MATHEMATICS OF COMPUTATION
Volume 00, Number 0, Pages 000–000
S 0025-5718(XX)0000-0

A STABILIZED NONCONFORMING FINITE ELEMENT METHOD FOR THE ELLIPTIC CAUCHY PROBLEM

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ABSTRACT. In this paper we propose a nonconforming finite element method for the solution of the ill-posed elliptic Cauchy problem. The recently derived framework from [8, 9] is extended to include the case of a nonconforming approximation space and we show that the use of such a space allows us to reduce the amount of stabilization necessary for convergence, even in the case of ill-posed problems. We derive error estimates using conditional stability estimates in the L^2-norm.

1. Introduction

We consider the Cauchy problem for Poisson's equation in a bounded domain. This problem is known to be severely ill-posed in the sense of Hadamard [16, 5, 2]. The ill-posedness makes numerical approximation challenging and different regularization methods have been proposed, such as Tikhonov regularization [26] or the quasi reversibility method introduced by Lattès and Lions [23].

Various finite element approaches for the solution of the elliptic Cauchy problem have been suggested in the literature. Some are based on standard Galerkin formulations, but rely on structured meshes or a special form of the continuous problem for stability [15, 24, 25]. Some use the above mentioned regularization techniques to ensure stability [3, 4, 6, 7, 13] a related approach is to recast the problem as a minimization problem [11, 18, 17], possibly with regularization.

The objective of the present work is to draw on the ideas of [8, 9] and propose a consistent stabilization of a nonconforming finite element method. The upshot is that the use of nonconforming elements allows us to reduce the stabilization. Indeed instead of penalizing the jump of the gradient as in [9], we may use the standard penalty operator acting on the jumps of the solution u_h, known from previous works on well-posed problems [19, 20, 10]. This shows that in spite of the ill-conditioning of the problem under study, the choice of the finite element spaces is of importance and leads to methods with different properties.

The structure of the method resembles to that introduced in [6], but the method proposed in [6], is of the form regularize first and then discretize, whereas we herein choose to discretize first and then regularize the discrete formulation. The idea is then to choose a stabilization/regularization that makes sense only for the discrete solution, indeed it is zero when applied to functions in $H^1(\Omega)$. The resulting method is a coupled primal/adjoint formulation where the adjoint solution of the exact (unperturbed) problem is zero, giving a large flexibility in the stabilization of the adjoint. The solution to the adjoint problem is the Lagrange multiplier of

2010 *Mathematics Subject Classification.* Primary 65N12, 65N15, 65N20, 65N21, 65N30.

an associated minimization problem (see [8]) and measures the sensitivities of the system. As we shall see below it plays an important role for the derivation of error estimates.

The fact that the stabilization is consistent allows us to derive error estimates using discrete stability and the conditional stability on data of the partial differential equation. We follow an approach similar to that suggested in [9], but in this case an inf-sup condition is necessary for the discrete stability. The error bound is in a posteriori form, using a residual quantity together with the conditional stability. Thanks to the primal/adjoint stabilization the residual terms can be shown to be optimally convergent independent of the stability of the underlying problem, for sufficiently smooth solutions.

We also show how perturbed data can be introduced in the analysis and discuss how a posteriori control of the mesh refinement may include the effect of perturbations, provided their magnitude is known.

2. The elliptic Cauchy problem

The problem that we are interested in takes the form: find $u : \Omega \mapsto \mathbb{R}$ such that

