Crossing the Bridge: From a Constructionist Learning Environment to Formal Algebra

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In the digital era, it is crucial to explore how digital technologies can be successfully integrated in the mathematics classroom and what their potential impact on learning is. This paper presents some reflections based on data gathered as part of the MiGen project (www.migen.org[1]) from studies aimed to investigate ways to support the transition to formal Algebra, through the use of a constructionist learning environment and carefully designed 'bridging' activities that consolidate, support and sustain students' algebraic ways of thinking. Our claim is that explicit links need to be made to Algebra through those specially designed activities so that such a digital tool can support students' learning of formal Algebra in order to be successfully integrated in the mathematics classroom.

Keywords: Generalisation, Microworlds, Transition, Algebra.

INTRODUCTION

In the last few decades, the number of appearances of digital technologies designed for mathematics learning keeps growing. Relevant research (e.g., EACEA Eurydice Report, 2011), though, has shown that these technologies are not always used to their full or intended potential and also, students rarely use ideas, concepts or strategies they seem to have acquired through their interactions with such technologies. For example, Gurtner (1992), referring to the Logo environment, demonstrated that the tool's features which are designed to support students when faced with complex mathematical problems may impede them from making connections between their work in Logo and any mathematical or geometrical ideas they are already familiar with and use when problems seem less complex. Also, the lack of information on why and how to build bridges to formal maths, which were not often made in standard Logo situations, led to the lack of connections to formal maths (Gurtner, 1992). In this paper, we discuss our approach to support students' transition of moving back and forth from paper-and-pencil to interacting with digital tools and therefore consider ways of facilitating the integration of digital technologies in the maths classrooms. In particular, our focus is on the transition to formal Algebra and

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how students ‘transfer’ their knowledge from their interactions with a digital tool, namely eXpresser, specially designed to support and address students’ difficulties with learning algebra, to paper-and-pencil (PaP) activities.

There is a lot of research on the issue of ‘transfer’ (e.g., DiSessa & Wagner, 2005). Our interpretation is closely aligned with Beach (2003) who has argued that the metaphor should be viewed as transition instead of transfer. Crossing boundaries from one location to another is in fact a process of transition and therefore people are the ones who move and not knowledge or learning. In the case of Logo, Gurtner (1992) considered “the type of connections generally expected, and very seldom observed, between Logo practice and mathematics” (p. 247) as *transfer* and suggested that there is a need for a long period of practicing in Logo, especially one which is rich in reflection, so that some transfer to mathematics can happen.

Going back to our focus on Algebra, the transition to formal Algebra has been investigated by various authors (e.g., Radford, 2014) and the literature is replete with examples of student difficulties (e.g., Stacey & Macgregor, 2002). Students struggle to understand the idea behind using letters to represent *any* value (Duke & Graham, 2007) and are inexperienced with mathematical vocabulary. Even students capable of expressing a general rule through the use of words, like ‘always’ or ‘every’, struggle to use letters and symbols and form algebraic expressions.

Similarly to Radford (2014), who claimed that there is a need for specially designed classroom activity to support students’ developmental path to formal Algebra, and to Gurtner (1992), who suggested presenting structured tasks, using appropriate microworlds and making explicit interventions during students’ interactions, we claim that a digital tool specially designed to support the development of algebraic ways of thinking (AWOT) together with carefully designed bridging activities should ‘smoothen’ the transition to formal algebra without rendering it impossible for students to reach the mathematical ‘bank of algebra’. Besides ‘learning’ the tool and developing expertise in using it, students should make the connections to the maths. The issue is to find out ways for supporting students to make such connections.

EXPRESSER AND THE TRANSITION TO ALGEBRA

The MiGen system is a pedagogical and technical environment that improves 11-14 year-old students’ learning of algebraic generalisation. Its core component consists of a microworld, eXpresser, which has been specially designed to help students develop AWOT through a series of generalisation tasks (Noss et al., 2012). In eXpresser, students construct figural patterns by expressing their structure through repeated building blocks of square tiles, and articulating the rules that underpin the calculation of the number of tiles in the patterns. A typical activity in eXpresser asks students to reproduce a dynamic model (or part of it) presented in a window that appears on the side of the activity screen.

