Building Bridges to Algebra through a Constructionist Learning Environment

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> Context • In the digital era, it is important to investigate the potential impact of digital technologies in education and how such tools can be successfully integrated into the mathematics classroom. Similarly to many others in the constructionism community, we have been inspired by the idea set out originally by Papert of providing students with appropriate "vehicles" for developing "Mathematical Ways of Thinking." **> Problem** • A crucial issue regarding the design of digital tools as vehicles is that of "transfer" or "bridging" i.e., what mathematical knowledge is transferred from students' interactions with such tools to other activities such as when they are doing "paper-and-pencil" mathematics, undertaking traditional exam papers or in other formal and informal settings. **> Method** • Through the lens of a framework for algebraic ways of thinking, this article analyses data gathered as part of the MiGen project from studies aiming at investigating ways to build bridges to formal algebra. **> Results** • The analysis supports the need for and benefit of bridging activities that make the connections to algebra explicit and for frequent reflection and consolidation tasks. **> Implications** • Task and digital environment designers should consider designing bridging activities that consolidate, support and sustain students' mathematical ways of thinking beyond their digital experience. **> Constructivist content** • Our more general aim is to support the implementation of digital technologies, especially constructionist learning environments, in the mathematics classroom. **> Key words** • **Algebraic generalisation and language, transition, exploratory learning, microworlds, bridging tasks.**

Introduction

« 1 » There is a growing concern that despite the increased availability of digital technologies designed for mathematics learning, students rarely use ideas, concepts or strategies that they could acquire through their interaction with such technologies in other contexts (cf. EACEA Eurydice Report 2011). As such, the full or intended potential of such technologies is diminished. One of the earliest (and most articulate) examples of these concerns is discussed in Jean-Luc Gurtner (1992). Referring to the Logo environment, Gurtner demonstrated that the tool's features, which are designed to support students when faced with complex mathematical problems, may impede them from making connections between their work in Logo and any mathematical or geometrical ideas they are already familiar with and use when problems seem less complex. One of the reasons behind this is that students might know how to use a digital tool procedurally, but can fail to understand conceptually the mathematical concepts and procedures that the tool was designed to help them with. Therefore the tool cannot immediately become an efficient mathematical instrument for them (Artigue 2002). Consequently, teachers can be hesitant to use such tools as it is hard for them to be convinced of the tools' value and they are reluctant to dedicate time to learning how to use them effectively and incorporate them into mathematics teaching practice (Clark-Wilson, Robutti & Sinclair 2014). However, a growing awareness of digital technologies' potential and limitations can support teachers using these technologies in the classroom (Abboud-Blanchard 2014). In this article, we discuss our approach to supporting students' transition in moving back and forth from paper-and-pencil to interacting with digital tools. We therefore consider ways of facilitating the integration of digital technologies into the mathematics classroom in an effort to shed some light on this issue that did not have the attention it deserves, as yet. In particular, our focus is on the transition to formal algebra and how students "transfer" their knowledge from their interactions with an algebraic microworld – a constructionist learning environment (eXpresser) specially designed to support and address students' difficulties with learning algebra (see Mavrikis et al. 2013) – to paper-and-pencil (PaP) activities.

« 2 » We are not the only ones concerned with the transition to formal algebra (e.g., Radford 2014), and the literature is replete with examples of student difficulties (e.g., Stacey & Macgregor 2002). Our view is similar to that of Luis Radford (2014), who claimed that there is a need for specially designed classroom activity to support students' developmental path to formal algebra, and to Gurtner (1992) and Stephen Godwin and Rob Beswetherick (2003), who suggested presenting structured tasks, using appropriate digital tools and making explicit interventions during students' interactions. We claim that a digital tool specially designed to support the development of algebraic ways of thinking (AWOT), together with carefully designed bridging activities, should scaffold the transition to formal algebra. Besides "learning" the tool and developing expertise in using it, students should be supported in making the connections to the maths.

http://www.univie.ac.at/constructivism/journal/10/3/321.geraniou

« 3 » Considering the issues discussed above, the research carried out in the MiGen project[1] has offered major gains in the understanding of students' development of AWOT through their interaction with exploratory learning environments. In this article, we present data gathered from 11–14-year-old students who worked on bridging activities carefully designed to support their transition from interacting with the MiGen constructionist learning environment, namely eXpresser, to traditional PaP algebraic generalisation tasks. General concluding remarks based on the initial stages of our data analysis are discussed on this transition to PaP tasks, as well as to algebra in general. Some research outcomes are shared regarding the successful integration of the eXpresser microworld, as well as the successful integration of similar digital tools, into the mathematics classroom.

Theoretical background

« 4 » Major transitions in a learner's life are much studied: the transition from counting to number (Cobb 1987), from number to arithmetic (Fuson 1990), from arithmetic to algebra (Davis 1985). The latter transition has been extensively researched and the mathematics education literature has revealed many examples of student difficulties in learning algebra (e.g., Stacey & Macgregor 2002). Students struggle to understand the idea behind using letters to represent *any* value (Duke & Graham 2007), and are inexperienced with using mathematical vocabulary to express generality (Hart 1981). Even students capable of expressing a general rule through the use of words such as "always" or "every" struggle to use letters and symbols to form algebraic expressions. As James Kaput (1992: 546) put it, students are routinely asked to "learn representation systems without anything to represent." Instead, the need to express and justify generality can be considered "the heart, root and purpose of algebra" (Mason 2005b: 2).

« 5 » In the digital era, where digital technologies are increasingly making their appearance in the mathematics classroom, students are faced with another transition, that of moving back and forth from PaP to interacting with digital tools. It is imperative, therefore, to investigate how and whether students "transfer" their knowledge from their interactions with digital tools to PaP activities. We put transfer in quotes because it refers to different constructs for different communities. There is of course a lot of research on "transfer" (e.g., diSessa & Wagner 2005). Our view is aligned with King Beach (2003) who has argued that the metaphor should be viewed as transition instead of transfer, as crossing boundaries from one location to another is in fact a process of transition and he considers that people are the ones who move and not knowledge or learning. Other authors claim that transfer "entails re-use of knowledge, demonstrated and / or acquired in one situation (or class of situations), in a 'new' situation (or class of situations)" (diSessa & Wagner 2005). Similarly, Robert Haskell (2001: xiii) claims "Transfer of learning is our use of past learning when learning something new and the application of that learning to both similar and new situations."

« 6 » When considering the "transfer" of knowledge, one needs to consider what type of knowledge is being transferred, as the educational literature has revealed many different types of knowledge (Beach 1999; diSessa & Wagner 2005). For example, James Hiebert and Patricia Lefeivre (1986) used the existence or lack of connections between internal networks (schemas) to introduce conceptual and procedural knowledge. Conceptual knowledge consists of a connection of networks and is rich in relationships. Procedural knowledge is defined as the learning of a series of actions, where the only apparent connections are those between successive actions in the procedure. The latter type of knowledge is the one that is usually observed with students who interact with digital tools, as they discover how the tool "works" and can rely on the immediate feedback they get from the tool but not necessarily reflect upon their actions nor verify their answers (Hieler, Gurtner & Kieran 1988). Moreover, students tend to ignore interesting perspectives on mathematics while interacting and gaining more and more experience with digital tools (e.g., Gurtner 1992; Godwin & Beswetherick 2003). Saying that though, it is worth revisiting the arguments in Seymour Papert (1980) that students who interact with Logo can visit mathematically rich areas that they would not have approached otherwise. Using digital tools that are specially designed to support students' difficulties and possible misconceptions on the topic of algebra, for example, should scaffold the transition to formal algebra without rendering it impossible for them to reach the mathematical bank of algebra. Besides "learning" the tool and becoming experts in using it, students should then be able to make the connections to the mathematics behind their digital interactions. The challenge is to find ways to support students to make these connections.