$$\left\{ \begin{array}{rcl} -\Delta u & = & f, \text{ in } \Omega \\ u & = & 0 \text{ on } \Gamma_D \\ \nabla u \cdot n & = & \psi \text{ on } \Gamma_N \end{array} \right. \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ is a polyhedral (polygonal) domain and Γ_N, Γ_D denote polygonal subsets of the boundary $\partial\Omega$, with union $\Gamma_B := \Gamma_D \cup \Gamma_N$ and that overlap on some set of nonzero $d-1$ measure, $\Gamma_C := \Gamma_D \cap \Gamma_N \neq \emptyset$. We denote the complement of the Dirichlet boundary $\Gamma_D' := \partial\Omega \setminus \Gamma_D$, the complement of the Neumann boundary $\Gamma_N' := \partial\Omega \setminus \Gamma_N$ and the complement of their union $\Gamma_B' := \partial\Omega \setminus \Gamma_B$. To exclude the well-posed case, we assume that the $d-1$-measure of Γ_B' is non-zero. The practical interest of (2.1) stems from engineering problems where the exact boundary condition is unknown on Γ_B', but additional measurements ψ of the fluxes are available on a part of the accessible boundary Γ_C. This results in an ill-posed reconstruction problem, that in practice most likely does not have a solution due to measurement errors in the fluxes [5]. However if the underlying physical process is stable, (in the sense that the problem where full boundary data are known is well-posed) we may assume that it allows for a unique solution in the idealized situation of unperturbed data. This is the approach we will take below. To this end we assume that $f \in L^2(\Omega)$, $\psi \in H^{\frac{1}{2}}(\Gamma_N)$ and that a unique $u \in H^2(\Omega)$ satisfies (2.1). We analyse this idealized situation using conditional stability, the condition being the existence of $u \in H^1(\Omega)$, and then use a perturbation argument to include the effect of measurement errors.

For the derivation of a weak formulation we introduce the spaces $V := \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$ and $W := \{v \in H^1(\Omega) : v|_{\Gamma_N'} = 0\}$, both equipped with the H^1-norm and with dual spaces denoted by V' and W'.

Using these spaces we obtain a weak formulation: find $u \in V$ such that

$$a(u, w) = l(w) \quad \forall w \in W, \tag{2.2}$$

where

$$a(u, w) = \int_\Omega \nabla u \cdot \nabla w \, \mathrm{d}x,$$

and

$$l(w) := \int_\Omega f w \, \mathrm{d}x + \int_{\Gamma_N} \psi \, w \, \mathrm{d}s.$$

We will use the notation $(\cdot,\cdot)_X$ for the L^2-scalar product over X and for the associated norm we write $\|x\|_X := (x,x)_X^{\frac{1}{2}}$. The H^s-norm will be denoted by $\|\cdot\|_{H^s(\Omega)}$ and we identify the norms on V and W with the H^1-norm, $\|\cdot\|_V = \|\cdot\|_W = \|\cdot\|_{H^1(\Omega)}$. Observe that we may not assume that the problem is well-posed for general $l(\cdot) \in W'$. Indeed since $u \notin W$ coercivity fails and inf-sup stability does not hold either in general [5].

Remark 2.1. The restriction to homogeneous Dirichlet conditions in (2.1) is made only to reduce notation and technical detail. It is straightforward to extend the formulation and the analysis below to include non-homogeneous Dirichlet data.

2.1. **Conditional stability.** The problem (2.1) is ill-posed and for our analysis we will only use a conditional stability result linking the size of some functional of the solution to the size of data. Consider a functional $j : V \mapsto \mathbb{R}$. Let $\Xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous, monotone increasing function with $\lim_{x\to 0^+} \Xi(x) = 0$. The conditional stability that we need for the analysis may be written on the following abstract form. We assume that the solution exists in some suitable Sobolev space and that, if for some sufficiently small $\epsilon > 0$, there holds

$$\|l\|_{W'} \leq \epsilon \text{ in (2.2) then } |j(u)| \leq \Xi(\epsilon). \tag{2.3}$$

It is known [2, Theorems 1.7 and 1.9 with Remark 1.8] that if there exists a solution $u \in H^1(\Omega)$, with $E := \|u\|_{H^1(\Omega)}$ to (2.1), a conditional stability of the form (2.3) holds for $0 < \epsilon < 1$ and

$$\begin{aligned} &j(u) := \|u\|_{L^2(\omega)},\ \omega \subset \Omega :\ \mathrm{dist}(\omega, \Gamma'_B) =: d_{\omega,\Gamma'_B} > 0 \\ &\text{with } \Xi(x) = C(E)x^\varsigma,\ C(E) > 0,\ \varsigma := \varsigma(d_{\omega,\Gamma'_B}) \in (0,1) \end{aligned} \tag{2.4}$$

and for

$$\begin{aligned} &j(u) := \|u\|_{L^2(\Omega)} \text{ with } \Xi(x) = C_1(E)(|\log(x)| + C_2(E))^{-\varsigma} \\ &\text{with } C_1(E), C_2(E) > 0,\ \varsigma \in (0,1). \end{aligned} \tag{2.5}$$