Figure 1 shows a model where a row of red tiles is surrounded by grey tiles. Students are asked to construct a model that works for *any number* of red tiles, and find a rule for the total number of tiles surrounding the red tiles. They can test generality by *animating the model:* that is, by letting the computer change the number of red tiles at random. The design of eXpresser capitalises on animated feedback and on the simultaneous representation of a *specific* and *general* model ('My Model' and 'Computer's Model' in Figure 2), built by combining patterns and on the close alignment of the symbolic expression, the *Model Rule* and the structure of the model. All numbers in eXpresser are *constants* by default, referred to as 'locked' numbers. When the user 'unlocks a number', it is possible to change its value; it becomes a *variable*. In the *Computer's Model*, a value of the variable ('Num of Red Tiles' in this example) is chosen automatically at random (it is '10' in Figure 2) which will generally be different from that in the specific model ('6' in Figure 2). So the *Computer's Model* indicates to students whether their constructions are *structurally* correct for the different values of the variable(s). Students also construct a *model rule* for the total number of tiles, and validation of its correctness is made evident by colouring: tilings are *only* coloured if the rule for the number required is correct.

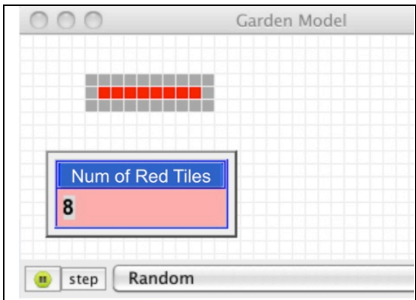


Figure 1: A model for 8 red tiles surrounded by grey tiles. Students must construct a general model and find the general rule

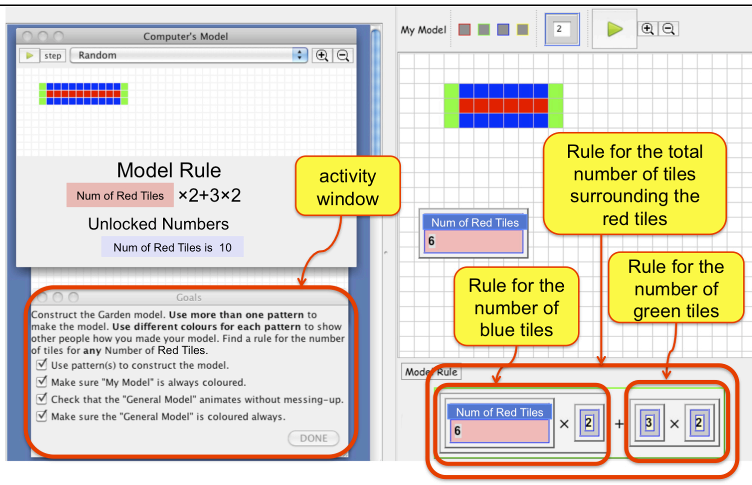


Figure 2: The eXpresser screen showing the general and specific models (Computer's Model on the left, and My Model on the right), and a correct rule for the total number of surrounding tiles. The task goals are shown in the "Activity window" (lower left-hand corner).

To make connections to formal Algebra feasible and support the transition from interactions with the eXpresser tool to PaP Algebra, we had to consider what characterizes formal Algebra and more specifically AWOT. Algebra involves a number of mathematical concepts, from numbers, to variables, from numerical expressions to expressions that involve the use of 'unknown' numbers and functions. Various authors have characterised algebra as 'generalised arithmetic' (e.g., Kieran & Chalouh, 1993). For example, Sfard and Linchevksi (1994) distinguished between the operational phase, where "the focus is on numerical processes and there is no hint of abstract objects rather than numbers" (p. 197) and the structural phase, which involves processes of manipulations of symbols. They argued, therefore, that there are "two crucial transitions: from the purely operational algebra to the structural algebra ‘of a fixed value’ (of an unknown) and then from here to the functional algebra (of a variable)” (p. 191). Leading on from these distinctions, Radford (2014) considered three conditions that characterise algebraic thinking: (i) *indeterminacy*, which is about recognising the use of ‘unknown’ values in the form of variables, parameters, etc.; (ii) *denotation*, involving the symbolisation of the undetermined values of the problem in question that can include the use of natural language, gestures, signs, as well as a mixture of these or symbols and (iii) *analyticity*, involving the skill of manipulating the indeterminate quantities like known values.

