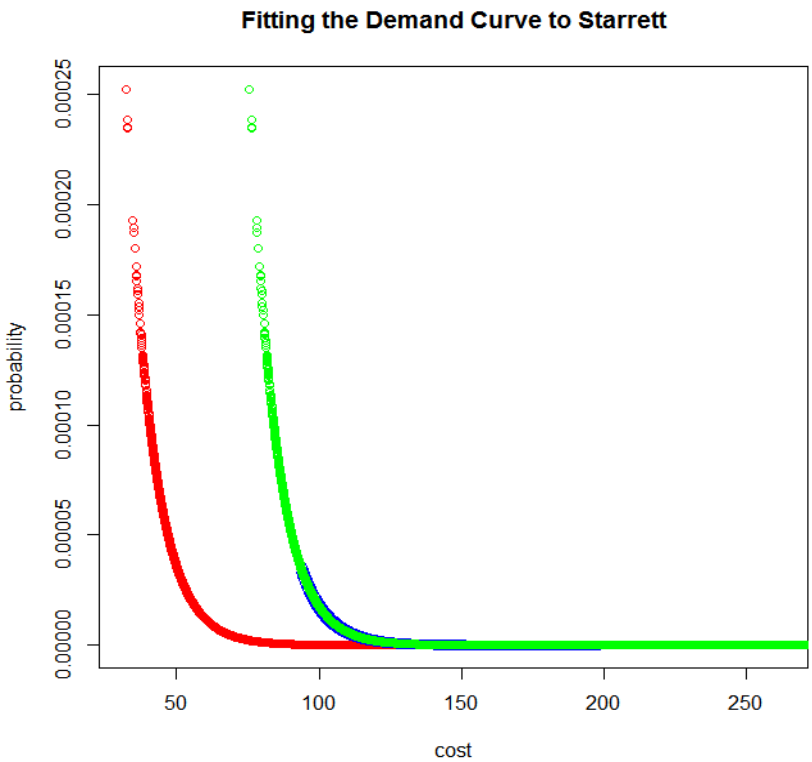


Diagram 2

References

Anas A., *"Discrete choice theory, information theory and the multinomial logit and gravity models"*, Transportation Research Part B: Methodological, Volume 17, Issue 1, pp 13–23, 1983

Batty, M., *"ARCADIA: Adaptation and Resilience in Cities: Analysis and Decision making using Integrated Assessment"*, EPSRC grant EP/G060983/1, 2009-2012

Callen, H. (1985) *Thermodynamics and an introduction to thermostatistics*. United States: John Wiley and Sons (WIE).

Evans, S., *"A Relationship Between The Gravity Model For Trip Distribution And The Transportation Problem In Linear Programming "*, Transportation Research, Vol. 7, pp. 39-61 , 1973

Fermi, E., *"Thermodynamics"*, Prentice Hall 1937, republished Dover, Mineola N.Y., 1956

Fujita, M. And Thisse, J.-F., *"Economics of Agglomeration: Cities, Industrial Location, and Globalization"*, Cambridge University Press; Cambridge, 2013

Galichon, A. and Salanie, B., *"Matching with Trade-offs: Revealed Preferences over Competing Characteristics"*, Cahier n°2010-10, Departement D'economie, Ecole Polytechnique, 2010, http://hal.archives-ouvertes.fr/docs/00/47/31/73/PDF/cahier_de_recherche_2010-10.pdf

Glaister, S., *Fundamentals of Transport Economics,* Blackwell, Oxford, 1981

Greenhut, M., Norman, G. and Hung, C.-S. (1987) *The Economics of Imperfect Competition: A Spatial Approach*. United Kingdom: Cambridge [Cambridgeshire] ; Cambridge University Press, 1987.

