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Doctoral Thesis

The Universal Coefficient Theorem and Quantum Field Theory

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Declaration of Authorship

I, Andrei T. Patrascu, declare that this thesis titled, 'The Universal Coefficient Theorem and Quantum Field Theory' and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
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Signed: Andrei T. Patrascu

Date: March 15th 2016

"Illusions commend themselves to us because they save us pain and allow us to enjoy pleasure instead. We must therefore accept it without complaint when they sometimes collide with a bit of reality against which they are dashed to pieces."

Sigmund Freud

Abstract

During the end of the 1950's Alexander Grothendieck observed the importance of the coefficient groups in cohomology. Three decades later, he presented his "Esquisse d'un Programme" to the main french funding body. This program also included the use of different coefficient groups in the definition of various (co)homologies. His proposal was rejected. Another three decades later, in the 21st century, his research proposal is considered one of the most inspiring and important collection of ideas in pure mathematics. His ideas brought together algebraic topology, geometry, Galois theory, etc. becoming the origin for several new branches of mathematics. Today, less than one year after his death, Grothendieck is considered one of the most influential mathematicians worldwide. His ideas were important for the proofs of some of the most remarkable mathematical problems like the Weil Conjectures, Mordell Conjectures and the solution of Fermat's last theorem. Grothendieck's dessins d'enfant have been used in mathematical physics in various domains. Seiberg-Witten curves, $N = 1$ and $N = 2$ gauge theories and matrix models are a few examples where his insights are relevant. In this thesis I try to connect the idea of cohomology with coefficients in various sheaves to some areas of modern research in physics. The applications are manifold: the universal coefficient theorem presents connections to the topological genus expansion invented by 't Hooft and applied to quantum chromodynamics (QCD) and string theory, but also to strongly coupled electronic systems or condensed matter physics. It also appears to give a more intuitive explanation for topological recursion formulas and the holomorphic anomaly equations. The counting of BPS states may also profit from this new perspective. Indeed, the merging of cohomology classes when a change in coefficient groups is implemented may be related to the wall-crossing formulas and the phenomenon of decay or coupling of BPS states while crossing stability walls. The *Ext* groups appearing in universal coefficient theorems may be regarded as obstructions characterizing the phenomena occurring when BPS stability walls are being crossed. Another important aspect is the existence of dualities. These are the non-perturbative analogue of symmetry transformations. Until now, they were discovered more by accident or by educated guesswork. I show in this thesis that there exists an underlying structure to the dualities, a structure that connects them the number fields used as coefficients in (co)homologies. This observation makes a nontrivial connection between number theory and physics.

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- A. T. Patrascu, On SU(2) anomaly and Majorana fermions, arXiv:1402.7283 (2015)

Contents

To my Parents

Chapter 1

Introduction

" 'Begin at the beginning', the King said, very gravely, 'and go on till you come to the end: then stop.' "

Lewis Carroll, Alice in Wonderland

The current understanding of high energy physics represents a vast development even with respect to what was known in the same field 50 years ago. However, its fundaments still lie mostly in perturbation theory, an idea that appeared much earlier in the context of astronomy and astrophysics [1-5]. Most of the present predictions of quantum electrodynamics rely on the fact that the coupling constant of this theory can be considered to be small and can be used as a perturbative expansion parameter. For other theories however, like quantum chromodynamics, the coupling constant is sufficiently small only in the high energy domain. In order to predict results for the low energy region one cannot directly rely on perturbation theory in the coupling constant anymore. Therefore non-perturbative techniques become relevant. These can be divided into two sections: numerical, lattice based methods on one side and analytic methods on the other side.

While it cannot be argued with the fact that perturbation theory led to a far better understanding of the universe at the fundamental level [6-9] one has to understand the effects of global properties as well. Fascinating new results originate from this area of research. In order to take a glimpse on the importance of this concept one should think about how science (or natural philosophy, in order to use the historical term) looked before the developments of Newton and Galilei. Indeed, at that time, the way of thinking was mainly global. By global, here I mean the construction of theorems concerned with the shapes and forms, lengths and angles of objects without considering the notion of differentiability, infinitesimal variation of a quantity or local variations, all invented only later by Newton and Leibniz [10-14]. Most of the results in Euclid's elements were global in their scope and rather universal in their applications. A volume of a pyramid depends on the area of the base and its height and this value remains unchanged regardless of the orientation, the angles between the sides and the base, etc. The same is valid for the length of the hypothenuse in a rectangular triangle as related to the lengths of the other sides, etc. All these observations, while accurate and certainly universal in the domain of the accepted axioms were far too abstract in order to solve more subtle practical problems. They were also too specific. It was undoubtedly difficult to imagine that the rectangular triangle that appeared in the construction of a pyramid could be used in approximating any geometrical shape. Hence, the road towards differential geometry appeared as vague at best. The cause for the occurrence of various phenomena was not understood so the ancients had to rely only on rough empirical observations. In this sense, the early philosophers tried to conceptualize the questions that made sense in their view [15-20]. The first to think about this was probably Aristotle with his construction of the four causes [21], as the four ways to answer questions about why phenomena occurred. In Aristotle's view the main four "why questions" are:

- material
- formal
- efficient
- final

The first explains the occurrence of phenomena in terms of the internal structure of the object that changes. Heaviness was, according to Aristotle caused by the heavy substance from which the object was made of. The cause for various phenomena occurring in nature was searched in the constituents of the objects under scrutiny. This way of thinking is still very popular in physics, mainly condensed matter and high energy physics. However, it was necessary for Grothendieck to come on the stage of science and to observe that many problems (the Weil conjectures, Fermat's last theorem, etc.) can be solved more easily if they are reformulated in a more general context [22]. Indeed, it was Grothendieck who observed that some problems that may appear as extremely complex in one context transform in simple, almost trivial questions when introduced in their suitable environment.

The second, formal cause, explains the occurrence of phenomena as a result of the way objects appear to be. The shape and the relation between various geometrical proportions was seen as a cause for some phenomena. This way of thinking, in an extended form, was rediscovered in the context of the various anomalies that arise due to topological obstructions [23-29].

The third cause is used in modern physics up to a certain point. It gives as a cause for a phenomenon an external agent that acts upon the object. The painter is the efficient cause of the painting, the force is the effective cause of the acceleration of a body, etc. To notice this however is not sufficient because the actions of the environment, described in such a direct way may soon become too complicated to analyze in any practical way. We call exactly solvable problems, those problems to which we can arrive at an exact solution by applying directly some mathematical tools without ever employing approximations. These problems are very few and generally not of practical interest. The phenomenologically important problems usually involve so many different external effective causes acting simultaneously and deforming the exact model that we must make use of some approximations.

Finally, the last cause, in Aristotle's terms, it is the final purpose for which the phenomenon occurs. For example, the final cause of a ball sitting at the top of a hill is its final velocity at the bottom of the hill, to be determined by the equation of motion, given an initial condition.

These types of questions had an important role in the construction of the medieval way of thinking, based on harmony, on ideal shapes and geometries and on large scale constructions. The more mundane phenomena were considered somehow impure and disregarded, while the natural philosophers were mostly concerned with the "celestial sphere" assumed to be perfect and unchanging [30]. While it appears strange to us today, it took almost 2 millennia for someone to actually look at the heavens carefully and patiently enough to observe that the highly regarded perfect world associated with the sky had various imperfections. The first to do this was Galileo Galilei who, between November 30 and December 18 of the year 1609 observed various irregularities on the moon [31]. These led him to the understanding that the apparent perfection of the moon is only the result of us ignoring the smaller, more detailed asperities in its structure. Following the accurate observations of the moons of Jupiter and of the phases of Venus, Galilei used simple logical arguments against the geocentric model in his work "Dialogo sopra i due massimi sistemi del mondo" [32]. While writing this, he put the basis of what we call today the Galilean group. He noticed the relativity of movement in his famous experiment concerning objects in uniform motion. Seventy years later, Isaac Newton started building on the observations of Galilei. His work had as motivation the desire to describe the continuous changes that occurs in nature, leaving aside the several millennia old way of thinking inspired by Aristotle and his followers. He described the change in terms of fluxions (in modern language differential calculus) and fluents (in modern language integral calculus). With the advent of differential calculus, the description of the "small irregularities" became possible and with this, natural philosophers were able to ask more detailed questions about how and why phenomena occur in nature and what are the causes for them. Moreover the understanding that physical, material effects represent the real cause for phenomena in general led to solutions of problems that were inaccessible before. More complex devices could be designed and studied and a completely new way of dealing with problems in physics became common: the ideal model used in the time of Aristotle and his followers was now "perturbed" by the interference of various "external" causes. The general laws by which objects interact were from that moment on, derived in ideal experiments and tested first under ideal conditions. At that stage the researcher was supposed to focus on a single, ideal aspect of the phenomenon. Then, by adding in a controlled fashion more complexity, models were developed that came closer and closer to the complexity of the natural world. In this way the inverse distance square law for the gravitational interaction was derived. The same has been observed for the electrostatic interaction. The magnetic interaction and its relation to electricity puzzled the scientists of the period for some time but Maxwell finally came to a simple solution [33], [34]. All these ideal phenomena, after they were accurately described in ideal situations, were thereafter perturbed by extra "complications" due to other objects around them. The motion of a planet around a star was now understood to be either an ellipse, a hyperbola or a parabola. Changes between the exact behaviors were now attributed to small perturbations appearing from the fact that various other objects existed in the system. While this way of thinking helped in dealing with the extremely large, cosmic objects, in the small distance region, an old doctrine re-emerged: atomism. It was assumed that material objects are built up from basic, elementary objects having certain properties and affecting in certain ways other "atoms" around them. If many such objects come together and the interactions between them allowed it, they would coagulate and form larger objects, visible at the human scale. The complicated interactions between them were to be associated (not always in the most direct way) with the macroscopic properties we can observe: temperature, pressure and later electric conductivity, etc.

History shows how the material and efficient cause were brought together by the newtonian way of thinking. The complexity and irregularity observed by Galilei was to be associated to the fact that fundamental objects obeying simple laws came together and combined in a complicated way, some perturbing the motion of others in various ways. In modern language this was the advent of perturbation theory. There, a hamiltonian H can be written as being constructed out of two model hamiltonians, one describing a perfect and exactly computable model H_E and the other describing the perturbation given by some "imperfections" H_P, in total

$$H = H_E + gH_P \tag{1.1}$$

where g is a small parameter. This way of looking at a problem generates physical quantities constructed as series of the form

$$Q = Q_0 + gQ_1 + g^2Q_2 + ... \tag{1.2}$$

If we can express the physical reality in such a way that the "perturbation" can be seen as small with respect to the ideal model, we can obtain a very good accuracy of our predictions. The main problem here is to be able to correctly separate the "ideal" world from the "irregular" world such that the effects of the irregularities are in some sense small. Usually when the physical reality is correctly described in this approximation, the calculations converge in reasonable times. This way of thinking was certainly very successful. The discovery of new truths about nature, like special and general relativity or quantum mechanics were soon incorporated in this perturbative way of thinking. In this way the many-electron atoms were simply perturbations of the ideal hydrogen like atoms [35], quantum amplitudes were series expansions in small couplings [36], special relativistic corrections were perturbations of the galilean solutions [37], general relativistic effects were corrections of special relativistic effects [38], etc. The sheer success of this approach during the last 400 years made us forget about the other two Aristotelian causes: formal and final. While certainly, the "final causes" are dealt with in any field of research at a certain level via the boundary conditions, the formal aspects, related to the global "form" of the problem considered, have been ignored for a long time. What started during the days of Plato with the ideal polygonal shapes [39] and the euclidean theorems regarding the invariance to small transformations, [40] has been almost forgotten. The idea of a topological space has been constructed only in the early decades of the 20th century [41]. While some observations have been already known to Euler in the 18th century [42], it was not until 1895 when Henry Poincare introduced notions like homotopy and homology and constructed the fundaments of what is today known as topology [43].

The main question asked by topology is also, in the tradition of Aristotle, a "why-question": why do some phenomena occur only in some ways when objects have certain shapes or are connected in a certain way and do not occur or occur differently in other situations? The main subject of topology is to describe properties that emerge out of a choice related to the way objects connect to each other [44]. Certainly, in order to define the connectivity of a space one has to show what objects are connected and in what way. In this sense, over a given space of points one defines a set of subsets, each subset containing points connected to each other. In this way two trivial situations occur: the case in which all points in the space belong to the same subset and the void set belongs to another subset. This trivial topology is called the indiscrete topology. On the other side, the discrete topology is the one in which every point in the space forms a subset on its own. The subsets that obey certain rules are called open sets and they define a topology. In order to understand what is the goal of topology one has to trace its origins in history. In fact, the topological way of thinking is far older than the perturbative way of thinking. In some sense, I like to argue that its origins can be found in the original mythical thinking of the primitive societies [45]. There, seemingly unrelated phenomena were connected by mythical links [46]. As the language was not developed enough to explain, say, lightening, one defined a “mythical link” between the observed phenomenon and a never observed character (say a god) whose changes of mood explained the phenomenon. This way of thinking is in some sense related to what we call today categorification and decategorification. How that? The standard mathematical tools we all learn in primary school are natural numbers and some operations associated to them: addition, multiplication, subtraction and division. In general, the habit in most schools is to postulate these objects. However, in some sense, they are a particularization of more general and at the same time more fundamental mathematical tools called categories. The basic idea related to categories is that we never reach the required level of generality specific to some problems if we discuss only about objects (algebraic groups, elements, sets, etc.). What we need is to discuss also the ways objects transform one into the other. Hence we need to add morphisms and form what is known as a category. In order to understand the process of categorification and decategorification I quote here a very suggestive explanation offered by J. Baez [47]:

> “To understand this, the following parable may be useful. Long ago, when shepherds wanted to see if two herds of sheep were isomorphic, they would look for an explicit isomorphism. In other words, they would line up both herds and try to match each sheep in one herd with a sheep in the other. But one day, along came a shepherd who invented decategorification. She realized one could take each herd and count it setting up an isomorphism between it and some set of ‘numbers’, which were nonsense words like ‘one, two, three, . . . ’ specially designed for this purpose. By comparing the resulting numbers, she could show that two herds were isomorphic without explicitly establishing an isomorphism! In short, by decategorifying the category of finite sets, the set of natural numbers was invented. According to this parable, decategorification started out as a stroke of mathematical genius. Only later did it become a matter of dumb habit, which we are now struggling to overcome by means of categorification. While the historical reality is far more complicated, categorification really has led to tremendous

progress in mathematics during the 20th century. For example, Noether revolutionized algebraic topology by emphasizing the importance of homology groups. Previous work had focused on Betti numbers, which are just the dimensions of the rational homology groups. As with taking the cardinality of a set, taking the dimension of a vector space is a process of decategorification, since two vector spaces are isomorphic if and only if they have the same dimension. Noether noted that if we work with homology groups rather than Betti numbers, we can solve more problems, because we obtain invariants not only of spaces, but also of maps. In modern parlance, the n-th rational homology is a functor defined on the category of topological spaces, while the n-th Betti number is a mere function defined on the set of isomorphism classes of topological spaces. Of course, this way of stating Noether insight is anachronistic, since it came before category theory. Indeed, it was in Eilenberg and Mac Lane subsequent work on homology that category theory was born! Decategorification is a straightforward process which typically destroys information about the situation at hand. Categorification, being an attempt to recover this lost information, is inevitably fraught with difficulties. One reason is that when categorifying, one does not merely replace equations by isomorphisms. One also demands that these isomorphisms satisfy some new equations of their own, called 'coherence laws'. Finding the right coherence laws for a given situation is perhaps the trickiest aspect of categorification."

I underline that there is no derogatory aspect related to "mythical thinking" implied here. We must understand this way of thinking in its historical context and this demands a certain level of mental availability to different ideas. In this mythical way of thinking, an unobserved object (the god) allowed the people of that age to formulate a theory that explained a certain phenomenon. The invisibility condition for the "link-object" was explained away by the fact that the respective god was placed in an inaccessible place, say Mount Olympus. In modern language let us consider Maxwell's equations for a magnetic monopole [48-50]. The equation $\nabla B \neq 0$ cannot coexist with the equation $A = \nabla B$ if A is to be nonsingular. However, we can introduce a semi-infinite very thin solenoid at the origin such that the magnetic potential becomes $A = A_{mon} + A_{sol}$, the sum of the potential associated to the monopole and the potential associated to the infinitesimal solenoid. This will change the divergence equation for B into

$$\nabla B = g * \delta(r) - g * \delta(r) = 0 \tag{1.3}$$

one being the contribution of the monopole and the other, the one of the solenoid. The position of the solenoid gives a singularity in the description of A that can be adjusted in order to exactly cancel the singularity that appears due to the magnetic monopole. The thin solenoid can always be gauged away via a simple gauge transformation. Hence the object is classically not real. It is however necessary as a concept in order to make our language capable of describing magnetic monopoles. We have to be sure, however, that all aspects related to the thin solenoid are invisible. If this were not so, the fact that we do not see thin solenoids would mean there are no independent magnetic monopoles in nature. While classically all observable effects have been annihilated when we allowed for gauge redundancies in the description, there are observable topological effects that a thin solenoid would have. These can be seen via the quantum Bohm-Aharonov experiment. In order to eliminate them as well one has to impose the fact that there will be no observable effects when we move around the thin solenoid an integer number of times. This condition is what produces the discreteness of the electrical charge.

In old mythological terms, our language was not sufficient to describe the magnetic monopole. The language we have chosen gave ambiguities in the form of singularities. We had to either decide to give up, as nobody ever observed a natural magnetic monopole or to see if a change of our language, while describing the rest of reality more accurately can allow a meaningful discussion about magnetic monopoles. This appeared to be the case: in order to speak about monopoles we need some extra words like "Dirac string" and we need to have them in such a way that they do not affect reality and do not refer to real objects. This way of thinking, as old as it might be, was always very useful in understanding nature. This example makes it clear, again, that the discovery of many laws of nature arose from our ability to change our way of describing reality in order to accommodate natural facts. In this sense we are not that far away from the ancient societies who tried to earn the benevolence of the invisible gods associated to lightening, thunder, rivers, etc. While their language lacked the subtlety required to describe the electrical phenomena of lightening, the sound-waves that appeared during thunder, the turbulent flow equations describing the water in the rivers and seas, they were able to derive certain primitive properties of these objects by associating to them words describing invisible objects that made sense to them at a basic level (in some sense, morphisms). These were the mythical figures of the ancient times.

The condition of invisibility of the objects related to some auxiliary words became more and more sophisticated with time. The presence of a god on the top of a very high mountain was certainly not enough as one could go there and verify whether there is a god or not. Fortunately, the theories we constructed in order to describe electrical and magnetic phenomena allow for basic redundancies. This means essentially that the same physical reality can be described using different mathematical tools, of different degrees of complexity. In general, with the construction of electromagnetism it was realized that the observable electric and magnetic fields were only an emergent part of reality while the electromagnetic potential function A_μ appeared to have a more fundamental nature. However, after writing the theory in a complete form, one realizes that the same physically observable configurations can be obtained by different potential functions. In principle, there were changes of the potential function not visible at the level of observable quantities. In the case of classical electrostatics one can use the electric field E or its potential V. However, potentials differing by a constant correspond to the same observable electric field. In general, in classical electromagnetism the situation is similar. Let the electric and magnetic field be defined in terms of the e.m. potential as

$$\begin{aligned} E &= -\nabla V - \tfrac{dA}{dt} \\ B &= \nabla \times A \end{aligned} \tag{1.4}$$

then the general gauge transformation is

$$\begin{aligned} A &\to A + \nabla f \\ V &\to V - \tfrac{\partial f}{\partial t} \end{aligned} \tag{1.5}$$

The fields remain the same after this transformation is performed and hence the Maxwell's equations are still obeyed. The main observation here is that the same reality as encoded by the measurable fields can be described using sets of equivalent functions. Again, any transformations are allowed as long as the observed effects remain compatible with reality. This way of thinking involves a certain level of subtlety. First, one has to be very clear about what is "reality" and to have a set of empirically proved situations that completely describe it. In some cases whatever we identify as "reality" is in contradiction with our previous understanding of it. This contradiction can manifest itself in various ways. One can obtain artificial singularities as the ones obtained in the naive introduction of magnetic monopoles. In other situations one observes phenomena that should not occur when looked at them in the context of the model we cherish. However it may be, a change in the language is needed in order to explain the new phenomena.

In general a quantum field theory is represented in the form of a Lagrangian with a certain set of fields. However, the actual fields are not a part of the measurable reality. Moreover, there are sets of transformations involving reparametrizations of fields, masses, charges, etc. in short, of various parameters of the theory that leave the measurable quantities unchanged. This would be of little relevance if the first theory invented were to be perfectly well defined. Unfortunately this was historically not the case. Due to our lack of imagination or due to the fact that our language and customs were far too remote from Nature, all of the early quantum field theories were ill-defined [51]. Indeed, before Feynman, the mere construction of a quantum theory of electromagnetism gave divergencies from the very beginning. While it was clear that the associated phenomena occur in nature and there are no physical problems with electrons interacting with photons, all quantum theories constructed in the form of perturbative corrections to some basic model were ill defined. It took over 20 years to observe that the "bare" parameters of the theory were not unique and that the divergencies arising there were easily removable if one allowed reparametrizations that would replace the bare quantities with modified ones. In principle, we were able to add so called "counter-terms" corresponding to terms existing already in the theory in order to eliminate the divergencies and to give a well defined meaning to the discussion. The "well-defined meaning" was essentially such that the parameters corresponded to the observed values at certain points in the parameter-space. The main result of this construction was that the parameters of the theory could not be fixed globally. Instead, they followed so called renormalization group equations derived from the fact that reality (defined in terms of the measurable S-matrix in this context) should not depend on the scale we chose to perform the measurements.

After this became clear, calculations were again possible, and they have also been done with astonishing success. Researchers obtained precisions of tens of decimal places [51] in perturbative expansions with more and more terms. The perturbative approach was strong again. However, the perturbative approach is limited to the domain where the expansion parameters are small. Outside of this domain there are still many mysteries. One is the existence of global anomalies i.e. deviations from the assumptions valid in a perturbation theory due to the global structure of the phenomena involved. It is the desire of this work to shed some light on the possibility of eliminating such anomalies by employing a homological algebraic way of thinking. In particular (co)homology depends on a choice of coefficients. A suitable choice of such coefficients makes some topological features become manifest while others may hide them. These coefficients may form various algebraic structures. One area where this becomes important is the topological genus expansion of QCD. This can be systematically improved by using homological algebra and by employing special coefficient groups in homology and cohomology. Also in topological string theory the use of the holomorphic anomaly equations can be used to derive a recursive equation relating all the higher topological genera with the two lowest ones. These are the subjects of chapters 8 - 11 of this work.

What strikes us when we speak about Grothendieck is his monumental contribution to understanding geometry at a completely new level. He introduced schemes and toposes in the study of shapes and he proved a large part of Weil's conjectures. His distinctive way of thinking, involving generalizations at a level never reached before, helped to the identification of solutions to some of the greatest problems in mathematics like Fermat's last theorem, to mention only one [22], [23], [166].

Understanding Grothendieck's thoughts is not a trivial endeavor and its application to physics is even more complicated because it is necessary to find a common language to start with. We, physicists, take for granted too many concepts far too easily. One of these concepts is the geometric point. For a concept that by definition has no further internal structure it is extremely important in the construction of all geometric objects. This is why, general topology also introduces notions like generic points, namely points that have closures that represent the whole of the topological space. This type of notion is defined by means of general topology [59]. We speak in physics about "point-like particles" and try to quantize them. We observe that in order to make justice to reality, we need to expand the notion of point-like particles and introduce fields and finally quantum fields. By introducing gravity we observe that the concept of holonomy becomes essential. If we try to quantize gravity we finally realize that truly point-like objects are insufficient. We therefore introduce string-like objects. In one way or another we need to ask ourselves what is a point. In order to describe the properties of such objects we need to be able to identify various global properties and to relate them in a well defined way. This is being done via homological algebra and general topology [55-58].

Part 1

The construction of this thesis is as follows. In the first part I give an introduction to the basic notions of topology as well as some results of homological algebra used in this work. What this first part lacks in originality, it gains in completeness. This being said, I wish to give a complete, albeit superficial review of some of the tools of topology together with historical references. In this way I follow the development of this discipline from its early stages to the most recent progresses. In general I repeat textbook definitions, theorems and proofs [57-59] adding original comments and explanations when necessary. I start with general topology or point-set topology. This is the earliest branch of topology which deals with the precise definition of the most fundamental concepts like open sets, continuity, connectedness and compactness. At this level one strives to obtain definitions for the intuitive notions of "proximity" or, the opposite term i.e. "far apart", in terms that do not necessarily involve metric related constructions. In fact, all these notions can be derived using only open sets. An extra structure like the metric, when added, simplifies these definitions. However, the real beauty of general topology is to deal with these notions in the absence of anything that can be defined as a "distance" from a metric point of view. This may become useful in situations where the metric notion of "distance" is meaningless, for example in some cases of relevance for statistics where distance may have a different meaning or in the construction of quantum mechanical entanglement. After the chapter discussing general topology, I shall deal with algebraic topology. This is a step forward in the direction of practical calculations. At that point, algebraic structure will be associated to the constructs resulting from the definition of open sets and will allow us to do practical calculations. We will use algebraic and categorial structures like groups, rings, sheaves, schemes etc. in order to characterize various spaces and to identify the situations where two spaces are identical up to a specified transformation (usually homeomorphisms). For this we must define so called topological invariants. The accuracy of these invariants in probing spaces will be of utmost importance in this work. The next mathematical field I will touch in this work is homological algebra. Historically it is a rather young field of research. While some of the ideas in mathematics, mostly those related to differential calculus date back to Newton and Leibniz, homological algebra appeared in its rigorous form at the end of the 19th century [56]. Only then, notions like Betti numbers [57] for a topological space or Hilbert's syzygy theorem [58] (1890) started to become popular in the mathematical circles. The main goal of homological algebra is to construct and solve what is considered to be the equivalent of equations for homological groups. The unknowns will be various groups and the operations and equalities will be the various arrow. Finding that some groups are isomorphic with others will be the equivalent of finding the solution of a specified equation.

Chapter 2

Elements of General Topology

“ ‘Would you tell me, please, which way I ought to go from here?’
‘That depends a good deal on where you want to get to.’
‘I don’t much care where -’
‘Then it doesn’t matter which way you go.’ ”

Lewis Carroll, Alice in Wonderland

Let me start this chapter with a simple why-question: Why general topology? What is the main problem it wishes to solve? The answer is deceivingly simple: general topology aims at analyzing and describing topological spaces. I will start this chapter by introducing the basic concepts of this field of research. I define notions like the axiomatic topology of a space, finite topological spaces, discrete spaces, indiscrete spaces, open, closed and clopen sets as well as some basic notions about limits and how various objects easily defined in calculus have to generalize in order to make sense in a general topological context. I mainly follow here reference [59] for a basic but very enlightening introduction. Topology, like most of the other branches of mathematics, can be described axiomatically [60]. In this sense, a topology can be defined as follows:

2.1 Definition Let X be a non-empty set. A collection τ of subsets of X is said to be a topology on X if

- X and the empty set belong to τ
- the union of any (finite or infinite) number of sets in τ belongs to τ and
- the intersection of any two sets in τ belongs to τ

The pair (X, τ) is called a topological space.

If X is a non-empty set and τ is the collection of all subsets of X then τ is called the discrete topology on the set X. The topological space (X, τ) is called a discrete space. The indiscrete topology on the other side is given by $\tau = \{X, \emptyset\}$ and then (X, τ) is called the indiscrete space. In both these cases each type of topology satisfies the condition in the general definition of the topology.

At this point I can remind the reader why an axiomatic definition of a notion is useful [61]. Axioms are a method of restraining the means used to define an object such that the validity of the object defined using them is as general as possible. By being able to axiomatize a definition we become capable of observing the appearance of the defining axioms even in some unexpected situations [62]. For example, in this case we can already see that a discrete space connects all the elements of a space to each other by defining an open set for each and every subset of the original space. The set of single elements-subsets will also be part of this discrete topology, hence the name "discrete". On the other side, we may think in terms of a coarse topology having only the empty set and the original set itself in it. This is an "indiscrete topology".

Instead of referring to "members of τ" we may give to these sets more appropriate names. Let us call them open sets. The complements of the open sets with respect to the space X are called "closed sets". This way of speaking leads to what is known as "open intervals" and "closed intervals" on the real number line. One observes that while any finite or infinite union of open sets is open, only finite intersections of open sets are open. Infinite intersections of open sets are not always open. I will show this in the next example:

2.2 Example Let $\mathbb{N}$ be the set of all positive integers and let τ consist of $\emptyset$ and each subset S of $\mathbb{N}$ such that the complement of S in $\mathbb{N}$, $\mathbb{N} - S$, is a finite set. It can be verified that τ is a topology on $\mathbb{N}$. It is called the finite-closed topology. For each natural number n, define the set S_n as

$$S_n = \{1\} \cup \{n+1\} \cup \{n+2\} \cup \{n+3\} \cup ... = \{1\} \cup \bigcup_{m=n+1}^{\infty} \{m\} \tag{2.1}$$

Clearly each S_n is an open set in the topology τ, since its complement is a finite set. However,

$$\bigcap_{n=1}^{\infty} S_n = \{1\} \tag{2.2}$$

As the complement of $\{1\}$ is neither $\mathbb{N}$ nor a finite set, $\{1\}$ is not open. So this shows that the intersection of the open sets S_n is not open.

It is important to observe that both inclusion and intersection must be verified in order to prove that a subset is open. Now that the open and closed sets are defined, one needs to notice that some open sets can also be closed at the same time. For example in a discrete space every set is both open and closed while in an indiscrete space (X, τ) all subsets of X except X and $\emptyset$ are neither open nor closed. Hence there is the

2.3 Definition A subset S of a topological space (X, τ) is said to be clopen if it is both open and closed in (X, τ).

In general in every topological space (X, τ) both X and $\emptyset$ are clopen, in a discrete space all subsets of X are clopen and in an indiscrete space the only clopen subsets are X and $\emptyset$.

In what follows I will discuss the notions that can be defined generally on a topological space. The analogy with the real line has its limits. First, on the real line we have a notion of "closeness". For example, if we have a sequence of the form

$$0.1, 0.01, 0.001, ..., \tag{2.3}$$

every element of this sequence is closer to zero than the previous. This means one can say that 0 is the limit point of this sequence. However, the interval $(0, 1]$ is not closed as it does not contain the limit of any sequence in it, in particular it does not contain the element 0.

A topological space is in some sense a general notion. For example we do not need to have notions like a metric over a topological space and the distance is therefore not always well defined. If we do not have a distance we must define the limit point differently, without considering the distance between two points as has been done in standard calculus.

Also, the topological spaces are defined by employing the concept of connectedness. This will also be defined in what follows. Let me start with a topological space (X, τ). The elements of this space are referred to as points. Let A be a subset of a topological space (X, τ). A point $x \in X$ is said to be a limit point (or accumulation point or cluster point) of A if every open set, U, containing x contains a point of A different from x.

In general a test whether a set is closed or not is the following

2.4 Proposition Let A be a subset of a topological space (X, τ). Then A is closed in (X, τ) if and only if A contains all of its limit points.

2.5 Proposition Let A be a subset of a topological space (X, τ) and A' the set of all limit points of A. Then $A \cup A'$ is a closed set.

2.6 Definition Let A be a subset of a topological space (X, τ). Then the set $A \cup A'$ consisting of A and all its limit points is called the closure of A and is denoted $\bar{A}$.

2.7 Definition Let A be a subset of a topological space (X, τ). Then A is said to be dense in X or everywhere dense in X if $\bar{A} = X$. As an example $\mathbb{Q}$ is a dense subset of $\mathbb{R}$.

As an example consider again the discrete topological space (X, τ). Then, every subset of X is closed (since its complement is open). Therefore the only dense subset of X is X itself, since each subset of X is its own closure.

2.8 Proposition Let A be a subset of a topological space (X, τ). Then A is dense in X if and only if every non-empty open subset of X intersects A non-trivially (that is, if $U \in \tau$ and $U \neq \emptyset$ then $A \cap U \neq \emptyset$).

In what follows we need the concept of neighborhood. Again, for topological spaces where a metric is not defined and there is no notion of distance, this concept will prove to be not only important for what follows, but also interesting from a logical point of view.

2.9 Definition Let (X, τ) be a topological space, N a subset of X and p a point in N. Then N is said to be a neighborhood of the point p if there exists an open set U such that $p \in U \subseteq N$.

As an example, the closed interval $[0, 1] \in \mathbb{R}$ is a neighborhood of the point $\frac{1}{2}$ since $\frac{1}{2} \in (\frac{1}{4}, \frac{3}{4}) \subseteq [0, 1]$.

2.10 Proposition Let A be a subset of a topological space (X, τ). A point $x \in X$ is a limit point of A if and only if every neighborhood of x contains a point of A different from x.

As a set is closed if and only if it contains all its limit points we deduce the following.

2.11 Corollary Let A be a subset of a topological space (X, τ). Then the set A is closed if and only if for each $x \in X - A$ there is a neighborhood N of x such that $N \subseteq X - A$.

2.12 Corollary Let U be a subset of a topological space (X, τ). Then $U \in \tau$ if and only if for each $x \in U$ there exists a neighborhood N of x such that $N \subseteq U$.

2.13 Corollary Let U be a subset of a topological space (X, τ). Then $U \in \tau$ if and only if for each $x \in U$ there exists a $V \in \tau$ such that $x \in V \subseteq U$.

The last corollary provides a practical test of whether a set is open or not. A set is open if and only if it contains an open set about each of its points. In what follows, a brief discussion about connectedness [63] will be given. Some simple definitions and facts are given in an informal way, mainly following reference [59] which is a source of inspiration for the major part of this section. Let therefore S be a set of real numbers. If there is an element $b \in S$ such that $x \leq b$, for all $x \in S$ then b is said to be the greatest element of S. Similarly if S contains an element a such that $a \leq x$ for all $x \in S$ then a is called the least element of S. A set S of real numbers is said to be bounded above if there exists a real number c such that $x \leq c$ for all $x \in S$, and c is called an upper bound for S. Similarly, the terms "bounded below" and "lower bound" are defined. A set which is bounded above and bounded below is said to be bounded [64].

2.14 Least Upper Bound Axiom Let S be a non-empty set of real numbers. If S is bounded above, then it has a least upper bound.

The upper bound also called the supremum of S, denoted $sup(S)$, may or may not belong to the set S. Indeed the supremum of S is an element of S if and only if S has a greatest element. Any set S of real numbers which is bounded below has a greatest lower bound which is also called the infimum and is denoted by $inf(S)$.

2.15 Lemma Let S be a subset of $\mathbb{R}$ which is bounded above and let p be the supremum of S. If S is a closed subset of $\mathbb{R}$, then $p \in S$.

Proof See appendix.

2.16 Proposition Let T be a clopen subset of $\mathbb{R}$. Then either $T = \mathbb{R}$ or $\mathbb{T} = \emptyset$.

Proof See appendix.

2.17 Definition Let (X, τ) be a topological space. Then it is said to be connected if the only clopen subsets of X are X and $\emptyset$. As an example, the topological space $\mathbb{R}$ is connected.

From the definition follows that a topological space (X, τ) is not connected (i.e. disconnected) if and only if there are non-empty open sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$. This fact is important because it constitutes the basis for the future generalizations to connected manifolds, groups, etc.

In what follows I will briefly discuss what means when we say that two structures are equivalent [65]. The distinction between objects implies two items: the objects themselves and the criteria by which the notion of "distinctiveness" is defined. In set theory, two sets are said to be equivalent from the perspective of set theory if there exists a bijective function which maps one set onto another. Two groups are equivalent, also said to be isomorphic, if there exists a homomorphism of one to the other which is one-to-one and onto. Two topological spaces are equivalent, also said to be homeomorphic if there exists a homeomorphism of one onto the other. Hence, first we need a definition for the objects we want to compare. Then we need to explain what means "equivalent" in our theory. I will start by defining the objects that are important in this context, and these objects are the topological spaces. Hence, we will want to compare subspaces of a given space.

2.18 Definition Let Y be a non-empty subset of a topological space (X, τ). The collection $\tau_Y = \{O \cup Y : O \in \tau\}$ of subsets of Y is a topology on Y called the subspace topology (or relative topology, or induced topology on Y by τ). The topological space (Y, τ_Y) is said to be a subspace of (X, τ).

One can check that τ_Y is indeed a topology on Y. Now we turn to the notion of equivalence defined for the topological spaces. We may start with an example

$$X = \{a, b, c, d, e\}, \quad Y = \{g, h, i, j, k\} \tag{2.4}$$

$$\tau = \{X, \emptyset, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}\} \tag{2.5}$$

and

$$\tau_1 = \{Y, \emptyset, \{g\}, \{i, j\}, \{g, i, j\}, \{h, i, j, k\}\} \tag{2.6}$$

It is intuitively clear that (X, τ) is equivalent to (Y, τ_1). The function $f : X \to Y$ defined by $f(a) = g$, $f(b) = h$, $f(c) = i$, $f(d) = j$ and $f(e) = k$, provides the equivalence.

2.19 Definition Let (X, τ) and (Y, τ_1) be topological spaces. Then we say they are homeomorphic if there exists a function $f : X \to Y$ which has the following properties:

- f is one-to-one (that is $f(x_1) = f(x_2)$ implies $x_1 = x_2$).
- f is onto (that is, for any $y \in Y$ there exists an $x \in X$ such that $f(x) = y$)
- for each $U \in \tau_1, f^{-1}(U) \in \tau$ and
- for each $V \in \tau$, $f(V) \in \tau_1$

Further, the map f is said to be a homeomorphism between (X, τ) and (Y, τ_1). We write $(X, \tau) \cong (Y, \tau_1)$.

It can be shown that $\cong$ is an equivalence relation and that all open intervals (a, b) are homeomorphic to each other. Length is not a topological property [66]. In particular, an open interval of finite length such as $(0, 1)$ is homeomorphic to one of infinite length such as $(-\infty, 1)$. In fact, all open intervals are homeomorphic with $\mathbb{R}$. There is an important aspect related to the methods of proof. In order to prove that two topological spaces are homeomorphic we have to find a homeomorphism between them. However, to prove that two topological spaces are not homeomorphic is often much harder as we have to show that no homeomorphism exists. In order to show this difficulty the next example is important.

2.20 Example We want to prove that the interval $[0, 2]$ is not homeomorphic to the subspace $[0, 1] \cup [2, 3]$ or $\mathbb{R}$. Let for this $(X, \tau) = [0, 2]$ and $(Y, \tau_1) = [0, 1] \cup [2, 3]$. Then $[0, 1] = [0, 1] \cap Y \Rightarrow [0, 1]$ is closed in (Y, τ_1) and $[0, 1] = (-1, 1\frac{1}{2}) \cap Y \Rightarrow [0, 1]$ is open in (Y, τ_1). Thus Y is not connected as it has $[0, 1]$ as a proper non-empty clopen subset.

Suppose that $(X, \tau) \cong (Y, \tau_1)$. Then there exists a homeomorphism $f : (X, \tau) \to (Y, \tau_1)$. So, $f^{-1}([0, 1])$ is a clopen subset of X, and hence X is not connected. This is false as $[0, 2] = X$ is connected. So we have a contradiction and thus the two topological spaces are not homeomorphic. Hence, we can observe the following

2.21 Proposition Any topological space homeomorphic to a connected space is connected.

This observation is extremely important in simplifying the proofs that objects (hence also topological spaces) are not homeomorphic with each other [67]. Instead of actually searching every possible homeomorphism and eliminating each of them, it is far easier to find one single property preserved by homeomorphisms which can be proven that one space has and the other does not. In this way, the "checking" of all possible homeomorphisms is avoided leading to a major simplification. There are several such properties preserved by homeomorphisms that can be used. However, when faced with a specific problem we may not be able to find the best property we would like to use. The art is to decide when it is easier to check all homeomorphisms and when it is easier to check all preserved properties [68]. One can however make statements about the real line for which we have the following

2.22 Definition A subset S of $\mathbb{R}$ is said to be an interval if it has the following property: if $x \in S$, $z \in S$ and $y \in \mathbb{R}$ are such that $x < y < z$ then $y \in S$.

Connectedness for the real line is easily prescribed by the following

2.23 Proposition A subspace S of $\mathbb{R}$ is connected if and only if it is an interval.

Up to now we discussed the objects and the equivalence relations. The next structure, specific to category theory is called the set of arrows [69]. They represent different things when analyzed in different branches of mathematics. In linear algebra we have as objects the vector spaces and as arrows the linear transformations. In group theory the objects are the groups while the arrows are the homomorphisms, while in set theory the objects are sets and the arrows are functions. In topology the objects are the topological spaces and the arrows are the continuous mappings. However, how can we define a notion such as "continuity" in a general topological space? Of course for functions from $\mathbb{R}$ to $\mathbb{R}$ this is simple: a function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous if for each $a \in \mathbb{R}$ and each positive real number ϵ, there exists a positive real number δ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. This construction however is very dependent on the definition of absolute value, subtraction and in general distance [70]. All these notions do not need to exist (although can certainly be defined for some cases) in general topological spaces [71]. Hence we need a different definition of continuity, more suitable for generalizations. We can see that $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for each $a \in \mathbb{R}$ and each interval $(f(a) - \epsilon, f(a) + \epsilon)$, for $\epsilon > 0$ there exists a $\delta > 0$ such that $f(x) \in (f(a) - \epsilon, f(a) + \epsilon)$ for all $x \in (a - \delta, a + \delta)$. This definition does not involve the notion of distance or of absolute value but it still involves the notion of subtraction which may not make sense in general i.e. the inversion of addition may not be defined [72]. In order to avoid subtraction completely we can introduce the following

2.24 Lemma Let f be a function mapping $\mathbb{R}$ into itself. Then f is continuous if and only if for each $a \in \mathbb{R}$ and each open set U containing $f(a)$, there exists an open set V containing a such that $f(V) \subseteq U$.

Proof See apendix.

One could use the property described in the above lemma to define continuity but the following lemma makes the definition more elegant.

2.25 Lemma Let f be a mapping of a topological space (X, τ) into a topological space (Y, τ'). Then the following two conditions are equivalent:

- for each $U \in \tau', f^{-1}(U) \in \tau$
- for each $a \in X$ and each $U \in \tau'$ with $f(a) \in U$, there exists a $V \in \tau$ such that $a \in V$ and $f(V) \subseteq U$.

Proof See appendix.

Hence the notion of continuity for a function between two topological spaces becomes

2.26 Definition Let (X, τ) and (Y, τ_1) be topological spaces and f a function from X into Y. Then $f : (X, \tau) \to (Y, \tau_1)$ is said to be a continuous mapping if for each $U \in \tau_1$, $f^{-1}(U) \in \tau$.

Now we can write the following

2.27 Proposition Let f be a mapping of a topological space (X, τ) into a space (Y, τ'). Then f is continuous if and only if for each $x \in X$ and each $U \in \tau'$ with $f(x) \in U$, there exists a $V \in \tau$ such that $x \in V$ and $f(V) \subseteq U$.

2.28 Proposition Let (X, τ), (Y, τ_1) and (Z, τ_2) be topological spaces. If $f : (X, \tau) \to (Y, \tau_1)$ and $g : (Y, \tau_1) \to (Z, \tau_2)$ are continuous mappings, then the composite function $g \circ f : (X, \tau) \to (Z, \tau_2)$ is continuous.

Of course, the next result shows that we can interchange closed sets with open sets in the definition of continuity

2.29 Proposition Let (X, τ) and (Y, τ_1) be topological spaces. Then $f : (X, \tau) \to (Y, \tau_1)$ is continuous if and only if for every closed subset S of Y, $f^{-1}(S)$ is a closed subset of X.

Proof See appendix.

There is a connection between continuous maps and homeomorphisms. If $f : (X, \tau) \to (Y, \tau_1)$ is a homeomorphism then it is a continuous map. Obviously not every continuous map is a homeomorphism.

2.30 Proposition Let (X, τ) and (Y, τ') be topological spaces and f a function from X to Y then f is a homeomorphism iff

- f is continuous
- f has an inverse
- f^{-1} is continuous

2.31 Proposition Let (X, τ) and (Y, τ_1) be topological spaces, $f : (X, \tau) \to (Y, \tau_1)$ a continuous mapping, A a subset of X and τ_2 the induced topology on A. Further, let $g : (A, \tau_2) \to (Y, \tau_1)$ be the restriction of f to A, that is $g(x) = f(x)$ for all $x \in A$. Then g is continuous.

An important result is given by the following

2.32 Proposition Let (X, τ) and (Y, τ_1) be topological spaces and $f : (X, \tau) \to (Y, \tau_1)$ surjective and continuous. If (X, τ) is connected then (Y, τ_1) is connected.

Proof See appendix.

Otherwise stated this proposition says that any continuous image of a connected set is connected. It also says that if (X, τ) is a connected space and (Y, τ') is not connected then there exists no mapping of (X, τ) onto (Y, τ') which is continuous. There exists a stronger definition of connectedness [73]:

2.33 Definition A topological space (X, τ) is said to be path-connected if for each pair of distinct points a and b of X there exists a continuous mapping $f : [0, 1] \to (X, \tau)$ such that $f(0) = a$ and $f(1) = b$. The mapping f is said to be a path joining a to b.

Every path connected space is connected. At this point, I can introduce Weierstrass' Intermediate value Theorem [74], an application of topology to the theory of functions of a real variable. The topological concept important for this is that of connectedness.

2.34 Theorem Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then for every number p between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = p$.

Proof See appendix.

2.35 Corollary If $f : [a, b] \to \mathbb{R}$ is continuous and such that $f(a) > 0$ and $f(b) < 0$ then there exists an $x \in [a, b]$ such that $f(x) = 0$.

2.36 Corollary (The fixed point theorem) Let f be a continuous mapping of $[0, 1]$ into $[0, 1]$. Then there exists a $z \in [0, 1]$ such that $f(z) = z$. The point is called a fixed point.

Proof See appendix.

This corollary is a special case for another theorem called the Brouwer fixed point theorem [75] which says that every continuous function from a convex compact subset $\mathcal{K}$ of a Euclidean space to $\mathcal{K}$ itself has a fixed point. Most proofs are of algebraic topological nature [76]. However, this theorem has many applications, from theoretical economics [77] to applied mathematics [78].

As I mentioned several times until now, the discussion in this first part was intentionally as general as possible. This implied the definition of notions like continuity such that they do not depend on notions related to metric spaces like distances, absolute values, etc. In what follows I will particularize the discussion a bit, making however as clear as possible that most of the interesting applications appear when the notions of metric and distance are not readily available. One may ask if there are situations when we do not wish to measure distances or distances are not well defined. Indeed, the basic question in topology is to describe structures that do not depend on continuous deformations, and obviously, distance is one concept that changes in continuous deformations. Therefore, topological notions are in the most general sense not dependent on structures like distance. Additional structure must be added to the topological structure so that we are capable of discussing about distances. However, there do exist situations where distance is not necessary, for example quantum entanglement is a correlation which does not, a-priori, depend on distance. Once topology itself becomes uncertain, the notion of distance will become even more ambiguous. Most of the applications of topology to analysis are via metric spaces [79]. Because of this I will start with a definition

2.37 Definition Let X be a non-empty set and d a real valued function defined on $X \times X$ such that for $a, b \in X$:

- $d(a, b) \geq 0$ and $d(a, b) = 0$ if and only if $a = b$
- $d(a, b) = d(b, a)$
- $d(a, c) \leq d(a, b) + d(b, c)$ for all $a, b, c \in X$

Then d is said to be a metric on X, (X, d) is a metric space and $d(a, b)$ is the distance between a and b.

Having a metric space (X, d) and r a positive real number we can define the open ball about $a \in X$ of radius r as the set

$$B_r = \{x : x \in X; d(a, x) < r\} \tag{2.7}$$

In what follows I wish to connect the metric spaces to the topological spaces. For this I will need the following

2.38 Lemma Let (X, d) be a metric space and a and b points of X. Further, let δ_1 and δ_2 be positive real numbers. If $c \in B_{\delta_1}(a) \cap B_{\delta_2}(b)$ then there exists a $\delta > 0$ such that $B_\delta(c) \subseteq B_{\delta_1}(a) \cap B_{\delta_2}(b)$.

2.39 Corollary Let (X, d) be a metric space and B_1 and B_2 open balls in (X, d). Then $B_1 \cap B_2$ is a union of open balls in (X, d).

2.40 Proposition Let (X, d) be a metric space. Then the collection of open balls in (X, d) is a basis for a topology τ on X. This is the topology induced by the metric d and (X, τ) is called the induced topological space[80] or the corresponding topological space.

As an example consider d the euclidean metric on $\mathbb{R}$. Then a basis for the topology τ induced by the metric d is the set of all open balls. But $B_\delta(a) = (a - \delta, a + \delta)$. From this it is easy to see that τ is the euclidean topology on $\mathbb{R}$. Hence the euclidean metric on $\mathbb{R}$ induces the euclidean topology on $\mathbb{R}$.

From the perspective of how a set of numbers can be completed, there exist other types of metrics. Among non-euclidean metrics one can cite the non-Archimedean metric which gives rise to the so called p-adic numbers. This is one of the three possible completions of the rationals, the other two being the real numbers and the complex numbers. The p-adic numbers do not obey the Archimedean axiom, one of the axioms introduced by Hilbert in his general approach to geometry. The basic formulation of Archimedes' axiom is that given two magnitudes having a ratio, one can find a multiple of either which will exceed the other. This multiple must be finite. By this one excludes the existence of differential objects. Just as the real numbers are a completion of the rationals with respect to the usual norm, the p-adic numbers are the completion of the rationals with respect to the p-adic norm.

Let me now consider d the discrete metric on a set X. Then for each $x \in X$, $B_{\frac{1}{2}}(x) = \{x\}$. So, all the singleton sets are open in the topology τ induced on X by d. As a consequence, τ is the discrete topology.

2.41 Definition Metrics on a set X are equivalent if they induce the same topology on X.

2.42 Proposition Let (X, d) be a metric space and τ the topology induced on X by the metric d. Then a subset U of X is open in (X, τ) if and only if for each $a \in U$ there exists an $\epsilon > 0$ such that the open ball $B_\epsilon(a) \subseteq U$.

Proof See appendix.

It was noticed that every metric on a set X induces a topology on the set X. However, the reverse is not always true i.e. not every topology on a set is induced by a metric.

2.43 Definition A topological space (X, τ) is said to be a Hausdorff space (or a T_2-space) if for each pair of distinct points a and b in X, there exist open sets U and V such that $a \in U$, $b \in V$ and $U \cap V = \emptyset$.

It can be seen that $\mathbb{R}$, $\mathbb{R}^2$ and all discrete spaces are Hausdorff [81]. However, any set with at least 2 elements which has the indiscrete topology is not a Hausdorff space. It may be relevant to note that $\mathbb{Z}$ with finite-closed topology is also not a Hausdorff space.

2.44 Proposition Let (X, d) be any metric space and τ the topology induced on the X by d. Then (X, τ) is Hausdorff.

Proof See appendix.

We can see out of this proposition that an indiscrete space with at least two points has a topology which is not induces by any metric. Also, $\mathbb{Z}$ with the finite-closed topology τ is such that τ is not induced by any metric on $\mathbb{Z}$.

2.45 Proposition A space (τ, X) is said to be metrizable if there exists a metric d on the set X with the property that τ is the topology induced by d.

For example the set $\mathbb{Z}$ with the finite-closed topology is not a metrizable space. One should not believe that any Hausdorff space is metrizable. In fact there exist Hausdorff spaces which are not metrizable [82].

In what follows, I will review briefly the notions surrounding the convergence of sequences. It is clear what a convergent sequence of real numbers is. In order to remind the reader, the definition is as follows. The sequence $x_1, x_2, ..., x_n, ...$ of the real numbers is said to converge to the real number x if given any $\epsilon > 0$, there exists an integer n_0 such that for all $n \geq n_0$, $|x_n - x| < \epsilon$. The generalization of this definition from $\mathbb{R}$ to any metric space is obvious

2.46 Definition Let (X, d) be a metric space and $x_1, ..., x_n, ...$ a sequence of points in X. Then the sequence is said to converge to $x \in X$ if given any $\epsilon > 0$ there exists an integer n_0 such that for all $n \geq n_0$, $d(x, x_n) < \epsilon$. This is denoted by $x_n \to x$. The sequence $y_1, y_2, ..., y_n...$ of points in (X, d) is said to be convergent if there exist a point $y \in X$ such that $y_n \to y$.

2.47 Proposition Let $x_1, x_2, ..., x_n, ...$ be a sequence of points in a metric space (X, d). Further, let x and y be points in (X, d) such that $x_n \to x$ and $x_n \to y$. Then $x = y$.

We say that a subset A of a metric space (X, d) is closed (resp. open) in the metric space (X, d) if it is closed (resp. open) in the topology τ induced on X by the metric d.

In fact, the topology of a metric space can be described entirely in terms of its convergent sequences.

2.48 Proposition Let (X, d) be a metric space. A subset A of X is closed in (X, d) if and only if every convergent sequence of points in A converges to a point in A. This means that A is closed in (X, d) if and only if $a_n \to x$ where $x \in X$ and $a_n \in A$ for all n, implies $x \in A$.

Proof See appendix.

This finishes the introduction in general topology required for this work. Further information on the subject can be found in [83-89]. While the results seem trivial, they by themselves are only marginally the reason for this chapter. I introduced this chapter mainly because the method of thinking derived from it reflects back to algebraic topology and more advanced mathematical subjects. In fact, during my independent research I started precisely with these constructions the formal study of topology. This proved very useful mainly because I understood the distinction between mathematical proofs and physical proofs. In general physicists tend to perform robust and numerically intensive calculations and to regard those as proofs in a very specific sense. The mathematical proofs, no matter how rigorous are often regarded with skepticism. On the other side, mathematically oriented researchers tend to see physical proves as inelegant, dull and sometimes plain inefficient. However, in what follows I show that the two ways of thinking may fruitfully coexist.

Chapter 3

Algebraic Topology

" Why, sometimes I've believed as many as six impossible things before breakfast."

Lewis Carroll, Alice in Wonderland

In the previous chapter I described basic topology and some of its most fundamental theorems and constructions. However, while the description above is rigorous it remains in a certain sense abstract. It is difficult if not impossible to perform actual operations and to characterize spaces using only the results of general (point-set) topology. In order to bring the concept closer to real calculations we need to introduce the analogue of "numbers" and "operations" such that we can perform calculations on the spaces we introduce. This is, in a symbolic sense, the reason for algebraic topology. The lecture notes of [252] give a very brief but comprehensive introduction to the necessary concepts. I will follow it along this chapter. For the missing proofs of some theorems or lemmas the reader should consult [252] as well as [113], [114]. Loosely speaking, algebraic topology is a systematic way of searching for holes in manifolds and of measuring the properties of various shapes. If we have to make an even more reducing analogy, the general topology is the analogue of set theory covered in primary school. Algebraic topology is the analogue of the understanding that the concepts of numbers and operations with numbers must be added to the sets in order to do practical calculations. Homological algebra then will be the analogue of solving equations and finding the unknowns satisfying certain relations. A polynomial equation for example becomes a certain type of commutative diagram in homological algebra, a diagram that determines some of the (co)homologies in it up to some kind of morphisms (either isomorphisms, epimorphisms, etc.)

At this point we can understand what topology wishes to obtain. The result of a topological measurement is available if we can answer to three types of questions:

- Is the space connected?
- Does the space contain holes?
- How are the holes in the space characterized?

The answer to these and more questions is given in terms of various topological invariants. These are defined such that they do not change when calculated over a space of a given topology. There are however some prerequisites for extracting information from a space by means of topological invariants. First we have to make a choice of a triangulation, call it $\mathcal{T}$ which means the original space X is represented in terms of oriented simplexes (points, intervals, triangles, etc.). This is a prescription that can be done even numerically. It allows us to introduce an algebraic structure in order to describe the space. This is done by the introduction of an object $C_*(X, \mathcal{T}, \mathbb{K})$ called the simplicial complex. This object is the complex of modules over a ring $\mathbb{K}$. The ring can be any of $\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ etc.

In this case the modules C_i count the simplexes of dimension i in the triangulation and the boundary maps between them encode the way these simplexes are connected to each other in order to form X. The modules C_i are not the most interesting objects if we want to accurately describe the space X. They involve not only the precise space X but also an auxiliary choice of the triangulation $\mathcal{T}$ of X. The important information is encoded in the homology $H_*(X, \mathcal{T}, \mathbb{K})$ of the complex C. Following from its construction, as will be seen later on, the homology is an invariant of the topological space X. It also in some sense gives a more tractable, more linear source of information regarding the space X. In general an abstract space described via its complex is a construction with little connection to linear algebra. By introducing the homology groups it is possible to arrive at an algebraic structure (based on numbers or on matrices) where the properties of linear algebra may be exploited. Indeed, this is often the case. This is true even if we calculate it with the extra information given by the triangulation of X. In fact the homology does not depend on the triangulation and hence we can re-write it as $H_*(X, \mathbb{K})$ i.e. the homology of the space X with coefficients in $\mathbb{K}$.

It is important to notice as a brief intermezzo, that a triangulation is certain to exist and to be unique for spaces of dimension up to 3. For higher dimensions including the 4-dimensional spacetime, a triangulation does not necessarily exist and if it exists it is not guaranteed to be unique or to correspond to a single space [90-93].

Consider two topological spaces X and Y. The associated invariants $I(X)$ and $I(Y)$ give us some information on how the spaces are related. They might be isomorphic or might be related via other morphisms. In general it is more of an art to chose the right invariants that can tell if two spaces are indeed related in a certain way. The applications of invariants are however, much broader. In order to introduce all required concepts I will start with explaining how to describe topological spaces in terms of simplexes and then how to use such descriptions in order to calculate the (co)homology of simplexes.

One branch of topology that deals with this problem is called combinatorial topology [94]. In fact this is the earlier term used for algebraic topology. It describes how a given topological space can be described using objects called simplexes. The way in which these simplexes connect together is given by a set of rules. But what are these simplexes? Depending on the dimension, the 0-simplex is usually understood to be a point, the 1-simplex is a closed interval, the 2-simplex is a triangle, the 3-simplex is a pyramid, etc. This is however only the case when we are discussing shapes. The beauty of algebraic topology is that this notion can be extended in rather unexpected ways. "Points" can become logical statements and simplexes can then be associated to theorems. Logics itself can be seen as a topology in some sense. Also, if the "points" are algebraic properties, new algebraic structures may appear depending on the possibility of defining the open sets in which they reside. I will show this in another chapter, discussing the notion of schemes and some of the ideas of Grothendieck. In order to systematize this approach one states the following

3.1 Definition (Standard and linear simplexes) The standard n-simplex is $\sigma_n \subseteq \mathbb{R}^{n+1}$. It is defined as the convex closure $\sigma_n = conv\{e_0, ..., e_n\}$ of the standard basis of $\mathbb{R}^{n+1}$. So, σ_n is the first "quadrant" $x_i \geq 0$ and there it is given by the hyperplane $\sum_i x_i = 1$. More generally we say that a linear i-simplex in a real vector space is the convex closure $conv(v_0, ..., v_i)$ of a set $V = \{v_0, ..., v_i\}$ of $i + 1$ vectors which lie in an i-dimensional affine subspace but do not lie in any $(i - 1)$-dimensional affine subspace. One says that V is the set of vertices of the simplex $conv(V)$ and we often denote by $\sigma_V = conv(V)$ the simplex with vertices V. So, an i-simplex has $(i + 1)$ vertices.

3.2 Lemma Any point x in the simplex can be written as $x = \sum_i x_i v_i$ with $x_i \geq 0$ and $\sum_i x_i = 1$. The x_i are called barycentric coordinates and they are unique.

Proof See appendix.

3.3 Lemma We can recover vertices from a linear simplex σ_V as the points with all but one coordinate zero. A bijection between vertices of two linear simplexes extends canonically to a homeomorphism e.g. an ordering of V gives a canonical identification $\sigma_V \cong \sigma_{|V|-1}$.

The facets of the simplex $conv(V)$ are the simplexes associated to subsets of the set of vertices. Any subset $W \subseteq V$ defines a facet of $conv(V)$ which is the simplex $conv(W)$. The facets are closed under intersections: $conv(W') \cap conv(W'') = conv(W' \cap W'')$. The facets of codimension 1 are called faces. The interior σ_V^0 of a linear simplex $conv(V)$ consists of the points with all $x_i > 0$. An ordering of the set of vertices V of a simplex σ_V gives an orientation of the simplex. Two orderings give the same orientation if they differ by an even permutation of vertices. It results that an orientation of a simplex is an orbit of the group of even permutations of vertices in the set of all orderings of vertices. The set of orientations of an i-simplex σ is denoted or_σ. Notice that it has two elements for $i > 0$ and one for $i = 0$. We denote by $\alpha \to \bar{\alpha}$ the operation of changing the triangulation of oriented simplexes. Notice the parallel between the notion of orientation in a vector space given by a basis ordered up to even permutations and the top form $dx^1 \wedge ... \wedge dx^n$ given by an ordering of coordinates up to even permutation.

3.4 Definition(Topological simplexes) A topological i-simplex is a pair (S, ϕ) of a topological space S and a homeomorphism $\phi : \sigma_V \to S$ with a linear i-simplex. For simplicity we usually omit ϕ from notation. Notice that the above notions of vertices, facets, coordinates, interior, orientation are defined for topological simplexes via ϕ. Facets of topological simplexes are again topological simplexes. One may denote the faces of a topological n-simplex S by S^i, $i \in V$ where S^i is obtained by throwing out the vertex i.

In what follows, the idea of triangulation needs a more precise definition. Formally, a triangulation is a method of presenting a given topological space as a combination of simple spaces, the simplexes [95]. After introducing it we can extract the information on X from the way the simplexes are patched together [96]. There are several ways of defining it.

There exists for example simplicial triangulation [97]. This is a notion of a triangulation with certain properties :

- The facet of simplexes in $\mathcal{T}$ are again simplexes in $\mathcal{T}$.
- If $\alpha, \beta \in \mathcal{T}$ and $\alpha \subseteq \beta$ then α is a facet of β
- For any $\alpha, \beta \in \mathcal{T}$ the intersection $\alpha \cap \beta$ is $\emptyset$ or a simplex in $\mathcal{T}$

These properties are making very easy to describe how simplexes fit together to form the space X. Everything is stated in terms of vertices. The information about the simplexes and how they glue is encoded in a combinatorial object called the simplicial complex. However, the price for these properties is that in practice one needs a large number of simplexes [98].

A more loose notion of a complex allows us to use fewer simplexes but makes the description of how they are glued together much more subtle. The formulation is now in terms of i-cells in X, i.e. maps $\sigma_i \xrightarrow{\phi} X$ such that the restriction to the interior is a homeomorphism onto the image

$$\sigma_i^o \xrightarrow{\cong} \phi(\sigma_i^o) \subseteq X \tag{3.1}$$

Now observe that

3.5 Lemma

- A non-empty intersection of simplexes α, β is a facet of both α, β.
- A simplex in $\mathcal{T}$ is determined by its vertices.

This means that the way the simplexes are attached will be completely described in terms of the combinatorics of the set of vertices $\mathcal{T}^0$ [99]. For that reason one can encode a simplicial triangulation as a combinatorial structure [100]: a set V (the set of all vertices in $\mathcal{T}$) endowed by a family $\mathcal{K}$ of subsets of V, the family of sets of vertices of all simplexes in $\mathcal{T}$. We saw that for each simplex $Y \in \mathcal{T}$ the set of its vertices is a subset of V and the mutual positions of two simplexes in $\mathcal{T}$ is recorded in the intersection of the sets of their vertices.

In this way it appears to be possible to describe some topological spaces in combinatorial terms. This will lead us to calculate their invariants purely algebraically using the combinatorics of the space rather than the space itself.

A simplicial complex is a set V together with a family $\mathcal{K}$ of finite non-empty subsets of V such that with any element $A \in \mathcal{K}$, family $\mathcal{K}$ also contains all subsets of A.

3.6 Lemma

- Any simplicial triangulation $\mathcal{T}$ defines a simplicial complex $\mathcal{K}(\mathcal{T})$.
- To any simplicial complex $\mathcal{K}$ we can associate a topological space $[\mathcal{K}]$ called its realization. It comes with a triangulation $\mathcal{T}$ such that $\mathcal{K}(\mathcal{T})$ is naturally identified with $\mathcal{K}$.

3.7 Theorem If we start with a triangulated topological space $(X, \mathcal{T})$ then the realization $|\mathcal{K}(\mathcal{T})|$ of the corresponding simplicial complex $\mathcal{K}(\mathcal{T})$ is canonically homeomorphic to X.

Our first goal is to encode a triangulation $\mathcal{T}$ algebraically. In order to pass from topological spaces to linear algebra we make a choice of a coefficient ring $\mathbb{K}$ so that we calculate in the linear algebra of $\mathbb{K}$-modules [101]. The set of simplexes will be encoded as a basis of a $\mathbb{K}$-module $C_*(X, \mathcal{T}, \mathbb{K})$ of chains in X. The boundary operator $\partial : C_*(X, \mathcal{T}, \mathbb{K}) \to C_*(X, \mathcal{T}, \mathbb{K})$ will encode the way the simplexes in $\mathcal{T}$ are glued in X.

The choice of $\mathbb{K}$, which may at this point appear trivial will have fundamental effects later on. Basically it is the structure that allows us to make use of linear algebras of certain kinds. The freedom which we gain by doing this has an effect upon what properties of the space we can detect. It is important to observe again that by passing from the space to the simplicial complex one obtains the triangulation as a supplemental structure. However, this structure is only auxiliary and directly related to the original space only in dimensions smaller than or equal to 3. By calculating the (co)homology we eliminate the dependence on the triangulation itself but we remain with the dependence on the coefficient group (or ring). This extra structure is defined by the methods we use in order to extract information about the space and the same space will look differently when analyzed with different coefficients. This fact is well known also in algebraic geometry where algebraic varieties appear very different depending on the coefficients of the polynomials used to describe them. Moreover, for the case where the dimension is 4 or larger, this freedom allows us to introduce compatibility statements about spaces as seen via different coefficient groups in (co)homology. This has important consequences on any attempts for a quantum description of gravity. If we return to the coefficient groups for a general simplicial complex, at this moment there is no problem in choosing this group as $\mathbb{K} = \mathbb{Z}$.

An oriented triangulation Σ on a topological space X is a pair $(\mathcal{T}, o)$ of a triangulation $\mathcal{T}$ of X and a choice o_α of an orientation of each simplex $\alpha \in \mathcal{T}$. The space of i-chains $C_i(X, \mathcal{T}, \mathbb{K})$ is defined for any triangulation $\mathcal{T}$ of a space X. A choice of orientation o for the triangulation will then give a simpler way of thinking of groups C_i. The space of i-chains for an oriented triangulation (X, Σ) is the free $\mathbb{K}$-module

$$C_i = C_i(X, \Sigma, \mathbb{K}) = \bigoplus_{\alpha \in \Sigma^i} k\alpha \tag{3.2}$$

with the basis given by the set of i-simplexes Σ^i in the oriented triangulation Σ. To define the space of i-chains for a triangulation $(X, \mathcal{T})$ we start with the free $\mathbb{K}$-module

$$\tilde{C}_i(X, \mathcal{T}; \mathbb{K}) = \bigoplus_{\alpha \in \mathcal{T}^i,\, o_\alpha \in or_\alpha} k\alpha \tag{3.3}$$

with the basis given by all i-simplexes α with all possible choices of orientations o_α. Then $C_i(X, \mathcal{T}; \mathbb{K})$ is the quotient of $\tilde{C}_i(X, \mathcal{T}, \mathbb{K})$ obtained by imposing $\bar{\sigma} = (-1)\sigma$ for oriented i-simplexes $\sigma = (\alpha, o_\alpha)$ with $i > 0$.

A choice of an orientation for a triangulation identifies the group $C_i(X, \mathcal{T}, \mathbb{K})$ with the same construction $C_i(X, \mathcal{T}, o, \mathbb{K})$ for the oriented triangulation $(\mathcal{T}, o)$ since the composition

$$C_i(X, \mathcal{T}, o, \mathbb{K}) \subseteq \tilde{C}_i(X, \mathcal{T}, \mathbb{K}) \to C_i(X, \mathcal{T}; \mathbb{K}) \tag{3.4}$$

is an isomorphism.

The boundary operator $\partial : C_i \to C_{i-1}$ for oriented simplexes in lower dimensions should also be defined. First, a point has no boundary hence $\partial\sigma_a = 0$. For an oriented segment σ_{ab} the boundary is given by

$$\partial\sigma_{ab} = \sigma_b - \sigma_a \tag{3.5}$$

For a triangle σ_{abc} with vertices a, b, c and the orientation given by ordering abc, the boundary is a triangle with the induced orientation hence

$$\partial\sigma_{abc} = \sigma_{ab} + \sigma_{bc} + \sigma_{ac} \tag{3.6}$$

However we can write is also as

$$\partial\sigma_{abc} = \sigma_{ab} - \sigma_{cb} + \sigma_{ac} \tag{3.7}$$

hence the boundary operator for an oriented simplex will be an algebraic sum of the form

$$\partial_i\sigma_{v_0,\dots,v_i} = \sum_{0 \le p \le i} (-1)^p \sigma_{v_0\dots\hat{v}_p\dots v_i} \tag{3.8}$$

where $\hat{v}_p$ means we omit v_p. This is a sum of all faces with orientations given by the ordering $v_0 \dots \hat{v}_p \dots v_i$ and the sign $(-1)^p$.

3.8 Lemma The above formula for ∂_i gives a well defined $\mathbb{K}$-map $\partial_i : C_i(X, \mathcal{T}; \mathbb{K}) \to C_{i-1}(X, \mathcal{T}; \mathbb{K})$.

Proof See appendix.

Another relatively simple observation is that a boundary has no boundary. This can be translated as $\partial^2 = 0$. In slightly more details, for a chain complex one has $\partial_i\partial_{i-1} = 0$. This observation is the origin of homological algebra. It brought us to the possibility of defining an algebraic notion of a complex. One speaks now about a complex of cochains as a sequence of $\mathbb{K}$ modules and maps of the form

$$\dots \xrightarrow{\partial^{-2}} C^{-1} \xrightarrow{\partial^{-1}} C^0 \to \dots \xrightarrow{\partial^0} C^1 \to \dots \tag{3.9}$$

such that $\partial^{i+1}\partial i = 0$, $i \in \mathbb{Z}$. From a complex of cochains we get three sequences of $\mathbb{K}$-modules.

- i-cocycles $Z^i = Ker(\partial^i) \subseteq C^i$
- i-coboundaries $B^i = Im(\partial^{i-1}) \subseteq C^i$
- i-cohomologies $H^i = Z^i/B^i$

Here we used $B^i \subseteq Z^i$ which follows from $\partial^2 = 0$. A complex of chains is the same thing but with maps going in the other direction

$$... \xleftarrow{\partial^{-2}} C^{-1} \xleftarrow{\partial^{-1}} C^0 \leftarrow ... \xleftarrow{\partial^0} C^1 \leftarrow ... \tag{3.10}$$

In this case we lower the indices and we talk about i-cycles $Z_i \subseteq C_i$, i-boundaries $B_i \subseteq C_i$ and i-homologies $H_i = Z_i/B_i$.

We have seen that any triangulation $\mathcal{T}$ of X associates to a topological space X the homology groups

$$H_i(X, \mathcal{T}, \mathbb{K}) = H_i[(C_*(X, \mathcal{T}; \mathbb{K}), \partial)] \tag{3.11}$$

It is possible to prove that these groups really are invariants of X itself and hence can be called the homology groups of X, denoted as $H_i(X, \mathbb{K})$.

It is important to observe that the definition of homology and cohomology depends on the coefficients used. Integers are considered the universal coefficient ring. This means that integral (co)homology (with integer coefficients) has a certain amount of information [102]. When passing to $\mathbb{Q}$ or $\mathbb{Z}/n\mathbb{Z}$ some information becomes easier to obtain while some other information is being erased. It is the main subject of this thesis to connect this fact with the idea of coarse graining, renormalization group and effective field theories in physics. This will be the main subject of another chapter.

Chapter 4

Homological Algebra

“ Mad Hatter: ‘Why is a raven like a writing-desk?’,
‘Have you guessed the riddle yet?’ the Hatter said, turning to Alice again.
‘No, I give it up’, Alice replied: ‘What’s the answer?’
‘I haven’t the slightest idea’, said the Hatter. ”

Lewis Carroll, Alice in Wonderland

The relevance of this chapter comes from the fact that it encompasses several very interesting mathematical concepts. Its main goal is to explain the ideas behind homological algebra. In some sense homological algebra represents the analogue of writing algebraic equations for homology groups. Finding the unknowns means finding a morphism between the unknown group and some other known group. The difference is that there is not only one single type of “equality”. The rules for solving equations transform into diagram chasing and the use of various lemmas (the five-lemma, the snake lemma, etc.). I will first introduce these explaining their practical utility in several situations and then will go on to a description of homological algebra per se, using a standard example. First, let me start with a brief introduction to the concept of category [103]. This notion represents one of the most elegant tools for mathematical thinking and allows the creation of concepts that would not arise otherwise. The notion itself appears to be surprisingly simple. It formalizes the idea that we can study certain objects endowed with specified structures and that it makes sense to go from one such object to another via morphisms i.e. transformations that preserve certain relevant structures. It is relatively easy to see that many well known constructions “categorify”, this means they have analogues in the language of categories i.e. by adding a new layer of morphisms new information can be derived and applied in different frameworks. The categories obtained in this way became important in the description of various phenomena and lead to the study of various "special categories", a term somehow analogous to the study of special classes of functions in analysis. In order to give a more systematic characterization I give the following

4.1 Definition A category C consists of

- a class $Ob(C)$ of elements called objects of C
- for any $a, b \in Ob(C)$ there exists a set $Hom_C(a, b)$ whose elements are called morphisms from a to b in C
- for any $a, b, c \in Ob(C)$ a function $Hom_C(b, c) \times Hom_C(a, b) \to Hom_C(a, c)$ exists and is called a composition
- for any $a \in Ob(C)$ an element $1_a \in Hom_C(a, a)$

such that the composition is associative and 1_a is a neutral element for composition.

Instead of $a \in Ob(C)$ we will usually just say that $a \in C$. As examples one can think of the following situations

- Categories of sets with additional structures: *Sets*, *Ab*, $m(\mathbb{K})$ for a ring $\mathbb{K}$
- If $\mathbb{K}$ is a field then we have $Vect(\mathbb{K})$, *Groups*, *Rings*, *Top*, *OrdSet* (i.e. the category of ordered sets), etc.
- To a category C one attaches the opposite category C^O so that objects are the same but the direction of the arrow reverses

$$Hom_{C^O}(a, b) = Hom_C(b, a) \tag{4.1}$$

- Any partially ordered set $(I, \leq)$ defines a category with $Ob = I$ and $Hom(a, b)$ is a point if $a \leq b$.
- Sheaves of sets on a topological space X, sheaves of abelian groups on X, etc

The universality brought upon us by the categorial thinking allows us to unify phenomena from different parts of mathematics. In a sense, adding a layer of morphisms in a given theory represents categorification. Of course, there is no single categorification given a certain theory and therefore, the process must be considered only when certain advantages are envisaged.

It could be interesting to ask if it is possible to express the quantum "compatibility" condition for observables at the level of topology. A method for doing this would be to design a compatibility relation for (co)homology groups via their coefficient groups. I made an attempt to an initial step in this direction in [104]. The main idea was to extend the notion of "compatibility" relevant to the observables of quantum mechanics to groups and algebraic structures. This project is at a very incipient stage but it may end up being relevant for quantum gravity.