In the case of the eXpresser tool in our previous work (Mavrikis, et al., 2013), we have identified two AWOT. The first one is:(i) *Perceiving structure and exploiting its power*, which is about noticing what stays the same and what is repeated in a figural sequence so as to understand how the sequence is ‘structured’, supporting therefore “the development of structural reasoning” and the habits of “breaking things into parts” by identifying “the building blocks of a structure” (Cuoco, Goldenberg & Mark, 1996, p. 69). This AWOT, especially as it is operationalized in eXpresser that encourages students to construct what they perceive and manipulate the various properties of their constructions, could relate to Radford’s (2014) indeterminacy and analyticity conditions, but also to the initial transition from the operational to the structural algebra of a fixed value of an unknown as described by Sfard and Linchevski (1994) above. The second AWOT is: (ii) *Recognising and articulating generalisations, including expressing them symbolically*, which is the process of translating the observed structure in an algebraic expression, using formal algebraic notation to write general rules for numerical sequences. This AWOT can be linked to Radford’s (2014) denotation condition as well as the second transition from the structural algebra to the more functional algebra of a variable (Sfard & Linchevski, 1994), as its focus is on the production of formal algebraic expressions.

BRIDGING ACTIVITIES

We designed a sequence of activities both to help students become familiar with the tool but also to facilitate the transition to algebra. The sequence starts with introductory and practice tasks that ask students to construct figural models. It continues with individual activities, such as the one described above (see figure 1). Students were asked to construct the task model in eXpresser using different patterns and combinations of patterns, depending on their perceptions of the task model’s structure and derive a general rule for the number of square tiles needed for any Model Number. In our initial studies, students were presented with off-computer tasks, immediately after the final eXpresser task in an effort to reveal their strategies on solving similar tasks on paper and whether eXpresser had an impact on those strategies or not. In later studies, though, and after close collaboration with teachers, we recognised the need of activities, which promote students’ reflections upon mathematical concepts and problem-solving strategies they used *throughout* their interactions with eXpresser and not just at the end. These we referred to as consolidation tasks. So, throughout their interactions with eXpresser and immediately afterwards, students were presented with four types of bridging activities (examples are given in Figure 3), which are designed to support their transition to paper-and-pencil tasks: (i) *Consolidation tasks*, which are usually short tasks that are used to intervene and encourage students to reflect on their interactions with eXpresser throughout a sequence of eXpresser tasks, (ii) *Collaborative tasks*, which are presented at the end of an eXpresser task and focus on students' justification strategies regarding rule equivalence, (iii) *eXpresser-like paper tasks*, which are figural pattern generalisation tasks on paper, and (iv) *text-book or exam like tasks*, which are the traditional generalisation tasks given to students on paper.