Strictly speaking [2] only considers the case $\Gamma_D \equiv \Gamma_N$, but the result also holds in the case with a pure Dirichlet or Neumann boundary part (one only needs to verify that the extension Theorem 6.2 still holds. See also discussion on page 11 of [2]). The constants above also depend on the geometry of the problem. Note that to derive these results $l(\cdot)$ is first associated with its Riesz representant in W (c.f. [2, equation (1.31)] and discussion.) It should also be noted that the interval $(0,1)$ for ϵ above can be extended provided that the constants in the estimates are rescaled accordingly.

3. The nonconforming stabilized method

Let $\{\mathcal{T}_h\}_h$ denote a family of shape regular and quasi uniform tessellations of Ω into nonoverlapping simplices, such that for any two different simplices κ, $\kappa' \in \mathcal{T}_h$, $\kappa \cap \kappa'$ consists of either the empty set, a common face or edge, or a common vertex. The diameter of a simplex κ will be denoted h_κ and the outward pointing normal n_κ. The family $\{\mathcal{T}_h\}_h$ is indexed by the maximum element-size of $\mathcal{T}_h$, $h := \max_{\kappa \in \mathcal{T}_h} h_\kappa$. We denote the set of element faces in $\mathcal{T}_h$ by $\mathcal{F}$ and let $\mathcal{F}_i$ denote the set of interior faces and $\mathcal{F}_\Gamma$ the set of faces in some $\Gamma \subset \partial\Omega$. To each face $F \in \mathcal{F}$ we associate the mesh parameter $h_F := \mathrm{diam}(F)$. We will assume that the mesh is fitted to the subsets of $\partial\Omega$ representing the boundary conditions Γ_D and Γ_N, so that the boundaries of these subsets coincide with the boundaries of subsets of element faces. To each face F we associate a unit normal vector, n_F. For interior faces its orientation is arbitrary, but fixed. On the boundary $\partial\Omega$ we identify n_F with the outward pointing normal of $\partial\Omega$. The subscript on the normal is dropped in cases where it follows from the context.

We define the jump over interior faces $F \in \mathcal{F}_i$ by $[v]|_F := \lim_{\epsilon\to 0^+}(v(x|_F - \epsilon n_F) - v(x|_F + \epsilon n_F))$ and for faces on the boundary, $F \in \partial\Omega$, we let $[v]|_F := v|_F$. Similarly we define the average of a function over an interior face F by $\{v\}|_F := \frac{1}{2}\lim_{\epsilon\to 0^+}(v(x|_F - \epsilon n_F) + v(x|_F + \epsilon n_F))$ and for F on the boundary we define $\{v\}|_F := v|_F$. The classical nonconforming space of piecewise affine finite element functions (see [12]) then reads

$$X_h^\Gamma := \{v_h \in L^2(\Omega) : \int_F [v_h] \;\mathrm{d}s = 0, \forall F \in \mathcal{F}_i \cup \mathcal{F}_\Gamma \text{ and } v_h|_\kappa \in \mathbb{P}_1(\kappa), \forall \kappa \in \mathcal{T}_h\}$$

where $\mathbb{P}_1(\kappa)$ denotes the set of polynomials of degree less than or equal to one restricted to the element κ and Γ denotes some portion of the boundary $\partial\Omega$ consisting of a union of a subset of boundary element faces. If Γ is omitted no constraints are set on the boundary elements. We may then define the spaces $V_h := X_h^{\Gamma_D}$ and $W_h := X_h^{\Gamma'_N}$. We recall the interpolation operator $r_h : H^1(\Omega) \to X_h$ defined by the relation