« 7 » In the case of Logo, Gurtner (1992: 247) considered "the type of connections generally expected, and very seldom observed, between Logo practice and mathematics" as *transfer* and suggested that "a rather long period of Logo practice (one that is rich in reflection) is necessary before transfer to mathematics can occur (Salomon & Perkins 1987)." He also used the "bridging" metaphor to describe the connections students or educators try to build between different domains or topics within the same domain or aspects of the school life and the everyday life. We valued and aligned our work with the bridge metaphor (as opposed to the notion of transfer) as it allows connections to be identified and made as early as possible between the domain of the digital tool and mathematics and hopefully to have a greater impact on students' learning.

« 8 » Relevant research (e.g., Gurtner 1992) and our anecdotal observations suggest that students rarely use ideas, concepts or strategies they seem to have acquired through their interactions with digital technologies in their mathematics classrooms. One way to build bridges to formal maths is through presenting structured tasks, using appropriate digital tools and making explicit interventions during students' interactions. Even though a lot of research

1| The MiGen project was funded by the ESRC/EPSRC Teaching and Learning Research Programme (Award no: RES-139-25-0381) and the MiGen follow-on project is funded by ESRC (Award no: ES/J02077X/1), http://www.migen.org

has been carried out on how to design such tools to address students' difficulties with mathematical concepts (e.g., Hoyles & Noss 1996; Noss et al. 2012), the issue of the integration of the tools in question needs to be investigated. This involves looking into what happens after students interact with a digital environment and what resources can render the transition to formal mathematics successful.

« 9 » Despite the advances in the technological infrastructure in schools and the plethora of digital tools specially designed to support the teaching and learning of mathematics, the integration of digital technologies in mathematics education has not always met expectations (e.g., Drijvers et al. 2010). One of the reasons seems to be teachers' usual practices (Clark-Wilson, Robutti & Sinclair 2014). The teachers' perspectives and their abilities to develop a new repertoire of appropriate teaching practices for technology-rich classrooms can play a crucial role in identifying the best strategies in supporting the successful integration of digital technologies in the mathematics classrooms (Ruthven 2007).

« 10 » Considering all the above issues, in this paper the focus is on bridging activities specially designed to support students' transition from their interactions with the MiGen constructionist learning environment to PaP activities outside the tool. We share some research outcomes in an effort to support students, and consequently their teachers, towards a successful integration of the MiGen tool and other similar tools in general into the mathematics classroom.

Methodology

« 11 » Using a Design-Based Research methodology (Design-Based Research Collective 2003), over the past seven years we carried out a number of studies in six different schools in London, worked with 11 mathematics teachers and collected data from 553 students aged 11–14 years old. Our data comprises interviews and transcripts that are one-to-one and with small groups of students, video (mostly screen recordings) and audio files from interviews, observations that are one-to-one, of small groups and of classrooms, detailed logs from students' interactions in the form of a database and interviews of teachers, and transcripts and bridging activities specially designed to gauge students' knowledge.

« 12 » We have presented results from our data analysis of our various studies in a number of papers (e.g., Mavrikis et al. 2013; Noss et al. 2012; Geraniou et al. 2011). In this article, however, we focus on the data collected from the bridging activities students worked on during, but mostly after their final interaction with the eXpresser tool. We present here our initial analysis offering examples of students' typical responses. We plan to do more in depth analysis for each type of bridging activity, as we started to do in the case of the collaborative activity presented later in the article (cf. Geraniou et al. 2011). Since the eXpresser tool has been specially designed to support students dealing with some well-known and researched misconceptions on algebra (Noss et al. 2012), the goal of this preliminary analysis was to identify the impact of those design decisions on students' understanding and reasoning about algebraic generalisation, and whether students use any of the strategies they were encouraged to employ while interacting with eXpresser on the PaP tasks and are successful in solving figural pattern generalisation tasks. We focused on two AWOT, as described in our previous work (Mavrikis et al. 2013):

- *Perceiving structure and exploiting its power*, which is about noticing what stays the same and what is repeated in a figural sequence so as to understand how the sequence is "structured," supporting therefore "the development of structural reasoning" and the habits of "breaking things into parts" by identifying "the building blocks of a structure" (Cuoco, Goldenberg & Mark 1996); and
- *Recognising and articulating generalisations, including expressing them symbolically*, which is the process of translating the observed structure in an algebraic expression, using formal algebraic notation to write general rules for numerical sequences. Students' answers were viewed several times and analysed using those two AWOT as an analytical framework for interpreting students' strategies.

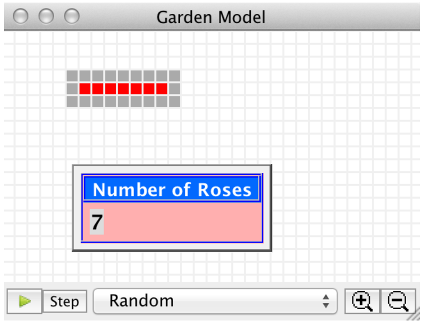


Figure 1 • A model for 7 red tiles surrounded by grey tiles. Students must construct a general model and find the general rule.

« 13 » In the results section, we share our insights gained from the initial analysis of this data on bridging activities and students' strategies to solve figural pattern generalisation tasks without the support or immediate feedback of eXpresser.

The eXpresser microworld and bridging activities

« 14 » The MiGen system is a pedagogical and technical environment that is designed to improve 11–14-year-old students' learning of algebraic generalisation, a specific and fundamental mathematical way of thinking. Its core component consists of a microworld, eXpresser, which is designed to support students in their reasoning and problem-solving of a class of generalisation tasks. Previous work (e.g., Küchemann 2010; Mason 2005b) has demonstrated in particular the potential of designing activities that can help students focus on the structure of the figural pattern by providing generic examples and challenging them to identify the rules that underlie them. In eXpresser, students construct figural patterns by expressing their structure through repeated building blocks of square tiles, and articulating the rules that underpin the calculation of the number of tiles in the patterns. A typical eXpresser activity asks the student to reproduce a dynamic model presented in a window that appears on the side of the activity screen.

« 15 » Figure 1 shows a model where a row of red tiles is surrounded by grey tiles. Students are asked to construct a model that

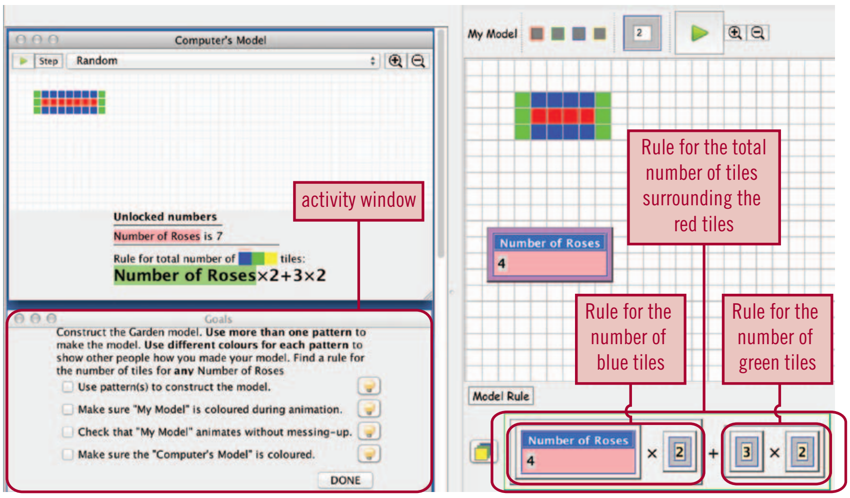


Figure 2 • The eXpresser screen showing the general and specific models (Computer's Model on the left, and My Model on the right), and a correct rule for the total number of surrounding tiles. The system has recognised that all the goals shown in the "Activity window" (lower left-hand corner) have been accomplished.