At this point it is advisable to start from the most basic concepts, for example, it is interesting to observe how the notion of equality can be generalized [105]. In a set, two elements can be equal or not equal. However, a set is a decategorification of a category that includes also morphisms. So, when the idea of equality extends to objects in a category the situation becomes somehow different. Two objects in a category can be

- the same
- isomorphic
- isomorphic by a canonical (given) isomorphism

It turns out that only the third possibility is the correct generalization and constitutes the analogue of the equality of elements in a set. In this sense, when we say "a=b" what we actually mean, in the language of categories, is that there exists a specific isomorphism $\phi : a \to b$. We therefore observe that objects in general do not stand alone. There are always somewhere, some morphisms implied when we speak in terms of categories. It is extremely important to notice this mainly because the most natural conclusion then, is that the information related to certain objects (even physical objects) is not always encoded exclusively in their internal structure but also in the way in which they can be mapped into other objects or in the way they relate with the framework where they are being analyzed. This idea became part of my work on the missing information in black holes thermodynamics [53]. In order to continue this discussion let me define the products of objects from a categorial perspective. We call a product of two objects a and b in a category C a triple (Π, p, q) where $\Pi \in C$ is an object and $p \in Hom_C(\Pi, a)$, $q \in Hom_C(\Pi, b)$ are maps such that for any $x \in C$ the function

$$Hom_C(x, \Pi) \ni \phi \to (p \circ \phi, q \circ \phi) \in Hom_C(x, a) \times Hom_C(x, b) \tag{4.2}$$

As can be seen from the above construction, it is important to observe that the object given as the product of two other objects is not sufficient to determine the actual existence of a product of two objects in a categorial sense. Maps from the product object to the objects that combine in order to generate the object must also exist. Moreover, these maps must be categorial morphisms and must obey certain rules.

As an example take $C = Sets$, the category of sets, and the product of sets $\Pi = a \times b$ together with the projections p, q which satisfy the required property. We can see that the categorial notion of a product is just the abstract formulation of properties of a product of sets. One could naively expect that a product of a and b should be a specific object built from a and b. However, this is not the categorial meaning of the notion. For two given a and b there can exist many triples (Π, p, q) satisfying the product property. However, any two such triples (Π_i, p_i, q_i) , with $i = 1, 2$ are related by a canonical isomorphism $\phi : \Pi_1 \to \Phi_2$ provided by the defining property of the product.

When we say "Π is a product of a and b" we abuse the language. We must always remember that there exists the additional data p and q. What we did above can be generalized into a standard construction of an object defined by a universal property.

In this case the universal property is that a map into a product is the same as a pair of maps into a and b. We also can think of the property as an object co-representing a functor.

A functor is another important concept in category theory. Essentially it maps one category into another in the same way in which a morphism maps one object into another. However, because of the fact that categories also contain morphisms, a functor is in some sense a more general construction that also maps morphisms into morphisms. A more formal definition is as follows:

4.2 Definition

Let C and D be categories. A functor F from C to D is a mapping that connects each object from the first category to an object from the second category. Due to the structure of categories (they also contain morphisms) a functor must associate to each morphism in the first category another morphism in the second category such that the identity in the first category is mapped into the identity in the second category and the composition of morphisms becomes the composition of the morphisms transformed by the functor.

Some functors turn morphisms around and reverse compositions. These are called contravariant functors and they satisfy the following properties

- to each object $X \in C$ a contravariant functor associates an object $F(X) \in D$
- to each morphism $f : X \to Y \in C$ a contravariant functor associates a morphism $F(f) : F(Y) \to F(X) \in D$ such that

– $F(id_X) = id_{F(X)}$ for every object $X \in C$

– $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f : X \to Y$ and $g : Y \to Z$

In other words a contravariant functor is a covariant (normal) functor on the opposite category C^O.

In our case, Π corepresents a contravariant functor $C \ni x \to F(x) = Hom_C(x,a) \times Hom_C(x,b) \in Sets$ in the sense that the functor F is identified with the functor $Hom(*,\Pi)$ that one gets from Π.

In general an object defined by some universal property P is

- not really a single object but a system of various objects related by compatible isomorphisms
- each of these objects does not come alone but is supplied with some additional data consisting of some morphisms such as p and q above
- a product of two objects a and b in a given category C need not exist

The next structure I wish to introduce is the sum. We can define the sum of the objects a and b in C as the triple (Σ, i, j) where $\Sigma \in C$ is an object while $i \in Hom_C(a,\Sigma)$, $j \in Hom_C(b,\Sigma)$ are maps such that for any $x \in C$ the function

$$Hom_C(\Sigma, x) \ni \phi \to (\phi \circ i, \phi \circ j) \in Hom_C(a,x) \times Hom_C(b,x) \tag{4.3}$$

is a bijection.

In *Sets* the sums exist and the sum of two objects a and b is the disjoint union $a \sqcup b$. We can also discuss about sums and products of families of objects. Like for the case of two objects, a product in C of a family of objects $a_i \in C$, $i \in I$ is a pair $(P,(p_i)_{i\in I})$ where $P \in C$ and $p_i : P \to a_i$ are such that the map

$$Hom_C(x,P) \ni \phi \to (p_i \circ \phi)_{i\in I} \in \Pi_{i\in I} Hom_C(x,a_i) \tag{4.4}$$

is a bijection. A sum of $a_i \in C$, $i \in I$ is a pair $(S,(j_i)_{i\in I})$ where $j_i a_i \to S$ gives a bijection

$$Hom_C(S,x) \ni \phi \to (\phi \circ j_i)_{i\in I} \in \Pi_{i\in I} Hom_C(a_i,x) \tag{4.5}$$

The notation becomes $\sqcup_{i\in I} a_i$ or $\oplus_{i\in I} a_i$.

4.3 Lemma For a ring $\mathbb{K}$ the category $m(\mathbb{K})$ has sums and products.

- The product $\Pi_{i\in I}M_i$ is (as a set) just the product of sets, so it consists of all families $m = (m_i)_{i\in I}$ with $m_i \in M_i$, $i \in I$.
- The sum $\oplus_{i\in I}a_i$ happens to be the submodule of $\Pi_{i\in I}M_i$ consists of all finite families $m = (m_i)$, $i \in I$, i.e. families such that $m_i = 0$ for all but finitely many $i \in I$.

Categorial thinking allows to extend the notion of limit from analysis to many other regions [106]. This generalization is often indispensable. In some instances it is clear what a limit of a family of objects means. Consider for example a sequence of increasing subsets

$$A_0 \subseteq A_1 \subseteq ... \tag{4.6}$$

of a set A. We can say that its limit $limA_i$ is the subset $\bigcup_{i\geq 0} A_i$ of A. Similarly, the limit of a decreasing sequence of subsets

$$B_0 \supseteq B_1 \supseteq ... \tag{4.7}$$

of A will be the subset $limB_i = \bigcap_{i\geq 0} B_i$ of A. Now we give a precise meaning to the constructions corresponding to these two examples

4.4 Definition (Inductive limits) An inductive system of objects of C over a partially ordered set $(I, \leq)$ consists of

- a family of objects $a_i \in C$, $i \in I$ and
- a system of maps $\phi_{ji} : a_i \to a_j$ for all $i \leq j$ in I

such that

- $\phi_{ii} = 1_{a_i}$, $i \in I$ and
- $\phi_{kj} \circ \phi_{ji} = \phi_{ki}$ when $i \leq j \leq k$.

Its inductive limit is a pair $(a, (\rho_i)_{i\in I})$ of an object $a \in C$ and a system of maps $\rho_i : a_i \to a$, $i \in I$ such that

- $\rho_j \circ \phi_{ji} = \rho_i$ for $i \leq j$ and moreover
- $(a, (\rho_i)_{i\in I})$ is universal with respect to this property in the sense that for any $(a', (\rho'_i)_{i\in I})$ that satisfies $\rho'_j \circ \phi_{ji} = \rho'_i$ for $i \leq j$, there is a unique map $\rho : a \to a'$ such that $\rho'_i = \rho \circ \rho_i$, $i \in I$.

Informally we write $lim_{\to I, \leq} a_i = a$.

4.5 Definition (Projective limits) A projective system of objects of C over a partially ordered set $(I, \leq)$ consists of

- a family of objects $a_i \in C$, $i \in I$ and
- a system of maps $\phi_{ij} : a_j \to a_i$ for all $i \leq j$ in I

such that

- $\phi_{ii} = 1_{a_i}$, $i \in I$ and
- $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ when $i \leq j \leq k$.

Its limit is a pair $(a, (\sigma_i)_{i\in I})$ of an object $a \in C$ and a system of maps $\sigma_i : a \to a_i$, $i \in I$ such that

- $\phi_{ji} \circ \sigma_j = \sigma_i$ for $i \leq j$ and moreover
- $(a, (\sigma_i)_{i\in I})$ is universal with respect to this property in the sense that for any $(a', (\sigma'_i)_{i\in I})$ that satisfies $\phi_{ij} \circ \sigma'_j = \sigma'_i$ for $i \leq j$, there is a unique map $\sigma : a' \to a$ such that $\sigma'_i = \sigma_i \circ \sigma$, $i \in I$.

Informally we write $lim_{\leftarrow I, \leq} a_i = a$. It is obvious now that limits are functorial.

In order to understand the next concepts, a definition of what a natural transformation of functor is, appears to be necessary. Given a functor $F : \mathcal{A} \to \mathcal{B}$ which transforms under a natural transformation η into another functor $G : \mathcal{A} \to \mathcal{B}$, we can say that the natural transformation η consists of maps $\eta_a \in Hom_{\mathcal{B}}(Fa, Ga)$, $a \in \mathcal{A}$ such that for any map $\alpha : a' \to a'' \in \mathcal{A}$ the following diagram commutes

$$\begin{array}{ccc} F(a') & \xrightarrow{F(\alpha)} & F(a'') \\ \downarrow{\scriptstyle \eta_{\alpha'}} & & \downarrow{\scriptstyle \eta(\alpha'')} \\ G(a') & \xrightarrow{G(\alpha)} & G(a'') \end{array} \tag{4.8}$$

meaning $\eta_{\alpha''} \circ F(\alpha) = G(\alpha) \circ \eta_{\alpha'}$. This means the map η relates values of functors on objects in a way compatible with the values of functors on maps. Any such natural choice of maps η_a will also be compatible in this sense [107].

Another very important concept is the one of adjoint functors [107]. In general an adjoint pair of functors is a pair of functors of the form $(A \xrightarrow{F} B, B \xrightarrow{G} A)$ together with the natural identifications

$$\zeta_{a,b} : Hom_B(Fa, b) \xrightarrow{\cong} Hom_A(a, Gb), a \in A, b \in B \tag{4.9}$$

Here "natural" means that ζ is a natural transformation of functors $\zeta : Hom_B(F-, -) \to Hom_A(-, G-)$ from $A^o \times B$ to $Sets$. Identification means here that each function $\zeta_{a,b}$ is a bijection. We say that F is the left adjoint of G and that G is the left adjoint of F i.e. in the identity of homomorphisms F appears on the left in Hom and G on the right [107]. If one considers the functors $\phi_* M = l \otimes_{\mathbb{K}} M$ and $\phi^* N = N$ we have canonical morphisms of functors

$$\begin{array}{ll} \alpha : \phi_* \phi^* \to 1_{m(l)} & \phi_* \circ \phi^*(N) = l \otimes_{\mathbb{K}} N \xrightarrow{\alpha_N} N = 1_{m(l)}(N) \\ \beta : 1_{m(\mathbb{K})} \to \phi^* \circ \phi_*, & \phi^* \circ \phi_*(M) = l \otimes_{\mathbb{K}} M \xleftarrow{\beta_M} = 1_{m(\mathbb{M})(M)} \end{array}$$

For any functor $F : \mathcal{A} \to \mathcal{B}$ there exists a unity $1_F : F \to F$ with $(1_F)_a = 1_{Fa} : Fa \to Fa$. For three functors F, G, H from $\mathcal{A}$ to $\mathcal{B}$ one can compose morphisms $\mu : F \to G$ and $\nu : G \to H$ such that they become $\nu \circ \mu : F \to H$. Given two categories $\mathcal{A}$ and $\mathcal{B}$ the functors from $\mathcal{A}$ to $\mathcal{B}$ form a category $Funct(\mathcal{A}, \mathcal{B})$.

4.6 Lemma

Functors (ϕ_*, ϕ^*) form an adjoint pair i.e. there is a canonical identification

$$Hom_{m(l)}(\phi_* M, N) \xrightarrow{\eta_{M,N}} Hom_{m(\mathbb{K})}(M, \phi^* N), M \in m(\mathbb{K}), N \in m(l) \tag{4.10}$$

If $l \otimes_{\mathbb{K}} M \xrightarrow{\sigma} N$, $M \xrightarrow{\tau} N$, then $Hom_{m(l)}(l \otimes_K M, N) \xrightarrow{\eta_{M,N}} Hom_{m(\mathbb{K})}(M, N)$ by $\eta(\sigma)(m) = \sigma(1 \otimes m)$ and $\eta^{-1}(\tau)(c \otimes m) = c\tau(m), m \in M, c \in l$.

The morphisms of functors $\phi_* \circ \phi^* \xrightarrow{\alpha} 1_{m(l)}$ and $\phi^* \circ \phi_* \xleftarrow{\beta} 1_{m(\mathbb{K}}$ are the same with the isomorphisms $Hom_{m(l)}(\phi_* *, *) \xrightarrow{\eta} Hom_{m(\mathbb{K})(*, \phi^* *)}$ i.e. two alternative but equivalent ways of describing adjointness.

It is often the case that an adjoint pair appears as consisting of an obvious functor A with an adjoint B which is an interesting construction. This last interesting construction B is related to the original simple functor A hence the properties of B can be derived from the properties of the original simpler construction A.

In general we call forgetful functors, those functors that drop a part of the structure of an object. Standard constructions that add to the structure of an object are often adjoints of forgetful functors. At this moment we have the basics of categorial theory and we defined what is generally understood by "categorial language". In what follows I will focus more on the actual subject of this subsection.

Homological algebra is, as its name already says, the algebra of homologies. The goal of this relatively new mathematical subdomain is to capture the information about a class of objects in terms of smaller or better behaved subclasses [108]. One problem of homological algebra is the construction of a dual for a module over a ring [109]. I will use this problem in order to construct a basic introduction to the goals of homological algebra.

It is known that the construction of a dual vector space over a field equally makes sense for modules over any ring [110]. This is so because we do not need all the operations defined for a field in order to define the dual of modules. However, the simple notion of duality is not very useful since it does not have the standard properties of the duality for vector spaces. In order to generalize the concept of duality we make the easy observation that the basic notion of duality still works well for some modules: the free modules. A free module is a module with a basis. In the same way, a free group is a group on which we can define a basis. The important and non-trivial observation is that any module can be represented (described) in terms of finitely generated free modules. This is achieved by the notion of a resolution. The correct notion of duality is then obtained by applying the naive duality not directly to the module but instead to its resolution, i.e. to a description in terms of free modules. The effect is that all computations are done with free modules and hence the new duality has the same properties as those of the old naive one.

Replacing modules by resolutions is done by passing from modules to complexes of modules [111]. These complexes represent a "larger world" [112] where we can find all the hidden parts of the naive constructions. There are two steps:

- abelian groups are thought as complexes in degree 0.
- some complexes are identified i.e. a module should be identified with its resolution

These steps mean that we change twice the categories in which we calculate:

$$m(\mathbb{K}) \xrightarrow{(1)} C^*(m(\mathbb{K})) \xrightarrow{(2)} D(m(\mathbb{K})) \tag{4.11}$$

We start in the category of $\mathbb{K}$-modules $m(\mathbb{K})$ and expand to the category of complexes of $\mathbb{K}$-modules $C^*(m(\mathbb{K}))$ and then we pass to a more subtle derived category $D(m(\mathbb{K}))$ of $\mathbb{K}$-modules.

The first step allows to think of any $\mathbb{K}$-module in terms of particularly nice modules (say free modules). The second step introduces the optimal setting $D(m(\mathbb{K}))$ which makes the identification of the right complexes exact.

Some of the more interesting classes of rings $\mathbb{K}$ are

- fields such as $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ or the finite fields $\mathbb{F}_q$ with q elements
- $\mathbb{Z}$ related to number theory
- smooth functions $C^\infty(M)$ on a manifold M related to differential geometry
- polynomial functions $\mathcal{O}(\mathbb{A}^n) = \mathbb{C}[x_1, ..., x_n]$ related to algebraic geometry
- Differential operators D_M on M related to linear differential operators

Let d be the naive notion of duality. One can define it as follows

4.7 Definition

The dual of a left $\mathbb{K}$-module M is the space $d(M) = M^*$ of linear functionals

$$M^* = Hom_{\mathbb{K}}(M, \mathbb{K}) =$$
$$\{f : M \to \mathbb{K};\ f(cm) = cf(m);\ f(m' + m'') = f(m') + f(m''),\ c \in \mathbb{K}, m, m', m'' \in M\} \tag{4.12}$$

The duality construction is a functor i.e. it is defined not only on $\mathbb{K}$-modules but also on maps of $\mathbb{K}$-modules; the dual of $f : M_1 \to M_2$ is the adjoint map $d(f) = f^* : M_2^* \to M_1^*$, $f^*(\nu)m =< \nu, fm >$, $m \in M_1$, $\nu \in M_2^*$.

4.8 Lemma(biduality)

For $M \in m^l((K))$ the canonical map $i_M : M \to (M^*)^*$ is well defined by $i_M(m)(\lambda) =< \lambda, m >$, $m \in M$, $\lambda \in M^*$.

4.9 Lemma If $\mathbb{K}$ is a field and $M \in m_{fd}(\mathbb{K})$ (i.e. M is a finite dimensional vector space over $\mathbb{K}$) the biduality map i_M is an isomorphism.

This last lemma is what we know generally about duality. However this is valid only for fields. We ask here how can this be true for modules over any ring $\mathbb{K}$. In general biduality is not always an isomorphism and now we distinguish a class of modules for which the isomorphism condition is preserved. We start with $M = \mathbb{K}$.

4.10 Lemma

For the module ${}_{\mathbb{K}}\mathbb{K} \in m^l(\mathbb{K})$ (${}_{\mathbb{K}}\mathbb{K}$ is the left module via the left multiplication):

- The map that assigns to $a \in \mathbb{K}_{\mathbb{K}}$ the operator of right multiplication $R_a :_{\mathbb{K}} \mathbb{K} \rightarrow_{\mathbb{K}} \mathbb{K}$,
  where by definition $x = xa$, gives an isomorphism of right $\mathbb{K}$-modules $\mathbb{K}_{\mathbb{K}} \xrightarrow{R} ({}_{\mathbb{K}}\mathbb{K})^*$.
- $i_{\mathbb{K}}\mathbb{K}$ is an isomorphism.

If we try to further extend these notions we get a nice class of modules for which $M \xrightarrow{\cong} (M^*)^*$.

4.11 Proposition

i_M is an isomorphism for any finitely generated free $\mathbb{K}$-module.

4.12 Lemma For two $\mathbb{K}$-modules P and Q

- $(P \oplus Q)^* \cong P^* \oplus Q^*$
- the map $i_{P \oplus Q}$ is an isomorphism iff both i_P and i_Q are isomorphisms

We observe that the duality operation $M \rightarrow M^*$ is not very good for arbitrary modules M of any ring $\mathbb{K}$. Even when $\mathbb{K}$ is a field, biduality is an isomorphism only for the finite dimensional vector spaces. Therefore for a general $\mathbb{K}$ the duality can have best properties only on the subcategory $m_{fg}(\mathbb{K})$ of finitely generated $\mathbb{K}$-modules. A more important problem appears when for example $\mathbb{K} = \mathbb{Z}$, the ring of integers and $M = \mathbb{Z}_n$ is a torsion module. For $\mathbb{K} = \mathbb{Z}$, the category of $\mathbb{Z}$-modules is just the category of abelian groups: $m(\mathbb{Z}) = (Ab)$. So, we have the notion of a dual of an abelian group $M^* = Hom_{\mathcal{A}b}(M, \mathbb{Z})$. However for $M = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ one has $M^* = 0$ so the duality loses all the information. On this example it is possible to develop the strategy of describing modules in terms of a subclass of free modules behaving well under duality.

Now we will see how to go from $\mathbb{Z}_n$ to its resolution P^*. We know that biduality works for the abelian group $M = \mathbb{Z}$ and $\mathbb{Z}_n$ is clearly related to $\mathbb{Z}$. The quotient map $\mathbb{Z} \xrightarrow{q} \mathbb{Z}_n$ relates $\mathbb{Z}_n$ to $\mathbb{Z}$ however it does not tell the whole story. The difference between $\mathbb{Z}_n$ and $\mathbb{Z}$ is the kernel $Ker(q) = n\mathbb{Z}$. However the inclusion $n\mathbb{Z} \subseteq \mathbb{Z}$ captures the definition of $\mathbb{Z}_n$ as $\mathbb{Z}/n\mathbb{Z}$ and since the abelian group $n\mathbb{Z}$ is isomorphic to $\mathbb{Z}$ by $\mathbb{Z} \ni x \rightarrow nx \in n\mathbb{Z}$ we will replace $n\mathbb{Z}$ by $\mathbb{Z}$ in this map. Then it becomes the multiplication map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. Now we can think of $\mathbb{Z}_n$ as encoded in the map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. For a more complicated $\mathbb{K}$-module such encoding will be more complicated, the proper setting will turn out to require to think of $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ as a complex $P^* = (... \rightarrow 0 \rightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \rightarrow 0 \rightarrow ...)$ with $\mathbb{Z}$ in degrees

-1 and 0. I showed that we can pass from $\mathbb{Z}_n$ to a complex P^*. Now we need to know how to dualize it. In what follows I will present the duality operation on complexes. Let $C^* = (... \to C^{-1} \to C^0 \to C^1 \to ...)$ be a complex of $\mathbb{K}$-modules. Its dual is dC^* obtained by applying d to the constitutive modules and maps. Since d is contravariant (i.e. it changes directions) the indexing will also change. As a result $(dC^*)^n = d(C^{-n})$ and $d^n_{dC^*}$ is the adjoint of $d^{-n-1}_{C^*}$. In order to calculate dP^* we need

4.13 Lemma

- $\mathbb{K}$-linear maps between left modules $\mathbb{K}^r$ and $\mathbb{K}^s$ can be described in terms of right multiplication by matrices. Precisely if we denote for $A \in M_{r,s}(k)$ by R_A the right multiplication operator $\mathbb{K}^r \ni x \to xA \in \mathbb{K}^s$ on row-vectors, then $M_{rs} \xrightarrow{R} Hom_{\mathbb{K}}(\mathbb{K}^r, \mathbb{K}^s)$ is an isomorphism.
- The adjoint of R_A is the left multiplication $L_{A^{tr}}$ with the transpose of A (acting on column vectors).

Biduality is an isomorphism on complexes over the subcategory of complexes over $m_{fg,free}(\mathbb{K}) \subseteq m(\mathbb{K})$. Let $\mathcal{P} = m_{fg,free}(\mathbb{K})$ be the category of all free finitely generated $\mathbb{K}$-modules.

The biduality map i_{C^*} is an isomorphism for any complex C^*.

On $\mathbb{K}$-modules we define the left derived duality operation Ld by $Ld(M) = dP^*$ for any resolution P^* of M by free modules.

Lets see what this means for $\mathbb{K} = \mathbb{Z}$ and $M = \mathbb{Z}_n$. When we identify $d\mathbb{Z}$ with $\mathbb{Z}$ then the adjoint of the map $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$ is again $\mathbb{Z} \xrightarrow{n} \mathbb{Z}$. From the point of view of complexes this says that dP^* is the complex $... \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to 0 \to ...$ but this time $\mathbb{Z}$ are in degrees 0 and 1. So $dP^* \cong P^*[-1]$ where one denotes by $C^*[n]$ the shift of the complex C^* by n places to the left. It is natural to identify any module N with a complex denoted also by N which has N in degree 0 and all other terms zero. So, since we have also identified $\mathbb{Z}_n$ with P^* we should identify the derived dual $Ld(\mathbb{Z}) = dP^* \cong P^*[-1]$ with $\mathbb{Z}[-1]$. So $Ld(\mathbb{Z}_n) = \mathbb{Z}_n[-1] =$ the shift of $\mathbb{Z}_n$ by one to the right. This is the complex which has $\mathbb{Z}_n$ in degree 1 and all other terms zero. The derived dual of $\mathbb{Z}_n$ is not a module but a complex in degrees ≥ 0. The fact that $H^0[Ld(\mathbb{Z}_n)] = 0$ corresponds to the fact that the naive definition of the dual gives $d(\mathbb{Z}) = 0$. So the simple definition does not see the hidden part of the dual which is $H^1[Ld(\mathbb{Z}_n)] = \mathbb{Z}_n$. Since the computation of the derived dual is in the setting of complexes of free finitely generated modules, the biduality works, hence the canonical map $\mathbb{Z}_n \to (Ld)(Ld)(\mathbb{Z}_n)$ is an isomorphism.

We can repeat for any module M what we have been able to do for $\mathbb{Z}_n$. We wish to describe a module M in terms of maps between some nicer modules P^n.

A complex of $\mathbb{K}$-modules $C^* = (... \to C^{-1} \to C^0 \to C^1 \to ...)$ is said to be exact if all of its cohomologies vanish i.e. if the inclusion $B^n \subseteq Z^n$ becomes equality.

We call a left resolution of a module M the exact complex

$$... \to P^{-2}_{-2} \to P^{-1}_{-1} \to P^0_0 \xrightarrow{q} M_1 \to 0 \to ... \tag{4.13}$$

The lower indices are the positions in the complex.

The complexes form a category. The morphisms are of the form $f : A^* \to B^*$. There exist a system of maps f^n of the corresponding terms in complexes which preserve the differential in the sense that in the diagram

$$\begin{array}{ccccccccccc} ... & \longrightarrow & A^{-2} & \longrightarrow & A^{-1} & \longrightarrow & A^0 & \longrightarrow & A^1 & \longrightarrow & ... \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ ... & \longrightarrow & B^{-2} & \longrightarrow & B^{-1} & \longrightarrow & B^0 & \longrightarrow & B^1 & \longrightarrow & ... \end{array} \tag{4.14}$$

all squares commute. This means that any two possible ways of following arrows gives the same result. Now we have a category of complexes of $\mathbb{K}$-modules $C^*[m(\mathbb{K})]$: objects are complexes and morphisms are maps of complexes.

4.14 Lemma This construction forms a category and the constructions Z^n, B^n and H^n are functors from $C^*(m(\mathbb{K}))$ to $m(\mathbb{K})$.

To each module M we can associate a very simple complex $M^\#$ which is M in degree 0 and zero in other degrees (so all maps are zero).

4.15 Lemma This gives a functor

$$m(\mathbb{K}) \to C^*(m(\mathbb{K})), M \to M^\# \tag{4.15}$$

which is fully faithful i.e. $m(\mathbb{K})$ is a full subcategory of $C^*(m(\mathbb{K}))$.

We can see resolutions as maps of complexes. We can also use the terminology "resolution" for the equivalent data of a complex $P^* = (... \to P^{-2} \to P^{-1} \to P^0 \to 0 \to ...)$ together with the map $q : P_0 \to M$. We can now think of resolutions in terms of complexes by looking at the map q as a morphism of complexes

$$\begin{array}{ccccccccccc} ... & \longrightarrow & P^{-2} & \longrightarrow & P^{-1} & \longrightarrow & P^0 & \longrightarrow & 0 & \longrightarrow & ... \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ ... & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & 0 & \longrightarrow & ... \end{array} \tag{4.16}$$

It remains to encode the exactness of $... \to P^{-1} \to P^0 \to M \to 0 \to ...$ in terms of complexes. For this we introduce the following

4.16 Definition(Quasi-isomorphisms) We say that a map of complexes $f : A^* \to B^*$ is a quasi-isomorphism if the induced maps of cohomology groups $H^n(f) : H^n(A^*) \to H^n(B^*)$, $n \in Z$ are all isomorphisms.

4.17 Lemma A left resolution of M is the same as a quasi-isomorphism of complexes $P^* \to M^\#$ such that $P^i = 0$ for $i > 0$.

Proof See appendix.

Now it is clear how to define a right resolution: as a quasi-isomorphism of complexes $M^\# \to I^*$ such that $I^i = 0$ for $i < 0$. A free resolution of M is a resolution P^* such that all P^i are free $\mathbb{K}$-modules.

4.18 Lemma Any module M has a free resolution.

There is a free module F and a surjective map $F \to M$ (a free cover of M). For this we choose any set $\mathcal{G} \subseteq M$ of generators of M (for instance $\mathcal{G} = M$) and let F be a free $\mathbb{K}-module$ with the basis $\mathcal{G}$. Let $P^0 \xrightarrow{q} M$ be the map $F \to M$. If q has no kernel we are ready. We can choose $P^k = 0$, $k < 0$. Otherwise we use again the previous observation to choose a free cover $P^{-1} \to Ker(q)$, then ∂^{-1} is the composition $P^{-1} \to Ker(q) \subseteq P^0$.

If we now want to make the definition of the derived version of duality

$$Ld(M) = d(P^*)\ P^* \in M \tag{4.17}$$

completely correct we have to deal with two problems

- show the existence of a free resolution P^* of M
- show the independence of the choice of a free resolution P^*

The first part is relatively simple. For the second one, remember that a resolution is a quasi-isomorphism $P^* \to M^\#$. Our problem would disappear if the quasi-isomorphism were an isomorphism since we would be replacing $M^\#$ with an isomorphic object. So, our problem will be resolved if we can find a setting in which all quasi-isomorphisms in $C^*(m(\mathbb{K}))$ become isomorphisms. Such settings exist and are the so called derived category of $\mathbb{K}$-modules $D(m(\mathbb{K}))$.

The passage from $C^*(m(\mathbb{K}))$ to $D(m(\mathbb{K}))$ requires inverting all quasi-isomorphisms in $C^*(m(\mathbb{K}))$. This can be done either by an universal abstract construction of inverting morphisms in a category or by using some convenient subcategory of $m(\mathbb{K})$. Both are useful in applications.

Finally I will introduce another basic result in homological algebra. Essentially its prove is by diagram chasing and it represented a good exercise for practicing the above concepts. It also will be used in the discussion about the universal coefficient theorem.

4.19 Lemma(Snake Lemma) In an abelian category (such as the category of abelian groups or the category of vector spaces over a field), if we consider the following commuting diagram

$$\begin{array}{ccccccc} & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{a} & & \downarrow{b} & & \downarrow{c} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \end{array} \tag{4.18}$$

with the rows exact, then there is an exact sequence relating the kernels and cokernels of a, b and c

$$Ker(a) \to Ker(b) \to ker(c) \xrightarrow{d} CoKer(a) \to CoKer(b) \to Coker(c) \tag{4.19}$$

If the morphism f is a monomorphism then so is the morphism $Ker(a) \to Ker(b)$ and if g' is an epimorphism then so is $CoKer(b) \to CoKer(c)$.

The idea of derived category is extremely relevant in various regions of mathematics and physics. The derived category $D(A)$ may have more important properties that the category A we started with. One case is when we have pairs of very different categories A and B such that their derived categories $D(A)$ and $D(B)$ are canonically equivalent. For example A and B could be the categories of graded modules for the symmetric algebra $S(V)$ and the exterior algebra $\wedge^* V^*$ for dual vector spaces V and V^*. There are other examples like the relation between linear differential equations and their solutions, mirror symmetry, etc.

Chapter 5

Connections: Topology and Analysis

" 'But I don't want to go among mad people', Alice remarked.
'Oh, you can't help that', said the Cat: 'we're all mad here. I'm mad. You're mad.'
'How do you know I'm mad?' said Alice.
'You must be,' said the Cat, 'or you wouldn't have come here'. "

Lewis Carroll, Alice in Wonderland

It is important to note that topology does not stand alone [113]. It strongly interferes with geometry and through it with analysis [114]. In order to explain this connection I will discuss a set of results, starting with the standard Riemann-Roch Theorems [115] and going to the results of Grothendieck [116] and Atiyah-Singer [117]. The Riemann-Roch theorem is historically the first in a set of results relating complex geometry and topology [118]. It deals with the computation of the dimension of the space of meromorphic functions which have a set of predefined zeroes and several poles of given order. I remind the reader that a meromorphic function [119] is a function that can be written as the fraction of two holomorphic functions. A holomorphic function is a complex function that is differentiable in a neighborhood of any point in its domain. This means that a holomorphic function is infinitely differentiable and equal to its own Taylor series. Hence, a meromorphic function has all the properties of a holomorphic function except for a set of isolated points which are the zeroes of the denominator function. This is the set of its poles. The meromorphic function itself must be describable by a Laurent series in each of the poles.

The Riemann-Roch theorem links the complex analysis of compact Riemann surfaces with some of the associated topological properties like the topological genus [120]. The initial form for this theorem was proved by Riemann [121] back in the year 1857. At that moment it was given in the form of an inequality as

5.1 Theorem Let X be a Riemann surface of genus g, then

$$dim(L(D)) \geq deg(D) + 1 - g \tag{5.1}$$

where here $L(D)$ is the space of meromorphic functions with poles bounded by a divisor D.

Gustav Roch, one of Riemann's students gave to this theorem its final form [122], eliminating the inequality and showing what the so called "error term" was:

5.2 Theorem Let S be a Riemann surface of genus g. Then for any divisor D and any canonical divisor K we have

$$dim(L(D)) - dim(L(K - D)) = deg(D) + 1 - g \tag{5.2}$$

For the beginning, a good definition of the notion of a divisor is required. In the case of 1-dimensional complex manifolds (or Riemann surfaces), the divisors [123] are the elements of the free abelian group over the points of the surface. One can otherwise say that the divisor is a finite linear combination of points of the surface with integer coefficients [124]. We call the sum of the coefficients of a divisor the degree of the divisor and we note it $deg(D)$. Given a meromorphic function f the divisor of it is defined as

$$(f) = \sum_{z_v \in R(f)} s_v z_v \tag{5.3}$$

Here $R(f)$ is the set of all zeroes or poles of the meromorphic function f and s_v is given by $s_v = \begin{cases} a & \text{if } z_v \text{ is a zero of order } a \\ -a & \text{if } z_v \text{ is a pole of order } a \end{cases}$

Hence one can see that the divisor of a meromorphic function is described on one side in terms of the poles and zeroes of that function on the underlying Riemann surface and on the other side in terms of the orders of these poles and zeroes. Because of this, the divisors are important in characterizing the functions in terms of their poles and zeroes and the respective degrees. It appears only natural to think that there is a connection between divisors of a Riemann surface and its topology. This idea connects finally the information encoded in the poles and zeroes of a meromorphic function with the topology of the surface on which it is constructed.

In order to clarify the scope of these results I will review here some notions about algebraic curves [124]. These are an important type of Riemann surfaces. One reason for their importance is the fact that we can construct a meromorphic function X from a finite number of points in X together with the tails of Laurent series such that at each of these points the Laurent series of the function starts with one of these tails. Also, given an algebraic curve, one can always find a non-constant meromorphic function on it. Let f be a non-constant meromorphic function on an algebraic curve X. Then one can also consider the field $\mathbb{C}(f)$ of rational expressions of f with coefficients in $\mathbb{C}$, which is a subfield of the field $\mathcal{M}(X)$ of meromorphic functions on X. One can use the Laurent series approximation theorem in order to compute the degree of the extension $\mathcal{M}/\mathbb{C}(f)$.

But let me first make the previous definitions more exact. I will follow for the rest of this chapter mostly reference [125]. The presentation there can be completed with original texts, like [126]-[129].

We call a function $f : X \to \mathbb{C}$ meromorphic at $p \in X$ if it is either holomorphic, has a removable singularity or has a pole at p.

Given a function $D : X \to \mathbb{Z}$ on a Riemann surface X, the set of all p such that $D(p) \neq 0$ is called the support of D. A divisor on X is a function $D : X \to \mathbb{Z}$ whose support is a discrete subset of X.

A divisor can be denoted as

$$D = \sum_{p \in X} D(p)p \tag{5.4}$$

In the case of a meromorphic function $f : X \to \mathbb{C}$, the divisor is defined by the order function

$$(f) = \sum_{p \in X} ord_p(f).p \tag{5.5}$$

The divisor of poles of f is

$$(f)_\infty = \sum_{p,\ ord_p(f)<0} (-ord_p(f)).p \tag{5.6}$$

We call a divisor "effective"[125], [126] if all the coefficients in its formal sum are non-negative. With this definition we can write $D \geq D'$ if the formal difference $D - D'$ is effective. The term $ord_p(f)$ is a valuation of the function f at the point p. A valuation gives a measure of the multiplicity of elements in a field. Here, the notion is related to the poles and zeroes of meromorphic functions. There the valuation is given by the degree of a pole or the multiplicity of a zero. However, this concept can be generalized considerably. For example the degree of divisibility of a number by a prime number in number theory is also a valuation associated to that number. The geometric concept of contact between two algebraic or analytic varieties is also an analogue in the context of algebraic geometry. Let $\mathcal{M}(X)$ be the field of meromorphic functions on X. The space of meromorphic functions with poles bounded by D is given by the following expression representing the set of meromorphic functions

$$L(D) = \{f \in \mathcal{M}/(f) \geq -D\} \tag{5.7}$$

where $(f) \geq -D$ means in fact $(f)(p) \geq -D(p)$ for every $p \in D$.

5.3 Proposition Let X be a compact Riemann surface and let D be a divisor on X. Then the space $L(D)$ is a finite dimensional complex vector space. When we write $D = P - N$, with P and N nonnegative divisors with disjoint supports then $dim(L(D)) \leq 1 + deg(P)$. In particular, if D is a nonnegative divisor then

$$dim(L(D)) \leq 1 + deg(D) \tag{5.8}$$

One defines a meromorphic differential [127] on an open set $V \subseteq \mathbb{C}$ as an expression of the form $\omega = f(z)dz$ where f is a meromorphic function on V. Let $\omega_1 = f(z)dz$ and $\omega_2 = g(w)dw$ two meromorphic differentials defined on V_1 and V_2. Let T be a holomorphic function from V_2 to V_1. It is said that ω_1 transforms to ω_2 under T if

$$g(w) = f(T(w))T'(w) \tag{5.9}$$

If X is a Riemann surface, a meromorphic 1-form on X is a collection of meromorphic differentials (ω_ϕ), one for each chart $\phi : U \to V$ defined on V such that if two charts $\phi_1 : U_1 \to V_1$ and $\phi_2 : U_2 \to V_2$ have overlapping domains then ω_{ϕ_1} transforms to ω_{ϕ_1} under the transition function $\phi_1 \circ \phi_2^{-1}$. The notion of holomorphic 1-form is defined analogously. Given a meromorphic 1-form ω on X for each p choose a local coordinate z_p centered at p. We may write $\omega = f(z_p)dz_p$ where f is a meromorphic function at $z = 0$. The order of p denoted by $ord_p(\omega)$ is the order of the function f at $z_p = 0$. One can express ω via the Laurent series of f in the coordinate z_p

$$\omega = f(z_p)dz_p = (\sum_{n=-M}^{\infty} c_n z_p^n)dz_p \tag{5.10}$$

where $c_{-M} \neq 0$ so that $ord_p(\omega) = -M$. The residue of ω at p, denoted by $Res_p(\omega)$ is the coefficient c_{-1} in a Laurent series for ω at p. One can then state the following

5.4 Theorem(The residue Theorem) Let ω be a meromorphic 1-form on a compact Riemann surface X. Then

$$\sum_{p\in X} Res_p(\omega) = 0 \tag{5.11}$$

The divisor of ω denoted by (ω) is the divisor defined by the order function

$$(\omega) = \sum_p ord_p(\omega).p \tag{5.12}$$

Any divisor of this form is called a canonical divisor.

5.5 Proposition If X is a compact Riemann surface of genus g which has a non-constant meromorphic function then there is a canonical divisor on X of degree $2g-2$.

If we consider that we have a divisor D on X the space of meromorphic 1-forms with poles bounded by D, denoted $L^{(1)}(D)$ is the set of meromorphic 1-forms

$$L^{(1)}(D) = \{\omega/(\omega) \geq -D\} \tag{5.13}$$

$L^{(1)}(D)$ is a complex vector space. The space of holomorphic 1-forms can be written as $L^{(1)}(0) = \Omega(X)$. The relation between the spaces $L(-)$ and $L^{(1)}(-)$ is given by the following

5.6 Proposition Let D be a divisor on X and let K be a canonical divisor on X. The spaces $L^{(1)}(D)$ and $L(D+K)$ are isomorphic.

In what follows I will prove the Laurent series approximation. We can always find a meromorphic function g on X such that $ord_p(g) = 1$. Then if we set $f = g^N$ we get the following

5.7 Lemma Let X be an algebraic curve and let $p \in X$. Then for any integer N there is a global meromorphic function f on X with $ord_p(f) = N$.

A Laurent polynomial $r(z) = \sum_{i=n}^{m} c_i z^i$ is called a Laurent tail of a Laurent series $h(z)$ if the Laurent series starts with $r(z)$.

5.8 Lemma Let X be an algebraic curve. Fix a point $p \in X$ and a local coordinate z centered at p. Fix any Laurent polynomial $r(z)$ in z. Then there exists a meromorphic function f on X whose Laurent series at p has $r(z)$ as a Laurent tail.

5.9 Lemma Let X be an algebraic curve. Then for any finite number of points $p, q_1, ..., q_n$ in X there is a meromorphic function f on X with a zero at p and a pole at each q_i.

5.10 Lemma Let X be an algebraic curve. Then for any finite number of points $p, q_1, ..., q_n$ in X and any $N \geq 1$ there is a global meromorphic function f on X with $ord_p(f-1) \geq N$ and $ord_{q_i}(f) \geq N$ for each i.

5.11 Theorem (Laurent Series Approximation) Suppose X is an algebraic curve. Fix a finite number of points $p_1, ..., p_n$ in X, choose a local coordinate z_i at each p_i and finally choose Laurent polynomials $r_i(z_i)$ for each i. Then there is a meromorphic function f on X such that for every i, f has r_i as a Laurent tail at p_i.

5.12 Corollary Let X be an algebraic curve. Fix a finite number of points $p_i, ..., p_n$ in X and a finite number of integers m_i. Then there exists a meromorphic function f on X such that $ord_p(f) = m_i$ for each i.

In what follows I will prove the Riemann-Roch theorem for algebraic curves and present two possible generalizations: the Hirzebruch Riemann Roch theorem and the Grothendieck Riemann Roch [128] theorem. In order to do this I will start with the study of the so called Laurent tail divisors. These are a set of sums that can be related to ordinary divisors via an operation called "truncation". In this way it is possible to define a group $\mathcal{T}[D](X)$ where X is an algebraic curve formed by Laurent tail divisors that are bounded by a divisor D. It is also possible to relate meromorphic functions to Laurent tail divisors. Every meromorphic function has a corresponding element in $\mathcal{T}[D]$, the correspondence being made via a group homomorphism $\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X)$. This map is important for the Riemann-Roch theorem. This theorem gives us a formula to compute the dimension of the space $L(D)$ which turns out to be the kernel of the homomorphism α_D. It is possible to prove a very important result called the Serre Duality in order to give a way to compute the dimension of the cokernel of α_D. In this way I can obtain a refined version of the Riemann-Roch theorem which allows the calculation of the dimension of $L(D)$ in terms of the degree of D and the genus of X only. In order to start let X be a compact Riemann surface. For each point $p \in X$ choose a local coordinate z_p centered at p. A Laurent tail divisor on X is a finite formal sum of the form

$$\sum_p r_p.p \tag{5.14}$$

where $r_p(z_p)$ is a Laurent polynomial in the coordinate z_p. The set of Laurent tail divisors forms a group under formal addition, denoted $\mathcal{T}$. There are some special subgroups of $\mathcal{T}(X)$: for any divisor D, we can define $\mathcal{T}[D](X)$ as the set of all finite formal sums $\sum_p r_p.p$ such that for all p with $r_p \neq 0$ the top term of r_p has degree strictly less than $-D(p)$. Note that $\mathcal{T}[D](X)$ is a subgroup of $\mathcal{T}(X)$. I shall construct group homomorphisms concerning the previous groups. Let there be a Laurent tail divisor

$\sum_p r_p.p$ and a divisor D. At every point p we have

$$r_p(z_p) = \sum_{i=n_p}^{m_p} a_i^p z_p^i \tag{5.15}$$

Let i_D be the smallest integer between n_p and m_p such that $i_D + 1 \geq -D(p)$. We can define a truncation of $r_p(z_p)$ as

$$\sum_{i=n_p}^{m_p} a_i^p z_p^i \to \sum_{i=n_p}^{i_D} a_i^p z_p^i \tag{5.16}$$

This mapping defines a group homomorphism

$$t_D : \mathcal{T} \to \mathcal{T}[D](X) \tag{5.17}$$

that sends each $\sum_p r_p.p$ to $\sum_p \hat{r}_p.p$ where $\hat{r}_p$ denotes the truncation of r_p whose top term is the largest integer between n_p and m_p strictly smaller than $-D(p)$. In other words, t_D is defined by removing from each $r_p(z_p)$ those terms of degree greater or equal to $-D(p)$. Now suppose we have two divisors D_1 and D_2 such that $D_1 \leq D_2$ or, equivalently, $D_1(p) \leq D_2(p)$ for every $p \in X$. Then $-D_2(p) \leq -D_1(p)$ for every $p \in X$. Let $\sum_p r_p.p \in \mathcal{T}[D_1](X)$. For each p with $r_p \neq 0$ consider the Laurent polynomial $r_p(z_p) = \sum_{i=n_p}^{m_p} a_i^p z_p^i$, where $m_p < -D_1(p)$. Suppose there exists an integer between n_p and m_p greater or equal than $-D_2(p)$ and let $i_{D_2} + 1$ be the smallest of such integers. Then we can truncate $r_p(z_p)$

$$\sum_{i=n_p}^{m_p} a_i^p z_p^i \to \sum_{i=n_p}^{i_{D_2}} a_i^p z_p^i \tag{5.18}$$

If such an i_{D_2} does not exist then just map $\sum_{i=n_p}^{m_p} a_i^p z_p^i$ to 0. This truncation defines a group homomorphism

$$t_{D_2}^{D_1} : [D_1](X) \to \mathcal{T}[D_2](X) \tag{5.19}$$

defined by removing from each $r_p(z_p)$ those terms of degree greater or equal than $-D_2(p)$. We shall call the maps $t_{D_2}^{D_1}$ truncation maps. Consider a meromorphic function f and a divisor D. Let $\sum_p r_p.p$ be a Laurent tail in $\mathcal{T}[D](X)$. Let $\sum_{i=n_p}^{\infty} a_i z_p^i$ be the Laurent series of f in the coordinate z_p and let $r_p(z_p) = \sum_{j=m_p}^{k} b_j z_p^j$. We take the product $(\sum_{i=n_p}^{\infty} a_i z_p^i)(\sum_{j=m_p}^{k} b_j z_p^j) = \sum_{i=n_p,j=m_p}^{\infty} a_i b_j z_p^{i+j}$ and truncate it by removing those terms of degree greater or equal to $-D(p) + (f)(p)$. This gives rise to a group homomorphism

$$\mu_f^D : \mathcal{T}[D](X) \to \mathcal{T}[D - (f)](X) \tag{5.20}$$

mapping each $\sum_p r_p.p$ to $\sum_p (fr_p).p$ where $f.r_p$ is the Laurent polynomial of the above truncation of the series $\sum_{i=n_p, j=m_p} a_i b_j z_p^{i+j}$. It is straightforward to check that μ_f^D has an inverse $\mu_{1/f}^{D-(f)}$. Now consider again the Laurent series of $f : \sum_{i=n_p}^{\infty} a_i z_p^i$. Given a divisor D there exists a smallest integer m_p such that $m_p + 1 \leq -D(p)$. Then we can truncate the previous series and get a Laurent polynomial $r_p(z_p) = \sum_{i=n_p}^{m_p} a_i z_p^i$. This way we get a map

$$\alpha_D : \mathcal{M}(X) \to \mathcal{T}[D](X) \tag{5.21}$$

which turns out to be a group homomorphism.

5.13 Proposition (Properties of α_D).

- α_D commutes with the truncation maps: if D_1 and D_2 are divisors with $D_1 \leq D_2$ then

$$\begin{gathered} \mathcal{M}(X) \xrightarrow{\alpha_{D_1}} \mathcal{T}[D_1](X) \xrightarrow{t_{D_2}^{D_1}} \mathcal{T}[D_2](X) \\ \mathcal{M}(X) \xrightarrow{\alpha_{D_2}} \mathcal{T}[D_2](X) \end{gathered} \tag{5.22}$$

- α_D is compatible with the multiplication operators: if f and g are meromorphic functions on X then

$$\mu_f^D(\alpha_D(g)) = \alpha_{D-(f)}(f.g) \tag{5.23}$$

  for any divisor D.

- $L(D) = Ker(\alpha_D)$

The proof of this result can be found in [125]. At this moment we have a result relating $Ker(\alpha_D) = L(D)$. In what follows I need some results about the co-kernel $coker(\alpha_D)$. Now let us define

$$H^1(D) := coker(\alpha_D) = \mathcal{T}[D](X)/Im(\alpha_D) \tag{5.24}$$

I shall prove that the space $H^1(D)$ is finite dimensional. Knowing that $L(D) = ker(\alpha_D)$ we have a short exact sequence

$$0 \to L(D) \to \mathcal{M}(X) \xrightarrow{\alpha_D} \mathcal{T}[D](X) \to H^1(D) \to 0 \tag{5.25}$$

Consider now the quotient space $\mathcal{M}/L(D)$. As α_D vanishes on $L(D)$ we can consider the map

$$\bar{\alpha}_D : \mathcal{M}(X)/L(D) \to \mathcal{T}[D](X) \tag{5.26}$$

given by $\bar{\alpha}_D(f + L(D)) = \alpha_D(f)$. Now it becomes clear that it is a well defined group monomorphism. Also note that $Im(\bar{\alpha}_D) = Im(\alpha_D) = ker(\mathcal{T}[D](X) \to H^1(D))$. Then we obtain the following short exact sequence

$$0 \to \mathcal{M}(X)/L(D) \to \mathcal{T}[D](X) \to H^1(D) \to 0 \qquad (5.27)$$

Now let D_1 and D_2 be two divisors satisfying $D_1 \leq D_2$. Then we have a truncation map $t^{D_1}_{D_2} : \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$. Also $L(D_1) \subseteq L(D_2)$. For each D_i there is a short exact sequence as above. We shall connect these sequences constructing two homomorphisms.

- Let $F : \mathcal{M}(X)/L(D_1) \to \mathcal{M}(X)/L(D_2)$ be the map given by $F(f + L(D_1)) = f + L(D_2)$. This map is well defined. For if $f - g \in L(D_1)$ then $(f) - (g) \geq -D_1 \geq -D_2$ hence $f - g \in L(D_2)$. Also it is clear that F is a group homomorphism.
- Define now the map $G : H^1(D_1) \to H^1(D_2)$ by $G(Z + Im(\alpha_{D_1})) = t^{D_1}_{D_2}(Z) + Im(\alpha_{D_2})$. It is possible to check that G is well defined. Suppose $Z - Z' \in Im(\alpha_{D_1})$. Then $Z - Z' = \alpha_{D_1}(f)$ for some meromorphic function f. By the first part of the previous proposition we have

$$t^{D_1}_{D_2}(Z) - t^{D_1}_{D_2}(Z') = t^{D_1}_{D_2}(Z) \circ \alpha_{D_1}(f) = \alpha_{D_2}(f) \in Im(\alpha_{D_2}) \qquad (5.28)$$

One can see that G is a group homomorphism.

5.14 Proposition The following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}(X)/L(D_1) & \xrightarrow{\bar{\alpha}_{d_1}} & \mathcal{T}[D_1](X) & \xrightarrow{\gamma_{D_1}} & H^1(D_1) & \longrightarrow & 0 \\ & & {\scriptstyle F}\downarrow & & {\scriptstyle t^{D_1}_{D_2}}\downarrow & & {\scriptstyle G}\downarrow & & \\ 0 & \longrightarrow & \mathcal{M}(X)/L(D_2) & \xrightarrow{\bar{\alpha}_{D_2}} & \mathcal{T}[D_2](X) & \xrightarrow{\gamma_{D_2}} & H^1(D_2) & \longrightarrow & 0 \end{array} \qquad (5.29)$$

Here γ_{D_1} and γ_{D_2} are the projection maps.

Proof see ref. [125]

Let now $H^1(D_1/D_2) := ker(G)$. We use this diagram to show that $H^1(D_1/D_2)$ is finite dimensional. This fact will be important in the proof of the finite dimensionality of $H^1(D)$. By making use of the above diagram and the Snake Lemma we obtain the following exact sequence

$$0 \to ker(F) \to ker(t^{D_1}_{D_2}) \to ker(G) \to coker(F) \to coker(t^{D_1}_{D_2}) \to coker(G) \to 0 \quad (5.30)$$

Observe that F is surjective. We shall see that so are $t^{D_1}_{D_2}$ and G. Let $Z = \sum_p r_p.p \in \mathcal{T}[D_2](X)$ where $r_p(z_p) = \sum_{i=n_p}^{m_p} a^p_i z^i_p$ and $m_p \geq n_p$ is the largest integer with $m_p <$

$-D_2(p) \leq -D_1(p)$. So $Z \in \mathcal{T}[D_1](X)$ and $Z = t^{D_1}_{D_2}(Z)$. Hence $t^{D_1}_{D_2}$ is surjective and this implies that so is G. On the other hand note that $ker(F) = L(D_2)/L(D_1)$. So we have

$$dim(ker(F)) = dim(L(D_2)) - dim(L(D_1)) \tag{5.31}$$

Now consider $\sum_p r_p.p \in ker(t^{D_1}_{D_2})$ where $r_p(z_p) = \sum_{i=n_p}^{m_p} a^p_i z^i_p$. Since $\sum_p a^p_i z^i_p$ is mapped to 0 we have $n_p \geq -D_2(p)$. Hence $ker(t^{D_1}_{D_2})$ is the space of all $\sum_p r_p.p$ such that the top term of r_p has order less than $-D_1(p)$ and the bottom term has order at least $-D_2(p)$. At each p we have $D_2(p) - D_1(p)$ possible monomials z^i_p, $-D_2(p) \leq i \leq -D_1(p)$ that are linearly independent. Hence

$$dim(ker(t^{D_1}_{D_2})) = \sum_p (D_2(p) - D_1(p)) = \sum_p (D_2(p)) - \sum_p (D_1(p)) = deg(D_2) - deg(D_1) \tag{5.32}$$

Remember that we have a short exact sequence

$$0 \to L(D_2)/L(D_1) \to ker(t^{D_1}_{D_2}) \to H^1(D_1/D_2) \to 0 \tag{5.33}$$

Since $H^1(D_1/D_2)$ is a free $\mathbb{C}$-module the previous sequence splits so

$$ker(t^{D_1}_{D_2}) \cong (L(D_2)/L(D_1)) \bigoplus H^1(D_1/D_2) \to 0 \tag{5.34}$$

It follows that

$$\begin{gathered} dim(ker(t^{D_1}_{D_2})) = dim(L(D_2)/L(D_1)) + dim(H^1(D_1/D_2)) \\ dim(H^1(D_1/D_2)) = deg(D_2) - deg(D_1) + dim(L(D_1)) - dim(L(D_2)) \end{gathered} \tag{5.35}$$

Hence it follows that $H^1(D_1/D_2)$ is finite dimensional. In conclusion we have

5.15 Lemma If D_1 and D_2 are divisors on a compact Riemann surface X with $D_1 \leq D_2$ then

$$dim(H^1(D_1/D_2)) = [deg(D_2) - dim(L(D_2))] - [deg(D_1) - dim(L(D_1))] \tag{5.36}$$

5.16 Proposition For any divisor D on an algebraic curve X, $H^1(D)$ is a finite dimensional vector space over $\mathbb{C}$.

5.17 Lemma Let f be a nonconstant meromorphic function on an algebraic curve X and let $D = (f)_\infty$. Then for any large m the dimension of $H^1(0/mD)$ is constant independent of m.

5.18 Lemma For any algebraic curve X and for any divisor A on X there is an integer M such that

$$deg(A) - dim(L(A)) \leq M \tag{5.37}$$

Note that the previous lemma implies that there exists a divisor A_0 on X such that the difference $deg(A_0) - dim(L(A_0))$ is maximal.

5.19 Lemma $H^1(A_0) = 0$

So far, these lemmas prove that for any two divisors D_1 and D_2 with $D_1 \leq D_2$ there is a short exact sequence

$$0 \to H^1(D_1/D_2) \to H^1(D_1) \to H^1(D_2) \to 0 \tag{5.38}$$

of finite dimensional vector spaces where $H^1(D_1) \cong H^1(D_1/D_2) \bigoplus H^1(D_2)$ since this sequence splits. It follows

$$dim(H^1(D_1/D_2)) = [deg(D_2) - dim(L(D_2))] - [deg(D_1) - dim(L(D_1))] \tag{5.39}$$

So we get

$$deg(D_2) - dim(L(D_2)) - deg(D_1) + dim(L(D_1)) = dim(H^1(D_1)) - dim(H^1(D_2))$$

$$dim(L(D_1)) - deg(D_1) - dim(H^1(D_1)) = dim(L(D_2)) - deg(D_2) - dim(H^1(D_2)) \tag{5.40}$$

Now let D_1 and D_2 be any divisors on X and let D be a common maximum of D_1 and D_2. Then

$$dim(L(D_1)) - deg(D_1) - dim(H^1(D_1)) = dim(L(D)) - deg(D) - dim(H^1(D)) =$$

$$= dim(L(D_2)) - deg(D_2) - dim(H^1(D_2)) \tag{5.41}$$

It follows that the integer $dim(L(D)) - dim(D) - dim(H^1(D))$ is constant and it is equal to

$$dim(L(0)) - deg(0) - dim(H^1(0)) = 1 - dim(H^1(0)) \tag{5.42}$$

and in consequence, putting all together we have

5.20 Theorem(The first form of the Riemann-Roch theorem) Let D be a divisor on an algebraic curve X. Then

$$dim(L(D)) - dim(H^1(D)) = deg(D) + 1 - dim(H^1(0)) \tag{5.43}$$

This first form of the Riemann-Roch theorem was not final. The associated formula considered three spaces : $L(D)$, $H^1(D)$ and $H^1(0)$. In what follows we wish to find expressions for $dim(H^1(D))$ and $dim(H^1(0))$. Such expressions will be consequences of a result called Serre Duality. This duality states that the space of meromorphic 1-forms with poles bounded by $-D$, $L^{(1)}(-D)$ and the dual space to $H^1(D)$ are isomorphic. We first construct a linear map $L^{(1)}(-D) \to H^1(D)^*$ which we shall call the Residue map. Suppose D is a divisor on X and ω a meromorphic 1-form on X in the space $L^{(1)}(-D)$ such that $ord_p(\omega) \geq D(p)$ for all $p \in X$. It follows that we can write

$$\omega = (\sum_{n=D(p)}^{\infty} c_n z_p^n) dz_p \tag{5.44}$$

in a local coordinate z_p about p for each p. We define the following linear map:

$$\begin{aligned} Res_\omega &= \mathcal{T}[D](X) \to \mathcal{C} \\ \textstyle\sum_p r_p.p &\to \textstyle\sum_p Res_p(r_p.\omega) \end{aligned} \tag{5.45}$$

Suppose f is a meromorphic function on X. Write $f = \sum_k a_k z_p^k$ in the coordinate z_p. Near p we have

$$f\omega = (\sum_k a_k z_p^k)(\sum_{n=D(p)}^{\infty} c_n z_p^n) dz_p \tag{5.46}$$

The coefficient of $1/z_p$ is given by $\sum_{n=D(p)}^{\infty} c_n a_{-n-1}$. So

$$Res_p(f\omega) = \sum_{n=D(p)}^{\infty} c_n a_{-n-1} \tag{5.47}$$

The expression $Res_p(f\omega)$ depends only on the Laurent tail divisor $\alpha_D(f) = \sum_p r_p.p$ since $\sum_{n=D(p)}^{\infty} c_n a_{-n-1}$ depends on the coefficients a_i for f with $i < -D(p)$. Then we have

$$Res_\omega(\alpha_D(f)) = \sum_p Res_p(f\omega) \tag{5.48}$$

By the Residue Theorem we get $Res_\omega(\alpha_D(f)) = 0$. This means that the previous map Res_ω descends to a map

$$Res_\omega : H^1(D) \to \mathbb{C} \tag{5.49}$$

in $H^1(D)^*$. Therefore we obtain a linear map

$$\begin{aligned} Res : L^{(1)}(-D) &\to H^1(D)^* \\ \omega &\to Res_\omega \end{aligned} \tag{5.50}$$

5.21 Theorem (Serre Duality) For any divisor D on an algebraic curve X the map

$$Res : L^{(1)}(-D) \to H^1(D)^* \tag{5.51}$$

is an isomorphism of complex vector spaces.

In what follows I will state a set of lemmas and propositions required for the understanding of the full Riemann-Roch theorem

5.22 Proposition

- If $\phi : \mathcal{T}[D](X) \to \mathbb{C}$ is a linear function that vanishes on $\alpha_D(\mathcal{M}(X))$ and f is any meromorphic function on X then $\phi \circ \mu_f^{D+(f)} : \mathcal{T}[D+(f)](X) \to \mathbb{C}$ is also a linear map vanishing on $\alpha_{D+(f)}(\mathcal{M}(X))$.
- Res_ω is compatible with the multiplication map μ_f: Suppose f is a meromorphic function on X and $\omega \in L^{(-1)}(-D)$. Then $f\omega \in L^{(1)}(-D-(f))$ and

$$Res_\omega \circ \mu_f^{D+(f)} = Res_{f\omega} \tag{5.52}$$

as functionals on $\mathcal{T}[D+(f)](X)$.

5.23 Lemma Suppose that ϕ_1 and ϕ_2 are two linear functionals on $H^1(A)$ for some divisor A. Then there is a positive divisor C and nonzero meromorphic functionals f_1 and f_2 in $L(C)$ such that

$$\phi_1 \circ t_A^{A-C-(f_1)} \circ \mu_{f_1} = \phi_2 \circ t_A^{A-C-(f_2)} \circ \mu_{f_2} \tag{5.53}$$

as functionals on $H^1(A-C)$.

5.24 Lemma Suppose that D_1 is a divisor on X with $\omega \in L^{(1)}(-D_1)$ so that $Res_\omega : \mathcal{T}[D_1](X) \to \mathbb{C}$ is well defined. Suppose that $D_2 \geq D_1$ and that Res_ω vanishes on the kernel of $t^{D_1}_{D_2} = \mathcal{T}[D_1](X) \to \mathcal{T}[D_2](X)$. Then $\omega \in L^1(-D_2)$

In order to continue now fix a canonical divisor $K = (\omega)$ for some meromorphic 1-form ω on X. We know that $L^{(1)}(-D)$ and $L(K-D)$ are isomorphic. Hence by Serre Duality we get

$$dim(H^1(D)) = dim(L^1(-D)) = dim(L(K-D)) \tag{5.54}$$

Finally we show that $dim(H^1(0)) = g$. Let K' be a canonical divisor on X of degree $2g-2$ which we know to exist. By Serre Duality we have

$$\begin{gathered} dim(H^1(0)) = dim(L(K'-0)) = dim(L(K')) \\ dim(H^1(K')) = dim(L(K'-K')) = dim(L(0)) = 1 \end{gathered} \tag{5.55}$$

By the first form of the Riemann Roch theorem and the previous equality we have

$$\begin{gathered} dim(L(K')) - dim(H^1(K')) = deg(K') + 1 - dim(H^1(0)) \\ dim(H^1(0)) - 1 = 2g - 2 + 1 - dim(H^1(0)) \\ dim(H^1(0)) = g \end{gathered} \tag{5.56}$$

In this way we obtained a more refined form of the Riemann-Roch theorem

5.25 Theorem(Riemann-Roch) Let X be an algebraic curve of genus g. Then for any divisor D and any canonical divisor K we have

$$dim(L(D)) - dim(L(K-D)) = deg(D) + 1 - g \tag{5.57}$$

There are several generalizations of the Riemann-Roch theorem. The first is a reformulation of the theorem for holomorphic line bundles. Let L be a holomorphic line bundle on a compact Riemann surface X of genus g and let $\Gamma(X, L)$ denote the space of holomorphic sections of L. This space is finite dimensional. Let K denote the canonical bundle on X i.e. $K = \Lambda^n(T^\wedge)$ where $T^\wedge$ is the cotangent bundle and $n = dim(X)$. Now consider a general section $\sigma : X \to L$. We can produce such a section by giving it locally and then glueing it together using a partition of unity. Locally, the line bundle L is trivial so it looks like $\mathbb{C} \times \Delta \to \Delta$ where Δ is the open unit disk. In this picture,

σ is just a map $\Delta \to \mathbb{C}$. Locally the inverse image of $0 \in \mathbb{C}$ under the map $\sigma : \Delta \to \mathbb{C}$ has a finite number of points. Each point $p \in X$ where σ intersects the zero section is called a zero of σ. Around each such point p the section σ is a map $\sigma : \Delta \to \mathbb{C}$ where $p = 0 \in \Delta$ and $\sigma(0) = 0$. The differential $T_p\sigma : T_0 \to T_0\mathbb{C}$ is a nonsingular two-by-two matrix. Let $sgn(p)$ denote the sign of the determinant of this matrix. The degree of L is defined by $deg(L) = \sum_{p \in X} sgn(p)$ where the sum is over all points p where a section σ is zero. The Riemann-Roch Theorem for holomorphic line bundles goes as follows

5.26 Theorem If L is a holomorphic line bundle on X and K is the canonical bundle on X then

$$dim(\Gamma(X, L)) - dim(\Gamma(X, L^{-1} \otimes K)) = deg(L) + 1 - g \tag{5.58}$$

The Hirzebruch-Riemann-Roch theorem is a result that contributes to the Riemann-Roch problem for complex algebraic varieties of all dimensions. The theorem applies to any holomorphic vector bundle E on a compact complex manifold X, to calculate the holomorphic Euler characteristic of E in sheaf cohomology, namely the alternating sum

$$\chi(X, E) = dim(H^0(X, E)) - dim(H^1(X, E)) + dim(H^2(X, E)) - ... \tag{5.59}$$

of the dimensions as complex vector spaces. Hirzebruch's theorem states that $\chi(X, E)$ is computable in terms of the Chern classes of E and the Todd classes of the tangent bundle of X.

Let for example E be a vector bundle of rank r. We can associate to it the Chern polynomial

$$c(E) = 1 + c_1(E) + c_2(E) + ... + c_r(E) \tag{5.60}$$

and the Chern roots, which are the formal roots of $c(E)$ i.e.

$$c(E) = \prod_{i=1}^{r}(1 + \alpha_i) \tag{5.61}$$

The Chern character of E is then rapidly defined as

$$ch(E) = \sum_{i=1}^{r} e^{\alpha_i} \tag{5.62}$$

The Chern character is symmetric in terms of the Chern roots and therefore it can be expressed in terms of Chern classes. It is possible to show that

$$\begin{aligned} ch(E) = r + c_1(E) + \tfrac{c_1^2(E)-2\cdot c_2(E)}{2} + \tfrac{c_1^3(E)-3c_1(E)\cdot c_2(E)+3c_3(E)}{6} + \\ + \tfrac{c_1^4(E)+4c_1(E)\cdot c_3(E)-4c_1^2(E)\cdot c_2(E)+2c_2^2(E)-4c_4(E)}{24} + ... \end{aligned} \quad (5.63)$$

The Chern character is a homomorphism from the Grothendieck K-group to cohomology. In that case it satisfies

$$ch(E \otimes F) = ch(E) \cdot ch(F) \quad (5.64)$$

This appears relatively straightforward. The Todd class is similarly defined as a formal power series in the Chern roots

$$Td(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - e^{-\alpha_i}} \quad (5.65)$$

It is also symmetric in the Chern roots, having an expression in terms of Chern classes given by

$$\begin{aligned} Td(E) = 1 + \tfrac{c_1(E)}{2} + \tfrac{c_1^2(E)+c_2(E)}{12} + \tfrac{c_1(E)\cdot c_2(E)}{24} \\ + \tfrac{-c_1^4(E)+4c_1^2(E)c_2(E)+c_1(E)c_3(E)+3c_2^2(E)-c_4(E)}{720} + ... \end{aligned} \quad (5.66)$$

The Todd class represents the information encoded in the tangent bundle. Otherwise stated the Todd class of a variety is the Todd class of its tangent bundle

$$Td(X) = Td(T_X) \quad (5.67)$$

Given a short exact sequence

$$0 \to E \to F \to G \to 0 \quad (5.68)$$

then $Td(F) = Td(E)Td(G)$.

5.27 Theorem(Hirzebruch-Riemann-Roch) Let E be a holomorphic vector bundle on a compact complex manifold X. Then using the Chern character $ch(E)$ in cohomology and the Todd class $Td(X)$ one can prove that

$$\chi(X, E) = \int_X ch(E)Td(X) \quad (5.69)$$

Finally there is another generalization of the Riemann-Roch theorem due to Grothendieck. Let X be a smooth quasiprojective scheme over a field. The Grothendieck group $K_0(X)$ of bounded complexes of coherent sheaves is canonically isomorphic to the Grothendieck group of bounded complexes of finite-rank vector bundles. Using this isomrophism, consider the Chern character as a functorial transformation

$$ch : K_0(X) \to A_d(X, \mathbb{Q}) \tag{5.70}$$

where $A_d(X, \mathbb{Q})$ is the Chow group of cycles on X of dimension d modulo rational equivalence, tensored with the rational numbers. Now consider a proper morphism $f : X \to Y$ between smooth quasi-projective schemes and a bounded complex of sheaves $\mathcal{F}^*$. Let $td(X)$ be the Todd genus of the tangent bundle of X. We denote the i-right derived functor of the pushforward f_* by $R^i f_*$. The Grothendieck-Riemann-Roch Theorem goes as follows:

5.28 Theorem $ch(f_!\mathcal{F}^*)td(Y) = f_*(ch(\mathcal{F}^*)td(X))$, where

$$f_! : \sum(-1)^i R^i f_* K_0(X) \to K_0(Y) \tag{5.71}$$

and $f_* : A(X) \to A(Y)$.

This theorem has various applications, mainly in enumerative geometry where it offers a tool for calculating the the number of lines on a general cubic. It also allows the calculation of relations among classes on the moduli space of curves.

Chapter 6

The Atyiah Singer Index Theorem

"If I had a world of my own, everything would be nonsense. Nothing would be what it is, because everything would be what it isn't. And contrary wise, what is, it wouldn't be. And what it wouldn't be, it would. You see?"

Lewis Carroll, Alice in Wonderland

Another important theorem relating topology and algebra is the Atyiah-Singer index theorem [129]. In order to be able to discuss about it several concepts need to be introduced at this point.

Let X be a topological space and let E be a vector bundle over X of dimension n. The "twisting" of the bundle E is measured by certain cohomology classes called "characteristic classes" [130-133]. There are four basic types of characteristic classes

- Stiefel-Whitney classes $w_1, ..., w_n$, $w_i \in H^i(X, \mathbb{Z}_2)$, E being real
- Chern classes $c_1, ..., c_n$, $c_i \in H^{2i}(x)$, E being complex
- Pontryagin classes $p_1, ..., p_{n/2}$, $p_i \in H^{4i}$, E being complex
- Euler class, $e \in H^n(X)$, E real and orientable

Let me show briefly what these mean and where they come from. It has been observed that certain quantities constructed by employing the curvature Ω on a differentiable principal bundle $P(G, M)$ defined in terms of a Lie group G and a manifold M are determined by the bundle only and do not depend on the specific curvature Ω used for their definition. They are therefore characteristic of the bundle, preserved by continuous transformations with continuos inverses on the bundle i.e. bundle diffeomorphisms. Each corresponds to a topological invariant associated with it [134]. Extending a local product structure $(U \times G, U \subset M)$ to a global product $(M \times G)$ structure may present some obstructions i.e. the extension doesn't occur unless additional (global) information is added [135]. The topological invariants may be used to measure such obstructions. This means they detect in how far a local structure cannot be extended for the whole bundle. The characteristic classes are given by various closed differential forms on M i.e. differential forms for which the exterior derivative is zero. These can be defined in terms of so called invariant polynomials in the curvature of a connection [136] or by the de Rham cohomology generators defined by these forms on the base manifold M [137]. The property of being closed differential forms implies they are locally exact. This is valid on each chart $U_i \subset M$. By Stokes theorem their integrals over M depend only on the transition functions which contain the topological information on the bundle [138]. The characteristic class refers to the cohomology class of the de Rham equivalent characteristic forms. By integrating a class over the base manifold we obtain a so called characteristic number.

The Chern [139], Euler [140] and Pontrjagin [141] classes are all examples of characteristic classes, each encoding the properties for manifolds of a certain type. The Stiefel-Whitney classes w_i [142] are not de-Rham cohomology classes. They are therefore, strictly speaking, not given in terms of the curvature Ω. These classes are however important. Their annihilation represents a necessary and sufficient condition for the respective manifold M to be orientable or, respectively, to admit a spin structure [143].

The characteristic classes can be obtained by the Chern-Weil procedure [144]. This procedure can be resumed as follows: We first need to introduce the set $I^l(G)$ of symmetric polynomials invariant to the adjoint representation of the linear group G, noted AdG. One shows that $I(G) = \sum_{l=0}^{\infty} I^l(G)$ has a graded ring structure. We secondly remark that these invariant polynomials can be used to define closed forms $P(\bar{\Omega})$ on M. Their cohomology classes do not depend on the connection. Thirdly and finally we construct the algebra Weil homomorphism $I(G) \to H_{DR}^{even}(M)$ [145] which associates de Rham cohomology classes with the closed forms on M previously obtained.

6.1 Theorem (AdG-invariant symmetric multilinear mappings)[132] Let AdG be the adjoint representation of the Lie group G on the associated Lie algebra $\mathcal{G}$. A symmetric l-linear mapping (a polynomial on $\mathcal{G}$)

$$P : \mathcal{G}^l = \mathcal{G} \times ... \times \mathcal{G} \to R \tag{6.1}$$

is AdG invariant if

$$P(Adg(X_1), ..., Adg(X_l)) = P(X_1, ..., X_l) \; X_1, ..., X_l \in \mathcal{G} \tag{6.2}$$

The polynomial is then called a characteristic or invariant polynomial.

Now, let me consider the vector space of all these AdG-invariant polynomials and let me call it $I^l(G)$. Then consider

$$I(G) = \sum_{l}^{\infty} I^l(G) \tag{6.3}$$

To $I(G)$ can be attached a graded ring structure by defining the product of two polynomials $P \in I^l(G)$, $P' \in I^k(G)$, $PP' \in I^{l+k}(G)$ by

$$(PP')(X_1, ..., X_{l+k}) = \frac{1}{(k+l)!} \sum_{perm(\sigma)} P(X_{\sigma(1)}, ..., X_{\sigma(l)}) P'(X_{\sigma(l+1)}, ..., X_{\sigma(l+k)}) \tag{6.4}$$

We now observe that there exists a connection between $I(G)$ and the de Rham cohomology ring $H^{even}_{DR}(M)$, here M being the base manifold of $P(G, M)$. This requires the proof of a

6.2 Lemma(Forms β on P projectable to forms $\bar{\beta}$ on M)[132]

Let β be a q-form on the total space P of a principal bundle. Then it projects to (it is the pull-back $\pi^*(\bar{\beta})$ of) a unique form β on the base manifold M.

- $R_g^* \beta = \beta$, β is invariant under the right action of G
- $\beta(Y_1, ..., Y_q) = 0$ if any of the arguments is vertical

The proof of this lemma can be found in [146] or [132]. With this we can state now the following

6.3 Theorem(Chern-Weil)

Let $P(G, M)$ be a principal bundle endowed with a connection ω of curvature Ω. Give an invariant polynomial $P \in I^l(G)$ construct the $2l$-differential form $P(\Omega)$ on by

$$P(\Omega)(Y_1, ..., Y_{2l}) = \frac{1}{2l!} \sum_{perm(\sigma)} sign(\sigma) P[\Omega(Y_{\sigma(1)}, Y_{\sigma(2)}), ..., \Omega(Y_{\sigma(2l-1)}, Y_{\sigma(2l)})], \; \forall Y_1, ..., Y_{2l} \in T_p(P) \tag{6.5}$$

where $sign(\sigma)$ is the signature of the permutation σ. Then

- for each $P \in I^l(G)$ the $2l$-form $P(\Omega)$ on P projects to a unique closed $2l$-form on M denoted $P(\bar{\Omega})$ i.e. $\pi^*(P(\bar{\Omega})) = P(\Omega)$
- the element of the de Rham cohomology group characterized by the cohomology class of the form $P(\bar{\Omega})$ on M is independent of the choice of the connection (and thus $P(\Omega)$ has topologically invariant integrals)
- the mapping $I(G) \to H^{even}_{DR}(M)$ defined by $P \to$ (cohomology class of $P(\bar{\Omega})$) is an algebra homomorphism, the Weil homomorphism.