$$\overline{\{r_h v\}}|_F := |F|^{-1} \int_F \{r_h v\} \;\mathrm{d}s = |F|^{-1} \int_F v \;\mathrm{d}s$$

for every $F \in \mathcal{F}$ and with $|F|$ denoting the $(d-1)$-measure of F. Observe that $r_h : W \mapsto W_h$ and $r_h : V \mapsto V_h$. It is convenient to introduce the broken norms

$$\|x\|_h^2 := \sum_{\kappa\in\mathcal{T}_h} \|x\|_\kappa^2, \quad \|x\|_{1,h}^2 := \|x\|_h^2 + \|\nabla x\|_h^2 \text{ and } \|x\|_{\mathcal{F}}^2 := \sum_{F\in\mathcal{F}} \|x\|_F^2.$$

The following inverse and trace inequalities are well known

$$\begin{aligned} &\|v\|_{\partial\kappa} \leq C_t(h_\kappa^{-\frac{1}{2}}\|v\|_\kappa + h_\kappa^{\frac{1}{2}}\|\nabla v\|_\kappa), \forall v \in H^1(\kappa) \\ &h_\kappa\|\nabla v_h\|_\kappa + h_\kappa^{\frac{1}{2}}\|v_h\|_{\partial\kappa} \leq C_i\|v_h\|_\kappa, \quad \forall v_h \in X_h. \end{aligned} \tag{3.1}$$

Using the inequalities of (3.1) and standard approximation results from [12] it is straightforward to show the following approximation results of the interpolant r_h

$$\begin{aligned} \|u - r_h u\| + h\|\nabla(u - r_h u)\|_h &\leq Ch^t|u|_{H^t(\Omega)},\ t \in \{1,2\} \\ \|h^{-\frac{1}{2}}(u - r_h u)\|_{\mathcal{F}} &\leq Ch^{t-1}|u|_{H^t(\Omega)},\ t \in \{1,2\} \\ \|h^{\frac{1}{2}} n \cdot \nabla(u - r_h u)\|_{\mathcal{F}} &\leq Ch|u|_{H^2(\Omega)}. \end{aligned} \tag{3.2}$$

It will also be useful to bound the L^2-norm of the interpolant r_h by its values on the element faces. To this end we prove a technical lemma.

Lemma 3.1. *For any function $v_h \in X_h$ there holds*

$$\|h^{-1}v_h\|_\Omega \leq c_{\mathcal{T}} \left(\sum_{F\in\mathcal{F}} h_F^{-1}\|\overline{\{v_h\}}\|_F^2\right)^{\frac{1}{2}}.$$

Proof. It follows by norm equivalence of discrete spaces on the reference element and a scaling argument (under the assumption of shape regularity) that for all $\kappa \in \mathcal{T}_h$

$$\|v_h\|_\kappa^2 \leq C \sum_{F \in \partial\kappa} h_F \|\overline{v}_h\|_F^2. \tag{3.3}$$

The claim follows by shape regularity and by summing over the elements of $\mathcal{T}_h$ and recalling that $\|\overline{v}_h\|_F^2 = \|\overline{\{v_h\}}\|_F^2$. □

Following [6, 8] we propose the formulation: find $(u_h, z_h) \in V_h \times W_h$ such that,

$$\begin{aligned} a_h(u_h, w_h) - s_W(z_h, w_h) &= l(w_h) \\ a_h(v_h, z_h) + s_V(u_h, v_h) &= 0 \end{aligned} \tag{3.4}$$

for all $(v_h, w_h) \in V_h \times W_h$. Here the bilinear forms are defined by

$$a_h(u_h, w_h) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \nabla u_h \cdot \nabla w_h \, \mathrm{d}x,$$

$$s_W(z_h, w_h) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \gamma_W \nabla z_h \cdot \nabla w_h \, \mathrm{d}x \tag{3.5}$$

or alternatively

$$s_W(z_h, w_h) := \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma'_N}} \int_F \gamma_W h_F^{-1} [z_h][w_h] \, \mathrm{d}s \tag{3.6}$$

and finally

$$s_V(u_h, v_h) := \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_D}} \int_F \gamma_V h_F^{-1} [u_h][v_h] \, \mathrm{d}s. \tag{3.7}$$