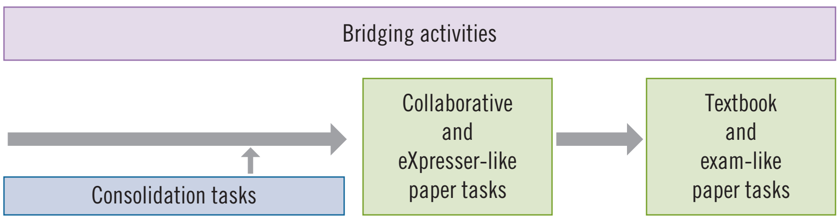


Figure 3 • The schematic presentation of the bridging activities.

works for *any number* of red tiles, and find a rule for the total number of tiles surrounding the red tiles. They can test generality by *animating the model*: that is, by letting the computer change the number of red tiles at random. The design of eXpresser capitalises on animated feedback and on the simultaneous representation of a *specific* and *general* model ("My Model" and "Computer's Model" in Figure 2), built by combining patterns and on the close alignment of the symbolic expression, the *Model Rule* and the structure of the model. In the *Computer's Model*, a value of the variable[2] ("Num of Red Tiles" in this example) is chosen automatically at random (it is "7" in Figure 2) and will generally be different from that in the specific model ("4" in Figure 2). So the *Computer's Model* indicates to students whether their constructions are *structurally* correct for the different values of the variable(s) assigned to the various properties. Students also construct a *model rule* for the total number of tiles, and validation of its correctness is made evident by colouring: tilings are *only* coloured if the rule for the number required is correct.

« 16 » In our studies, students are presented with a sequence of activities. Initially, they are familiarised with eXpresser in two lessons through a number of introductory and practice tasks, asking students to construct figural models. Afterwards they are given individual activities, such as the one described above (Figure 1). Students are asked to construct the task model in eXpresser using different patterns and combinations of patterns, depending on their perceptions of the task model's structure, and derive a general rule for the number of square tiles needed for any Model Number. There are progressively harder tasks students can work on in eXpresser.[3] In our initial studies, students were presented with off-computer tasks, immediately after the final eXpresser task, in an effort to reveal their strategies for solving similar tasks on paper and whether eXpresser had an impact on those strategies or not. In later studies, though, and after close collaboration with teachers, we recognised the need for activities that promote students' reflections upon mathematical concepts and the problem-solving strategies they used *throughout* their interactions with eXpresser and not just at the end. These we referred to as consolidation tasks and were used with 175 students out of the total 553 students we worked with. Throughout their interactions with eXpresser and immediately afterwards, students are presented with a number of bridging activities (Figure 3), which are designed to support their transition to PaP tasks. We designed 4 types of bridging activities:

1 | consolidation tasks, which are used to intervene and encourage students to reflect on their interactions with eXpresser throughout a sequence of eXpresser tasks;[4]
2 | collaborative tasks, which are presented at the end of an eXpresser task and focus on students' justification strategies regarding the equivalence or non-equivalence of their derived rules;[5]
3 | eXpresser-like paper tasks, which are figural pattern generalisation tasks that are presented on paper; and

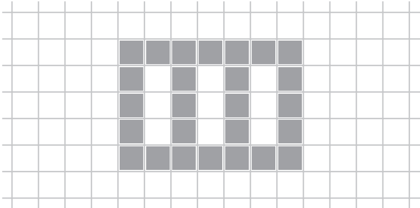
2 | All numbers in eXpresser are *constants* by default, referred to as "locked" numbers. When the user "unlocks a number," it is possible to change its value; it becomes a *variable*.

3 | Some eXpresser tasks can be found on http://expresser.lkl.ac.uk

4 | The consolidation activities are designed to accompany eXpresser activities and therefore all models, e.g., the grey train-track model presented in Figure 4, are animated in eXpresser when shown to students.

5 | For these tasks, students' models are not necessarily presented for the same Model Number. Instead students are encouraged to explore and trial different values for the Model Number and quite often they were observed using the same value for their Model Number so as to compare the two models and rules (see Geraniou et al. 2011).

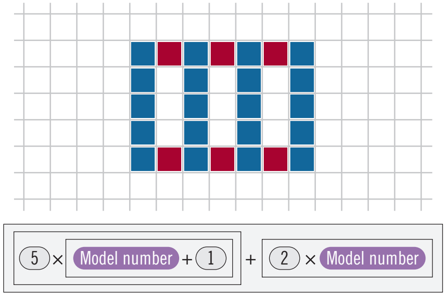
Consolidation Task – Train-track



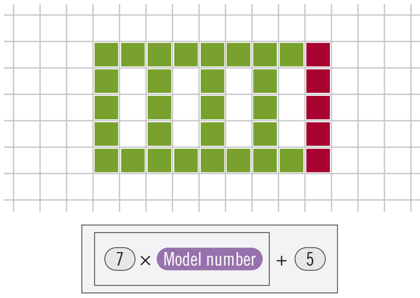
1. How many tiles are needed for Models 4, 8, 1 and 100?
2. If we use “M” to stand for the Model Number, how many tiles are needed for Model M?
3. Use the space below to explain the different parts of your rule – use the diagrams left or your own if it helps.

(Note: Task models for these tasks are presented animated in eXpresser)

Collaborative Task – Equivalent expressions



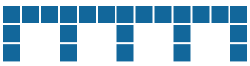
$5(n+1)+2n$



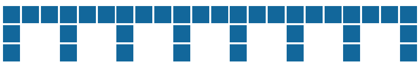
$7n+5$

1. Convince each other that your model and rules are correct.
2. Can you explain to each other why the rules look different but are equivalent? Discuss and write your explanations.

eXpresser-like Paper Task – Bridges



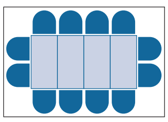
Model number 4



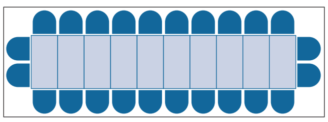
Model number 7

1. Find the rule for the number of tiles for any Model Number.
2. Find the number of tiles for Model Numbers 5, 10 and 100.

Text-book Paper Task – Tables and Chairs



4 tables



10 tables

1. Find the general rule for the number of chairs for any number of tables.
2. Use your rule to find the number of chairs for 20 tables and for 200 tables.
3. If I have 26 chairs, how many tables do I need?

Figure 4 • Examples of the 4 types of Bridging Activities.