For the proof, I refer again to [132], [146].

We also have the Chern-Simons forms given by the following

6.4 Definition(The Chern-Simons form of $P(\Omega)$) [147] The Chern-Simons $(2l+1)$-form associated with the symmetric polynomial $P(\Omega)$ is the projection on M of the form $Q^{(2l-1)}(\omega, \omega_0)$ given by

$$\begin{aligned} P(\Omega, ..., \Omega) - P(\Omega_0, ..., \Omega_0) &= \int_0^1 dt \tfrac{d}{dt} P(\Omega_t, ..., \Omega_t) = \\ = l \int_0^1 dt P(\tfrac{d\Omega_t}{dt}, \Omega_t, ..., \Omega_t) &= l \int_0^1 dt P(\mathcal{D}_t \eta, \Omega_t, ..., \Omega_t) = \\ = l \int_0^1 dt \mathcal{D}_t P(\eta, \Omega_t, ..., \Omega_t) &= l \int_0^1 dt\, dP(\eta, \Omega_t, ..., \Omega_t) = \\ = d[l \int_0^1 dt P(\eta, \Omega_t, ..., \Omega_t)] &= dQ^{(2l-1)}(\omega, \omega_0) \end{aligned} \tag{6.6}$$

where here η is the difference between two connections on $P(G, M)$ $\eta = \omega - \omega_0$. We can define an interpolating one-parameter family of connections $\omega_t = \omega_0 + t\eta$. Also, here $\mathcal{D}_t$ is the exterior covariant differentiation and Ω_t is the curvature associated with ω_t. Its definition form can be written as

$$\Omega_t = d\omega_t + \frac{1}{2}[\omega_t, \omega_t] = \Omega_0 + t\mathcal{D}_0\eta + \frac{t^2}{2}[\eta, \eta] \tag{6.7}$$

The last term can be written as $t^2 \eta \wedge \eta$ and

$$\frac{d\Omega_t}{dt} = \mathcal{D}_0\eta + t[\eta, \eta] = d\eta + [\omega_t, \eta] = \mathcal{D}_t \eta \tag{6.8}$$

therefore $\frac{d\Omega_t}{dt}$ is the covariant derivative $\mathcal{D}_t$ of the tensorial form η.

The Chern-Simons form is then given by

$$Q^{2l-1}(\bar{\omega},\bar{\omega}_0) = TP(\bar{\omega},\bar{\omega}_0) = l\int_0^1 dtP(\bar{\omega}-\bar{\omega}_0,\bar{\Omega}_t,...,\bar{\Omega}_t) \tag{6.9}$$

we shall refer to this as the transgression formula.

Let me interpret this from a physical point of view. For this let me start from a basic example, namely the electromagnetic coupling. Consider $I[A,z] = \int_M j^\mu A_\mu d^4x$ as the electromagnetic coupling term with j^μ the current and A_μ the field. One important property of this coupling term is that it is gauge invariant, namely under transformations of the form $A_\mu(x) \to A_\mu(x) + d\Omega(x)$ the coupling term does not change. This of course is possible only if the current j^μ is conserved i.e. $\partial_\mu j^\mu = 0$. Therefore we remark the fact that the condition of gauge invariance for the minimal coupling of a gauge potential and a charged source requires that the source satisfies a conservation law. The coupling term can further be written as $\int_M *j \wedge A$ where $*j = e\cdot\delta(\Gamma)d\xi^1\wedge d\xi^2\wedge d\xi^3$ where here the components ξ^i represent transverse directions to the path Γ. In an even more simplified formulation the coupling term becomes $I[A,z] = e\int_\Gamma A$, where $A = A_\mu(z)dz^\mu$ where dz is the tangent vector to the path Γ and e is the coupling. Will this be gauge-invariant? It is obvious that A is not. When $A(x) \to A(x)+d\Omega(x)$ we have $I[A] \to I[A]+\Omega^{\infty}_{-\infty}$. Again gauge invariance requires an additional condition, this time imposed on the boundary conditions: $\Omega(\infty) = \Omega(-\infty)$. Therefore it is possible that

$$\delta I = e\int_\Gamma \delta A = e\int_\Gamma d\Omega = 0 \tag{6.10}$$

We observe that it is possible to have a non-gauge invariant integrand but to obtain a gauge invariant integral if the right boundary conditions are satisfied. However, in general we don't have only sources as the ones we encounter in electromagnetism. If we want to generalize the problem for non-abelian groups or for higher dimensional sources, this is where the Chern-Simons forms appear. The last century showed that all fundamental interactions in nature share the same form: they are all gauge theories, they all have a fiber bundle ($\mathcal{F}$) which locally can be written as $\mathcal{F} = M \times G$ where M is the manifold and G is the fiber's group. We also have a connection i.e. a Lie algebra valued 1-form $A_\mu = A^a_\mu J_a$ where A^a_μ is the interaction field and J_a is the Lie algebra generator. In the case of simple electrodynamics the action functional is

$$I[A] = \int_M (\frac{1}{2}F\wedge *F - j\wedge A) \tag{6.11}$$

where the field strength is

$$F = dA = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)dx^\mu \wedge dx^\nu \tag{6.12}$$

For Yang-Mills interactions (electro-weak and strong) we have the action

$$I[A] = \int_M (\frac{1}{2} F \wedge *F - j \wedge A) \tag{6.13}$$

where now A takes values in a non-abelian Lie algebra and

$$F = dA + A \wedge A = [\partial_\mu A^a_\nu + f^a_{bc} A^b_\mu A^c_\nu] J_a dx^\mu \wedge dx^\nu = F^a J_a \tag{6.14}$$

It is useful at this moment to enumerate some actions and the ways they generalize.

$$\begin{aligned} &I[A]_{YM/EM} = \tfrac{1}{4K} \int_{M^D} \sqrt{g} g^{\mu\alpha} g^{\nu\beta} \gamma_{ab} F^a_{\mu\nu} F^b_{\alpha\beta} d^D x \\ &I[A]_{CS} = K \int_{M^3} < A \wedge dA + \tfrac{2}{3} A \wedge A \wedge A > \\ &I[A]_{general} = K \int_{M^{2n+1}} < A \wedge (dA)^n + \alpha_1 A^3 \wedge (dA)^{n-1} + ... + \alpha_n A^{2n+1} > \end{aligned} \tag{6.15}$$

where in the last row α_i are fixed rational numbers. The Chern-Simons form is then

$$C_{2n+1} =< A \wedge (dA)^n + \alpha_1 A^3 \wedge (dA)^{n-1} + ... + \alpha_n A^{2n+1} >=< \tilde{C}_{2n+1} > \tag{6.16}$$

Their exterior derivatives are then the invariant polynomials (characteristic classes)

$$dC_{2n-1}(A) = P_{2n}(F) \tag{6.17}$$

Given a connection A under a Lie algebra then, by means of gauge transformations we have

$$A \to A' = g^{-1} A g + g^{-1} dg \ , \quad F \to F' = g^{-1} F g \tag{6.18}$$

Then the polynomial $P_{2n}(F) =< F^n >$ is invariant i.e. $P_{2n}(F') = P_{2n}(F)$ and closed i.e. $dP_{2n}(F) = 0$. Under a gauge transformation the Chern-Simons forms change like an abelian connection:

$$dC_{2n-1}(A) = P_{2n}(F) \to \delta C_{2n+1}(A) = d\Omega_{2n} \tag{6.19}$$

The Chern-Simons forms are also important because they provide us with a generalization of the coupling between a point charge and the electromagnetic field to higher dimensional objects like branes and non-abelian gauge fields. In order to obtain a generalization of the coupling term lets start by replacing the 1-form by a p-form and we obtain

$$I[A, z] = \int_M j_{\mu_1 ... \mu_p} A^{\mu_1 ... \mu_p} \to I[A, z] = \int_{\Gamma^p} A \tag{6.20}$$

For an abelian field $A \to A' = A + d\Omega$ we have $A_\mu \to A'_\mu = g^{-1}(A_\mu + \partial_\mu)g$ for $g \in G$. For a non-abelian field however once we have the gauge transformation in this form $A \to A' = g^{-1}Ag + g^{-1}dg$ it is not trivial to find an analogue for a p-form

$$A_{\mu_1 \dots \mu_p} \to A'_{\mu_1 \dots \mu_p} = g^{-1}(A_{\mu_1 \dots \mu_p} + X_{\mu_1 \dots \mu_p})g \tag{6.21}$$

In fact there is no natural expression for X. More precisely the p-form does not define a natural connection (covariant derivative). We also lack a non-abelian curvature, analogue for $F = dA + A \wedge A$. There appears to be no clear relation between current conservation and nonabelian gauge transformations. Fortunately, a gauge invariant coupling for any group G even for higher dimensional Γ exists and is given by a Chern-Simons form $C_{2n-1}(A)$

$$dC_{2n-1}(A) = P_{2n}(F) \tag{6.22}$$

Under a gauge transformation $C_{2n-1}(A)$ changes by a closed form

$$0 = \delta dC = d\delta C \tag{6.23}$$

and hence $\delta C = d\Omega$ i.e. we have local exactness. Following this discussion, it appears that the minimal coupling between a particle and the electromagnetic field can be generalized for arbitrary dimensions and non-abelian groups by the relation

$$I[A] = \int_M < *j_{2p+1}\tilde{C}_{2p+1}(A) > \tag{6.24}$$

This formula helps to the description of the coupling between an extended object like a $2p$-brane and a non-abelian connection. The coupling is invariant under gauge transformations if the current is conserved i.e. $D * j_{2p+1} = 0$.

Let me now continue the theoretical introduction started before this physical digression.

6.5 Definition(Chern forms, Chern classes) [132],[148] Let $\bar{\Omega}^i{}_j$ be the components of the curvature two form on the manifold M. Then the Chern forms $c_l(\bar{\Omega})$ on M are the closed $2l$-forms given by

$$c_l(\bar{\Omega}) = \frac{1}{l!}(\frac{i}{2\pi})^l \epsilon^{i_1 \dots i_l}_{j_1 \dots j_l} \bar{\Omega}^{j_1}_{\ i_1} \wedge \dots \wedge \bar{\Omega}^{j_l}_{\ i_1} \tag{6.25}$$

We call the total Chern form

$$c(\Omega) = det(1 + \frac{i}{2\pi}\Omega) \tag{6.26}$$

The Chern classes $c_l(E)$ of a complex vector bundle E are the elements of $H^{2l}_{DR}(M, R)$ determined by the cohomology classes of the closed Chern $2l$-forms $c_l(\bar{\Omega})$ on M that are the projections of the forms $P_l(\Omega)$ on P, $\pi^* c_l(\bar{\Omega}) = P_l(\Omega)$. The sum of all Chern classes is the total Chern class $c(E)$. When E is the tangent bundle $\tau(M)$ of a complex manifold, the Chern classes $c_l(\tau(M))$ are called Chern classes of the manifold M and are written $c_l(M)$.

6.6 Definition(Chern numbers) [132], [150] The Chern numbers are those obtained by integrating characteristic polynomials of degree $dim(M)$ over the entire manifold M.

For instance if $dim(M) = 4$ there are only two independent Chern numbers given by

$$C_1 = \int_M c_1(\bar{\Omega}) \wedge c_1(\bar{\Omega}) \tag{6.27}$$

and

$$C_2 = \int_M c_2(\bar{\Omega}) \tag{6.28}$$

6.7 Definition(Chern Characters) [132],[151] The Chern character forms are the closed $2l$-forms on M given by

$$ch_l(\bar{\Omega}) = \frac{1}{l!}(\frac{i}{2\pi})^l Tr(\bar{\Omega} \wedge ... \wedge \bar{\Omega}) \tag{6.29}$$

where again, $ch_l(\bar{\Omega}) = 0$ if $2l > dim(M)$.

It is not difficult to express the Chern character forms in terms of the Chern forms [132],[152]. Let therefore $X = (\frac{i}{2\pi})$ be a $n \times n$ complex matrix from the lie algebra $\mathcal{G}l(n, C)$. Let me assume that X has been written in diagonal form with eigenvalues $\lambda_1, ..., \lambda_n$. Then the expansion that leads to the Chern form is

$$\begin{aligned} det(1+X) = \Pi_{i=1}^n (1+\lambda_i) = 1 + \textstyle\sum_i \lambda_i + \sum_{i<j} \lambda_i \lambda_j + ... + \lambda_1 ... \lambda_n \\ = c_0(\lambda_i) + c_1(\lambda_i) + ... + c_n(\lambda_i) \\ = 1 + Tr(X) + \tfrac{1}{2}\{(TrX)^2 - TrX^2\} + ... + det(X) \end{aligned} \tag{6.30}$$

obviously the last equality does not require X to be in diagonal form. Similarly the invariant polynomial $Tr(exp(X))$ gives

$$\begin{aligned} Tr(exp(X)) = \textstyle\sum_{i=1}^n exp(\lambda_i) = \sum_{i=1}^n (\sum_l \frac{1}{l} \lambda_i^l) = \\ = n + c_1(\lambda_i) + \tfrac{1}{2}[c_1(\lambda_i)^2 - 2c_2(\lambda_i)] + ... \end{aligned} \tag{6.31}$$

Since the polynomials are invariant we can use for them the expression in terms of X. We get

$$\begin{aligned} ch_0(\bar{\Omega}) &= n \\ ch_1(\bar{\Omega}) &= c_1(\bar{\Omega}) \\ &\dots \\ ch_2(\bar{\Omega}) &= \frac{1}{2}[c_1(\bar{\Omega})^2 - 2c_2(\bar{\Omega})] \end{aligned} \tag{6.32}$$

In this way it is possible to see how the algebra of characteristic classes of a bundle is in general generated by the corresponding Chern classes. This also means that any characteristic class of P is a polynomial in the Chern classes [153].

Now we finally have everything we need in order to be able to state and to discuss index theorems for manifolds without boundaries [132], [155]. Indeed, the use of the plural is meaningful. There exists a whole class of index theorems, all connected by the same basic way of thinking.

Essentially they relate an analytically calculated quantity known as index with another quantity which is of a topological nature and which is given by a characteristic class. The Atiyah-Singer index theorem essentially claims that the analytical index and the topological index are equal. To see how this can be understood I will employ mostly the methods of ref. [132]. Let me therefore first reformulate the Hodge-de-Rham theory as an example for the index theorem.

To start, consider a compact manifold with no boundary M on which we define a Riemannian metric. The exterior derivative d and the codifferential δ are adjoints. Consider $(\, , \,)$ as positive definite. The differential forms $\bigwedge(M)$ on M form a ring and the action of d induces a sequence

$$0 \rightarrow \overset{0}{\bigwedge}(M) \xrightarrow{d_0} \overset{1}{\bigwedge}(M) \xrightarrow{d_1} \dots \xrightarrow{d_{i-1}} \overset{i}{\bigwedge}(M) \xrightarrow{d_i} \overset{i+1}{\bigwedge}(M) \rightarrow \dots \xrightarrow{d_{n-1}} \overset{n}{\bigwedge}(M) \xrightarrow{d_n} 0 \tag{6.33}$$

where d_i acts on i-forms.

In a similar way, if we act with the co-differential operator δ we obtain the sequence with the arrows oriented in the opposite direction

$$\dots \leftarrow \overset{i-1}{\bigwedge}(M) \xleftarrow{\delta_i} \overset{i+1}{\bigwedge}(M) \xleftarrow{\delta_{i+1}} \dots \tag{6.34}$$

where now the subscript i of δ_i refers to the order of the form resulting from its action. With this notation now we have

$$(d_i\alpha, \beta) = (\alpha, \delta_i\beta), \quad \alpha \in \bigwedge^i(M), \quad \beta \in \bigwedge^{i+1}(M) \tag{6.35}$$

We may also write $d_i^+ = \delta_i$ and $\delta_i^+ = d_i$ where the dagger indicates adjoint with respect to the product of forms (,).

We do not have exact sequences, i.e. $ker(d_i) \neq Im(d_{i-1})$ (the same for δ) although clearly $Im(d_{i-1}) \subset ker(d_i)$. This property, or, equivalently the fact that $d^2 = 0$ defines the sequence as a complex, in this case a de-Rham complex. In fact the de-Rham cohomology group measures the lack of exactness of the first sequence since $H^i_{DR}(M, R) = ker(d_i)/Im(d_{i-1})$. Similarly $\alpha \in \bigwedge^i(M)$ was defined as being co-closed (co-exact) if $\alpha \in ker(\delta_{i-1})$ (resp $\alpha \in Im(\delta_i)$).

The operator Δ is defined as the homogeneous Hodge-de-Rham operator. By using the differential and the co-differential we have

$$\Delta_i = \delta_i d_i + d_{i-1}\delta_{i-1} \tag{6.36}$$

Lets now assume that the $(i+1)$-form β can be expressed as $\delta_{i+1}\beta'$, $\beta' \in \bigwedge^{i+2}(M)$. If this is so, the product $(d_i\alpha, \delta_{i+1}\beta')$ is zero. Proceeding similarly we conclude that

$$Im(d_{i-1}) \perp Im(\delta_i) \perp Ker(\Delta_i) \tag{6.37}$$

or, in other words that $\bigwedge^i(M)$ has an unique splitting of the form

$$\bigwedge^i(M) = Im(d_{i-1}) \oplus Im(\delta_i) \oplus Ker(\Delta_i) \tag{6.38}$$

and consequently, since $Ker(\Delta_i) = Harm^i(M)$ we have

$$\alpha_i = d_{i-1}\alpha_{i-1} + \delta_i\alpha_{i+1} + h_i, \; \alpha \in \bigwedge^i(M) \tag{6.39}$$

where h_i is an harmonic i-form $\Delta_i h = 0$. Every i-th de Rham cohomology class is represented by one and only one harmonic form

$$H^i_{DR}(M, R) = Ker(\Delta_i) = Ker(d_i)/Im(d_{i-1}) \tag{6.40}$$

If the analytic index of the de Rham complex is now the integer defined by the alternating sum

$$index(\bigwedge(M), d) = \sum (-)^i dim(Ker(\Delta_i)) \tag{6.41}$$

we find

$$index(\bigwedge(M), d) = \sum_{i=0}^{N} (-)^i b^i(M) = \chi(M) = \int_M e(\tau(M)) \tag{6.42}$$

The right-hand side of this expression is topological, namely it is a topological index. Remembering the form of the Gauss-Bonnet theorem [132] we now can understand that it is the prototype of an index theorem: the index of the de Rham complex over the manifold M is the Euler-Poincare characteristic of M.

The last equation has precisely the form of an index theorem. The left hand side is expressed in terms of the number of zero-frequency solutions of the Laplace equation i.e. the number of harmonic, linearly independent forms on the manifold. The right hand side is the Euler-Poincare characteristic, which is formulated in terms of the Betti numbers of the manifold M.

The Laplacian depends on the Riemannian metric and so depends the space of the harmonic forms. The Euler characteristic and the Betti numbers however do not depend on the metric being topological invariants.

If $dim(M) = odd$ then, $index(\bigwedge(M^{odd}), d) = 0$ since $\chi(M^{odd}) = 0$. This remains true for index theorems associated to other differential operators.

We may note that Δ and $d + \delta$ have the same Kernel (a harmonic form is closed and co-closed).

Following [132] let me split $\bigwedge(M)$ into the sum $\bigwedge(M) = \bigwedge^{even}(M) \oplus \bigwedge^{odd}(M)$ of even and odd forms $\bigwedge^{even}(M) = \sum_{i\oplus} \bigwedge^{2i}(M)$, $\bigwedge^{odd}(M) = \sum_{i\oplus} \bigwedge^{2i+1}(M)$. Let D_+ and D_- be the operators defined by

$$D_+ = D = \textstyle\sum_i (d_{2i} + \delta_{2i-1}), \quad D_- = D^+ = \sum_i (d_{2i-1} + \delta_{2i}) \tag{6.43}$$

Then D is a mapping $D : \bigwedge^{even}(M) \to \bigwedge^{odd}(M)$, meaning

$$D(\alpha_{(0)}, \alpha_{(2)}, \alpha_{(4)}, ...) = (d_0\alpha_{(0)} + d_1\alpha_{(2)}, d_2\alpha_{(2)} + d_3\alpha_{(4)}, ...) \tag{6.44}$$

Its adjoint D^+ is a mapping $D^+ : \bigwedge^{odd}(M) \to \bigwedge^{even}(M)$. The associated Laplacians are given by

$$\Delta_+ = D^+D = \textstyle\sum_i \Delta_{2i}, \quad \Delta_- = DD^+ = \sum_i \Delta_{2i-1} \tag{6.45}$$

Thus we can replace the definition of the analytical index of the de-Rham complex by

$$index(\bigwedge(M), D) = dim(Ker(\Delta_+)) - dim(Ker(\Delta_-)) \tag{6.46}$$

or equivalently by

$$index(\bigwedge(M), D) = dim(Ker(D)) - dim(Ker(D^+)) \tag{6.47}$$

since $Ker(\Delta_+) = Ker(D^+D) = Ker(D)$ and $Ker(\Delta_-) = Ker(DD^+) = Ker(D^+)$. In fact, $Im(D)$ (resp. $Ker(D)$) is the orthogonal complement of $Ker(D^+)$ (resp. $Im(D^+)$).

We also have that $coker(D) = \bigwedge^{odd}/Im(D) = ker(D^+)$ and hence the analytic index may be given as

$$index(\bigwedge(M), D) = dim(Ker(D)) - dim(coker(D)) \tag{6.48}$$

The de-Rham complex has an analytic index which is quite general. It has many aspects in common with the index theorem for other complexes.

The de Rham sequence could have been written

$$0 \to \Gamma(M, E_0) \to \Gamma(M, E_1) \to ... \to \Gamma(M, E_i) \xrightarrow{D_i} \Gamma(M, E_{i+1}) \to ... \to \Gamma(M, E_n) \xrightarrow{D_n} 0 \tag{6.49}$$

where $\Gamma(M, E_i)$ is the module of cross sections of the vector bundle $E_i = \wedge^i \tau^*(M)$. We can write the above sequence only if there exists a differential operator of degree 1 acting on a sequence of sections of vector bundles E_i such that $D_{i+1} \circ D_i = 0$. Only when this happens we can speak about a complex when referring to that sequence.

Now, the fact that the expression

$$index(\bigwedge(M), d) = \sum(-)^i dim(ker(\Delta_i)) \tag{6.50}$$

was a well defined one was guaranteed by the nature of the Laplacian operator $\Delta = (d + \delta)^2$, the kernel of which is finite dimensional. This is a consequence of the fact that Δ is an elliptic operator. To formalize this assertion consider a differential operator $\mathcal{D} : \Gamma(M, E) \to \Gamma(M, E')$, $\mathcal{D} : s(x) \to s'(x)$ acting on the cross sections of a vector bundle E. Let x^i, $i = 1, ..., n$ be local coordinates on $U \subset M$. Then a general differential operator $\mathcal{D}$ of degree K over U may be expressed through its action on sections as

$$[\mathcal{D}s(x)]^{l'} = \textstyle\sum_{\tilde{\alpha}} A_l^{\alpha l'}(x) D_\alpha s^l(x), \quad D_\alpha = (\frac{\partial}{\partial x^1})^{\alpha_1} ... (\frac{\partial}{\partial x^n})^{\alpha_n} \tag{6.51}$$

where $l' = 1, ..., m'$, $l = 1, ..., m$ are the respective fibre indices, $(A^{\alpha})^{l'}_{\ l}$ is an $m' \times m$ matrix on U, α is a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, $\alpha_i \leq 0$, $\tilde{\alpha} = \alpha_1 + \alpha_2 + ... + \alpha_n$ and the sum over $\tilde{\alpha}$ is a sum over the multi-index α which includes all possibilities for $\tilde{\alpha} \leq K$.

6.8 Definition(Leading or principal symbol of $\mathcal{D}$) Let $\xi = (\xi_1, ..., \xi_n)$ be a real n-tuple. The leading symbol $\sigma_x(\xi, \mathcal{D})$ associated with the local representation of $\mathcal{D}$ at the point x is the linear map from the fibre F_x to F'_x defined by the matrix

$$\sigma_x(\xi, \mathcal{D})^{l'}_{\ l} = \sum_{\tilde{\alpha}=K} A^{\alpha l'}_{l}(x)\xi_\alpha \tag{6.52}$$

$(\mathcal{D}_\alpha \to \xi_\alpha)$ the entries of which are polynomials in $\xi_1, ..., \xi_n$. Notice that only the highest order K derivatives in $\mathcal{D}$ enter in the definition.

Let now $E = E'$. The ellipticity of a differential operator is given by

6.9 Definition(Elliptic operator) A differential operator $\mathcal{D} : \Gamma(M, E) \to \Gamma(M, E)$ is said to be elliptic on M if $\forall x \in M$ the linear mapping given by the $m \times m$ matrix $\sigma_x(\xi, \mathcal{D})$ is an isomorphism for every $\xi \neq 0$ i.e. if the matrix is invertible.

An elliptic operator defined on a compact manifold has a finite-dimensional kernel and cokernel and expressions of the type of

$$index(\bigwedge(M), D) = dim(ker(D)) - dim(coker(D)) \tag{6.53}$$

are well defined for them. For instance if the fibre is one-dimensional and $\mathcal{D}$ is given by

$$\mathcal{D} = A^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + B^i\frac{\partial}{\partial x^i} + C \tag{6.54}$$

the leading symbol is given by the one-dimensional matrix $A^{ij}\xi_i\xi_j$ and $\mathcal{D}$ is elliptic if it is non-zero for $\xi \neq 0$. This is the case if the quadratic form is positive or negative definite i.e. if $A^{ij}\xi_i\xi_j = c$ is an ellipsoid (hence the name of elliptic operator; other possibilities for the quadratic form lead to hyperbolic - maximal rank, mixed signature - or parabolic operators). As a result the operator $d\delta + \delta d$ is elliptic for a Riemannian metric (it is not for the Minkowski signature). Indeed at a point x a Riemannian manifold can be made Euclidean up to first order by using geodesic coordinates and then $\Delta = -\sum_{i=1}^{n}(\partial/\partial x^i)^2$ gives $\sigma_x(\xi, \Delta) = -(\xi_1^2 + ... + \xi_n^2)$. In contrast, for a pseudo-Riemannian metric of signature (p, q), the quadratic form $\xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2$ is not invertible on the light cone. An important property of elliptic operators is that since the leading symbol of a composite operator is the composite of symbols, the composites of elliptic operators (and hence powers and roots) will also be elliptic.

Looking at the de Rham complex on the compact boundaryless manifold M we conclude that it is an elliptic complex since its associated Laplacians are elliptic. This suggests a more general

6.10 Definition(Elliptic complex)[132] Let E be a vector bundle. An elliptic complex (E, D) is a finite sequence of differential operators $D_i : \Gamma(M, E_i) \to \Gamma(M, E_{i+1})$ acting on smooth sections such that $D_{i+1} \circ D_i = 0$ and the Laplacians of the complex $\Delta_i = D_i^+ D_i + D_{i-1} D_{i-1}^+$ where D_i^+ is the adjoint operator with respect to the scalar product on the fibres with a smooth density on M, are elliptic on $\Gamma(M, E_i)$.

Since $(D_{i+1} \circ D_i)^+ = D_i^+ \circ D_{i+1}^+$, it follows that if the complex $(\Gamma(M, E_i), D_i)$ is elliptic, so is the complex $(\Gamma(M, E_{i+1}), D_i^+)$ where the arrows point in the opposite direction. To relate this picture to the form of the index for a de Rham complex we have to reduce the elliptic complex to a two-term elliptic complex (to roll up the complex) and see that the new complex has the same index as the original one $(\Gamma(M, E), D)$. This is where the comparison with the previous equations for the index comes in. Defining the even and odd bundles $E^{even} = \sum_{i\oplus} E_{2i}$, $E^{odd} = \sum_{i\oplus} E_{2i+1}$

$$\begin{aligned} \Gamma(M, E^{even}) &= \textstyle\sum_{i\oplus} \Gamma(M, E_{2i}), \quad \Gamma(M, E^{odd}) = \textstyle\sum_{i\oplus} \Gamma(M, E_{2i+1}) \\ D &= \textstyle\sum_{i\oplus}(D_{2i} + D_{2i-1}^+), \qquad D^+ = \textstyle\sum_{i\oplus}(D_{2i-1} + D_{2i}^+) \end{aligned} \tag{6.55}$$

and the associated Laplacian

$$\Delta_i = D_i^+ D_i + D_{i-1} D_{i-1}^+, \quad \Delta_+ = \textstyle\sum_i \Delta_{2i} = D^+ D, \quad \Delta_- = \textstyle\sum_i \Delta_{2i-1} = DD^+ \tag{6.56}$$

6.11 Definition(Analytical index of an elliptic complex) The analytical index of an elliptic complex $(\Gamma(M, E), D)$ is defined to be the integer

$$index(\Gamma(M, E), D) = \sum_i (-)^i dim(ker(\Delta_i)) = dim(Ker(\Delta_+)) - dim(Ker(\Delta_-)) \tag{6.57}$$

We note [132] that the differential operator defining a complex, the Riemannian scalar product defining its adjoint and the ellipticity property which guarantees that the rhs of the equation above is well defined (an integer) are the ingredients for the definition of an index of a compact manifold. In order to have a non-trivial index the operator D cannot be self-adjoint. The Atiyah-Singer index theorem states that the analytic index is equal to the topological index of the complex, which is given by the rhs in the formula of the Atiyah-Singer index theorem

6.12 Theorem(Atiyah and Singer) [129] Let $(\Gamma(M,E),D)$ be an elliptic complex over a compact boundary-less manifold M of even dimension n. Then the index of the complex is given by

$$index(\Gamma(M,E),D) = (-)^{n(n+1)/2} \int_M ch(\sum_{i=0}^{n} \oplus (-)^i E_i) \frac{Td(\tau(M)^C)}{e(\tau(M))} \tag{6.58}$$

where in the integrand only n-forms are retained. If the manifold is odd-dimensional, the index of the differential operator D is zero.

$Td(\tau(M)^C)$ is the Todd class of the complexified tangent bundle $\tau(M)^C$ and $e(\tau(M))$ is the Euler class.

Chapter 7

Universal Coefficient Theorems

"Who in the world am I? Ah, that's the great puzzle."

Lewis Carroll, Alice in Wonderland

As a continuation of the last chapter, I give here various facts relevant for the construction of universal coefficient theorems. I also show some results stemming from this type of theorems, mainly referring to [96] but also to [156]. This way of thinking may solve several problems found in quantum field theories [157].

It is important to remind the fact seen already in chapter 5 and 6 that the topological invariants most widely used, namely homology and cohomology, do not depend only on the topological space X which we wish to characterize but also on the "type of numbers" we use i.e. the numerical fields used as coefficients in the simplicial complex expansion. This dependence, as has been shown in [158], has an important effect on the definition of characteristic classes and finally on the index theorem. Classes can merge or dissociate. These processes depend on several factors, including on what number fields we use as coefficients in the topological invariants. It may be of some interest to review some situations where this dependence on the "numerical probing device" becomes important.

In physics this kind of merging and dissociation of equivalence classes has been known for a long time [159]. However it has mainly been correlated to the spatial or temporal resolution of a measuring device. Processes visible at a very fine grained scale become averaged-over at coarser scales leading to changes in the values of certain parameters with respect to those measured at previous scale. Because of this, the sensitivity of a measuring device was usually related to a specific property that could be detected at certain scales of length [160]. The appearance of effective theories was primarily due to the fact that scales separated by large transformations could be analyzed independently up to some point, only after the results for one scale were properly considered and their effects properly introduced in the effective model of the other scale. The effective theories could therefore be used to connect very distant scales of a theory. When scales were contiguous, the renormalization group equations and the renormalization group flow were used to show how certain parameters depend on the renormalization point i.e. the scale where a measurement is performed [161]. This may not be the only way in which dissociation or merging of classes can occur. In fact, by thinking only in terms of spatial or temporal scales we ignore other ways in which a theoretical measuring device may fail to acknowledge the existence of separated structure. Of course, classes can be related by other equivalence relations and the transition from one class to the other can be triggered by other transformations than scale changes. Indeed, if we consider so called "topological measurements" (see section 8 or ref. [54]) we need to think in terms of coefficient groups in (co)homology. These constitute a part of a framework where certain properties can be defined. If chosen in a specific way, some properties may remain undefined in that specific framework. The universal coefficient theorem is the result that makes the connection between the use of various coefficient groups in (co)homology and hence plays the role of the "renormalization group equations" for topology.

On the mathematical side there exists the so called "Grothendieck's relative point of view" [250]. The idea here was to shift the focus from the study of single objects in a given category to the study of a family of objects depending on various parameters. Basically the idea was implemented such that one did not focus on objects X in a given category C but instead on morphisms

$$f : X \to S \tag{7.1}$$

where S is a fixed object. This makes use of the context or the framework where the specific structure is inserted. At the same time, one wishes to describe the properties of a structure by the way in which it can map into another structure of some well known properties.

There appears to be a subtle connection between this idea of Grothendieck and the renormalization group theory. This connection is the main subject of the current thesis.

It is important to observe that in a physical theory the parameters are not completely determined. Indeed, by changing the scale where we chose to evaluate the theory, various parameters (like mass, charge, etc.) can change [162].

This basic fact can be generalized in an unexpected way. First, the existence of an object (a theory), depending on several parameters including the scale, is considered as given.

The characterization of a theory is not done focusing only on itself, but also on the way in which the theory can be mapped into another well known object. In the context of topology we have topological invariants like (co)homology. However, the sensibility of (co)homology depends on coefficient groups. Therefore, mutatis mutandis, we have a theory depending on various group structures. We then have the groups as "parameters" for the topological description. The question will be how to characterize the theory when various coefficient groups are being considered. This would be equivalent to finding an "effective" theory for a given scale. The physics itself should remain unchanged. However, the parameters characterizing charge, mass, etc. will vary accordingly. What we need is an analogue of the renormalization group equations in the case of (co)homology with coefficients i.e. a way of answering to the question of how the properties of the theory change when we chose different coefficient groups. In the case of effective field theories the analogue is the prediction of effective values like magnetization, electric susceptibility, etc. and the variations thereof in various circumstances.

By making various choices for the numerical fields of coefficients and for the group structures used, one can transform one algebraic variety in another. I will make use of this in the last chapter of this thesis in order to give a possible way for solving problems in QCD.

It is my goal to answer to the question: what is the analogue of the renormalization group equations in the case of topology and how do we have to change our viewpoint accordingly? It appears to me that the perfect analogue in this situation is what is known in algebraic topology as "The Universal Coefficient Theorem" (UCT). This will be the subject of this chapter. I follow closely ref. [25], [96], [108], [251]. The lecture notes of ref. [96] were particularly useful for the understanding of the ideas behind the derivation of the universal coefficient theorem. These are the source of many standard definitions and theorems in this chapter.

Some of the simplest notions required in the construction of universal coefficient theorems are the tensor products, the adjoint functors and the Hom group. For the sake of the completeness of this chapter I will give some exact definitions here.

7.1 Definition(Tensor Products) Let A and B be modules over a commutative ring R. The tensor product of A and B is the R-module $A \otimes_R B$ defined as the quotient

$$\frac{F(A \times B)}{R(A \times B)} \tag{7.2}$$

where $F(A \times B)$ is the free R-module with basis $A \times B$ and $R(A \times B)$ the submodule generated by

- $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$
- $(a, b_1 + b_2) - (a, b_1) - (a, b_2)$
- $r(a, b) - (ra, b)$
- $r(a, b) - (a, rb)$

The image of a basis element (a, b) can be written in $A \otimes_R B$ as $a \otimes b$.

It is useful to observe that one has the relations

- $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
- $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$
- $(ra \otimes b) = r(a \otimes b) = (a \otimes rb)$

Otherwise we can say that $A \otimes_R B$ is the largest R-module generated by the set of symbols $\{a \otimes b\}_{a \in A, b \in B}$, satisfying the above product relations. Any element of $A \otimes B$ can be expressed as a finite sum $\sum_{i=1}^{n} a_i \otimes b_i$, but it may not be possible to take $n = 1$. The representation in form of a sum is not necessarily unique.

In what follows I will briefly introduce the adjoint functors. Note that an R-bilinear map $\beta : A \times B \to C$ is the same as an element of $Hom_R(A, Hom_R(B, C))$.

7.2 Proposition(Adjoint property of Tensor Products)[107] There is an isomorphism of R-modules

$$Hom_R(A \otimes_R B, C) \cong Hom_R(A, Hom_R(B, C)) \tag{7.3}$$

natural in A, B, C given by $\phi \leftrightarrow (a \to (b \to \phi(a \otimes b)))$.

In order to introduce Hom, for an R-module A, define $A^* = Hom_R(A, R)$. The module A^* is often called the dual of A. For an R-module map $f : A \to B$ the dual map $f^* : B^* \to A^*$ is defined by $f^*(\phi) = \phi \circ f$. Hence taking duals defines a contravariant functor from the category of R-modules to itself. More generally, for R-modules A and M, $Hom_R(A, M)$ is the R-module of homomorphisms from A to M. It is contravariant in its first variable and covariant in its second variable. For an R-map $f : A \to B$ we have

$$Hom_R(f, M) : Hom_R(B, M) \to Hom_R(A, M) \tag{7.4}$$

defined by $\phi \to \phi \circ f$. Usually, f^* means the same as $Hom_R(f, M)$.

It will prove useful to have an axiomatic conceptualization of the notion of (co)homology. This has been given in terms of the so called Eilenberg-Steenrod axioms

7.3 Definition (The Eilenberg-Steenrod axioms for homology)[283]

An (ordinary) homology theory is a covariant functor

H_*\{(space, subspace), pairs, continuous maps of pairs\} $\to$ \{graded R-modules; homomorphisms\}

In other words a collection of covariant functors H_q for each non-negative integer q which assign an R-module $H_q(X, A)$ to a pair (X, A) of topological spaces and a homomorphism $f_* = H_q(X, A) \to H_q(Y, B)$ to every continuous function of pairs $f : (X, A) \to (Y, B)$. These are required to satisfy the following axioms

- There exist natural connecting homomorphisms

$$\partial : H_q(X, A) \to H_{q-1}(A) \tag{7.5}$$

  for each pair (X, A) and each integer q so that the sequence

$$\begin{gathered} ... \to H_q(A) \to H_q(X) \to H_q(X, A) \xrightarrow{\partial} H_{q-1}(A) \to ... \\ ... \to H_1(X, A) \xrightarrow{\partial} H_0(A) \to H_0(X) \to H_0(X, A) \to 0 \end{gathered} \tag{7.6}$$

  is exact (Long exact sequence of pairs).

- If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps on homology are equal, $g_* = f_* : H_q(X, A) \to H_q(Y, B)$. (Homotopy invariance)

- If $U \subset X$ and $\bar{U} \subset Int(A)$, then $H_q(X - A, A - U) \to H_q(X, A)$ is an isomorphism for all q. (Excision)

- If "*pt*" denotes the one-point space, then $H_q(pt) = 0$ when $q \neq 0$. (Dimension Axiom)

7.4 Theorem (Existence)

For any R-module M there is a homology theory with $H_0(pt) = M$

Similarly, for cohomology we have

7.5 Definition (The Eilenberg-Steenrod axioms for cohomology)[282]

An (ordinary) cohomology theory is a contravariant functor

$H^*\{$(space, subspace), pairs, continuous maps of pairs$\} \to \{$graded, R-modules, homomorphisms$\}$

In other words a collection of contravariant functors H^q for each non-negative integer q which assign an R-module $H^q(X, A)$ to a pair (X, A) of topological spaces and a homomorphism $f^* = H^q(Y, B) \to H^q(X, A)$ to every continuous function of pairs $f : (X, A) \to (Y, B)$. These are required to satisfy the following axioms

- There exist natural connecting homomorphisms

$$\delta : H^q(A) \to H_{q+1}(X, A) \tag{7.7}$$

  for each pair (X, A) and each integer q so that the sequence

$$\begin{gathered} 0 \to H^0(X, A; M) \to H^0(X; M) \to H^0(A; M) \xrightarrow{\delta} H_1(X, A; M) \to ... \\ ... \to H^{q-1}(A; M) \xrightarrow{\delta} H^q(X, A; M) \to H^q(X; M) \to ... \end{gathered} \tag{7.8}$$

  is exact (Long exact sequence of pairs).

- If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then $g^* = f^* : H_q(Y, B) \to H_q(X, A)$. (Homotopy invariance)
- If $U \subset X$ and $\bar{U} \subset Int(A)$, then $H^q(X, A) \to H^q(X - U, A - U)$ is an isomorphism for all q. (Excision)
- If "*pt*" denotes the one-point space, then $H^q(pt) = 0$ when $q \neq 0$. (Dimension Axiom)

7.6 Theorem (Existence)

For any R-module M there is a cohomology theory with $H^0(pt) = M$

As the *Ext* and *Tor* constructions are relevant for the construction of the universal coefficient theorem I will try to introduce them now more carefully in an axiomatic way. I mainly follow for this introduction reference [108]. First, let me call R-Mod the R-module and $F : R - Mod \to R - Mod$ a functor which takes short exact sequences to short exact sequences. We can call a covariant functor $F : R - Mod \to R - Mod$ right exact if $F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence. Similarly a contravariant functor is called right exact if $F(C) \to F(B) \to F(A) \to 0$ is a short exact sequence.

The functors $* \otimes_R M$, $Hom_R(M, *)$ and $Hom_R(*, M)$ are not exact in general. For example taking $R = \mathbb{Z}$, $M = \mathbb{Z}/2$ and the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \tag{7.9}$$

we obtain

$$\begin{array}{ccccccc}
\mathbb{Z} \otimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z} \otimes \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 \otimes \mathbb{Z}/2 & \longrightarrow & 0 \\
\cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow & & \\
\mathbb{Z}/2 & \xrightarrow{\times 2} & \mathbb{Z}/2 & \xrightarrow{Id} & \mathbb{Z} & \longrightarrow & 0
\end{array} \tag{7.10}$$

$$\begin{array}{ccccccc}
0 & \longrightarrow & Hom(\mathbb{Z}/2, \mathbb{Z}) & \longrightarrow & Hom(\mathbb{Z}/2, \mathbb{Z}) & \longrightarrow & Hom(\mathbb{Z}/2, \mathbb{Z}) \\
 & & \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2
\end{array} \tag{7.11}$$

and

$$\begin{array}{ccccccc}
0 & \longrightarrow & Hom(\mathbb{Z}/2, \mathbb{Z}/2) & \longrightarrow & Hom(\mathbb{Z}, \mathbb{Z}/2) & \longrightarrow & Hom(\mathbb{Z}, \mathbb{Z}/2) \\
 & & \cong\big\downarrow & & \cong\big\downarrow & & \cong\big\downarrow \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \xrightarrow{\times 2} & \mathbb{Z}/2
\end{array} \tag{7.12}$$

One of the goals of homological algebra is to find natural functors which measure the failure of another functor to preserve short exact sequences. One may try to take for $* \otimes_R M$ the kernel of $A \otimes M \to B \otimes M$ as the value of this functor. However, this does not behave nicely with respect to morphisms. To construct these functors the only things we will use are the left/right exactness properties and the observation that for any module M there is a surjective map from a free module to M. I have shown that exactness is a very important property. It essentially means that the objects and morphisms in the sequence are arranged such that the image of one morphism is the kernel of the next. When we speak about short exact sequences we also have the first map being an injection and the second a surjection. I used this property for example in

[54] and it is of major importance in [53] as well. Hence, preserving exactness when we apply functors is crucial. There will be necessary to prove a

7.7 Theorem (Existence)

- There exist functors

$$Tor_n^R : R - Mod \times R - Mod \to R - Mod \tag{7.13}$$

  for all $n = 0, 1, 2, ...$, with $(M_1, M_2) \to Tor_n^R(M_1, M_2)$ covariant in M_1 and M_2 satisfying the following axioms:

  - $Tor_0^R(M_1, M_2) = M_1 \otimes_R M_2$
  - If $0 \to A \to B \to C \to 0$ is any short exact sequence of $R - modules$ and M is any $R - module$ then there is a natural long exact sequence

$$\begin{gathered} ... \to Tor_n^R(A, M) \to Tor_n^R(B, M) \to Tor_n^R(C, M) \to Tor_{n-1}^R(A, M) \to ... \\ ... \to Tor_1^R(C, M) \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0 \end{gathered} \tag{7.14}$$

  - $Tor_n^R(F, M) = 0$ if F is a free module and $n > 0$.

  The functor $Tor_n^R(*, M)$ is called the n^{th} derived functor of the functor $* \otimes_R M$.

- There exist functors $Ext_R^n : R - Mod \times R - Mod \to R - Mod$ for all $n = 0, 1, 2, ...$ with $(M_1, M_2) \to Ext_R^n(M_1, M_2)$ contravariant in M_1 and covariant in M_2 satisfying the following axioms:

  - $Ext_R^0(M_1, M_2) = Hom_R(M_1, M_2)$
  - If $0 \to A \to B \to C \to 0$ is any short exact sequence of $R - modules$ and M is any $R - module$ then there is a natural long exact sequence

$$\begin{gathered} ... \to Hom_R(C, M) \to Hom_R(B, M) \to Hom_R(A, M) \to Ext_R^1(C, M) \to ... \\ ... \to Ext_R^q(B, M) \to Ext_R^q(A, M) \to Ext_R^{q+1}(C, M) \to ... \end{gathered} \tag{7.15}$$

  - $Ext_R^n(F, M) = 0$ if F is a free module and $n > 0$.

  The functor $Ext_R^n(*, M)$ is called the n^{th} derived functor of the functor $Hom_R(*, M)$

It is important to note that these axioms characterize the functors Tor and Ext.

7.8 Theorem (Uniqueness) Any two functors satisfying the above first axioms are naturally isomorphic. Any two functors satisfying the above last axioms are also naturally isomorphic.

Proof We will show that values of $Tor_n^R(M_1, M_2)$ are determined by the axioms by induction on n. This is true for $n = 0$ by the first axiom. Next we note that for any module M_1 there is a surjection $F \xrightarrow{\phi} M_1 \to 0$ where F is a free module. For example let $S \subset M_1$ be a set which generates M_1 as an $R-Module$ (e.g. $S = M_1$) and let $F = F(S)$ the free module with basis S. There is an obvious surjection ϕ. Let $K = ker(\phi)$. Apply the second axiom to the short exact sequence

$$0 \to K \to F \to M_1 \to 0 \tag{7.16}$$

Then by the second and the third axioms concerning Tor we have

$$Tor_1^R(M_1, M_2) \cong ker(K \otimes_R M_2 \to F \otimes_R M_2) \tag{7.17}$$

and

$$Tor_n^R(M_1, M_2) \cong Tor_{n-1}^R(K, M_2) \tag{7.18}$$

for $n > 1$. The values of Tor_{n-1}^R are known by induction. The proof for Ext is similar using the other axioms.

The technique used in the proof above is called dimension shifting and is useful in various homological computations. As an example, if F is a free module and

$$0 \to K \to F' \to M \to 0 \tag{7.19}$$

is a short exact sequence with F' free then

$$Tor_1^R(M, F) \cong ker(K \otimes F \to F' \otimes F) \tag{7.20}$$

but this is equal to zero. Thus $Tor_1^R(*, F)$ is identically zero. But $Tor_n^R(M, F) \cong Tor_{n-1}^R(K, F)$ for $n > 1$ so inductively we see $Tor_n^R(*, F)$ is zero for $n > 0$. To compute $Ext^1_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z})$ we apply the second axiom for Ext to the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2 \to 0 \tag{7.21}$$

to get the exact sequence

$$\begin{array}{ccccccc} Hom(\mathbb{Z},\mathbb{Z}) & \xrightarrow{(\times 2)*} & Hom(\mathbb{Z},\mathbb{Z}) & \longrightarrow & Ext^1(\mathbb{Z}/2,\mathbb{Z}) & \longrightarrow & Ext^1(\mathbb{Z},\mathbb{Z}) \\ \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow \\ \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & Ext^1(\mathbb{Z}/2,\mathbb{Z}) & \longrightarrow & 0 \end{array} \tag{7.22}$$

so $Ext^1_{\mathbb{Z}}(\mathbb{Z}/2,\mathbb{Z}) \cong \mathbb{Z}/2$.

In what follows I will give some examples of some useful computations.

7.9 Proposition Let R be a commutative ring and $a \in R$ a non-zero divisor (i.e. $ab = 0 \Rightarrow b = 0$). Let M be an $R-module$. Let $M/a = M/aM$ and ${}_aM = \{m \in M | am = 0\}$. Then

- $R/a \otimes M \cong M/a$
- $Tor_1(R/a, M) \cong {}_aM$
- $Hom(R/a, M) \cong {}_aM$
- $Ext^1(R/a, M) \cong M/a$

Proof Since a is not a divisor of zero there is a short exact sequence

$$0 \to R \xrightarrow{\times a} R \to R/A \to 0 \tag{7.23}$$

Apply the functors $* \otimes M$ and $Hom(*, M)$ to the above short exact sequence. By the axioms we have exact sequences

$$0 \to Tor_1(R/a, M) \to Hom(R, M) \to Hom(R, M) \to Ext^1(R/a, M) \to 0 \tag{7.24}$$

The middle maps in the exact sequence above can be identified with

$$M \xrightarrow{\times a} M \tag{7.25}$$

which has kernel ${}_aM$ and cokernel M/a.

In particular if n is a non-zero integer and $R = \mathbb{Z}$ the four functors Tor_1, $\otimes$, Hom and Ext^1 applied to the pair $(\mathbb{Z}/n, \mathbb{Z}/n)$ are all isomorphic to $\mathbb{Z}/n$. If m and n are relatively prime integers then applied to the pair $(\mathbb{Z}/m, \mathbb{Z}/n)$ they are all zero.

In what follows, the concept of an Principal Ideal Domain (PID) will be useful. The PID are the abstract objects that generalize in a sense the notion of "integers", mainly with respect to divisibility. In essence they are the domains of a ring that are integral (i.e. contain only the non-zero commutative parts which have the property that the products of any two nonzero elements are also non-zero) and principal (i.e. can be generated by acting via the associated operation on a single element). This means essentially that any element of a PID has a unique decomposition in "prime elements" and any two elements of a PID have a common greatest divisor.

Here are the results where this concept is necessary

7.10 Proposition

- If R is a field then $Tor_n^R(*,*)$ and $Ext_R^n(*,*)$ are zero for $n > 0$.
- If R is a P.I.D. then $Tor_n^R(*,*)$ and $Ext_R^n(*,*)$ are zero for $n > 1$.

Proof

- All modules over a field are free so this result follows from the third Tor axiom and respectively the third Ext axiom.
- A submodule of a free module over a P.I.D. is free so for any module M there is a short exact sequence
  $$0 \to F_1 \to F_0 \to M \to 0 \tag{7.26}$$
  with F_1 and F_0 free. Then by the second and third Tor and Ext axioms for $n > 1$, $Tor_n^R(M,*)$ and $Ext_n^R(M,*)$ are in the long exact sequence flanked by zero and hence must vanish.

One may ask what is the behavior of these functors with respect to exact sequences in the second variables. Is it possible to write $Tor_n(A,B) \cong Tor_n(B,A)$? It could appear natural because $A \otimes B \cong B \otimes A$. The same would not be valid for Ext since $Hom(A,B) \ncong Hom(B,A)$. The following theorem gives the results related to these questions

7.11 Theorem

- There exist functors

$$Tor_n^R : R - Mod \times R - Mod \to R - Mod \tag{7.27}$$

for all $n = 0, 1, 2, ...$, satisfying the following axioms:

- $Tor_0^R(M_1, M_2) = M_1 \otimes_R M_2$
- If $0 \to A \to B \to C \to 0$ is any short exact sequence of $R - modules$ and M is any $R - module$ then there is a natural long exact sequence

$$\begin{gathered} ... \to Tor_n^R(M, A) \to Tor_n^R(M, B) \to Tor_n^R(M, C) \to Tor_{n-1}^R(M, A) \to ... \\ ... \to Tor_1^R(M, C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0 \end{gathered} \tag{7.28}$$

- $Tor_n^R(M, F) = 0$ if F is a free module and $n > 0$.

- There exist functors $Ext_R^n : R - Mod \times R - Mod \to R - Mod$ for all $n = 0, 1, 2, ...$ satisfying the following axioms:
  - $Ext_R^0(M_1, M_2) = Hom_R(M_1, M_2)$
  - If $0 \to A \to B \to C \to 0$ is any short exact sequence of $R - modules$ and M is any $R - module$ then there is a natural long exact sequence

$$\begin{gathered} 0 \to Hom_R(M, A) \to Hom_R(M, B) \to Hom_R(M, C) \to Ext_R^1(M, A) \to ... \\ ... \to Ext_R^q(M, B) \to Ext_R^q(M, C) \to Ext_R^{q+1}(M, A) \to ... \end{gathered} \tag{7.29}$$

  - $Ext_R^n(M, I) = 0$ if I is an injective module and $n > 0$.

7.12 Corollary The functors $Tor_n^R(A, B)$ and $Tor_n^R(B, A)$ are naturally isomorphic.

Proof The functor $(A, B) \to Tor_n^R(B, A)$ satisfies the Tor axioms 1, 2 and 3 above and thus by the uniqueness theorem it must be naturally isomorphic to $(A, B) \to Tor_n^R(A, B)$.

These observations are related to what are called projective modules and projective resolutions. The functors Ext_R^n can also be defined using injective resolutions.

Tor and Ext are higher derived versions of $\otimes_R$ and Hom so they have analogous properties. For example

- $Tor_n^R(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha Tor_n^R(A_\alpha, B)$
- $Ext_R^n(\oplus_\alpha A_\alpha, B) \cong \Pi_\alpha Ext_R^n(A_\alpha, B)$ and
- $Ext_R^n(A, \Pi_\alpha B_\alpha) \cong \Pi_\alpha Ext_R^n(A, B_\alpha)$

With these notions we can finally introduce the universal coefficient theorems. This will be the main task for the following part of this section.

Let (C_*, ∂) be a chain complex over a ring R. Then there is an evaluation map

$$\begin{gathered} Hom_R(C_q, M) \times C_q \to M \\ (f, z) \to f(z) \end{gathered} \tag{7.30}$$

This pairing passes to the Kroneker pairing

$$<,>: H^q(C_*; M) \times H_q(C_*) \to M \tag{7.31}$$

of cohomology with homology. This pairing is bilinear and its adjoint is a homomorphism

$$H^q(C_*, M) \to Hom(H_q(C_*); M) \tag{7.32}$$

This adjoint need not be an isomorphism. The understanding of the kernel and cokernel of this map is a subtle question. Universal coefficient theorems provide among other things a measure of how this adjoint fails to be an isomorphism in terms of the derived functors Ext^q and Tor_q. The answer is usually not simple. This type of questions can be answered completely when R is a P.I.D. and C_* is a free chain complex. In this case $H^q(C_*, M) \to Hom(H_q(C_*); M)$ is surjective with kernel $Ext(H_{q-1}(C_*), M)$. This will cover the topological situation when the coefficients are integers or belong to a field since the singular and cellular complexes of a space are free. I can therefore start with the statement of the following

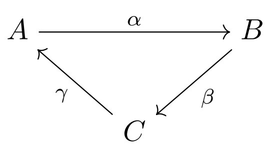
7.13 Theorem(The universal coefficient theorem for cohomology) [279,280] Let R be a principal ideal domain. Suppose that M is a module over R and (C_*, ∂) is a free chain complex over R (each C_q is a free $R - module$). Then the sequence

$$0 \to Ext_R(H_{q-1}(C_*), M) \to H^q(C_*; M) \to Hom(H_q(C_*), M) \to 0 \tag{7.33}$$

is exact and natural with respect to chain maps of free chain complexes. Moreover, the sequence splits albeit not naturally.

The proof will require the concept of an exact triangle.

7.14 Definition An exact triangle of $R - modules$ is a diagram of $R - modules$



(7.34)

satisfying $ker(\beta) = Im(\alpha)$, $\ker(\gamma) = Im(\beta)$ and $ker(\alpha) = Im(\gamma)$. One can define an exact triangle of graded R-modules A_*, B_*, C_*. In this case we require that the homomorphisms α, β, γ, each to have a degree. For example if α has degree 2 then $\alpha(A_q) \subset A_{q+2}$. The basic example of an exact triangle of graded R-modules is the long exact sequence in homology

$$\begin{array}{ccc} H_*(A) & \xrightarrow{\quad i_* \quad} & H_*(X) \\ & {\scriptstyle\partial}\nwarrow \qquad \swarrow{\scriptstyle j_*} & \\ & H_*(X, A) & \end{array} \tag{7.35}$$

For this exact triangle i_* and j_* have degree 0 and ∂ has degree -1.

If we consider the diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow E & \xrightarrow{j} & A & & \xrightarrow{\quad \alpha \quad} & & B & \xrightarrow{k} & F \longrightarrow 0 \\ & & & {\scriptstyle\gamma}\nwarrow & & \swarrow{\scriptstyle\beta} & & & \\ & & & & C & & & & \end{array} \tag{7.36}$$

as a diagram with the top row exact and the triangle exact it is possible to show that there exists a short exact sequence

$$0 \to F \xrightarrow{\beta \circ k^{-1}} C \xrightarrow{j^{-1} \circ \gamma} E \to 0 \tag{7.37}$$

At this moment it is possible to prove the universal coefficient theorem

Proof There is a short exact sequence of graded free R-modules

$$0 \to Z_* \xrightarrow{i} C \xrightarrow{\partial} B_* \to 0 \tag{7.38}$$

where Z_q denotes the q-cycles and B_q denotes the q-boundaries. The homomorphism i has degree 0 and ∂ has degree -1. This sequence is in fact a short exact sequence of chain complexes where Z_* and B_* are given the zero differential. Since the sequence above is an exact sequence of free chain complexes applying the functor $Hom(*, M)$ gives again a short exact sequence of chain complexes.

$$0 \to Hom(B_*, M) \xrightarrow{\partial^*} Hom(C_*, M) \xrightarrow{i^*} Hom(Z_*, M) \to 0 \tag{7.39}$$

Applying the zig-zag lemma we obtain a long exact sequence in homology which, since the differentials for the complexes $Hom(B_*, M)$ and $Hom(Z_*, M)$ are zero reduces to the exact triangle

$$\begin{array}{ccc} Hom(Z_*, M) & \xrightarrow{\delta} & Hom(B_*, M) \\ & {}_{i^*}\nwarrow \quad \swarrow_{\partial^*} & \\ & H^*(C_*; M) & \end{array} \tag{7.40}$$

There is also a short exact sequence of graded R-modules

$$0 \to B_* \xrightarrow{j} Z_* \xrightarrow{\partial} H_* \to 0 \tag{7.41}$$

coming from the definition of homology i.e.

$$Z_* = ker\partial : C_* \to C_*$$
$$B_* = Im\partial : C_* \to C_* \tag{7.42}$$
$$H_* = H_*(C_*) = Z_*/B_*$$

Observe that in the sequence

$$0 \to B_* \xrightarrow{j} Z_* \xrightarrow{\partial} H_* \to 0 \tag{7.43}$$

B_* and Z_* are free since R is a P.I.D. and these are submodules of the free module C_*. Thus using the second Ext axiom and the fact that $Ext(Z_*, M) = 0$ we obtain an exact sequence

$$0 \to Hom(H_*, M) \to Hom(Z_*, M) \xrightarrow{j^*} Hom(B_*, M) \to Ext(H_*, M) \to 0 \tag{7.44}$$

7.15 Corollary If R is a field, M is a vector space over R and C_* is a chain complex over R then

$$H^q(C_*; M) \cong Hom(H_q(C_*), M) \tag{7.45}$$

and the Kronecker pairing is non-degenerate. Applying the universal coefficient theorem to the singular or cellular complexes of a space or a pair of spaces one obtains the following

7.16 Corollary If (X, A) is a pair of spaces $A \subset X$; R a P.I.D., M a module over R then for each q the sequence

$$0 \to Ext_R(H_{q-1}(X, A; R), M) \to H^q(X, A; M) \to Hom(H_q(X, A; R), M) \to 0 \quad (7.46)$$

is a short, exact and natural and it splits although not naturally.

An important special case of the universal coefficient theorem for cohomology is the use of it for the computation of $H^q(X)$ i.e. cohomology with integer coefficients. For an abelian group A we have denoted the torsion subgroup (i.e. the subgroup of finite order elements) by $torsion(A)$. Let $free(A) = A/torsion(A)$. Then for a space X whose homology is finitely generated in every dimension (e.g. a finite CW-complex), the universal coefficient theorem shows that

$$H^q(X) \cong free(H_q(X)) \oplus torsion(H_{q-1}(X)) \quad (7.47)$$

It is also possible to define the dual of an abelian group A by $A^* = Hom(A, \mathbb{Z})$ and the torsion dual $A^\wedge \cong Hom(A, \mathbb{Q}/\mathbb{Z})$ then the universal coefficient theorem says that

$$H^q(X) \cong H_q(X)^* \oplus (torsion(H_{q-1}(X)))^\wedge \quad (7.48)$$

The r.h.s. is a contravariant functor in X but the isomorphism is not natural.

There are other formulations of the universal coefficient theorem for various rings and other algebraic structures. Their knowledge can accelerate the calculations significantly. For example one may have the following

7.18 Theorem If R is a P.I.D., M is a finitely generated R-module and C_* is a free chain complex over R then there is a split short exact sequence

$$0 \to H^q(C_*) \otimes M \to H^q(C_*, M) \to Tor_1^R(H^{q+1}(C_*, M)) \to 0 \quad (7.49)$$

There are universal coefficient theorems in homology as well. For example the following universal coefficient theorem measures the difference between first tensoring a complex with a module M and then passing to homology versus first passing to homology and then tensoring with M.

7.19 Theorem(Universal coefficient theorem for homology)[279, 280, 281] Suppose that R is a P.I.D., C_* is a free chain complex over R and M is a module over R. Then there is a natural short exact sequence

$$0 \to H_q(C_*) \otimes M \to H_q(C_* \otimes M) \to Tor_1^R(H_{q-1}(C_*), M) \to 0 \tag{7.50}$$

which splits although not naturally.

Proof The proof is very similar to the proof of the universal coefficient theorem in cohmology. In the same way, there is a short exact sequence

$$0 \to Z_* \xrightarrow{i} C \xrightarrow{\partial} B_* \to 0 \tag{7.51}$$

which remains exact when tensoring with M since B_* is free. Applying again the zig-zag lemma to the tensored sequence one obtains the exact triangle

$$\begin{array}{ccc} B_* \otimes M & \longrightarrow & Z_* \otimes M \\ \nwarrow & & \swarrow \\ & H_*(C_*; M) & \end{array} \tag{7.52}$$

The short exact sequence of graded R-modules

$$0 \to B_* \to Z_* \to H_*(C_*) \to 0 \tag{7.53}$$

gives using the second axiom of Torsion an exact sequence

$$0 \to Tor(H_*(C_*), M) \to B_* \otimes M \to Z_* \otimes M \to H_*(C_*) \otimes M \to 0 \tag{7.54}$$

Assembling the triangle and the exact sequence one obtains the short exact sequence

$$0 \to H_*(C_*) \otimes M \to H_*(C_*; M) \to Tor(H_*(C_*), M) \to 0 \tag{7.55}$$

If one takes the grading into account one finishes the proof.

7.20 Corollary If (X, A) is a pair of spaces $A \subset X$, R a P.I.D., M a module over R then for each q the sequence

$$0 \to H_q(X, A; R) \otimes M \to H_q(X, A; M) \to Tor_1^R(H_{q-1}(X, A; R), M) \to 0 \tag{7.56}$$

is short exact, natural and splits, again not naturally.

There is another universal coefficient theorem for homology. It addresses the question of how a different version of the Kronecker pairing fails to pass to a perfect pairing on (co)homology. In this case the pairing

$$Hom_R(C_*, R) \times (C_* \otimes M) \to M \tag{7.57}$$

is defined by

$$(f, z \otimes m) \to f(z)m \tag{7.58}$$

This pairing passes to a pairing on homology

$$H^q(C_*; R) \times H_q(C_* \otimes M) \to M \tag{7.59}$$

Taking the adjoint produces the homomorphism

$$\alpha : H_q(C_* \otimes M) \to Hom_R(H^q(C_*), M) \tag{7.60}$$

The kernel of this homomorphism is calculated for R a P.I.D. and C_* with a finitely generated homology by the following

7.21 Theorem Let R be a P.I.D., C_* a free chain complex over R such that $H_q(C_*)$ is finitely generated for each q and let M be an R-module. Then the sequence

$$0 \to Ext^1_R(H^{q+1}(C_*), M) \to H_q(C_*, M) \xrightarrow{\alpha} Hom(H^q(C_*), M) \to 0 \tag{7.61}$$

is a short exact sequence and splits.

As an application of the universal coefficient theorems we can identify the different versions of the Betti numbers of a space. The q-th Betti number $\beta_q(X)$ of a space X is the rank of $H_q(X; \mathbb{Z})$. Since $\mathbb{Q}$ and $\mathbb{R}$ are flat abelian groups, $Tor(*, \mathbb{Q})$, $Tor(*, \mathbb{R})$, $Ext(*, \mathbb{Q})$, $Ext(*, \mathbb{R})$ all vanish. This implies that

7.22 Corollary The following numbers are all equal: the Betti number $\beta_q(X)$, $dim_{\mathbb{Q}} H_q(X; \mathbb{Q})$, $dim_{\mathbb{R}} H_q(X; \mathbb{R})$, $dim_{\mathbb{Q}} H^q(X; \mathbb{Q})$ and $dim_{\mathbb{R}} H^q(X; \mathbb{R})$.

In particular if X is a compact smooth manifold by the above corollary and DeRham cohomology we see the q-th Betti number is the dimension of the real vector space of closed q-forms modulo exact q-forms.

Chapter 8

BV and BRST quantization, quantum observables and symmetry

"I know who I WAS when I got up this morning, but I think I must have been changed several times since then"

Lewis Carroll, Alice in Wonderland

8.1 BV-BRST quantization

Gauge redundancy has been a guiding principle for most of the theories about nature. Starting with quantum electrodynamics, continuing with Yang-Mills theories and Quantum Chromodynamics and reaching into the realms of supergravity, all theories appear to obey this principle. The existence of a gauge redundancy therefore appears to be ubiquitous. When performing path integral quantization, gauge fixing is a natural requirement. The existence of unphysical degrees of freedom would otherwise make practical calculations impossible. If we describe a physical system by its action functional S and the associated theory has gauge freedom, then the path integral would imply the integration over spurious gauge trajectories resulting in a divergence

$$\int DA_\mu e^{iS} = \infty \tag{8.1}$$

The redundant gauge variables will have to be removed from the theory. However, once gauge freedom is eliminated, it becomes harder to identify gauge independence. One should not forget that the gauge independence is one of the most important features of physical observables, and therefore must be obeyed in order to have physical meaning. One solution to this problem is the BRST-antifield formalism which allows us to apply path integral formalisms to any type of quantum gauge theory while preserving the spacetime covariance. In the quantum domain we remain with a global invariance under the so called BRST transformation. This operation plays the role of a differential and can be used in the definition of a cohomology. This cohomology plays an important role in identifying the connection to physics. In what follows I will introduce the reader into the main aspects of BRST quantization. I will therefore follow references [262-264] and present the basic facts related to gauge invariance and the nilpotency of the BRST operator from a geometric perspective. According to how the algebra of the gauge transformation behaves off-shell we can classify theories in the following way [265]: if the algebra of the gauge transformation closes off-shell i.e. the commutator of the gauge transformation is again a gauge transformation of the same type, without using the equations of motion, we have a theory similar to the Yang-Mills theories. QCD is such an example. If the theory closes only on-shell in a natural way i.e. it requires the equations of motion for closure, we call it an on-shell closed theory. One particular example for such a case is given by various theories of supergravity. Another class of gauge transformation is given by the so called reducible gauge transformations. If one considers a theory of an abelian $2-$form $B_{\mu\nu} = -B_{\nu\mu}$ with the field strength given by the form

$$H_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} + \partial_{\nu} B_{\rho\mu} + \partial_{\rho} B_{\mu\nu} \tag{8.2}$$

and a lagrangean of the form

$$L = -\frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \tag{8.3}$$

invariant under the gauge transformation

$$\delta_{\Lambda} B_{\mu\nu} = \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu} \tag{8.4}$$

where Λ is a gauge transformation parameter then these transformations vanish for a class of parameters $\Lambda_{\mu} = \partial_{\mu}\epsilon$. This brings us to the conclusion that not all gauge parameters are independent. That means the associated gauge transformations are reducible. Two-forms define a natural generalization of electromagnetism $A_\mu \to B_{\mu\nu}$.

The last class of transformations I mention here is formed by the so called on-shell reducible gauge transformations. This essentially means that the reducibility of the gauge transformation is possible only on-shell. If in the previous Lagrangean we introduce an auxiliary field, we obtain

$$L = \frac{1}{12} A_\mu \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} - \frac{1}{8} A_\mu A^\mu \tag{8.5}$$

By introducing an equation of motion for the field A_μ of the form

$$A_\mu = \frac{1}{3} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma} \tag{8.6}$$

the second Lagrangean reduces to the first one. The gauge transformations associated to the initial and the auxiliary fields are now defined as

$$\begin{gathered} \delta_\Lambda B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \\ \delta_\Lambda A_\mu = 0 \end{gathered} \tag{8.7}$$

What is important to notice at this moment is that BV-BRST are intrinsically cohomological theories. The nilpotent linear operator associated to the gauge transformation becomes the differential operator of the cohomological theory. In order to apply path integral quantization to a gauge theory, a gauge fixing procedure is needed. This leads to a loss of the gauge invariance. The BRST construction allows us to recover this lost gauge invariance in the form of a rigid symmetry called the BRST symmetry, present even after the gauge is fixed. However, in order to do this, additional auxiliary fields must be added into the theory: a ghost field and an additional conjugate antifield. The BRST operator (let me call it s) acts on the enlarged field space of fields, ghosts and antifields. The extended action involving all these fields is constructed such that it is BRST invariant. The operator s is called the BRST differential and plays the role of the differential in the cohomological construction. It is nilpotent ($s^2 = 0$) and therefore allows the construction of cohomological groups $H^k(s)$. The gauge invariant observables are given by the $0-$order cohomology group $H^0(s)$ in ghost number zero

$$H^0(s) \cong \mathcal{O} \tag{8.8}$$

where $\mathcal{O}$ is the set of gauge invariant observables. This means that if one identifies two BRST invariant functions that differ by a BRST exact function one just finds at ghost number zero the gauge invariant functions. The derivation of the gauge-fixed Lagrangean is done in two steps. In the first step one replaces the original local gauge invariance with an equivalent global symmetry called the BRST symmetry in such a way that the BRST invariance replaces the gauge invariance. This does not require any gauge fixing. Second, one chooses certain gauge fixing conditions and computes the gauge fixed action in a way that incorporates the BRST invariance. The BRST symmetry operator s acts as a graded odd derivative on the original fields (let me call them ϕ^i) but also on some extra fields i.e. for any A and B where B has the Grassmann parity ϵ_B one finds

$$\begin{gathered} s(AB) = A(sB) + (-1)^{\epsilon_B}(sA)B \\ s^2 = 0 \end{gathered} \tag{8.9}$$

The grading of s is the ghost number and we have

$$\begin{aligned} &gh(sA) = gh(A) + 1 \\ &\epsilon(sA) = \epsilon_A + 1 \qquad\qquad (mod\ 2) \end{aligned} \tag{8.10}$$

We can see the BRST symmetry as a canonical transformation in an appropriate bracket structure $(,)$. We have

$$sA = (A, S) \tag{8.11}$$

where we see S as the canonical generator of s. These requirements completely determine S up to a canonical transformation and capture the BRST symmetry [266].

In order to construct a nilpotent symmetry for which the gauge invariant observables form a set isomorphic to the zero-order cohomology group of the associated operation, it is necessary to see how the gauge invariant functions (observables) are described. For a manifestly relativistic covariant description, let us first assume that there is no gauge invariance. The observables are then realized as the phase space functions $F(q,p)$. But a phase space point refers to the state of the system at a given instant of time. As (q,p) at $t = t_0$ completely determines $(q(t), p(t))$ through the Hamiltonian equations one can alternatively view phase space as the space of all solutions of the equation of motion. If we eliminate the momenta and consider only the solutions $q(t)$ of the equations of motion these q usually take a manifestly covariant form. The space of all solutions of the equations of motion is known as the covariant phase space. In quantum field theory the situation is very similar. Observables can be viewed as functions $f(\phi^i)$ of the solutions

ϕ^i of the equations of motion $\frac{\delta S_0}{\delta \phi^i} = 0$. Let me call I the infinite dimensional functional space of all possible field history. A point of I will be an arbitrary entire history that may not solve $\frac{\delta S_0}{\delta \phi^i} = 0$. In I, the equations of motion $\frac{\delta S_0}{\delta \phi^i} = 0$ determine a submanifold Σ called the stationary surface. This submanifold is called the covariant phase space in the absence of a gauge invariance. The formalism and the basic reasoning follows reference [267]. The observables are the functions defined on Σ namely the elements of $C^{\infty}(\Sigma)$. Any function f on Σ can be extended off of Σ to a function $F(\phi^i)$ defined on I namely to an element of $C^{\infty}(I)$. Two different extensions F and F' differ by a function that vanishes on Σ. These functions form an ideal $\mathcal{N}$ as FF' vanishes on Σ whenever F or F' does. The algebra $C^{\infty}(\Sigma)$ of the smooth functions on Σ is thus the quotient algebra $C^{\infty}(I)/\mathcal{N}$ of the smooth functions on I by the functions that vanish on Σ. The same considerations are applicable in quantum mechanics as the observables can be identified with the operator valued functions of $\hat{q}$ and $\hat{p}$ at a given moment of time. In order to consider all the solutions of the equations of motion, not just the ones corresponding to a set of initial data, the space of histories I should not be restricted by boundary conditions at the initial and final times t_i and t_f. The stationary surface Σ contains then all the possible dynamical states of the system. Therefore the space I is not the space $I_{i \to f}$ over which one integrates in the path integral representation of a definite quantum mechanical amplitude between given in and out states. The space I is actually the union over all possible pairs of the in and out states of the spaces $I_{i \to f}$. Moreover, we do not vary the boundary data at t_1 and t_2 in the action principle. Now let me consider the case when there is gauge freedom. In that case the observables should also be gauge invariant. The gauge transformation $\delta_{\epsilon}\phi^i = R^i_{\alpha}\epsilon^{\alpha}$ is integrable when the equations of motion are hold. In that case they generate well defined orbits on Σ. This surface has a dimension equal to the number of independent R^i_{α}. The gauge invariant functions are constant along the gauge orbits and induce definite functions on the quotient space Σ/G of the stationary surface by the gauge orbits. The space of observables therefore is written as $C^{\infty}(\Sigma/G)$ i.e. the space of smooth functions on Σ/G. The gauge invariant observables are therefore reached in two steps. First one goes from I to Σ then one goes from Σ to Σ/G. One must now find a nilpotent operator s that implements these two steps through its cohomology $H^0(s) = C^{\infty}(\Sigma/G)$. This operator contains two nilpotent components, each inducing one of the two steps above. The first differential δ induces the so called Koszul-Tate resolution of $C^{\infty}(\Sigma)$ such that $H_0(\delta) = \frac{Ker(\delta)}{Im(\delta)} = C^{\infty}(\Sigma)$. It implements the first step leading from I to Σ. The second component d is the vertical exterior derivative along the gauge orbits and implements the second step, from Σ to Σ/G giving $H^0(d) = C^{\infty}(\Sigma/G)$. The BRST derivative is in general the formal sum between the two $s = \delta + d$. In order to construct the BRST formalism we need to implement the restriction from I to Σ. So, one needs to define a differential δ that acts as a nilpotent graded derivative on polynomials in some generators with coefficients that are functions on I, in the same way in which d in the standard exterior calculus acts on polynomials in dx, dy, dz, ... with coefficients that are functions on the manifold. Then δ must compute $C^{\infty}(\Sigma)$ through its homology. The grading of δ is called the antighost number. As δ decreases the antighost number by one unit, it behaves like a boundary operator. We need also that δ computes $C^{\infty}(\Sigma)$ through its homology, therefore

$$H_0(\delta) = \frac{Ker(\delta)}{Im(\delta)}_0 = C^{\infty}(\Sigma) = C^{\infty}(I)/\mathcal{N} \tag{8.12}$$

Moreover, this relation contains all the homology of δ. This means $H_k(\delta) = 0$, for $k \neq 0$. This is necessary in order for the BRST cohomology at ghost number zero to be given by the gauge invariant functions and for being able to prove the BRST symmetry itself. A differential complex with the properties above is said to provide a resolution of the quotient algebra $C^{\infty}(I)/\mathcal{N}$. In the present context the relevant resolution is given by the Koszul-Borel-Tate resolution. In the absence of gauge invariance the construction of δ is relatively simple. We simply define it so that

$$\begin{gathered} Ker(\delta)_0 = C^{\infty}(I) \\ Im(\delta)_0 = \mathcal{N} \end{gathered} \tag{8.13}$$

and we therefore put $\delta\phi^i = 0$. This implies by Leibnitz rule that $\delta F(\phi^i) = 0$ for any functions on I and hence $Ker(\delta)_0 = C^{\infty}(I)$. To implement $Im(\delta)_0 = \mathcal{N}$ we notice that due to the regularity assumptions the elements of $\mathcal{N}$ are given by the combinations of the field equations

$$G(\phi^i) \in \mathcal{N} \leftrightarrow G(\phi^i) = \lambda^j(\phi^i)\frac{\delta S_0}{\delta\phi^j} \tag{8.14}$$

Therefore we introduce as many new generators ϕ_i^* as there are field equations and set

$$\delta\phi_i^* = -\frac{\delta S_0}{\delta\phi^i} \tag{8.15}$$

These new generators are called the antifields associated to the original fields. To preserve the grading properties of δ one must impose $\epsilon(\phi_i^*) = 1$ as we assume the fields to be bosonic and $antigh(\phi_i^*) = 1$. The action of δ on a general polynomial in ϕ^i and ϕ_i^* is obtained by using the Leibnitz rule and the nilpotency can easily be checked. To see whether δ provided a resolution of $\frac{C^{\infty}(I)}{\mathcal{N}}$ it remains to compute $H_k(\delta)$. But in this case we do not have a gauge invariance. The equations of motion are then independent so that the number of new objects ϕ_i^* in degree one is exactly equal to the number of independent equations of motion. Using this property one proves $H_k(\delta) = 0$, $k \neq 0$.