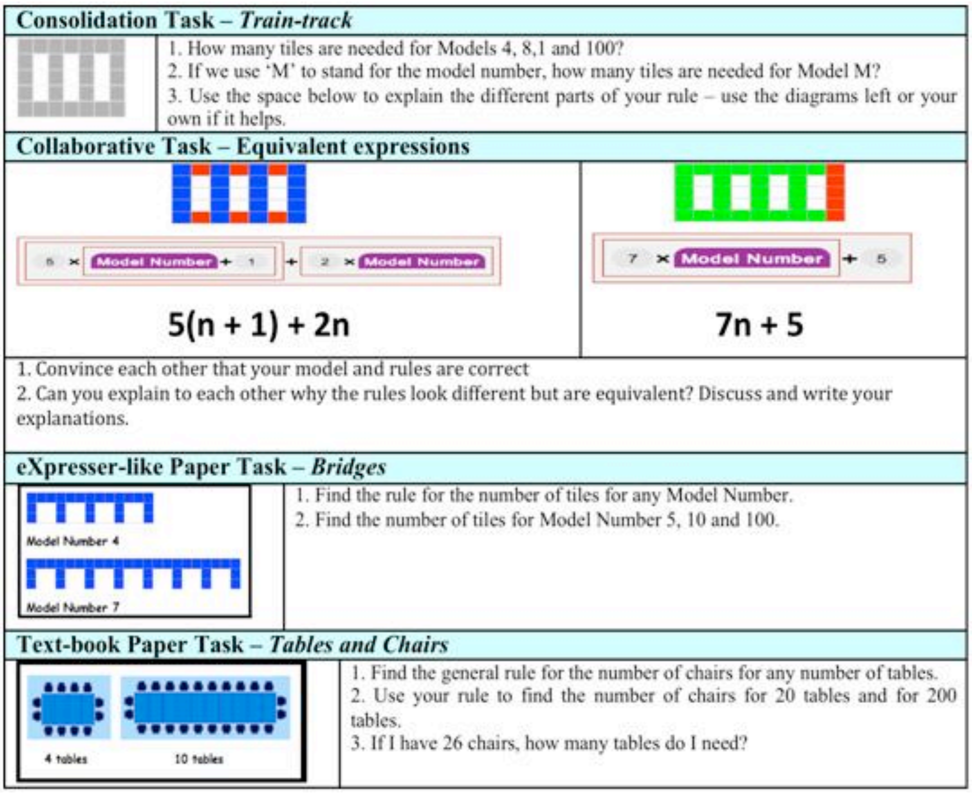


Figure 3: Examples of Bridging Activities.

STUDENT DATA

Over the past 7 years, we have carried out studies in 6 different schools in London, worked with 11 maths teachers and collected data from 553 students aged 11-14 years old. Each study was carried out over the course of four consecutive lessons, during which students became familiarised with the tool through some simple tasks, worked on one or two main activities and then were given bridging activities. A sample of students was interviewed at the end of their interactions with eXpresser. All students had been introduced to Algebra at school before their interactions with eXpresser, but of course their experience varied based on their age. Our data comprise one-to-one and small groups of students' and teachers' interviews and transcripts, video and audio files from interviews, one-to-one, small groups and classroom observations, detailed logs from students' interactions in the form of a database and bridging activities. Results from our studies are presented in a number of papers (e.g., Mavrikis et al., 2013; Noss et al., 2012; Geraniou et al., 2011). In this paper, we focus on the data collected from the bridging activities students worked on independently (or in pairs/groups of 3 for the collaborative tasks) during, but mostly after their final interaction with eXpresser. Using the two AWOT described in Mavrikis et al. (2013), as an analytical framework for interpreting students' strategies, we present our initial results under those two headings.

(i) Perceiving structure and exploiting its power. For the consolidation tasks, which were used with 175 students as their necessity was identified later in our studies, most of the 175 students demonstrated on the model figures presented on paper how they visualised the structure of the given model. In Figures 4, 5, 6 and 7, we present some examples of students' answers on the four bridging activities presented in Figure 3. Students clearly marked the different parts that would remain the same in any instance of the pattern and the parts, which, repeated every time, create the different instances of the pattern. Especially for the collaborative task, students verbally identified their building blocks in their models and rules and compared them to conclude about their equivalence. An example of two students' collaboration and its outcome is presented in Figure 5. Students demonstrated a variety of ways to visualise the task patterns and it was evident how influenced they were by the eXpresser's features as they were using the eXpresser terminology, e.g., number of building blocks or models. For example, in Figures 6 [F], [G] and [H], students drew the 2 building blocks that they could use if they were solving this task in eXpresser, that of a column of 3 square tiles and that of an 'L'- shaped one of 5 tiles. For example, Janet named her independent variable as "number of red BBs" (BBs stands for Building Blocks), and even though Nancy, named hers as 'Nancy', she used eXpresser's terminology in her discussions with Janet.