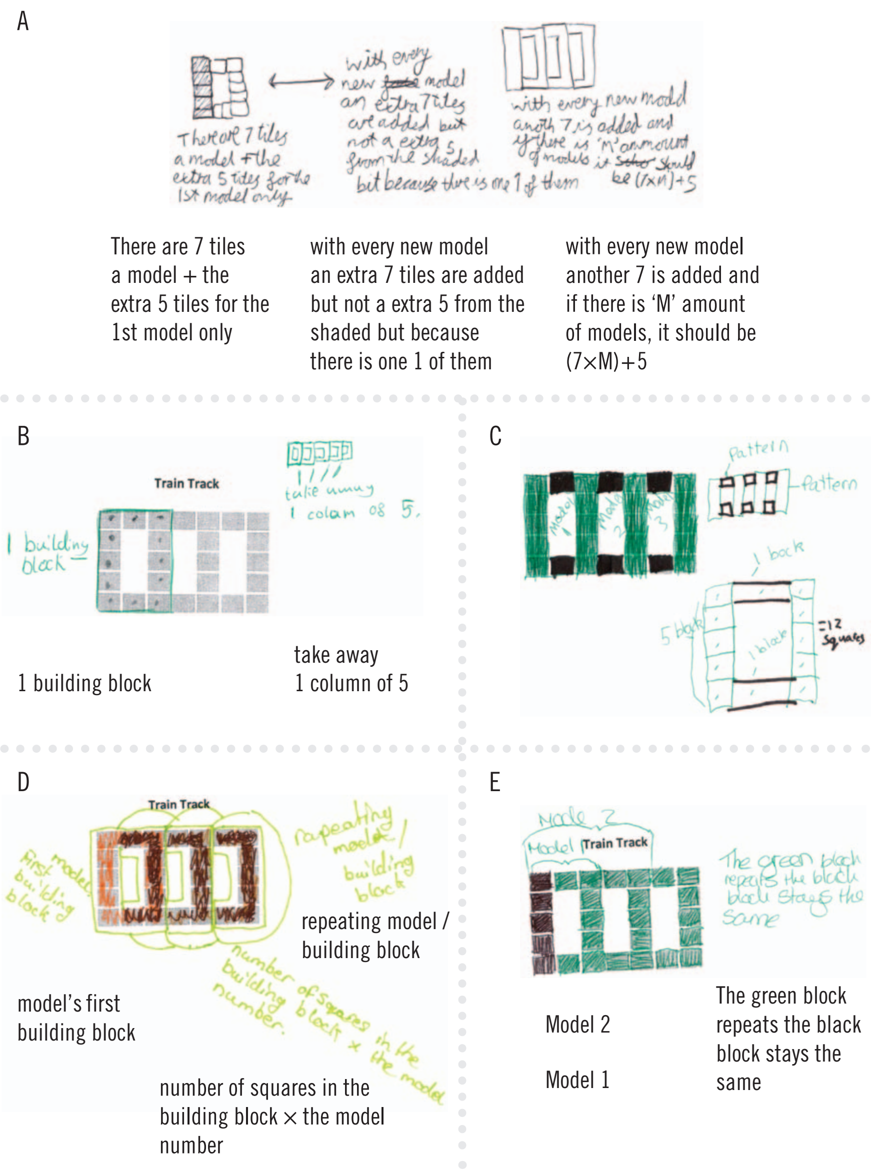


Figure 5 • Examples of 13-year-old students' answers on the Train-track consolidation bridging activity.

Results

« 17 » Students were asked to work independently on all the types of bridging activities, except for the collaborative one, for which they worked in pairs or groups of three. Using the two AWOTs mentioned earlier as an analytical framework for interpreting students' strategies when undertaking the bridging activities, we present our initial results under those two headings.

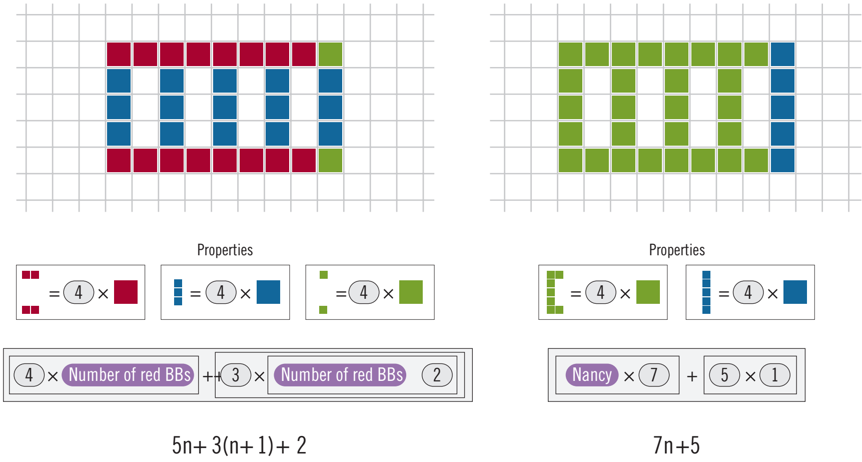
Perceiving structure and exploiting its power

« 18 » For the consolidation tasks, most of the 175 students demonstrated on the model figures presented on paper how they visualised the structure of the given model. In Figures 5, 6, 7 and 8, we present some examples of students' answers on the "Train-track" consolidation task, the "Equivalent expressions" collaborative task, the "Bridges" eXpresser-like paper task and the "Tables and Chairs" textbook-like task respectively. Students clearly marked the different parts that would remain the same in any instance of the pattern and the parts, which, repeated every time, create the different instances of the pattern. Some of them, perhaps influenced by the colouring feature of eXpresser, used coloured pens to identify these different building blocks. Students demonstrated a variety of ways to visualise the task patterns and they seemed to be as influenced by the eXpresser's features as they were using the eXpresser terminology, e.g., number of building blocks or models. For example, in Figures 7 H, 7 I and 7 J, students drew the two building blocks that they could use if they were solving this task in eXpresser, that of a column of three square tiles and that of an "L"-shaped one of five tiles. For example, Janet named her independent variable as "number of red BBs" (BBs stands for Building Blocks), and even though Nancy named hers as "Nancy," she used eXpresser's terminology in her discussions with Janet (see Figure 6). Especially for the collaborative task, most students verbally identified their building blocks in their models and rules and compared them to conclude about their equivalence. An example of two students' collaboration and its outcome is presented in Figure 6.

4 | text-book or exam-like tasks, which are similar to the traditional generalisation tasks given to students on paper and involve patterns presented, usually non-sequentially, so as to assess students' approaches in terms of perceiving the patterns' structure. All bridging activities were carefully designed to prevent students from looking for the term-to-term rule in a sequence. For example, the models are either presented animated in eXpresser (consolidation tasks) or non-sequentially (eXpresser-like paper tasks) and the task question is open-ended ("Find the rule for the number of tiles for any Model Number"). Examples of the four types of bridging activities are presented in Figure 4.

« 19 » In Figure 7 J, the student has derived two "theories" as they describe the two ways in which they were able to construct the Bridges model. In both these cases, they clearly identified the building blocks that are repeated to form the task model and used different colours to indicate which blocks, for example, need to be removed from the constructed model so that the task model is produced (two grey tiles for "Theory 1") and which building block would be added as an extra one to complete the task model in "Theory 2" (the green building block of three tiles). For their "Theory 2," they also indicated clearly that the blue tiles are used for the "repeating model," whereas the green tiles are used for the "not repeated model."

« 20 » In all these examples, it is evident how many different ways students have visualised the task models. There were also a few students who managed to derive a correct general rule, but they did not demonstrate on paper the structure in which they possibly visualised the pattern. Such an example is given in Figure 7 F.



Nancy: Yeah. It's one red building block plus one blue building block so that would actually kind of make the…
Janet: yeah, it would make the same shape…
Nancy: because one red building block added to one blue building block…
Janet: and that's the same as one of my green building blocks.

Figure 6 • An example of two 12-year-old students' discussion on the collaborative bridging activity.

Recognising and articulating generalisations, including expressing them symbolically

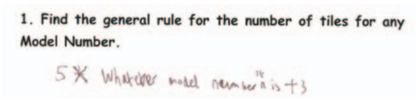
« 21 » Students seemed to rely on the structure of the given task model in order to articulate a general rule. Most of them provided clear explanations to justify their derived rules and share their solution and revealed some fluency in using the formal algebraic language. They identified what stayed the same and translated that into a constant in their rule. For example, in Figure 7 J, the student had even annotated his / her rule (5 × M) + 3. He / she showed that the coefficient 5 is the number of building blocks in his/her second building block, which, as he/she claimed, is repeated. The constant 3 is the number of tiles in their first building block, which is not repeated, and "M is the "model number." Similarly, the student in Figure 7i successfully identified two building blocks that can produce the task model, and indicated which building block stays the same and which is repeated.

« 22 » Students' answers revealed their ability to articulate general statements, such as "with every new model, another 7 is added and if there's "M" amount of models, it should be (7 × M) + 5" (Figure 5 A) or "there are always 2 chairs to the ends of the single tables, then 2 chairs on the end of all tables put together" (Figure 8 K). But the crucial step was their ability to translate that generalisation in parallel to their visualised structures into general rules, as well as to argue about similarities (or differences) between their models and derived general rules when discussing rule equivalence (e.g., Figure 6). Most students used the eXpresser language and terms such as "model number" to represent the variable in their rule (e.g., "5 × whatever model number n is +3," Figure 7 F, as an intermediate step before expressing their derived rules in a formal algebraic expression (e.g., "(5 × M) + 3," Figure 7 J). eXpresser seems to have influenced this outcome, as it encourages students to name their variables ("unlocked" numbers) based on what its various values represent and therefore allows students to give meaning to that variable. This step eased students' transition to the formal algebraic language and seems to have given meaning to the use of letters to represent "unknown" values.