For cases where the construction of the spaces V_h and W_h with Dirichlet conditions set on different parts of the boundary is inconvenient we propose the following formulation using weak imposition of the boundary conditions in a fashion reminiscent of Nitsche's method. Find $(u_h, z_h) \in X_h \times X_h$ such that,

$$\begin{aligned} a_h(u_h, w_h) - b_h(u_h, w_h) - s_W(z_h, w_h) &= l(w_h) \\ a_h(v_h, z_h) - b_h(v_h, z_h) + s_V(u_h, v_h) &= 0 \end{aligned} \tag{3.8}$$

for all $(v_h, w_h) \in X_h \times X_h$. The boundary term $b_h(\cdot, \cdot)$ is defined by

$$b_h(v_h, w_h) := \sum_{F \in \mathcal{F}_{\partial\Omega}} \left(\int_{F \cap \Gamma'_N} n \cdot \nabla v_h \, w_h \, \mathrm{d}s + \int_{F \cap \Gamma_D} n \cdot \nabla w_h \, v_h \, \mathrm{d}s \right) \tag{3.9}$$

and we modify the stabilization $s_W(\cdot, \cdot)$ so that the stabilization parameter may be chosen differently in the interior and on the boundary,

$$s_W(z_h, w_h) := \sum_{\kappa \in \mathcal{T}_h} \int_\kappa \gamma_W \nabla z_h \cdot \nabla w_h \, \mathrm{d}x + \sum_{F \in \mathcal{F}_{\Gamma'_N}} \int_F \gamma_{W,bc} h_F^{-1} z_h w_h \, \mathrm{d}s, \tag{3.10}$$

or alternatively

$$s_W(z_h,w_h) := \sum_{F\in\mathcal{F}_i}\int_F \gamma_W h_F^{-1}[z_h][w_h]\ \mathrm{d}s + \sum_{F\in\mathcal{F}_{\Gamma'_N}}\int_F \gamma_{W,bc} h_F^{-1} z_h w_h\ \mathrm{d}s. \tag{3.11}$$

For the stabilization term $s_V(\cdot,\cdot)$ of equation (3.7) we may consider a similar distinction between the penalty parameter in the interior γ_V and on the boundary $\gamma_{V,bc}$. To reduce the notation we will never make a distinction between the penalty parameter in the bulk and on the boundary in the analysis, but their choice will be discussed in the numerical section. The penalty parameters γ_V, γ_W, $\gamma_{W,bc}$ and $\gamma_{V,bc}$ are all strictly positive and independent of the mesh size h. They play a role similar to the regularization parameter in standard Tikhonov regularization, but the operators they are associated with makes sense only when applied to the discrete approximation space. Indeed the jump penalty vanishes when applied to the exact solution.

We also propose the compact form: find $(u_h,z_h)\in\mathcal{V}_h := X_h\times X_h$ such that,

$$A_h[(u_h,z_h),(v_h,w_h)] = l(w_h)$$

for all $(v_h,w_h)\in\mathcal{V}_h$. The bilinear form is then given by

$$\begin{aligned} A_h[(u_h,z_h),(v_h,w_h)] := a_h(u_h,w_h) - b_h(u_h,w_h) - s_W(z_h,w_h)\\ + a_h(v_h,z_h) - b_h(v_h,z_h) + s_V(u_h,v_h). \end{aligned}$$

If (u_h,z_h) and (v_h,w_h) are restricted to $V_h\times W_h$ in (3.8) we recover the formulation (3.4), since $V_h\times W_h$ is in the kernel of the operator $b_h(\cdot,\cdot)$. We therefore present the analysis for (3.8) however a similar analysis is valid for (3.4), simply by omitting the contributions from $b_h(\cdot,\cdot)$ and instead using the zero average property to treat boundary terms. Observe that for (3.5) and (3.10), by Poincaré's inequality there exists $c_1,c_2>0$ so that

$$c_1\gamma_W^{\frac12}\|w_h\|_{1,h} \le s_W(w_h,w_h)^{\frac12} \le c_2\gamma_W^{\frac12}\|w_h\|_{1,h}, \forall w_h\in W_h.$$