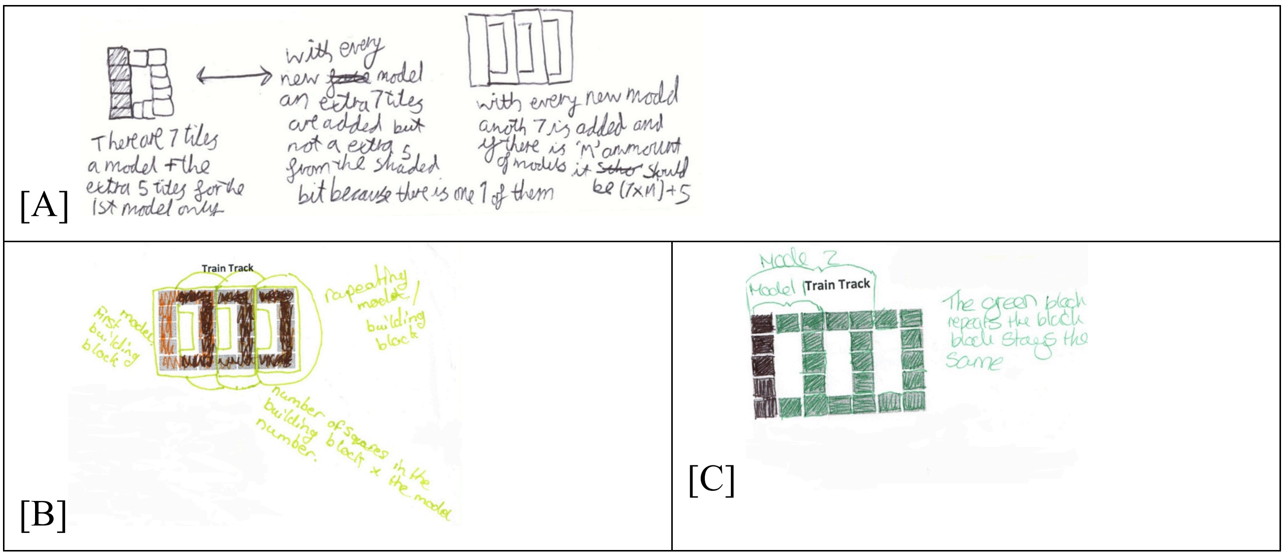
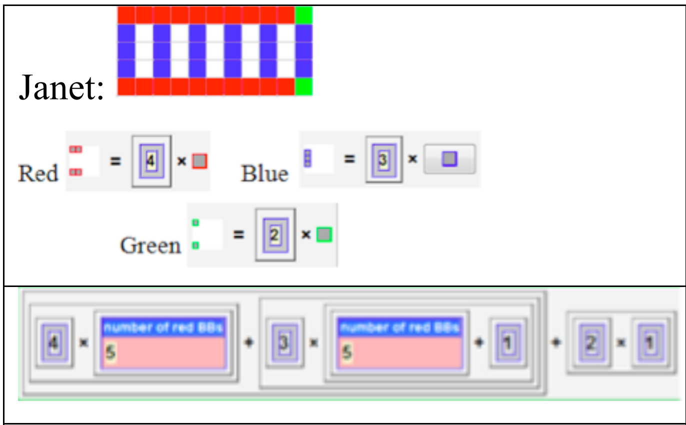


Figure 4. 13 year-old students' answers on the Train-track consolidation activity.

(ii) *Recognising and articulating generalisations, expressing them symbolically.* Students seemed to rely on the structure of the given task model in order to articulate a general rule. Most of them provided clear explanations to justify their derived rules.



4n + 3(n+1) + 2x1	7n + 5x1

Nancy: Yeah it's one red building block plus one blue building block so that would actually kind of make the…

Janet: yeah, it would make the same shape...

Nancy: because one red building block added to one blue building block…

Janet: and that's the same as one of my green building blocks.

Figure 5. 12 year-old students' discussion on the Collaborative bridging activity.