« 23 » Some students evaluated their rules by using specific values for their variable and used these examples to justify further their derived rules. For example, in Figure 7 J, the student calculated the number of blocks for Model Number 50 and found that there are 253 blocks. As they had derived a second rule, i.e., 5 (m + 1) – 2 (top corner of the Figure 7 J), they could see that for the same Model Number 50, both rules give the same number of building blocks. In Figures 8 K and 8 L, even though both students justified in words how they derived their rules, they even used numerical examples, choosing 7 chairs and 50 chairs respectively.

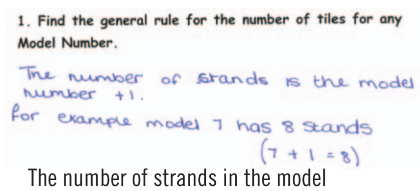
« 24 » Of course challenges remain, and even though the presentation of each of the bridging activities was carefully designed to prevent students from looking for the term-to-term rule in a sequence, there were some cases of students, especially in the text-book like bridging activities, who reverted to their past experiences and worked out the answers for each consecutive term in a sequence. For example, in Figure 8 N, the student calculates the number of chairs when having 1 table, then 2 tables, then 3 tables, etc. He / she focused on the term-to-term rule and managed to spot the correct general rule and wrote "Chairs = tables × 2 + 4."

F



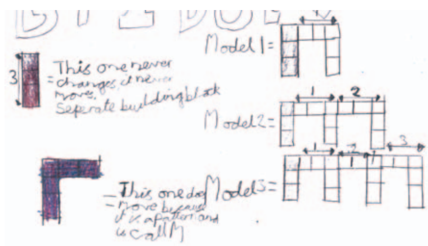
5 x whatever model number it is + 3

G



The number of strands in the model number +1
For example model 7 has 8 strands (7+1)=8

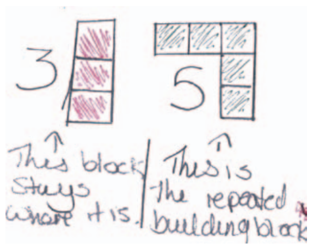
H



This one never changes, it never moves. Separate building block

This one does move because it is a pattern and is call M

I



This block stays where it is

This is the repeated building block

J

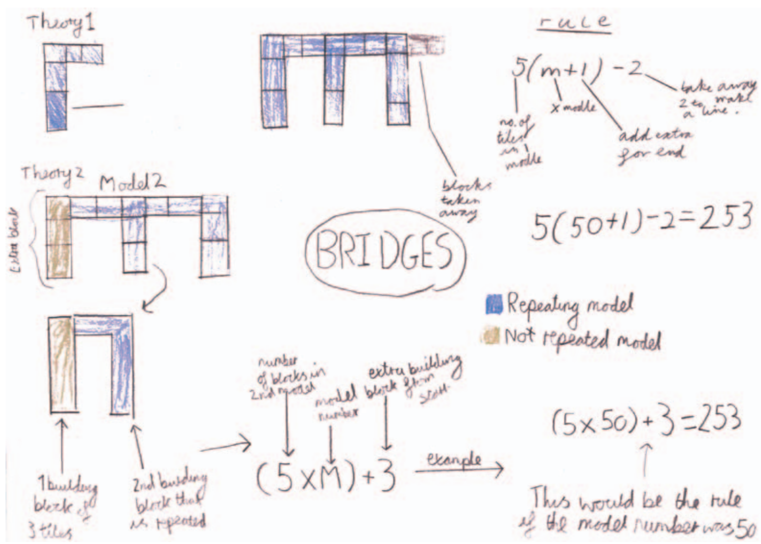


Figure 7 • Examples of 12-year-old students' work on the eXpresser-like Bridges bridging activity.

This example reveals that the student either solely relied on a term-to-term approach or they used this empirical approach to start with but then switched to a functional view, writing "every table adds in 2 more chairs." There are of course cases in which students do not transfer the skills which the eXpresser was designed to help them develop. Perhaps there is a need for more interventions through bridging activities to enrich students' interactions with constant reflections on their strategies and greater emphasis to be given to supporting students who have well-known and researched difficulties. It would have been interesting to investigate whether this student would have used the same strategy when faced with a more complex figural pattern generalisation task, in which such a "term-to-term rule" strategy would not have been easy (e.g., quadratic sequences, which were occasionally given as additional challenging tasks).

Conclusion

Researcher: "So, what did you learn from interacting with the eXpresser tool?"
Student: "I learned how to put tiles and make nice patterns in different colours"
(Excerpt from an early pilot study with eXpresser)

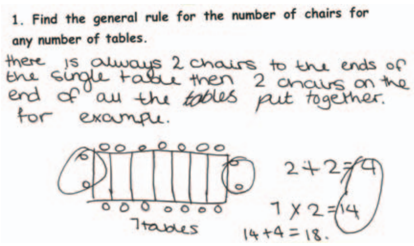
« 25 » Students nowadays are asked to interact with a number of ICT tools that have been carefully designed and developed to engage them and support their learning of mathematics. Quite often though, students learn how to interact with the tool, create beautiful productions, such as colourful patterns as suggested by the student in the above quote, and often get to the right answer or "an" answer without necessarily reflecting on and consolidating their knowledge during their interactions. They may know how to use the tool procedurally, but may fail in understanding conceptually the mathematical concepts and procedures that the tool was designed to help them with. Consequently, teachers can be hesitant to use such tools in their lessons as it is hard for them in their busy work lives to be convinced of the tools' short- and long-term value and can be reluctant to incorporate the use of ICT tools into mathematics instruction.

« 26 » In the case of the eXpresser tool, and as has been revealed in the few examples presented in the previous section, most students seemed to have successfully "transferred" their gained knowledge or crossed the "bridge" from the eXpresser algebra to formal algebra. They have demonstrated their conceptual understanding of deriving a general rule and allowed us to claim they can generalise and adopt AWOT when solving paper and pencil figural pattern generalisation tasks. Our experience from the various studies for the MiGen project so far has supported the need for bridging activities, whose objective is to make the connections to algebra explicit. The need and value of such activities have been claimed by Gurtner (1992: 253) too, who claimed that "the do-math-without-noticing-it philosophy of Logo can be abandoned in favour of techniques that *explicitly* present looking for connections between Logo and mathematics as an objective of a task." We also recognized the need for constant reflections by students when interacting with the eXpresser tool, which was achieved through the consolidation tasks. Similarly to our research outcomes, Godwin and Beswetherick (2003), when investigating the use of Omnigraph in the classroom, claimed that there is a need for tasks that encourage more focused interactions by students in an effort to help them formalise and concretise their generalisations, notice relevant properties and develop mathematical ways of thinking. There also seems to be a need for a long period of practice with the eXpresser tool, and any mathematics digital tool, rich in reflection and consolidation, before transfer to mathematics can be deemed possible. This view has been supported by other researchers in the past (e.g., Noss et al. 2012; Gurtner 1992; Godwin & Beswetherick 2003).

« 27 » Finally, we do not claim that our approach to bridging activities is the only way to encourage transfer, neither that eXpresser is the only environment to help students develop AWOT. However, as we have elaborated in Mavrikis et al. (2013), the interaction with eXpresser provides a substrate of activity and experience for the teaching and learning of algebraic generalisations that is difficult to achieve with traditional paper-based activities (perhaps with notable exceptions of concentrated research efforts, e.g., Küchemann 2010). This is the case for other areas of mathematical learning as well, and although digital tools like eXpresser provide part of the answer, the article demonstrated how carefully designed bridging activities may be of value.