When a gauge freedom exists one can still find $H_0(\delta) = C^\infty(\Sigma)$. However $H_k(\delta)$ does not look the same. Using the Noether identity

$$\frac{\delta S_0}{\delta\phi^i} R^i_\alpha = 0 \tag{8.16}$$

one finds non-trivial δ-closed polynomials in degree one. These are given by $R^i_\alpha \phi^*_i$. One checks that the $R^i_\alpha \phi^*_i$ are δ-closed

$$\delta(R^i_\alpha \phi^*_i) = -R^i_\alpha \frac{\delta S_0}{\delta\phi^i} = 0 \tag{8.17}$$

and exhaust all non-trivial δ-closed polynomials of degree one

$$\delta(\lambda^i \phi^*_i) = 0 \rightarrow \lambda^i \phi^*_i = \mu^\alpha R^i_\alpha \phi^*_i + \delta(\frac{1}{2}\epsilon^{ij}\phi^*_i \phi^*_j) \tag{8.18}$$

Moreover, $R^i_\alpha \phi^*_i$ are not exact and hence $H_1(\delta) \neq 0$. This problem can be solved. Let us first assume that the gauge transformations are independent, so that all the non-trivial cycles are independent as well. We can recover $H_1(\delta) = 0$ and at the same time $H_k(\delta) = 0 \ \forall k \neq 0$ by simply adding one new generator ϕ^*_α for each cycle and define

$$\delta\phi^*_\alpha = R^i_\alpha \phi^*_i \tag{8.19}$$

Now, because $\delta(R^i_\alpha \phi^\alpha_i) = 0$ one has $\delta^2\phi^*_\alpha = 0$. Furthermore, taking $antigh(\phi^*_\alpha) = 2$ and $\epsilon(\phi^*_\alpha) = 0$ and extending δ as a graded derivation to any polynomial in ϕ^i, ϕ^*_i and ϕ^*_α one maintains $\delta^2 = 0$. With the introduction of the antifields ϕ^*_α the cycles $R^i_\alpha \phi^*_i$ that were not exact become exact. Therefore now, $H_1(\delta) = 0$. Using the assumed irreducibility of the gauge transformation one can now show that $H_k(\delta) = 0 \ \forall k > 0$.

The interpretations of the various constructions of the BV formalism are summarized in the following table

Table 8.1: Physical meaning of cohomological objects in gauge theory

Physics	Mathematical concept	degree of ghost	degree of anti-ghost
field	section of bundle	0	0
anti-field	Koszul generator	0	1
ghost	Chevalley Eilenberg generator	1	0
anti-ghost	Tate generator	0	2

The paragraph above justified from a field theoretical perspective the BRST construction and identified the gauge invariant observables with elements of the zero-degree BRST cohomology. In order to have a better idea of what means to discuss about coefficient groups for such a cohomology we need first to make some algebraic aspects about Lie algebra cohomology clearer.

Therefore I will introduce the Chevalley-Eilenberg cohomology of a Lie algebra which will represent one half of the BRST cohomology. Using mainly reference [268], the way this will be done will manifest the main characteristics of coefficient groups. As we already saw, the natural algebraic structure associated to a smooth manifold M is its algebra $C^\infty(M)$ of smooth functions on it. It is a commutative, associative unital algebra which encodes information about M. A symplectic structure on M gives to $C^\infty(M)$ additional structure. The Poisson bracket turns $C^\infty(M)$ into a Lie algebra and for any $f \in C^\infty(M)$, $\{f, *\}$ is a derivation over the commutative multiplication. This turns $C^\infty(M)$ into a Poisson algebra. If we define a closed embedded submanifold M_0 of M we induce an ideal $I \subset C^\infty(M)$ consisting of those functions which vanish on M_0. We call this the vanishing ideal of M_0. If $M_0 = \Phi^{-1}(0)$ is the zero locus of a smooth function $\Phi : M \to \mathbb{R}^k$ where $0 \in \mathbb{R}^k$ is a regular value, then the ideal I is precisely the ideal generated by the components ϕ_i of Φ relative to any basis for $\mathbb{R}^k$. Every smooth function on M restricts to a smooth function on M_0 and two such functions restrict to the same function if and only if their difference belongs to the ideal I. Conversely every smooth function on M_0 can be extended but possibly not uniquely to a smooth function on M. That means there is an isomorphism

$$C^\infty(M_0) \cong C^\infty/I \tag{8.20}$$

Note that vector fields are derivations of the algebra of functions $X(M) = Der(C^\infty(M))$. The isomorphism above assures us that a derivation of $C^\infty(M)$ gives rise to a derivation of $C^\infty(M_0)$ provided it preserves the ideal I. One can see that

$$Der(C^\infty(M_0)) = \{\xi \in Der(C^\infty(M)) | \xi(I) \subset I\} \tag{8.21}$$

Vector fields in $TM_0^\perp$ are the hamiltonian vector fields which arise from functions in I. Therefore $TM_0^\perp \subset TM_0$ is the condition that the vanishing ideal is closed under the Poisson bracket $\{I, I\} \subset I$. Such ideals are called coisotropic. The functions on $\tilde{M}$ are those functions on M_0 which are constant on the leaves of the foliation. Since the leaves are connected and the tangent vectors to the leaves are the hamiltonian vector fields of functions in I, we have an isomorphism $C^\infty(\tilde{M}) = \{f \in C^\infty(M_0) | \{f, I\} = 0\}$, where $\{f, I\} = 0$ on M_0. Extending f to a function on M, the isomorphism becomes

$$C^{\infty}(\tilde{M}) = \{f \in C^{\infty}(M) | \{f, I\} \subset I\}/I \tag{8.22}$$

and it does not depend on the extension because I is closed under the Poisson bracket. This means that

$$C^{\infty}(\tilde{M}) = N(I)/I \tag{8.23}$$

where $N(I)$ is the Poisson normalizer of I in $C^{\infty}(M)$. The algebraic formalism continues to make sense in situations where the geometry may become singular [269]. The aim of the BRST construction is to construct a complex of Poisson superalgebras and a differential which is a Poisson superderivative so that its cohomology (at least in degree zero) is isomorphic as a Poisson algebra to $N(I)/I$. In this context it is important to understand the basics of equivariant cohomology. The most important aspect of this type of cohomology is that the spaces dealt with by it are subject to a group action. Therefore the equivariant cohomology of a space X with action of a topological group G is defined as the ordinary cohomology with coefficient W of $E \times_G X$ namely $H^*_G(X, W) = H^*(EG \times_G X, W)$. If G acts freely on X then the canonical map $EG \times_G X \to X/G$ is a homotopy equivalence, and therefore $H^*_G(X, W) = H^*(X/G, W)$. The most common situation appears when X is a manifold, G is a compact Lie group and W is either the field of real numbers or that of complex numbers. Then the equivariant cohomology can be computed by means of the Cartan model.

8.2 The Harmonic oscillator

Now that we have a cohomological interpretation for the BV-BRST quantization prescription, we can apply it to two simple examples: the free particle and the the harmonic oscillator. The Harmonic oscillator is a particularly simple example. In order to formulate such a problem in a cohomological sense, revealing the BRST quantization prescription, one has to generate and to fix an artificial set of fictitious symmetries. Following reference [267-269] this is being done with no additional difficulty. Indeed, the resulting cohomology will describe the correct harmonic oscillator. Then I will apply a change in coefficient structures for the BV-BRST cohomology and explore the consequences. It should be noted that the harmonic oscillator as well as the free particle are trivial problems from the perspective of the coefficient structure in (co)homology when there is no ambiguity related to the topology of the associated space (or spacetime). To be fully consistent, let me start with the conventional canonical quantization. Let the time coordinate be x and the oscillator degree of freedom be $\phi(x)$. The action will be of the form

$$S = \frac{1}{2}\int dx(\phi^{'}(x)^2 - \omega^2\phi(x)^2) \tag{8.24}$$

where $\phi^{'} = \frac{d\phi}{dx}$. By using the Euler Lagrange operator we obtain the Euler Lagrange equation

$$\epsilon(x) = -\frac{\delta S}{\delta\phi(x)} = \phi^{''}(x) + \omega^2\phi(x) = 0 \tag{8.25}$$

If we are interested in the Hamiltonian formalism we define the canonical momenta as $\pi(x) = \phi^{'}(x)$ and we construct the formalism such that it obeys the equal time commutation relations

$$[\phi(x), \pi(x)] = i \quad [\phi(x), \phi(x)] = [\pi(x), \pi(x)] = 0 \tag{8.26}$$

Having now the pairs of variables (ϕ, π), we can span a phase space $\mathcal{P}$. Hamilton's equations will therefore govern the time evolution of any function $F(\phi, \pi)$ over $\mathcal{P}$ in the sense that it will satisfy the equation

$$F^{'}(x) = i[H, F(x)] \tag{8.27}$$

with the Hamiltonian given by

$$H = \frac{1}{2}(\pi^2 + \omega^2\phi^2) \tag{8.28}$$

The Euler Lagrange equations then become simply

$$\phi^{'}(x) = \pi(x) \quad \pi^{'}(x) = -\omega^2\phi(x) \tag{8.29}$$

We can chose a basis in $\mathcal{P}$ given by the creation and annihilation operators

$$a^{\dagger} = \tfrac{1}{\sqrt{2\omega}}(\omega\phi - i\pi) \quad a = \tfrac{1}{\sqrt{2\omega}}(\omega\phi + i\pi) \tag{8.30}$$

with the canonical commutation relations

$$[a, a^\dagger] = 1 \tag{8.31}$$

The Hamiltonian obviously becomes

$$H = \omega a^\dagger a \tag{8.32}$$

and time evolution is simply given by

$$a'(x) = -i\omega a(x) \quad (a^\dagger)'(x) = i\omega a^\dagger(x) \tag{8.33}$$

In order to have a quantum image about the phenomenon, let $|0\rangle$ be the vacuum state, mapped into zero by the annihilation operators. In this context a basis for the Hilbert space is therefore

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle \tag{8.34}$$

where

$$H|n\rangle = n\omega |n\rangle \tag{8.35}$$

and in the dual space

$$\langle n| = \frac{1}{\sqrt{n!}} \langle 0| a^n \tag{8.36}$$

Obviously $\langle m| |n\rangle = \delta_{mn}$. This same approach can be achieved by treating the dynamics as a constraint in the phase space. By introducing so called virtual histories $\phi(x)$ with canonical momenta $\pi(x) = -\frac{\delta}{\delta\phi(x)}$. One will have to satisfy the Heisenberg algebra

$$[\phi(x), \pi(x')] = i\delta(x - x') \quad [\phi(x), \phi(x')] = [\pi(x), \pi(x')] = 0 \tag{8.37}$$

These vanish whenever $x \neq x'$. The Euler Lagrange equation has two solutions ϕ_ω and $\phi_{-\omega}$ which in the real space are

$$\begin{aligned} \phi(x) &= \phi_\omega e^{i\omega x} + \phi_{-\omega} e^{-i\omega x} \\ \pi(x) &= i\omega\phi_\omega e^{i\omega x} - i\omega\phi_{-\omega} e^{-i\omega x} \end{aligned} \tag{8.38}$$

To describe the harmonic oscillator in this history space we must eliminate the additional variables. If we regard the dynamics as a constraint in this history space we must transform the Euler Lagrange equation into a dynamical constraint

$$\epsilon_k : (k^2 - \omega^2)\phi_k \cong 0 \tag{8.39}$$

The definition of momentum becomes the momentum constraint

$$\mathcal{M}_k : \pi_k - ik\phi_k \cong 0 \tag{8.40}$$

where $\cong$ means equality provided constraints. Using a vectorial notation for ϵ_k and $\mathcal{M}_k$ we write

$$\chi_k = \begin{bmatrix} \epsilon_k \\ \mathcal{M}_k \end{bmatrix} = \begin{bmatrix} (k^2 - \omega^2)\phi_k \\ \pi_k - ik\phi_k \end{bmatrix} \tag{8.41}$$

Then for the Poisson bracket we have the matrix

$$\Delta_{kk'} = [\chi_k, \chi_{k'}] = \begin{bmatrix} 0 & i(k^2 - \omega^2) \\ -i(k^2 - \omega^2) & 2k \end{bmatrix} \delta_{k+k'} \tag{8.42}$$

For $k \neq \pm\omega$ the matrix is non-singular and the constraint $\chi_k \cong 0$ is second class. Define the Dirac brackets as

$$[F, G]^* = [F, G] - \sum_k \sum_{k'} [F, \chi_k] \Delta^{kk'} [\chi_{k'}, G] \tag{8.43}$$

here $\Delta^{kk'}$ is the inverse of $\Delta_{kk'}$. Now we can eliminate the constraints. This leads to $\phi_k \cong \pi_k \cong 0$. For $k = \pm\omega$, the dynamics constraint $\epsilon_\omega = \epsilon_{-\omega}$ vanishes, $\Delta_{kk'}$ is singular and we can only impose the momentum constraints $\mathcal{M}_\omega \cong \mathcal{M}_{-\omega} \cong 0$. The non-singular part of the Poisson bracket is now

$$\Delta_{\omega,-\omega} = [\mathcal{M}_\omega, \mathcal{M}_{-\omega}] = 2\omega \tag{8.44}$$

The associated Dirac brackets commute with the momentum constraints so we can eliminate two of the four variables ϕ_ω, $\phi_{-\omega}$, π_ω, $\pi_{-\omega}$ in terms of the two others. The independent solutions are then

$$
\begin{aligned}
a_\omega &= \tfrac{1}{\sqrt{2\omega}}(\pi_\omega + i\omega\phi_\omega) \cong \sqrt{\tfrac{2}{\omega}}\pi_\omega \cong i\sqrt{2\omega}\phi_\omega \\
a_{-\omega} &= \tfrac{1}{\sqrt{2\omega}}(\pi_{-\omega} - i\omega\phi_{-\omega}) \cong \sqrt{\tfrac{2}{\omega}}\pi_{-\omega} \cong -i\sqrt{2\omega}\phi_{-\omega}
\end{aligned}
\tag{8.45}
$$

Since $[a_\omega, a_{-\omega}] = -1$ we can identify $a_\omega = a^\dagger$, $a_{-\omega} = a$. The hamiltonian is simply the generator of rigid time translations

$$H = \int dx\phi^{'}(x)\pi(x) = \sum_{k=-\infty}^{\infty} ik\phi_k\pi_{-k} \tag{8.46}$$

Once the constraints are in place we obtain again the harmonic oscillator hamiltonian, as expected

$$H \cong i\omega(\phi_\omega\pi_{-\omega} - \phi_{-\omega}\pi_\omega) = \omega a_\omega a_{-\omega} \tag{8.47}$$

What is important to notice at this point is that one can look at the constraints in various different ways. One can for example divide them into first class constraints and gauge fixation constraints and this then suggests the possibility of applying the BRST-cohomology. Unphysical variables are expected and in order to eliminate them one has to introduce fermionic antifields ϕ_k^* with canonical momenta π_k^*, subject to the canonical anticommutation relations

$$\{\phi_k^*, \pi_{k'}^*\} = \delta_{k+k'}, \quad \{\phi_k^*, \phi_{k'}^*\} = \{\pi_k^*, \pi_{k'}^*\} = 0 \tag{8.48}$$

We then will have an extended history phase space $\mathcal{HP}^*$ and the functions over such a space will belong to the space called $C(\mathcal{HP}^*) = C(\phi, \phi^*, \pi, \pi^*)$. The dynamics constraint is implemented by the BRST charge

$$Q_D = \sum_{k=-\infty}^{\infty} \epsilon_k\pi_{-k}^* = \sum_{k=-\infty}^{\infty} (k^2 - \omega^2)\phi_k\pi_{-k}^* \tag{8.49}$$

which acts like $\delta_D F = [Q_D, F]$

$$
\begin{aligned}
\delta_D\phi_k &= 0, & \delta_D\pi_k &= i(k^2-\omega^2)\pi_k^* \\
\delta_D\phi_k^* &= (k^2-\omega^2)\phi_k, & \delta_D\pi_k^* &= 0
\end{aligned}
\tag{8.50}
$$

The cohomology are as a result generated by all eight variables with $k = \pm\omega$. The physical phase space is generated only by the two variables a and $a^\dagger$. To eliminate this over-counting, additional terms must be added. Let us therefore add bosonic antifields θ_ω and $\theta_{-\omega}$ with canonical momenta χ_ω and $\chi_{-\omega}$ satisfying the commutation relations

$$[\theta_k, \chi_{k'}] = i\delta_{k+k'}, \quad [\theta_k, \theta_{k'}] = [\chi_k, \chi_{k'}] = 0 \tag{8.51}$$

where k, k' are either $\pm\omega$. The antifield constraint $\phi^*_\omega \cong \phi^*_{-\omega} \cong \pi^*_\omega \cong \pi^*_{-\omega} \cong 0$ can be implemented by the BRST operator

$$Q_A = \phi^*_\omega \chi_{-\omega} + \phi^*_{-\omega}\chi_\omega \tag{8.52}$$

acting in the extended phase space $(\phi_k, \pi_k, \phi^*_k, \pi^*_k, \theta_{\pm\omega}, \chi_{\pm\omega})$. The two BRST generators commute $\{Q_D, Q_A\} = 0$. The antifield constraints act similarly $\delta_A F = [Q_A, F]$ where

$$\begin{array}{cc} \delta_A\phi^*_\omega = 0, & \delta_A\phi^*_{-\omega} = 0 \\ \delta_A\pi^*_\omega = \chi_\omega, & \delta_A\pi^*_{-\omega} = \chi_{-\omega} \\ \delta_A\theta_\omega = -i\phi^*_\omega, & \delta_A\theta_{-\omega} = -i\phi^*_{-\omega} \\ \delta_A\chi_\omega = 0, & \delta_A\chi_\omega = 0 \end{array} \tag{8.53}$$

We notice that $Ker(\delta_A) = Im(\delta_A) = C(\phi^*_\omega, \phi^*_{-\omega}, \chi_\omega, \chi_{-\omega})$. This gives a trivial cohomology and therefore all supplemental variables vanish in cohomology as expected. We remain with the four variables $\phi_\pm$, $\pi_{\pm\omega}$ only. Two still have to be eliminated. We therefore use the momentum constraints

$$\mathcal{M}_\omega = \pi_\omega - i\omega\phi_\omega, \quad \mathcal{M}_{-\omega} = \pi_{-\omega} + i\omega\phi_{-\omega} \tag{8.54}$$

satisfying

$$[\mathcal{M}_\omega, \mathcal{M}_{-\omega}] = 2\omega \tag{8.55}$$

These constraints are second class. We introduce two canonically conjugate fermionic antifields β_ω and $\beta_{-\omega}$

$$\{\beta_\omega, \beta_{-\omega}\} = 1, \quad \{\beta_\omega, \beta_\omega\} = \{\beta_{-\omega}, \beta_{-\omega}\} = 0 \tag{8.56}$$

The associated BRST charge becomes

$$Q_M = \mathcal{M}_\omega \beta_{-\omega} \tag{8.57}$$

and again $\delta_M F = [Q_M, F]$ with

$$\begin{array}{cc} \delta_M \phi_\omega = 0, & \delta_M \phi_{-\omega} = -i\beta_{-\omega} \\ \delta_M \pi_\omega = 0, & \delta_M \pi_{-\omega} = \omega\beta_{-\omega} \\ \delta_M \beta_\omega = \mathcal{M}, & \delta_M \beta_{-\omega} = 0 \\ \delta_M \mathcal{M}_\omega = 0, & \delta_M \mathcal{M}_{-\omega} = 2\omega\beta_{-\omega} \end{array} \tag{8.58}$$

$Ker(Q_M)$ is generated by $(\phi_\omega, \pi_\omega, \beta_{-\omega}, \pi_{-\omega} - i\omega\phi_{-\omega})$. $Im(Q_M)$ is generated by $(\mathcal{M}_\omega = \pi_\omega - i\omega\phi_\omega, \beta_{-\omega})$. The cohomology $H(Q_M)$ is thus generated by

$$\begin{array}{c} a_\omega = \frac{1}{\sqrt{2\omega}}(\pi_\omega + i\omega\phi_\omega) + x\mathcal{M}_\omega \\ a_{-\omega} = \frac{1}{\sqrt{2\omega}}(\pi_{-\omega} - i\omega\phi_{-\omega}) \end{array} \tag{8.59}$$

where x is an arbitrary constant. They have the non-zero brackets

$$[a_\omega, a_{-\omega}] = -1 \tag{8.60}$$

In particular $a_{-\omega}$ commutes with $\mathcal{M}_\omega$ so the bracket is independent of the parameter x, as is necessary because it must be well defined in cohomology. If we make the particular choice $x = 0$, a_ω also commutes with $\mathcal{M}_{-\omega}$ and we can identify $a_\omega = a^\dagger$, $a_{-\omega} = a$. We therefore started with the space of functions over the extended phase space $C(\phi_k, \pi_k, \phi_k^*, \theta_{-\omega}, \chi_{-\omega}, \chi_\omega, \beta_\omega, \beta_{-\omega})$. With the BRST charge $Q = Q_D + Q_A + Q_M$ which is nilpotent we can form a cohomology with the degree zero groups given by

$$H^0_{classic}(Q) = C(a_\omega, a_{-\omega}), \quad H^n_{classic}(Q) = 0, \forall n \neq 0 \tag{8.61}$$

We have therefore obtained the resolution of the classical phase space $C(a_\omega, a_{-\omega})$. In the extended history phase space we can define a natural hamiltonian as the generator of rigid time translations

$$H = \sum_{k=-\infty}^{\infty} k(i\phi_k\pi_{-k} + \phi_k^*\pi_{-k}^*) + i\omega\theta\chi_{-\omega} - i\omega\theta_{-\omega}\chi_\omega + \omega\beta_\omega\beta_{-\omega} \tag{8.62}$$

The BRST charge commutes with the hamiltonian $[H, Q] = 0$. Hence the Hamiltonian acts in a well defined manner on the cohomology groups. We verify that the hamiltonian can be written as

$$H = \omega a_{\omega} a_{-\omega} + \{Q, O\} \tag{8.63}$$

where

$$O = \sum_{k^2 \neq \omega^2} \frac{ik}{(k^2 - \omega^2)} \phi_k^* \pi_{-k} + i\omega(\theta_{\omega}\pi_{-\omega}^* - \theta_{-\omega}\pi_{\omega}^*) + \frac{1}{2}\beta_{\omega}\mathcal{M}_{-\omega} \tag{8.64}$$

In particular on the zero degree cohomology $H^0_{classic}(Q) = C(a_{\omega}, a_{-\omega})$ it is equivalent with the operator $H = \omega a_{\omega} a_{-\omega}$.

The cohomology therefore encodes the physical states. We did not yet perform a quantization. A consistent method of doing this is to quantize the theory in the extended history phase space and then to restrict the theory by means of constraints while going to cohomology. Let there be a vacuum state $|0\rangle$ annihilated by all negative frequency operators. Therefore one has

$$\begin{aligned} \phi_{-k}|0\rangle = \pi_{-k}|0\rangle = \phi^*_{-k}|0\rangle = \pi^*_{-k}|0\rangle = \\ = \theta_{-\omega}|0\rangle = \chi_{-\omega}|0\rangle = \beta_{-\omega}|0\rangle = 0 \end{aligned} \tag{8.65}$$

Since all negative frequency operators annihilate the vacuum, quantization leaves us with the state space $H^0_{quant}(Q) = C(a_{\omega})$, $H^n_{quant}(Q) = 0, \forall n \neq 0$. A basis for the state space $H^0_{quant}(Q)$ is thus given by

$$|n\rangle = \frac{1}{\sqrt{n!}} a^n_{\omega}|0\rangle \tag{8.66}$$

An operator A is physical if $[Q, A] = 0$ and two physical operators A and A' are equivalent if $A' = A + [Q, B]$, for arbitrary B. The operator cohomology for an arbitrary state is therefore given by $H^0_{quant}(Q) = C(a_{\omega}, a_{-\omega})$. The hamiltonian commutes with the BRST charge $[H, Q] = 0$ and the energy of the n-quanta state is given by $H|n\rangle = n\omega|n\rangle$. If we look now at the hermitian conjugates of the BRST operator we note that $Q_D^{\dagger} = Q_D$ and $Q_A^{\dagger} = Q_A$ but $Q_M^{\dagger} = \mathcal{M}_{-\omega}\beta_{\omega} \neq Q_M = \mathcal{M}_{\omega}\beta_{-\omega}$ we notice that the full BRST operator is not yet self-adjoint. This appears because the $\mathcal{M}_{\pm\omega}$ constraints are second class. Making them first class means to employ again a pair of canonically conjugate variables, this time bosonic $\alpha_{\pm\omega}$ with non-zero brackets $[\alpha_\omega, \alpha_{-\omega}] = -2\omega$. The new momentum constraints become then

$$\begin{aligned} \mathcal{M}'_\omega &= \mathcal{M}_\omega + \alpha_\omega = \pi_\omega - i\omega\phi_\omega + \alpha_\omega \\ \mathcal{M}'_{-\omega} &= \mathcal{M}_{-\omega} + \alpha_{-\omega} = \pi_{-\omega} + i\omega\phi_{-\omega} + \alpha_{-\omega} \end{aligned} \tag{8.67}$$

which are first class $[\mathcal{M}'_\omega, \mathcal{M}'_{-\omega}] = 0$. Now, we let the antifields $\beta_{\pm\omega}$ anticommute and introduce their canonical momenta $\gamma_{\pm\omega}$. The brackets then become

$$\begin{aligned} \{\beta_\omega, \gamma_{-\omega}\} &= \{\beta_{-\omega}, \gamma_\omega\} = 1 \\ \{\beta_{\pm\omega}, \beta_{\pm\omega}\} &= \{\gamma_{\pm\omega}, \gamma_{\pm\omega}\} = 0 \end{aligned} \tag{8.68}$$

and then the BRST charge becomes

$$Q'_M = \mathcal{M}'_\omega \gamma_{-\omega} + \mathcal{M}'_{-\omega}\gamma_\omega \tag{8.69}$$

The $Q = Q_A + Q'_M$ part of the BRST charge acts on the modes with $k = \omega$ as

$$\begin{aligned} \delta\phi_\omega &= -i\gamma_\omega, & \delta\pi_\omega &= -\omega\gamma_\omega \\ \delta\phi^*_\omega &= 0, & \delta\pi^*_\omega &= \chi_\omega \\ \delta\beta_\omega &= \mathcal{M}'_\omega, & \delta\gamma_\omega &= 0 \\ \delta\theta_\omega &= -i\phi^*_\omega, & \delta\chi_\omega &= 0 \\ \delta\alpha_\omega &= 2\omega\gamma_\omega \end{aligned} \tag{8.70}$$

Observe that $\delta\phi_\omega$, $\delta\pi_\omega$ and $\delta\alpha_\omega$ are proportional to γ_ω. Two linearly independent combinations of these variables belong to the kernel, namely $\pi_\omega + i\omega\phi_\omega$ and $\mathcal{M}'_\omega$. Also in the kernel are ϕ^*_ω, γ_ω and χ_ω. $Im(Q)$ is generated by $(\mathcal{M}'_\omega, \phi^*_\omega, \gamma_\omega, \chi_\omega)$. Therefore $dimH^*(Q) = 1$. But the action on modes with $k = -\omega$ is completely analogous and Q_D eliminates all modes with $k^2 \neq \omega^2$ the generators of $H^*(Q)$ is

$$\begin{aligned} a_\omega &= \tfrac{1}{\sqrt{2\omega}}(\pi_\omega + i\omega\phi_\omega) + x\mathcal{M}'_\omega \\ a_{-\omega} &= \tfrac{1}{\sqrt{2\omega}}(\pi_{-\omega} - i\omega\phi_{-\omega}) + y\mathcal{M}'_{-\omega} \end{aligned} \tag{8.71}$$

with x and y arbitrary constants. With these new terms the BRST charge $Q = Q_D + Q_A + Q'_M$ is self-adjoint i.e. $Q^\dagger = Q$. The hamiltonian must still commute with the BRST charge and therefore must include also the new modes. Its form will be

$$H = \sum_{k=-\infty}^{\infty} k(i\phi_k \pi_{-k} + \phi_k^* \pi_{-k}^*) + i\omega\theta_\omega \chi_{-\omega} - i\omega\theta_{-\omega}\chi_\omega$$
$$+\omega\beta_\omega\gamma_{-\omega} - \omega\beta_{-\omega}\gamma_\omega + \tfrac{1}{2}\alpha_\omega\alpha_{-\omega} \tag{8.72}$$

Then $[H, Q] = 0$ and $H = \omega a_\omega a_{-\omega} + \{Q, O\}$ where

$$O = \sum_{k^2 \neq \omega^2} \tfrac{ik}{(k^2-\omega^2)} \phi_k^* \pi_{-k}$$
$$+i\omega(\theta_\omega \pi_{-\omega}^* - \theta_{-\omega}\pi_\omega^*) + \tfrac{1}{2}(\beta_\omega \alpha_{-\omega} + \beta_{-\omega}\alpha_\omega) \tag{8.73}$$
$$+x(\beta_\omega \mathcal{M}_{-\omega} - \beta_{-\omega}\mathcal{M}_\omega)$$

Therefore we have a Hamiltonian in the cohomology.

In this thesis the goal of the universal coefficient theorem is to re-express results in one theory in terms of another theory. This is what physicists call a duality. Indeed, I will show in subsection 8.4 how the results associated to the description of the harmonic oscillator can be re-expressed in terms of a different theory, obtained via an extension. Indeed, there I will show that the change in the coefficient structure of a theory defined (co)homologically amounts to a change in the extension of the field/string theory. This relates for example the algebra of the harmonic oscillator to that of a string theory. At this point I will present a simpler use of the universal coefficient theorem. The change in the coefficient structure will not be so dramatic as what will be done in section 8.4. However, it will be sufficient to allow us to follow step by step the changes that will occur when the coefficient groups in (co)homology are changed (at least in a very basic way) and how will that affect the physical result. As showed previously in this section, given the BRST charge $Q = Q_D + Q_A + Q_M$ in the form of a nilpotent operator, the associated cohomology will be given by

$$H^0_{classic}(Q) = C(a_\omega, a_{-\omega}), \quad H^n_{classic}(Q) = 0, \ \forall n \neq 0 \tag{8.74}$$

The classical phase space, $C(a_\omega, a_{-\omega})$ is then well defined. The algebra of the harmonic oscillator is

$$[a_\omega, a_{-\omega}] = -1 \tag{8.75}$$

As showed previously the BRST charge commutes with the hamiltonian $[H, Q] = 0$ and therefore we can define the Hamiltonian up to a bracket

$$H = \omega a_\omega a_{-\omega} + \{Q, O\} \tag{8.76}$$

where O has been defined above. For the zero degree cohomology one obtains here the standard operator $H = \omega a_\omega a_{-\omega}$. Therefore, it appears that the physical states can be extracted from the cohomology groups. However, if we consider the coefficient groups in cohomology it appears that the particular cohomology groups change: the coefficient choices modify the zero order cohomology and we don't get the same cohomologies as before. More radical changes in the coefficient structures may alter the higher degree cohomologies as well, leading to more drastic modifications. Therefore, for the same BRST charge, the cohomology groups become

$$H^0_{classic}(Q,\mathbb{K}) = C^*(a_\omega, a_{-\omega}), \quad H^n_{classic}(Q,\mathbb{K}) = \Delta_n, \ \forall n \neq 0 \tag{8.77}$$

where Δ_n denotes the modification of the higher degree (co)homology groups. $\mathbb{K}$ is the new coefficient group. Consider for the moment that the associated extension in the universal coefficient theorem is trivial for this choice. Indeed, this choice can be made and is actually the standard choice. For non-standard choices which may lead to more non-trivial connections between theories (for example harmonic oscillator - Virasoro algebra) see section 8.4. The same quantization prescription of above can be employed here as well. One quantizes the theory in the extended phase space. The physical states will be given by the cohomology. However, at this moment the cohomology is different. One must not forget that the change in coefficients in this example still implies a trivial extension. Therefore the short exact sequence of the universal coefficient theorem

$$0 \to Ext_R(H_{q-1}(Q),\mathbb{K}) \to H^q(Q;\mathbb{K}) \to Hom(H_q(Q),\mathbb{K}) \to 0 \tag{8.78}$$

becomes

$$0 \to 0 \to H^q(Q;\mathbb{K}) \xrightarrow{f} Hom(H_q(Q),\mathbb{K}) \to 0 \tag{8.79}$$

where the map f will here be a bijection. The physical states will be encoded in the right hand side of this sequence. This particular form of $\mathbb{K}$ allows us to practically work with a modified algebraic structure, namely the structure of homomorphisms from $H_q(Q)$ to $\mathbb{K}$. Therefore, in this particular case, the physical states defined by means of the (co)homology in the new coefficient structure become the homomorphic maps between the homology of the physical states in the original coefficient group and the new coefficient group. One may ask how this affects the physically measurable results like the ground state energy. Indeed, these will be the same as before. The only modifications will appear in the unphysical fields. The change in the higher degree cohomology is encoded here by Δ_n. However, by changing the focus from states to maps between states and elements of a group, we gain much more flexibility in dealing with global problems. The choice I made here, namely to introduce a coefficient group for which the particular *Ext* group is trivial is a particularization. As there is a bijection between the two groups in this case, the physical states will remain unchanged. However, this is not a stringent requirement. It is definitely more comfortable to work with the universal coefficient theorem with trivial extension. However, if we decide to use a coefficient group which does not trivialize the extension, the full Universal Coefficient Theorem must be used and the effects will be those shown in section 8.4. The general way of looking at this is to notice that the original Hamiltonian is modified by means of a potential V resulting in $H = H_{phys} + V$. The dynamics induced by the modified Hamiltonian leaves the physical Fock space unaltered, in the same way in which adding additional ghost and ghost of ghost fields (in the proper way) doesn't affect the physical dynamics in the BRST or BV quantization. As the dynamics of H_{phys} does not depend on the coupling with V we have that $\langle n| V |n'\rangle = 0$, $\forall n, n' \in H_{phys}$. We do obtain a new set of coupled oscillators which, however, are equivalent to the uncoupled physical oscillators obtained with a trivial coefficient structure. The departure from the BV-BRST interpretation appears due to the non-perturbative nature of the changes induced by the coefficient groups. Indeed, while the BRST construction strongly relies on perturbative calculations, the coefficient structure in (co)homology allows us to add global effects not visible in any perturbative approach. I analyzed possible effects of such global modifications in [52]. There I showed that it is possible to modify a theory not only in order to introduce a BRST-anti-BRST symmetry as done until now, but to also introduce a dual-BRST-anti-BRST symmetry. The role of this new symmetry was to implement a (non-physical) global structure whose role was to annihilate a global anomaly. In fact it is important to notice that any theory can be reformulated in terms of a gauge theory of some sort. Usually, if the theory is easily representable in a non-gauge form, its gauge extension appears to be trivial. However, being trivial does not mean it is irrelevant, quite the opposite. For completeness, the main idea is to add auxiliary fields to the theory by means of a trivial procedure like

$$\begin{aligned} \delta A^l &= B^l \\ \delta B^l &= 0 \end{aligned} \tag{8.80}$$

The original theory defined by a Hamiltonian, Lagrangian or simply by an action, does not depend on these fields, therefore one may shift A^l with no physical effect. This shift would therefore manifest itself as a local symmetry and the fields B^l will be the associated ghost fields. This observation allows us to redefine the field structure of a theory in many different ways. These new fields however allow us a new perspective on the mathematical properties that can be added through them in the theory. For example it becomes possible to move undesirable aspects of the theory to the auxiliary field sector and to transfer desirable properties to the physical field structure while using the unphysical sector in order to compensate the unphysical changes. In particular, if there are more symmetries available due to the extra fields, the interplay between them at the level of the BRST (-anti-BRST-dual-(anti)-BRST) transformations introduces additional freedoms that I used in [52] to avoid certain global anomalies.

Particularly, a global anomaly can be lifted if a suitable "measuring device" presenting a similar (compatible) global "anti-anomaly" is employed in the process of gauge fixing. This "measuring device" must be non-local in nature and is described either in terms of fields associated to the BRST-dual-BRST quantization prescription or in terms of (co)homology with torsion coefficient groups. The resulting theory is either equivalent to the original theory or may contain aspects of the original theory encoded in it. It is important to remember that both the auxiliary fields and the coefficient groups in cohomology are arbitrary constructions that do not change the physical content of the theory. In the case of the BRST-dual-BRST quantization the integration over the artificial fields reconstructs the original theory. In the case of the interpretation using (co)homology with torsional coefficient groups, the universal coefficient theorem tells us that the choice of coefficients is to a large extent arbitrary. One example of a situation where the coefficient groups in (co)homology are modified in order to obtain a "diluted" cohomology isomorphic with the Chech cohomology group with integer coefficients is [310]. These two ways of thinking (BRST-dual-BRST extension of a theory and the use of "exotic" coefficient groups in (co)homoloy) are to a large extent isomorphic.

We can start with a largely arbitrary gauge theoretic action $S[A]$. At this moment we can even assume it is a set of basic harmonic oscillators. For example free electrodynamics may be seen as described by means of a set of harmonic oscillators. Starting with the field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8.81}$$

we can write the Maxwell equations in empty space as

$$\partial_\mu F_{\mu\nu} = \Box A_\mu = 0 \tag{8.82}$$

Equivalently

$$(\partial_t^2 - \partial_x^2)A = 0 \tag{8.83}$$

with planar wave solutions

$$A(x) = a_k(t)e^{i\vec{k}\cdot\vec{x}} \tag{8.84}$$

where

$$(\partial_t^2 + \omega_k^2)a_k(t) = 0, \quad \omega_k = |\vec{k}| \tag{8.85}$$

which is the equation of motion for the harmonic oscillator. A similar harmonic oscillator representation can be imagined for the free scalar field theory. Indeed take the action

$$S[\phi] = \frac{1}{2}\int_{\mathbb{R}^4} \partial_\mu\phi\partial^\mu\phi - m^2\phi^2 \tag{8.86}$$

with the Klein-Gordon equation of motion

$$(\partial_\mu \partial^\mu + m^2)\phi = 0 \tag{8.87}$$

which also looks like a harmonic oscillator wave equation. Once interactions and perturbations arise, the harmonic oscillator approximation is of course insufficient and we need the full mechanism of Feynman diagrams to find solutions. However, at this point it makes sense to think of these theories as describing collections of harmonic oscillators. I denote from now on the connection for the gauge theory by A. It can be used to define a covariant derivative as

$$D_\mu^{(A)} = \partial_\mu - [A_\mu, \cdot] \tag{8.88}$$

We insist on enforcing the Schwinger Dyson equations now. In general the Schwinger Dyson equations are the quantum equations of motion. They are derived as a consequence of the generalization to path integrals of the invariance of an integral under a shift of the integration variable by a quantity a i.e. $x \to x + a$. For fields this might be expressed as $\phi(x) \to \phi(x) + a(x)$. While looking trivial, they were among the first results capable of bringing us beyond perturbative calculations. This has been realized by noticing that they specify non-perturbative relations between quantum correlation functions. In order to enforce the Schwinger-Dyson equations as a result of the BRST algebra we may introduce a collective field

$$A_\mu(x) \to A_\mu(x) - a_\mu(x) \tag{8.89}$$

The transformed action $S[A_\mu - a_\mu]$ has two independent gauge symmetries. Due to the redundancies introduced by the collective field we can write the two symmetries in different ways. One way of doing it is

$$\begin{aligned} \delta A_\mu(x) &= \Theta_\mu(x) \\ \delta a_\mu(x) &= \Theta(x) - D_\mu^{(A-a)}\epsilon(x) \end{aligned} \tag{8.90}$$

We may choose the original symmetry of the original field structure to be carried entirely by the collective field. The transformation of the original gauge field is always just a shift. $\Theta(x)$ includes arbitrary deformations. However, it only leaves the transformed field invariant. The action is also invariant under the original gauge transformations of the transformed field itself. This is why two independent gauge transformations are being included. These two gauge symmetries have to be gauge fixed in the standard BRST fashion. We therefore introduce a suitable multiplet of ghosts and auxiliary fields. The shift symmetry of A_μ requires a vector ghost field $\psi_\mu(x)$. One original gauge ghost field $c(x)$ will also be necessary. Gauge fixing the shift symmetry of A_μ by removing the collective field a_μ leads to the introduction of a corresponding antighost $A^*_\mu(x)$ and of an auxiliary field $b_\mu(x)$.

The nilpotent BRST algebra now becomes

$$\begin{aligned}
&\delta A_\mu(x) = \psi_\mu(x) \\
&\delta a_\mu(x) = \psi_\mu(x) - D^{(A-a)}_\mu c(x) \\
&\delta c(x) = -\tfrac{1}{2}[c(x), c(x)] \\
&\delta \psi_\mu(x) = 0 \\
&\delta A^*_\mu(x) = b_\mu(x) \\
&\delta b_\mu(x) = 0
\end{aligned} \tag{8.91}$$

By adding

$$-\delta[A^*_\mu(x)a^\mu(x)] = -b_\mu(x)a^\mu(x) - A^*_\mu(x)\{\psi^\mu - D^\mu_{(A-a)}c(x)\} \tag{8.92}$$

to the Lagrangian we fix $a_\mu(x)$ to zero. At this point we can make the choice of integrating over pairs of ghosts and anti-ghosts. Hence we can integrate over $\psi_\mu(x)$ and $A^*_\mu(x)$ while keeping $c(x)$ unintegrated at this point. The extended but not yet fully gauge fixed action is

$$S_{ext} = S[A_\mu - a_\mu] - \int dx\{b_\mu(x)a_\mu(x) + A^*_\mu(x)[\psi^\mu(x) - D^\mu_{(A-a)}c(x)]\} \tag{8.93}$$

with the partition function

$$Z = \int dA_\mu da_\mu d\psi_\mu dA^*_\mu db_\mu exp[\frac{i}{\hbar}S_{ext}] \tag{8.94}$$

In order to continue, we first integrate out a_μ and b_μ and then, integration over A^*_μ leaves a trivial ψ_μ integral. In this way we obtain back the starting point, namely the original gauge-theoretic action $S[A_\mu]$ integrated over the original measure.

We must insist that the Schwinger-Dyson equations involving the field $c(x)$ i.e. equations of the form

$$0 = \int dc \frac{\delta^l}{\delta c(x)}[Fe^{\frac{i}{\hbar}[S]}] \tag{8.95}$$

are satisfied automatically when employing the full, unbroken BRST algebra. In order to achieve this we have to introduce yet another collective field, say $\tilde{c}(x)$. We now shift the original gauge ghost

$$c(x) \rightarrow c(x) - \tilde{c}(x) \tag{8.96}$$

From this shift results a new fermionic gauge symmetry which we have to fix via the introduction of a new BRST ghost-antighost pair and an auxiliary field. We let the transformation of the new collective field $\tilde{c}(x)$ carry the BRST transformation of the original ghost.

$$\begin{aligned} &\delta c(x) = C(x) \\ &\delta \tilde{c}(x) = C(x) + \tfrac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)] \\ &\delta C(x) = 0 \\ &\delta c^*(x) = B(x) \\ &\delta B(x) = 0 \end{aligned} \tag{8.97}$$

Now, in order to gauge fix $\tilde{c}(x)$ to zero we add the term

$$-\delta[c^*(x)\tilde{c}(x)] = B(x)\tilde{c}(x) - c^*(x)\{C(x) + \frac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)]\} \tag{8.98}$$

to the Lagrangian. This leads to the fully extended action

$$\begin{aligned} S_{ext} = S[A_\mu - a_\mu] - \textstyle\int dx\{ &b_\mu(x) a_\mu(x) + A^*_\mu(x)[\psi^\mu(x) - D^\mu_{(A-a)}\{c(x) - \tilde{c}(x)\}] \\ &-B(x)\tilde{c}(x) + c^*(x)(C(x) + \tfrac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)])\} \end{aligned} \tag{8.99}$$

In the partition function all fields appearing above are being integrated except the field $c(x)$ for which another antighost $\bar{c}$ must still be introduced when the original gauge symmetry will be fixed eventually. The extended action and the functional measure is invariant under the following transformations

$$\begin{aligned} &\delta A_\mu(x) = \psi_\mu(x), & &\delta\psi_\mu(x) = 0 \\ &\delta a_\mu(x) = \psi_\mu(x) - D^{(A-a)}_\mu[c(x) - \tilde{c}(x)], & &\delta c(x) = C(x) \\ &\delta A^*_\mu(x) = b_\mu(x), & &\delta b_\mu(x) = 0 \\ &\delta\tilde{c}(x) = C(x) + \tfrac{1}{2}[c(x) - \tilde{c}(x), c(x) - \tilde{c}(x)], & &\delta C(x) = 0 \\ &\delta c^*(x) = B(x), & &\delta B(x) = 0 \end{aligned} \tag{8.100}$$

The fields $A^*_\mu(x)$ and $c^*(x)$ are the antighosts of the collective fields which enforce the Schwinger-Dyson equations through shift symmetries.

Therefore, this shows that additional shift symmetries can be used in order to encode the Schwinger-Dyson equations directly via the BRST algebra. It can be seen that by judiciously using artificial symmetries and gauge fixing, additional properties can be added to the original field structure. This is being done such that, by carefully integrating over the supplemental fields we obtain the same theory again. This extension of the field space corresponds to the usual (co)homology and the associated exact sequence for the universal coefficient theorem is again the one inducing a bijection in the place of the last non-trivial arrow

$$0 \to 0 \to H^q(Q;\mathbb{K}) \xrightarrow{f} Hom(H_q(Q),\mathbb{K}) \to 0 \tag{8.101}$$

There is another possibility for the extension of the field space. Instead of introducing a shift by means of one auxiliary field, one can use two such fields. This enlargement of the field space is equivalent to a modified coefficient structure, namely a coefficient structure that allows two irrelevant fields to start with. This will lead to a modified set of non-physical fields that will have to be eliminated via (co)homology and therefore a modified (co)homology theory. However, we still keep the case when the *Ext* group is trivial.

Let me show how to introduce the Schwinger-Dyson equations as Ward identities in this case. Following the procedure by Batalin and Vilkovisky I extend the previous example and I insert now two auxiliary fields

$$A_\mu \to A_\mu - \phi_1 - \phi_2 \tag{8.102}$$

These encode a trivial gauge symmetry representing a shift. The Jacobian associated to the above transformation is trivial. However, this symmetry involves additional freedoms which can be employed for various purposes. The new symmetry has to be gauge-fixed. In doing so via the BRST-anti-BRST formalism the Schwinger-Dyson equation emerges again as a Ward identity. The field multiplets introduced are the ghosts (π_1, A_2^*) and the antighosts (A_1^*, π_2). The BRST and anti-BRST transformations are as follow:

$$\begin{aligned} &\delta_1 A = \pi_1 && \delta_2 A = \pi_2 \\ &\delta_1\phi_1 = \pi_1 - A_2^* && \delta_2\phi_1 = -A_1^* \\ &\delta_1\phi_2 = A_2^* && \delta_2\phi_2 = \pi_2 + A_1^* \\ &\delta_1\pi_1 = 0 && \delta_2\pi_2 = 0 \\ &\delta_1 A_2^* = 0 && \delta_2 A_1^* = 0 \end{aligned} \tag{8.103}$$

Here δ_1 and δ_2 are respectively the BRST and anti-BRST transformations. The next step is to impose gauge fixing. This is done in the standard way by adding more bosonic fields, call them B and λ. The BRST transformation rules extend according to

$$\begin{array}{ll} \delta_1\pi_2 = B & \delta_2\pi_1 = -B \\ \delta_1 B = 0 & \delta_2 B = 0 \\ \delta_1 A_1^* = \lambda - \frac{B}{2} & \delta_2 A_2^* = -\lambda - \frac{B}{2} \\ \delta_1\lambda = 0 & \delta_2\lambda = 0 \end{array} \tag{8.104}$$

These rules imply the nilpotency conditions:

$$(\delta_2\delta_1 + \delta_1\delta_2)A = 0 \tag{8.105}$$

$$(\delta_2\delta_1 + \delta_1\delta_2)\phi_1 = 0 \tag{8.106}$$

$$\delta_1^2 = \delta_2^2 = 0 \tag{8.107}$$

One can chose the gauge fixing condition such that both auxiliary fields are fixed to zero by adding the BRST-anti-BRST closed term

$$S_{col} = \frac{1}{2}\delta_1\delta_2[\phi_1^2 - \phi_2^2] \tag{8.108}$$

By using the BRST-anti-BRST transformations above this becomes

$$S_{col} = -(\phi_1 + \phi_2)\lambda + \frac{B}{2}(\phi_1 - \phi_2) + (-1)^a A_a^*\pi_a \tag{8.109}$$

which makes the gauge fixed action

$$S_{gf} = S_0[A - \phi_+] - \phi_+\lambda + \frac{B}{2}\phi_- + (-1)^a A_a^*\pi_a \tag{8.110}$$

where $\phi_\pm = \phi_1 \pm \phi_2$. Here the index $a = 1,2$ represents the field-antifield index and summation over it is implied. Now the theory is well defined. At this moment the Schwinger-Dyson equation is encoded via an emerging Ward identity $\langle\delta_1[A_1^* F[A^\mu]]\rangle = 0$. Alternatively this can be written as

$$\begin{array}{l} 0 =< \delta_1[A_\mu^* F(A^\mu)] >= \\ = \int d\mu[A_{\mu 1}^* \frac{\delta^l F}{\delta A_\mu}\pi_1 + (\lambda - \frac{B}{2})F(A_\mu)]e^{\frac{i}{\hbar}S_{gf}} \end{array} \tag{8.111}$$

Here F is a general functional on the fields A_μ and $\frac{\delta^l}{\delta A_\mu}$ is the left functional derivative. It gains a sign with respect to the right derivative when acting on fermionic fields. The introduction of the BRST-anti-BRST symmetry in this way is related to a new way of looking at the cohomology. I will use as notation for the eventual field strengths, the matter fields and the anti-fields the symbol χ. In calculation of the cohomology of particular importance are the invariant polynomials in χ denoted $\alpha_J(\chi)$. They represent the coefficient structure in the following construction. Indeed, one can define the form $\bar{a} = \sum_J \alpha_J(\chi)\omega_J$ where ω_J represents a basis of the Lie algebra cohomology of the Lie algebra of the gauge group. The coefficients of this expansion are the invariant polynomials in the field strengths, fields and anti-fields. If $\alpha_J(\chi)$ is a closed form with respect to the differential operator of the cohomology i.e. $d\alpha = 0$ then one knows from the fact that d is a differential operator associated to a cohomology that $\alpha = d\beta$ for some β. If α does contain anti-fields as well, then one may deduce that β is also an invariant polynomial. However, if α does not contain anti-fields this is not the case. One can therefore state that the cohomology of the differential operator d in form degree $< n$ is trivial in the space of coefficients defined by means of invariant polynomials in χ with strictly positive anti-ghost number. This is reflected on the coefficient structure of the cohomology as well. Therefore the invariant polynomials in field strengths, fields and anti-fields defined as coefficients for the forms and the cohomology are associated to the BRST-anti-BRST quantization prescription. Let now α be a representative of the cohomology $H^*(\gamma)$ given by

$$\alpha = \sum \alpha_J(\chi)\omega_J \tag{8.112}$$

where γ can be considered the longitudinal differential operator for a original gauge sector. As $d\gamma + \gamma d = 0$, we have that d induces a well defined differential on $H^*(\gamma)$. One may notice this in the following way. The derivative $d\alpha_J$ is an invariant polynomial in χ. Thus $d\alpha = \pm\sum(d\alpha_J)\omega_J + \gamma(\sum \alpha_J\hat{\omega}_J)$ defines an element of $H^*(\gamma)$, namely the class of $\sum(d\alpha_J)\omega^J$. $\hat{\omega}$ is another choice of a basis. One may therefore have a cohomology of the differential d on $H^*(\gamma)$. This is indeed $H_k^{g,l}(d, H^*(\gamma)) = 0$ for $k \geq 1$ and $l < n$. Here g is the ghost number, l is the form degree and k is the anti-ghost number. The coefficients are defined by the forms obtained in the cohomology of the other differential. The proof of this statement can be found in ref. [329].

One can see that both the previous simple BRST method and the BRST-anti-BRST method presented here can equally be integrated and lead to the original theory defined by $S[A]$. There is not much to calculate at this point, as the prescription simply amounts to un-doing everything that has been done in the constructions of BRST and respectively BRST-anti-BRST quantizations. The advantage however is that additional symmetries may be used in the extended systems. I called in [52] this method "symmetry out of cohomology". The coefficients employed in the various cases therefore are functions, defined over the functional space (as is typical for the BRST cohomology, see section 8.1). They differ by the number and type of supplemental fields used as coefficients in the form polynomials. In the simple BRST case the whole system contained only the fundamental number of fields, ghosts and anti-ghosts, while in the second case

(BRST-anti-BRST) we had form polynomials with coefficients containing the associated anti-fields for the anti-BRST sector. The differential operator associated to this cohomology is therefore extended accordingly, becoming $d = \delta_1 + \delta_2$ satisfying the generalized nilpotency conditions showed above with respect to the BRST and anti-BRST sectors.

Up to this point, I showed on one side that the extension of the BRST method to a field-anti-field method is equivalent to the change of the coefficient structures of the forms and of the cohomology associated to the theory.

An even more advanced structure becomes important when dual transformations are being considered [328]. Indeed, the (co)homological analogue in this case is the non-triviality of the bijection map on the right side of the universal coefficient theorem for trivial *Ext* groups. As dual transformations may be used to encode global fictitious field structures the map towards the homomorphisms between the homology with trivial coefficients and the chosen coefficient group becomes more important. The total symmetry to be considered here is not merely BRST-anti-BRST but also the associated dual symmetry. Additional global structures like the Hodge duality induced discrete symmetry [52] may appear. These are easier to be analyzed from the "dual" perspective i.e. $H^q(Q;\mathbb{K}) \xrightarrow{f} Hom(H_q(Q),\mathbb{K})$.

The Hodge decomposition operators (d, δ, Δ) can be represented as some symmetries of a given BRST invariant Lagrangian of a gauge theory. In general, the Hodge decomposition theorem states that on a compact manifold any n-form $f_n (n = 0, 1, 2, ...)$ can be uniquely represented as the sum of a harmonic form $h_n(\Delta h_n = 0, dh_n = 0, \delta h_n = 0)$, an exact form de_{n-1} and a co-exact form δc_{n+1} as

$$f_n = h_n + de_{n+1} + \delta c_{n+1} \tag{8.113}$$

where here d is the exterior derivative, δ is its dual and Δ is the Laplacian operator $\Delta = d\delta + \delta d$. In order to identify the dual BRST transformation, one has to observe that while the direct BRST transformations leave the two form $F = dA$ in the construction of a gauge theory invariant (and therefore the basic harmonic oscillator was sufficient to discuss it) and transforms eventual Dirac fields like a local gauge transformation, the dual-BRST transformations leave the previous gauge fixing term invariant and transform the Dirac fields like a chiral transformation. So, as a practical example, we can start from a BRST invariant Lagrangian for QED.

$$L_B = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu\psi + B(\partial A) + \frac{1}{2}B^2 - i\partial_\mu\bar{C}\partial^\mu C \tag{8.114}$$

$F^{\mu\nu}$ being the field strength tensor, B is the Nakanishi-Lautrup auxiliary field and C, $\bar{C}$ are the anti-commuting ghosts. The BRST transformations that leave this Lagrangian invariant are

$$\begin{array}{ll} \delta_B A_\mu = \eta \partial_\mu C & \delta_B \psi = -i\eta e C\psi \\ \delta_B C = 0 & \delta_B \bar{C} = i\eta B \\ \delta_B \bar{\psi} = i\eta e C\bar{\psi} & \delta_B F_{\mu\nu} = 0 \\ \delta_B(\partial A) = \eta \Box C & \delta_B B = 0 \end{array} \tag{8.115}$$

where η is an anti-commuting space-time independent transformation parameter. Particularizing for the 2 dimensional case the Lagrangian becomes

$$L_B = -\frac{1}{2}E^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi + B(\partial A) + \frac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C \tag{8.116}$$

and this can be rewritten after introducing another auxiliary field $\mathcal{B}$ as

$$L_{\mathcal{B}} = \mathcal{B}E - \frac{1}{2}\mathcal{B}^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu A_\mu \psi + B(\partial A) + \frac{1}{2}B^2 - i\partial_\mu \bar{C}\partial^\mu C \tag{8.117}$$

The dual BRST symmetry operators to be associated to the theory above in the 2 dimensional case are

$$\begin{array}{ll} \delta_D A_\mu = -\eta \epsilon_{\mu\nu}\partial_\nu \bar{C} & \delta_D \psi = -i\eta e \bar{C}\gamma_5 \psi \\ \delta_D C = -i\eta \mathcal{B} & \delta_D \bar{C} = 0 \\ \delta_D \bar{\psi} = i\eta e \bar{C}\gamma_5 \bar{\psi} & \delta_D F_{\mu\nu} = \eta \Box \bar{C} \\ \delta_D(\partial A) = 0 & \delta_D B = 0 \\ \delta_D \mathcal{B} = 0 & \end{array} \tag{8.118}$$

Moreover, as noted in reference [328] the interacting Lagrangian in 2 dimensions is invariant under the following transformations

$$\begin{array}{ll} C \to \pm i\gamma_5 \bar{C} & \bar{C} \to \pm i\gamma_5 C \\ \mathcal{B} \to \mp i\gamma_5 B & A_0 \to \pm i\gamma_5 A_1 \\ A_1 \to \pm i\gamma_5 A_0 & B \to \mp i\gamma_5 \mathcal{B} \\ E \to \pm i\gamma_5(\partial A) & (\partial A) \to \pm i\gamma_5 E \\ e \to \mp ie & \psi \to \psi \\ \bar{\psi} \to \bar{\psi} & \end{array} \tag{8.119}$$

Reference [328] shows that these are the analogues of the Hodge duality $(*)$ for this particular example and that they induce a discrete symmetry. One can also verify that

$$*\,(*\Phi) = \pm\Phi \tag{8.120}$$

where for (+) the generic field Φ is ψ, $\bar{\psi}$ and for (−) Φ represents the rest of the fields. One can also observe that for the direct and dual BRST symmetries

$$\delta_D \Phi = \pm * \delta_B * \Phi \tag{8.121}$$

is valid. Therefore we have direct and dual differential operators, inducing a more advanced cohomology structure.

Up to this point I employed as a basic tool for the current construction, auxiliary fields of various types. Connections with the coefficients in cohomology were briefly presented on the way. These concepts have a direct analogy in the domain of (co)homology with torsion coefficient groups as has been shown before in this chapter. In categorial terms the connection between the construction using auxiliary fields and the construction using non-trivial coefficient groups can be written as the following diagram

$$\begin{array}{ccc} \mathfrak{F}_S^n(M) & \xrightarrow{h} & \mathfrak{F}_S^{n'}(M') \\ \downarrow i & & \downarrow j \\ H^p(C,\mathbb{Z}) & \xrightarrow{h^*} & H^p(C,\mathbb{G}) \end{array} \tag{8.122}$$

Here, $\mathfrak{F}_S^n(M)$ is the space of physical solutions of the theory containing an initial number n of fields while $\mathfrak{F}_S^{n'}(M')$ is the space of physical solutions for the theory obtained via the introduction of new auxiliary fields such that the required global properties emerge. This space contains the required topological particularity introduced via the employment of the auxiliary fields. The diagram commutes when the *Ext* groups are trivial. It must be specified that the morphism in the lower arrow requires the use of the universal coefficient theorem where in this case the *Ext* group is trivial. The upper arrow morphism is valid when we talk about the physical domain of the theory. $\mathbb{G}$ is the new coefficient group. If *Ext* and/or *Tor* are trivial, the horizontal arrows become isomorphisms. If not, the horizontal arrows must be adjusted and the diagram must be re-designed including the appropriate universal coefficient theorem.

Of course, in the simple context of a harmonic oscillator presented here, there is no global anomaly to alleviate via this method and therefore one can move back and forth between the situations with additional auxiliary fields and therefore non-trivial coefficient groups in cohomology. The reader may however consult ref. [52] and mainly pages 14 - 21 therein for nontrivial examples for this method.

Even with this relatively mild modification in the coefficient structure, some problems considered hard may become tractable. By changing the viewpoint from the states themselves to the homomorphisms from the states to certain groups global aspects may become visible.

This concludes the most basic example related to a change in coefficient groups. In this case the coefficients simply lead to various ghost-anti-ghost-ghost-of-ghost representations of the theory. For more advanced and more relevant connections see section 8.4.

8.3 The free particle

A similar prescription to that of the previous chapter, again taking [267] as a basic model is the free particle. In order to employ the cohomological construction for the description of the free particle moving in d dimensions, one may think at it as to d harmonic oscillators with zero frequency. We denote the independent time parameter by t and the histories of the particles in the phase space by $q^{\mu}(t)$ and $p_{\mu}(t)$. These are subject to the commutation relations

$$[q^{\mu}(t), p^{\nu}(t')] = i\delta^{\mu}_{\nu}\delta(t-t') \tag{8.123}$$

Considering the Minkowski space with a flat background metric $\eta_{\mu\nu}$, we can use it to raise and lower indices in the standard manner $q_{\mu}(t) = \eta_{\mu\nu}q^{\nu}(t)$. The equations of motion then become

$$\epsilon^{\mu}(t) = \ddot{q}^{\mu}(t) \cong 0 \tag{8.124}$$

In cohomology, for implementing these equations we introduce the fermionic antifields $q^{\mu}_{*}(t)$ with momenta $p^{*}_{\mu}(t)$, satisfying non-zero brackets

$$\{q^{\mu}_{*}(t), p^{*}_{\nu}(t')\} = \delta^{\mu}_{\nu}\delta(t-t') \tag{8.125}$$

There still remain redundancies in the equations of motion, of the form

$$\int dt\epsilon^{\mu}(t) = \int dtt\epsilon^{\mu}(t) = 0 \tag{8.126}$$

These require the introduction of two new bosonic antifields, call them θ^{μ}_{1} and θ^{μ}_{0} together with the momenta χ^{μ}_{1} and χ^{μ}_{0}. If the integrand is a total derivative, the integrals above vanish. We identify the momenta and velocities by means of the constraint

$$\mathcal{M}_\mu(t) = p_\mu(t) - \dot{q}_\mu(t) \tag{8.127}$$

Most of the momentum constraints do not commute with the dynamics since

$$[\mathcal{M}_\mu(t), \epsilon^\nu(t')] = -i\delta_\mu^\nu \delta(t-t') \tag{8.128}$$

Integrating by parts we obtain

$$\begin{aligned} \mathcal{M}_\mu^0 &= \int dt \mathcal{M}_\mu(t) = \int dt p_\mu(t) \\ \mathcal{M}_\mu^1 &= \int dt t \mathcal{M}_\mu(t) \cong \int dt t p_\mu(t) \end{aligned} \tag{8.129}$$

These expressions do commute with $\epsilon^\nu(t')$. One may use here the fact that

$$\int dt t \dot{q}^\mu(t) = \delta(-\frac{1}{2}\int dt t^2 q_*^\mu(t)) \tag{8.130}$$

is BRST exact in view of the dynamics constraint and thus can be ignored. The momentum constraints become now first class

$$[\mathcal{M}_\mu^0, \mathcal{M}_\nu^1] = 0 \tag{8.131}$$

Now that we have implemented the dynamics and antifields constraints, we still have four degrees of freedom: the two solutions of $\ddot{q}^\mu(t) = 0$ and their associated momenta. Momentum constraints identify velocities and momenta and thus cut down the number of degrees of freedom to two. This would then be realized by means of two second class constraints. However, here $\mathcal{M}_\mu^0$ and $\mathcal{M}_\mu^1$ are first class and count twice. Hence only one of them is implemented in cohomology. The total BRST operator is then $Q_q = Q_d + Q_a + Q_m$ where

$$\begin{aligned} Q_d &= \int dt \ddot{q}^\mu(t) p_\mu^*(t) \\ Q_a &= \chi_\mu^1 \int dt q_*^\mu(t) + \chi_\mu^0 \int dt t q_*^\mu(t) \end{aligned} \tag{8.132}$$

$$Q_m = \mathcal{M}_\mu^0 \gamma_0^\mu, \quad Q_m = \mathcal{M}_\mu^1 \gamma_1^\mu \tag{8.133}$$

Let me now expand the fields in a Laurent series in t and define the modes q^μ_m and p^m_μ by

$$q^\mu(t) = \sum_{m=-\infty}^{\infty} q^\mu_m t^m$$
$$p_\mu(t) = \sum_{m=-\infty}^{\infty} p^m_\mu t^{-m-1} \tag{8.134}$$

The Laurent modes satisfy commutation relations

$$[q^\mu_m, p^n_\nu] = -i\delta^\mu_\nu \delta^n_m \tag{8.135}$$

But at the same time we have

$$\delta(t-t') = -\tfrac{1}{t'}\delta(1-\tfrac{t}{t'})$$
$$\delta(1-s) = \sum_{m=-\infty}^{\infty} s^m \tag{8.136}$$

and by using them we confirm the previous results. The modes can be recovered from the fields by taking moments e.g.

$$q^\mu_m = \int dt t^{-1-m} q^\mu(t) \tag{8.137}$$

We note that q^μ_{-1} is the residue. The dynamics constraint then becomes

$$\epsilon^\mu_m = (m+2)(m+1)q^\mu_{m+2} \tag{8.138}$$

Hence we introduce an antifield $q^{*\mu}_m$ with momentum $p^m_{*\mu}$ and nonzero brackets $\{q^{*\mu}_m, p^n_{*\nu}\} = \delta^\mu_\nu \delta^n_m$ and then we define the dynamics constraints as

$$Q_d = \sum_{m=-\infty}^{\infty} \epsilon^\mu_m p^m_{*\mu} = \sum_{m=-\infty}^{\infty} m(m-1)q^\mu_m p^{m-2}_{*\mu} \tag{8.139}$$

Q_d is an operator which acts like

$$\delta_d q^\mu_m = 0 \qquad \delta_d p^m_\mu = -im(m-1)p^{m-2}_{*\mu}$$
$$\delta_d q^{*\mu}_m = (m+2)(m+1)q^\mu_{m+2} \qquad \delta_d p^m_{*\mu} = 0 \tag{8.140}$$

Here, $Ker(Q_d)$ is generated by all q^μ_m, $q^{*\mu}_{-2}$, $q^{*\mu}_{-1}$, p^0_μ, p^1_μ and $p^m_{*\mu}$. On the other side, $Im(Q_d)$ is generated by all q^μ_m except $m=0$ and $m=1$ and by all $p^m_{*\mu}$ except $m=-2,-1$. Therefore the cohomology $H^*(Q_d)$ is generated by q^μ_0, q^μ_1, $q^{*\mu}_{-2}$, $q^{*\mu}_{-1}$, q^0_μ, q^1_μ, $q^{-2}_{*\mu}$, $q^{-1}_{*\mu}$. The antifields constraints correspond to

$$q^{*\mu}_{-1} = \int dt q^\mu_*(t), \quad q^{*\mu}_{-2} = \int dt t q^\mu_*(t) \tag{8.141}$$

We need to introduce new bosonic antifields θ^μ_1 and θ^μ_0 together with their momenta χ^1_μ, χ^0_μ. The antifield BRST charge is then

$$Q_a = q^\mu_{-1}\chi^1_\mu + q^{*\mu}_{-2}\chi^0_\mu \tag{8.142}$$

It will act on the relevant fields as

$$\begin{aligned} \delta_a q^{*\mu}_{-1} = \delta_a q^{*\mu}_{-2} = 0, \quad \delta_a p^{-1}_{*\mu} = \chi^1_\mu, \qquad \delta_a p^{-2}_{*\mu} = \chi^0_\mu \\ \delta_a \theta^\mu_1 = q^{*\mu}_{-1}, \qquad\qquad \delta_a \theta^{*\mu}_0, \qquad \delta\chi^1_\mu = \delta_a \chi^0_\mu = 0 \end{aligned} \tag{8.143}$$

It is clear that the antifields cancel in quadruplets and the cohomology $H^*(Q_d + Q_a)$ is generated by q^μ_m and p^m_μ, $m = 0, 1$. The modes of the momentum constraints are then

$$\mathcal{M}^m_\mu = p^m_\mu + m\eta_{\mu\nu}q^\nu_{m+1} \tag{8.144}$$

for $m = 0, 1$. They commute with ϵ^μ_m as in

$$[\mathcal{M}^m_\mu, \epsilon^\nu_n] = im(m-1)\delta^\nu_\mu \delta^m_{n+2} = 0 \tag{8.145}$$

Since $q^\mu_{-1} = \delta(1/2q^{*\mu}_{-3}) \cong 0$ we can take the momentum constraints to be $\mathcal{M}^0_\mu = p^0_\mu$ or $\mathcal{M}^1_\mu = p^1_\mu$. They commute and therefore they are first class constraints. Hence only one will be implemented in the cohomology. Let us consider therefore only $\mathcal{M}^0_\mu \cong 0$. Now we introduce a pair of antifields β^0_μ and γ^μ_0 and the momentum part of the BRST operator then becomes

$$Q_m = Q_m^0 = p_\mu^0 \gamma_0^\mu \tag{8.146}$$

acting on the relevant fields like

$$\begin{aligned} \delta_m q_0^\mu = i\gamma_0^\mu, \quad \delta_m p_\mu^0 = \delta_m q_1^\mu = \delta_m p_\mu^1 = 0 \\ \delta_m \beta_0^\mu = p_\mu^0, \qquad\qquad \delta_m \gamma_\mu^0 = 0 \end{aligned} \tag{8.147}$$

The fields q_0^μ, p_μ^0, β_0^μ and γ_μ^0 cancels and the cohomology becomes a group generated only by

$$\begin{aligned} u_\mu = q_\mu^1 + x p_\mu^0 \\ s_\mu = p_\mu^1 \end{aligned} \tag{8.148}$$

for some constant x. Now $[s_\mu, u_\nu] = i\eta_{\mu\nu}$ is well defined in the cohomology.

8.4 Harmonic oscillator via cohomology with nontrivial coefficients

In the past section we saw that the BRST prescription can be employed for the description of two simple cases: the harmonic oscillator and the free particle. Indeed, the solutions were trivial and all too well known. As BRST is finally a cohomological construction, and as the oscillators have been represented via cohomology, one could ask if there are some possible advantages one could get by modifying the associated coefficient structure. The methods used previously made the roles of the BRST charges and the BRST cohomology manifest. In what follows I will show how this description depends on choices of coefficient groups in cohomology. This will validate the method by showing that standard physical results are not changed by employing different coefficient structures. However, the theories may become widely different and the particular situations above may map into very special, limit-situations of more general theories. Indeed, it will appear that the coefficient structure in cohomology for Lie groups and Leibniz groups control the center of the considered algebra and provide us with relations between various possible central extensions. The harmonic oscillator algebra appears as a particular case during this analysis. The main references for this section are [270-278]. First we need to understand the cohomology of Lie algebras. It is defined in order to give information about the algebraic structure of the Lie algebra via the low degree interpretations of the cohomology spaces. It also gives geometric information about the corresponding Lie group. It is well defined for finite and infinite dimensional Lie algebras. Let g be a Lie algebra over a characteristic zero field k. One could consider finite dimensional examples, like matrix Lie algebras $g = gl(n, k)$, $sl(n, k)$, $so(n, k)$, or $sp(2n, k)$. Two g-modules will be of relevance now. First, the trivial module, which has an action $x \cdot \lambda = 0$ for all $\lambda \in k$ and all x. Second, the adjoint module g acting on g by the adjoint action. This means, it acts by the bracket of the algebra like $x \cdot y = [x, y]$ for all $x, y \in g$. We call an universal enveloping algebra Ug of g an associative algebra of the form

$$Ug = Tg/(x \otimes y - y \otimes x - [x, y], \forall x, y \in g) \tag{8.149}$$

This means that Ug is the quotient of the tensor algebra Tg on g by the ideal generated by the elements

$$x \otimes y - y \otimes x - [x, y] \; \forall x, y \in g \tag{8.150}$$

The Lie algebra g may be regarded as included in Ug. Given an associative algebra A and a Lie algebra morphism $\phi : g \to A$ into the underlying Lie algebra of A, there is a unique morphism of associative algebras $\Phi : Ug \to A$ such that $\Phi|_g = \phi$.