This norm equivalence is important for stability when there are perturbations in data (see Lemma 5.5). For the weaker adjoint stabilization (3.6) only the upper bound holds. For the part of the analysis considering unperturbed data the stability obtained by (3.6) is sufficient and the analysis is identical. However in Section 5.1 where perturbed data are considered, the two approaches lead to slightly different estimates. The operator (3.6) has the advantage of being adjoint consistent, but since duality arguments are not used herein this has no impact on the results presented below. The stabilization (3.10) will be considered in the analysis, but we will outline in remarks how the arguments change if (3.6) is used. We will then compare the behavior of the two operators numerically.

4. Stability estimates

The issue of stability of the discrete formulation is crucial since we have no coercivity or inf-sup stability of the continuous formulation (2.2) to rely on. By taking $v_h=u_h$ and $w_h=-z_h$, and defining the semi-norm

$|v_h|_{s_V} := s_V(v_h,v_h)^{\frac12}$, $\forall v_h\in V_h$ and the norm $\|w_h\|_{s_W} := s_W(w_h,w_h)^{\frac12}$, $\forall w_h\in W_h$ we obtain the stability estimate

$$|u_h|_{s_V}^2 + \|z_h\|_{s_W}^2 = -l(z_h) \tag{4.1}$$

showing that we have control of z_h and of the nonconforming part of the approximation of u_h. If the stabilization operator (3.6) is used, $\|\cdot\|_{s_W}$ is a semi-norm similar to $|\cdot|_{s_V}$. The stability (4.1) is of course insufficient for any useful analysis, however we will use it here as a starting point for an inf-sup argument that implies existence of a unique discrete solution. To this end we introduce a mesh-dependent norm

$$|||v_h|||_V := \gamma_V^{\frac{1}{2}}\|h\nabla v_h\|_h + \gamma_V^{\frac{1}{2}}\|h^{\frac{1}{2}}[n\cdot\nabla v_h]\|_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} + |v_h|_{s_V}. \tag{4.2}$$

Observe that the first term on the right hand side of (4.2), scales differently than the two other terms. The reason for this is that stability of the first term is obtained using a Poincaré inequality, involving a constant with the dimension of a length scale depending on Ω. To simplify the presentation we have assumed that this constant is $O(1)$ and do not track it. The following approximation estimate is an immediate consequence of (3.2),

$$|||v - r_h v|||_V \le C\gamma_V^{\frac{1}{2}} h|v|_{H^2(\Omega)}, \quad \forall v \in H^2(\Omega). \tag{4.3}$$

We will also use the composite norm

$$|||(u_h, z_h)||| := |||u_h|||_V + \|z_h\|_{s_W}. \tag{4.4}$$

When the formulation (3.4) is used Dirichlet boundary conditions are set weakly on Γ_D in V_h and on Γ_N' in W_h and when formulation (3.8) is used the corresponding penalty term on Γ_N' is included in $s_W(\cdot,\cdot)$, therefore $|||(u_h, z_h)|||$ is a norm, when (3.10) is used. When using (3.6), it is only a semi norm, however in that case the jump of ∇z_h and $\|hz_h\|_{1,h}$ can be included in the norm (4.4) above. We now prove a fundamental stability result for the discretization (3.8) (valid also for (3.4) after minor modifications).