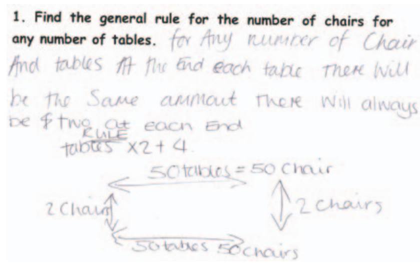
« 28 » In a series of projects related to the eXpresser tool[6] we have engaged with a number of teachers, trainee teachers or mathematics educators, and encouraged them to co-design activities around eXpresser and to use them for teaching. We have seen several creative approaches directly or indirectly aiming towards the objective of making links between students' experience with eXpresser and algebra. For example, teachers have designed activities that invite students themselves to design eXpresser tasks and challenge their peers or create posters to share their views of eXpresser, its activities and what they believe they have learned during their interactions and to identify similarities to traditional algebra. There are several examples of this in the MiGen follow-on package resource (http://link.lkl.ac.uk/migen-package), and the M C Squared project (http://www.mc2-

K



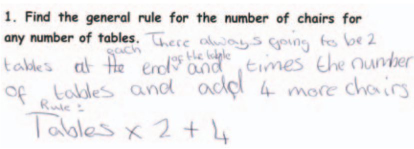
there is always 2 chairs to the ends of
the single table then 2 chairs on the
end of all the tables put together
For example:

L



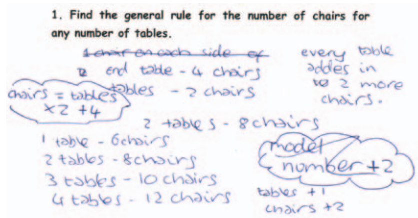
for any number of Chair
And tables At the End each tble there will
be the Same amount there will always
be two at each End
RULE
Tables x 2 + 4

M



There always going to be 2
tables at each end of the table
and times the number
of tables and add 4 more chairs
Rule: Tables x 2 + 4

N



every table adds in 2 more chairs
End table – 4 chairs
Tables – 2 chairs
Chairs = tables x 2 + 4

Figure 8 • Examples of 12-year-old students' work on the Tables and Chairs textbook-like bridging activity.

6| These projects include the ESRC funded "follow-on" project (ES/J02077X/1), the EU-funded METAFORA project, http://www.metafora-project.org, and the M C Squared project, http://mc2-project.eu.

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project.org) involves a community of interest that is designing a c-book around these ideas.

« 29 » There are of course several tools, ranging from mathematical games to elaborate production or programming tools, where students are given the opportunity to use or develop skills in order to solve puzzles or create and share artefacts. But the concern for us (and teachers we interact with daily) is the same as mentioned above: what is the residual knowledge that gets noticed by the interaction with such a tool and how can we make it more explicit and support the learning of mathematics in constructionist learning environments?

« 30 » A successful integration in our view involves the transition from interacting with a digital tool to the awareness of the knowledge that can potentially be transferred from students' interactions with digital technologies to PaP activities, and identifying ways to encourage explicitly the sustainability of such knowledge. Taking into consideration this vision, our aim remains to investigate further the issues of "transfer" and "bridging" and support the implementation of digital tools in the classroom through carefully designed and innovative bridging activities that consolidate, support and sustain students' mathematical ways of thinking.

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Open Peer Commentaries

on Eirini Geraniou & Manolis Mavrikis's "Building Bridges to Algebra through a Constructionist Learning Environment"

Proposing a Framework for Exploring "Bridging"

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> Upshot • Geraniou and Mavrikis raise the important issue of "transfer," when students transition from activity in technological tools to paper-and-pencil tasks. In this commentary, I contribute to the conversation by focusing on the relationship between task design and students' development of knowledge.

« 1 » Education researchers have tried to describe the situated nature of knowledge that students construct as they interact with contextual problems and digital tools. Examples include the notions of "abstraction in context" (Hershkowitz, Schwarz & Dreyfus 2001), "situated generalizations" (Nemirovsky 2002) and "situated abstractions" (Hoyles & Noss 1992). What is common among these notions is the idea of constructing knowledge that is situated in the specific context in which that construction takes place. Although powerful for developing advanced mathematical ideas, "there is, however, little evidence that students can abstract beyond the modeling context" (Doerr & Pratt 2008: 272). Eirini Geraniou and Manolis Mavrikis make a significant contribution to the field by raising the issue of the "transfer" of students' constructed knowledge from the eXpresser digital environment to paper-and-pencil activities.

« 2 » The authors study "transfer" through a series of what they refer to as "bridging activities" to assist the "transition" to formal algebra. Unlike Harry Broudy's (1977) notion of "applicative knowing" or the ability of students to apply their prior knowledge in order to solve new problems, Geraniou and Mavrikis's goal is not just about replicating or applying knowledge but rather about placing students "on a trajectory towards expertise" (Bransford & Schwartz 1999: 68) by scaffolding the transition to formal algebra. In order to make this distinction, they take a view of transfer that is aligned with King Beach's conception of transfer as "transition," since "people are the ones who move and not knowledge or learning" (Beach 2003: 3). Although the authors describe the notions of "transfer," "bridging" and "transition" as separate but inter-related, at points throughout the article these are used interchangeably. In my view, Geraniou and Mavrikis's goal of transfer aligns with the metaphor of "bridging" as "a process of abstraction and connection making," as described by David Perkins and Gavriel Salomon (1988: 28). Consequently, bridging can be seen as expanding what Dave Pratt and Richard Noss (2002, 2010) refer to as students' *contextual neighbourhood*, or the range of contexts and variety of circumstances in which the students' knowledge is made relevant and accessible.

« 3 » By using a design-based research methodology, Geraniou and Mavrikis designed a series of "bridging activities" to explore how students use the situated knowledge they constructed through their activity in eXpresser to solve similar paper-and-pencil (PaP) tasks, namely:

- Consolidation tasks
- Collaborative tasks
- eXpresser-like paper tasks
- Textbook paper tasks.

Since context plays such a vital role in this study, in this commentary I raise some design issues that can be considered while studying the notion of bridging. First, I would like to challenge the authors to define what they mean by "context." If the goal of the design-based research is to develop a contextual variety of bridging activities aimed at broadening the scope of the mathematical abstractions in the eXpresser activities, then the choices of context in the four PaP tasks need to be made explicit.

« 4 » The choice of context is influenced both by the initial eXpresser context and by the two algebraic ways of thinking (AWOT) that the authors aim to explore, namely

- Perceiving structure and exploiting its power; and
- Recognizing and articulating generalizations, including expressing them symbolically.

Design studies involve "'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb et al. 2003: 9). Likewise, the process of "bridging" can be "systematically" studied by making the iterations of the task design process explicit and presenting how students' development of AWOT informed the design of those iterations. In other words, it can be studied by exposing the study's iterative nature, where researchers formulate conjectures about student learning and then revisit and refine them throughout the study (Confrey

& Lachance 2000). A mechanism showing the process of designing those tasks (e.g., nature, sequence) that makes explicit the choices of context, the iterative cycles of design, if any, and the conjectures the authors had about students' progression of thinking while working with those tasks could provide a stronger framework for studying the "bridging" process.

« 5 » Simon's (2013) design approach to learning through activity may offer a guide to structure this bridging framework through a sequence of four steps:

1 | Assess students' relevant mathematical conceptions;
2 | Articulate a learning goal;
3 | Specify an activity that students currently have available that can be the basis for developing the abstraction specified by the learning goal; and
4 | Design a task sequence and postulate a related learning process.

Geraniou and Mavrikis constructed a model of students' thinking in eXpresser, clearly described the two AWOT they have as learning goals, developed a sequence of tasks for reaching those goals and began their task design by having students' activity with eXpresser as the basis. What needs further investigation is the hypothetical learning process (Simon 2013, 2014), which takes the form of conjectures about student thinking and how the specific engineering of the task design and sequence may assist students in developing their knowledge and reach the AWOT goals. Questions that may guide this process include:

- What schemes and operations of AWOT were provoked in the initial context of eXpresser?
- How can similar schemes and operations be provoked in the new contexts?
- What could be the thinking of the student in those tasks that would explain "bridging"?