Let now g be a Lie algebra and M be a g-module. Define the space of p-cochains on g with coefficients in M to be

$$C^p(g, M) = Hom_k(\Lambda^p g, M) \tag{8.151}$$

the space of p-linear alternating maps from g to M where for $p = 0$ we set $C^0(g, M) = M$ Let $c \in C^p(g, M)$. Define $dc \in C^{p+1}(g, M)$ by

$$\begin{aligned} dc(x_1, ..., x_{p+1}) = &\sum_{1 \leq i \leq j \leq p+1} (-1)^{i+j} c([x_i, x_j], x_1, ..., \hat{x}_i, ..., \hat{x}_j, ..., x_{p+1}) + \\ &+ \sum_{i=1}^{p+1} (-1)^{i+1} x_i \cdot c(x_1, ..., \hat{x}_i, ..., x_{p+1}) \end{aligned} \tag{8.152}$$

The cochain complex $(C^*(g, M), d)$ is called the Chevalley-Eilenberg complex. It is used to define the Lie algebra cohomology. In the standard way, define the space of p-cocycles

$$Z^p(g, M) = \{c \in C^p(g, M) | dc = 0\} \tag{8.153}$$

and the space of p-coboundaries

$$B^p(g, M) = \{c \in C^p(g, M) | \exists c' \in C^{p-1}(g, M) : c = dc'\} \tag{8.154}$$

Then the cohomology space of g with coefficients in M is the quotient vector space

$$H^p(g,M) = Z^p(g,M)/B^p(g,M) \tag{8.155}$$

For infinite dimensional Lie algebras the topological Lie algebra may be employed (i.e. Lie algebras of vector fields). Let now G be a connected Lie group with Lie algebra g and suppose M is trivial i.e. $M = \mathbb{R}$ with trivial action. Let $\omega \in \Omega^p(G)$ be a differential p-form on G. Then the Cartan formula for the exterior differential reads

$$d\omega(X_1,...,X_{p+1}) = \sum_{1\leq i<j\leq p+1} (-1)^{i+j}\omega([X_i,X_j],X_1,...,\hat{X}_i,...,\hat{X}_j,...,X^{p+1})+$$
$$+\sum_{i=1}^{p+1}(-1)^{i+1}X_i \cdot \omega(X_1,...,\hat{X}_i,...,X_{p+1}) \tag{8.156}$$

We consider here X_i to be vector fields on G and the bracket is the bracket of vector fields. The action $X_i \cdot \omega(X_1,...,\hat{X}_i,...,X_{p+1})$ denotes the Lie derivative. This formula for the exterior differential is essential for the Lie algebra cohomology. Define a p-form $\omega \in \Omega^p(G,M)$ i.e. the space of differential forms with values in the vector space M. Such a p-form is called equivariant if for all $g \in G$, $\lambda_g^*\omega = \rho(g)\circ\omega$. Here λ_g is left translation on the group G and $\rho : G \times M \to M$ is the smooth group action of G on M. The subspace of equivariant forms is denoted $\Omega^p(G,M)^{eq}$. The evaluation at $1 \in G$, $ev_1 : \Omega^p(G,M)^{eq} \to C^p(g,M)$ defines an isomorphism of the de Rham complex of equivariant M-valued differential forms on G to the complex of M-valued Lie algebra cochains.

Considering G a connected compact Lie group, we have $H^*_{dR}(G) \cong H^*(g,\mathbb{R}) \cong Inv_G\Lambda^p(g^*)$. These have been introduced in order to continue on the path of the derived functor approach to cohomology. Indeed, one can see the Lie algebra cohomology as the derived functor of the functor of invariants

$$M \to M^g = \{m \in M | \forall x \in g : x \cdot m = 0\} \tag{8.157}$$

Considering this, it can be described as an *Ext* functor

$$H^*(g,M) = Ext^*_{Ug}(k,M) \tag{8.158}$$

because $M^g = Hom_{Ug}(k,M)$ and the Koszul complex is a resolution of the trivial g-module k. Otherwise stated the augmentation map $Ug \to k$ induces a quasi-isomorphism $\Lambda^*g \otimes Ug \to k$. The Koszul complex arises then as a projective resolution of the trivial g-module k. Applying the functor $Hom_{Ug}(*,M)$ to the resolution $Ug \otimes \Lambda^*g \to k$ one obtains that

$$Hom_{Ug}(Ug \otimes \Lambda^* g, M) \cong Hom_k(\Lambda^* g, M) \tag{8.159}$$

and simplifying one obtains the Chevalley Eilenberg complex for the Lie algebra cohomology. Cohomology spaces are interesting invariants. Indeed, in the following part of this section I will use them as invariants in order to detect the harmonic oscillator Fock space [319]. The various degrees of the cohomology have various interpretations. In degree zero,

$$H^0(g, M) = Z^0(g, M) = \{m \in M | dc = 0\} = \{m \in M | \forall x \in g : x{\cdot}m = 0\} = M^g \tag{8.160}$$

In degree one, we have

$$H^1(g, M) = Z^1(g, M)/B^1(g, M) = Der(g, M)/PDer(g, M) \tag{8.161}$$

where we define the category of derivators as

$$Der(g, M) = \{f \in Hom_k(g, M) | \forall x, y \in g : f([x, y]) = x \cdot f(y) - y \cdot f(x)\} \tag{8.162}$$

and the category of prederivators as

$$PDer(g, M) = \{f \in Hom_k(g, M) | \exists m \in M : \forall x \in g : f(x) = x \cdot m\} \tag{8.163}$$

In order to compute cohomology spaces it is important to know how these spaces decompose when g or the coefficients M decompose. In this work, the focus was on the coefficients, therefore consider the decomposition of the coefficient structure

$$0 \to M' \to M \to M'' \to 0 \tag{8.164}$$

be a short exact sequence of g-modules. This leads to a long exact sequence in cohomology

$$... \to H^i(g, M'') \to H^{i+1}(g, M') \to H^{i+1}(g, M) \to H^{i+1}(g, M'') \to ... \tag{8.165}$$

It is also possible to show that the short exact coefficient sequence induces a short exact sequence of complexes

$$0 \to C^*(g, M') \to C^*(g, M) \to C^*(g, M'') \to 0 \tag{8.166}$$

and one then can apply the connecting homomorphism. The study of the effect of various coefficients in the Lie algebra cohomology can be used for the description of the harmonic oscillator. We can express the massless harmonic oscillator algebra $\phi(n)$ as a Lie subalgebra of the Schrodinger algebra $sch(n)$ and finally calculate the Leibniz homology using the Pirasvili spectral sequence [318, 319]. Such a homology theory gives an infinite number of harmonic oscillator invariants and therefore allowing us to obtain information about the harmonic oscillators [319]. These invariants allow us to identify and count the irreducible representations of the harmonic oscillator algebra and therefore to classify those irreducible representations with important results in the analysis of the Fock spaces. This might become useful in quantum chemistry and the study of condensed matter systems. Moreover, a cohomological description allows us to access this information from various perspectives, given by the coefficients in cohomology and by the universal coefficient theorem. Indeed, one may start with various central extensions and generalizations of the algebra of oscillators and obtain information in different ways, all related via universal coefficient theorems.

A Leibniz algebra is a module over a commutative ring with a bilinear product $[*, *]$ satisfying the Leibniz identity

$$[[a, b], c] = [a, [b, c]] + [[a, c], b] \tag{8.167}$$

Therefore, it is a generalization of the Lie algebra. Indeed, if $[a, b] = -[b, a]$ the Leibniz identity becomes a Jacobi identity i.e. $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$ and the Leibniz algebra becomes a Lie algebra. The homology of the Leibniz algebra has been defined in [320]. The Schrodinger algebra sch is generated by the following set

$$\{X_{ij}, a_n, b_n, c_n, x_i \frac{\partial}{\partial x^{n+1}}, x_i \frac{\partial}{\partial x^{n+2}}; 1 \leq i \leq n\} \tag{8.168}$$

where

$$\begin{array}{lll} X_{ij} = -x_i \frac{\partial}{\partial x^j} + x_j \frac{\partial}{\partial x^i} & 1 \leq i < j \leq n & \text{(Rotations)} \\ a_n = -x_{n+1} \frac{\partial}{\partial x^{n+1}} + x_{n+2} \frac{\partial}{\partial x^{n+2}} & & \text{(Dilation)} \\ b_n = x_{n+1} \frac{\partial}{\partial x^{n+2}} & & \text{(Time translation)} \\ c_n = -x_{n+2} \frac{\partial}{\partial x^{n+1}} & & \text{(Conformal transformation)} \\ x_i \frac{\partial}{\partial x^{n+1}} & 1 \leq i \leq n & \text{(Galilean boosts)} \\ x_i \frac{\partial}{\partial x^{n+2}} & 1 \leq i \leq n & \text{(Space translations)} \end{array} \tag{8.169}$$

The non-trivial brackets are

$$[X_{ij}, X_{ik}] = X_{jk}, \quad [a_n, b_n] = -2b_n, \quad [a_n, c_n] = 2c_n, \quad [b_n, c_n] = a_n \tag{8.170}$$

$$[X_{ij}, x_i \frac{\partial}{\partial x^{n+1}}] = x_j \frac{\partial}{\partial x^{n+1}}, \quad [X_{ij}, x_i \frac{\partial}{\partial x^{n+2}}] = x_j \frac{\partial}{\partial x^{n+2}} \tag{8.171}$$

$$[a_n, x_i \frac{\partial}{\partial x^{n+1}}] = x_i \frac{\partial}{\partial x^{n+1}}, \quad [a_n, x_i \frac{\partial}{\partial x^{n+2}}] = -x_i \frac{\partial}{\partial x^{n+2}} \tag{8.172}$$

$$[b_n, x_i \frac{\partial}{\partial x^{n+1}}] = -x_i \frac{\partial}{\partial x^{n+2}}, \quad [c_n, x_i \frac{\partial}{\partial x^{n+2}}] = x_i \frac{\partial}{\partial x^{n+1}} \tag{8.173}$$

There will be an isomorphism of graded vector spaces

$$HL_*(\phi(n); \mathbb{R}) \cong T^*(\mathcal{U}(n)) \otimes < \tilde{\beta}_{n,*} > \tag{8.174}$$

where $T^*(\mathcal{U}(n))$ is the tensor algebra on a 1-degree generator and $< \tilde{\beta}_{n,*} >$ is a graded vector space generated by tensor powers of even degree. When dealing with a Hilbert space we obtain the description of Fock spaces for quantum mechanics. Generalizations to quantum field theory and string theory are also known. Let $\mathcal{U}(n)$ and $\mathcal{I}$ be the Lie subalgebras of $sch(n)$ generated respectively by

$$\{a_n\}, \quad \{x_i \frac{\partial}{\partial x^{n+1}}, x_i \frac{\partial}{\partial x^{n+2}}; \ 1 \leq i \leq n\} \tag{8.175}$$

The harmonic oscillator algebra will then consist of the dilation, the Galilean boosts and the space translations and will be isomorphic to $\mathcal{U}(n) \ltimes \mathcal{I}_n$. There is a short exact sequence of Lie algebras

$$0 \to \mathcal{I}_n \xrightarrow{i} \phi(n) \xrightarrow{\pi} \mathcal{U}(n) \to 0 \tag{8.176}$$

i being an inclusion and π a projection

$$\phi(n) \to (\phi(n)/\mathcal{I}_n) \cong \mathcal{U}(n) \tag{8.177}$$

$\mathcal{U}(n)$ acts on $\mathcal{I}_n$ via matrix multiplication on vectors. Such an action is extended to $\phi(n) \otimes \mathcal{I}_n^{\wedge k}$ by

$$\begin{aligned}[g \otimes \alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_k] = [g, X] \otimes \alpha_1 \wedge ... \wedge \alpha_k + \\ + \sum_{i=1}^{k} g \otimes \alpha_1 \wedge \alpha_2 \wedge ... \wedge [\alpha_i, X] \wedge ... \wedge \alpha_k\end{aligned} \tag{8.178}$$

for $g \in \phi(n)$ and $X \in \mathcal{U}(n)$. It can also be extended to $\mathcal{I}_n^{\wedge k}$ by

$$[\alpha_1 \wedge \alpha_2 \wedge ... \wedge \alpha_k, X] = \sum_{i=1}^{k} \alpha_1 \wedge \alpha_2 \wedge ... \wedge [\alpha_i, X] \wedge ... \wedge \alpha_k \tag{8.179}$$

for $\alpha_i \in \mathcal{I}_n$, $X \in \mathcal{U}(n)$. For any Lie algebra g over a ring k and any g-module M the Lie algebra homology of g with coefficients in the module M, written as $H_*^{Lie}(g; M)$ is the homology of the Chevalley-Eilenberg complex $M \otimes \wedge^*(g)$, namely

$$M \xleftarrow{d} M \otimes g^{\wedge^1} \xleftarrow{d} M \otimes g^{\wedge^2} \xleftarrow{d} ... \xleftarrow{d} M \otimes g^{\wedge^n} \tag{8.180}$$

where $g^{\wedge^n}$ is the nth exterior power of g. The differential is

$$\begin{aligned} d(v \otimes g_1 \wedge ... \wedge g_n) = \sum_{1 \leq j \leq n} (-1)^j [v, g_j] \otimes g_1 \wedge ...\hat{g}_j ... \wedge g_n + \\ + \sum\nolimits_{1 \leq 1 < j \leq n} (-1)^{i+j-1} v \otimes [g_i, g_j] \wedge g_1 \wedge ...\hat{g}_i ... \hat{g}_j ... \wedge g_n \end{aligned} \tag{8.181}$$

where $\hat{g}_i$ represents the deleted variable and $v \in M$. For any g-module M, the submodule M^g of g-invariants is defined by

$$M^g = \{m \in M | [m, g] = 0, \forall g \in g\} \tag{8.182}$$

We have the natural vector space isomorphisms

$$\begin{aligned} H_*^{Lie}(\phi(n); \mathbb{R}) \cong H_*^{Lie}(\mathcal{U}(n); \mathbb{R}) \otimes [\wedge^*(\mathcal{I}_n)]^{\mathcal{U}(n)} \\ H_*^{Lie}(\phi(n); \phi(n)) \cong H_*^{Lie}(\mathcal{U}(n); \mathbb{R}) \otimes H_*^{Lie}([\phi(n) \otimes \wedge^*(\mathcal{I}_n)]^{\mathcal{U}(n)}; \mathbb{R}) \end{aligned} \tag{8.183}$$

The Leibniz homology of g with coefficients in $\mathbb{R}$ is denoted $HL_*(g, \mathbb{R})$ and is the homology of the Loday complex $T^*(g)$ namely,

$$\mathbb{R} \xleftarrow{0} g \xleftarrow{[,]} g^{\otimes^2} \xleftarrow{d} ... \xleftarrow{d} g^{\otimes^n} \leftarrow ... \tag{8.184}$$

where $g^{\otimes^n}$ is the n-th tensor power of g over $\mathbb{R}$. The derivative is

$$\begin{aligned} & d(g_1 \otimes g_2 \otimes ... \otimes g_n) \\ & = \sum_{1 \leq i < j \leq n} (-1)^j g_1 \otimes g_2 \otimes ... \otimes g_{i-1} \otimes [g_i, g_j] \otimes g_{i+1} \otimes ...\hat{g}_j ... \otimes g_n \end{aligned} \tag{8.185}$$

There exists a projection $\phi(n) \otimes \phi(n)^{\wedge s} \xrightarrow{\pi} \phi(n)^{\wedge(s+1)}$ for $s \geq 0$ acting as a map of chain complexes

$$\pi : \phi(n) \otimes \wedge^*(\phi(n)) \rightarrow \wedge^{*+1}(\phi(n)) \tag{8.186}$$

which induces an $\mathbb{R}$-linear map on the homology

$$\pi_* : H_*^{Lie}(\phi(n); \phi(n)) \rightarrow H_{*+1}^{Lie}(\phi(n); \mathbb{R}) \tag{8.187}$$

It has been shown in [319] that there exists $\mathrm{HR}_*(\phi(n))$ which is the homology of the complex

$$CR_s(\phi(n)) = (Ker(\pi_*))_s = Ker[\phi(n) \otimes \phi(n)^{\wedge(s+1)} \rightarrow \phi(n)^{\wedge(s+2)}], \quad s \geq 0 \tag{8.188}$$

Also, for $s > 1$ the maps

$$\begin{aligned} &\pi_{2s-1} : H^{Lie}_{2s-1}(\phi(n); \phi(n)) \to H^{Lie}_{2s}(\phi(n); \mathbb{R}) \\ &\pi_{2s} : H^{Lie}_{2s}(\phi(n); \phi(n)) \to H^{Lie}_{2s+1}(\phi(n); \mathbb{R}) \end{aligned} \tag{8.189}$$

are isomorphisms mapping the corresponding classes $\hat{\beta}_{n,2s}$ to $\beta_{n,2s}$ and $a_n \otimes \beta_{n,2s}$ to $a_n \wedge \beta_{n,2s}$ respectively. The map

$$\pi_1 : H^{Lie}_1(\phi(n); \phi(n)) \to H^{Lie}_2(\phi(n); \mathbb{R}) \tag{8.190}$$

maps $\hat{\beta}_{n,2}$ to $\beta_{n,2}$ and $a_n \otimes a_n$ to 0. Therefore there is the natural isomorphism [319]

$$HR_{k-3}(\phi(n); \mathbb{R}) \cong \begin{cases} < a_n \otimes a_n >, & k = 3 \\ 0, & k > 3 \end{cases} \tag{8.191}$$

Considering

$$(Ker(\tilde{\pi}_*))_n = Ker[\phi(n)^{\otimes(n+2)} \to \phi(n)^{\wedge(n+2)}], \;\; n \geq 0 \tag{8.192}$$

the relative homology $H^{rel}(\phi(n))$ is defined as the homology of the complex

$$C^{rel}_n(\phi(n)) = (Ker(\tilde{\pi}_*))_n \tag{8.193}$$

There exist graded vector space isomorphisms

$$\begin{aligned} &H^{Lie}_*(\phi(n); \mathbb{R}) \cong H^{Lie}_*(\mathcal{U}(n); \mathbb{R}) \otimes < \beta_{n,*} > \\ &HL_*(\phi(n); \mathbb{R}) \cong T^*(\mathcal{U}(n)) \otimes < \tilde{\beta}_{n,*} > \end{aligned} \tag{8.194}$$

where $\mathrm{T}^*(\mathcal{U}(n))$ is the tensor algebra, the $\beta_{n,2k}$ and $\tilde{\beta}_{n,2k}$ are exterior and tensor power of even degree. $\tilde{\beta}_{n,2k}$ is the antisymmetrization of $\beta_{n,2k}$.

These relations calculate the Leibniz homology of the oscillator algebra with real coefficients [319]. They determine the invariant structure for the foliations of the harmonic oscillator. Indeed, one obtains a tensor structure similar to that defined for Fock spaces. Other, maybe easier or more relevant invariants may be calculated by using other coefficients. For the classification of irreducible representations of the harmonic oscillator (or analysis of Fock spaces for various systems) such invariants may be determined by employing different coefficients in the Leibniz homology. Up to this point it is clear that the Leibniz homology determines the tensor algebra structure associated to the Fock space. I will now show how changing the coefficient structure in a consistent manner modifies the theory leading in general to central extensions of the original algebra. However, the particular case obtained above, namely the harmonic oscillator algebra, will be identified as a particular case in those more complicated descriptions. This is probably an archetypal example for the use of the universal coefficient theorems in the analysis of physical problems. One obtains a relation between a simple theory and a more advanced one. The simple theory can be recovered in the advanced, more complicated one, as a particular case. It is important to understand the role of the universal coefficient theorem here. For Leibniz (co)homology in dimension 2 such a universal coefficient theorem has been derived [320] and the results are of particular interest. The concept of universal envelopes of Leibniz algebras has also been established and the Leibniz (co)homology has been interpreted in terms of Tor- respectively Ext- functors. From their very definitions we know that Lie algebras are naturally Leibniz algebras.

$$\begin{aligned} HL^*(g,M) &\cong Ext^*_{UL(g)}(U(g_{Lie}),M) \\ HL^*(g,A) &\cong Tor_*^{UL(g)}(U(g_{Lie}),A) \end{aligned} \tag{8.195}$$

where A is a co-representation of g and M is a representation of g. The central extensions of Leibniz algebras are also relevant. In [321] it has been proved that the Virasoro algebra Vir is a universal central extension of $Der(\mathbb{C}[t,t^{-1}])$ both in the Lie and in the Leibniz framework. Indeed, it has been shown in [322] that to every short exact sequence of Leibniz algebras

$$(e): n \xrightarrow{\chi} g \xrightarrow{\pi} q \tag{8.196}$$

there corresponds a natural exact sequence in trivial coefficient homology of Leibniz algebras

$$HL_2(g) \xrightarrow{HL_2(\pi)} HL_2(q) \xrightarrow{\theta_*(e)} n/[n,g] \xrightarrow{\chi'} HL_1(g) \xrightarrow{\pi_{ab}} HL_1(q) \xrightarrow{0} \tag{8.197}$$

Also, given the short exact sequence of Leibniz algebras above, and having $\psi : n_{ab} \to A$ a morphism of q-modules, we have that

$$\theta_*(\theta^*(\psi)) = \psi_c \circ \theta_*(e) \tag{8.198}$$

This is so because the map θ_* is natural in the above sequence in homology. Then ψ gives rise to an extension $({}^{\psi}ab(e)) : A \to p \to q$ as well as to maps of extensions $(\psi \circ ab, \cdot, 1) : (e) \to ({}^{\psi}ab(e))$ thus the naturality of θ_* gives a commutative square

$$\begin{array}{ccc} HL_2(q) & \xrightarrow{\theta_*(e)} & n/[n,g] \\ {\scriptstyle 1}\downarrow & & \psi\downarrow \\ HL_2(q) & \xrightarrow{\theta_*({}^{\psi}ab)(e)} & A/[A,p] \end{array} \tag{8.199}$$

By the definition of θ^*, we have $\theta^*(\psi) = \psi_*(ab(e)) = [{}^{\psi}ab(e)]$ and therefore $\theta_*(\theta^*(\psi)) = \theta_*({}^{\psi}ab(e))$. For the Leibniz algebra one can now define a universal coefficient theorem. Let q be a Leibniz algebra and let A be a vector space regarded as a trivial q-module. The sequence

$$0 \to Ext(HL_1(q), A) \xrightarrow{\psi} HL^2(q, A) \xrightarrow{\theta_*} Hom(HL_2(q), A) \to 0 \tag{8.200}$$

represents the universal coefficient theorem for Leibniz cohomology. In this sequence the action of ψ is given by $\psi([e]) = [e_{ab}]$ and the action of θ is $\theta_*([e]) = \theta_*(e)$. The sequence is natural, split short exact. Looking at the action of ψ it appears that the universal coefficient theorem gives us control over the center subalgebra of the central extension. Therefore if I define the Lie algebra (according to [323]) $\mathcal{L}$ as the center-less, twisted, Schrodinger-Virasoro-Lie algebra with $\mathbb{C}$ basis $\{L_n, Y_n, M_n | n \in \mathbb{Z}\}$ and non-zero Lie brackets

$$[L_n, L_{n'}] = (n' - n)L_{n+n'}, \quad [L_n, L_p] = pM_{n+p} \tag{8.201}$$

$$[L_n, Y_m] = (m - \tfrac{n}{2})Y_{n+m}, \quad [Y_m, Y_m] = (m' - m)M_{m+m'} \tag{8.202}$$

The twisted Schrodinger-Virasoro Lie algebra has an infinite-dimensional subalgebra S with $\mathbb{C}$-basis $\{Y_n, M_n | n \in \mathbb{Z}\}$ and a center-less Virasoro subalgebra $\mathcal{V}$ with $\mathbb{C}$-basis $\{L_n | n \in \mathbb{Z}\}$. The universal coefficient theorem gives us access to the center of the Schrodinger algebra. Moreover, it allows us to derive the Leibniz cohomology group of the center-less twisted Schrodinger-Virasoro Lie algebra $\mathcal{L}$ defined above. The calculations have been done in [323]. The sets of generators given by the cohomology classes of the cocycles for both the original and the twisted sectors are to be found in [324]. As I mentioned earlier, the Virasoro algebra can be seen as a universal extension both in the Lie and in the Leibniz framework. Finally, there exists an oscillator representation of Virasoro algebras. This has been established in [325, 326]. By gathering together the results above, it follows that the oscillator algebra can be obtained by means of the Virasoro algebra starting with the homology of a Leibniz algebra with coefficients modified such that the resulting Virasoro algebra becomes center-less. One starts with the Virasoro operators, written in the oscillator representation as

$$L_n(\lambda) = (a_0 + \lambda n)a_n + \sum_{j=1}^{\infty} a_{-j}a_{n+j} + \tfrac{1}{2}\sum_{j=1}^{n-1} a_j a_{n-j}$$

$$L_{-n}(\lambda) = (a_0 - \lambda n)a_{-n} + \sum_{j=1}^{\infty} a_{-n-j}a_j + \tfrac{1}{2}\sum_{j=1}^{n-1} a_{-j}a_{-n+j} \tag{8.203}$$

$$L_0(\lambda) = \tfrac{1}{2}(a_0^2 - \lambda^2) + R = \tfrac{1}{2}(a_0^2 - \lambda^2) + \sum_{j=1}^{\infty} a_{-j}a_j$$

where λ is a complex parameter, R is the level operator in the Fock space, and the operators a_n, $(n = 0, \pm 1, \pm 2, ...)$ satisfy the hermiticity relation $a_n^{\dagger} = a_{-n}$ and the commutation relations

$$[a_n, a_m] = n\delta_{n+m,0} \tag{8.204}$$

The $L_n(\lambda)$ operators themselves satisfy the commutation relations of the Virasoro algebra. We may employ the standard quantum notation writing $a_0 = p_0$ and bring a coordinate q_0 as the conjugate of p_0

$$[q_0, a_n] = i\delta_{n,0} \tag{8.205}$$

The vacuum is defined by

$$(a_n - \lambda\delta_{n,0})\,|0;\lambda\rangle = 0, \;\; n \geq 0 \tag{8.206}$$

It is also a conformally invariant vacuum, therefore

$$L_n(\lambda)\,|0;\lambda\rangle = 0, \;\; n \geq -1 \tag{8.207}$$

For a general ground state differing from the vacuum by a momentum shift t, the ket-vector becomes $|0; t+\lambda\rangle$ and we have

$$L_0(\lambda)\,|0;t+\lambda\rangle = (\tfrac{t^2}{2} + \lambda t)\,|0;t+\lambda\rangle$$

$$L_n(\lambda)\,|0,t+\lambda\rangle = 0, \;\; n \geq 1 \tag{8.208}$$

The state $|0, t+\lambda\rangle$ is a primary state with the conformal weight

$$h = h_0(t) = \frac{t^2}{2} + \lambda t \tag{8.209}$$

For certain special values of $t = t_{(r,s)}$

$$t_{r,s} = \frac{1+r}{2}t_+ + \frac{1+s}{2}t_- \tag{8.210}$$

$$t_\pm = -\lambda \pm \sqrt{\lambda^2+2} \tag{8.211}$$

we have a null state

$$|\chi\rangle = \Psi \left|0; t_{(r,s)} + \lambda\right\rangle \tag{8.212}$$

at the level $N = rs$ with the conformal weight

$$h_0(t_{(-r,s)}) = h_0(t_{(r,s)}) + rs \tag{8.213}$$

The operator Ψ consists of the oscillators a_{-n}. The null state $|\chi\rangle$ satisfies

$$L_n(\lambda)\,|\chi\rangle = 0, \quad n \geq 1 \tag{8.214}$$

The central charge c and the conformal weight usually determines a representation of the Virasoro algebra. When we go to the oscillator representation of the Virasoro algebra the important numbers are λ and t. We should notice that the correspondence between (c, h) and (λ, t) is degenerate with both λ and $-\lambda$ producing the same c and with λ fixed with t and $-2\lambda - t$ giving the same weight $h = h_0(-2\lambda - t)$. We also have parity as an operator $Pa_nP = -a_n$ and $Pq_0P = -q_0$. The Virasoro operators then are transformed like

$$PL_nP = L_n(-\lambda), \quad n = 0, \pm 1, \pm 2, ... \tag{8.215}$$

$$P\,|0; t+\lambda\rangle = |0; -t-\lambda\rangle \tag{8.216}$$

In what follows I will continue to repeat the arguments of [326] until I arrive at the relation between the oscillator representation and the Virasoro operators. Consider the states of level N. There are $p(N)$ independent states at this level in the Fock space, with the degeneracy being the number of ways in which one can rewrite N as a sum of positive integers

$$a^{-J}\,|0; t+\lambda\rangle\,, \quad J = 1, 2, ..., p(N) \tag{8.217}$$

having

$$\begin{gathered} a^J = const \times a_1^{n_1} a_2^{n_2} ... a_N^{n_N} \\ a^{-J} = const \times a_{-N}^{n_N} a_{-N+1}^{n_{N-1}} ... a_{-1}^{n_1} \end{gathered} \tag{8.218}$$

$$\sum_{k=1}^{N} k n_k = N, n_k = 0, 1, 2, ... \tag{8.219}$$

The normalization implies

$$\langle 0;t+\lambda|\, a^I a^{-J}\, |0;t+\lambda\rangle = \delta^{IJ} \tag{8.220}$$

For the Virasoro operators for the same level N we have

$$\begin{aligned} L^J(\lambda) &= L_1^{n_1}(\lambda)L_2^{n_2}(\lambda)...L_N^{n_N}(\lambda) \\ L^{-J}(\lambda) &= L_{-N}^{n_N}(\lambda)L_{-N+1}^{n_{N-1}}(\lambda)...L_{-1}^{n_1}(\lambda) \end{aligned} \tag{8.221}$$

$$\sum_{k=1}^{N} kn_k = N, \quad n_k = 0,1,2,... \tag{8.222}$$

A relation between the two states $a^{-J}\,|0;t+\lambda\rangle$ and $L^{-J}(\lambda)\,|0;t+\lambda\rangle$ is expressed as

$$L^{-I}(\lambda)\,|0;t+\lambda\rangle = \sum_J C_{IJ}(a_0,\lambda)a^{-J}\,|0;t+\lambda\rangle \tag{8.223}$$

The determinant of the matrix C_{IJ} decides if the equation above is invertible and hence if the equation can be resolved in reverse expressing the oscillators in terms of the Virasoro operators. It turns out that this is possible in certain situations, as has been shown in reference [326]. The details are not important here. What is important is that there exists a representation of the harmonic oscillator in the Virasoro algebra. The central charge of the Virasoro algebra corresponds by Noether theorem to the center of the central extension of the original group. Therefore we have the exact sequence

$$0 \xrightarrow{i} h \xrightarrow{j} e \xrightarrow{s} g \xrightarrow{\sigma} 0 \tag{8.224}$$

where $e = h \oplus g$ as a vector space direct sum. We also have $Im(j) = Ker(s) \subset Z(e)$ where $Z(e)$ is the center of e. Then e is called the central extension of g by h or the extension by a 2-cocycle. The classes of the second cohomology of the original algebra g with coefficients in some group A are precisely the possible extensions of that original algebra, therefore

$$[\phi] \in H^2(g,A) \tag{8.225}$$

The extensions are therefore defined by the 2-cocycles and to each pair of non-cohomological 2-cocycles i.e. not related via a co-boundary, there is a pair of non-equivalent extensions. The universal coefficient theorem in this context describes a change in the center of the extension. Take therefore two extensions

$$0 \xrightarrow{i} h \xrightarrow{j} e \xrightarrow{s} g \xrightarrow{\sigma} 0$$
$$0 \xrightarrow{i} h' \xrightarrow{j} e' \xrightarrow{s} g' \xrightarrow{\sigma} 0 \tag{8.226}$$

then we have the commutative square diagram

$$\begin{array}{ccc} HL_2(g') & \xrightarrow{\theta_*(e)} & h'/[h',e'] \\ {\scriptstyle 1}\big\downarrow & & {\scriptstyle \psi}\big\downarrow \\ HL_2(g) & \xrightarrow{\theta_*(^{\psi}ab)(e)} & h/[h,e] \end{array} \tag{8.227}$$

As the cocycles controlling the extension correspond to classes in the second cohomology of the original group g, it appears that a change in coefficients will lead to a change of the central extension of the Lie algebra, as well as to a deformation of the equivalence classes forming the cohomology. Therefore the universal coefficient theorem shows in how far theories based on a certain central extension are to be represented in terms of different coefficient modules. It has been shown in [321] that the Virasoro algebra Vir is a universal central extension of $Der(\mathbb{C}[t,t^{-1}])$ both in the Lie and in the Leibniz framework. Therefore the non-trivial Leibniz 2-cocycles on the infinite dimensional Lie algebras of differential operators over $\mathbb{C}[t,t^{-1}]$ (the algebra of Laurent polynomials over the complex numbers), $\mathbb{C}((t))$ (the algebra of the formal Laurent series over complex numbers) and the quantum q-torus are related via universal coefficient theorems [327].

Consider now the cochain complex $(C^*(L,M),d)$ defined by

$$C^n(L,M) = Hom_F(L^{\otimes n}, M), \quad n \geq 0$$
$$d^n : C^n(L,M) \to C^{n+1}(L,M) \tag{8.228}$$

defined in the standard way and its cohomology as being the cohomology of the Leibniz algebra L with coefficients in the representation M i.e. $HL^*(L,M) = H^*(C^*(L,M),d)$. A Leibniz 2-cocycle on this algebra L is a field F valued form ψ satisfying the condition

$$\psi(a,[b,c]) = \psi([a,b],c) - \psi([a,c],b), \;\; \forall a,b,c \in L \tag{8.229}$$

As is the case for Lie algebras, the one-dimensional Leibniz central extensions of a Leibniz algebra L are uniquely determined by Leibniz 2-cocycles on L. If a Leibniz 2-cocycle ψ is induced by a linear function f on L i.e. $\psi = \alpha_f$ having

$$\alpha_f(x, y) = f([x, y]), \;\; \forall x, y \in L \tag{8.230}$$

then ψ is trivial. The corresponding one-dimensional Leibniz extension is also trivial i.e. isomorphic to $L \oplus F_C$ i.e. a direct sum of Leibniz ideals. A canonical construction for a Leibniz central extension of L implies

$$[x + \lambda c, y + \mu c]_0 = [x, y] + \alpha(x, y)c, \;\; \forall x, y \in L, \;\; \lambda, \mu \in F \tag{8.231}$$

where $[,]$ is the Leibniz bracket on the original algebra and $[,]_0$ is the Leibniz bracket on $L \oplus F_C$. Every 1-dimensional Leibniz central extension of L can be obtained in this way. In particular, the oscillator representation can be recovered in various ways, by going to the center-less representation via transformations of coefficients. This discussion showed how the universal coefficient theorem for Lie algebras and generally for Leibniz algebras can be used to identify new connections: as the map ψ of the universal coefficient theorem controls the central extensions, the universal coefficient theorem becomes a tool that allows us to recover particular subalgebras (say the harmonic oscillator algebra) as appearing in various extensions in different ways. A similar way of thinking might be used for identifying new dualities in gauge/string theory. Although this is not attempted in this work, as far as I know, I am the first to notice such a connection and to identify this as a new possibility of deriving gauge/string dualities.

BRST denotes in general a quantization prescription for a classical system with constraints by means of some odd variables known as ghost fields. In a classical theory starting with symplectic manifolds M that possess a symplectomorphic group action by a Lie group G, we construct the map $\mu : M \to g^*$ defined by being equivariant under the coadjoiunt action of G on g, with $d(\mu(*)(g)) = \omega(\rho(g), *)$ with ω as the symplectic form. If the action of G represents a gauge symmetry, we would like to obtain $\tilde{M} = M/G$ containing no redundancies. We may define a submanifold M_0 and observe that the poisson algebra of functions on $\tilde{M}$ fulfills

$$C^\infty(\tilde{M}) = H^0(g, C(M_0)) \tag{8.232}$$

The zeroth cohomology of a Lie algebra with coefficients in a module consists of precisely the elements of the module that are invariant under the group action. At this moment it is important to notice what means to change the coefficient structure in such a cohomology. The physical quantum states are the elements of the cohomology. In the case of the Lie algebra cohomology associated with the BRST complex, this change represents a change in symplectic manifold of the pre-quantum construction. The cases for the harmonic oscillator and the free particle have been discussed previously. The main results derived in this chapter can be recovered in the following theorems

8.1 Theorem (Summarizing previous results for the harmonic oscillator)

In the BV formalism for the simple harmonic oscillator, changing the coefficients corresponds to the homomorphic maps between the homology of the physical states in the original coefficient group and the new coefficient group. The physical quantities, however, remain unchanged.

Proof For the first part of the proof one has to look again at the universal coefficient theorem

$$0 \to Ext_R(H_{q-1}(Q), \mathbb{K}) \xrightarrow{f} H^q(Q, \mathbb{K}) \xrightarrow{g} Hom(H_q(Q), \mathbb{K}) \to 0 \tag{8.233}$$

Having the form of a short exact sequence, the map f is injective and the map g is surjective. Defined in this way for every homomorphism $Hom(H_q(Q), \mathbb{K})$ there exist one or more classes in the cohomology $H^q(Q, \mathbb{K})$ associated to the physical states of the system. Making the choice of $\mathbb{K}$ such that $Ext_R(H_{q-1}(Q), \mathbb{K})$ is trivial we obtain the short exact sequence

$$0 \to 0 \xrightarrow{f} H^q(Q, \mathbb{K}) \xrightarrow{g} Hom(H_q(Q), \mathbb{K}) \to 0 \tag{8.234}$$

But this being an exact sequence with the first map becoming $0 \xrightarrow{f} H^q(Q, \mathbb{K})$, the second map becomes injective as well. Now g is surjective and injective, therefore bijective. This simply means that to any physical state given by the cohomology there is one and only one state in the group of homomorphism to the trivializing coefficient group $Hom(H_q(Q), \mathbb{K})$. Of course, other choices of coefficients can be made but then the surjective nature of the second map must be kept in mind. Such choices have been presented in section 8.4. On the side of the observables, one might start with the action

$$S = \frac{1}{2} \int dx(\phi'(x)^2 - \omega^2 \phi(x)^2) \tag{8.235}$$

and introduce additional fictitious gauge symmetries. These gauge symmetries must then be gauge fixed by procedures similar to BRST. Of course, there are multiple options, as has been showed before. We can employ direct BRST, BRST-anti-BRST, or more complex methods. Integrating back the additional fields arising via those methods will lead us back to the original formulation and the identical physical results as has been shown before. This already represents a change in the coefficient structure of Lie group cohomology i.e. a change in the field structure of the theory. If we prefer to work with the Hamiltonian formalism

$$H \cong \omega a_\omega a_{-\omega} \tag{8.236}$$

This can be deformed by means of adding a bracket

$$H = \omega a_\omega a_{-\omega} + \{Q, O\} \tag{8.237}$$

with O satisfying

$$O = \sum_{k^2 \neq \omega^2} \frac{ik}{(k^2-\omega^2)} \phi_k^* \pi_{-k} + i\omega(\theta_\omega \pi_\omega^* - \theta_{-\omega}\pi_\omega^*) + \frac{1}{2}(\beta_\omega \alpha_\omega + \beta_{-\omega}\alpha_\omega) + x(\beta_\omega \mathcal{M}_{-\omega} - \beta_{-\omega}\mathcal{M}_\omega) \tag{8.238}$$

This already represents a trivial change in the coefficients in the sense of altering the field structure of the gauge theory i.e. addition of fields, antifields and ghosts. The detailed calculation and the definitions are the same as those explained previously in this chapter. As said before, the physical operators (observables) are defined as elements of the cohomology and two operators A and A' are equivalent if they differ by means of

$$A' = A + [Q, B] \tag{8.239}$$

for arbitrary B. This is precisely what happens in the case of the Hamiltonian. Namely one obtains

$$H = \omega a_\omega a_{-\omega} \leftrightarrow H = \omega a_\omega a_{-\omega} + \{Q, O\} \tag{8.240}$$

This should summarize the main application to the harmonic oscillator.

The idea of employing invariants in order to count states, as done in the description of the oscillator algebra for the identification of the associated representations is not new. In fact, counting BPS states is of major importance for black hole thermodynamics [232-242]. Various anomalies described by wall crossing formulas [253, 254, 257, 258] have been noticed and discussed. Counting curves on various varieties is in itself an interesting mathematical problem [255].

Part 2

Up to this point all notions related to algebraic and general topology have been introduced. Also, some necessary concepts related to group theory and category theory have been presented. At this moment it is important to take a step forward and introduce some new ideas interlinking quantum field theory, gauge theory and some of Grothendieck's ideas related to (co)homology groups with coefficients. While the discussion in the previous chapters was based on existing results and its goal was to bring the reader up to date with a set of ways of thinking and concepts, at this moment I insist on the original ideas I am introducing in this work. Certainly, many concepts used here are well known and I refer to the original works when necessary. A brief review of some basic facts and their historical background will complement this original work.

Chapter 9

Universal Coefficient Theorem and Quantum Field Theory

"Curiouser and curiouser."

Lewis Carroll, Alice in Wonderland

9.1 A categorial Viewpoint on Quantum Field Theories

During the 1950's physics was still struggling with the standard model of elementary particles [163], renormalization [164] and experimental confirmation of the major results of non-abelian gauge theories [165]. At the same time mathematics took a completely different approach [166]. Only more than 30 years later did the two disciplines come together again in a fruitful encounter started by E. Witten and C. Vafa [167]. The approach taken by mathematicians was focused on discovering original ways of thinking. The idea of category theory was essential for this scope. The understanding that a simple mathematical concept like the set of natural numbers can be seen as the decategorification [168] of a deeper concept involving not only objects but also associated morphisms has led to various changes in the ways of understanding concepts like groups, sets, etc. Lifting a property from a given category to another via functors [169] i.e. preserving the underlying structure while changing the framework where the property is analyzed, has led to concepts like quantum groups, the Riemann-Roch theorems, etc. Moreover, it was Grothendieck [248] who noticed that it is possible to characterize a space not only by describing its properties. An alternative method is to describe all the possible ways in which the space we wish to characterize can be mapped into another, known space. This idea, related to the classifying space was important in understanding that properties hard to detect when dealing with a single space may become obvious when the maps between two spaces are being considered. All this plethora of mathematical structure has been largely ignored for almost 40 years. This is in some sense understandable because physics was deeply rooted in the perturbative way of thinking. If we think about quantum field theories with interactions we already see that starting from the simplest interactions it becomes impossible to talk about exact results using the analytical tools available to the physicist. Moreover, a great amount of partial success can be achieved by using only perturbative tools so, at an early stage, thinking about such subtleties as the topology of the field space or the coefficient structure of homology was considered at best a nuisance. However, keeping only a perturbative viewpoint can lead to absurd conclusions [170]. Mathematicians knew about the perils of speaking about global structures in local terms or about the dangers of naively lifting properties from one algebraic structure to another. These dangers are best described as groups encoding the obstructions to constructions that appear to make sense when looked upon locally. Probably Cech cohomology is the best example for this. Soon, physicists also realized that taking a naive way of thinking and pursuing it without a mathematically well defined structure may lead to wrong results. These manifested themselves as anomalies. The first observations were related to the quantum non-conservation of classically conserved currents [171]. As conservation laws are related by Noether's theorem to continuous symmetries the first interpretation of anomalies was a breaking of symmetries due to the quantization prescription. Today we know that such a prescription essentially means to take into account all possible paths of a particle on a manifold in order to arrive via the Feynman path integral to the quantum mechanical expectation values. But this prescription forces us to take into account the topology of the manifold of these paths. This is how it became clear that quantum mechanics lies at the boundary between topology and analysis, being rooted somehow in both these disciplines. Doing quantum mechanics without thinking correctly about the topology of the underlying space is an action bound to fail sooner or later. While this has been understood [172], the idea of anomaly can be defined in a broader sense, by means of category theory. In principle, I speak about an anomaly whenever a property valid for an algebraic construction does not make sense directly when used in another algebraic construction. Several changes in the way we interpret the property must be performed in order for it to have a meaning in the new construction. Understanding in what sense these changes have to be performed is of major importance, not only in physics but also in computer science or in the theory of language [173]. For the layperson but not only, an explanation related to the theory of language would be the following anecdote: two people discuss without agreeing at the beginning what is the subject of their discussion. One is thinking at her dog and the other at her child. They describe the respective object by the impersonal "it" such that it never becomes clear what they are talking about. The discussion starts rather innocently: both agree they take "it" for a walk every day, they feed "it" and they also agree about quite a variety of actions that can be performed identically on both the subjects of the discussion. At a certain moment however, one of them is making a rather strange remark: "Oh, it was so happy, it even started wagging its tail". At this moment, the other partner of the discussion may start asking questions: how is it possible for a child to have a tail? Was that only an analogy or a metaphor? How should the other person change this statement such that it has meaning when applied to the subject of her discussion? Probably the answer would be that tail wagging should translate into a smiling face. However, probably the best thing to do in this discussion is to remark that the two persons are speaking about different things, hence using different "frameworks" with sometimes coinciding words. While, in a natural discussion, sooner or later this revelation will occur, when we speak about theoretical physics and mathematics we can fool ourselves practically indeterminately.

And so we did. Quantization of gravity is probably the best example where lifting the ideas that defined quantum mechanics to a situation where the space-time has itself quantum properties became rather difficult to be done meaningfully [174]. There were several reasons and explanations for why we did not succeed in this area but, apart of all excuses, the main reason is that we do not know how to speak meaningfully about the quantum, topological and geometrical properties of the underlying manifold on which we construct our theory. It must be said that the problems do not appear when the gravitational field is weak. We can speak meaningfully about quantum field theories on curved space-times and hence "combine quantum mechanics and gravity" [174-180]. We can also discuss about quantum field theories on space-times with variable metric. Such constructions have been made in [181] and used by Hawking in [182] in order to derive the fact that radiation is emitted by black holes and to derive its spectrum in a first approximation. This is all possible due to the Bogoliubov transformation [183] and the realization that the notion of vacuum, while well defined in standard flat spacetime, becomes relative i.e. observer dependent, when spacetime ceases to be flat. At a deeper level, what we use is the concept of differential forms. This is essentially a more general notation that makes the work with curved spacetimes easy. The ability to re-derive the symbols in a way that is general enough to be used perfectly at any level of formalization on curved or flat manifolds has brought quite some insight into general relativity. In the case of quantum gravity it is far more important to take into account variable or uncertain topologies. One could ask if it is possible to derive an even more general "framework" that would allow us to consider the change in topology in a universal way. This idea of generalizing concepts such that they become meaningful for the most variate situations was a constant in Grothendieck's approach to various problems [22],[23]. It is important to remark at this point something known as the "triangulation conjecture".

This can be formulated in terms of the "Hauptvermutung" of combinatorial topology [249]. The main conjecture is that any two triangulations of a triangulable space have a common refinement, i.e. a single triangulation that is a subdivision of both. This assumption (conjecture) is known to be false for dimensions 4 or larger. In another form the triangulation conjecture states that any n-dimensional topological manifold is homeomorphic to a simplicial complex. This is true in dimensions at most 3, but false in dimension 4 or larger, as proved in the work of Casson and Freedman [184]. It is important to notice however that the universal coefficient theorem and the therein used Tor and Ext groups show in what sense the notions employed in a simplicial complex description are being "measured" via (co)homologies and up to what extent the measurable properties encoded in (co)homologies are depending on the coefficient groups of (co)homology. This is essentially the subject of this section and it expresses the original work associated to this thesis.

9.2 Coefficients in (co)homology and quantum gravity

In this section I will show in what way the coefficient groups in cohomology can be useful in quantum gravity. Plausibility arguments will be given for the usefulness of standard and non-trivial cohomology coefficients in quantum gravity. The main reference is [284] but I will also refer to [285] and [286]. First, one should notice that quantum gravity is currently understood via two rather distinct and in many aspects incompatible [289] theories. It has even been argued that if one is correct, the other one cannot be, and vice-versa [289] (see conclusions). I will keep my neutrality with respect to such statements. The two theories are loop quantum gravity and string theory. String theory appears in various forms. It is basically trying with certain success, to quantize fundamental objects which are not naturally zero-dimensional. Therefore, the major change is to replace the fundamental point-like objects used in basic quantum mechanics with a fundamental string-like object. In any cohomological theory, the coefficient structure represents the zero degree cohomology of a point space. In the axiomatic definition, this cohomology group is given by the group of integer numbers $\mathbb{Z}$. This encodes the standard way of looking at a point, namely a structureless, zero dimensional, infinitely small object. This is also the basics of what is now known as the classical geometry. Grothendieck however, was the first to notice that additional structure can be added in general to a point. Therefore, it is possible to define cohomology theories where the degree zero cohomology group of a single point space is far from being only $\mathbb{Z}$. Indeed, one can introduce coefficients belonging to twisted groups, but also cohomologies with coefficient groups given by elliptic functions. These are particularly important in string and M-theory, both theories of quantum gravity. To make this argument clearer we may start by looking at the generalized elliptic cohomologies. We define an elliptic spectrum [284, 285] as consisting of

- an even periodic homotopy commutative ring spectrum E with formal group P_E over the coefficient group (associated to the single point space pt) $E^0(pt)$
- a generalized elliptic curve ϵ defined over the coefficient group $E^0(pt)$
- an isomorphism t of P_E with the formal completion $\hat{\epsilon}$ of ϵ

In general a 2-periodic ring spectrum can be seen as a generalized cohomology theory E with an orientation, of the complex line bundle $\mathbb{CP}^{\infty}$, which when restricted to $\mathbb{CP}^1$has an inverse in E_*. The theory is said to be even when $E_{2n+1} = 0$ for all n.

This definition involves the notion of generalized elliptic curves over a ring R [284, 285]. This means a marked curve over the scheme $Spec(R)$ which is locally isomorphic to a Weierstrass curve given for example by

$$\epsilon : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4 + a_6 \tag{9.1}$$

where $a_1, a_2, ..., a_6 \in R$.

A well known example of a cohomology theory disregarding the standard axiom of dimension is the so called K-theory. There, the cohomology of the point space may not only have a different group structure, but it may also have non-trivial values for higher degrees. From a physical perspective the elliptic spectrum appears for example in the discussion of Ramond-Ramond fields in K-theory. A K-theoretic description of the RR-fields for example in type IIA appears in [285, 287]. Moreover, the generalized cohomology theories that are being dealt with in string theory and M-theory have the correct mathematical structure required to incorporate the interactions between $M2$-branes and $M5$-branes. When speaking about cohomology theories with coefficient groups defined by means of prime numbers (for example $\mathbb{Z}/p\mathbb{Z}$) the prime 2 appears to be important in string theory. This happens because the 2-torsion appears in the fields and can be seen for example in the K-theory calculation of the IIA partition function [284, 288]. One can also consider cohomology theories whose formal group laws are elliptic, i.e. obtained by Taylor expansion of the group law on the elliptic curve over some commutative ring. Such theories are the complex oriented elliptic cohomology theories. Given the expression

$$\epsilon : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6 \tag{9.2}$$

introduce the new variables $t = \frac{-x}{y}$ and $s = \frac{-1}{y}$. Solving iteratively for s with respect to t and rewrite x and y only in terms of t yields:

$$x = t^{-2} - a_1 t^{-1} - a_2 - a_3 t - (a_4 + a_1 a_5) t^2 + ... \tag{9.3}$$

Now it is obvious that x and y are power series expansions in the variable t and with coefficients in $\mathbb{Z}[a_1, ..., a_6][[t]]$. The group law can be obtained near $t = 0$ where t is the local parameter near the origin $(t, s) = (0, 0)$ in E. If $(t_1, s_1) + (t_2, s_2) = (t_3, s_3)$ on an elliptic curve E in the (t, s) plane then $t_3 = \Phi(t_1, t_2)$ is

$$t_3 = t_1 + t_2 - a_1 t_1 t_2 - a_2(t_1^2 t_2 + t_1 t_2^2) - 2a_3(t_1^3 t_2 + t_1 t_2^3) + ... \tag{9.4}$$

therefore $\Phi(t_1, t_2) \in \mathbb{Z}[a_1, ..., a_6][[t_1, t_2]]$. If the coefficients a_j of E lie in a ring R then $t_3 = \Phi(t_1, t_2)$ is in $R[[t_1, t_2]]$. The formal series $\Phi(t_1, t_2)$ arising form the group law on E is a formal group law given by

$$\hat{C}(x, y) = x + y - a_1 xy - a_2(x^2 y + xy^2) - (2a_3 x^3 y - (a_1 a_2 - 3a_3) x^2 y^2 + 2a_3 xy^3) + ... \tag{9.5}$$

The formal group law can be defined as the way line bundles behave under tensor product i.e.

$$\hat{G}_E(x, y) = e(L_1 \otimes L_2) \in E^*(\mathbb{CP}^\infty \times \mathbb{CP}^\infty) \cong \pi_* E[[x, y]] \tag{9.6}$$

with e the Euler class. There are different models for complex oriented cohomology E which are described by their coefficient rings $E^*(pt)$. Choosing a complex oriented elliptic cohomology theory corresponds to choosing coordinates on the elliptic curve. The Weierstrass curve is the universal generalized elliptic curve given by the equation

$$y^2 x + a_1 xyz + a_3 y^3 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3 \tag{9.7}$$

over the ring $\mathbb{Z}[a_1, ..., a_6]$ where the parameters a_i are generalized modular forms. This curve has automorphisms and thus the corresponding theory with $E_* = \mathbb{Z}[a_1, ..., a_6][u, u^{-1}]$ is not universal. The elliptic curves are physical parts of spacetime and one can look at them as they were Riemann surfaces. Thus, they are defined over the set of complex numbers. The elliptic curves that show up in elliptic cohomology are defined over various fields or rings like the finite fields $\mathbb{F}_p$, integral or p-integral polynomial rings, etc. There is a standard procedure to get curves over the latter coefficients starting with curves over the complex numbers. One method, described also in [284] is to go from an elliptic curve over $\mathbb{C}$ to one over $\mathbb{Z}$ by means of a curve over the rationals $\mathbb{Q}$. One can get the curve over $\mathbb{Z}[x_1, x_2, ...]$ or $\mathbb{Z}[t]$ if one can define the curve over some finite extension of $\mathbb{Q}$. The resulting curve would have the same Weierstrass form except that the coefficients a_i instead of taking complex values now take values in the new field or ring. As an example for $\mathbb{Z}[t]$ one has $a_i(t) \in \mathbb{Z}[t]$ and

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t) \tag{9.8}$$

giving an elliptic curve E over $\mathbb{Z}[t]$. However, in these examples only the standard groups or fields for the coefficients have been used. Other possibilities imply additional torsion inside these groups as well as twisted groups for the coefficients. At this moment, I showed that the coefficient groups in (co)homology have important effects at least in the string theoretical interpretation of quantum gravity.

The main way in which (co)homology with various coefficients enters in the study of physical phenomena is the following. We start with a quantum state $\mathcal{H}$ carrying a representation of a large symmetry algebra, $\mathcal{G}$ of the problem. The constraints will naturally force us to introduce additional degrees of freedom, the "ghost states" which will form the space $\mathcal{H}^{gh}$. Then the constraints the constraints forming a subalgebra of $\mathcal{G}$ are imposed simultaneously on the enlarged space $\mathcal{H} \otimes \mathcal{H}^{gh}$ via the nilpotent BRST operator Q. The quotient space $Ker(Q)/Im(Q)$ will contain the physical states only. This space can carry its own symmetry, namely any algebra represented on $\mathcal{H} \otimes \mathcal{H}^{gh}$ with the property of being BRST invariant is a symmetry of the quotient space itself. The BRST approach and its associated semi-infinite cohomology have been discussed for example in [272].

9.3 Quantization Holography and the Universal coefficient theorem

I start my presentation with a method of quantization using cohomology groups extended via coefficient groups of different types. This is possible according to the Universal Coefficient Theorem (UCT). I also show that by using this method new features of quantum field theory not visible in the previous treatments emerge. The main argument is that several constructions considered as absolute until now may appear as relative, depending on individual choices of group structures needed to probe a topology. The universal coefficient theorem also gives information about how these structures as measured by different choices of groups, relate to each other. This may result in the formulation of new dualities and a deeper understanding of the relation between quantum field theories and quantum gravity (string theory). As already presented, the quantization of gravity is a major unsolved problem [185]. The equivalence principle [186], the black hole information paradox [187], the holographic conjecture [188], emergence of space-time [189] or coarse graining of observables [190] are only a few concepts that followed from it. I present here a method that makes use of a theorem of algebraic topology and homological algebra (the universal coefficient theorem) in order to suggest that some theoretical constructions used in previous descriptions of quantum gravity may not have an absolute meaning independent of some arbitrary choices of groups of coefficients. These choices of coefficients may induce different topological structures, therefore assuming independence of coefficient groups implies a form of independence of topology. This means that the observable topology is a property similar to the curvature in general relativity: by using the general notation of differential forms we encapsulate in the notation itself all the extra information required in order to deal with curved manifolds. The notation itself is however manifestly independent on the curvature of the manifold. The same happens with a new notation defined here, that makes use of coefficient groups in (co)homology.

The reason for considering this invariance as important in a quantum theory of gravity is the fact that there exist arbitrary choices that may make the connectivity of a space change.

One can cite the formation of a black hole that makes matter in a region of spacetime collapse onto itself. After the collapse passes the horizon, there is no method of avoiding the central region where quantum effects like spacetime topology change may appear.

Another example is the choice of making extremely accurate length measurements in space. This implies adding energy in a given region. This may in the end generate horizons which imply the collapse of matter towards a region where quantum gravity and changes of topology are assumed to be possible.

At this moment it is probably important to understand that with certain choices of coordinates, singularities may appear in the metric at the horizon. Such singularities are not physical. In order to quote Hawking on this matter, such a singularity is similar to the singularity appearing at the north pole of a sphere. It simply means that the concept of meridians doesn't have a meaning at the poles and there is nothing "at the north of the north pole". Such a singularity can be eliminated by another choice of a coordinate. However, in the case of a sphere, it is important to notice that there is no unique continuous coordinate system that may cover the entire sphere. The existence of coordinate singularities doesn't mean that some catastrophe happens where the singularity appears. Indeed, on the surface of the sphere nothing happens at such a singularity. However, the simple existence of such singularities tells us some information about the shape we want to cover. Indeed, no matter what unique continuous coordinate system we may use on a sphere, there will somewhere be a coordinate singularity. In order to eliminate such singular situations, for a sphere we use two patches with which we may cover the whole sphere. The global structure of the sphere therefore is connected to the existence of coordinate singularities for a single uniform patch which tries to cover the whole sphere.

For a black hole the situation is similar. There is nothing special at the horizon, but we cannot use a single patch of coordinates to map the whole black hole. Indeed, we can change the exterior coordinate patch such that there is no singularity at the horizon, and therefore we may make the connection with the interior patch. However, the simple fact that we cannot cover the whole black hole with just one patch tells us something about the topology of the spacetime with black holes. Indeed, like for a simple sphere, the horizon indicates that the global structure of a spacetime with a black hole is not trivial.

Moreover, the simple existence of a horizon makes some restriction on the possible choice of topologies. If we define a topology using open sets and we define them as sets of points joined together if they can be connected in both directions by light then, the appearance of a horizon makes a strong restriction on possible choices of topologies i.e. we cannot define open sets of the above type containing points from both sides of the horizon. I note also that the metric may well be defined on both sides and still, the problem of a choice of open sets persists. It is possible to define the metric with a "one-way" topology defined with "one-way" open sets from the exterior to the interior. However a "back-and-forth" topology with open sets containing elements joined in both directions by light is impossible. Hence one can ask the question: how does a choice of topology relate to another? What formal description can we introduce that could take care of any such transformations in a natural way? To these and more questions I will answer here.

One may assume that a full theory of quantum gravity may not depend on arbitrary choices of the kind mentioned above in the same way in which the formal aspects of general relativity should not depend on a choice of a coordinate system.

The applicability of the universal coefficient theorem is not restricted to space-time itself but can be used generally to field-spaces, groups, various manifolds or discrete spaces. Its use in these different situations will be made implicitly. The main idea here is that the identification of relevant physical observables in the QFT context is strongly dependent on the choice of coefficient groups associated to (co)homology groups of the field space.

The (co)homological structure of a field theory can be described with various coefficient groups, each inducing some indexation over the field space. It is well known that some choices are better than other. In general one uses a $\mathbb{Z}_2$-group when orientation is not relevant or a $\mathbb{R}$-coefficient structure when continuum properties of the analyzed space appear to be relevant. However, there are more subtle applications of the coefficient groups.

I show here that the choice of one coefficient group instead of another can hide a set of physically relevant observables in the quantization procedure. Also, the logical assignment of observables in an equivalence class dictated by the availability of a practical measurement of its spectrum by an observer may allow, by using the axiom of choice, the construction of predictors for the spectrum of other observables in the same equivalence class [191].

As a result, it appears to be impossible to assign an absolute topology to a space (be it "physical" spacetime or the space of field configurations) in the absence of an arbitrary choice of a coefficient group.

I start with a field theoretical context. At this level already some aspects must be clarified. When quantizing a one particle theory one may use for example Feynman's path integral formulation. This implies the existence of an "expectation catalogue" for positions in space-time indexed in some way. As no information about the intermediate steps is available one uses the principle of quantum mechanics that states that no actual state can be assigned to an object unless that state can be actually empirically confirmed to be realized. In this case the integration that gives rise to the quantum amplitude must be a sum over all possible configurations.

An extension of this principle was necessary due to the Lorentz group. As one was not able to discuss in the context of special relativity about a predefined or fixed number of particles, quantum fields had to be introduced. These are simply extensions of the "expectation catalogues" of simple one-particle quantum mechanics. They are not "measurable" in any physical sense individually, but their interference and their topology is probed statistically by the rules of quantum mechanics.

It should be well known that the statistics of an experiment (say Bohm-Aharonov) depends on the topology of the field space (the regions where the wavefunction is defined). In the end, the statistics must probe all connected components of all possible configurations. In the case of quantum gravity there are different approaches on how a quantum field theoretical formulation should look like. It is however clear that such a formulation should exist. I refer here to the works on string field theory, for example [205, 207, 208]. There the "quantum field" becomes a world-sheet-string-field "expectation catalogue" which is expanded even more with respect to the previous situations. While a string-field theoretical approach exists, it is not clear how the various configurations interrelate and what configurations can exist in various situations.

Dualities are supposed to help in this aspect by identifying configurations and simplifying the overall problem. The four-dimensional problem appears even more complicated as the triangularization conjecture is not valid. This means there exists no homeomorphism between the space we wish to probe and a simplicial complex i.e. there can be more simplicial complexes associated to one topological space, more topological spaces associated to a simplicial complex or there may be no simplicial representation at all. This is very important because the universal coefficient theorem expressly demands that various choices of coefficients may modify the perceived topology of the space via (co)homology groups while starting from the same simplicial complex. In this sense, the construction I introduced here is more general and closer to a true description of quantum gravity in four space-time dimensions. The "ambiguity" in the choice of coefficient groups appears hence to be natural.

Moreover, it appears to me that there exists a general method of constructing dualities based on the ideas presented in this article. It also appears to me that the constructed dualities will have an applicability restricted to specific arbitrary choices of group-structures in topology. This is conjectured to be valid also for the holographic principle. It is the universal coefficient theorem that will in the end provide a description of what configurations can be simultaneously known and what configurations will interfere at the level of the "catalogue of expectations". It also appears that the change of topology is of major importance in quantum gravity as one expects a change in the topology of spacetime during the formation of a black hole. However, the form of the laws of nature should not depend on a specific topology. I partially follow in this introduction reference [192,193].

First construct a functor $\mathfrak{E}$ from the category of spacetimes (Loc) to the category of local convex vector spaces (Vec).

This functor associates to each spacetime M a configuration space $\mathfrak{E}(M)$ of fields defined on it. The isometric embeddings $\chi : M \to N$ are mapped into pullbacks $\chi^* : \mathfrak{E}(N) \to \mathfrak{E}(M)$. The space of the observables called $\mathfrak{F}$ will be the space of the functionals $F : \mathfrak{E}(M) \to \mathbb{R}$. It is at this point that one also has to define the topological structure of the space (or space-time M). Physically this remains uncertain unless a choice of a coefficient group in (co)homology is made. This will define the topology and will allow a specific definition of the observables. Essentially the "experimental setup" (or a coefficient group choice) tells spacetime how to connect. This connection tells quantum mechanics how the correlations between "expectation catalogues" should be constructed (what observables make physical sense). What follows is standard quantum mechanics which (via the universal coefficient theorem) tells the experimentalist how to connect the results obtained with one group structure to possible results obtained by other observers using other group structures.

This is important when one compares, for example, the observations made when falling towards a black hole to those of a far away observer. Finally, accurate measurements and probing of spacetime at small scales implies adding energy in a small region of space which in the end may alter the topology of spacetime itself. One can observe that in principle a topology induced by a choice of a coefficient group (via a particular experimental setup) results in a modified set of observables and a modified algebra for the resulting quantum (field) theory. Also, the geometry of the (field) space imposes restrictions on possible topologies (for example extreme curvature may imply restrictions over the allowed topologies). One can summarize this as

$$Topology \begin{pmatrix} \text{probed by quantum mechanics} \\ \text{induced by a choice of a coefficient group} \end{pmatrix} \leftrightarrows Geometry \begin{pmatrix} \text{well defined local quantum observables} \\ \text{quantum operator algebras} \end{pmatrix}$$

In this context the main question for quantum gravity is "how do different geometries correlate?" To this question one can give an answer when one considers the topology of the field space and the fact that this topology is not given in an absolute sense. The acceptance of the non-universality of topology (as proved clearly by the universal coefficient theorem) leads to different "counting rules" for different contexts. In what follows one defines the class of functionals called "local functionals" as

$$F(\phi) = \int_M dvol_M f(j_x(\phi)) \tag{9.9}$$

where $j_x(\phi) = (x, \phi(x), \partial\phi(x), ...)$ is the jet of ϕ at the point x.

Let L be a suitably defined Lagrangian. We can define an associated action functional $S[L[\phi]]$. The field equation becomes in this context $S'_M(\phi) = 0$ where the prime denotes the Euler-Lagrange derivative. The space of solutions of this equation forms a subspace of $\mathfrak{E}(M)$ called $\mathfrak{E}_S(M)$. In the context of classical field theory one is interested in the space of local functionals over $\mathfrak{E}_S(M)$ called $\mathfrak{F}_S(M)$. This space can be defined as the quotient $\mathfrak{F}_S(M)$=$\mathfrak{F}(M)/\mathfrak{F}_0(M)$ where $\mathfrak{F}_0(M)$ is the space of functionals that vanish on-shell (on $\mathfrak{E}_S(M)$).

A (co)homological interpretation for the $\mathfrak{F}_S(M)$ space is required. For this one needs a vector field structure on the configuration space. The action of the vector fields $X[.]$ on the space of smooth functionals $C^\infty(\mathfrak{E}(M))$ is

$$\partial_X F[\phi] =< F[\phi], X[\phi] > \tag{9.10}$$

One can associate to the action functional a map from the set of test functions over the spacetime manifold to the space of observable functionals $\delta_S : \mathfrak{D}(M) \to \mathfrak{F}(M)$ such that

$$\phi \mapsto < S'_M[\phi], X[\phi] >= \delta_S(X)(\phi) \tag{9.11}$$

where S'_M is the Euler-Lagrange derivative of the action. Suppose there is an action S such that $\mathfrak{F}_0(M) = \delta_S(\mathfrak{D}(M))$. Then

$$\mathfrak{F}_S(M) = \mathfrak{F}(M)/\mathfrak{F}_0(M) = \mathfrak{F}(M)/Im(\delta_S) \tag{9.12}$$

From this one can construct the chain complex

$$0 \to \mathfrak{D}(M) \xrightarrow{\delta_S} \mathfrak{F}(M) \to 0 \tag{9.13}$$

This can be associated with the Batalin-Vilkovisky complex used in the geometric quantization. The 0-order homology of this complex is $\mathfrak{F}_S(M) = \mathfrak{F}(M)/\mathfrak{F}_0(M)$. The set of critical points of the action functional

$$\{\phi \in \mathfrak{D}(M) | \delta_S[\phi] = 0\} \tag{9.14}$$

contains connected components that can be identified by the first homotopy group

$$\pi_0(\{\phi \in \mathfrak{D}(M) | \delta_S[\phi] = 0\}) \tag{9.15}$$

The functionals on the classes of this group are the gauge invariant observables. One can see that the correct identification of possible maps as well as homotopically equivalent structures is extremely important for the correct construction of the field space in the phase preceding actual quantization. Probably the best mathematical formalization of quantum mechanics is offered by what is known as "geometric quantization" [193]. In this formulation one starts with a classical theory and follows a set of steps that assure the consistency of the resulting quantum theory. Obviously, not every classical theory can be lifted to a quantum level. In some cases it is necessary to re-interpret various properties appearing at the classical level from a quantum perspective, as I explained previously. Indeed the method presented here may explain what changes in our ways of thinking are required for this. One may start with a general classical action depending on a set of fields $S[\phi]$. This implies the existence of a symplectic manifold. The main idea is to realize the symplectic form of this manifold as the curvature of a $U(1)$ principal bundle with a connection. We obtain the pre-quantum Hilbert space as the Hilbert space of square integrable sections of the principal line bundle. One has to pick for each point in this space a certain subspace of the complexified tangent space at that point. One defines the quantum Hilbert space to be the space of all square integrable sections of the line bundle that give 0 when differentiated covariantly at that point in the direction of any vector of the tangent space. As basic quantum mechanics teaches us there exist two sets of variables that become non-commutative operators when quantizing. These may be called "positions" and "momenta" although their physical meaning may be rather different. The next step is the choice of a polarization i.e. the choice of "positions" and "momenta". This choice is not unique. Once a polarization is available one can form a Hilbert space of states as the space of sections of the associated line bundle. The last step would be to associate to the classical variables actual quantum operators on the quantum Hilbert space. This amounts to the quantization of observables while mapping Poisson brackets to commutators. This procedure is in general not well defined for all operators. Strictly speaking the method of geometric quantization is not properly defined in the context of quantum gravity. The definition of a field-space or a space of configurations is extremely complicated and the integration over such a structure appears to be ill-defined. However, it is precisely the method presented in this article that may add some extra structure to this space (for example via the identification of new dualities) such that its rigorous definition might become possible. Several attempts of using geometric quantization in the context of string theory are known [193,244] but the subject remains open for future research.

Given a BV complex and some quantum observables in the context of a choice of a coefficient structure I repeat here the universal coefficient theorem. At this point it plays the role of a lemma, which I intend to use for further constructions related to quantum field theories.

9.1 Lemma (The Universal Coefficient Theorem)

If C is a chain complex of free abelian groups, then there are natural short exact sequences

$$0 \to H_n(C) \otimes G \to H_n(C;G) \to Tor(H_{n-1}(C),G) \to 0 \tag{9.16}$$

$\forall\ n$, G, and these sequences split. Here $Tor(H_{n-1}(C),G)$ is the torsion group associated to the homology. In this way homology with arbitrary coefficients can be described in terms of homology with the "universal" coefficient group $\mathbb{Z}$

This is also valid for cohomology groups where it is formulated as

$$0 \to Ext(H_{i-1}(C),G) \to H^i(C;\mathbb{Z}) \otimes G \xrightarrow{h} H^i(C;G) \xrightarrow{r} Hom(H_i(C),G) \to 0 \tag{9.17}$$

where now the Tor group on the right is replaced by the Ext group on the left. Moreover, this theorem is a property of algebraic topology independent of the existence of an underlying manifold structure for the spaces or groups on which it may be applied. I also repeat here the example showing how the choice of the coefficient group can affect the correct identification of the homotopy type of a function.

9.2 Example (Homotopy and coefficient group)

Take a Moore space $M(\mathbb{Z}_m, n)$ obtained from S^n by attaching a cell e^{n+1} by a map of degree m. The quotient map $f : X \to X/S^n = S^{n+1}$ induces trivial homomorphisms on the reduced homology with $\mathbb{Z}$ coefficients since the nonzero reduced homology groups of X and S^{n+1} occur in different dimensions. But with $\mathbb{Z}_m$ coefficients the situation changes, as we can see considering the long exact sequence of the pair (X, S^n), which contains the segment

$$0 = \tilde{H}_{n+1}(S^n; \mathbb{Z}_m) \to \tilde{H}_{n+1}(X; \mathbb{Z}_m) \xrightarrow{f_*} \tilde{H}_{n+1}(X/S^n; \mathbb{Z}_m) \tag{9.18}$$

Exactness requires that f_* is injective, hence non-zero since $\tilde{H}_{n+1}(X; \mathbb{Z}_m)$ is $\mathbb{Z}_m$, the cellular boundary map

$$H_{n+1}(X^{n+1}, X^n; \mathbb{Z}_m) \to H_n(X^n, X^{n-1}; \mathbb{Z}_m) \tag{9.19}$$

being exactly

$$\mathbb{Z}_m \xrightarrow{m} \mathbb{Z}_m \tag{9.20}$$

One can see that a map $f : X \to Y$ can have induced maps f_* that are trivial for homology with $\mathbb{Z}$ coefficients but not so for homology with $\mathbb{Z}_m$ coefficients for suitably chosen m. This means that homology with $\mathbb{Z}_m$ coefficients can tell us that f is not homotopic to a constant map, information that would remain invisible if one used only $\mathbb{Z}$-coefficients.

It is important to observe that this example shows that (co)homology classes can merge or dissociate according to specific choices of coefficient groups. By this it becomes possible to redefine quantum field theories previously constructed on non-trivial field-manifolds, on simply connected manifolds. The corrections appear then as Tor or Ext groups. As the final step of this introduction I state here the main theorems of this thesis as well as a conjecture.

9.3 Theorem (Relativity of Observables) There exist observables visible using some choices of coefficient groups and invisible using other choices.

9.4 Theorem (Relativity of distinguishability)

There exists no unequivocal measure of distinguishability of quantum states that is independent of the choice of the coefficient group. Distinguishability is relative.

9.5 Theorem (Relativity of Symmetry)

A particular choice of a coefficient group makes a specific symmetry structure in the field space manifest. There exists no absolute symmetry.

9.6 Conjecture (Relativity of Holography)

There is no general unequivocal mapping of any consistent geometric structure in a space-time volume to its surface. In the full context of quantum gravity the existence of a holographic principle is an undecidable statement depending on particular choices of the coefficient groups. "Strong-weak" dualities can however be constructed and generalized in a case-by-case way

The proofs of the theorems as well as validity arguments for the conjecture are provided in the following subsections. The method of proof is as follows: I make a choice of a coefficient group in cohomology (i.e. a choice of topology). I try to construct standard quantum mechanics (eventually using geometric quantization). If geometric quantization is impossible I can always switch to a different topology where this method is possible and see how it relates to the topology where it was impossible via the universal coefficient theorem. This may bring new insights about the geometric quantization prescription. I construct a set of observables and physical states using a particular choice of the coefficient group. I obtain a set of physical states obeying some properties (distinguishability, etc.). I make another choice of the group structure where the above stated properties are not valid any more. By the Universal Coefficient Theorem it follows that the considered properties are relative i.e. cannot be associated to a full theory of quantum gravity.