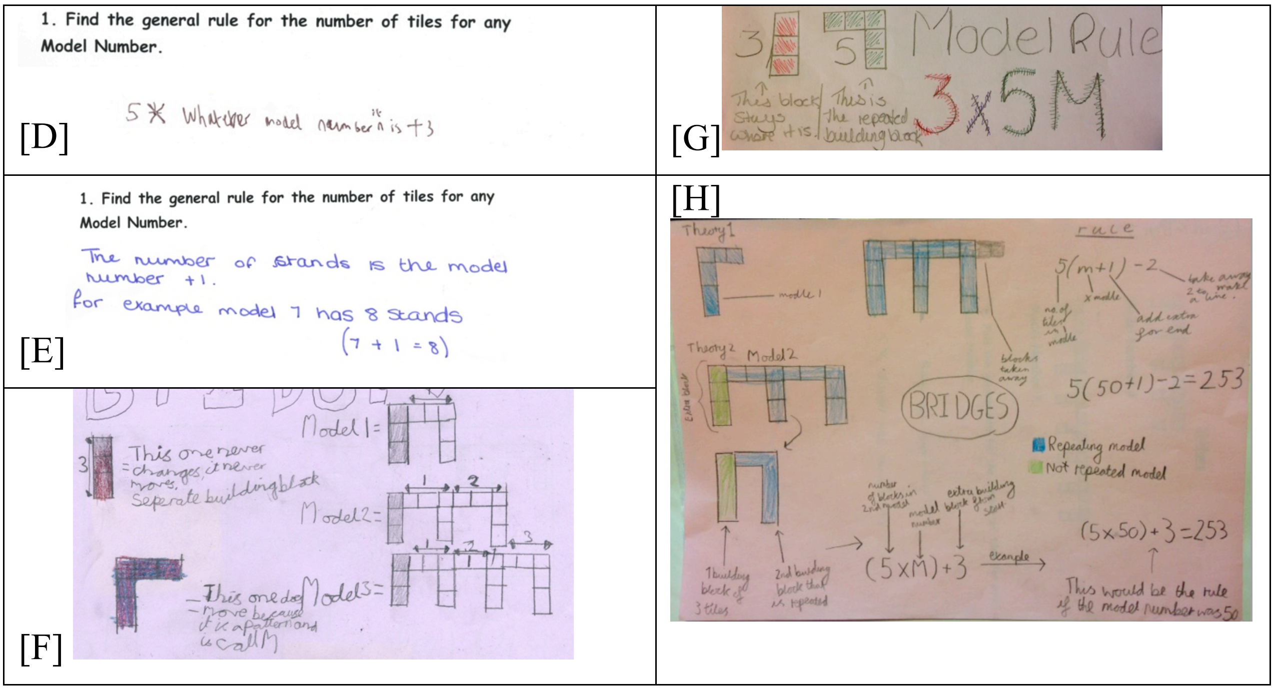


Figure 6. 12 year-old students' work on the eXpresser-like Bridges bridging activity.

Their work revealed some fluency in using the formal algebraic language. They identified what stayed the same and translated that into a constant in their rule. For example, in Figure 6 [H], the student annotated their rule (5xM)+3 and showed that the coefficient 5 is the number of repeated building blocks in their second building block. The constant 3 is the number of tiles in their first building block, which is not repeated. Similarly, the student in Figure 6 [G] successfully identified 2 building blocks, that produce the task model, and indicated which building block stays the same and which is repeated.

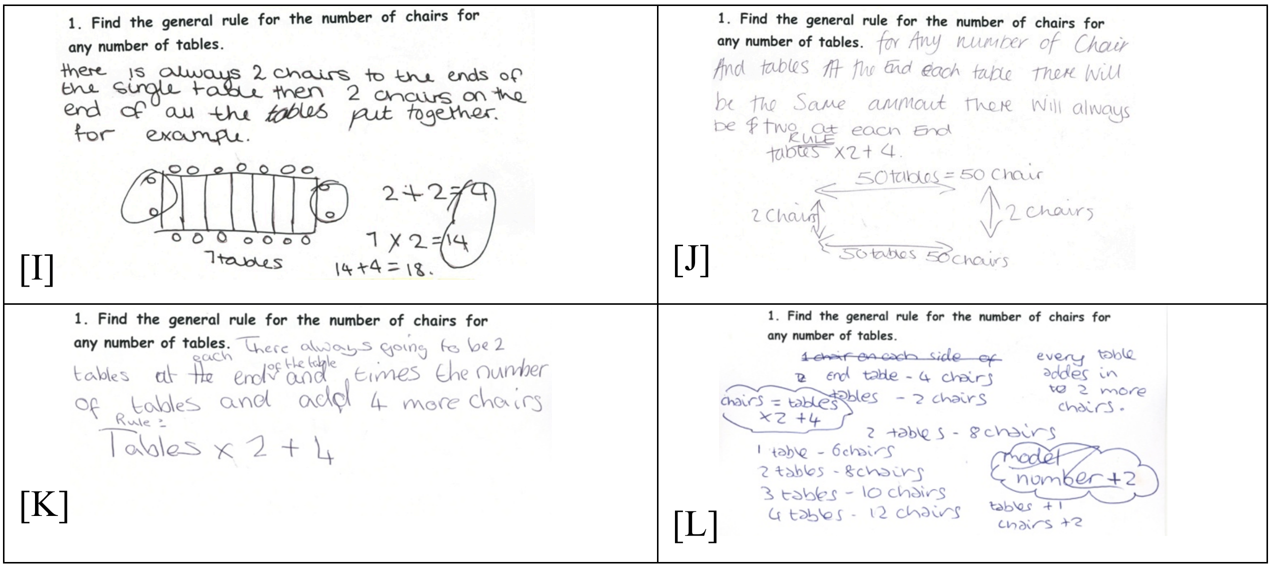


Figure 7. 12 year-old students' work on Tables and Chairs textbook-like activity.