« 6 » Subsequently, the "bridging" process can be described by constructing models of how students' thinking developed through the research process (Cobb & Steffe 1983; Thompson 1982). These models will portray a trajectory of students' development of AWOT that consists of an explanation of students' initial schemes, explanations of changes in those schemes, and analysis of the contribution of the activities involved in those changes (Steffe 2003, 2004). A description of students' *intermediate changes of thinking* from the initial to the final AWOT would show the dynamic perspective of "bridging" as a process that evolves through design. The authors provide an example in their discussion of the development of the second AWOT, where they present the "intermediate step" of students' use of the eXpresser language to represent variables in the rule before they express their derived rules in a formal algebraic expression. "Bridging" would then be described as the process of how students' knowledge has been developed, modified, adapted or even refined during the learning process by identifying those "intermediate steps" as landmarks that build up to algebraic generalization.

« 7 » In this commentary, I have tried to contribute to the conversation by raising some issues that I consider essential to the "bridging" design and also presenting some suggestions of how students' thinking during the bridging process can be described and studied. My goal was to initiate a conversation of how a mechanism that explains the relationship between task design and students' development of knowledge can provide a framework for "bridging."

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Building Bridges that are Functional and Structural

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> Upshot • In their article, Geraniou and Mavrikis describe an environment to help children explore algebraic relationships through pattern building. They report on transfer of learning from the computer to paper, but also implicit is transfer from concrete to abstract contexts. I make the case that transfer from abstract to concrete contexts should complement such approaches.

« 1 » In their target article, Eirini Geraniou and Manolis Mavrikis investigated how knowledge developed in a microworld environment, called MiGen, might transfer outside of that environment. They describe a sequence of "bridging" activities to aid students' transition from the computer to paper-based tasks. Students start with learning about the environment and constructing and describing generalised patterns within it, and then move on to paper-based activities that at first resemble the MiGen environment before taking the form of "textbook or exam-like tasks" (Figure 2). That is, the digital environment provides scaffolding to help students construct knowledge and the bridging activities provide fading to where "attention is purely on the mathematical notation and the mathematics of solving equations" (Hewitt 2014: 26).

« 2 » The MiGen environment needs to be learned and experienced for a sustained time. The authors report that students received two lessons designed to familiarise them with the environment, and conclude that students need "a long period of practice […] before transfer to mathematics can be deemed possible" (§26). Moreover, Geraniou and Mavrikis state that the literature and their own experiences "suggest that students rarely use ideas, concepts or strategies they seem to have acquired through their interactions with digital technologies in their mathematics classrooms" (§8). Microworlds take a lot of work, and success, in terms of transfer to non-digital contexts, is far from guaranteed. So is it worth the trouble?

« 3 » Constructionists argue that microworlds provide a powerful resource for immersing learners in mathematics. Abstract objects and concepts become tangible, allowing trial and error experimentation, mental reflection and discussion (Papert 1980). Students might then discover and explore ideas that are otherwise be inaccessible to them, and can be challenged in ways not always supported by typical classroom activities. Some readers of this journal will have experienced and studied this enabling power of microworlds. In my own research, students working in the SumPuzzles environment interacted with formal arithmetic equations in distinctly algebraic ways, focussing on structure not calculation, and did so with minimal explicit instruction (Jones & Pratt 2012). However, when the plug is pulled, is the knowledge constructed by the student switched off along with the computer? Work such as that by Geraniou and Mavrikis is important for exploring how students might be bridged to working with formal mathematics on paper, and helping to evaluate whether the scaffolding and fading payoff is worthwhile.

« 4 » Another form of transfer, or perhaps more accurately transition, is implied in the research; namely, the shift from arithmetic to algebraic ways of thinking. The authors report that many students were successful with the final bridging task, and so claim that students "can generalise and adopt [algebraic ways of thinking] when solving paper and pencil figural pattern generalisation tasks" (§26). However, there were exceptions in which students "reverted to their past experiences and worked out the answers for each consecutive term in a sequence" (§24). Researchers working in the early algebra field will be unsurprised by this. Years of learning arithmetic using conventional notation has been shown to develop "operational patterns" (McNeil & Alibali 2005), such as the expectation of a numeric answer and a propensity to perform calculations even when they are irrelevant to the task goal. Moreover, operational patterns are stubborn and can be triggered unhelpfully by traditional paper-based tasks (McNeil 2008). Carefully designed microworlds can free students from operational patterns in order to explore algebraic ways of thinking, but operations are likely to be prioritised again for some students when returning to more traditional presentations of mathematical tasks.

« 5 » At the heart of the MiGen philosophy is another important aspect of transfer, the shift from concrete to abstract knowledge. This has been a contentious issue of late, with a high-profile paper by Jennifer Kaminski, Vladimir Sloutsky and Andrew Heckler (2008) claiming mathematical ideas should be introduced in abstract contexts to ensure better transfer, and others challenging their finding (e.g.. De Bock et al. 2011). The use of generalised patterns to support algebraic ways of thinking has been termed "functional approaches" (Kirshner 2001). Appeals are made to children's experiences of pattern and regularity, and tasks are designed such that formal algebra offers a powerful medium for describing and generalising patterns. Alternatives, which are perhaps less visible in the literature, are "structural approaches." These start with the abstract (that is, formal symbols and their structural relationships, with no concern for real-world referents) and seek to nurture conceptual understanding that can be transferred to new contexts, be they abstract or concrete. Structural approaches perhaps have a tarnished reputation, sometimes being associated with "meaningless" arithmetic and algebraic drill. However, carefully designed tasks can enable interactions with formal notation and associated transformation rules in a rich, meaningful and educationally valuable way (Dörfler 2006). Microworlds that take this approach have been found to motivate engagement with algebraic ways of thinking about formal notation systems (Hewitt 2014; Jones & Pratt 2012).

« 6 » There are two potential reasons to consider structural approaches as complements to functional approaches. First, whereas functional approaches typically end with the production of a formal expression or equation used to describe a concrete referent (typically a pattern), structural approaches enable the exploration of how formal expressions can be transformed; the notation becomes a medium for *doing* mathematics rather than *describing* mathematics. Second, structural microworlds start with formal notation, a virtual and manipulable symbol system that closely resembles that typically seen in textbooks and classrooms. Therefore, transfer from a digital to a paper-based domain might be relatively natural and intuitive for many students.

« 7 » We can assume that constructionist approaches to introducing formal algebra naturally align with both functional and structural approaches. Indeed, both approaches have been shown to lend themselves to the design of microworlds that enable tangible exploration and testing of conjectures such that formal symbol systems become a natural and useful medium of mathematical learning. Ideally, we might want learners to shift flexibly between thinking about concrete referents such as generalisable patterns, and thinking with formal symbols and their transformation rules. Such a fluid and dialectic mixed-approach might be expected to strengthen algebraic experience and understanding, and so promote transfer in the broadest sense of the term.

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Bringing Reflection to the Fore Using Narrative Construction

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> Upshot • In striving to support transition or bridging between arithmetic and algebra through software, Geraniou & Mavrikis come up against the need for learners not simply to "reflect" on what they have been doing, but to withdraw from action every so often, consider what actions have been effective, and construct their own narrative to hold together actions and goals and connections to past experience with other topics.

« 1 » The reason we give learners mathematic tasks to do is, I conjecture, that we want them, through their subsequent activity, to encounter mathematical themes, make use of mathematical concepts, employ mathematical procedures, and experience the use and development of their mathematical powers. We want them to "learn" something, which must mean to integrate into their functioning new actions, or variations on already integrated actions. Such actions may be visceral or virtual, taking place in the material world (including e-screens), in the imagination, and with symbols.

« 2 » Unfortunately mere experience is not sufficient for learning, for integrating into one's functioning, or for making more and more effective actions available to be enacted in the future.

"One thing we do not seem to learn from experience, is that we do not often learn from experience alone." (Mason 2002: 8)

Something more is required. Put another way,

"A succession of experiences does not add up to an experience of succession." (Mason & Davis 1989: 275)

The sentiment behind these slogans has, of course, been articulated before: George Santayana and Edmund Burke are both associated with doubt about learning from history, and William James wrote:

"A succession of feelings does not add up to a feeling of succession." (James 1950: 628)

How then are we to learn from experience? George Pólya (1945) delineated four phases of problem solving, the last of which, *looking back*, is, as Jim Wilson (1984 private communication) suggested, "more honoured in the breach."