One method of quantization is given by what is known under the name of "Feynman path integral". For an introduction I partly follow Feynman's original paper [206]. I assume that the standard prescription of computing quantum probabilities using quantum amplitudes is well known. If P_{ac} is the quantum probability of measuring event c when it follows the measurement of event a then the probability must be calculated as $P_{ac} = |\varphi_{ac}|^2$ where $\varphi_{ac} = \sum_b \varphi_{ab}\varphi_{bc}$ where the sum is over the possible intermediate states b which, I emphasize, following Feynman have no meaningful independent value. In a 1-space and 1-time dimensional context a succession of measurements may represent a succession of the space-coordinate x at successive times $t_1, t_2, ...,$ where $t_{i+1} = t_i + \epsilon$. Let the observed value at t_i be x_i. Classically the successive values of $x_1, x_2, ...$ define a path $x(t)$ when $\epsilon \to 0$. If the intermediate positions are actually measured one may talk about such a path with a well defined set of observed positions $x_1, x_2, ...$ and the probability that the specified path $P(...x_i, x_{i+1}, ...)$ lies in a region R is given by the classical formula

$$P = \int_R P(...x_i, x_{i+1}, ...)...dx_i dx_{i+1}... \tag{9.21}$$

where the integral is taken over the ranges of the variables which lie within the region R. If the intermediate positions are not measured then one cannot assign a value to them. In this case the probability of finding the outcome of a measurement in R is $|\varphi(R)|^2$ and

$\varphi(R)$, i.e. the probability amplitude, is calculated as

$$\varphi(R) = \lim_{\epsilon \to 0} \int_R \Phi(...x_i, x_{i+1}, ...) \tag{9.22}$$

where $\Phi(...x_i, x_{i+1}, ...)$ defines the path. In the given limit this object becomes a path functional. There should be no mystery nowadays that the probability amplitude should be calculated as

$$\varphi(R) = \lim_{\epsilon \to\ 0} \int_R \exp[\frac{i}{\hbar} \sum_i S(x_{i+1}, x_i)] ... \frac{dx_{i+1}}{A} \frac{dx_i}{A} ... \tag{9.23}$$

where S is the action functional for the given path segment. In order to go a step further and define the wavefunction in this context I will continue to follow Feynman's paper [206]. The region R considered above can be divided into future and past with respect to a choice of a time position t. One can define the region R' as the past and the region R'' as the future. The probability amplitude connecting these regions will be

$$\varphi(R', R'') = \int \chi^*(x,t)\psi(x,t)dx \tag{9.24}$$

where

$$\psi(x_k, t) = \lim_{\epsilon \to 0} \int_{R'} \exp[\frac{i}{\hbar} \sum_{i=-\infty}^{k-1} S(x_{i+1}, x_i)] \frac{dx_{k-1}}{A} \frac{dx_{k-2}}{A} ... \tag{9.25}$$

and

$$\chi^*(x_k, t) = \lim_{\epsilon \to 0} \int_{R''} \exp[\frac{i}{\hbar} \sum_{i=k}^{\infty} S(x_{i+1}, x_i)] \frac{1}{A} \frac{dx_{k+1}}{A} \frac{dx_{k+2}}{A} ... \tag{9.26}$$

In this way one can separate the "past" and the "future" via the functions ψ and χ. One may also construct a closer equivalence to the matrix representation of quantum mechanics by introducing matrix elements of the form

$$\begin{aligned} &< \chi_{t''}|F|\psi_{t'} >_S = \lim_{\epsilon \to\ 0} \int ... \int \chi^*(x'', t'')F(x_0, ...x_j)\times \\ &\times \exp[\tfrac{i}{\hbar} \sum_{i=0}^{j-1} S(x_{i+1}, x_i)]\psi(x', t') \tfrac{dx_0}{A} ... \tfrac{dx_{j-1}}{A} dx_j \end{aligned} \tag{9.27}$$

In the limit $\epsilon \to 0$, F is a functional of the path $x(t)$. At this moment one can define various equivalences between functionals. These are to be associated to operator equations in the matrix formulation. One can of course define $\frac{\partial F}{\partial x_k}$ and one can calculate the associated matrix element using an action functional S. Using the fact that the action functional appears as $\exp(\frac{i}{\hbar}S)$ one obtains matrix equations as, say

$$< \chi_{t''}|\frac{\partial F}{\partial x_k}|\psi_{t'} >_S = -\frac{i}{\hbar} < \chi_{t''}|F\frac{\partial S}{\partial x_k}|\psi_{t'} >_S \tag{9.28}$$

which can be stated as a functional relation defined for an action S as

$$\frac{\partial F}{\partial x_k} \leftrightarrow -\frac{i}{\hbar}F\frac{\partial S}{\partial x_k} \tag{9.29}$$

Using the fact that $S = \sum_{i=0}^{j-1} S(x_{i+1}, x_i)$ one can rewrite

$$\frac{\partial F}{\partial x_k} \leftrightarrow -\frac{i}{\hbar}F[\frac{\partial S(x_{k+1}, x_k)}{\partial x_k} + \frac{\partial S(x_k, x_{k-1})}{\partial x_k}] \tag{9.30}$$

In the case of a simple 1-dimensional problem one can write

$$\frac{\partial S(x_{k+1}, x_k)}{\partial x_k} = -m(x_{k+1} - x_k)/\epsilon \tag{9.31}$$

and

$$\frac{\partial S(x_k, x_{k-1})}{\partial x_k} = +m(x_k - x_{k-1})/\epsilon - \epsilon V'(x_k) \tag{9.32}$$

Neglecting terms of order ϵ one obtains

$$m\frac{(x_{k+1} - x_k)}{\epsilon}x_k - m\frac{(x_k - x_{k-1})}{\epsilon}x_k \leftrightarrow \frac{\hbar}{i} \tag{9.33}$$

The important aspect here is that the order of terms in a matrix operator product corresponds to the order in "time" of the corresponding factors in a functional. The order of the factors in the functional is of no importance as long as the indexation of these factors is reflected in the ordering of the operators in the matrix representation. This means the left-most term in the above equation must change order so that one obtains the well known commutation relation

$$px - xp = \frac{\hbar}{i} \tag{9.34}$$

One may observe that the choice of a specific indexation of the measurement outcomes, according to a time index (i.e. $\mathbb{Z}$-group), leads to the well known commutation relations. The ideas behind path integral quantization are kept intact when going to the relativistic context. However, when we have to go to a gravitational context the sum over configurations (geometries) becomes non-trivial. One first attempt of dealing with a path integral formulation for gravity has been studied by S. Hawking in [316]. There, it is argued that the contribution to the action functional given by solutions of Einstein's equation with trivial topology i.e. $\mathbb{R}^3 \times S^1$ is trivial. Non-trivial topologies, like those associated to the Schwarzschild solutions are expected however to give the simplest non-trivial contributions. It is also known that perturbation theory breaks down in quantum gravity beyond the one loop calculations as the theory, as formulated by Einstein is not renormalizable by standard means. As the topology becomes relevant, the (co)homology together with other topological invariants become relevant. In this sense one has to construct the (co)homology structure of the space and one has to deal with the universal coefficient theorem. This theorem states that a specific framework, constructed through the choice of a coefficient group in (co)homology is, up to (extension) torsion in (co)homology, equivalent with the choice of an integer coefficient group. However, some choices of coefficient groups may make some observables manifest while others may hide them. Moreover, simple order relations as the ones used in the proof above are no longer uniquely defined. What was identified by Feynman as a natural choice (time ordering) may in fact be just the result of a given coefficient group. Other ordering relations (like radial ordering in the case of CFT's) are also known. It is visible in this context that the construction of a path integral prescription using another coefficient group will change the quantization prescription (as formulated via the algebra of operators). Quantization doesn't mean only algebra of operators, as has been made obvious in the definition of geometric quantization. In an ideal situation one would expect a physical motivation that determines the operator algebra. This might appear in the context of the application of universal coefficient theorems.

The group structure imposed over the configuration space can be chosen for example as $\mathbb{R}/\mathbb{Z}$ case in which one arrives at a continuous cyclic structure. This will present a somehow altered operator algebra. One may ask what is the physical meaning of the coefficient group? In fact, it is an extra layer of information that has to be dealt with when performing quantization. It appears that it is not sufficient to simply integrate over non-equivalent field configurations as done in non-gravitational models. The coefficient structure adds new "degrees of freedom" to the problem. These must be considered when performing path integral quantization in order to obtain suitable unitary results. From this point of view, the extra-structure appears to be a step forward towards the unambiguous solution of the unitarity problem (also known as "information paradox"). In a less formal tone, the "information" describing the system is encoded not only in the actual system but also in the set of rules one chooses in order to "read" that information. I stressed in the above digression that the intermediate states in the path integral formulation must be added to the amplitude while keeping all possible outcomes, mainly because one cannot assign an outcome before a measurement is performed. The same considerations are valid when dealing with coefficient groups. While one can certainly prepare an experiment that involves a special choice of a coefficient group one will obtain a result dependent of this choice. When no practical choice is made one cannot assign any "physical" value to the choices of coefficients but one must consider them when calculating quantum amplitudes. From this perspective the question of the existence of a "Planck scale topology" is void of meaning. "Microscopic geometries" are to be associated to choices of coefficient groups and these choices are arbitrary. However, the universal coefficient theorem generates classes of topologies that can be identified in the sense of having the same Ext and Tor groups. This may lead to an overall simplification of the path integral formulation as many configurations will appear as connected by dualities. One should notice that both string theory and loop quantum gravity assume special choices of topology as being absolute (Lie group topology for string theory as the "string worldsheet" and discrete topology for LQG). I consider these choices as an epistemological issue. In string theory one starts by postulating a fundamental string. This implies a continuous group structure and a well defined topology. By the universal coefficient theorem however, this is simply a convention. Using that convention one arrives at an algebra of operators (say, Virasoro algebra). It should be clear now that this choice has nothing fundamental to it. In loop quantum gravity one starts the other way around: one fixes the canonical quantization prescription involving the standard algebra and obtains in the end a particular topology (a discrete topology). Again, one arbitrary choice determines the other. There is nothing fundamental to it either. One cannot assign a precise topology to any space unless one makes a choice of a coefficient group in cohomology. In order to do this one must consider the universal coefficient theorem and its Tor and Ext groups. Any fixation on an absolute topology would be equivalent with the postulation of the "ether" in special relativity i.e. void of meaning.

One may notice that quantum gravity cannot be defined using a fixed (non-dynamical) spacetime manifold. In fact, analysis in terms of the universal coefficient theorem makes the spacetime highly dynamical allowing even changes of topology. Indeed, for example topology is not fixed by the condition of Ricci flatness, i.e. $R_{\mu\nu} = 0$. The standard example is given by the Calabi-Yau manifolds which have a non-trivial structure of Hodge numbers while being Ricci flat.

Aside from its dynamical structure, spacetime is also non-triangulable. This also means there is no single spacetime to be associated to a chain complex and hence there exists an intrinsic uncertainity when speaking about spacetime topology. All these can be seen if one considers for example coefficient groups of finite torsion degrees. The larger (but finite) the torsion degree of the group the more "non-local" will the associated "observables" look. The "non-local" behaviour in extreme conditions (black holes) is essentially the result of a specific choice of topology. This will persist until clearer information about the group structure imposed by a particular experiment is given. When this happens is for the experiment to decide. The situation is similar to the supposed "objective collapse of the wavefunction" which is assumed (wrongly) to actually happen at some scale. This mistake vanishes when one understand that the wavefunction is to be interpreted as a "expectation catalogue". In the same way, when information about the connectivity of spacetime and of the "field-space" becomes manifest one will have to adopt the local structure at hand. Of course, topologically disconnected macroscopic black holes may retain (from the perspective of an observer lying outside) some apparent non-local aspects as their internal structure is inaccessible.

One may ask if my method identifies the different representations for the same algebra of operators. This is not the case. As can be seen from Feynman's example the specific ordering of the events generates some commutation relations which define the algebra of operators. If one generalizes this to different choices of coefficient groups for probing the field space one can see that the algebra of operators will not be preserved. Indeed, one can use coefficients in a continuous group. In this case one can recover the string-theoretical case where a continuous line-like object appears as "fundamental" and in fact the algebra of its operators is rather different. The associated group is generally not easily connected to the local algebra as the exp map is not always easily defined. Continuous group coefficients are useful. It is well known that one uses continuous coefficient groups when one wishes to avoid unnecessary complications due to the low-scale behaviour of the space to be studied. In fact, a claimed advantage of working with string-like objects is its so called "UV-completeness". Of course, from the perspective of coefficient-group-extended quantization this property is just a trade-off between using continuous groups in order to have UV-completeness and the complications that appear in the BRST-cohomology treatment of string theory.

9.4 Relativity of Observables

As shown in the previous subsection, the physical observables are to be identified with the functionals over the classes of the homotopy group associated to the critical points of the action functional. Example 9.2 already showed how this identification is relativized by the UCT. I give here a more detailed proof. Take a set of observables obtained after geometric quantization

$$\mathcal{A} = \{A_1, A_2, ..., A_n\} \tag{9.35}$$

where $\mathcal{A} \subset \mathfrak{F}_S$. While in the classical case $\mathfrak{F}_S$ is to be associated with a space of local functionals, in the case of quantum gravity the locality condition may be relaxed. One can observe that the BV-complex

$$0 \to \mathfrak{D}(M) \xrightarrow{\iota} \mathfrak{F}(M) \xrightarrow{\gamma} \mathfrak{F}_S(M) \to 0 \tag{9.36}$$

with $\mathfrak{F}_S(M) = \mathfrak{F}(M)/\mathfrak{F}_0(M)$ and $\delta_S = \gamma \circ \iota$ can be represented as the complex of example 9.2

$$0 \to \tilde{H}_{n+1}(X; \mathbb{Z}_m) \xrightarrow{f_*} \tilde{H}_{n+1}(X/S^n; \mathbb{Z}_m) \to ... \tag{9.37}$$

In the last case f_* is the induced map over the homology groups of the map $f : X \mapsto X/S^n$ over the analyzed spaces. In the case of the BV-complex the original maps would be the functionals $F : \mathfrak{E}_S \mapsto \mathfrak{E}_S$ which are to be associated to the physical observables of the quantum theory. In the same way as in example 2 one can define the map as a function of degree m.

One may remark that observables that cannot be distinguished in $\mathbb{Z}$ will be visible if the choice of coefficients is $\mathbb{Z}_m$.

In order to have a correct representation of the actual set of observables one must redefine $\mathcal{A}$ as

$$\tilde{\mathcal{A}} = \{[A_1], [A_2], ..., [A_n]\} \tag{9.38}$$

where each term $[A_i]$ may be a set of observables on its own, the elements of which may not be discernible given a specific choice of coefficients. To show that the sequence in homology from example 9.2 is applicable I will have to show that the set of observables is representable in the form of a (co)homology. I showed previously that indeed the observables are related to a homotopy group. In following reference [262, 309] I will show that in fact there is a homological representation of the observables and indeed one arrives at a sequence comparable to the sequence above in homology. In order to have a better understanding of this statement it is useful to understand the meaning of (co)homology in the standard BV formalism. Using the vector fields on $\mathfrak{E}(M)$ it is possible to characterize the elements of $\mathfrak{F}_0(M)$. This space includes elements of the form

$$\phi \to < S'_M(\phi), X(\phi) >= \delta_S(X)(\phi) \tag{9.39}$$

where S'_M is the Euler Lagrange derivative of the original action. This notation underlines that one can associate to the action S a map $\delta_S = \mathfrak{D}(M) \to \mathfrak{F}(M)$. This map is a differential and its image is contained in $\mathcal{F}_0(M)$. If it also holds that $\mathfrak{F}_0(M) = \delta_S(\mathfrak{D}(M))$ we say that $\mathfrak{F}_0$ is generated by the equations of motion. This is the case for many physical situations including the Yang-Mills theory and gravity. The problem is then reduced to the finite dimensional case [262]. One may use locality and make it sufficient to characterize the local functionals that vanish on-shell. These can be written as an integration $\int_M dvol_M (j_x^\infty)^*(\phi) f(x)$ of a function on the jet space that depends only on a finite number of derivatives $j_x^k(\phi)$ of configuration fields $\phi \in \mathfrak{E}(M)$ at a given point $x \in M$. This reduces the problem locally to a finite dimensional one [309]. Assuming that the action

S is given such that $\mathfrak{F}_0(M) = \delta_S(\mathfrak{D}(M))$ holds we can write

$$\mathfrak{F}_S(M) = \mathfrak{F}(M)/\mathfrak{F}_0(M) = \mathfrak{F}(M)/Im(\delta_S) \tag{9.40}$$

and this can be written in terms of homological algebra. Take the chain complex

$$0 \to \mathfrak{D}(M) \xrightarrow{\delta_S} \mathfrak{F}(M) \to 0 \tag{9.41}$$

The zero-degree homology of this complex is $\mathfrak{F}(M)/\mathfrak{F}_0(M) = \mathfrak{F}_S(M)$. In order to find the full homological interpretation of $\mathfrak{F}_S(M)$ we need a resolution of $\mathfrak{F}_S(M)$. Given a graded algebra A and a differential operator δ a resolution of A is $(\mathcal{A}, \delta)$ such that $H_0(\delta) = A$ and $H_n(\delta) = 0$ for $n > 0$. Now, starting from the chain above, one can start to construct the resolution of $\mathfrak{F}_S(M)$ considering that the space of multivector fields $\Lambda\mathfrak{D}(M)$ is a graded commutative algebra with respect to the exterior product. We also have a natural bracket structure $\{*, *\}$. Having $\delta_S(X) = \{X, L_M(f)\}$ for $f = 1$ on the support of $X \in \mathfrak{D}(M)$, it is possible to extend δ_S to $\Lambda\mathfrak{D}(M)$ by using the graded bracket. This results in the complex

$$... \to \Lambda^2\mathfrak{D}(M) \xrightarrow{\delta_S} \mathfrak{D}(M) \xrightarrow{\delta_S} \mathfrak{F}(M) \to 0 \tag{9.42}$$

Here the map δ_S is the Koszul map. In what follows we want to calculate the homology $H_1(\Lambda\mathfrak{D}(M), \delta_S)$. It will be important to identify the elements of $Ker(\delta_S)_{\mathfrak{D}(M)\to\mathfrak{F}(M)}$. These are the symmetries associated to the theory. But how does this approach in general relate to the BV formalism? Vector fields $\mathfrak{D}(M)$ correspond to functionals of the antifields. The action functional of a vector field on a functional can be written as

$$\partial_X F(\phi) =< F(\phi), X(\phi) >= \int_M dvol_M X(\phi)(x) \frac{\delta F(\phi)}{\delta\phi(x)} \tag{9.43}$$

The functional derivatives $\frac{\delta}{\delta\phi(x)}$ are to be identified with the antifields $\phi^\dagger$ and the elements of $\mathfrak{D}(M)$ are

$$X(\phi) = \int_M dvol_M X(\phi)(x) \frac{\delta}{\delta\phi} = \int_M dvol_M X(\phi)(x)\phi^\dagger \tag{9.44}$$

The antibracket is then

$$\{X, Y\} = -\int dx(\frac{\delta X}{\delta\phi(x)} \frac{\delta Y}{\delta\phi^\dagger(x)} + (-1)^{|X|} \frac{\delta X}{\delta\phi^\dagger(x)} \frac{\delta Y}{\delta\phi(x)} \tag{9.45}$$

The most important part for characterizing the BV formalism is to show how it deals with symmetries. Moreover this will allow us to calculate the homology and present the (co)homological interpretation of the whole formalism. The symmetries are vector fields on $\mathfrak{E}(M)$ that describe directions in the configuration space in which the action S

is constant. This can be expressed as the condition that for every $\phi \in \mathfrak{E}(M)$

$$0 = \delta_S X(\phi) =< S'_M(\phi), X(\phi) >= \partial_X(S_M)(\phi) \tag{9.46}$$

If a symmetry vanishes on-shell i.e. $X(\phi) = 0$ for all $\phi \in \mathfrak{E}_S(M)$ it is called trivial. Call the Lie subalgebra of $\mathfrak{D}(M)$ consisting of all symmetries as $\mathfrak{s}(M)$ and the trivial ones call them $\mathfrak{s}_0(M)$. The space of non-trivial symmetries then is just the quotient

$$\mathfrak{s}_{ph}(M) = \mathfrak{s}(M)/\mathfrak{s}_0(M) \tag{9.47}$$

therefore a set of equivalence classes of vector fields on $\mathfrak{E}(M)$ with the equivalence relation connecting elements which are on shell. The algebra of symmetries $\mathfrak{s}(M)$ has a natural action on $\mathfrak{F}(M)$ by derivations. The aim of the BV formalism is to determine the construction of the spaces $\mathfrak{s}(M)$ and $\mathfrak{s}_{ph}(M)$. The trivial symmetries are in the image of δ_S i.e. $(Im\delta_s)_{\Lambda^2\mathfrak{D}(M)\to\mathfrak{D}(M)}$. They do not contribute to the homology $H_1(\Lambda\mathfrak{D}(M), \delta_S)$. Therefore the first homology of the Koszul complex above is trivial if S doesn't possess any non-trivial local symmetries. We therefore get

$$\begin{aligned} &H_0(\Lambda\mathfrak{D}(M), \delta_S) = \mathfrak{F}_S(M) \\ &H_k(\Lambda\mathfrak{D}(M), \delta_S) = 0, \qquad k > 0 \end{aligned} \tag{9.48}$$

The complex $(\Lambda\mathfrak{D}(M), \delta_S)$ is the Koszul resolution of $\mathfrak{F}_S(M)$.

This shows that the set $\mathfrak{F}_S(M)$ can be represented as the degree zero homology of the Koszul complex which indeed contains the observables. In general however, additional symmetry may be introduced in the system, pushing the non-trivial homology to higher degrees. Indeed, the various degrees of the homology correspond to the ghost numbers of the respective formulation of the physical theory.

Therefore we can rewrite the sequence

$$0 \to \mathfrak{D}(M) \xrightarrow{\iota} \mathfrak{F}(M) \xrightarrow{\gamma} \mathfrak{F}_S(M) \to 0 \tag{9.49}$$

in terms of the homology with a certain coefficient structure and obtain

$$0 \to H_*(\mathfrak{D}(M), \mathbb{G}) \to H_*(\mathfrak{F}(M), \mathbb{G}) \xrightarrow{f^*} H_*(\mathfrak{F}_S(M), \mathbb{G}) \to 0 \tag{9.50}$$

not forgetting that $\mathfrak{F}(M)/\mathfrak{F}_0(M) = \mathfrak{F}_S(M)$. $\mathbb{G}$ is a generic notation for a group structure which might induce the nonzero homology in different dimensions. If this happens, the same argument of example 9.2 will also apply here, and the observables will merge.

There are various physical confirmations for this theoretical observation. Indeed, it has been noted in reference [195] that for example classes of microscopical observables of black holes may be inaccessible to independent measurement due to large energies or long times required for accurate probing. While this is certainly possible, I showed here that the same can happen due to certain choices of coefficient groups. While it is certainly always possible to change the coefficient group with which one probes the field space this change may involve a change in the physical experimental setup. This would make a simultaneous use of two coefficient groups in the same experiment impossible. As indiscernability of observables (coarse graining) may imply emergent locality (as shown in [195]) it may look like the UCT assures some form of locality at all levels. However, I am cautious in calling this "locality" with its proper name. I am also cautious when speaking about "emergent locality" or even more drastically, "emergence of space-time" (see ref. [195]) The reasons for this caution are expressed in the following subsection.

9.5 Relativity of distinguishability

Ongoing research in quantum information has led to various alternative definitions of distinguishability of quantum states. One recent paper [195] argues that physical criteria like extreme energy requirements or long waiting times would make some distinctions between quantum states impractical. I show here that in fact distinguishability of quantum states is mainly related to choices of the coefficient groups of (co)homology. There exist possible predictors that allow "guesses" concerning the presence of different physical states in the same equivalence classes associated to some observers [191]. Using quantum information tools one observes that given a set of observables $\mathcal{A}$ one cannot distinguish a random pure micro-state in a microcanonical ensemble H_E of dimension d_E from the maximally entangled state $\Omega_E = \frac{I_E}{d_E}$ unless the number of different outcomes of the operator $N(\mathcal{A})$ scales as $\sqrt{d_E}$. Whenever $N(\mathcal{A}) \sim \sqrt{d_E}$ one would require a long time or very large energies to achieve the accuracy that would allow the distinction of these states. These statements presented also in [195] are partially correct. While one can follow the standard path of constructing normed or semi-normed spaces that would predict how "far away" quantum states are in a given configuration I show here that these measures must be relative considering the fact that the arbitrary choice of a coefficient group may make the difference between distinguishability and indistinguishability of two quantum states relative. This statement is in full agreement with the uncertainty principle and in the spirit of quantum mechanics as it extends the concept of uncertainty to the arbitrary choice of a coefficient group. In this subsection I follow ref. [195] in order to introduce the concepts I require. Consider a finite dimensional subspace $H_E \subset H$ of dimension d_E consisting of all pure states $\psi = |\psi><\psi|$ that live in a microcanonical ensemble of energy $[E-\delta E, E+\delta E]$. I may assume that the Hamiltonian describing the unitary time evolution of the system has non-degenerate energy gaps. Consider again the set of observables $\mathcal{A} = \{A_1, A_2, ..., A_n\}$. One may ask what are the necessary conditions for such a set to distinguish a random pure state $\psi \in H_E$ from a maximally mixed state in H_E. One can follow two obvious paths and one less obvious path to quantify the difference between quantum states $\psi \in H_E$. What one obviously could do is to measure the expectation value of some operator $A \in \mathcal{A}$. However, the measurement of expectation values of an observable is not sensitive enough to distinguish any different quantum states. A quantum measurement in general offers a set of eigenvalues a appearing with some probabilities p_a. Most of the information about the quantum system is encoded in the probability spectrum $\{p_a\}$. Hence in order to distinguish two quantum states ρ and σ using a particular observable A one can define a measure as

$$D_A(\rho,\sigma) = \frac{1}{2}\sum_a |tr(|a><a|\rho) - tr(|a><a|\sigma)| \tag{9.51}$$

$|a>$ being the eigenvectors of A. This measure is defined so that it encodes the information of the entire spectrum $\{p_a\}$. One can extremize the definition in order to define a measure over a whole set of observables

$$D_{\mathcal{A}}(\rho,\sigma) = \max_{A\in\mathcal{A}} D_A(\rho,\sigma) \tag{9.52}$$

If $\mathcal{A}$ includes the entire set of observables in the Hilbert space one may define the distinguishability of two quantum states in general as

$$D(\rho,\sigma) = \frac{1}{2} tr|\rho-\sigma|_{\mathcal{A}} \tag{9.53}$$

where $|\rho-\sigma|_{\mathcal{A}}$ is the maximal difference in probability spectra over the entire set of available observables. If I continue to use this language it will be impossible to identify the restrictions due to the universal coefficient theorem. In fact one has to go a step back and to remember that quantization implies summation over inequivalent field configurations and this implies the construction of (co)homology groups. Physical observables are identified with the functionals over the classes of these groups. Different choices of coefficient groups in the (co)homology may lead to identification of functionals (they may appear as homotopic to the identity) while using other groups may make them appear in different classes (i.e. being different observables). Considering that special features of the field space induced by mappings of finite degree cannot be ignored in the procedure of quantization one may have for a complex like

$$0 \to \tilde{H}_{n+1}(X;\mathbb{Z}_m) \xrightarrow{f_*} \tilde{H}_{n+1}(X/S^n;\mathbb{Z}_m) \to ... \tag{9.54}$$

a set of observables $\mathcal{A} = \{A_1, A_2, ..., A_n\}$ while under

$$0 \rightarrow \tilde{H}_{n+1}(X;\mathbb{Z}) \xrightarrow{f_*} \tilde{H}_{n+1}(X/S^n;\mathbb{Z}) \rightarrow ... \tag{9.55}$$

another set $\tilde{\mathcal{A}} = \{[A_1...A_{i_1}], [A_{i_2}...A_{i_3}]..., [A_{i_k}...A_{i_n}]\}$ where the observables in the square brackets represent the classes of observables that cannot be distinguished in the given coefficient setup. One may imagine that the choice of a coefficient group induces a forgetful functor between the category of observables $\mathcal{A}$ and $\tilde{\mathcal{A}}$. This functor also maps the discernability measure from

$$D_{\mathcal{A}}(\rho, \sigma) = \max_{A \in \mathcal{A}} D_A(\rho, \sigma) \tag{9.56}$$

towards

$$D_{\tilde{\mathcal{A}}}(\rho, \sigma) = \max_{A \in \tilde{\mathcal{A}}} D_A(\rho, \sigma) \tag{9.57}$$

One may observe that although the definition is still valid, the set of available observables changed significantly. In the previous subsection I invited to caution in using terms like locality in relation to indiscernability of observables and entanglement. Indeed, the prescription of maximization used in the definition of the measure above is not trivial. Following the universal coefficient theorem, in order to establish the maximum over the set of observables, one will always have to pick one element from an equivalence class. One may not be aware of the existence of more than one element in the given class but the class exists and a choice has to be made in order to be able to compare in the end representatives from various classes. In order to be able to do this (as the elements of one class are supposed to be indiscernable so one cannot define a choice function) one has to invoke the axiom of choice. However, associating probability theory and the axiom of choice in the context of quantum mechanics is probably the most non-trivial task in mathematical logics. Examples of how the axiom of choice reflects on the mathematics of coordinated inference can be found in [191]. What I may add here is that the indexation of operators in $\mathcal{A}$ and $\tilde{\mathcal{A}}$ may give an order relation in terms of, for example, energy. In this sense one may define the order over the operators in $\mathcal{A}$ as

$$A_1 \prec A_2 \prec ... \prec A_n \tag{9.58}$$

This ordering implies the visibility at a given energy. However, the deformation of some observables such that they enter a single homotopy class after the application of a new coefficient group may alter this order. In fact, one will have to define an order relation between equivalence classes where the choice of representatives is not unambiguously defined in the absence of the axiom of choice

$$[A_{i_1}] \preceq [A_{i_2}] \preceq ... \preceq [A_{i_n}] \tag{9.59}$$

Nothing stops this new ordering to invert the previous one in some instances such that observables invisible at some energy and choice of coefficients become visible under another choice of coefficients [194]. It follows that new "strong-weak" dualities can be constructed using the method of coefficient groups. Their applicability goes beyond quantum gravity to subjects like condensed matter or many particle systems. Everything one has to do is to re-quantize the theory using a different coefficient setup and to take into account possible torsion groups in homology. While theoretically this is possible it remains to be seen if there are practical difficulties. Another aspect that might be important in this context is the similarity of these problems with the "hat problems" discussed in [191]. The main idea is that although it may look unlikely, there might exist predictors that after a finite set of trials are always capable of assigning the equivalence class of an operator and determine an order of occurrence. These predictors however, depend on the availability of the axiom of choice. However, their existence may suggest that exact locality may be dependent of some very particular choices. One may also ask if the renormalization prescription is affected by the indiscernability of states induced by choices of (co)homology. Possible emergence of new "topological" Ward identities (i.e. having their origin in some remaining "invariance" under change of topology, prescribed by the UCT) may have important roles in a possible renormalization of gravity.

9.6 Relativity of Symmetry

Symmetries are of major importance in physics in general and in quantum field theories in particular. They manifest themselves in the quasi-invariance of an action under the transformations of a group. The fact that one has quasi-invariance (i.e. invariance up to a total derivative) of the action under a group may be irrelevant classically, however, it is important in quantum mechanics as it allows the construction of group-invariant quantum equations (Schrodinger-equations when the group is the non-relativistic Galilei group for example). One may notice that the existence of a quantum formulation of the laws of physics is related to the existence of non-trivial (phase) factors (i.e. additive terms in the composition rule of the group operation, see [200]) that cannot be reduced to zero for all group elements (i.e. they form non-trivial classes in the second cohomology of the transformation group). One also observes that the existence of basic quantum effects is a result of the global (topological) properties of the groups associated to the supposed "natural" symmetries (Galilei group, Lorentz group, conformal group, etc.).

These properties are probed via group (co)homologies. Information about a group (or in general a space) is not only encoded in the group (space) itself but also in the way in which the group (space) acts (is mapped) into some reference module (space). This is why one can study group properties by analyzing the actions of the group on an associated space. On that space one can construct a CW complex and analyze it via combinatorial techniques. Moreover, information about a group (space) may also be encoded in the way in which one probes that group (space). One can classify the various ways in which information about a group fails to be encoded geometrically (i.e. non-topologically)[1] by using cohomology groups of different orders. For example the classes of the second cohomology group $H^2(G, U(1))$ i.e. the cohomology group of the maps between the analyzed group G and the unitary 1-dimensional group $U(1)$ encode the global character of the factors in the composition rule of the group-operation in G i.e. the way in which they fail to vanish globally [200]. The non-trivial third cohomology group $H^3(G, U(1))$ encodes the failure of the associativity property of the composition rule [200]. Also, the existence of not globally vanishing (phase) factors induces super-selection rules. They are induced in standard quantum mechanics by the presence of non-trivial operators that commute with all the observables and thus belong to any complete set of commuting observables. As a result, these operators decompose the Hilbert space of all possible states of a system into coherent subspaces characterized by their eigenvalues. The superposition principle holds only inside these superselection subspaces and no observable may have non-zero matrix elements between states of different superselection eigenvalues. As an example one may consider the mass of particles in a space acted upon by a Galilei group. Bargmann superselection rules arising due to the topology of the Galilei group forbid for example mass decay (i.e. physical subspaces corresponding to different mass are incoherent). Of course, this is not true as one has to consider the Lorentz group as a "true" group of nature. What one must remember here is that the existence of such superselection rules is a result of the existence of non-trivial second group cohomologies of the transformation groups i.e. a result of non-trivial topology of the symmetry group as mapped over a space. Further properties can be encoded by higher cohomology groups. However, as showed before, it is important to notice that the topology of a space (or group) cannot be probed in an absolute sense (regarding all the properties one may wish). Topology is only defined together with the algebraic structure that makes computable, i.e. topological invariants and the associated coefficient groups. In some sense this is an extension of the quantum uncertainty that involves the topology of the space. One may quote the existence of super-selection rules in order to avoid solutions like Wheeler's bags of gold. I will show later on that these expectations may be misleading. In order to extract useful properties from cohomology

[1] I contrast here geometrical and topological results although they might be related, see for example Gauss-Bonnet theorem, etc.

one must make a choice of a coefficient structure. Various choices may make classes inside the cohomology merge or become separated. The actual "nature" of them being "separated" or "merged" depends on the actual type of "topological measurement" (i.e. the choice of a coefficient group). Because of this, physical properties depending on classes of (co)homology or being defined as non-trivial function(als) over such classes must have a relative nature. As symmetries map various states into equivalence classes one may conclude that symmetries are in general relative. What I wrote above is visible also in the path-integral formulation. It is well known that anomalies are failures of a symmetry that is manifest at the "classical" level i.e. in the initial action, to exist after one proceeds to a path-integral quantization. This failure is associated to the non-invariance of the measure of the path integral to the transformation prescribed by the given group. There are of course physical anomalies (like chiral anomalies) that manifest themselves experimentally and there are gauge anomalies that must in principle be avoided. In any sense, as seen in [201], relevant anomalies (that cannot be set to zero via "local" transformations) are again given by the non-trivial BRST cohomology classes at ghost number one on the space of local functionals. They are of course topological in nature and dependent on the way in which the topology of the given space (or group) is analyzed. In this sense, setting a (global) group structure for the coefficients may prove useful in avoiding gauge anomalies while making use of only a limited number of extra dimensions (or none at all). These effects are purely quantum-gravitational in nature and refer to the situation when the probing of the topology of a space-time region (or a space or group in general) becomes uncertain and various choices of coefficient groups in (co)homology become relevant. Please note that this doesn't have to happen only at very high energies or low distances. A basic example supporting this claim is given in reference [310]. As showed in [311] a cohomological interpretation of anomalies due to the nilpotency of BRST operator exists. A functional $a = a(A, \omega)$ locally depending on the gauge potential A and linear in the ghost field ω is an anomaly if it satisfies the Wess-Zumino consistency condition $\delta a = 0$ but there is no local functional $\Lambda_{loc}(A)$ such that $-\delta\Lambda_{loc}(A) = a$. Physically, this means there is no redefinition $\Gamma \to \Gamma + \Lambda_{loc}$ of the effective action Γ which directly cancels the anomaly. The mathematical setup requires the cohomology of the Lie algebra of infinitesimal gauge transformations with local functionals of the gauge potentials as coefficients. After realizing that the anomalies have a cohomological interpretation, one understands that they must be related with the secondary characteristic classes of Chern and Simons. As characteristic classes are related with the topology of fiber bundles one may ask what topological structure controlled them? Abelian anomalies have been understood in terms of the Atiyah-Singer index theorem. The chiral asymmetry of the zero modes of the Dirac operator was breaking the conservation of the abelian chiral current. For non-abelian anomalies the explanation appeared later, relating them with the twisting of the infinite dimensional bundle of gauge orbits [312, 313]. These were due to the non-triviality of the determinant line bundle of the index bundle for the family of Dirac operators parametrizing the gauge potentials [310, 314]. When the index bundle has a non-trivial first Chern class, treating zero modes in the computation of the determinant of the Dirac operator becomes a global problem. The phase of the determinant is not single valued and therefore prevents the existence of a well defined effective action $\Gamma = \Gamma(A)$ either local or global such that $a = \delta\Gamma$. This usually destroys the consistency of quantum field theories. However, one may change the coefficient group in the cohomology describing these anomalies. One example was presented in [310] where the coefficient structure was given by the full space of complex valued smooth functionals. As stated in [310] such an enlargement of coefficients "dilutes the cohomology in such a way that the relevant BRST cohomology group turns out to be isomorphic with the first Chech cohomology group $H^1(G, \mathbb{Z})$". I performed a more general calculation in [52] where I show that global anomalies can be shifted towards the *Ext* groups by means of the universal coefficient theorem allowing the consistent construction of some extended quantum field theories with no manifest global anomalies. I have performed another calculation in [315] where I relate quantum entanglement with a special topology. I also showed how the coefficient structure in cohomology may affect the notion of entanglement.

One should notice that in the case when symmetries are preserved during quantization they are mapped into Ward identities involving Green functions. They have the role of identifying various Feynman diagrams in the perturbative expansion allowing in this way various proofs of renormalizability for theories that may naively look non-renormalizable (see Yang-Mills or QCD). One may wonder if suitable splitting of equivalence classes due to various choices of coefficient groups may add supplemental (maybe topological) Ward identities that may prove renormalizability of gravity.

9.7 A conjecture: Relativity of Holography

Probably the most important result presented here is the possibility that the Holographic principle is dependent on the choice of the coefficient group. The holographic principle states that the non-equivalent degrees of freedom inside a volume can be mapped unambiguously on the surface encapsulating that volume [188]. The key word here is "non-equivalent". I proved in theorem 2 that discernability (or equivalence) are relative concepts. Following this line of thought the number of non-equivalent degrees of freedom may depend on arbitrary choices. In fact one may make a choice of a coefficient group where the number of degrees of freedom in a volume largely exceeds the accessible number of degrees of freedom on the encapsulating surface. One cannot argue that they are not in the "observable-super-selection" sector associated to a measurement because, as showed before, there are situations when there exists a topological measurement ambiguity (i.e. arbitrary choice of coefficient groups) that makes the existence of such super-selection sectors relative. Indeed one may expect that in a complete theory of quantum gravity one cannot count the independent degrees of freedom in the same way as in a classical or non-quantum gravitational theory. I definitely agree with this. The only difference with respect to the usual interpretation is that there might not be an unequivocal prescription of counting degrees of freedom that is independent of an arbitrary choice of coefficients. Let me underline that I do not claim that the holographic principle is wrong (or absolutely right by that matter). It appears to me that a choice of a coefficient group in (co)homology imposes one form of counting of degrees of freedom (it identifies some as being in the same equivalence class). It is very likely that for some choices a strict holographic principle emerges. In fact, for a black hole, any group structure that misses the region behind the horizon will satisfy the standard holographic principle. However, this may not be an absolute property of quantum gravity. I can claim this simply because a general theory of quantum gravity should be independent of the choice of coefficients (i.e. topologically covariant) in the same way in which general relativity is diffeomorphism covariant or some quantum field theories are gauge invariant. Somehow surprising, on the *classical* side there exist solutions of the Einstein field equations that violate the entropy law allowing essentially for an infinite number of degrees of freedom to be present inside a compact region of space-time. The solutions are called "Wheeler's bags of gold" [197] and are assumed to be eliminated via some quantum mechanism mainly in order to obtain results compatible with the Ad-S/CFT conjecture. However, it appears to me that the "bags of gold" may have some effects after all in a full theory of quantum gravity. They become obvious when one adopts a topological definition of entropy in the context presented in this subsection. In order to improve on clarity I start by reminding the standard definition of entropy as being given by the logarithm of the number of microstates associated to the same macrostate $S = k_B log[\Omega]$ or, when considering a general quantum case the definition becomes $S = -k_B Tr[\rho Log[\rho]]$ where ρ is the density matrix operator. The entropy can be defined as the failure of macroscopic states to reveal all the microscopic details. Otherwise stated it may be interpreted as the uncertainty that remains after a macroscopic state is fully described. The concept of entropy evolved from the practical inability of probing classical microstates to the inherent inability of probing quantum microstates. An extension would be towards the inability of probing the topological structure of the analyzed space and this appears to be precisely the case when dealing with quantum gravity and coefficient structures in (co)homology. One may observe that entropy can in general be extracted from the (co)homology of the space of microstates. In fact the cohomology measures precisely the failure of probing topological structures using local considerations. Because of this, it is a perfect tool for identifying the topological uncertainty i.e. the topological component of the entropy. I showed before that this has a measurable effect when a topology is chosen and contributes to the statistics when such a topology is left unspecified. Let me call C the space of microstates available to a specific microscopic probing of a topological space. This may be represented as a linear combination of simplexes with various coefficients. Let δ be an operator that realizes a form of "coarse graining" in the sense of partitioning the microstates into classes according to the macrostates they can encode and taking into account the topology of the associated space (i.e. as a boundary operator). Then one can define a chain complex for cohomology as

$$... \xrightarrow{\delta^{n-1}} C^*_{n-1} \xrightarrow{\delta^n} C^*_n \xrightarrow{\delta^{n+1}} ... \tag{9.60}$$

or for homology

$$... \xrightarrow{\delta^{n+1}} C^n \xrightarrow{\delta^n} C^{n-1} \xrightarrow{\delta^{n-1}} ... \tag{9.61}$$

The star in the above description is a notation that makes the difference between homology and cohomology groups manifest. The argument here is purely formal. I simply prove that this concept exists. In general the (co)homology group is defined as the group obtained by taking the quotient between the kernel of δ^n and the image of δ^{n-1}. In the present context the kernel of δ^n represents the number of microstates that are mapped into the identity class of the space of macroscopic states and the image of δ^{n-1} represents the result of the application of the operator over the initial microstates. The (co)homological structure in this case represents the division of the kernel in partitions defined by the image. The non-topological entropy may be identified with the number of microstates in a class. Indeed, the class structure is not visible macroscopically and contains all the microstates associated to a macrostate. However, this definition offers the advantage of taking into account the additional topological uncertainty in a more complete way. Different coefficient groups in cohomology may merge or dissociate classes. In this sense entropy is defined only up to a choice of a coefficient structure over the (co)homology. While the properties of standard entropy remain unchanged if the "topological uncertainty" is irrelevant, when this is not the case (i.e. in the case of strong quantum gravity but not only) entropy can be defined only up to a choice of probing the topology. Certain choices of coefficients are known to merge the equivalence classes increasing the total number of equivalent microstates. However, each choice of coefficients, once made must remain consistent with itself i.e. no violation of the second law is allowed for any choice. While a maximum bound may exist for each choice, it may be a relative notion, depending on the actual choice made. One must also note that the classification of the topologically distinct features is now encoded in the Ext-group (in the case of cohomology) or in the Tor-group (in the case of homology) via the universal coefficient theorem. The map $Ext(H_{i-1}(X), A) \rightarrow H^i(X; Z) \otimes A$ is an injection. This means all elements in Ext must have a corresponding element in $H^i(X;Z) \otimes A$ but the reverse is not true in general. This means the Ext category offers a more accurate classification of "topologically inequivalent phases" than would be offered simply from cohomological considerations alone. I will not insist on this now but it may prove important in the classification of topological phases. As a practical example, I will focus here on the classical solution of Einstein's field equations known as "Wheeler's bag of gold". In general, the ADM (Arnowitt, Deser, Misner [198]) theory for general relativity allows the foliation of the spacetime manifold into a series of space-like hypersurfaces. The next step would be to re-express the Lagrangian in terms of a pure spatial metric (g_{ij}), a lapse function N and a shift vector that represents shifts along the tangent to the surface of constant time-coordinate. One can now find the conjugate momenta associated to these terms and obtain a Hamiltonian equivalent of the problem. In this context solutions to Einstein equations imply the definition of initial data which means the specification of the 3-dimensional Riemanian metric (g_{ij}) and its conjugate momentum (π^{ij}). These have to satisfy constraints of the form

$$^{(3)}R - (I/g)(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) = 0 \tag{9.62}$$

$$\nabla_i \pi^{ij} = 0 \tag{9.63}$$

where $^{(3)}R$ is the 3-scalar curvature of g_{ij} and $g = det(g_{ij})$ while $\pi^2 = (Tr\pi^{ij})^2$. ∇_i is the covariant derivative corresponding to g_{ij}. Some solutions to these equations possess a "moment of time symmetry" i.e. a point where $^{(3)}R = 0$ [196-199]. It has been proved [199] that the total energy of an axisymmetric, moment of time symmetry initial data is positive. One can also write a general expression for an axisymmetric 3-metric of the form

$$ds^2 = e^{2q}(d\rho^2 + dz^2) + \rho^2 d\theta^2 \tag{9.64}$$

However, a metric can be deformed by a conformal transformation of conformal factor ϕ leading to another possible solution. Suppose now one starts with a smooth conformal factor which is positive at infinity but becomes negative at some point. Obviously it must pass through at least a point where it is identical to zero. In that point of time all the points on the constant time coordinate surface S are transformed into a single point and must be identified. The space becomes the union of an asymptotically flat manifold and a compact manifold. These two are joined at a single point. This solution is called the "Wheeler bag of gold" due to the singularity appearing at the intersection point. That such a solution is indeed a valid solution of Einstein's field equations has been shown in reference [197].

In fact one can prove that the energy on one side may become $+\infty$ while on the other side $-\infty$. This formal divergence may be only a classical artifact not to be recovered in a full quantum description. However, some relevant quantum effects exist. In order to find them one has to integrate over inequivalent geometric configurations defined by the action

$$S = \frac{1}{2k} \int R\sqrt{-g}d[vol_M] \tag{9.65}$$

where

$$g = det(g_{\mu\nu}) \tag{9.66}$$

R is the Ricci scalar, $g_{\mu\nu}$ is the space-time metric, $k = 8\pi G c^{-4}$, G being the gravitational constant, c the speed of light in vacuum and the configuration space $\mathfrak{C}(M) = (T^*M)^{2\otimes} = T_2^0 M$ is a space of rank $(0, 2)$ tensors. It is generally argued that although the classical solutions exist they may be suppressed once the correct measure of integration is used in the quantization prescription. However, this solution is particularly interesting from the perspective of the universal coefficient theorem. Let me consider a quantum-gravity probing device with an internal group structure that can detect the asymptotically flat manifold (say, for example $\mathbb{Z}$). This trivial manifold can be mapped into a ball which has non-vanishing homology with coefficients in $\mathbb{Z}$ only for the zero dimension. Now attach to this space a sphere S tangent to it at a single point. Depending on the group structure used to perform the measurement the sphere may or may not be visible. However, the quantum gravity properties of this structure will remain encoded in the possible Ext groups appearing in the UCT sequence. In some sense the information will be encoded in the topology of possible maps of the group chosen to perform the measurement and the group of the physical spacetime involving a "bag of gold". This Ext group is obviously non-trivial (i.e. the equivalence in standard quantum mechanical language would be "non-commuting observables"). This requires for the quantization prescription to take the correct Ext group into account when performing the "sum over histories". This allows these types of solutions to indirectly influence the quantum results via the topologies of the Ext and Tor groups. Of course I do not expect infinite energy in the region covered by the bag of gold as prescribed in classical general relativity but I also do not expect to have solutions of this type being completely irrelevant in the context of quantum gravity. In some sense it is known that processes described by single Feynman diagrams may look non-physical and are certainly unobservable, however, it is the cross section calculated with them that makes physical sense. The same situation appears to happen for the bag of gold solutions. While I share the common belief that this solution is unlikely to appear as a physical outcome in the sense predicted by classical general relativity (infinite entropy, infinite energy), it appears to me that it should be considered in a full theory of quantum gravity simply due to the non-triviality of the extension group it generates. Its overall effect may be the cancellation of some other inconsistent object so it might as well never arise as a physical configuration. One could ask if they may somehow correlate to the cosmological horizons?

9.8 Information, measurement and quantum gravity

As seen in the subsections above, the common ideas that appeared to be absolute in the classical (non-quantum-gravitational) approach to physics i.e. observables, symmetries, discernibility, entropy, etc. become relative. It is possible that a quantum theory of gravity may not be expressible in terms of local observables and that quantum gravity observables must have a rather special form. Analyzing the algebraic-topological aspects of gravity it appears that one has to expand the algebraic structures in order to obtain relevant information. For example in order to probe topologically non-trivial space-times one has to use coefficient groups in cohomology. These may play the role of an experimental probing device (an apparatus). In this sense an abstract representation of an apparatus in quantum gravity may be seen as a group structure. Next, one may ask what procedure has to be performed in order to make a quantum-gravitational measurement. It appears that one has to provide a coefficient group (apparatus) as an input. The choice of the group structure is not "predefined" in the same sense in which the choice of the z-axis in the quantum measurement of a spin 1/2 particle is not defined a priori. Once the z axis is defined one may obtain a statistics of the outcomes. In the same sense, once a group structure is defined one obtains a (co)homology sequence and an Ext resp. Tor group. The (co)homology obtained in this way will encode the topological properties that can be obtained using the given coefficient group. The Ext respectively Tor groups will encode the failure of the coefficient groups to encode the full information about the space as well as a means to classify various choices of coefficient groups i.e. sequences with identical Ext or Tor will form the analogue of symmetry equivalence classes. One may also notice that this way of thinking may become useful in the classification of topological phases of matter, apart of the obvious applications to quantum gravity. One may imagine the quantum gravity measurement device as an extended object that encodes a group structure. The actual measurement is the process of obtaining the (co)homology (or homotopy) of the given space as an output of the apparatus (i.e. with the coefficient group of the apparatus). One can regard the UCT as a statement about how much the outcome differs when using an apparatus with a given group structure with respect to the case when one simply tensors the outcome of an apparatus using a trivial group structure with the previous group structure. This difference is encoded in Tor respectively Ext and may be seen as the equivalent of the failure of observables in standard quantum mechanics to commute.

9.9 Quantization and topological properties of symmetry groups

There are several important ideas that come together here. On one side I observed that the probing of the topology of a given space or group may be fundamentally limited by specific incompatible choices of coefficient structures in the (co)homology. The probing of the topology of a space appears to be limited not only by a lack of energy or of time as mentioned in some earlier work [195] but also by the fact that certain "global-measurements" associated to different coefficient groups in cohomology cannot be performed simultaneously in a perfect sense. Some information visible using one choice will be lost when dealing with the other choice. This fact relativizes certain objects and has various other important effects. The choice of the coefficient structure may determine the topological features that can be observed. In this subsection I show with some simple examples (following mainly [200]) how some topological properties are relevant in the construction of group invariant quantum theories and how quantum effects are actually to be related to the specific behavior of a theory under some symmetry groups. In order to keep the discussion as simple as possible I will give the examples using the Galilei group. Its elements can be parametrized by

$$g = (B, A, V, R) \tag{9.67}$$

where B refers to time, A refers to space, V refers to boosts and R refers to rotations. The associated group law is

$$g'' = g' * g = (B' + B, A' + R'A + V'B, V' + R'V, R'R) \tag{9.68}$$

The action of the group on space-time is obviously

$$x' = Rx + Vt + A, \quad t' = t + B \tag{9.69}$$

In classical mechanics one can define a Lagrangian as

$$L = \frac{1}{2}m\dot{x}^2 \tag{9.70}$$

This is considered as quasi-invariant as its transformed form differs from the original form only by a total derivative

$$L \to L' = L + \frac{d}{dt}m(xV + \frac{1}{2}V^2 t) = L + \frac{d}{dt}\Delta(t, x; V) \tag{9.71}$$

There is no way of removing the function $\Delta(t, x; g)$ for all transformations g of the Galilei group by adding a total derivative to L. The classical equation of motion (Lagrange equation) is not affected by this change and $\Delta(t, x; g)$ may appear as unimportant although it is relevant when defining conserved quantities. However, it will reappear in the quantum case in an interesting fashion. When going to quantum mechanics one identifies the analogue of energy conservation with the Schrodinger equation and in order to keep quantum mechanics Galilei-invariant one must assure that Schrodinger's equation has the same form in reference frames related via Galilei transformations. One may observe that there is no way of implementing Galilei invariance by using a transformation directly on the wavefunction

$$\psi'(x', t') = \psi(x, t) \tag{9.72}$$

However, one may observe that pure states are in fact described by rays where the set of rays is defined as

$$\{rays\} = H/R \tag{9.73}$$

where R is the equivalence relation that identifies vectors ψ and ψ' of the Hilbert space H which differ only in an unobservable phase. Thus one may enforce Galilei invariance by allowing spacetime dependent phase factors as in

$$\psi'(x', t') = \exp(\frac{i}{\hbar}\Delta(t, x))\psi(x, t) \tag{9.74}$$

One can determine Δ by imposing Galilei invariance as

$$\Delta(t, x) = m(xV + \frac{1}{2}V^2 t) = \Delta(t, x; g), g \in G \tag{9.75}$$

The exponential is the same as the one appearing in the transformation rule of the Lagrangian. These two functions are caused by related effects. They are in fact related to the non-trivial cohomology of the Galilei group.

The transformation law given above allows us to find the composition law of two successive transformations

$$\psi'(x') = [U(g)\psi](gx) = \exp(\frac{i}{\hbar}\Delta(x; g))\psi(x) \tag{9.76}$$

where $x' = gx$. If $x'' = g'x' = g'gx$ we may write similarly

$$[U(g'g)\psi](x'') = \exp(\frac{i}{\hbar}\Delta(x; g'g))\psi(x) \tag{9.77}$$

To compare $U(g'g)$ with $U(g')U(g)$ we first notice that

$$\begin{gathered}[U(g')U(g)\psi](x'') = [U(g')(U(g)\psi)](g'x') = \\ = \exp(\tfrac{i}{\hbar}\Delta(x';g'))(U(g)\psi)(x') = \\ = \exp(\tfrac{i}{\hbar}\Delta(gx;g'))\exp(\tfrac{i}{\hbar}\Delta(x;g))\psi(x)\end{gathered} \tag{9.78}$$

Then we obtain

$$U(g')U(g) = U(g'g)\exp\{\frac{i}{\hbar}(\Delta(gx;g') + \Delta(x;g) - \Delta(x;g'g))\} \tag{9.79}$$

which can be rewritten using

$$\xi(g',g) = \Delta(gx;g') + \Delta(x;g) - \Delta(x;g'g) \tag{9.80}$$

as

$$U(g')U(g) = \exp\{\frac{i}{\hbar}\xi(g',g)\}U(g'g) = \omega(g',g)U(g'g) \tag{9.81}$$

where $\omega(g',g)$ are the unimodular factors. This rule defines a projective (or ray) representation of the group G and ξ defines a two-cocycle on G. The fact that ξ cannot be made zero for all group elements of the Galilei group (i.e. the projective representation of the Galilei group used in quantum mechanics cannot be transformed into an ordinary one) is expressed by saying that ξ is a non-trivial cocycle on the Galilei group. Since pure states are represented by rays, symmetry operators may be realized by unitary ray operators. These may form equivalence classes bringing together all operators which differ by a phase that can be locally eliminated. The classes of inequivalent two-cocycles define the second cohomology group $H^2(G,U(1))$. As another interesting example of topological effects on groups is the group extension. The simplest case may be considered the Weyl-Heisenberg group which defines essentially the quantization prescription. It is a three-dimensional (or in general $(2n+1)$-dimensional) manifold (q,p,ζ) with the group law given by

$$\begin{gathered}q'' = q' + q \\ p'' = p' + p \\ \zeta'' = \zeta'\zeta\exp\{\tfrac{i}{2\hbar}(q'p - p'q)\} \\ (\zeta;q,p)^{-1} = (\zeta^{-1};-q,-p)\end{gathered} \tag{9.82}$$

The two-cocycle is here given by

$$\xi(g', g) = \frac{1}{2\hbar}(q'p - p'q) \tag{9.83}$$

This two-cocycle is only one representative of its class. One may add two-coboundaries and obtain different but equivalent Lie algebra commutation relations. However, preserving the topological structure of the group one cannot globally eliminate these cocycles. One may ask what if the probing of the topological structure of the transformation group (manifold) may be affected by different choices of coefficients? Would it be possible to merge the identity class with the class of the above cocycle? In that case would it be possible to arrive at 't Hooft's conclusion (for example [202]) about "pre-quantization"? Of course, in this case one must consider possible Ext-groups for the cohomology exact sequence of the UCT that may return all quantum effects in another way. I will not follow here this line of thought but one must acknowledge G. 't Hooft for his work related to this subject albeit he was probably not aware of the algebraic-topological interpretation I present here. I must also underline that the possibility mentioned above is in essence a quantum effect that merely introduces an ambiguity into the way in which topological properties of groups and spaces can be probed. Standard quantum mechanics remains valid in each equivalence class. The only difference is that due to further (quantum) uncertainty some equivalence classes may merge when strong gravitational effects are present or when special ambiguities in the experimental topological setup are being introduced. I also stress that the "validity" of quantum mechanics is not altered and this remains a fact, independent of the energy scales, distance scales, etc. What I show is only that one may "abelianize" the commutation rules of quantum mechanics with the cost of introducing Tor or Ext groups in the chain complex. The quantum effects are simply "shifted" towards these constructions that must be taken in account in the end of the calculations.

9.10 Topology of spacetime and anomalies

One may ask if my construction is dependent on a purely geometrical interpretation of space-time that may indeed not be valid in the case of quantum gravity. In fact there have been several attempts to define quantum-gravity spacetime using a discrete topology (causal sets [203]) or some form of superposition of "microscopic geometries" [204] related to Mathur's "Fuzzballs" (essentially fundamental strings that in my representation would be the result of choosing a continuous group of coefficients). My approach is a description of why all these approaches are in some sense plausible but still incomplete.

Considering this, string theory already makes an assumption about the topology of space by introducing the "worldsheet" or the "fundamental string" in the non-field theoretical approach. This might be possible but one has to take into account that by doing this one selects a topology via a group, (say $\mathbb{R}/\mathbb{Z}$ but not necessarily) which selects the length of the string or the fact that it connects two points. As a consequence string theory can only make predictions for "experiments" that are designed in such a way that this configuration makes sense. Indeed it appears that this offers an UV-completion of the theory and the prediction of the graviton. However, due to its topological non-covariance it must contain an enormous amount of irrelevant and/or fictitious information which my idea helps to uncover. About loop quantum gravity it is known that it introduces a discrete topology of space-time due to its choice of the operator algebra. This too, is an artificial construction and focuses the description on "experiments" that can probe such a discrete structure. In this case we may speak about the $\mathbb{Z}_n$ group and one has to pay attention what fictitious constructions this group generates. Again, the universal coefficient theorem and its exact sequence (with the first injective map) may give an image about what dualities one may expect and what objects are non-physical. There is certainly a whole range of alternatives: closed strings, open strings, n-p-branes etc. but the reader may notice that all of them imply choices of topologies hence specific experimental situations that should be probed. They cannot be fundamental for a theory of quantum gravity.

In fact I argue that the topological structure of space-time may be subject to some form of ambiguity in its accurate definition due to the impossibility of probing the full information encoded in topology via (co)homology in an unequivocal way. In this sense the question "what is the precise topology of space-time at extremely low scales" may have no precise answer unless one provides a specific method of probing that topology. In some sense the problem is similar to the double slit experiment of standard quantum mechanics. There, the question "through what slit did the electron go" must change the topological setup of the experiment forcing us to obtain a non-interference pattern. If the precise trajectory of the electron is of no concern to us the topological setup allows interference patterns. Unlike this case where we can actually control the topological setup of the experiment and have a precise definition of it, in quantum gravity this might be fundamentally impossible. One cannot any longer keep all topological features independent of the choice of a coefficient structure (i.e. independent of an actual probing of the topology, be it the topology of the space-time itself, the topology of the field space or the topological properties of the symmetry groups acting on a given object). One can notice that anomalies in the construction of a quantum theory of fields may be common and gauge anomalies may appear. This is indeed dangerous for a consistent quantum field theory. However, it has been shown that the gauge anomalies are to be associated with classes of the BRST cohomology [201]. Of course, if the topology of the space becomes uncertain the associated topology of the field space will follow. It can be possible that some choices of group coefficients in (co)homology may make the anomalous cohomology classes equivalent to the identity (i.e. they become trivial). This doesn't mean that any field theory can be directly quantized but that in the extreme case of quantum gravity a choice of coefficients might exist that makes the anomalies cancel in a trivial way. I will continue here by analysing the effect on symmetries of the fact that topological properties of groups and spaces depend on choices of coefficient groups in (co)homology. Symmetries can in principle be seen as equivalence classes over a space. Different choices of coefficient groups may merge symmetry classes and change the structure of the sets of states to be considered equivalent in certain situations. One can prove that an anomaly is a loop effect in the Feynamn diagram description. In fact it appears because of the non-invariance of the path integral measure and is encoded in the Jacobian of the symmetry transformation. This can be shown to be a loop effect due exclusively to quantization. It is well known that one can add in general counter-terms to the classical action as long as they are of higher order in the coupling constant. This is because they are corrections to unspecified loop terms invisible in the classical theory. This procedure leads to renormalization as long as the added terms are local. Let us start with a classical action

$$S_{cl} = \int d^4x(-\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\alpha\mu\nu} + L_{matter}[A, \psi, \bar{\psi}]) \tag{9.84}$$

where ψ, $\bar{\psi}$ are the matter fields, A is the gauge field and $F_{\mu\nu}$ is the field strength tensor (also for a non-abelian theory). Suppose there exists a gauge anomaly and suppose one adds a local counter-term of order 3 in the coupling constant g called $\Delta\Gamma$ such that

$$S_{cl} \to S_{cl} + \frac{1}{6}\int d^4pd^4q\Delta\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}(-p-q,p,q)A^{\alpha}_{\mu}(-p-q)A^{\beta}_{\nu}(p)A^{\gamma}_{\rho}(q) \tag{9.85}$$

At order g^3 such a term modifies the 3-point vertex function as

$$\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma} \to [\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}]_{new} = \Gamma^{\mu\nu\rho}_{\alpha\beta\gamma} + \Delta\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma} \tag{9.86}$$

If one can find a local $\Delta\Gamma$ such that $(p_\mu + q_\mu)[\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}]_{new}(-p-q,p,q) = 0$ then one says the anomaly is irrelevant. Whenever such a local counter-term does not exist the anomaly is relevant. One may notice that the "relevance" of anomalies is due to their failure to be cancelled locally. As stated in the main paper, relevant anomalies can be associated to non-trivial BRST cohomology classes at ghost number one. Let now $[\Gamma^{\mu\nu\rho}_{\alpha\beta\gamma}]_{new} \to [c]$. The arrow maps the transformed 3-point vertex function to a (co)homology class of the group $H^n(X)$ where X is the associated space. The description here is formal; only the reasoning is of importance. Using the UCT one can see that the cohomology group is determined via the short exact sequence:

$$0 \to Ext(H_{i-1}(X), A) \to H^i(X; Z) \otimes A \xrightarrow{h} H^i(X; A) \xrightarrow{r} Hom(H_i(X), A) \to 0 \quad (9.87)$$

One can now chose A such that the map $X \to X/([c] \sim id)$ becomes trivial. In this case one cannot distinguish the class of the previously "relevant" anomaly from the identity over X. This assures that there exists a coefficient structure over the cohomology that trivializes the anomaly. This comes at a cost. One must introduce the extension group on the left $Ext(H_{i-1}(X), A)$. The extension group is generally defined in association with the Ext functor. Its definition is not particularly involved: let R be a ring and let Mod_R be the category of modules over R. Consider $B \in Mod_R$, take a fixed $A \in Mod_R$ and define $T(B) = Hom_R(A, B)$ as the set of homomorphisms over R from A to B. The Ext functor is defined as

$$Ext^n_R(A, B) = (R^n T)(B) \quad (9.88)$$

This can easily be calculated considering the injective resolution

$$0 \to B \to I^0 \to I^1 \to ... \quad (9.89)$$

and computing

$$0 \to Hom_R(A, I^0) \to Hom_R(A, I^1) \to ... \quad (9.90)$$

where we excluded $Hom_R(A, B)$ from the complex. Then the extension $(R^n T)(B)$ is the homology of this complex. So, in the particular case above, the existence of anomalies is "shifted" into the way in which one can non-trivially map a general group into an abelian group. The relevant information is in this case encoded not in one of the two groups but in the topology of the maps between them. This facilitates calculations for field theories quantized over cohomologies with particular coefficient groups while preserving the non-trivial information related to quantization in the Ext part of the sequence above. One should notice that the second arrow in the UCT formula above is an injection i.e. while all the elements of the Ext group must have a correspondence in $H^i(X; Z) \otimes A$, the latter group might have different elements with no correspondence in Ext. This may suggest that Ext may be a better measure for the true (physical) anomalies. Indeed, in the standard model gauge anomalies introduced by chiral fermions cancel naturally when all the fermions are included. However, there appears to be a more general rule suggesting a more accurate method of predicting "true" particles while avoiding to fall in the trap of considering fictitious objects, “needed” in order to cancel anomalies, as “physical particles”.

9.11 Beyond the Holographic principle

Finally one may ask what this idea brings new with respect to the interpretation of the holographic principle. In order to answer this I may turn again to the idea of performing a quantum-gravity experiment. Assume one has a topological (global) measuring device using a particular choice of a group structure for the coefficients. It might be represented by some non-local observables [317] (the topological properties are in general not detectable via local measurements). It remains to be seen how such a device can be implemented practically. Assume also one performs the measurements at a scale where quantum gravity is irrelevant and in a region where there are no black holes to talk of. In this case the choice of the coefficient group is irrelevant. The extension and torsion is always trivial and one obtains the same results known from simple quantum mechanics. One can chose a complete set of commuting observables and start making predictions considering also the effects of possible non-commuting observables as it is the custom in standard quantum mechanics. Now consider a different region of space-time where either because one excites gravitational modes that can alter the topology of spacetime or because one has a black hole somewhere, the topology of the space-time stops being trivial. In this case one has to perform a topological measurement with an apparatus that will provide information about how the (co)homology or homotopy of the region looks like when seen through the specific choice of the coefficient group. According to this measurement one has to design restrictions on the observables allowed by classical quantum mechanics. The Ext and Tor parts of the chain will not be trivial and will have to be considered when designing further lower-scale experiments using the space-time measured via the coefficient groups. Not all observables will exist in this situation (due to merging of equivalence classes). A somehow metaphorical way of looking at this is considering the group choice as a choice of coefficients in a polynomial. Classical quantum measurements after a choice is made are metaphorically equivalent to finding solutions of these equations. If one chooses for example rational coefficients, the number e (the basis of the natural logarithm) will be transcendental (i.e. no polynomial with rational coefficients can have e as a root).

9.12 Experimental verification

The idea of adding uncertainty to the topology of space-time itself has, as I showed before, many implications. Unfortunately most of these are not easily verifiable. I try here to pinpoint some possible experiments where this subject may become useful. It is known that topology is not only associated to space-time itself. As I showed before, one may probe via (co)homology or homotopy with coefficients (of course in an abstract sense) also field-spaces, groups and other abstract spaces. A more accessible experiment where topological features are important is the Bohm-Aharonov experiment. There, one may observe the effects of a non-trivial topology generated by a magnetic field, in a region where the given magnetic field vanishes. If one could manage to create a magnetic field in a state of quantum superposition between a situation with trivial topology and one with non-trivial topology one could check if the measurement of the shift of the observed interference pattern will fix the degrees of freedom of the system or if new quantum restrictions may appear due to the quantization of the topology itself. One should notice that the topological superposition should ideally be obtained without an entanglement with a local object (like the spin of an electron, etc). Also, possible verifications could be provided by the study of the topological phases of matter. I expect the procedure given by the UCT to be particularly important for the classifications of these phases and for the possible discovery of new ones. The fractional quantum hall effect may also have an interpretation in terms of rational Ext groups. One may ask what happens with the theoretical prediction of magnetic monopoles in the context of uncertain topology. Are they still possible? If future experiments will succeed in proving the fundamental limitations of topological measurements one can safely extend this principle towards space-time itself. Up to this point I presented an aspect of quantization that has been probably overlooked but that may have major implications not only in the description of quantum gravity but also in the theory of quantum information. On the quantum information side problems like the "hat problems" may have some interesting quantum representations. Also possible new "strong-weak" dualities may result to be important in fields like condensed matter or many particle physics.

Chapter 10

The Universal Coefficient Theorem and Black Holes

"I'm not strange, weird, off, nor crazy, my reality is just different from yours."

Lewis Carroll, Alice in Wonderland

In this chapter I intend to give a more practical application of the theorems proved in the previous one. It represents my original research and is based on several of my observations. While most of it is new, there are some technical aspects that have been taken over from various sources. The main references for the theoretical background are [200] and [212].

Probably the best method to start this discussion is to remark that the prescriptions of general relativity and quantum mechanics are taking away most of the absoluteness associated to choices of coordinates, trajectories followed by particles and states of physical systems in the absence of any accessible information about them. It is my observation that there still remains an epistemological defect associated to these ideas. Not to all arbitrary conventions have been taken their absolute status away. In fact the connectivity of space is probably the last convention that still is considered absolute by many physicists. It is my observation that one cannot assign an absolute topology to spacetime in the absence of a method for detecting such a topology. This obstruction is at the origin of several paradoxes and inconsistencies in the formulation of quantum gravity, notably the "information paradox" for black holes. Because of this, in order to construct a consistent formulation of physics in a general context, it appears to be necessary for the laws of nature to be specified in a topology-covariant way. The attempt of doing so is the main subject of this paper. In a more practical tone, one of the problems arising in the discussion of black holes in a quantum field theoretical context is the fact that the quantum prescription of unitarity may be lost in processes involving the thermal radiation of black holes [209]. In fact it can be shown that in a semi-classical approximation, each process involving the presence of a horizon may lead to outgoing thermal radiation [210]. An in-falling pure quantum state is then mapped into the external radiation which presents a thermal spectrum thus violating unitarity. I analyze here the origin of this problem and find that the semi-classical approximation is insufficient for a correct quantum description of phenomena involving space-time horizons. In fact, the solution appears to be related to topological properties of the transformation groups considered as acting on the given space. The covariant formulation with respect to some transformations and the related ideas leading to equivalence principles (Galilei, Lorentz, Poincare) are important in this context. In particular, it is possible to relate the existence of a simple manifest covariant formulation and, in a more extended way, of an equivalence principle, to some topological properties of the transformation groups employed in the theory. If a topological covariant formulation of a theory is required, (as I assume to be the case in the context of black holes and horizons) its existence will depend on the structure of the torsion (Tor) and extension (Ext) groups associated to a coefficient structure in cohomology. The main requirement will be for the measurable physical properties to be independent of choices of coefficients in cohomology and hence independent of the apparent topology induced by these choices. This condition will introduce a set of factors (distinct factors for distinct extensions) in the canonical quantization conditions and in the Bogolyubov transformations [215]. It can be shown that these factors will change the thermal nature of the emergent radiation in a way that can appear only when analyzing its topological properties (the cohomology). The practical conclusion of this chapter is that the violation of unitarity is an artifact generated by the semi-classical nature of the approximations used until now. Once one takes various topological effects into account, in a manifestly topologically covariant way, the black hole radiation is corrected with non-thermal terms and an avoidance of unitarity breaking becomes possible. As a side remark, it will also be visible that the information can be seen as encoded in the cohomology of the space in a dimension smaller by one unit. This is in agreement with the present formulation of the holographic principle and one of its realizations (the AdS/CFT correspondence [216]).

The main developments of the past century (special relativity, general relativity and quantum mechanics) have brought to our attention the fact that abstract mathematical conventions should not stand at the fundaments of a description of reality. In general, the role of conventions is to facilitate the comprehension of physical reality and not to assign physical reality to conventional constructions. This idea was noted probably for the first time by Einstein and incorporated in his theory of special relativity as the weak equivalence principle: "the laws of nature should not depend on the arbitrary choice of an inertial reference frame". This law was further generalized to the statement that "the general laws of nature are to be expressed by equations which hold good for all systems of co-ordinates, that is, are co-variant with respect to any substitutions whatsoever (generally co-variant)" [211]. This statement can be translated in modern terminology by using (co)homological algebraic notations. In order to do this let me follow reference [212] and define

$$\mathcal{P} = Tr_4 \circ L \tag{10.1}$$

to be the Poincare group where Tr_4 is the four dimensional translation group and L the Lorentz group and

$$G = Tr_4 \circ L_G \tag{10.2}$$

to be the Galilei group where again Tr_4 is the four dimensional translation group and L_G is the group of galilean boosts and rotations. In contrast to the Poincare group, due to the absoluteness of time, the Galilei group admits several semi-direct structures. One can use for example the decomposition

$$G = (((Tr_3 \otimes B_3) \circ T)) \circ \mathcal{R} = H \circ \mathcal{R} \tag{10.3}$$

where Tr_3 is the 3 dimensional translation group, B_3 is the 3 dimensional boost group, T represents time translations and $\mathcal{R}$ represents rotations. This allows one to define the mechanical evolution space as the homogeneous space parametrized by $(t, x, \dot{x})$. This evolution space is however not a homogeneous space for the Poincare group, because of the different cohomological properties of the Galilei and Poincare groups: while $H_0^2(G, U(1)) = R$ for the Galilei group, for the Poincare groups $H_0^2(\mathcal{P}, U(1)) = 0$. This difference in the cohomological structures of the Galilei and Poincare groups has as consequence the absence of any simple 'covariant' formulation of Newtonian mechanics, as opposed to the Poincare case [212]. In this way, the existence of a special topological structure of the symmetry group of a theory is related to the existence of a simple enough covariant formulation. This is not to say that a covariant formulation for the Newtonian mechanics is impossible. In fact, it is possible, after certain choices regarding the probing of topological properties are made. It is important to notice how this argument can be extended when one deals not only with invariance with respect to a symmetry group but with invariance to a change in the measurement technique for the topology of space-time. I showed in a previous chapter that the observed cohomological structure depends on choices of arbitrary coefficient groups. The difference in the algebraic prescriptions induced by different choices of coefficient groups is generally encoded in universal coefficient theorems (UCT). These theorems allow, via the same association that connected the group cohomology with the existence of equivalence principles, the construction of new equivalence principles, at the higher level of (for example) group extensions. These equivalence principles allow the formulation of the laws of nature in a topologically covariant way and the restoration of the fundamental prescription of unitarity required by quantum mechanics (albeit in a modified form) even in the case when topology changing events may occur (as is the case for the formation of black holes). In this way, the observation that the existence of a simple covariant formulation of a theory depends on topological properties of the groups associated to the transformations considered, will become relevant not only for the Galilei and Poincare groups but also for more general situations when a change in topology occurs. Hence, this chapter aims towards an extension of the equivalence principles as formulated by Einstein in a form that suits better the prescriptions of quantum mechanics.

10.1 Independence of topology and the Universal Coefficient Theorem

As argued in the previous chapter, the laws of physics should not depend on unobservable properties of spacetime. Specifically the choice of a particular coordinate system or a particular coefficient group in cohomology should not be relevant for the formulation of the laws of physics. I showed in the previous chapter, that specific choices of coefficient groups in cohomology may affect the observable connectedness of space-time (or generally of an abstract space or group) as measured by topological techniques. Here I focus on a different aspect, namely what changes should be made in a theory in order for it to describe the physical reality independent on the way one choses to regard the topology? As has been shown in [212] and as I argued in the previous section, the existence of a trivial second group cohomology associated to a symmetry group implies the existence of a straightforward covariant formulation of the associated theory. The triviality of the cohomology in a given dimension however, is controlled by the choice of a coefficient structure in the cohomology. The effect of this choice is on its turn, encoded in the UCT via the Ext or Tor groups.