Students' answers revealed their ability to articulate general statements, such as "with every new model, another 7 is added and if there's 'M' amount of models, it should be (7xM)+5" (Figure 4[A]) or "there is always 2 chairs to the ends of the single tables, then 2 chairs on the end of all tables put together" (Figure 7[I]). But the crucial step was their ability to translate that generalisation in parallel to their visualised structures into general rules and argue about similarities (or differences) between their models and derived general rules, when discussing rule equivalence (e.g., Figure 5). Most students used the eXpresser language and terms such as 'model number' to represent the variable in their rule (e.g., "5xwhatever model number n is+3", Figure 6[D]), as an intermediate step before expressing their derived rules in a formal algebraic expression (e.g., "(5xM)+3", Figure 6[H]). During collaboration, most students seemed to reach similar conclusions. Janet and Nancy for example recognized that the simplified general rule for their models is 7n+5 and that 'n' represents any model number. eXpresser seems to have played a crucial role in this outcome, as it encourages students to name their variables ('unlocked' numbers) based on what their values represent and therefore allows students to give meaning to that variable, thus easing students' transition to formal algebraic language.

Even though the bridging activities have been carefully designed to prevent students from looking for the term-to-term rule in a sequence, there were some students, especially in the text-book like bridging activities, who reverted to their past experiences and worked out the answers for each consecutive term in a sequence. For example, in Figure 7 [L], the student calculates the number of chairs when having 1 table, 2 tables, 3 tables, etc. Despite, their focus on the term-to-term rule, they spotted the correct general rule and wrote "Chairs=tablesx2+4". Such an outcome though may be ephemeral and more work is needed to support the sustainability and longevity of any AWOT formed soon after interacting with eXpresser.

CONCLUSION

When solving problems, mathematicians do not need to stop and think, but instead get into a "mechanical mode" (Sfard & Linchevski, 1994). Similarly, students who become experts in a digital tool may learn how to interact with it procedurally and provide right answers, but not necessarily reflect on and consolidate their knowledge during their interactions. Consequently, they may fail in developing a robust understanding of the mathematical concepts (and procedures) the tool was designed to help them with and may not be able to offer mathematically valid justifications for their actions. Such an outcome can discourage teachers from using digital tools in their mathematics lessons, as they are not convinced of their value.

In the case of eXpresser, the examples presented above reveal how students seem to successfully cross the 'bridge' from eXpresser algebra to formal algebra. Students demonstrated a conceptual understanding behind the development of general rules and generalised and adopted AWOT when solving PaP generalisation tasks. eXpresser, through the use of its language, supported students in their transition from numbers to 'unknown' numbers and variables and made the transition to symbolic thinking successful. In our experience, for such transitions to be successful, there is a need for bridging activities making the connections to algebra explicit. Their need and value have been mentioned by Gurtner (1992) too, who argued that 'the do-math-without-noticing-it' philosophy of Logo can be abandoned in favour of techniques that *explicitly* present looking for connections" (p. 253). We also recognised, similarly to Gurtner's (1992) research that "In contrast to the more classical transfer model […] useful bridges can be built from the beginning, as soon as work has started in both domains" (p. 265). This was addressed by the consolidation tasks. There also seems to be the need for a long period of practice with eXpresser, rich in reflection and consolidation, before transfer to mathematics can be possible.

We have investigated the initial transition from a constructionist learning environment to the PaP algebraic generalisation tasks, and we have only started looking at the further transition to tasks that focus on abstract algebra, as described by Sfard and Linchevski (1994). The main concern is to identify and make more explicit the residual knowledge that gets noticed particularly by the interaction with constructionist learning environments. A successful integration in our view involves the successful transition from interacting with a digital tool to the awareness of the knowledge that can potentially be transferred to PaP activities and identifying ways to encourage the sustainability of such knowledge. Our aim remains to investigate further the issues of 'Transfer' and 'Bridging' and support the implementation of digital tools in the classroom through carefully designed and innovative bridging activities that consolidate and sustain students' mathematical ways of thinking.

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