« 3 » An important part of learning is constructing your own narrative, whether inner incantations to bring a sequence of actions to the fore when carrying out a procedure, articulations in your own words of the meaning of concepts and connections between concepts, or expressing a generality as seen through one or more particular examples. Teachers often urge students to use mathematical vocabulary, but this is only successful when learners have something that they want to express and when they have the technical vocabulary to express it.

« 4 » Enter the MiGen project, which offers not a means of doing calculations, nor of achieving some virtual task, but rather an expressive medium with manipulative, iconic, and symbolic elements. MiGen is an attempt to provide learners with a supportive but undirective environment in which to encounter and express generality, and to bridge the gap between informal expressions (in words and actions) and formal use of symbols. Its intention is to encourage and enable generalisation, to capture invariant relationships between a term of a sequence and the number of that term.

« 5 » What Eirini Geraniou and Manolis Mavrikis are looking for in their chapter is evidence of transfer, or in the language of situated cognition, evidence of the broadening of the scope of situatedness within which mathematically useful actions come to be enacted. In their study, they discovered the necessity of prompting learner reflection in mid-action (consolidation tasks), not simply at the end as implied in most interpretations of Polya's fourfold framework. This is an important part of reflection.

« 6 » It is absolutely vital for learners to withdraw from the action and *reflect*. Although reflection has been worked on and elaborated by many authors (too many to begin listing), it is rarely evident in classroom practice. My conjecture is that this is largely why the use of digital technology has not resulted in widespread improvement in mathematical thinking: the medium is the message (McLuhan 1964) in that it is so fully engaging, so accomplishment-driven that it is difficult to remember to learn from the experience. Turning off or away from a motion-colour-sound-rich medium is even harder than putting down an absorbing book: there is a sudden hiatus or vacuum before attention re-enters the world outside the medium, and in that hiatus intentions, desires, insights, and experiences can evaporate all too readily. The notion of *situated abstraction* (Noss & Hoyles 1996) is one attempt to articulate the gap between actions enacted in one context but not in others. Returning to James again,

"each of us literally chooses, by his way of attending to things, what sort of a universe he shall appear to himself to inhabit." (James 1950: 424)

« 7 » In modern parlance, this comes out as "we are our attention; we are where our attention is." Virtual e-screen worlds are inhabited very differently from the material world, or even the world of mental imagination. People in the same situation think, feel, and act differently; indeed, the same person may feel, act, and think differently at different times in what seems to be the same or similar situation, much to the consternation of teachers. Continued and engaged presence in a particular micro-world is likely to enculturate people into the vocabulary, the discourse of that micro-world, which is why it is incumbent on teachers to enrich learners' experience with technical vocabulary that provides access to experiences below a surface level of description. As Geraniou and Markolis report, this certainly happened with MiGen, with many subjects continuing to use the same vocabulary in paper and pencil tasks. Yet some reverted to more established ways of thinking (term-to-term rather than direct expression of relationship between term member and term value) in subsequent tasks. Of course the well-honed mathematical thinker whose awareness has been educated is flexible, using whatever actions seem most appropriate, while the procedure-mastering learner whose behaviour has only been trained is more hidebound, more routine in the actions they enact.

« 8 » Fostering and sustaining a constructive stance to learning involves more than providing engaging tasks, more than encounters with pervasive mathematical themes, more than experience of one's own use of natural powers in a mathematical context. It requires immersion in and prompts use of a vocabulary that captures those experiences and enables learners to become aware of what has been effective and what has not, not only at the end of a piece of work, but throughout. It is the construction of a personal narrative, with on-going improvements and refinements, that constitutes learning in the fullest sense of the word.

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Authors' Response: Let's Cross that Bridge… but Don't Forget to Look Back at Our Old Neighborhood

Eirini Geraniou
& Manolis Mavrikis

> **Upshot** • This response addresses the main points from the three commentaries, focusing particularly on additional terms and concepts introduced to the bridging metaphor. We further clarify our call for future research in the area and conclude with reflections about the practical implications emerging from our target article and the commentaries.

« 1 » In her commentary, **Nicole Panorkou** explicitly reminds us, among other relevant literature, of the concept of *contextual neighbourhood* from Pratt & Noss (2010), which of course permeates our research due to the direct and indirect influence of the research of the authors in our work. In §3, **Panorkou** challenges us to define our context explicitly. Revisiting Pratt & Noss (2010), we are reminded that in design-based research, the determination of contextual neighbourhoods is sometimes implicit, both in the case of software design, as in eXpresser, and in the case of task design, as in our paper-and-pencil activities. The challenge with making context explicit is that it is in the eye of the beholder. We therefore have at least three contexts to elaborate on — researcher, student, and teacher or schooling contexts. In brief, the framework of algebraic ways of thinking (inspired by Seymour Papert's 1972 reference to mathematical ways of thinking) gave us, as researchers, the lens through which to examine students' activities and learning as well as a way to map those to the teacher and the schooling parlour (e.g., in our case, to the national curriculum). These contexts of course overlap, and perhaps the distinction is mainly academic, but we are primarily interested in the context as perceived by the students and its influence in the knowledge or ways of thinking that they develop.

« 2 » We see therefore the rest of **Panorkou**'s review as a call for future research, particularly her excellent suggestions, in §§5f, on how students' learning trajectories between contexts, facilitated by bridges, can be studied. We see our article as a first step towards this investigation. The "bridging" activities were a design-based research outcome after carrying out a number of studies; they therefore served a purpose in the research context rather than the object of investigation itself. We agree, however, that future research should be structured in ways that bring out the dynamic nature of the bridging activities and (sticking with the metaphor) help investigate what situated abstractions (Hoyles & Noss 1992) or other learning takes place on the two sides of the bridge.

« 3 » Along the same line of thought, **Ian Jones**'s review first brings to our attention a recent paper by Dave Hewitt (2014) that can also help in future research by thinking in terms of scaffolding and fading. In §2, he raises an important question that has troubled us and the team behind the original MiGen project that designed the eXpresser microworld and its tasks: Is the time investment in scaffolding students through one microworld, designed with specific algebraic ways of thinking in mind, worth the trouble?

« 4 » We think that an answer to this conundrum comes on the back of more than 40 years of research in constructionism and endless debates since. Avoiding opening a can of worms in such a short response, our other papers on eXpresser have demonstrated its potential (e.g., Mavrikis et al. 2013), and **Jones**'s eloquent summary of functional and structural approaches in §§5f provides claims towards the potential of a microworld to support flexibility. Additionally, we rely on anecdotal teacher reports and our experience of the potential of using eXpresser and other microworlds in so called "blended-learning" scenarios, recently popularised by advocates of "flipped learning." We have seen first hand the potential of giving students eXpresser homework or group projects that can act as substrate for a teacher-led plenary, or subsequent engagement with traditional algebra in the classroom.

« 5 » An additional answer to the point above lies between the lines of the third commentary by **John Mason**, whose research on mathematics education and his contributions, in particular on algebra learning (Mason 2005a; Mason et al. 1985), have heavily influenced the design of eXpresser and its associated tasks. Mason refers to the engaging potential of digital technology that can paradoxically lead to a situation that is not conducive to learning *per se*. **Mason** invokes George Polya's "looking back," which so elegantly frames the aim of our bridging activities. We want to help students to take a step back from the microworld in which they have immersed themselves and remember to learn.

« 6 » Putting all the commentaries together brings us to the title of this article. Engineering (in the sense of Cobb et al. 2003) eXpresser activities interspersed with bridging activities at appropriate time points can answer **Jones**'s question with respect to efficiency, achieve **Mason**'s call to encourage students to capture those experiences and become aware of their work by looking back to their interactions throughout the eXpresser tasks, and achieve what **Panorkou** saw as expansion of contextual neighbourhoods.

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