This observation is general and doesn't relate only to the Poincare group. In fact, one can bring the same arguments in the case of the Weyl-Heisenberg group. This encodes the quantization prescriptions and allows a central extension structure. Moreover, its group cohomological properties when analyzed from the perspective of particular coefficient groups allow for the covariant formulation required by the quantum prescription of unitarity. Indeed, this prescription is not preserved in the same form when one changes the coefficient structure used to probe the group topological properties. This is the reason for the paradoxes one encounters when discussing the unitarity in processes involving black holes. I will continue here with a presentation of the topological properties of the Weyl-Heisenberg group, followed by an analogy between the general (or special) relativity covariant formulation and unitarity prescriptions in quantum mechanics. The conclusion of this chapter shows how to use specific formulations of the universal coefficient theorem in order to restore unitarity when dealing with black holes and event horizons. I will also show how a thermal density matrix appears to be modified when a different choice of a coefficient structure in the (group)-cohomology is made. The final result shows that the notion of density matrix has to be extended such that it incorporates relevant group topological information. Also, entanglement can be connected to the existence of non-trivial group-cohomological classes. Hence, the universal coefficient theorem can show how entanglement is relativized when different coefficient structures are being chosen. This will make subsystems that look completely uncorrelated when analyzed with one coefficient structure, appear entangled when analyzed with another coefficient structure. The information however will always be there, in one situation, encoded in the group law of the actual cohomology and in the other situation in the special form of the extension or torsion that appears in the UCT. It has been brought as an argument for the information paradox that a relatively ordered initial situation (dust or a star) leading to a black hole has as an inescapable final state the thermal radiation. Unless some "emission of negative entropy" [210] by the black hole occurs, information should be lost. However, I showed in the previous chapter that the definition of entropy in a situation where several different coefficient groups are required, must change. In fact, the entropy will have to include topological information as well. It will not be defined uniquely. Instead it will have different forms when regarded via different coefficient groups. This allows the changes in entropy required to restore unitarity in a global (topological) way. In order to start this project, I remind the reader that projective representations in physics are required since standard quantum mechanics represents pure states as rays. Because of this, symmetry operators are represented as classes of unitary ray operators $\bar{U}$. Unless otherwise stated, in what follows G and K represent general groups. The operation over these classes is defined as

$$\bar{U}(g')\bar{U}(g) = \bar{U}(g'g), \ g', g \in G \tag{10.4}$$

Actual operators in each class differ by a phase. Let us make a choice of operators in each of the classes. Let $g, g' \in \Pi$ where Π is a neighborhood of the identity $e \in G$. Now select a representative $U(g'g)$ in the class $\bar{U}(g'g)$. The composition rule becomes then

$$U(g')U(g) = \omega(g', g)U(g'g), \ |\omega(g', g)| = 1 \tag{10.5}$$

$\omega(g', g)$ are the local factors that can be written in terms of local exponents as

$$\omega(g', g) = exp(i\xi(g', g)) \tag{10.6}$$

Different representatives from each class U' will select new local factors $\omega'(g', g)$. When $U'(g)$ and $U(g)$ belong to the same class they will be related by a phase for each g

$$U'(g) = \gamma(g)U(g), \ |\gamma(g)| = 1 \tag{10.7}$$

and this generates a relation between the local factors

$$\omega'(g', g) = \omega(g', g)\gamma^{-1}(g'g)\gamma(g')\gamma(g) \tag{10.8}$$

If it is possible to select $\gamma(g)$ such that the factors become the identity one says that the local exponents are equivalent to 1. It is however not always possible to extend the choice of representatives around the identity to the whole group. When this can be done the ray representation can be replaced with an ordinary (vector) representation. In general the local factors ω can be seen as mappings

$$\omega : G \times G \rightarrow U(1) \tag{10.9}$$

satisfying the normalization condition $\omega(e, e) = 1$ and the two-cocycle condition

$$\omega(g'', g')\omega(g''g', g) = \omega(g'', g'g)\omega(g', g) \tag{10.10}$$

which is nothing but the associativity property of the factors. Two cocycles ω and ω' are equivalent when there exists a two-coboundary

$$\omega_{cob}(g', g) = \gamma^{-1}(g'g)\gamma(g')\gamma(g) \tag{10.11}$$

such that the two-cocycles are related by

$$\omega'(g', g) = \omega(g', g)\omega_{cob}(g', g) \tag{10.12}$$

The classes of inequivalent two-cocycles define the second cohomology group $H^2(G, U(1))$. It is important to notice that due to the identification of pure states with classes in the second cohomology group, the fact that states are pure is dependent on the choice of the coefficients used to probe the desired space, hence dependent on the coefficient group in cohomology. This has a major impact on the identification of the thermal final state in the case of a black hole. The "appearance" of the radiation as thermal (or the states as mixed) depends on a specific choice of coefficient groups. A topologically covariant formulation however can show that the "locally-thermal" radiation will in fact contain global, topological information. The operators inside a class $\bar{U}(g)$ can be written as $e^{i\theta}U(g)$. In this way I introduced a new variable θ. In this case the transformation rule becomes

$$e^{i\theta'}U(g')e^{i\theta}U(g) = e^{i(\theta'+\theta)}e^{i\xi(g',g)}U(g'g) = e^{i\theta''}U(g'') \tag{10.13}$$

One can use the notation ($\zeta = e^{i\theta}$, $\omega(g',g) = exp(i\xi(g',g))$) and form a new group $\tilde{G}$ with the parameters (ζ, g) such that $\tilde{G}$ contains $U(1)$ as an invariant subgroup and $\tilde{G}/U(1) = G$ i.e. $\tilde{G}$ is a (central) extension of G by $U(1)$. Following the rationale of this article, the next step is to formulate a quantum analogue. For this we construct the Weyl-Heisenberg group as a manifold (q, p, ζ) with the composition law given by

$$\begin{aligned}
&q'' = q' + q \\
&p'' = p' + p \\
&\zeta'' = \zeta'\zeta exp(\tfrac{i}{2\hbar}(q'p - p'q)) \\
&(\zeta^{-1}; q, p)^{-1} = (\zeta^{-1}; -q, -p)
\end{aligned} \tag{10.14}$$

Here, the two-cocycle is given by

$$\xi(g', g) = \frac{1}{2\hbar}(q'p - p'q) \tag{10.15}$$

Again, this group can be seen as a $U(1)$ extension of the $2n$ dimensional abelian (p, q) group. The two-cocycle has the role of a commutator, encoding the extent to which the commutativity property is obstructed. A three-cocycle for example would encode an obstruction of the associativity property, resulting in a non-trivial associator $[*, *, *] : R \times R \times R \to R$ (see for example [132]). The standard quantum construction in terms of the Dirac bra-ket formalism relies on the possibility of formulating the quantization prescription in a covariant form. This depends on the second group cohomology of the associated symmetry transformation.

More practically let for example (C_*, ∂) be a chain complex over a ring R and let M be the associated module. The chain groups are C_*. Then there is a map

$$Hom_R(C_q, M) \times C_q \to M \tag{10.16}$$

that evaluates like

$$(f, z) \to f(z) \tag{10.17}$$

This is a general formulation of a structure that has analogues in the covariant and contravariant objects in general relativity but also in the bra-ket notation of standard quantum mechanics. In quantum mechanics the amplitudes are characterized by complex numbers. The adjoint is defined naturally via hermitian conjugation giving rise to the bra-ket formalism and allowing the construction of theories preserving overall unitarity. In general relativity adjoints are constructed as dual 1-forms that appear as "covariant" indices and together with their contravariant counterparts assure that the theory can be formulated in a diffeomorphism invariant form despite the possible intrinsic curvature of spacetime. In principle the 1-forms take the value of a vector and produce a scalar. If $\tilde{P}$ is a 1-form and $\vec{V}$ is a vector then $< \tilde{P}, \vec{V} >= \tilde{P}(\vec{V}) = \vec{V}(\tilde{P})$. The existence of such a covariant formulation and the associated equivalence principle is related to the triviality of the second cohomology group associated to the considered symmetry of the theory. This symmetry might be described by a (possibly central) extension of the original group. Up to now, the statements regarding equivalence principles have been constructed only at the level of symmetries generated by operators forming groups or semigroups. In a more physical language, the statements of Galilei and Einstein, namely that the laws of nature should be written in a form that remains unchanged to a change of coordinates imply the construction of covariant formulations in terms of vectors, tensors, spinors, etc. In the context of the Galilei group the existence of a covariant formulation is obstructed by the fact that its second group cohomology is non-trivial. This is due to the fact that the time component is absolute. Going to a relative time alters the group structure in a way that makes a covariant formulation manifest and trivializes the group cohomology, leading to the Poincare group. However, there are physical and logical indications that the laws of nature should also be written in a form that is independent on arbitrary choices of coefficient groups in (co)homologies. This statement implies that the laws of nature should not depend on a particular choice of probing the topological properties of a space or a group. However, in order to construct a theory of this form, it appears to be necessary to go beyond symmetry groups of a given, fixed cohomological structure when formulating the equivalence principles. One method to do so is given by the universal coefficient theorems. These theorems state that a specific framework, constructed by the choice of a coefficient group in (co)homology is (up to (extension) torsion in (co)homology) equivalent with the choice of an integer coefficient group. One result of this theorem is that distinct classes in (co)homology under one coefficient group may appear as identified under another coefficient group. Suppose M is a module over R then the sequence:

$$0 \to Ext_R(H_{q-1}(C_*), M) \to H^q(C_*; M) \to Hom(H_q(C_*), M) \to 0 \tag{10.18}$$

is exact. Here Ext is the group extension. It appears then that cohomology groups that look non-trivial given a coefficient structure become trivial under another one. There are several ways in which possible pairings as the ones discussed above can be mapped into the realm of universal coefficient theorems. One possible pairing defined in the way described above is

$$<,>: H^q(C_*; M) \times H_q(C_*) \to M \tag{10.19}$$

which relates homology with cohomology. This pairing is bilinear and its adjoint is a homomorphism

$$H^q(C_*, M) \to Hom(H_q(C_*); M) \tag{10.20}$$

Universal coefficient theorems, among other things, provide a measure of how this adjoint fails to be an isomorphism in terms of Ext^q and Tor_q [96]. Here q represents the dimension of the space for which the (co)homology is calculated. Particularizing this statement in the previous cases, one may find that the bra-ket formulation of standard quantum mechanics as well as the covariant formulation of general relativity must be adapted in the situation when the measurement dependence of the topology of the space(time) becomes relevant i.e. when the effects depending on the way in which the coefficient structure is chosen in the cohomology are important.

10.2 Black Holes and the unitarity problem

The previous sections showed that when using equivalence principles and covariant formulations of theories, one usually relies on specific topological properties of the symmetry groups. Especially the second group-cohomology, when trivial, allows for a simple covariant formulation as the one used in the bra-ket formalism or in the tensorial construction of general relativity. However, not in all situations is the second group-cohomology trivial. The nature of the second cohomology depends on one side on the manifold acted upon by the group and on the other side on the coefficient structure chosen in order to describe the cohomology itself. When the second cohomology of the required group is non-trivial one can still formulate a covariant theory provided one considers the universal coefficient theorem and the specific extensions and/or torsions. In this section, I present some physical arguments for the necessity of a coefficient independent construction and, implicitly, of theories that do not depend on arbitrary changes of topology. If in the case of general relativity and quantum mechanics the covariance had to be implemented with respect to a symmetry group, in order to implement the topological covariance one has to consider the coefficient structures in (co)homology. The scalars, vectors and tensors of general relativity will have their equivalents in the various morphisms between extensions or torsions in the universal coefficient theorems. Probably the most important object for which the current discussion is relevant is a black hole. The problem of information conservation was discussed in the context of quantized fields over a given background in [209], [214]. I partially follow the discussion presented therein, pinpointing the aspects where an extension of that treatment is necessary due to some ignored topological aspects. Considering, in agreement with [209] a massless Hermitian scalar field and an uncharged non-rotating black hole, after quantization one obtains a scalar field operator ϕ which satisfies the wave equation

$$\Box\phi = 0 \tag{10.21}$$

Given the background metric associated to the Schwarzschild spacetime [213] where the considered black hole is present one can rewrite this as

$$(-g)^{\frac{1}{2}}\partial_\mu[(-g)^{\frac{1}{2}}g^{\mu\nu}\partial_\nu\phi] = 0 \tag{10.22}$$

One can also define a conserved scalar product of the form

$$(\phi_1, \phi_2) = i\int d^{n-1}x|g|^{1/2}g^{0\nu}\phi_1^*(x,t)\overleftrightarrow{\partial_\nu}\phi_2(x,t) \tag{10.23}$$

the integral being over a constant t hypersurface. When ϕ_1 and ϕ_2 are solutions of the field equation above and vanish at spatial infinity, then (ϕ_1, ϕ_2) is conserved. The existence of a flow of particles originating at a small affine distance from the event horizon has been derived in [210]. One particularity of this derivation is that the average number of outgoing particles in each mode is distributed in accordance with a thermal spectrum. Moreover, the full probability distribution, not just the average, of the emitted particles is that of thermal radiation.

This observation creates a conflict with standard quantum mechanics when one considers the process of an in-falling object together with the radiation emitted on the external part of the horizon. The main issue is that this process does not preserve unitarity. If the in-falling system is in a pure quantum state, the out-coming radiation is in a naturally mixed state. The full information related to the in-falling object is forever hidden behind the horizon. This result, however, appears only when one does not consider the process as described in a topologically covariant way. Using some of the observations in [54] I show here that there exists a special choice of coefficients in the field space cohomology for which there exists a unitary connection between the supposed thermal radiation and the in-falling system. This suggests that the quantum information is in fact conserved, albeit not in the obvious way, directly in the fields, but in the topology (more precisely in the higher cohomology) of the automorphisms of the field space. In order to show this I continue the derivation of the spectrum of the Hawking radiation underlining the modifications in the way of thinking that must be considered in order to obtain the correct result. This method is in agreement with the AdS/CFT solution but its construction allows for a higher degree of generality. Let me now take the quantum fields used in the field equation above and decompose them as

$$\phi = \int d\omega (a_\omega f_\omega + a_\omega^+ f_\omega^*) \tag{10.24}$$

where f_ω and f_ω^* form a complete set of solutions of the field equation and are normalized according to

$$(f_{\omega_1}, f_{\omega_2}) = \delta(\omega_1 - \omega_2) \tag{10.25}$$

The a_ω operators are time independent. The standard method of quantization would be

$$\begin{gathered}[a_{\omega_1}, a_{\omega_2}^+] = \delta(\omega_1 - \omega_2) \\ 0 = [a_{\omega_1}^+, a_{\omega_2}^+] = [a_{\omega_1}, a_{\omega_2}]\end{gathered} \tag{10.26}$$

Let me chose the f_ω such that at early times and large distances they form a complete set for the incoming positive frequency solutions of energy ω. It is possible to compute the spectrum of the created particles by making an expansion of the field in terms of the late time positive frequency solutions. Let p_ω be the solutions of the field equation that have zero Cauchy data on the event horizon and are asymptotically out-coming with positive frequency. Again, consider that in this domain p_ω and p_ω^* form a complete set of solutions. The normalization condition is

$$(p_{\omega_1}, p_{\omega_2}) = \delta(\omega_1 - \omega_2) \tag{10.27}$$

There must also be an in-coming component of the solution at the event horizon at late times. Let me call this set of solutions q_ω. The superposition of these components at late times is localized on the horizon and has zero Cauchy data on the distant region. The components q_ω and q_ω^* form a complete set on the horizon and are normalized as

$$(q_{\omega_1}, q_{\omega_2}) = \delta(\omega_1 - \omega_2) \tag{10.28}$$

The two components, being defined in disjoint regions are assumed to have null scalar product

$$(q_{\omega_1}, p_{\omega_2}) = 0 \tag{10.29}$$

The expansion of the fields in terms of the above components is then

$$\phi = \int d\omega \{b_\omega p_\omega + c_\omega q_\omega + b_\omega^+ p_\omega^* + c_\omega^+ q_\omega^*\} \tag{10.30}$$

where b_ω and c_ω are the associated annihilation operators. The commutation relations are now

$$[b_{\omega_1}, b^+_{\omega_2}] = \delta(\omega_1 - \omega_2)$$
$$[c_{\omega_1}, c^+_{\omega_2}] = \delta(\omega_1 - \omega_2) \tag{10.31}$$

all other commutators are vanishing. The spectrum of the outgoing particles is determined by the coefficients of the Bogolyubov transformation [215] relating b_ω to $a_{\omega'}$ and $a^+_{\omega'}$. One may define the operators c_ω and c^+_ω as the annihilation and creation operators for particles falling into the black hole. However, this definition is ambiguous due to the fact that positive frequency components for the in-falling matter is not well defined. The physical meaning of these operators should therefore be takes as symbolic. Using the complete set given by f_ω and f^*_ω one can write

$$p_\omega = \int d\omega'(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}) \tag{10.32}$$

where α and β are complex numbers independent of the coordinates. We can therefore calculate

$$b_\omega = (p_\omega, \phi) \tag{10.33}$$

and expressing ϕ and p_ω in terms of $f_{\omega'}$ and $f^*_{\omega'}$ one can obtain

$$b_\omega = \int d\omega(\alpha^*_{\omega\omega'} a_{\omega'} - \beta^*_{\omega\omega'} a^+_{\omega'}) \tag{10.34}$$

and the invariant becomes

$$(p_{\omega_1}, p_{\omega_2}) = \int d\omega'(\alpha^*_{\omega_1\omega'} \alpha_{\omega_2\omega'} - \beta^*_{\omega_1\omega'} \beta_{\omega_2\omega'}) \tag{10.35}$$

It is worthwhile noticing that the coefficients can be expressed as

$$\beta_{\omega\omega'} = -(f^*_{\omega'}, p_\omega)$$
$$\alpha_{\omega\omega'} = (f_{\omega'}, p_\omega) \tag{10.36}$$

The discussion up to this point is unsurprising. The calculation of the coefficients above can be used in order to derive the average number of created particles observed at later times. However the exact form in which the previous calculations are being performed does not take the fact into account that the topology as encoded by cohomology groups changes when a black hole forms. In order to show this one has to recall the abstract formulation of the bracket notation used in the previous chapter. While the curvature of spacetime is correctly taken into account in the previous discussion, there are certain modifications required for the pairing operations used above to be isomorphically translated from the language of flat or curved spacetime to the language of spacetime with a horizon. Let the pair (X, τ) be a topological space given by X with a topology τ. It is important to notice that there are several possible choices of topologies over a space. One possible choice would be to consider any two points joined together in a subset for a specific topology if they can be connected by light in both directions. The space made up of low density dust before the formation of a black hole has every point connected in such a topology. Once a horizon forms the topology defined in the above way changes. Moreover, after the horizon is formed, any topology that, prior to the formation of the horizon, connected two points on different sides of what is now the horizon, must change in order to consider the new situation.

Because of this, each of the constructions defined above has to be redefined. Consider first the conserved scalar product over the field space:

$$(\phi_1, \phi_2) = i \int d^{n-1}x|g|^{1/2} g^{0\nu} \phi_1^*(x,t) \overleftrightarrow{\partial_\nu} \phi_2(x,t) \tag{10.37}$$

Such a scalar product depends on the topology of the field space at least at the level of the second cohomology. The two fields appearing in the above inner product are solutions of the wave equation. Their form is correct and the corrections due to topology changes are formally absorbed by the metric when this notation is used. However, the general pairing of such fields in the context of a non-trivial topology must take into account the universal coefficient theorem. In this way, a sampling of the field space is needed. This sampling requires the introduction of simplexes (basic building blogs that generate the analyzed space, see chapter 11 or chapter 3).

One may ask what happens if a structure of this form is used in order to map a space before and after the collapse of a dust cloud into a black hole. While all the simplexes can be defined in the initial case, after the formation of a horizon some subtleties arise. If the definition of the topology is such that points separated by a horizon are not defined to belong in the same open set then the simplex structure must be altered. However, there is no physical difficulty in extending the metric of spacetime beyond the horizon. Also, particles can fall through the horizon. In order to maintain a topological covariant description, the change must therefore be made via the coefficients used in the description of the space as a linear combination of simplexes. Because of this, several concepts required in the construction of the Hawking radiation and the derivation of its distribution function will have to be adapted. First, any pairing that is required for the definition of an invariant structure must be constructed via the universal coefficient theorem. It is the extension that controls the pairing and the UCT provides the information about what is "lost" when one makes a change in the topology via the coefficient structure.

This will have some effect on the definition of entanglement, the construction of density matrices, etc. Second, I will show that the correction given by the "lost information" encoded in the extension group appears in the form of an extra factor in the composition rule. In order to do this I will refer again to [212]. Let $f_i : G_i \rightarrow G_{i+1}$ be a collection of group homomorphisms, then the sequence

$$... \rightarrow G_i \xrightarrow{f_i} G_{i+1} \xrightarrow{f_{i+1}} G_{i+2} \rightarrow ... \tag{10.38}$$

is called exact if

$$Im\ f_i = ker\ f_{i+1} \tag{10.39}$$

As a result, for any exact sequence $f_i \circ f_{i-1} = 0$. Using this formulation, let G and K be two abstract groups. A group $\tilde{G}$ is said to be an extension of G by K if K is an invariant subgroup of $\tilde{G}$ and $\tilde{G}/K = G$. In terms of exact sequences this means that

$$1 \rightarrow K \rightarrow \tilde{G} \rightarrow G \rightarrow 1 \tag{10.40}$$

is exact i.e. K is injected into $\tilde{G}$ and $\tilde{G}$ is projected onto G by the canonical homomorphism so that $G = \tilde{G}/K$. However, the mere knowledge of K and G does not define $\tilde{G}$ uniquely. In order to be able to discern extensions one has to define two exact sequences

$$1 \rightarrow K \xrightarrow{i_1} \tilde{G}_1 \xrightarrow{\pi_1} G \rightarrow 1 \tag{10.41}$$

$$1 \rightarrow K \xrightarrow{i_2} \tilde{G}_2 \xrightarrow{\pi_2} G \rightarrow 1 \tag{10.42}$$

If the two group extensions are related via an isomorphism $\tilde{f}$:

$$\tilde{f} : \tilde{G}_1 \rightarrow \tilde{G}_2 \tag{10.43}$$

and the injective maps $i_{1,2}$ and the projections $\pi_{1,2}$ satisfy

$$\begin{aligned} i_2 &= \tilde{f} \circ i_1 \\ \pi_1 &= \pi_2 \circ \tilde{f} \end{aligned} \tag{10.44}$$

then the extensions are equivalent. Consider now the two group extensions, defined by two different two-cocycles ξ_1 and ξ_2 with their group laws defined separately with simple brackets (...) for the first group and square brackets [...] for the second group:

$$(\theta', g')(\theta, g) = (\theta' + \theta + \xi_1(g', g), g'g), \quad [\theta', g'][\theta, g] = [\theta' + \theta + \xi_2(g', g), g'g] \tag{10.45}$$

If there exists an isomorphism $\tilde{f}$ as defined above and if we can rewrite

$$(\theta, g) = (\theta, e)(0, g) \tag{10.46}$$

$(0, e)$ being the identity of this law, $\tilde{f}$ is completely determined when the images of (θ, e) and $(0, g)$ are given. From the conditions on the injection and projection above one obtains

$$\begin{aligned} \tilde{f} \circ i_1 = i_2 \;\Rightarrow\; \tilde{f}(\theta, e) = [\theta, e] \\ \pi_2 \circ \tilde{f} = \pi_1 \;\Rightarrow\; \tilde{f}(0, g) = [\eta(g), g] \end{aligned} \tag{10.47}$$

This implies a general form for $\tilde{f}$ namely

$$\tilde{f}(\theta, g) = [\theta + \eta(g), g] \tag{10.48}$$

The knowledge of η determines the knowledge of $\tilde{f}$. However, $\tilde{f}$ is also a homomorphism hence

$$\tilde{f}(\theta' + \theta + \xi_1(g', g), g'g) = [\theta' + \theta + \xi_1(g', g) + \eta(g'g), g'g] \tag{10.49}$$

must be equal to

$$\begin{aligned} \tilde{f}(\theta', g')\tilde{f}(\theta, g) = [\theta' + \eta(g'), g'][\theta + \eta(g), g] = \\ = [\theta' + \theta + \xi_2(g', g) + \eta(g') + \eta(g), g'g] \end{aligned} \tag{10.50}$$

and hence

$$\begin{aligned} \xi_1(g', g) = \xi_2(g', g) + \eta(g') + \eta(g) - \eta(g'g) = \\ = \xi_2(g', g) + \xi_{cob}(g', g) \end{aligned} \tag{10.51}$$

where the notation $\xi_{cob}(g', g)$ is used for the two-coboundary generated by $\eta(g)$. The calculation above gives a condition for the equivalence of extensions. One can see that proportional two-cocycles $\xi_2 = \lambda\xi_1$ may define equivalent groups but inequivalent extensions. In order to make the connection with the bracket construction and to classify the extensions one has to rely on a fiber bundle definition of the extension. Let therefore G and K be abstract general groups and $\tilde{G}$ be the extension of G by K. One can relate the cosets of K in $\tilde{G}$, each defining an element $g \in G$ with the fibers over g of a fiber bundle that defines the extension. The fiber through $\tilde{g}_0 \in \tilde{G}$ is given by

$$\pi^{-1}(\pi(\tilde{g}_0)) = \{\tilde{g} | \tilde{g} = k\tilde{g}_0, k \in K\} \tag{10.52}$$

A section of $\tilde{G}(K, \tilde{G}/K = G)$

$$s : G \to \tilde{G}, \quad s : (g) \to s(g) \tag{10.53}$$

selects an element in $\tilde{G}$ in each fiber. Now, given a fiber

$$\pi(s(g'')) = \pi(s(g')s(g)) \tag{10.54}$$

thus there exists a factor $\omega(g', g) \in K$ such that

$$s(g')s(g) = \omega(g', g)s(g', g) \tag{10.55}$$

and this relation defines the factor $\omega(g', g)$. One can define $\omega(g', e) = \omega(e, g) = s(e)$ and take $s(e) = \tilde{e} \in \tilde{G}$. Thus, one obtains the normalized section. Similarly one can obtain, for a normalized section, also a normalized factor:

$$\omega(g, e) = \omega(e, g) = \omega(e, e) = e \in K \tag{10.56}$$

As a general statement, relative to any normalized trivializing section $s : G \to \tilde{G}$ one can associate a factor system $\omega : G \times G \to K$ satisfying

$$\omega(g'', g)\omega(g''g', g) = ([s(g'')]\omega(g', g))\omega(g'', g'g) \tag{10.57}$$

where $[s(g)]k = s(g)ks(g)^{-1}$ $\forall k \in K$. According to this fiber bundle representation of the extensions, the group law of the group extension can be defined in terms of the factor system as

$$(k'', g'') = (k', g') *_s (k, g) = (k'[s(g')]k\omega(g', g), g'g) \tag{10.58}$$

Returning to the physical problem, the invariant bracket defined above,

$$(\phi_1, \phi_2) = i \int d^{n-1}x|g|^{1/2} g^{0\nu} \phi_1^*(x, t) \overleftrightarrow{\partial_\nu} \phi_2(x, t) \tag{10.59}$$

must be extended in order to obtain a topologically covariant description. The change in topology can be considered as the effect of a specific choice of the coefficient structure in (co)homology. The definition of the adjoint of the topological bracket can be identified as the right hand side of the universal coefficient theorem. When a choice of coefficients is considered such that the horizon of the black hole becomes visible one obtains a correction to the bracket as given by the factor that characterizes the extension of the homology group in a dimension smaller by one unit. It will be this extension that will generate the algebra to be used in the physical situation defined by the coefficient structure where the horizon is visible. The bracket is defined now with a correction in the group operation associated to its defining symmetry. Hence a topological factor would be missing in the construction used in [210]. I underline that this factor is purely topological and in a sense of a quantum nature. Hence one has to extend the scalar bracket when a topological covariance is required:

$$(\phi_1, \phi_2) = i \int d^{n-1}x|g|^{1/2} g^{0\nu} \phi_1^*(x,t) \overleftrightarrow{\partial_\nu} \phi_2(x,t) \tag{10.60}$$

must be transformed into

$$(\phi_1, \phi_2)' =< \phi_1, \phi_2 > \omega(\phi_1, \phi_2)(\phi_1, \phi_2) \tag{10.61}$$

where the $< ... >$ notation refers to the topological invariant and $\omega(\phi_1, \phi_2)$ refers to the factor system that characterizes the extension and depends on the choice of the coefficient structure. This factor will appear also in the coefficients defining the probability of particle detection far from the black hole horizon. The fact that an object can fall behind the horizon while nothing can travel from behind the horizon to the outside will imply the change in the topology used to define the considered phenomenon in the presence of a black hole. This change will be encoded in the factor system. It will however not be visible in any perturbative analysis. By topology I mean here the general topological notion i.e. a choice related to the points that are considered to be connected in some sense or "assembled" together. To make these considerations more accurate I will follow again [209].

Consider therefore the vacuum state at the infinite past as

$$|0_- >= \sum \sum \lambda_{AB} |A_I > |B_H > \tag{10.62}$$

where $|A_I >$ is the outgoing state with n_{ja} particles in the jth outgoing mode and $|B_H >$ is the horizon state with n_{kb} particles in the kth mode going into the hole. Otherwise stated

$$\begin{aligned} |A_I >&= \textstyle\prod_j (n_{ja}!)^{-1/2} (b_j^+)^{n_{ja}} |0_I > \\ |B_H >&= \textstyle\prod_k (n_{kb}!)^{-1/2} (c_k^+)^{n_{kb}} |0_I > \end{aligned} \tag{10.63}$$

One can chose an observable at the far future, composed only of $\{b_j\}$ and $\{b_j^+\}$ and operating only on the vectors $|A_I >$. The expectation value of this observable can be written as

$$< 0_-|Q|0_- >= \sum \sum \rho_{AC} Q_{CA} \tag{10.64}$$

where $Q_{CA} =< C_I|Q|A_I >$ is the matrix element of the observable in the Hilbert space of the outgoing states. The density matrix is

$$\rho_{AC} = \sum \lambda_{AB}\bar{\lambda}_{CB} \tag{10.65}$$

and is associated to measurements in the far future but not to measurements of systems falling into the black hole. It is at this point where several extensions of the standard prescription are necessary. This density matrix does not encode the full information that can be obtained in the far future. It does encode however everything that can be obtained from non-topological considerations. In order to see this one has to observe the fact that the information can be encoded not only directly, as considered here, but also via the cohomology groups associated to the field space. I showed in the previous chapter that quantum observables are relative, depending on the particular choice of a coefficient group in the cohomology of the field space. A particular form of the universal coefficient theorem is

$$0 \to Ext^1_R(H_{i-1}(X;R),G) \to H^i(X;G) \xrightarrow{h} Hom_R(H_i(X;R),G) \to 0 \tag{10.66}$$

This can be interpreted in a form that resembles the interpretation of the non-commutativity of some physical observables: the third arrow

$$H^i(X;G) \xrightarrow{h} Hom_R(H_i(X;R),G) \tag{10.67}$$

maps the cohomology with coefficients in the group G into the homomorphisms between the homology with coefficients in R and the group G. The sequence is exact, hence this map is a surjection. This means there are no elements in the set of homomorphisms from the homology with coefficients in R to the group G not represented in the cohomology with coefficients in G. However, there are elements in the cohomology that can be mapped into the same element of Hom. The second arrow

$$Ext^1_R(H_{i-1}(X;R),G) \to H^i(X;G) \tag{10.68}$$

is an injection. Hence the extension encodes the way in which the use of a coefficient structure instead of another changes the classes of the cohomology.

One can extend the uncertainty principle from the non-commuting observables to the mutually incompatible coefficient structures in cohomology. Indeed, the universal coefficient theorem shows that physical observables in a quantum field theory on a topological space are relative, depending on a particular choice of the coefficient group in the cohomology. Observables visible when using one coefficient structure for the probing of the functional space may become indistinguishable when another coefficient structure, incompatible with the first, is used. This fact can be translated in terms of density matrices. Indeed one can construct a density matrix in the form given above

$$\rho = \sum_i \rho_i |\Psi_i >< \Psi_i| \tag{10.69}$$

which can be represented in an arbitrary basis as

$$\rho = \sum_{i,a,b} \rho_i |\phi_a >< \phi_a|\Psi_i >< \Psi_i|\phi_a >< \phi_a| = \sum_{ab} |\phi_a >< \phi_b|\rho_{ab} \tag{10.70}$$

The expectation value of an observable can be defined as

$$\bar{F} = tr[\rho F] = \sum_{ab} F_{ba} = \sum_{ab} \rho_{ab} < \phi_b|F|\phi_a > \tag{10.71}$$

I showed in the previous chapter (Theorem 2) that the discernibility of quantum states is relative in the sense that it depends on the choice of a coefficient group in the cohomology. Here, I show a consequence of this. Indeed, let now take a system composed of two subsystems identified by the variables q_1 and q_2. Suppose the entire system is in a pure state and let that state be $|\Psi_{12} >$. If this state can be factorized into a product of pure states from subsystem 1 and subsystem 2 as

$$|\Psi_{12} >= |\Psi_1 > \otimes |\Psi_2 > \tag{10.72}$$

then the subsystems are said to be unentangled. Otherwise the systems are said to be entangled. However, this notion cannot be described unambiguously in the presence of horizons because there exists at least one choice of coefficients in cohomology where the subsystems are entangled and one choice where the subsystems are independent while both choices being compatible with the region inside and outside the horizon. It is always possible to traverse the horizon towards the interior of the black hole, hence the physics should not change due to a choice of topology or a choice of a coefficient group.

The condition for this is translated in the isomorphism condition for the extensions, formulated in the previous section. In terms of cocycles this leads to the fact that the density matrices must differ in an additive coboundary. One should not have a difference between the two density matrices as seen via one coefficient group and the other of the form $\xi_1 = \lambda\xi_2$ as this relation cannot insure the isomorphism of the extensions. Hence, the density matrix must be extended additively, leading to terms that break the factorization into pure states. Otherwise stated, pure states can be seen as classes in the second cohomology group $H^2(G, U(1))$ associated to the above mentioned group. The universal coefficient theorem implies that the classes can merge or dissociate according to the coefficient groups used to map the analyzed space (or group). Hence, as the notion of "entangled" or "unentangled" is well defined in flat or curved space, it becomes a relative notion when the necessity of a topologically covariant description arises. Another way of looking at this is to see that the non-trivial commutation relations appear as two-cocycles in the cohomology associated to the Weyl-Heisenberg group in particular, as shown in the previous section, and in general, the non-commuting property of two general observables which leads to the block structure of the density matrix depends on the choice of coefficient groups in the associated cohomology. Hence, the "uncertainty principle" introduced at the level of topological information via the universal coefficient theorem can be mapped directly into block diagonal elements of the density matrix. Hence, quantum correlation arises as a global topological property when a horizon that enforces two different choices of coefficient groups appears. Of course, this observation may have implications not only for black holes but also for entangled states in topological condensed matter systems.

10.3 Coefficient groups, the ER-EPR duality and topology

In the previous section I argued that entanglement depends on the choice of coefficient structures in cohomology. I will support the formal derivations made therein with more concrete algebraic homology calculations in this section. Particularly I will show that the presence of entanglement is dependent on the perceived topology of the underlying space. I will also prove by means of a direct example that one particular perceived topology, associated to a coefficient structure in cohomology is associated to the presence of entanglement, while another coefficient structure makes the previous entanglement disappear, trivializing the cohomology with that coefficient group. The discussion in this section therefore brings the claims related to entanglement made in the previous section on more concrete fundaments. Historically, the origin of entanglement lies within basic quantum mechanics [290]. However, there is no doubt today that there is a connection between entanglement of vacuum and the emergence of spacetime [291], [292].

At a very intuitive level the statement behind the newly discovered ER-EPR duality [293] is very appealing. The connection between spacetime topology and entanglement however remains an unproved conjecture. The ideas behind it have their origin far deeper into the past. They have already been mentioned in [294], [295] and several conclusions have been extracted in [296], [297]. The new formulation of the ER-EPR duality basically reminds us that the statistical connection between space-like separated regions of vacuum associated to generic quantum field theories may have a topological interpretation as well.

However, the algebraic topological implications of the ER-EPR statement have only marginally been explored [298]. There has been a strong temptation to connect quantum entanglement to the topological entanglement defined in terms of linked loops, knots or braids in three-dimensional space [299], [300]. Indeed, quantum entanglement is a global structural feature of a quantum system which may sometimes have the appearance of non-locality. Topology generally deals with global features of spaces and topological entanglement is indeed given by a global structure, for example braids or links. Here, I will show that this analogy is insufficient and quite restrictive. While linking or braiding is certainly a form of topological entanglement that may have an analogue in quantum entanglement, it cannot possibly cover all possible forms of quantum entanglement. A more general situation is given by the Mayer-Vietoris theorem [301]. Indeed, this theorem has a strong unifying character, connecting topological features of various spaces with properties of chains defined on subspaces of the original space. This induces new restrictions on the observables when the topology of the spacetime is non-trivial.

In order for this article to be self-contained, a discussion about the meaning of entanglement in quantum field theory is required. Indeed, like in basic quantum mechanics, a relatively good indicator for entanglement is the violation of Bell's inequalities. This must however be formulated in the context of generic quantum field theories. Two mathematically rigorous formulations exist: one based on quantum fields satisfying the Wightman axioms and the other one based on local algebras satisfying the Haag-Kastler-Araki axioms. Both allow consistent descriptions of entanglement.

In the local algebraic description of quantum field theory, Bell's inequalities concern results of correlation experiments involving measurements on two subsystems. Such experiments can be characterized according to [302] by the so-called correlation dualities.

These represent a set of three objects, $(\hat{p}, \mathcal{A}, \mathcal{B})$. $\mathcal{A}$ and $\mathcal{B}$ being real vector spaces with a specific vector ordering defined on them and having a well defined identity $id = 1$. $\hat{p}$ is a bilinear function $\hat{p} : \mathcal{A} \times \mathcal{B} \to \mathbb{R}$. The observables of one such subsystem are represented by partitions of the identity in the respective subsystem i.e. $\{a_i | i \in I\}$, $\sum_i a_i = 1$, $a_i \geq 0$, $\forall i \in I$. Every $i \in I$ is interpreted as a possible outcome of the measurement of an observable a_i. The probability of the joint occurrence of two outcomes $i \in I$ and $j \in J$ in the respective two subsystems will then be by definition $\hat{p}(a_i, b_j)$. Using this definition the Bell correlation is defined as

$$\beta(\hat{p}, \mathcal{A}, \mathcal{B}) = \frac{1}{2} sup(\hat{p}(x_1, y_1) + \hat{p}(x_1, y_2) + \hat{p}(x_2, y_1) - \hat{p}(x_2, y_2)) \tag{10.73}$$

the supremum being taken over all $x_i \in \mathcal{A}$ and $y_i \in \mathcal{B}$. The expression for the Bell equality is then $\beta(\hat{p}, \mathcal{A}, \mathcal{B}) = 1$ which we expect to be violated. When the vector spaces $\mathcal{A}$ and $\mathcal{B}$ modeling the observables of the considered subsystems are in fact C^* algebras (like in quantum mechanics) the Bell correlation satisfies the inequality $\beta(\hat{p}, \mathcal{A}, \mathcal{B}) \leq \sqrt{2}$.

When dealing with relativistic quantum field theory the basic structure is an assignment to each open space-time region $\mathcal{O} \in \mathbb{R}^4$ of a C^*-algebra $\mathcal{A}(\mathcal{O})$ of norm-closed bounded operators on some Hilbert space. This assignment must satisfy certain axioms originating in physics.

First if there are two regions of space-time $\mathcal{O}_1 \subseteq \mathcal{O}_2$ then the associated algebras also satisfy $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$. Therefore, each $\mathcal{A}(\mathcal{O})$ is a subalgebra of the C^*-algebra $\mathcal{A}$ generated by $\bigcup_{\mathcal{O} \subset \mathbb{R}^4} \mathcal{A}(\mathcal{O})$.

Second, in order to define the flat relativistic space-time, Poincare covariance must be obeyed. Therefore, for flat space-times there must exist a representation $\{\alpha_\lambda | \lambda \in \mathcal{P}_+^\dagger\}$ of the identity connected component $\mathcal{P}_+^\dagger$ of the Poincare group by a group of automorphisms on $\mathcal{A}$ such that $\alpha_\lambda(\mathcal{A}(\mathcal{O})) = \mathcal{A}(\mathcal{O}_\lambda)$ where $\mathcal{O}_\lambda$ is the image of $\mathcal{O}$ under the transformation corresponding to λ.

Third, if $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$ then every element of the algebra $\mathcal{A}(\mathcal{O}_1)$ commutes with every element of the algebra $\mathcal{A}(\mathcal{O}_2)$. This assures the existence of a notion of locality. It is important to make a clear distinction between what I call locality in this article, namely the property that observables in spacelike separated regions commute, and another, weaker definition of locality used sometimes in quantum information theory, focusing mostly on the quantum fields or their simpler analogues, the wavefunctions. Indeed, apparent non-local effects resulting from wavefunction superpositions or quantum field correlations are not truly non-local according to the definition of this article. They may naively look non-local but, according to the operational definition of locality they are local.

Finally, there exists a physical, faithful i.e. one-to-one representation π of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$ such that on $\mathcal{H}$ there is a nontrivial strongly continuous unitary representation $U(\mathcal{P}_+^\dagger)$ of the universal covering group of the Poincare group $\mathcal{P}_+^\dagger$ satisfying first, $U(\lambda)\pi(A)U(\lambda)^{-1} = \pi(\alpha_\lambda(A))$ for each $A \in \mathcal{A}$, $\lambda \in \mathcal{P}_+^\dagger$, and second, the generators $\{P_\mu\}_{\mu=0}^3$ of the translation subgroup satisfy the condition $P_0^2 - P_1^2 - P_2^2 - P_3^2 \geq 0$ and $P_0 \geq 0$ where P_0 is the generator of time translations. Self adjoint elements $A \in \mathcal{A}(\mathcal{O})$ of the local algebras are interpreted as observables which are measurable in the corresponding space-time region $\mathcal{O} \subset \mathbb{R}^4$. A positive, normalized linear functional ϕ on the C^*-algebra $\mathcal{A}$ is supposed to correspond to a physical state of the system whose local observables are represented by the net $\{\mathcal{A}(\mathcal{O})\}$. For such a state ϕ and an observable $A \in \mathcal{A}(\mathcal{O})$, $\phi(A)$ is considered to be the expected value of the observable A of the statistical system that has been prepared in the state ϕ.

If $\mathcal{A}$ and $\mathcal{B}$ are commuting C^*-algebras and ϕ is a state on a C^*-algebra $\mathcal{C}$ containing both $\mathcal{A}$ and $\mathcal{B}$ then $(\phi, \mathcal{A}, \mathcal{B})$ determines a correlation duality $\hat{p}(A, B) = \phi(AB)$ for each $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Therefore if ϕ is a state on an algebra $\mathcal{A}$ generated by a net of local algebras $\{\mathcal{A}(\mathcal{O})\}$ and if $\mathcal{O}_1$ and $\mathcal{O}_2$ are any two spacelike separated regions in Minkowski space then $(\phi, \mathcal{A}(\mathcal{O}_1), \mathcal{A}(\mathcal{O}_2))$ is a correlation duality.

In the alternative formulation based on the Wightman axioms we employ so called quantum fields i.e. operator valued distributions ϕ on space-time which act on the physical state space. These fields are then integrated with test functions f having support in a given region $\mathcal{O}$ of space-time $\phi[f] = \int d^4x f(x)\phi(x)$. The resulting objects form under the operations of addition, multiplication and hermitian conjugation a polynomial *-algebra $\mathcal{P}(\mathcal{O})$ of unbounded operators.

Both approaches however assume Poincare invariance and therefore must be replaced with local Lorentz invariant formulations when space-time is curved. Moreover, if we want to connect quantum field theory to quantum information theory, we need a sufficiently accurate description of a qubit. Given a Hilbert space, a qubit can be physically realized as any two dimensional subspace of that Hilbert space. Such realizations however will often not be localized in space. We can restrict ourselves to approximately well localized realizations and represent the qubit as a two dimensional quantum state attached to a single point in space. If we want to ensure relativistic invariance we notice that there are no finite dimensional faithful unitary representations of the Lorentz group. For flat space-time we can go to the Wigner representations. These provide us with unitary and faithful but still infinite dimensional representations of the Lorentz group. These representations strongly rely on the symmetries of Minkowski space and in particular on the inhomogeneous Poincare group. The basis states are taken to be eigenstates of the four-momentum operator such that $\hat{P}^\mu |p, \sigma\rangle = p^\mu |p, \sigma\rangle$ where σ refers to some discrete degree of freedom i.e. a spin or a polarization. To obtain a physical two-dimensional quantum state we may restrict ourselves to a specific momentum eigenstate $|p, \sigma\rangle$ of fixed p. The remaining degrees of freedom will then be discrete. However, when we go from flat to curved space-time we loose the translational symmetry and therefore the momentum eigenstates $|p, \sigma\rangle$. We still have local Lorentz invariance. A qubit must still be understood as a two-level quantum system with the property of being spatially well localized. The history of such a localized quantum system is a sequence of two dimensional quantum states $|\psi(\lambda)\rangle$ each associated to a point $x^\mu(\lambda)$ on the worldline parametrized by λ. Each quantum state in this sequence $|\psi(\lambda)\rangle$ must be thought as belonging to a distinct Hilbert space $\mathcal{H}_{x(\lambda)}$ attached to each point $x^{\mu}(\lambda)$ of the trajectory. The parallel transport is then a sequence of infinitesimal Lorentz transformations acting on the quantum state and this sequence is in general path dependent. Therefore, in general it is not possible to compare quantum states associated with distinct points in space-time. As a consequence it is not meaningful to say that two quantum states associated to distinct points in space-time are the same. We may however use quantum teleportation and entangled states to define what means "the same" in the context of curved space-time. Therefore the whole sequence of quantum states attached to points along a worldline describing the history of $|\psi(\lambda)\rangle$ will be called a quantum field theoretical qubit. One can of course take a localized qubit in a superposed state and split it up into a spatial superposition transported simultaneously along two or more distinct worldlines and make it recombine at some future space-time region to produce quantum interference phenomena [330]. Such spatial superpositions will still be considered to be localized if the components of the superposition (the two elements of the expectation catalogue) are each well localized around space-time trajectories [331], [332], [333]. Moreover, any qubit can be written as a superposition of states by means of the Hadamard matrix. Therefore any qubit can be written in terms of topological cycles. The classification of such cycles is then naturally based on a (co)homology theory.

Taking into account the topology of the space, various qubits can be classified according to the possible deformations such worldline cycles may support. For a simply connected space the situation is straightforward. Any such cycle can be continuously deformed to a single worldline without leaving the space. For a p-connected space-time with $p \geq 2$ there exist certain classes of worldlines cycles that cannot be continuously mapped into simple worldlines i.e. cannot be rotated back by simple one-qubit Hadamard matrices. Such classes depend on the connectivity of the space and are precisely defined by (co)homology groups. A cycle can also be constructed by taking the tensor product of two qubits. In particular two-qubit states may correspond to two worldline segments which may be connected in various ways. If the two worldlines combined belong to a non-trivial (co)homology group then there exists an obstruction in expressing them independently on the given space-time topology and therefore they may not be considered as separable. At this point the connection between space-time topology and superposed quantum states starts being clear. The rest of this article will go further and connect entanglement to topology by a similar way of thinking.

It is not new [303], [304] that the partition of a quantum system into subsystems is dictated by the set of operationally accessible measurements. Given a Hilbert space $\mathcal{H}$ it is possible to either look at it as a bipartite space i.e. $\mathcal{H}_1 \otimes \mathcal{H}_2$ or as an irreducible space $\mathcal{H}$. If the space can be seen as a bipartite space then a tensor product structure exists and this may support entangled states i.e. states that cannot be represented as a direct product of separate states on each of the partitions of the Hilbert space. But what induces the partitioning of a given Hilbert space? It has been argued by [305] that this partitioning is due to the experimentally accessible observables. Therefore a maximally entangled state is only defined as such when the particular experimental setup capable of detecting the associated properties is specified.

However, I proved previously that observables, when described in terms of (co)homology groups are dependent on the coefficient groups used. Indeed, given a certain choice of coefficients in (co)homology, observables merge together becoming undistinguishable.

Another important aspect is that the use of certain coefficient groups may mask the topological properties of an underlying space. Therefore, topology can only be defined up to the coefficient groups in (co)homology. In order to be more specific, take the torus T_1. Its homology in dimension 1 is $H_1(T_1) = \mathbb{Z} \oplus \mathbb{Z}$ and the 0-dimensional and 2-dimensional homology groups are each isomorphic to $\mathbb{Z}$. However, the first cohomology group $H^1(T_1; \mathbb{G})$ with coefficients in a group $\mathbb{G}$ is isomorphic to the group of homomrophisms from $\mathbb{Z} \oplus \mathbb{Z}$ to the group $\mathbb{G}$. This group $Hom(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{G})$ is trivial if $\mathbb{G}$ is a torsion group. If not, it is a direct sum of copies of $\mathbb{G} \oplus \mathbb{G}$. Hence the torsion of the coefficient group in cohomology determines the visibility of a torus as such. Maintaining the chain of cohomology groups with the new coefficients, but reversing the arrows and allowing for a dualizing operation we return to the homology chain, defined this time with the new coefficient structure and preserving the same "accuracy" in detecting topological spaces. The supplemental information remains only encoded in the extension Ext that appears in universal coefficient theorems used when switching the coefficient groups.

Therefore, from the perspective of (co)homology with coefficients and implicitly of quantum states or quantum observables, there exists a duality between toruses and spheres, the relation between the two shapes being given by a particular choice of coefficients.

It is therefore pertinent to ask what will happen with the entanglement when coefficient groups in cohomology are being chosen such that the space appears to be a torus i.e. when an ER bridge emerges.

Consider therefore a space-time and let it contain a compact region Ω with non-trivial topology (i.e. the topology of an n-torus, a Klein bottle, etc.). As a simplification, the asymptotic regions may be compactified such that the whole picture appears to be isomorphic to a torus. I will consider the compactified and non-compactified objects similarly and I will not start any speculations about the topology of the outer regions (i.e. the large scale topology of the universe) here. For all practical purposes of this article, the ER-bridge will look like $\Omega \cong \mathbb{R} \times \Sigma$ where Σ is a 3-manifold with non-trivial topology (i.e. torus, Klein bottle, etc.). When looking at the hypersurfaces Σ we have to see them as spacelike in this context. As a slight simplification I will discuss the case of a two-dimensional torus embedded in a three dimensional space in this article. This doesn't affect the generality of the discussion. Going to higher dimensions and to spaces with higher genera will be the subject of a future article where multipartite entanglement will be the main focus. Here, the main subject will be recovering bipartite entanglement from topological considerations alone and therefore the torus T_2 is sufficient.

I shall call an ER-bridge as being topologically a torus. The important feature that leads me to this name is that the space-time in this case contains a worldtube (the time evolution of a closed surface) that cannot be continuously deformed into a world line (the time evolution of a point). This is the homotopical definition of a torus. This deformation can however be done on a sphere, and it generates the homotopical definition of a sphere which is equivalent to that of a plane i.e. on both, any closed curve can be homotopically deformed into a point. This is the context in which I will use the terms "torus" and "sphere" in this article.

I showed previously that the quantum field theoretical analogues of qubits in curved space-times are to be associated to worldlines. When the qubit is in a superposed state such a worldline can be seen as a cycle. Let me therefore call $|\Psi\rangle$ a qubit associated to the geodesic relating the exterior of the black hole to its interior, which avoids the intrinsic singularity and is continuable indefinitely with respect to its natural length. This would be a qubit state in the context of an ER bridge. It doesn't take too much effort to notice that such a worldline (qubit) is not continuously deformable into a worldline which never enters the horizon in the first place. Also, a superposed qubit which splits between the interior and the exterior of the ER-bridge forms a cycle which cannot be reduced to a point i.e. a cycle which is not a boundary. Also, connecting two non-superposed worldlines we may obtain a worldline around a large cycle of the above defined torus. Such a cycle will also belong to a non-trivial (co)homology. The worldline segments remaining only inside the wormhole or only outside will form elements in a trivial (co)homology. Therefore, qubits, seen as worldlines are classified in terms of (co)homology groups and pairs of qubits may belong to non-trivial (co)homology groups. With this, the connection between quantum information, qubits and homological algebra is established.

One main result connecting algebraic topology and homological algebra is the so called Mayer-Vietoris sequence. Its main underlying idea of the is that the (co)homology of a given space may be obtained via the (co)homology of some subspaces defined on that space together with the intersection of those subspaces. Otherwise stated, the following sequence is exact

$$... \to H_{n+1}(X) \xrightarrow{\partial_*} H_n(A \cap B) \xrightarrow{(i_*, j_*)} H_n(A) \oplus H_n(B) \xrightarrow{k_* - l_*} H_n(X) \xrightarrow{\partial_*} H_{n-1}(A \cap B) \to$$

$$... \to H_0(A) \oplus H_0(B) \xrightarrow{k_* - l_*} H_0(X) \to 0 \tag{10.74}$$

Here $H_{n+1}(X)$ is the homology of the original space X, A and B are the subspaces of X chosen to describe the topological properties of the whole space X, $H_n(A \cap B)$ is the n-th homology of the intersection of the two considered subspaces and finally $H_n(A) \oplus H_n(B)$ is the direct sum of the n-th homologies of the considered subspaces. The associated maps are defined as follows: the map i includes $A \cap B$ into A, $i : A \cap B \hookrightarrow A$, the map j includes $A \cap B$ into B, $j : A \cap B \hookrightarrow B$, the map k includes A into X, $k : A \hookrightarrow X$ and the map l includes B into X, $l : B \hookrightarrow X$. The map ∂_* is a boundary map lowering the dimension of the given group. The symbol $\oplus$ denotes the direct sum of abelian groups.

This is a purely mathematical result. However, its implications for physics and most importantly for the construction of a quantum theory of space-time (and implicitly gravity) cannot be ignored. The main statement of Mayer-Vietoris is that the (co)homology of a space with a more complicated topology can be calculated by dividing that space into pieces of known (co)homology and assembling them together in a controlled way. The main goal of this article is to show that the formation of a space-time torus induces entanglement via the various maps appearing in the Mayer-Vietoris sequence. Reciprocally, entanglement of two qubits induces a superposition which results in a p-connected space-time when the coefficient structures of the associated (co)homology groups are modified accordingly.

The main idea behind the ER-EPR duality is that a non-trivial space-time topology can be associated to the entanglement of two patches of space-time in a trivial topology. I will not insist on the particular geometry of the space-time as the main idea behind ER-EPR is about topology. As a basic example one can consider a situation in which a black hole forms in a certain region of space-time and it is continued via a hyper-cylinder to another region of space-time where another black hole forms. The process that leads to the formation of such a structure alters the topology significantly. In fact, one may start with a topologically trivial space-time and end up with a topologically non-trivial one. The final configuration in the present context is conventionally called an Einstein-Rosen bridge (short ER bridge). Obviously, no actual information transfer is possible as the wormholes are non-traversable.

This space can be described as a simple tensorial product of circles, similar to any generic torus. Concretely the space can be written as $T^n = S^1 \times ... \times S^1$ i.e. the n-fold product of a circle. Quantum states however, as I have shown in the previous chapters are to be searched in the (co)homology of a given space. I caution the reader again that effects induced by quantum states should not be directly used to define locality or violation of locality if the case appears. As noted before, the correlations may appear to be non-local while in fact they are local in the sense of commuting observables. In order to compute the cohomology associated to quantum states on a non-trivial topology one needs theorems similar to Mayer-Vietoris. But we should remember that the long exact sequence describing the homology of our final space depends on the direct sum between the homologies of the two subspaces and on the homology of the intersection of these.

The first step in the construction is to find an open cover of S^1 (one of the constituent circles of the torus) by two (hyper)-intervals I_1 and I_2 such that the intersection $I_1 \cap I_2$ is equal to the disjoint union $J_1 \bigsqcup J_2$ of two smaller intervals. Now, by employing the Mayer-Vietoris sequence for the open cover $U = I_1 \times T^{n-1}$, $V = I_2 \times T^{n-1}$ and $U \cap V = (J_1 \bigsqcup J_2) \times T^{n-1}$. This leads by induction to the homology of the torus

$$H_k(T^n) = \mathbb{Z}^{\binom{n}{k}} \tag{10.75}$$

where $\binom{n}{k}$ is the binomial coefficient of n choose k. What is important to notice in this otherwise standard calculation is the physical interpretation: when the spacetime deforms itself so strongly that the topology changes, in order to calculate the associated homology and hence the associated quantum states, we may have to split the space in easily computable shapes. These are to be associated with unentangled systems in standard quantum mechanics. However, these are never sufficient to compute the actual cohomology. Therefore looking for example only at the two black holes we always miss important topological information. This information is retrieved if we correctly make use of the Mayer-Vietoris theorem and therefore include the (co)homology of the intersection of the two open covers used in the first place. This intersection may have non-trivial topology and represents the entanglement when looked upon from a quantum mechanical perspective.

Therefore one arrives at three relevant results:

First, the inclusion map relating the homology of the intersection of two subspaces of the full topological space X to the direct sum of the homologies of the same two subspaces induces a Hadamard-matrix operation which affects the qubit associated to the branch it acts upon. The map which includes the direct sum above into the full topological space X is a c-NOT operation on the branches associated to the two qubits. The global effect of these two maps arising in the Mayer-Vietoris sequence for a torus is the entanglement of the qubits described by the worldlines on the two branches of X.

Second, two entangled qubits correspond each to worldlines which, combined, induce the (co)homology of a not simply connected space. Superpositions of the qubits are equivalent to combinations of (co)homology groups as presented via the Mayer-Vietoris theorem for the torus. The apparently disconnected components can be considered (not simply) connected if the coefficient structures in the (co)homologies associated to the respective qubits becomes torsional or cyclical.

Finally, in general the ER-EPR conjecture is true, the entanglement being in all situations induced by the inclusion maps appearing in the Mayer-Vietoris sequences.

In what follows I give proofs of the first theorems and partial evidence for the final corollary. First I will revise some known facts about the entanglement of vacuum and some basic entanglement measures [306], [307], [308]. This will prepare the stage for the discussion in terms of homological deformations of the covering domains and the connectivity of the space-time itself. Finally, the maps of the Mayer-Vietoris sequence required for the construction of a torus will be interpreted in terms of quantum information gates.

As seen when referring to relativistic algebraic quantum field theory, a good measure for entanglement is the Bell inequality. More explicitly, states which violate Bell's inequalities are necessarily entangled although states which are entangled may not violate Bell's inequality. Given a quantum system, we may define a pair of algebras associated to the observables of two subsystems, say $(\mathcal{M}, \mathcal{N})$. A physical state may be defined as $\phi : \mathcal{A} \to \mathbb{C}$ where $\mathcal{A}$ is an observable algebra, while ϕ takes values over the complex numbers. A given such state is called a product state across the pair of algebras $(\mathcal{M}, \mathcal{N})$ if $\phi(MN) = \phi(M)\phi(N)$ for all $M \in \mathcal{M}$ and $N \in \mathcal{N}$. In such states the observables of the two subsystems are not correlated and the subsystems are in a sense independent. A state ϕ on $\mathcal{M} \bigvee \mathcal{N}$ is separable if it is in the norm closure of the convex hull of the normal product states across $(\mathcal{M}, \mathcal{N})$ i.e. it is a mixture of normal product states. If this is not so, we call ϕ an entangled state across $(\mathcal{M}, \mathcal{N})$. Only if both algebras are non-commutative i.e. quantum, can we have entangled states on the composite system. A consequence of the Reeh-Schlieder theorem is that for any non-empty spacelike separated observables $\mathcal{O}_1$ and $\mathcal{O}_2$ with non-empty causal complements, independent on the distance between them, there exist more than one projection $P_i \in \mathcal{R}(\mathcal{O}_i)$ which are positively correlated in the vacuum state such that $\phi(P_1P_2) > \phi(P_1)\phi(P_2)$. This shows that the vacuum is not a product state across $(\mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2))$. In order to determine if it is entangled we need a different measure called the maximal Bell correlation, defined for the pair $(\mathcal{M}, \mathcal{N})$ in the state ϕ, as

$$\beta(\phi, \mathcal{M}, \mathcal{N}) = sup\frac{1}{2}(M_1(N_1 + N_2) + M_2(N_1 - N_2)) \tag{10.76}$$

where the supremum is taken over all self adjoint operators $M_i \in \mathcal{M}$ and $N_j \in \mathcal{N}$ with norm less or equal to one. Bell inequality in the case of algebraic quantum field theory can be formulated as

$$\beta(\phi, \mathcal{M}, \mathcal{N}) \leq 1 \tag{10.77}$$

If ϕ is separable across $(\mathcal{M}, \mathcal{N})$ then $\beta(\phi, \mathcal{M}, \mathcal{N}) = 1$. Under quite general physical assumptions, in a vacuum representation of a local net of observables, $\beta(\phi, \mathcal{R}(W), \mathcal{R}(W')) = \sqrt{2}$ which maximally violates the Bell inequality. Moreover, for any spacelike separated double cones whose closures intersect i.e. tangent double cones, $\beta(\phi, \mathcal{R}(\mathcal{O}_1), \mathcal{R}(\mathcal{O}_2)) = \sqrt{2}$.

Now that a construction capable of measuring entanglement has been designed and the observables and quantum states have been assigned each to their own space-time regions, it remains to be shown that it is possible to define entanglement as being generated by the maps of the Mayer-Vietoris sequence for a torus.

Indeed, I showed previously that qubits can be associated to worldlines in quantum field theory and that one- or two-qubit states can be classified in terms of (co)homology groups. Such groups will be represented by means of the basis $\{|a\rangle, |b\rangle\}$. The (co)homology would then be defined by the linear combinations of elements in this basis each such combination satisfying the topological properties defining their respective (co)homology. The coefficients of such a linear combination belong to the coefficient structure of the cohomology. Therefore in order to work in the context of quantum mechanics the homology with complex coefficients $H_n(X;\mathbb{C})$ will be constructed by means of vectors $|\Psi\rangle = c_1 |a\rangle + c_2 |b\rangle$ with $c_1, c_2 \in \mathbb{C}$. This is a more suitable representation for qubits. Two-qubit states will also be classified by means of (co)homology groups but this classification may not be trivial i.e. two independent states belonging to trivial (co)homology groups may become two-qubit states belonging to non-trivial (co)homologies. This would appear as a result of the application of an entangler gate e.g. Hadamard gate on one branch followed by a two-qubit c-NOT gate.

Now, by looking at the Mayer-Vietoris sequence one notices the appearance of direct sums of homology groups like

$$H_n(A;\mathbb{C}) \oplus H_n(B;\mathbb{C}) \tag{10.78}$$

Whenever the objects involved in such direct sums appear in finite numbers and represent abelian structures (like the complex numbers), the direct sums are isomorphic to the direct products and hence

$$H_n(A;\mathbb{C}) \oplus H_n(B;\mathbb{C}) \cong H_n(A;\mathbb{C}) \times H_n(B;\mathbb{C}) \tag{10.79}$$

As a basic example one may consider $\mathbb{R} \times \mathbb{R} \cong \mathbb{R} \oplus \mathbb{R}$ which both represent the cartesian plane. I will continue to use however the $\oplus$ notation for the sake of generality as, for example in the case of infinite direct sums or in the case of topological spaces with no additional structures, such an isomorphism will not apply. For all the considerations relevant to the present discussion however one may assume that $\oplus \cong \times$.

What remains to be seen in what follows is that patching together a torus by means of the Mayer-Vietoris sequence implies the appearance of entanglement. To see this, one has to understand the basics of the Mayer-Vietoris method. Its original use was to detect the (co)homology of an unknown topological space X by means of known (co)homologies of subspaces of X which were wisely chosen such that by patching them together, the full space X could be obtained. The maps capable of doing this patching formed a long exact sequence called the Mayer-vietoris sequence. In this discussion this procedure is somehow reversed, as now we know the full space is a T_2 torus and its homology is also known. We consider the two patches A and B on the left and the right side of the torus and form the Mayer-Vietoris sequence paying attention at the particular forms the respective maps can take. The two patches will intersect (by convention) in the upper and lower regions of the torus. The qubits belong respectively to the homologies of the patches A and B and, after connecting A and B and including them into the torus they will represent entangled qubits on the torus.

It is important to notice that in the Mayer-Vietoris theorem the two groups $H_n(A \cap B; \mathbb{C})$ and $H_n(A; \mathbb{C}) \oplus H_n(B; \mathbb{C})$ are isomorphic as groups but the inclusion maps between them do obviously not induce isomorphisms. If we look again at the Mayer-Vietoris sequence, mainly at the map $H_n(A \cap B; \mathbb{C}) \xrightarrow{(i_*, j_*)} H_n(A; \mathbb{C}) \oplus H_n(B; \mathbb{C})$ we notice that the map (i_*, j_*) is induced in homology by the inclusions $i : A \cap B \hookrightarrow A$ and $j : A \cap B \hookrightarrow B$ and is not an isomorphism neither when acting on the space, nor in its homology induced form.

This map is in fact fundamental to the understanding of the dependence of entanglement on the topology, therefore we need to have it expressed in more comfortable terms. Consider therefore the standard two dimensional torus T_2 and let's start computing its second homology group by means of the Mayer-Vietoris sequence. On this path I will make the connections to entanglement as manifest as possible. For $n = 2$ we have the Mayer-Vietoris sequence in the form

$$... \to H_2(A; \mathbb{C}) \oplus H_2(B; \mathbb{C}) \to H_2(T_2; \mathbb{C}) \xrightarrow{\partial} H_1(A \cap B; \mathbb{C}) \xrightarrow{(i_*, j_*)} H_1(A; \mathbb{C}) \oplus H_1(B; \mathbb{C}) \to ... \tag{10.80}$$

In this part of the long sequence we can calculate all groups except the one of the torus (which however we assume it is known or at least it is not our concern to calculate it). We therefore may already write down the known parts

$$... \to 0 \to H_2(T_2; \mathbb{C}) \xrightarrow{\partial} \mathbb{C} \oplus \mathbb{C} \xrightarrow{(i_*, j_*)} \mathbb{C} \oplus \mathbb{C} \to ... \tag{10.81}$$

Notice that here too, the map (i_*, j_*) is not an isomorphism. Take therefore 1-cycles generating the homologies of A, B and $A \cap B$ respectively in this way: for each cylinder formed by the intersection $A \cap B$ chose your cycle as the equatorial circumference. Let the associated homology classes be α and β. These cycles will each generate $\mathbb{C}$ and we will have

$$(i_*, j_*) : \mathbb{C}[\alpha] \oplus \mathbb{C}[\beta] \hookrightarrow \mathbb{C}[\alpha] \oplus \mathbb{C}[\beta] \tag{10.82}$$

but $\alpha = \beta$ when we are in $H_n(A; \mathbb{C})$ and $H_n(B; \mathbb{C})$ therefore

$$(i_*, j_*)(\alpha, 0) = (i_*, j_*)(0, \beta) = (\alpha, \beta) \tag{10.83}$$

Applying a global twist in the torus (i.e. keeping the upper intersection circle unchanged and rotating the lower intersection circle around an axis perpendicular to its center by π) will not affect the physical situation but will generate the map (i_*, j_*) which can then be written (considering the normalization factor imposed by hand in advance) as the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} : \mathbb{C} \oplus \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$$

This matrix resulted solely from the Mayer-Vietoris theorem, a twist in the torus and a specific choice of basis but, in terms of quantum entanglement it is a standard Hadamard matrix. This matrix is used to map the qubit $|0\rangle$ into the superposition of two states with equal weight i.e. $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. In terms of quantum field theoretical qubits this encodes the representation of a worldline qubit in the form of a cycle qubit. In order to better show the analogy with quantum mechanics I detail the maps arising in the Mayer-Vietoris sequence and connect them to the hadamard-CNOT entangler gate for a bipartite system. In particular I show how the Hadamard map created by the (i_*, j_*) inclusions is combined with the other maps arising from the Mayer-Vietoris sequence in order to produce entangled states on the two branches of a torus. The general situation is as follows. Take two qubits $|\Psi_1\rangle$ and $|\Psi_2\rangle$ each defined in terms of quantum field theory on curved space-time as specific worldlines. In the quantum information approximation they can be seen as unit vectors each in $\mathbb{C} \times \mathbb{C}$. For the beginning, the two states will encode both the $|0\rangle$ state. Start now with an ER space-time configuration (torus). Take the subspaces of the torus covering each one of the two handles on the left and on the right side of the torus. The intersections between these two covers occur by convention on opposing regions of the torus, let me call them the upper and the lower intersection. Let me also call the left region of the torus by A and the right region by B. Starting from the intersections of the two covers, the two qubits are being mapped respectively onto the two handles of the torus by means of the inclusion maps $H_1(A \cap B; \mathbb{C}) \hookrightarrow H_1(A; \mathbb{C})$ and respectively $H_1(A \cap B; \mathbb{C}) \hookrightarrow H_1(B; \mathbb{C})$. The upper intersection will be mapped on the left and on the right side by the map (i_*, j_*) producing a rotated state on the upper half of the torus as if acted upon by the Hadamard gate (the normalization is introduced by hand according to the principles of quantum mechanics). The result will be $|\Psi_1\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle + |1_B\rangle)$. In general, on the lower side of the torus one can obtain similarly $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|0_A\rangle - |1_B\rangle)$. However, to obtain the Hadamard gate (the minus sign in the last entry of the matrix) on the upper side, we used a twisted torus. This amounts basically to a change of basis. This twist will untwist the action of (i_*, j_*) on the lower half of the torus (which would otherwise by itself try again to twist the torus) and therefore the final state on the lower torus will remain $|\Psi_2\rangle = |0\rangle$. This untwisting operation on the lower half leads to a lower map of the form

$$(i_*, j_*) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which acting on the state $|0\rangle$ leaves it unchanged (considering the convention of having $|0\rangle$ in the form of a column vector with the upper entry 1).

Therefore at this moment, after applying the first map of the Mayer-Vietoris sequence we obtained two qubits on the upper and lower halves of the torus

$$\frac{1}{\sqrt{2}}(|0_A\rangle + |1_B\rangle), \quad |0\rangle \tag{10.84}$$

In order to obtain the torus, the direct sum of the two homologies must be mapped in the total homology of the space. This map acts on the upper and lower components i.e. it acts on the two qubits above. This means it must be a two-qubit gate. The map is $H_1(A) \oplus H_1(B) \xrightarrow{(k_* - l_*)} H_1(T_2)$. The notation $(k_* - l_*)$ is formal. It can be interpreted as a formal difference for the cycles of the torus but when acting on qubits it will act as a CNOT gate, as will be seen soon. The patches have to be continuously embedded into the whole torus. But the lower side adds an extra twist via the map $(k_* - l_*)$ which compensates the twist on the upper intersection (the upper intersection is not twisted by this map but it was twisted by the previous one). Therefore this map flips the second (lower) qubit when the initial first qubit has been flipped by the previous map (generating the superposition). But as the initial state was $|0\rangle$ it will only flip the lower qubit when the upper state is $|1\rangle$. Moreover, it brings us the actual homology of the torus back. Therefore what we obtained is a CNOT gate acting on two qubits, namely

$$(k_* - l_*) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(obviously, when acting on an actual qubit the proper normalization constants will be added) Together with the previously introduced Hadamard gate the resulting state is now

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle) \tag{10.85}$$

which is defined over the whole torus and therefore I can drop the indices A and B. Summarizing, the quantum states after the action of the first Mayer-Vietoris map (i_*, j_*) for the torus, are

$$\left.\begin{matrix} |0\rangle \\ |0\rangle \end{matrix}\right\} \xrightarrow{(i_*,j_*)} \left\{\begin{matrix} \frac{1}{\sqrt{2}}(|0_A\rangle + |1_B\rangle) \\ |0\rangle \end{matrix}\right.$$

As has been seen before in order to obtain an entangled state we also need the CNOT map. This map has two roles: first it has to include a second qubit in the superposed states above, second it has to switch the state of the second qubit when the first qubit is in the state $|1\rangle$ such that a truly entangled state of the two qubits emerges and third, it has to restore the whole torus from the two patches A and B. I have shown above that such a map arises naturally from the Mayer-Vietoris sequence for a torus. For a better understanding one may have a careful look at the Mayer-Vietoris sequence

$$... \to H_1(A \cap B; \mathbb{C}) \xrightarrow{(i_*,j_*)} H_1(A; \mathbb{C}) \oplus H_1(B; \mathbb{C}) \xrightarrow{(k_*-l_*)} H_1(T_2; \mathbb{C}) \xrightarrow{\partial} H_0(A \cap B; \mathbb{C}) \to ... \tag{10.86}$$

We are now interested in the map, $(k_* - l_*)$. This one takes as input the sates on the two sheets covering the two handles of the torus and maps them together into a formal difference, generating the homology of the torus i.e. the vector space where the resulting entangled states will reside. While the map (i_*, j_*) was injective, this map is surjective in order to preserve the exactness of the sequence. Merging together elements of the two sheets such that they connect in a continuous way obviously takes two qubits as an input and performs an operation on one, depending on the state of the other. These are all properties desirable for maps in the category of the CNOT map of quantum computing. The final construction I am deriving from the Mayer-Vietoris sequence is shown in fig. 1. Notice first that the maps k and l basically map the regions A and B into the whole of $X = T_2$ after the map (i_*, j_*) has been applied. They take the superposed state obtained

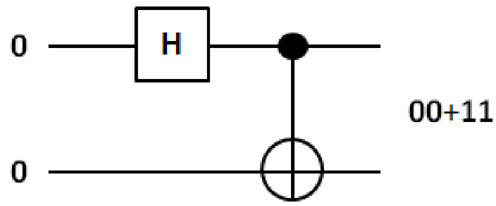


Figure 10.1: The standard Hadamard entangler gate

after the application of the Hadamard-type map (normalization is assumed) and map it into the torus as a whole. Two aspects are important. First this will bring together the new superposition state and the original state $|0\rangle$. This basically implies tensoring the superposed qubit in the upper half with the original qubit in the lower half. Second, the two sheets must generate a torus and therefore the combination between the two maps k and l must be taken such that this will be the case. Formally we have

$$\left.\begin{matrix}\frac{1}{\sqrt{2}}(|0_A\rangle + |1_B\rangle) \\ |0\rangle \end{matrix}\right\} \xrightarrow{(k_*,l_*)} \frac{1}{\sqrt{2}}(|0_A\rangle + |1_B\rangle) \otimes |0\rangle \xrightarrow{(k_*-l_*)} \frac{1}{\sqrt{2}}(|0\rangle\,|0\rangle + |1\rangle\,|1\rangle)$$

The $ER \Rightarrow EPR$ part of the duality has been derived by analyzing the form and the actions of the maps in the Mayer-Vietoris sequence of a torus. In order to make the reciprocal affirmation $EPR \Rightarrow ER$ plausible we have to explain how the entanglement of disconnected spaces (and the states defined on them) may result in a connected space. In general it is verified that spaces of different topology exist in mutually orthogonal sectors of the associated Hilbert space and therefore the paradox is particularly stringent. The connectivity of a space is determined by means of the (co)homology which, in the case of complex coefficients also represents the qubit states. However, when we alter the algebraic structure of the coefficients in cohomology, the information about the connectivity of a space may appear to change. Could therefore a specific non-trivial choice of coefficients lead to a non-trivial superposition of disconnected topological spaces that may result in connected topological spaces? We will start with two circular spaces S^1 and show that by means of a particular change in coefficients the two circular spaces representing together a disconnected space, will become a space homeomorphic to a single circle and hence a connected (although not simply connected) space. Then the resulting not simply connected space will be mapped by means of another change in coefficients into a simply connected space homeomorphic to a single point (see Fig. 2). The particular choice of coefficients must contain a certain twisted cyclicality. In this subsection I will discuss the process in terms of integer and twisted cyclical integer

FIGURE 10.2: Two circles merging, as seen by using various torsions in the coefficient groups of (co)homology. The change in the coefficient structure brings us from two independent circles to the wedge sum between two circles tangent at a common point, then to a single circle and finally to a simple point. The information is presented as seen by homology with various coefficients

coefficients. In the next subsection a short discussion of the acyclicity of the circle will imply the use of complex coefficients [334].

In order to begin, consider a circle space S^1 and an abelian group A. Let then $\rho : \pi_1 S^1 \to Aut(A)$ a representation of the fundamental group of the circle into the abelian group A.