Harberger, A.C., *"The Measurement of Waste"*, The American Economic Review, vol.54, no.3, pp58-76, 1964

Hotelling, H., *"Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions"*, Journal of Political Economy, vol. 40, no. 5, pp 577-616, 1932

Koopmans, T.C. and Beckmann, M., *"Assignment Problems and the Location of Economic Activities"*, Econometrica, Vol. 25, No. 1, pp. 53-76, 1957

Kozlik, A., *"Note on the Integrability Condition for Interrelated Demand"*, The Review of Economic Studies, vol 10-1, 1942

Lind, R.C., *"Spatial Equilibrium, the Theory of Rents, and the Measurement of Benefits from Public Programs"*, The Quarterly Journal of Economics, Vol. 87, No. 2 pp. 188-207, 1973

McInnis, A., *"Land-value capture has merit"*, Montreal Gazette, 5th March, 2015, http://montrealgazette.com/opinion/editorials/editorial-land-value-capture-has-merit

Mishan, E., *"Rent as a measure of welfare change"*, The American Economic Review, v49(3) pp. 386-394, 1959

Morphet, R., *"Von Thune's Legendre Transform"*, Working Paper 193, Centre for Advanced Spatial Analysis, CULL, 2013 http://www.bartlett.ucl.ac.uk/casa/publications/working-paper-193

Morphet, R., *A Note on the Calculation and Calibration of Doubly Constrained Trip Distribution Models*, Transportation 4 (1): 43-53, 1975

Neuburger, H. , "*User Benefit in the Evaluation of Transport and Land Use Plans*", Journal of Transport Economics and Policy, Vol. 5, No. 1, 1971, pp. 52-75

Neuburger, H. and Wilcox, J., *"The Economic Appraisal of Land-Use Plans"*, Journal of Transport Economics and Policy, Vol. 10, No. 3 (Sep., 1976), pp. 227-236

Ott, J.B. and Boerio-Goates, J., *"Chemical Thermodynamics:Principles and Applications"*, Academic Press, London, 2000

Pickford, J., *"London project to use risky funding model"*, Financial Times, 8th April, 2013 http://www.ft.com/cms/s/0/fcda4910-9f64-11e2-b4b6-00144feabdc0.html#ixzz3YPIjRU5j

Prest, A.R. and Turvey, R., *"Cost-Benefit Analysis: A Survey"*, in *"Surveys of Economic Theory, vol 3"*, Macmillan, London, 1968

Roy, J. R. ,*Spatial Interaction Modelling: A Regional Science Context*, Springer, Berlin and NY, 2004

Snickars, F. and Weibull, J., *"A Minimum Information Principle: Theory and Practice"*, Regional Science and Urban Economics 7, pp 137-168, 1977

SACTRA , *"Transport and the Economy"*, The Standing Advisory Committee on Trunk Road Assessment, Mackay E., OHMS, ISN'T-11-753507-9, 1998

Sanford, M., *"Local Government in England: capital finance"* , House of Commons Library, 30th April 2014, http://www.parliament.uk/business/publications/research/briefing-papers/SN05797/local-government-in-england-capital-finance

Starrett, D., *"Principals of Optimal Location in a Large Homogeneous Area"*, Journal of Economic Theory, 9, pp 418-448, 1974

Starrett, D., *"Foundations of Public Economics"*, Cambridge University Press, Cambridge, 1988

Tavasszy, L.A. , Burgess, A. and Renes G., *"Final Publishable Report: Conclusions and recommendations for the assessment of economic impacts of transport projects and policies"*, IASON (Integrated Appraisal of Spatial economic and Network effects of transport investments and policies) Deliverable D10. Funded by 5th Framework RTD Programme. TNO Inro, Delft, Netherlands, March 2004

Williams, H. and Senior M., *"Accessibility, Spatial Interaction and the Spatial Benefit Analysis of Land Use-Transportation Plans"*, in "Spatial Interaction Theory and Planning Models" eds. Karlquist, A., Lundqvist,L.,Snickars, F., and Weibull, J., North Holland, Amsterdam, 1978, pp 253-286

Williams, H., *"Travel Demand Models, Duality Relations and User Benefit Analysis"*, Journal of Regional Science, vol. 16, no. 2, pp147-166, 1976

Wilson A.G., *"A statistical theory of spatial distribution models"*, Transportation Research, Volume 1, Issue 3, pp 253-269, 196