Then, the homology of the circle with coefficients in the group A twisted by the map ρ is $H_k(S^1, A_\rho)$. As a simple example one can consider the group $A = \mathbb{Z}_3$ and the the map $\rho : \mathbb{Z} \to Aut(\mathbb{Z}_3)$ as being

$$\rho = \begin{cases} 0 \to 0 \\ 1 \to 2 \\ 2 \to 1 \\ 3 \to 0 \\ 4 \to 2 \\ ... \end{cases} \tag{10.87}$$

The cellular chain complex associated to the homological representation of the circle is then

$$0 \to \mathbb{Z}[t, t^{-1}] \xrightarrow{\delta} \mathbb{Z}[t, t^{-1}] \to 0 \tag{10.88}$$

δ is the boundary map which by definition represents the multiplication with $(t - 1)$. Therefore t and t^{-1} define the required ring structure for the circular space. We therefore have an isomorphism $\mathbb{Z}[\pi_1 S^1] \cong \mathbb{Z}[t, t^{-1}] \cong \mathbb{Z}[\mathbb{Z}]$ which will slightly simplify the calculation without affecting the final result. Let me now tensor with $\mathbb{Z}_3$ in order to obtain the homology with the desired coefficients over $\mathbb{Z}[t, t^{-1}]$. Then I obtain

$$\mathbb{Z}_3 \xrightarrow{\cong} \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}_3 \xrightarrow{\delta \otimes Id} \mathbb{Z}[t, t^{-1}] \otimes_{\mathbb{Z}[t,t^{-1}]} \mathbb{Z}_3 \xrightarrow{\cong} \mathbb{Z}_3 \tag{10.89}$$

The first map is $a \to 1 \otimes a$ and the last map is $1 \otimes a \to a$. It is required to reduce to $1 \otimes a$ before applying the last map. The result therefore is

$$a \to 1 \otimes a \to (t-1) \otimes a = 1 \otimes (ta - a) \to ta - a \tag{10.90}$$

The boundary map obtained after tensoring with $\mathbb{Z}_3$ is then

$$D : \mathbb{Z}_3 \to \mathbb{Z}_3 \tag{10.91}$$

$$\begin{aligned} &D(0) = 0 \\ &D(1) = t \cdot 1 - 1 = 2 - 1 = 1 \\ &D(2) = t \cdot 2 - 2 = 1 - 2 = 2 \end{aligned} \tag{10.92}$$

and hence is the identity on $\mathbb{Z}_3$. Therefore the homology groups of S^1 with coefficients in $\mathbb{Z}_3$ twisted by the nontrivial map ρ are all trivial

$$H_0(S^1; \mathbb{Z}_3)_\rho \cong H_1(S^1; \mathbb{Z}_3)_\rho \cong ... \cong 0 \tag{10.93}$$

This shows how a circle can be mapped into a point via a controllable change of coefficients in homology provided all information obtained about the space is obtained via (co)homology. Let me further apply a similar procedure that will merge two disjoint circles into one single circle. In order to do this the coefficient group A will now be $\mathbb{Z}_2$ and the twisting will have the form

$$\rho = \begin{cases} 0 \to 1 \\ 1 \to 0 \\ 2 \to 1 \\ 3 \to 0 \\ ... \end{cases}$$

The analyzed space will now be a disjoint union of circles S^1 namely $X = S^1 \sqcup S^1$. By a simple application of Mayer-Vietoris theorem it results that $H_q(S^1) \cong H_q(S^1) \oplus H_q(S^1)$. Now, by using the twisted coefficients as described above, the homology won't be able to distinguish the two circles and hence we arrive at the single circle case.

It appears that the "quantum superposition" of topological spaces may be governed by a deeper form of entanglement, one in which the role of the linear superposition is altered by the structure of the coefficient ring in (co)homology. Therefore, by employing different coefficient structures one may entangle topologically disconnected pieces of space-time producing (not necessarily simply) connected space-times if certain restrictions on the coefficient structures are being imposed. This new form of entanglement is governed by the universal coefficient theorem in the sense that it allows us to switch from the information which can be obtained by means of one coefficient structure to the information obtainable via the other coefficient structure. Like in the case of normal entanglement, some questions about the topological space cannot be meaningfully answered when one relies exclusively on one coefficient structure. Therefore, entanglement as a linear combination of topological spaces in this case admits extra-flexibility due to the various possible choices of coefficient rings and the global effects such choices entail. This cannot be ignored because in this case the coefficient rings may alter the topological information which can be extracted from the given spaces. Therefore, finally, I briefly extend the analogy between qubits and homological algebra by going to a (co)homology theory with twisted complex coefficients. The key property of twisted (co)homology is the twisted acyclicity of the circle [334]. This property tells us that a twisted homology of a circle with coefficients in $\mathbb{C}$ which have a non-trivial monodromy must vanish. Subsequently a twisted homology theory of this kind completely ignores the parts of the space it wishes to describe which are formed by circles along which the monodromy of the coefficient system is non-trivial. The implications to physics are important mainly because, as I argued before the use of coefficient systems of various forms and of the universal coefficient theorem amounts to a prescription of finding new dualities in physics i.e. different analytical tools used to describe the same phenomena. In this case the duality is between entanglement and topology. In general for a homology theory, the dimension of $H_0(X;\mathbb{C})$ is equal to the number of path-connected components in X. Also, in classical homology theory (based on the standard Eilenberg-Steenrod axioms) $H_0(X;\mathbb{C})$ does not vanish unless X is empty. For twisted homology this last property is not valid anymore. Particularly when we analyze a circle $X = S^1$, we consider the map $\mu : H_1(S^1) \to C^\times$ taking the generator $1 \in \mathbb{Z} = H_1(S^1)$ to $\zeta \in \mathbb{C}^\times$. By this twist we then have the acyclicity of the circle in the sense that $H_*(S^1;\mathbb{C}^\mu) = 0$ if and only if $\zeta \neq 1$. Moreover, let X be a path connected space and $\mu : H_1(S^1 \times X) \to \mathbb{C}^\times$ be a homomorphism. Then let ζ be the image under μ of the homology class realized by a fiber $S^1 \times pt$. Then $H_*(S^1 \times X;\mathbb{C}^\mu) = 0$ if $\zeta \neq 0$. The proof of these results can be found in [334]. Physically this means that we may consider quantum states on a region of our space as belonging to the homology with complex coefficients $|\Psi\rangle \in H_1(X;\mathbb{C})$. X is in this case is the direct sum of two disconnected regions $X = A \sqcup B$. The homology of such a space will be the direct sum of the homologies of the two disjoint regions $H_1(X;\mathbb{C}) = H_1(A;\mathbb{C}) \oplus H_1(B;\mathbb{C})$. We can choose A and B to be spacelike separated. The state $|\Psi\rangle$ is entangled over A and B although the space itself doesn't show any topological features at this moment. The same properties will remain valid when we change the coefficient structure $\mathbb{C} \to \mathbb{C}^\mu$ where the twisting induced by μ is such that the coefficients form a twisted system with a non-trivial monodromy around any circle connecting region A and B. But with such coefficients $H_1(X;\mathbb{C})$ becomes trivial and hence the two regions become trivially identified i.e. in a sense similar to quantum teleportation. However, we can now modify the space X, by introducing the required circles which will make it look like a torus. This cannot affect the homology with twisted coefficients as it is not sensitive to circular components. However, if we now move back to untwisted coefficients we need to carefully employ the universal coefficient theorem and we will obtain the standard homology of a torus in complex coefficients, particularly $H_1(X;\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$. Summarizing, we started with a flat space and an entangled state and by changing the coefficient structure to a twisted one, making some undetectable changes to the space which left the homology intact and then changing back to the original coefficients we obtained the homology of a torus in complex coefficients. Of course the last transformation cannot be performed without penalizing some bijective maps due to the universal coefficient theorem. However, the physically relevant states remain unchanged, the only modifications being at the level of the Tor and Ext functors arising in the universal coefficient theorem for homology respectively cohomology. But how can it be that the physical states obtained when we go back to complex coefficients do not match the original states (as we do not have an absolute bijection because of the Tor and Ext functors)? First one should notice that Ext and Tor encode precisely the deviations introduced by adding the circular components. Therefore, this collapse of the bijection is simply because to begin with we made an assumption which cannot hold after a proper topological analysis, namely that in the original case we have a flat, topologically trivial space-time and entangled states. The whole point of this article is to show that such a situation is impossible, as entanglement automatically has to imply non-trivial space-time topologies. The main result is that entanglement is precisely encoded in the homology of a torus and a torus precisely encodes entanglement but entanglement cannot exist in topologically trivial space-time. It is obviously interesting to interpret this result in the case of basic quantum entanglement experiments where, apparently, the topology of space-time changes. How should such a change be interpreted in terms of basic entanglement experiments and apparently flat space-time remains a mystery, although mathematically it is possible to have a flat, topologically non-trivial space-time.

One may ask if locality is preserved in this situation. Indeed, the problem of locality when unitarity is restored appears to be fundamental to the AdS/CFT solution of the information paradox [214]. The information, in the approach of this chapter, is encoded in the global topological structure of the field in such a way that it is not accessible by any local measurements. One has to remember that the quantum field is not a measurable quantity. There is no physically observable “quantum field” in the same way in which there is no physically observable wavefunction. Nevertheless, the global, topological properties of the fields (and wavefunctions) are important and encode relevant information. Any local measurement can be seen as a “small” (weak) measurement. Can such a measurement reveal the global information? The correct answer to this question is no. Any weak measurement will reveal a weak information that will not provide any access to the information encoded globally and retrievable only via a statistical topological measurement. If one choses a coefficient structure for which the global non-triviality is invisible, locality is regained. Information is conserved but only in the factors appearing due to the use of the extension group. Hence unitarity is still preserved but in a "hidden" form (in the extension). If one choses a suitable coefficient structure the global information becomes accessible due to the manifest visibility of the global non-triviality. However, one cannot recover the information unless one performs a probing of the topology. This may look non-local in a sense but the information obtained in this way concerns topologically non-trivial field (wavefunction) structures hence this "non-locality" is not a physical one but rather one related to a choice of performing certain measurements.

In this chapter I have shown that topological corrections to the thermal radiation of a black hole as given by the requirement of topological covariance of the laws of physics can account for a factor in the coefficients defining the thermal radiation. This factor imposes non-trivial changes in the form of the distribution function that amount to non-thermal corrections. This observation confirms the suspicions that the solution of the unitarity problem relies on non-perturbative effects and on topological properties of the quantum groups involved in the derivation of the radiation distribution function. While the results are mostly formal and qualitative in nature, they do show that the method based on my idea has valid physical substrate.

Chapter 11

From Grothendieck's schemes to QCD

" 'Do you know, I always thought unicorns were fabulous monsters, too? I never saw one alive before!'
'Well, now that we have seen each other', said the unicorn, 'if you'll believe in me, I'll believe in you' "

Lewis Carroll, Alice in Wonderland

11.1 Rings and ringed spaces

In order to have a self-contained discussion about universal coefficient theorems, coefficient groups and their effects on quantum field theories some supplemental concepts must be introduced. I suppose that the concept of ring is well understood. Basically it represents a set of elements for which we can define two operations: multiplication and addition. The set is then a group for addition and a monoid for multiplication while the multiplication is distributive with respect to addition. The set can contain not only numbers but various other objects. In the theory of rings we can define the so called ideal of a ring. For a ring $(R, +, \cdot)$ we consider $(R, +)$ to be its additive group. We call a subset I its ideal if it is an additive subgroup of R that absorbs through multiplication by elements of R all the other elements. Formally this can be written as

- $(I, +)$ is a subgroup of (R,+)
- $\forall x \in I$, $\forall r \in R$: $x \cdot r \in I$

- $\forall x \in I$, $\forall r \in R$: $r \cdot x \in I$

For a ring ideal we can define the prime ideal which is a subset of a ring that extends the notion of prime numbers defined only on the ring of integers. The prime ideals for the integers are the sets that contain all the multiples of a given prime number together with the zero ideal.

As a formal definition we say that an ideal P of a commutative ring R is prime if it has the following two properties

- If a and b are two elements of R such that their product ab is an element of P then a is in P or b is in P,
- P is not equal to R

In order to make the connection with algebraic geometry, it may be useful to note that the zero set of ideals in polynomial rings define the geometric varieties.

Another useful concept is that of schemes, introduced by Grothendieck in his attempts to confer a more general perspective to algebraic geometry [166]. I will describe them briefly, following mainly [217] and the lecture notes [245]. There are two important structures that come together in order to define a scheme: firstly, a topological space, i.e. open sets and a definition of how to put elements together in these open sets and, secondly, the structure sheaf. An easy analogy, useful in the construction of schemes is the idea of a manifold. Indeed, from a topological point of view a manifold is a space that preserves the local property of being euclidean. This means that there exists an open cover $\{U_i\} \subset M$ such that each U_i is homeomorphic to $\mathbb{R}^n$. The coordinate patches must also satisfy compatibility conditions that allow us to define the notion of a smooth function. We call the spectrum of a commutative ring R to be the set of its prime ideals and we denote it by $Spec(R)$. We define a topology for $Spec(R)$. The closed sets are of the form

$$V(I) = \{p \in Spec(R) | I \subseteq p\} \tag{11.1}$$

where I is any ideal in R. Hence, from a geometric point of view we define a way in which various algebraic varieties are connected to each other. This leads us to think about what changes may be necessary i.e. what constraints we have to impose on the various fields or rings we employ in order to "transform" one variety into another. This whole construction will prove to be relevant in various mathematical and physical problems. Most importantly, it may relate various terms in the topological genus expansion of string theory or quantum chromodynamics. At this point we have a topology but we still need a method of discussing about functions on the topological space. One of the central ideas of algebraic geometry is thinking of rings as sets of functions on certain spaces, namely on their spectra. In analogy with smooth manifolds we must think about the functions more than about the space since the functions may not be completely determined by the space. We therefore have to think about R as functions on $Spec(R)$. In order to be able to meaningfully discuss about them we need a sheaf of rings. We define it as O_X on a topological space X as an assignment of a ring $O_X(U)$ to each open set U in X together with, for each inclusion $U \subseteq V$ a restriction homomorphism $res_{V,U} : O_X(V) \to O_X(U)$ subject to the following conditions

- $res_{U,U} = id_U$
- if $U \subseteq V \subseteq W$ then $res_{V,U} \circ res_{W,V} = res_{W,U}$
- For each open cover $\{U_\alpha\}$ of $U \subseteq X$ and for each collection of elements $f_\alpha \in O_X(U_\alpha)$ such that for all α, β if $res_{U_\alpha,U_\alpha \cap U_\beta}(f_\alpha) = res_{U_\beta,U_\alpha \cap U_\beta}(f_\beta)$, then there is a unique $f \in O_X(U)$ such that for all α, $f_\alpha = res_{U,U_\alpha}(f)$.

We may think of the elements of $O_X(U)$ as functions defined on U. The restriction homomorphisms correspond to restricting a function on a big open set to a smaller one. Intuitively, the axioms say that the elements behave as functions, hence

- restricting a function to its original domain does nothing at all
- Restricting and then restricting again is the same thing as restricting all at once
- If we have functions defined on some different open sets and these functions agree on the overlaps then we can glue them all together to get a unique function on the union of these open sets, and if we restrict this glueing to one of the open sets we get the corresponding function back.

A topological space equipped with a sheaf of rings on it is called a ringed space.

If we consider the manifold category and let M be a smooth manifold then for each open set U of M we have $C(U)$, the set of real valued continuous functions on U. Under point-wise addition and multiplication, this is a ring. If $V \subseteq U$ then we have the restriction homomorphism $C(U) \to C(V)$ given by actually restricting functions. It is easy to verify that this in fact is a sheaf. This is one of the prototypical examples of a sheaf.

Considering again M a smooth manifold and the sets $C^\infty(U)$ of C^∞ real valued functions on U, where U is an open set in M, these are still closed under point-wise addition and multiplication and the same restriction maps from above still hold. It can be verified that this still is a sheaf.

It is interesting to note that manifolds can also be defined in the opposite way: instead of defining them as a topological space with certain open cover satisfying some conditions and then deriving a sheaf from that, we can simply define them as a space together with a sheaf satisfying a similar property. This definition is equivalent to the coordinate charts definition [245].

11.1 Definition [245] A smooth manifold is a topological space together with a sheaf of real-valued continuos functions subject to the condition that there exists an open covering $\{U_\alpha\}$ with the restriction sheaf to each U_α isomorphic to $\mathbb{R}^n$. Moreover, here $\mathbb{R}^n$ is equipped with the sheaf of standard differentiable functions.

In what follows we denote a space just by X or Y while understanding that we mean a space together with a given sheaf of rings. There are various sorts of sheaves, for instance sheaves of abelian groups or, more generally, of R-modules over some ring R, defined in exactly the same way.

11.2 Definition [245] Let X be a topological space and O_X, O'_X be two sheaves of rings on X. Then a morphism $\phi : O_X \to O'_X$ is a collection of ring homomorphisms $\phi(U) : O_X(U) \to O'_X(U)$, one for each open set $U \subseteq X$, which commute with the restriction maps. That is, if $V \subseteq U \subseteq X$ are open sets, then the following diagram commutes

$$\begin{array}{ccc} O_X(U) & \xrightarrow{\phi(U)} & O'_X(U) \\ {\scriptstyle res_{U,V}}\big\downarrow & & {\scriptstyle res_{U,V}}\big\downarrow \\ O_X(V) & \xrightarrow{\phi(V)} & O'_X(V) \end{array} \tag{11.2}$$

It is also possible to create a sheaf from an existing one.

Given a sheaf O_X on a space X and U an open subset of X we can define a sheaf O_U on U by taking $O_U(V) = O(V)$, for any open subset V of U and by keeping the same restriction maps.

Sheaves can also be pushed along continuous functions

11.3 Definition [245] Let X and Y be topological spaces, O_X a sheaf on X and $f : X \to Y$ be a continuous function. We define the pushforward sheaf f_*O_X on Y by declaring $f_*O_X(U) = O_X(f^{-1}(U))$ for any open set U in Y with the obvious restriction maps. It is easy to check that this will be a sheaf.

If we have a sheaf on X and a basis of open sets for X then the sheaf is completely defined by its values on the basis.

11.4 Definition [245] The structure sheaf of $Spec(R)$ is the scheme $O_{Spec(R)}$ defined by

$$O_{Spec(R)}(Spec(R)_f) = R_f \tag{11.3}$$

i.e. the localization of R at the element f. We say just $Spec(R)$ when we mean the set of prime ideals in R together with the Zariski topology and this sheaf of rings on it. Spectra of rings are the first example of a scheme and will play the same part in defining general schemes as $\mathbb{R}^n$ does in defining smooth manifolds.

11.5 Definition [245] A spectrum of a ring with the sheaf of rings defined above is called an affine scheme.

11.6 Example [245] If k is a field then $Spec(k)$ is the one point space with $O_{Spec(k)}(*) = k$

11.7 Example [245] $Spec(\mathbb{Z})$ is one point for each prime number corresponding to the maximal ideal (p) as well as one non-closed point (0).

It is interesting to note that when the ring has nilpotent elements, functions are no longer determined by their values. Let us define for example k to be a field and $R = k[x]/x^2$. Then R has only one prime ideal, namely (x), so $Spec(R)$ is one point, with $k[x]/(x^2)$ at that point. The function x is then everywhere zero but is not the zero function.

A question that arises is : what do we mean when we think about elements on a ring R as functions on $Spec(R)$? This can be made rigorous as follows

11.8 Definition [245] For a point $p \in Spec(R)$ we have the following canonical map

$$R \to R/(p) \to k(p) \tag{11.4}$$

where $k(p)$ is the fraction field of $R/(p)$. For an element $f \in R$ we define $f(p)$ to be the image of f under this map. This definition does not always yield actual functions.

11.8 Example [245] Let $X = Spec(\mathbb{Z})$ and consider the element $f = 7 \in \mathbb{Z}$. Then $f((2)) = 1$ in the ring $\mathbb{Z}/2\mathbb{Z}$, $f((5)) = 2$ in the ring $\mathbb{Z}/5\mathbb{Z}$, and $f((7)) = 1$ in the ring $\mathbb{Z}/7\mathbb{Z}$. In particular note that the values of f lie in different fields.

The set $\{p \in Spec(R)|f(p) = 0\}$ still does make sense though. Also, if k is an algebraically closed field, and $R = k[x_1, ..., x_n]$ then for all maximal ideals m, $k(m) = k$ since it is a finite extension of an algebraically closed field. Therefore they are really functions in the classical sense.

With the notion of affine scheme and isomorphism of sheaves we can define a general scheme

11.9 Definition [245] A scheme is a topological space X, together with a sheaf of rings which is locally affine in the following sense: there is an open covering $\{U_\alpha\}$ of X such that the restriction of O_X to each U_α is isomorphic to an affine scheme.

This is just like the smooth manifold category where we can define a smooth manifold to be a topological space with a sheaf of differentiable functions on it (the sheaf of rings), that is locally isomorphic (in the sense above) to some $\mathbb{R}^n$ with its standard sheaf of differentiable functions.

11.2 Topological strings and the holomorphic anomaly equations

In what follows I intend to show that using different coefficient groups in (co)homology, the genus expansion can be transformed such that all topological terms of genus higher than 1 can be described in terms of spheres. This is a drastic statement, as, from any classical topological point of view, a torus is not a sphere. However, it appears that by a surprising choice of coefficients in (co)homology, a torus can, for all practical purposes, i.e. from the perspective of the integration over ribbon-graph diagrams on it, be treated like a sphere. The universal coefficient theorem, then, will encode the additional information related to the degeneration of the torus into a sphere in the form of Tor and Ext groups. This cannot however happen unless we allow some degeneration of surfaces as well as some singularities. In order to make this consistently, I need to enlarge some concepts. For example, I cannot speak only in terms of a single manifold category, as the morphisms in that category must be continuous and p-times differentiable. I propose here a different mechanism that relates algebraic varieties, namely the coefficient ring (or group) in cohomology. In this way I extend the mapping of one algebraic variety into another (as introduced by Grothendieck [22]) to algebraic topology with (co)homology groups with torsion coefficients. The main application on which I would like to insist is quantum chromodynamics (QCD), namely the theory of quarks and gluons. The results on this theory are however more speculative in nature. Therefore I will start by introducing a theory where a recursive relation between manifolds of different genera exists and is well understood. This is the theory of topological strings and the recursive relation is known as the holomorphic anomaly equation. For this introduction I follow mainly [235] and [246]. However, it appears that the universal coefficient theorem is also capable of encoding general anomalies that appear when an algebraic variety degenerates into another one. Holomorphic anomaly equations can then be seen as a particular case to more general relations that can be derived from universal coefficient theorems.

A major simplification in dealing with string theory was brought by the discovery of the topological phase of string theory. General string theory and particularly superstring theory has many possible formulations. The worldsheet may be described by an $\mathcal{N} = 1$ supersymmetric Ramond-Neveu-Schwarz (RNS) formalism. It may also be described by a Green-Schwarz (GS) formalism. For any quantum theory, be it of strings or of fields the most important task is the calculation of scattering processes. This is in general very difficult. In order to do this, one has to obtain in one form or another, excitations that correspond to every kind of elementary particle. While bosonic string theory is obviously insufficient, if string worldsheet supersymmetry is extended to $\mathcal{N} = 2$ some geometrical properties arise that make the calculations somehow easier. The field theoretical equivalent of this construction can be shown to be a topological field theory.

For example a $(2, 2)$ sigma model contains bosonic fields. These can be seen as maps from a Riemann surface to a Kahler manifold. Exact supersymmetry can exist only when the Riemann surface is flat. However, a curved surface prohibits in general the existence of a covariant constant spinor that can be considered a supersymmetry transformation parameter and hence supersymmetry is broken. By a topological twist however, it is possible to modify the theory such that some fermionic symmetry remains. This also generates a scalar supercharge which is preserved on any Riemann surface. A flat sigma model can be transformed into a curved one by changing the flat metric into a curved one and the partial derivative into a covariant one. As the scalar supercharge is nilpotent in any Riemann surface we can define an associated cohomology. The physical spectrum of the topological theory is given by this cohomology. We therefore obtain a "topological sigma model". The procedure of introducing the topological twist is a defining property of topological models. The correlation functions of this theory are independent of the metric on the Riemann surface. When we couple the theory to gravity on the Riemann surface we obtain the associated topological string theory and the Riemann surface becomes a worldsheet of string theory. As long as we do not allow the variation of the worldsheet metric we talk about a sigma model. Topological string theory does not have local degrees of freedom. Topological string theory is exceptionally solvable [235]. The reason for this is the existence of various differential equations satisfied by the topological correlation functions. These result from topological Ward identities derived using the independence of the worldsheet theory from the worldsheet metric. It is because of the topological nature of the theory that we can integrate these equations up to some integration constants specified by the classical data of the underlying model. A particularly interesting special case is the $\mathcal{N} = 2$ superconformal field theory with central charge $\hat{c} = 3$ because they give many non-trivial topological amplitudes and can be related to four dimensional physics. The most important differential equation in this case is the holomorphic anomaly equation of Bershadsky, Cecotti, Ooguri and Vafa [246],

[247] which controls the amplitudes as functions over the coupling space. The origin of the holomorphic anomaly can be related to the unitarity and CPT invariance of the underlying conformal field theory [235]. In general we are interested in the perturbative topological string amplitudes. We call them $\mathcal{F}^{(g,h)}$ and define them as the integral over the moduli space $\mathcal{M}^{(g,h)}$ of oriented Riemann surfaces of genus g and with h boundary components. $\mathcal{F}^{(g,h)}$ are functions over the coupling space which is a complex manifold. In general a Riemann surface is not compact. In that case, we can make a statement regarding the anti-holomorphic derivative $\bar{\partial}\mathcal{F}^{(g,h)}$. In particular we can say that this derivative is zero. However, in order to compactify the Riemann surface we have to add some boundaries. In particular, we add to $\mathcal{M}^{(g,h)}$ certain objects $\partial\mathcal{M}^{(g,h)}$ that allow us to make $\mathcal{M}^{(g,h)}$ compact. The anti-holomorphic derivative $\bar{\partial}\mathcal{F}^{(g,h)}$ receives corrections precisely from these terms. Otherwise stated the additional terms arise from the singularities or "contact terms" that appear in the integrand of $\mathcal{F}^{(g,h)}$ when the Riemann surface degenerates. Corrections of this type must encode all the possible ways in which such a "degeneration" may occur. The boundary term itself is not a holomorphic function over the coupling space. For a closed string $(g,h) = (g,0)$ the Riemann surface can degenerate in two distinct ways encoded by the following formula

$$\partial_{\bar{i}}\mathcal{F}^{(g)} = \frac{1}{2}C_{\bar{i}\bar{j}\bar{k}}e^{2K}G^{\bar{j}j}G^{\bar{k}k}\Big(\sum_{g_1+g_2=g}\mathcal{F}^{(g_1)}_j\mathcal{F}^{(g_2)}_k + \mathcal{F}^{(g-1)}_{jk}\Big) \tag{11.5}$$

Here, $\mathcal{F}^{(g)}$ with various subscripts i, j, etc. are the amplitudes with insertions i.e. derivatives of $\mathcal{F}^{(g)}$ in holomorphic directions on the moduli space. $C_{ijk} = \mathcal{F}^{(0)}_{ijk}$ is the three point function on the sphere and it is holomorphic. The sum is a recursive relation for the topological amplitudes, genus by genus.

The open string will have $h \neq 0$. We need to choose some boundary conditions i.e. to specify some D-brane configuration. Consider the Yukawa coupling, i.e. the first non-vanishing amplitude at tree-level. The sphere 0, 1, and 2 point functions vanish and all higher-point functions on the sphere can be computed by simply taking the derivatives. Consider $\Omega(z) \in H^{3,0}(Y)$ as the holomorphic 3-form as a function of the complex structure moduli of the Calabi-Yau manifold Y. Let me define the Gauss-Manin connection first. This represents a generalization of the concept of parallel transport in the context of a vector bundle made from a family of algebraic varieties. The base space is considered the space of the parameters defining the family and the fibers are taken to be the de-Rham cohomology group H^k_{dR}. Flat sections of the bundle are described by differential equations like for example the Picard-Fuchs equation for the family of elliptic curves. Calling the Gauss-Manin connection ∇ we have

$$C_{ijk} = - < \Omega, \nabla_i\nabla_j\nabla_k\Omega > \tag{11.6}$$

where the bracket $< *, * >$ represents the symplectic pairing on $H^3(Y)$ i.e.

$$< *, * >= \int_Y * \wedge * \tag{11.7}$$

In the case of open strings the invariant holomorphic data that characterizes a topological D-brane at tree level is a Poincare normal function. Such a normal function ν is defined by a three-chain $\Gamma \subset Y$ whose boundary is a holomorphic curve. Such a three chain does not specify an element in $H^3(Y)$ completely but since the boundary is holomorphic, integration against Γ gives a well defined pairing with cohomology classes in $H^{3,0}(Y)$ and $H^{2,1}(Y)$. This can be written as

$$\mathcal{T} =< \Omega, \nu >= \int_\Gamma \Omega \tag{11.8}$$

which physically can be identified with the domainwall tension. The hallmark of a normal function is

$$< \Omega, \nabla\nu >= 0 \tag{11.9}$$

All the information about ν is then contained in the disk two point function which is an infinitesimal invariant

$$\mathcal{F}^{(0,1)}_{ij} = \Delta_{ij} =< \Omega, \nabla_i \nabla_j \nu > \tag{11.10}$$

Mathematically, the infinitesimal invariant is a cohomology class whose representative depends on a lift of ν to $H^3(Y;\mathbb{C})$. But given

$$\mathcal{T} =< \Omega, \nu >= \int_\Gamma \Omega \tag{11.11}$$

and its physical interpretation as a domainwall tension, we have a preferred lift obtained by imposing ν to be real i.e.

$$\nu \in H^3(Y;\mathbb{R}) \subset H^3(Y;\mathbb{C}) \tag{11.12}$$

But the reality of ν is not compatible with the holomorphic dependence on the parameters. This is precisely the holomorphic anomaly of the disk two-point function.

Hence the anomaly leading to the recursive relation for the amplitudes of different genera originates in the fact that we restrict the complex cohomology to a real cohomology. Reality of the cohomology however is in conflict with the notion of holomorphicity.

We already saw to begin with that the non-zero terms contributing to the anti-holomorphic derivative of $\mathcal{F}$ are related to the degeneration of the Riemann surface and its change of genus. Can this be interpreted in the sense of the universal coefficient theorem? Apparently yes, although the interpretation might be somehow different with respect to what we are used with.

11.3 A giant leap: Quantum Chromodynamics

Before arriving at the conclusion of this thesis, I present in this section a new way of analyzing Feynman diagrams. The main application originates from the universal coefficient theorem, but in general, homological algebra may prove itself extremely useful in deriving new, non-perturbative results for various quantum field theories [259-260].

Quantum Chromodynamics is a non-abelian gauge theory with the gauge group $SU(3)$ where the fermions live in the direct representation while the gluons live in the adjoint representation. It is a very accurate theory that can be analyzed perturbatively in the high energy regime [218] where the first obvious expansion parameter (the coupling) can be considered as small [219]. Its results in that domain are accurate enough to validate it as the correct theory of strong interactions [220]. However, due to the property of confinement (which is not systematically proved but for which we have strong experimental evidence [221]) the perturbative approach becomes less and less reliable at low energy [222]. This is because there is a strong contribution towards the coupling constant from low energy corrections [223]. These make the perturbative approach break down at energies comparable to our immediate observations [224], [225] i.e. nuclear physics, predictions for the masses of protons, neutrons, hadrons, etc. This is why non-perturbative methods are being employed at that level. Today we rely in this area mostly on lattice approaches which are notoriously computationally intensive [226]. There are however different other methods. One method tries to connect regions unaccessible to a perturbative approach in QCD to regions of other theories that can easily be solved either via perturbative methods, as in the standard way, for example like in [227] or via methods originating from, say, general relativity [228]. These methods are spin-offs from the study of dualities [229] and are impressively promising today. Another, somehow related method is the large N expansion [230]. This appeared due to the search for alternative expansion parameters when the most obvious choice (the coupling constant) became unreliable. For a simplified alternative model (the Gross-Neveu Lagrangian [231])

$$L = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g^2}{2N}(\bar{\psi}\psi)^2 \tag{11.13}$$

the $1/N$ expansion parameter appears naturally in the interaction term. This can be understood with an analogy to simple quantum mechanics [231] : a state $|\psi\rangle$ which can be written as a sum of N orthonormal states with equal amplitudes

$$|\psi\rangle = \alpha(|1\rangle + |2\rangle + |3\rangle + ... + |N\rangle) \tag{11.14}$$

has as normalization condition

$$N|\alpha|^2 = 1 \tag{11.15}$$

and hence $(\bar{\psi}\psi)^2$ has as coefficient $1/N$. The limit $N \to \infty$ then gives the desired approximation. QCD is more complicated than this Gross-Neveu model. In principle, QCD is a $SU(3)$ theory with gluons in the adjoint representation of the gauge group. In order to define an accessible expansion parameter we will consider $3 \to N \to \infty$. In this way we will write $SU(N)$ instead of $SU(3)$ for the gauge group, the case $N = 3$ being a restriction to the physical situation. There will be a covariant derivative

$$D_\mu = \partial_\mu + i\frac{g}{\sqrt{N}}A_\mu \tag{11.16}$$

where the gauge field will be of the form $A_\mu = A_\mu^A T^A$ with the T^A matrices normalized according to

$$TrT^AT^B = \frac{1}{2}\delta^{AB} \tag{11.17}$$

The indices A and B refer to the adjoint representation of the gauge group. The coupling constant is considered to be $g/\sqrt{N}$ instead of g in order to obtain a sensible and non-trivial large N limit [231]. One can define the field strength then as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{g}{\sqrt{N}}[A_\mu, A_\nu] \tag{11.18}$$

the Lagrangian being

$$L = -\frac{1}{2}TrF_{\mu\nu}F^{\mu\nu} + \sum_{k=1}^{N_F}\bar{\psi}_k(i\not{D} - m_k)\psi_k \tag{11.19}$$

The large N limit can be taken with the number of flavors N_F fixed. The quark propagator has the form

$$< \psi^a(x)\bar{\psi}^b(y) >= \delta^{ab}S(x-y) \tag{11.20}$$

and is represented in the Feynman diagrams as a single line. The gluon propagator is

$$< A_\mu^A(x)A_\nu^B(y) >= \delta^{AB}D_{\mu\nu}(x-y) \tag{11.21}$$

where A and B are the indexes of the adjoint representation. A gluon can be seen as a $N \times N$ matrix with two indices, $(A_\mu)_b^a = A_\mu^A(T^A)_b^a$ and the propagator becomes

$$< A_{\mu b}^a(x)A_{\nu d}^c(y) >= D_{\mu\nu}(x-y)(\frac{1}{2}\delta_d^a\delta_b^c - \frac{1}{2N}\delta_b^a\delta_d^c) \tag{11.22}$$

One can redraw the gluon lines from the standard Feynman diagrams in a way that explicitly shows these two indices on each line [230]. In this way one arrives at something

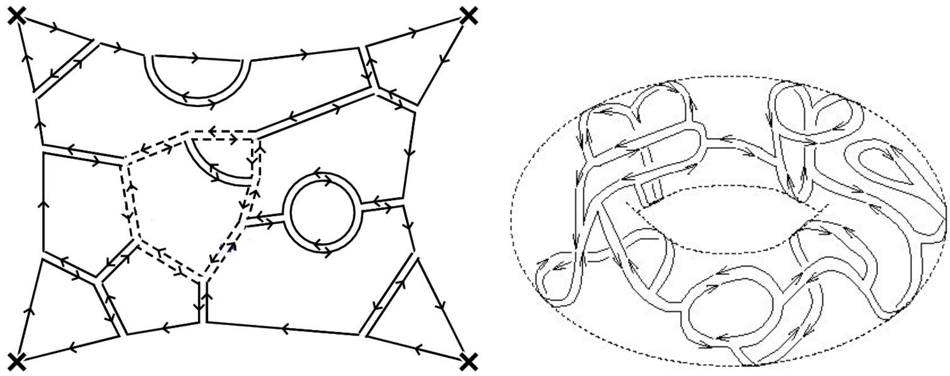


FIGURE 11.1: (a) ribbon graphs on a planar topological space (or sphere), (b) ribbon graphs on a toroidal topological space [230]

called a "ribbon graph" where each gluon line is now represented as a double arrow (in the case of an $SU(N)$ theory; an $SO(N)$ theory for example would have only double lines, with no orientation). Putting together various processes one arrives at a situation where these double arrow graphs fill a topological space. Indeed, they can be classified with respect to the homotopy class of the topological space they can fill: sphere or planar diagrams, non-planar diagrams of genus 1 (a single hole torus) or of genus 2 (a double hole torus) etc [233]. In QCD planar diagrams with quark lines are represented as planar diagrams with the quark line as the boundary or as spheres with holes represented by the quark loops. These appear with order N in the expansion. Diagrams with no quark lines (gluon lines only) form closed spheres and appear to be of order N^2. Some diagrams however cannot be drawn in a plane with only quark lines at the boundaries or on a sphere with no boundaries. In that case, these diagrams behave as powers of $\frac{1}{N}$ and are represented on toruses of various genera [231]. See figure 1 for a graphical representation. In a more general sense, increasing the size of the group to $SU(N)$ where $N \to \infty$ makes the theory simplify considerably. This is remarkable because one would not expect from a theory with an infinite number of degrees of freedom to manifest itself in a simpler way than a theory with a finite, and rather small number of degrees of freedom. It is possible to prove that in some theories, terms in the series expansion in $\frac{1}{N}$ are related to homotopy classes of diagrams. The fact that diagrams become less relevant in the large N limit is classified by the topological space they can fill.

The series expansion in this limit is not the classical expansion in the coupling constant but instead in $\frac{1}{N}$ and the diagrams arrange themselves such that each term of this expansion corresponds to a set of diagrams on a topological space characterized by a given genus.

In order to see the way in which the situation simplifies one should go back to what is known as the 't Hooft model. There, one considers a theory similar to QCD but on only two spacetime dimensions and in the large N limit. The theory is confining and there are several simplifications. What is important however, is that in this simplified model all the diagrams appearing in the theory can be represented on the surface of a plane or on a sphere. This is not a general fact of QCD. Due to this fact, specific to the case of 2 dimensional QCD in the 't Hooft model, we may solve this theory exactly. Complete planarity makes a theory exactly solvable.

It is therefore important to consider the topological properties of the space formed by the ribbon graph diagrams and particularly to see in what cases non-planar diagrams behave like planar diagrams. One observes that each color index loop forms a polygon (simplified, it encompasses a 2-simplex) that can be glued together with another index loop in order to form a surface. The topology of a space can be determined at various homological "resolutions" with various homological-algebraic tools. Here "resolution" refers to the visibility of a certain topological feature. It has nothing to do with the scale at which one looks and with the analogy to a "magnifying glass". Maybe the best analogy would be a detector sensible to certain topological properties if suitably tuned. In general, what the physicist wishes is to integrate over such a topological space in order to obtain the answers encoded in the diagrams defined on it. For this, one needs an integration measure. This measure is in general sensitive to the (co)homology of the space. The (co)homologies are invariants of the topological space of a certain accuracy (strength). They are in general defined in terms of chain complexes [96]. The chain complexes are defined starting from the q-simplexes Δ^q [96]

$$\Delta^q = \{(t_0, t_1, ..., t_q) \in \mathbb{R}^{q+1} | \sum t_i = 1,\, t_i \leq 0 \forall i\} \tag{11.23}$$

together with face maps

$$f^q_m : \Delta^{q-1} \to \Delta^q \tag{11.24}$$

defined as

$$(t_0, t_1, ..., t_{q-1}) \to (t_0, ..., t_{m-1}, 0, t_m, ..., t_{q-1}) \tag{11.25}$$

This abstract construction must be mapped into a realistic space X. In order to do this a continuous map is required (see fig. 2)

$$\sigma : \Delta^q \to X \tag{11.26}$$

Considering this, any space can be constructed as a chain

$$\{X\} \;=\; \sum_{i=1}^{l} r_i \sigma_i \tag{11.27}$$

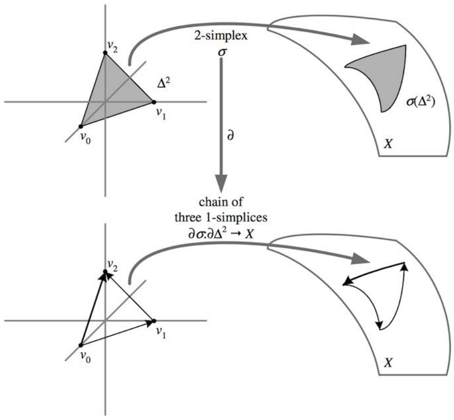


FIGURE 11.2: Mapping a simplex on an arbitrary manifold [232]

where $\{r_i\}$ is the set of coefficients belonging in general to a ring R. The space X as seen via the basis formed from the q-simplexes defined above is denoted $S_q(X;R)$. One defines a boundary map as

$$\partial : S_q(X;R) \to S_{q-1}(X;R) \tag{11.28}$$

such that

$$\partial(\sigma) = \sum_{m=0}^{q} (-1)^m \sigma \circ f_m^q \tag{11.29}$$

One can extend the above definition by introducing the covariant functor $S_*(-;R)$. This means that given a continuous map

$$f : X \to Y \tag{11.30}$$

this will induce a homomorphism

$$f_* : S_*(X;R) \to S_*(Y;R) \tag{11.31}$$

with the definition

$$f_*(\sigma) = f \circ \sigma \tag{11.32}$$

Then, the complex $(S_*(X;R), \partial)$ is called the simplicial chain complex of the space X with coefficients in R. The homology of this chain complex with coefficients in R is then

$$H_q(X;R) = \frac{ker\ \partial}{Im\ \partial} \tag{11.33}$$

where ker represents the kernel of the considered map and Im represents its image.

Hence the homology groups depend on the coefficient rings R used to define them. For simplicity one can also restrict the rings to groups. I showed in the previous chapters of this thesis that the universal coefficient theorems can express the (co)homology groups of a space with a certain coefficient group in terms of (co)homology groups of the same space with a different coefficient group. I also showed in the previous chapters of this thesis that some information visible when a certain group is used becomes invisible when another group is used. The main idea is that the information about the homotopy class of a function may be accessible when a certain coefficient group is used, while not visible from the (co)homological perspective when another coefficient group is used. However, the universal coefficient theorem, written as

$$0 \to Ext(H_{n-1}(C), G) \to H^n(C; G) \to Hom(H_n(C), G) \to 0 \tag{11.34}$$

for cohomology or as

$$0 \to H_n(C) \otimes G \to H_n(C; G) \to Tor(H_{n-1}(C), G) \to 0 \tag{11.35}$$

for homology gives us the extra information related to the homotopy classes, just that in this case encoded in the "homological obstruction" given by the Ext respectively Tor groups. Here $H_*(C)$ is the *-th dimension homology of the chain complex C, $H^*(C; G)$ is the *-th dimensional cohomology of the chain complex C measured with coefficients in G, $Hom(H_*(C), G)$ is the group of all homomorphisms from $H_*(C)$ to the coefficient group G, $H_*(C; G)$ is the *-th dimensional homology group with coefficients in the group G and the Ext and Tor functors are here the extensions and the torsions of the respective homologies. These behave as obstructions to the exactness of the short sequence where they would be absent.

Translated in terms of $\frac{1}{N}$ expansions, this would mean that, when using a certain coefficient group, the integration measure may "see" the non-planar graphs as planar while the formal differences between the two types can be found only in the form of Ext and Tor groups and modified group operations. In this way, one can relate theories containing non-planar diagrams, considered hard to solve today, to theories containing only planar diagrams and homological-algebraic corrections to some composition rules. These corrections will differ for each topological genus they originate from. This would make many theories exactly solvable if the above mentioned corrections are correctly understood. In some sense, this amounts, loosely speaking, to a renormalization procedure: the "non-solvability" due to non-planar contributions is eliminated, maintaining the relevant, computable "non-planar" contributions only in the form of modifications of group laws as specified by the Ext and/or Tor groups. Applications to QCD should be obvious: exact calculations, new dualities, a new systematic approach to non-perturbative QCD. In the context of string theory this would allow to probe regions beyond the perturbative string expansion in a systematic way in the same way in which the renormalization group prescription in renormalizable quantum field theories allows us to go beyond the strict perturbative regime.

In a sense, until now, the lack of strength of topological invariants (their inability to discern some topological spaces) was certainly not seen as a desirable property. However, several physical and finally natural phenomena can also not make the distinction between some topological spaces defined in terms of chain complexes. For example, naturally a measure of an integral over a topological space is related more to the cohomology of the space than to its actual mathematically perfectly described shape. Natural phenomena are also defined in terms of integrations over spaces with certain measures. This reminds us of the coarse graining in the problems analyzed via renormalization groups. However, in this case it is not the large scale that hides features but the topological invariants and other homological tools that we naturally use. In this sense I see this idea as a generalization of "renormalization group approaches". The renormalization group transformations now become changes in the group structures used in (co)homology. The regularization step becomes the identification of the problems originating from the non-planar nature of the corrections and the translation of these into the language of derived functors (Ext and Tor). The standard example of how a function that looks homotopic to a constant in the cohomology with a set of coefficients, is in fact homotopically non-trivial when analyzed with another set of coefficients, has been presented in [54] and also in chapter 9.

Let me be more accurate and translate this into notions related to integration. This has important consequences in the way we calculate the terms in the topological expansion of QCD but also in practical calculations of integrals over topologically non-trivial manifolds in general. In principle the homology groups, $H_k(C)$ relate to the shape of the manifold. The cohomology groups, $H^k(C)$ relate to the differential forms defined over the manifold. Hence, if there is a manifold M characterized by a sequence of homology groups, then, one can define the integral

$$\int_M \omega \tag{11.36}$$

characterized by the differential form ω and by the manifold M. In QCD the differential form may encode the integration over all internal bands of a genus term. Integration can be seen as the pairing

$$H_k(M,\mathbb{R}) \times H^k(M,\mathbb{R}) \to \mathbb{R} \tag{11.37}$$

such that

$$([M],[\omega]) \to \int_M \omega \tag{11.38}$$

where this pairing is constructed with real coefficients and this coefficient structure characterizes also the measure of integration and implicitly the differential form ω. Here, $[M]$ represents a class in homology and $[\omega]$ represents a class in cohomology. The pairing above however is an isomorphism (one-to-one relation) only when this particular choice of coefficients is made. For other coefficients this pairing may fail to be an isomorphism. The correction is encoded in a term controlled by the Ext and Tor groups

$$H_k(M, \mathbb{G}) \times H^k(M, \mathbb{G}) \to \mathbb{G} \tag{11.39}$$

where the map becomes

$$([M], [\omega]) \to \int_{\{M\}} \omega \oplus C_{Ext(H_{n-1}(C),\mathbb{G})} \tag{11.40}$$

Here, the first integral is over the surface $\{M\}$ visible when the coefficient structure $\mathbb{G}$ is used and the correction appears as $C_{Ext(H_{n-1}(C),\mathbb{G})}$ depending on the extension group constructed from the homology with general integer coefficients over a lower dimension. Here I simply used the universal coefficient theorem in cohomology. The non-trivial topology however is not visible from the lower dimension hence the simplification.

In this way I show that properties defined on some more complex topological objects may be acceptably described on simpler topological objects if controlled changes in the groups used to describe them are being employed.

The observations above may appear relevant for the large N expansions in quantum chromodynamics. As has already been noticed by 't Hooft [230], QCD bears strong similarities with string theory. Therefore, going beyond the perturbative expansion in this case may even prescribe how to talk about quantum gravity and finally how to unify physics in general. The name given to this approach is M-theory.

Indeed, one of the first motivations for string theory was the description of mesons [222], namely states of quarks and antiquarks together with the color "bond" between them. This bond has a very peculiar behavior. While the electromagnetic interaction is mediated via a field that becomes weaker and more diffuse at larger distances (hence at lower energies), the potential associated to color becomes denser and denser, increasing the strength of the interaction. This effect called confinement gives to the strong interaction an especially important behavior at low energies.

We can still try to develop QCD in the same way as we did with the other interactions: try to construct the associated Feynman diagrams, to apply the renormalization prescriptions and to finally solve the problem. In doing so we obtain a series expansion. This series expansion is meaningful only in the high energy domain. In the low energy domain something very interesting occurs. It has been noted by 't Hooft [239] that the gluon propagators can be represented using double arrows instead of simple lines, one arrow for each of the indexes of the field matrix in the adjoint representation of the gauge group.

When doing this, the Feynman diagrams start looking more like rubber bands, having their own thickness. These bands are now associated to the propagators. They can also be linked together such that in the end they fill a surface of a specific topology.

After this is done, one obtains a classification of diagrams in terms of the topological genus. A theory containing only diagrams that in the large N limit can be represented on a plane or a sphere can at least in principle be solved exactly [231].

The next corrections depend on the genus of the torus associated to the surface on which the diagrams reside. An exact solution of the full theory including these corrections remains unknown.

This chapter opens a new research avenue, one that brings together on one side algebraic topology and homological algebra (mainly some early results by A. Grothendieck on (co)homology with coefficient groups) and, on the other side, some outstanding problems in physics. The mathematics invented by Grothendieck, while certainly brilliant and universally appreciated by mathematicians, remained somehow esoteric for physicists. In this last chapter of this thesis I intend to change this by showing a way in which his results can have an impact on physical research. While this chapter does not offer a final resolution of all the physical puzzles, it offers a new way of thinking that largely extends the applicability of several methods, well known to the physicists e.g. renormalization prescription, renormalization group, regularization, etc.

In standard quantum field theory it is possible to split a divergent parameter of a theory (e.g. the coupling) into a renormalized component and a singular counter-term. However, this partitioning is based on an arbitrary reparametrization. One can make a different choice by transferring a finite amount from the renormalized parameter to the divergent counter-term without changing physics. Suppose one has a graph with a renormalized value which is too large to allow the computation of its associated quantity in the lower orders of a perturbative expansion. It is however possible to adjust the splitting into the renormalized part and the divergent counter-term such that the counter-term cancels not only the initial divergence but also the excessively large piece of the graph's finite part. The large piece now comes in the lowest orders of the perturbation expansion instead of appearing at high orders [224].

The renormalization procedure is based on a regularization prescription (a method that allows the identification of the "problems", e.g. the divergencies) and a renormalization prescription i.e. a method of canceling systematically the divergencies at each order by employing terms of the same form as those already existing in the Lagrangian. This leads to corrections in the terms controlling the mass, charge, etc. defined in the original theory [224].

The whole renormalization program appeared because in any process there are contributions from quantum fluctuations originating at every scale of the theory. In essence, a process that looks simple at one renormalization point may appear as very complicated at another point. In order to disentangle this mixture of scales and complicated graphs one may use the renormalization group when one has to deal with contiguous scales or one may use the tools of effective field theory when more distant scales are involved.

What if there exists a third approach and something similar would be possible for the topological expansion? This would imply not the scale of the theory but the visible (co)homological properties. The "disentanglement of scales" would translate into reparametrizations relating different (co)homological properties. Topology tells us that a torus and a sphere, or two toruses of different genera are not homeomorphic, hence cannot be smoothly transformed into one another. This is however not the only possibility of relating the two objects. In fact, homological algebra gives various tools by which we can find what are the obstructions to transformations we would like to perform. These obstructions may in some cases be eliminated by less obvious changes in the way we look at our theories. In this chapter, I show that there exists a systematic prescription that, used properly, maps the information defined on a torus of any genus into a sphere. There are however strong obstructions to this mapping that can be eliminated by employing various coefficient groups in the cohomology associated to the space. These obstructions, if properly considered, may have an effect analogous to the renormalization group flow or the passage to an effective theory. The difference with respect to these other approaches is that the obstructions appear as corrections to the algebraic structure of the theory and not as corrections to specific parameters.

We saw while analyzing the holomorphic anomaly equation that a torus can degenerate into a sphere in certain situations. Here I show that such a situation can be modeled via the universal coefficient theorem as interpreted above. Figure 1 shows how obviously different a torus is from a sphere. In fact we can detect the differences naturally, by looking at them. Mathematically however, we need to define certain invariant objects that on one side should change when a relevant change in the topology of what we intend to measure occurs and on the other side, must remain invariant to any other variation that does not change the topology. Whenever such an invariant is incapable to detect a certain change that would result in a change of topology, we say it is "blind" to a

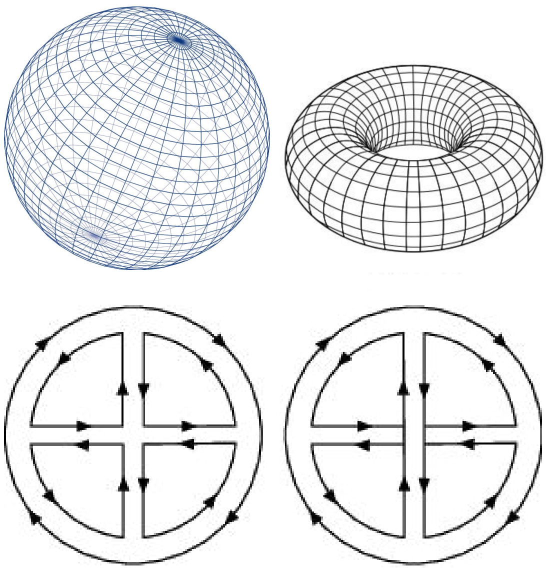


FIGURE 11.3: A sphere and a torus. They appear to be topologically different. This difference is measured by topological invariants that are defined such that they are the same on surfaces that are topologically identical. Homology and cohomology are such invariants. However, the invariants are not always ideal. In order to define them correctly one needs additional information, brought to them by what is known as the "coefficient structure". There exist coefficient groups with enough torsion such that the cohomology group that may make the distinction between the two becomes incapable of doing so. This makes the integration over the cohomology (cohomology being the only construction that is useful in diagrammatic calculations) equivalent for the two cases. Representative Feynman ribbon-graph diagrams are presented below each of the topological objects.

specific topological transformation. This makes that invariant of a rather low quality if our desire is an accurate description of a shape.

Homologies and cohomologies are in general relatively good topological invariants. They are easily computable and probe the topology relatively well. However there are well known situations when objects with the same homology have different homotopical and topological properties (see for example Poincare's homological sphere [232]). While this is a way in which (co)homologies may fail in their ability to discern topologies, it is not this the method I wish to insist upon here. Instead, I remind the reader that homologies and cohomologies, in order to be calculated and measured, must be defined including a set of coefficients (the coefficient group) [23]. I explained above how these enter in the construction of (co)homology. Enough to say now that they determine the sensitivity of the (co)homology to various features that would otherwise remain invisible. Figure 2 shows another set of two surfaces. Both unorientable and both undetectable by (co)homology as such when an unadapted coefficient group is employed. In the same way in which homology may be insensitive to unorientability for various shapes, the cohomology may become insensitive to the presence of a torus instead of a sphere. For example [261] take the torus T_1. Its homology in dimension 1 is $H_1(T_1) = \mathbb{Z} \oplus \mathbb{Z}$ and the

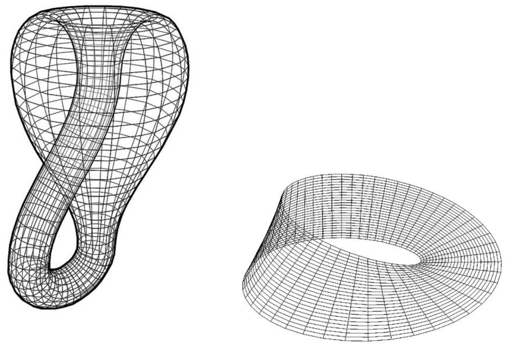


FIGURE 11.4: A Klein bottle and a Moebius strip. Two unorientable surfaces that cannot be detected by homology with arbitrary coefficients. The coefficient groups, in order to make the (co)homology capable of detecting these shapes must have a certain level of torsion. If one uses real or rational coefficients these surfaces cannot be identified as such. However, the supplemental information will be encoded by the universal coefficient theorem when one attempts to change the coefficients in homology or cohomology to a coefficient group with torsion. Then, obstructions in the exact sequence describing the universal coefficient theorem will appear, in the form of Tor resp. Ext groups.

0-dimensional and 2-dimensional homology groups are each isomorphic to $\mathbb{Z}$. However, the first cohomology group $H^1(T_1;\mathbb{G})$ with coefficients in a group $\mathbb{G}$ is isomorphic to the group of homomorphisms from $\mathbb{Z} \oplus \mathbb{Z}$ to the group $\mathbb{G}$. This group $Hom(\mathbb{Z} \oplus \mathbb{Z}, \mathbb{G})$ is trivial if $\mathbb{G}$ is a torsion group. If not, it is a direct sum of copies of $\mathbb{G} \oplus \mathbb{G}$. Hence, the torsion of the coefficient group in cohomology determines the visibility of a torus as such. Otherwise, the information remains only encoded in the extension Ext that appears in the universal coefficient theorem used when one has to change the coefficient groups used in cohomology.

One can bring a similar argument for homology. There too one has coefficient groups, only that this time they appear in the formal expansion in terms of the simplexes defining the chain complex. It is important to mention that specific choices of the coefficient groups in homology and cohomology may completely alter the visibility of topological features. When considering the topological expansion for QCD this amounts to the mapping of the diagrams that can be represented on a simple torus into diagrams that can be represented on a sphere. The same is valid for the upper genera tori which may become indiscernible from their lower genus counterparts and finally from a sphere.

This amounts to great simplifications in the calculation of the large N expansions and may lead to new, topological factorization theorems. It is important to understand the role of the universal coefficient theorem in this construction. It essentially gives the homological algebraic obstruction to the visibility of a torus from the perspective of a (co)homology group with a given set of coefficients. The coefficient groups behave like an instrument that, if tuned appropriately, gives us the desired information about a specific topological property. If however, tuned differently they can mask some information. This masked information will reveal itself in the Ext and Tor groups of the universal coefficient theorem. If we go back to the representation of QCD as a genus expansion, we observe that the obstructions given by Ext or Tor manifest themselves as controlled, order by order deformations of the algebras of the gauge groups. Using for example $SO(N)$ as a gauge group instead of $SU(N)$, would transform the double arrows of the ribbon-graph representation of the gluon matrix into an un-oriented double line ribbon-graph. In this case, the gluing of diagrams would allow also Klein bottles as surfaces and the difference with respect to the oriented $SU(N)$ theory would be encoded in the form of the Tor obstruction in the universal coefficient theorem.

It is important to notice that the integration required to calculate the classes of diagrams summarized by a topological surface implies a measure of integration that depends only on the cohomology group.

Hence, what we are interested in is not the chain complex description of the space but, instead, the cohomology of the surface with a certain set of coefficients and the way it transforms under the universal coefficient theorem. Indeed, these two aspects encode the full information about the theory [23]. Therefore it appears that there exist a vast generalization of the notion of "renormalization" and "renormalization group" that may transform the difficult summation over topological genera into the calculation of mere planar or spherical graphs in QCD together with controllable deformations of the algebras encoding the higher genus diagrams. This may lead to an exact solution for the strong coupling regime of QCD and a solution of the confinement problem. Also, the same method can prove to be of major use in the domain of strongly correlated electrons and in strongly coupled condensed matter systems. On the theoretical side, this idea is based on the observation of Grothendieck [22-23] that in order to completely characterize a space one cannot rely only on the abstract notion of (co)homology but instead, one also has to consider the coefficient groups in (co)homology. In this way, this article proposes an application of several concepts from homological algebra to practical physical situations. The applications are not limited to QCD. Indeed, they can be applied to go beyond the perturbative domain of string theory and to better characterize the still mysterious M-theory. Due to the application of the large-N expansion to condensed matter, several other practical applications can be envisaged.

Chapter 12

Conclusions

"If you drink much from a bottle marked 'poison' it is certain to disagree with you sooner or later."

Lewis Carroll, Alice in Wonderland

The structure of this thesis was based on two parts. The first part may look somehow old-fashioned. Indeed, it represents work done either one hundred years ago or during the second half of the 20th century. However, how can we make any progress if we ignore the work of the predecessors? In fact, while working for the original research which stands at the fundaments of this thesis I reviewed many books written by the originators of the ideas related to general topology, homological algebra or algebraic topology [22],[24],[25],[26],[28],[29],[41],[43],[44]. It is important to understand where we stand now in order to be able to make any progress. It sometimes happens that physicists have the tendency to misinterpret mathematical notions in more than a single way. One of the mistakes physicists tend to make, and I include myself therein, is that we stop with the "understanding" process whenever we get the most superficial grasp of what a concept means. We often are not aware of the axiomatic definitions given by mathematicians and therefore use concepts that we understand only partially or not in their full complexity. Usually, the physicist's understanding is restricted to the most obvious applications. It often happens that our most commonly used concepts have far deeper meanings which, most of the time we do not see. Category theory is therefore a very important tool that can make us understand how concepts understood in a specific framework can also be understood and transferred to another framework.

At this moment, we do not have a categorial theory for quantum mechanics. In the same way, we do not have a precise axiomatic definition of quantum field theory that would allow us to derive certain properties of, say, QCD. The standard reply is "why do we need such a definition when, simply using the ad-hoc prescriptions, we get all the results we need?". The answer to this would be "how can we be sure that we obtain all the results we need?" We may not even come close to the full extent of the applications of axiomatization. The only difference is that most of the mathematicians are aware of this weakness while physicists tend not to care about it. I hope, with this thesis, I make the reader aware of the fact that we are unaware of what generalizations quantum field theories may support and what effects these may have on the way we understand reality.

This is mainly why this thesis also has a second part. There, I use the Universal Coefficient Theorem in order to get a new view on some results in physics and also to open some new paths for thinking.

We use to think about numbers in a very limited way. In some sense, we use them as artifacts with very little consideration for the way they have been developed. However, for a very long time human civilization knew only about integer numbers. Then rational numbers were invented. Irrational numbers and real numbers came as a surprise. For the pythagoreans the mere thought about irrationals was a blasphemy. However, today, we understand the role of numbers and number fields in far greater details. It would therefore be a pity not to employ this knowledge for the simplification of various physical problems. This was the original goal of this thesis, which, I hope, was reached in the pages above.

This thesis is not a closed work. Many future applications of the ideas presented here are possible. The main idea that has led to this research was based on the question "how does a numerical group of coefficients used to describe a problem affect the solvability of the problem"? I presented effects of the change in coefficients in various fields. Notably, observations related to the effects of finite fields brought me to observations of phenomena envisaged in string theory via the homological anomaly equations and the topological recursion relations. The same observations led me to a different way of understanding the wall-crossing formulas and the counting rules of BPS states. Future work may relate to other subjects of human inquiry. For example cryptography is based on the fact that some problems may look harder when solved in one direction and easier in the opposite direction. The discrete logarithm problem for example is based on the fact that, for some groups, the determination of the period of an element is a hard problem i.e. the determination of k such that $a^k = b$ is hard while the exponentiation can be computed easily. Cyclic subgroups of elliptic curves over finite fields generate something called "elliptic curve cryptography". The observation that changing the coefficient fields defining the elliptic curves can lead to equivalent but easier problems may be of major importance. Also the fact that a change in the coefficient group of the (co)homology has controllable effects defined by the universal coefficient theorem may lead to observations related to how difficult it really is to solve some discrete logarithm problems.

The homological algebraic side of the problem is also important. It is possible that some morphisms between group homologies may prove that some groups allow easier solutions for the discrete logarithm problem while being related in some way to groups for which the problem appears hard. This may lead to new domains where problems considered hard may appear in fact simple.

Hence, this thesis is at the intersection of several interesting subjects of modern research: quantum field theory, string theory, elliptic curves but also modular forms, modular arithmetics and cryptography. It may also be related to the vast domain of dualities between string theories and gauge field theories. The path it opens may lead to simpler solutions to some of the problems considered today to be so difficult that encryption protocols are being designed on the assumption that they cannot be solved efficiently. I hope, this path will not remain deserted in the future.

Chapter 13

Appendix: Some relevant proofs

This Appendix contains the proofs of the theorems, propositions and lemmas which were only mentioned in the main text. The expert reader may follow the main text only. The details and main arguments for the statements therein can be found here.

2.15 Lemma Let S be a subset of $\mathbb{R}$ which is bounded above and let p be the supremum of S. If S is a closed subset of $\mathbb{R}$, then $p \in S$.

Proof Suppose $p \in \mathbb{R} - S$. As $\mathbb{R} - S$ is open there exist real numbers a and b with $a < b$ such that $p \in (a, b) \subseteq \mathbb{R} - S$. As p is the least upper bound for S and $a < p$ it is clear that there exists an $x \in S$ such that $a < x$. Also $x < p < b$ and so $x \in (a, b) \subseteq \mathbb{R} - S$. But this is a contradiction since $x \in S$. Hence our supposition is false and $p \in S$.

2.16 Proposition Let T be a clopen subset of $\mathbb{R}$. Then either $T = \mathbb{R}$ or $\mathbb{T} = \emptyset$.

Proof Suppose $T \neq \mathbb{R}$ and $T \neq \emptyset$. Then there exists an element $x \in T$ and an element $z \in \mathbb{R} - T$. Without loss of generality, assume $x < z$. Put $S = T \cap [x, z]$. Then S, being the intersection of two closed sets, is closed. It is also bounded above, since z is obviously an upper bound. Let p be the supremum of S. By the previous Lemma, $p \in S$. Since $p \in [x, z]$, $p \leq z$. As $z \in \mathbb{R} - S$, $p \neq z$ and so $p < z$. Now T is also an open set and $p \in T$. So there exist a and b in $\mathbb{R}$ with $a < b$ such that $p \in (a, b) \subseteq T$. Let t be such that $p < t < min(b, z)$, where $min(b, z)$ denotes the smaller of b and z. So, $t \in T$ and $t \in [p, z]$. Thus $t \in T \cap [x, z] = S$. This is a contradiction since $t > p$ and p is the supremum of S. Hence out supposition is false and consequently $T = \mathbb{R}$ or $T = \emptyset$.

2.24 Lemma Let f be a function mapping $\mathbb{R}$ into itself. Then f is continuous if and only if for each $a \in \mathbb{R}$ and each open set U containing $f(a)$, there exists an open set V containing a such that $f(V) \subseteq U$.

Proof Assume that f is continuous. Let $a \in \mathbb{R}$ and let U be any open set containing $f(a)$. Then there exist real numbers c and d such that $f(a) \in (c, d) \subseteq U$. Put ϵ equal to the smaller of the two numbers $d - f(a)$ and $f(a) - c$, so that

$$(f(a) - \epsilon, f(a) + \epsilon) \subseteq U \tag{13.1}$$

As the mapping f is continuous, there exists a $\delta > 0$ such that $f(x) \in (f(a)-\epsilon, f(a)+\epsilon)$ for all $x \in (a-\delta, a+\delta)$. Let V be the open set $(a-\delta, a+\delta)$. Then $a \in V$ and $f(V) \subseteq U$ as required.

Reversely, assume that for each $a \in \mathbb{R}$ and each open set U containing $f(a)$ there exists an open set V containing a such that $f(V) \subseteq U$. We have to show that f is continuous. Let $a \in \mathbb{R}$ and ϵ be any positive real number. Put $U = (f(a) - \epsilon, f(a) + \epsilon)$. So U is open an open set containing $f(a)$. Therefore there exists an open set V containing a such that $f(V) \subseteq U$. As V is an open set containing a, there exist real numbers c and d such that $a \in (c, d) \subseteq V$. Put δ equal to the smaller of the two numbers $d-a$ and $a-c$, so that $(a - \delta, a + \delta) \subseteq V$. Then for all $x \in (a - \delta, a + \delta)$, $f(x) \in f(V) \subseteq U$ as required. So, f is continuous.

2.25 Lemma Let f be a mapping of a topological space (X, τ) into a topological space (Y, τ'). Then the following two conditions are equivalent:

- for each $U \in \tau'$, $f^{-1}(U) \in \tau$
- for each $a \in X$ and each $U \in \tau'$ with $f(a) \in U$, there exists a $V \in \tau$ such that $a \in V$ and $f(V) \subseteq U$.

Proof Assume that the first condition is satisfied. Let $a \in X$ and $U \in \tau'$ with $f(a) \in U$. Then $f^{-1}(U) \in \tau$. Put $V = f^{-1}(U)$ and we have that $a \in V$, $V \in \tau$ and $f(V) \subseteq U$. So, the second condition is satisfied. Reversely, assuming that the second condition is satisfied, let $U \in \tau'$. If $f^{-1}(U) = \emptyset$ then $f^{-1} \in \tau$. If $f^{-1} \neq \emptyset$, let $a \in f^{-1}(U)$. Then $f(a) \in U$. Therefore there exists a $V \in \tau$ such that $q \in V$ and $f(V) \subseteq U$. So for each $a \in f^{-1}(U)$ there exists a $V \in \tau$ such that $a \in V \subseteq f^{-1}(U)$. This implies that $f^{-1}(U) \in \tau$. So the first condition is satisfied.

Putting these two lemmas together we can see that $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for each open subset U of $\mathbb{R}$, $f^{-1}(U)$ is an open set.

2.29 Proposition Let (X, τ) and (Y, τ_1) be topological spaces. Then $f : (X, \tau) \to (Y, \tau_1)$ is continuous if and only if for every closed subset S of Y, $f^{-1}(S)$ is a closed subset of X.

Proof The result is direct if one observes that

$$f^{-1}(compl(S)) = compl(f^{-1}(S)) \tag{13.2}$$

where *compl* is the complement.

2.32 Proposition Let (X, τ) and (Y, τ_1) be topological spaces and $f : (X, \tau) \to (Y, \tau_1)$ surjective and continuous. If (X, τ) is connected then (Y, τ_1) is connected.

Proof Suppose (Y, τ_1) is not connected. Then it has a clopen subset U such that $U \neq \emptyset$ and $U \neq Y$. Then $f^{-1}(U)$ is an open set, since f is continuous, and also a closed set, that is, $f^{-1}(U)$ is a clopen subset of X. Now, $f^{-1}(U) \neq \emptyset$ as f is surjective and $U \neq \emptyset$. Also $f^{-1}(U) \neq X$ since if it were, U would equal Y by the surjectivity of f. Thus (X, τ) is not connected. This is a contradiction. Therefore (Y, τ_1) is connected.

2.34 Theorem Let $f : [a, b] \to \mathbb{R}$ be continuous and let $f(a) \neq f(b)$. Then for every number p between $f(a)$ and $f(b)$ there is a point $c \in [a, b]$ such that $f(c) = p$.

Proof As $[a, b]$ is connected and f is continuous we have that $f([a, b])$ is connected. This implies that $f([a, b])$ is an interval. now $f(a)$ and $f(b)$ are in $f([a, b])$. So, if p is between $f(a)$ and $f(b)$ then $p \in f([a, b])$, that is, $p = f(c)$ for some $c \in [a, b]$.

2.36 Corollary (The fixed point theorem) Let f be a continuous mapping of $[0, 1]$ into $[0, 1]$. Then there exists a $z \in [0, 1]$ such that $f(z) = z$. The point is called a fixed point.

Proof If $f(0) = 0$ or $f(1) = 1$ the result is obviously true. Thus it suffices to consider the case when $f(0) > 0$ and $f(1) < 1$. Let $g : [0, 1] \to \mathbb{R}$ be defined by $g(x) = x - f(x)$. Clearly g is continuous $g(0) = -f(0) < 0$ and $g(1) = 1 - f(1) > 0$ Consequently there exists a $z \in [0, 1]$ such that $g(z) = 0$ that is, $z - f(z) = 0$ or $f(z) = z$.

2.42 Proposition Let (X, d) be a metric space and τ the topology induced on X by the metric d. Then a subset U of X is open in (X, τ) if and only if for each $a \in U$ there exists an $\epsilon > 0$ such that the open ball $B_\epsilon(a) \subseteq U$.

Proof Assume $U \in \tau$. Then for any $a \in U$ there exists a point $b \in X$ and a $\delta > 0$ such that

$$a \in B_\delta(b) \subseteq U \tag{13.3}$$

Let then $\epsilon = \delta - d(a, b)$. Then

$$a \in B_\epsilon(a) \subseteq U \tag{13.4}$$

Reversely, assume that U is a subset of X with the property that for each $a \in U$ there exists an ϵ_a such that $B_{\epsilon_a}(a) \subseteq U$. Then U is an open set.

2.44 Proposition Let (X, d) be any metric space and τ the topology induced on the X by d. Then (X, τ) is Hausdorff.

Proof Let a and b be any points of X with $a \neq b$. Then $d(a, b) > 0$. Put $\epsilon = d(a, b)$. Consider the open balls $B_{\epsilon/2}(a)$ and $B_{\epsilon/2}(b)$. Then these are open sets in (X, τ) with $a \in B_{\epsilon/2}(a)$ and $b \in B_{\epsilon/2}(b)$. So, to show τ is Hausdorff we have to prove that $B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b) = \emptyset$.

Suppose $x \in B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b)$. Then $d(x, a) < \frac{\epsilon}{2}$ and $d(x, b) < \frac{\epsilon}{2}$. Hence

$$d(a, b \leq d(a, x) + d(x, b) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{13.5}$$

This says $d(a, b) < \epsilon$ which is false. Consequently there exists no $x \in B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b)$; that is, $B_{\epsilon/2}(a) \cap B_{\epsilon/2}(b) = \emptyset$, as required.

2.48 Proposition Let (X, d) be a metric space. A subset A of X is closed in (X, d) if and only if every convergent sequence of points in A converges to a point in A. This means that A is closed in (X, d) if and only if $a_n \to x$ where $x \in X$ and $a_n \in A$ for all n, implies $x \in A$.

Proof Assume that A is closed in (X, d) and let $a_n \to x$ where $a_n \in A$ for all positive integers n. Suppose that $x \in X - A$. Then, as $X - A$ is an open set containing x, there exists an open ball $B_\epsilon(x)$ such that $x \in B_\epsilon(x) \subseteq X - A$. Noting that each $a_n \in A$, this implies that $d(x, a_n) > \epsilon$ for each n. Hence the sequence $a_1, a_2, ..., a_n...$ does not converge to x. This is a contradiction. So, $x \in A$, as required. Conversely, assume that every convergent sequence of points in A converges to a point of A. Suppose that $X - A$ is not open. Then there exists a point $y \in X - A$ such that for each $\epsilon > 0$, $B_\epsilon(y) \cap Q \neq \emptyset$. For each positive integer n, let x_n be any point in $B_{1/n}(y) \cap A$. Then we claim that $x_n \to y$. To see this let ϵ be any positive real number, and n_0 any integer greater than $\frac{1}{\epsilon}$. Then for each $n \geq n_0$

$$x_n \in B_{1/n}(y) \subseteq B_{1/n_0}(y) \subseteq B_\epsilon(y) \tag{13.6}$$

So $x_n \to y$ and by our assumption, $y \in A$. This is a contradiction and so $X - A$ is open and thus A is closed in (X, d).

3.2 Lemma Any point x in the simplex can be written as $x = \sum_i x_i v_i$ with $x_i \geq 0$ and $\sum_i x_i = 1$. The x_i are called barycentric coordinates and they are unique.

Proof In general we can always translate a simplex into another one with $v_0 = 0$. Now, if $v_0 = 0$ then $v_1, ..., v_i$ have to be independent. So, $x = \sum_{p=0}^{i} x_p v_p = \sum_{p=1}^{i} x_p v_p$ hence x_p, $p > 0$ are determined by x and then so is $x_0 = 1 - \sum_{p=1}^{i} x_p$.

3.8 Lemma The above formula for ∂_i gives a well defined $\mathbb{K}$-map $\partial_i : C_i(X, \mathcal{T}; \mathbb{K}) \to C_{i-1}(X, \mathcal{T}; \mathbb{K})$.

Proof

- First it is verified that the formula only depends on the orientation. For instance for two orderings xyz and zxy which give the same orientation one has $\partial\sigma_{zxy} = \sigma_{xy} - \sigma_{zy} + \sigma_{zx}$ and $\partial\sigma_{xyz} = \sigma_{yz} - \sigma_{xz} + \sigma_{xy}$ coincide.
- Second, it is also verified that the map descends to $C_i \to C_{i-1}$ i.e. that the opposite orientations produce opposite results.

The two verified requirements signify that for any permutation one has the action of the boundary operator expressed in terms of the sign of the permutation.

4.16 Definition(Quasi-isomorphisms) We say that a map of complexes $f : A^* \to B^*$ is a quasi-isomorphism if the induced maps of cohomology groups $H^n(f) : H^n(A^*) \to H^n(B^*)$, $n \in Z$ are all isomorphisms.

4.17 Lemma A left resolution of M is the same as a quasi-isomorphism of complexes $P^* \to M^\#$ such that $P^i = 0$ for $i > 0$.

Proof If (P^*, q) is a resolution of M then the only non-zero cohomology group of P^* is $H^0(P^*) \cong M$, the same being true for $M^\#$. Moreover the morphism of complexes $P^* \to M^\#$ is given by $q : P^0 \to M$ which induces isomorphisms of $H^0(P^*) = P^0/dP^{-1}$ onto $M = H^0(M^\#)$.

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