Theorem 4.1. *Assume that $(\gamma_V\gamma_W) \le 1$. Then there exists a positive constant c_s independent of γ_V, γ_W and h, but not of the mesh geometry, such that there holds*

$$c_s|||(x_h, y_h)||| \le \sup_{(v_h,w_h)\in\mathcal{V}_h} \frac{A_h[(x_h, y_h), (v_h, w_h)]}{|||(v_h, w_h)|||}.$$

Proof. First we recall the positivity

$$|x_h|_{s_V}^2 + \|y_h\|_{s_W}^2 = A_h[(x_h, y_h), (x_h, -y_h)].$$

Then observe that by integrating by parts in the bilinear form $a_h(\cdot,\cdot)$, using the equality $ab-cd = \frac{1}{2}(a-c)(b+d)+\frac{1}{2}(a+c)(b-d)$ and the zero mean value property of the jump of x_h on interior faces we have

$$a_h(x_h, w_h) = \sum_{F\in\mathcal{F}} \int_F [n_F\cdot\nabla x_h]\{w_h\}\, \mathrm{d}s.$$

Define the function $\xi_h \in W_h$ such that for every face $F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}$

$$\overline{\{\xi_h\}}|_F := \gamma_V h_F [n_F\cdot\nabla x_h]|_F.$$

This is possible in the nonconforming finite element space since the degrees of freedom may be identified with the average value of the finite element function on an element face. Using Lemma 3.1 we have

$$\|h^{-1}\xi_h\|_\Omega^2 \le c_{\mathcal{T}}^2 \sum_{F\in\mathcal{F}_i\cup\mathcal{F}_C} \gamma_V^2\|h_F^{\frac{1}{2}}[n_F\cdot\nabla x_h]\|_F^2. \tag{4.5}$$

Testing with $w_h = \xi_h$ and $v_h = 0$ we get

$$\gamma_V \|h^{\frac{1}{2}}[n_F \cdot \nabla x_h]\|^2_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} = A_h[(x_h, y_h), (0, \xi_h)] + b_h(x_h, \xi_h) + s_W(y_h, \xi_h). \tag{4.6}$$

To bound the second term in the right hand side we proceed as follows,

$$\begin{aligned} b_h(x_h, \xi_h) \leq & \left(\sum_{F \in \mathcal{F}_{\Gamma_D}} \|h^{\frac{1}{2}} \nabla \xi_h\|^2_F \right)^{\frac{1}{2}} \|h^{-\frac{1}{2}} x_h\|_{\Gamma_D} \\ & \leq C_i C_t c_{\mathcal{T}} \gamma_V^{\frac{1}{2}} \|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} |x_h|_{s_V} \\ & \leq (C_i C_t c_{\mathcal{T}})^2 |x_h|^2_{s_V} + \frac{1}{4} \gamma_V \|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|^2_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}}. \end{aligned}$$

For the stabilization term in the right hand side of (4.6) we have the upper bounds, using the inverse inequality (trace inequality if (3.6) is used) (3.1)(ii) and (4.5)

$$\begin{aligned} s_W(y_h, \xi_h) \leq \|y_h\|_{s_W} \|\xi_h\|_{s_W} \leq C_i \|y_h\|_{s_W} \gamma_W^{\frac{1}{2}} \|h^{-1} \xi_h\|_\Omega \\ \leq C_i c_{\mathcal{T}} \|y_h\|_{s_W} (\gamma_V \gamma_W)^{\frac{1}{2}} \gamma_V^{\frac{1}{2}} \|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} \\ \leq (C_i c_{\mathcal{T}})^2 \|y_h\|^2_{s_W} + \frac{1}{4} \gamma_V \|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|^2_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}}. \end{aligned}$$

Where we used that $\gamma_V \gamma_W < 1$ and the arithmetic-geometric inequality for the third bound. The consequence of this is that for $\alpha = \frac{1}{2} + (C_i c_{\mathcal{T}})^2 \max(1, C_t^2)$ we have

$$\begin{aligned} \frac{1}{2} \left(|x_h|^2_{s_V} + \|y_h\|^2_{s_W} + \gamma_V \|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|^2_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} \right) \\ \leq A_h[(x_h, y_h), (\alpha x_h, -\alpha y_h + \xi_h)]. \end{aligned} \tag{4.7}$$

To include the control of the gradient of x_h we use a well-known discrete Poincaré inequality for piecewise constant functions [14]

$$\|\nabla x_h\|^2_h \leq C \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} h_F^{-1} \|[\nabla x_h]\|^2_F.$$

The right hand side is now upper bounded by decomposing the jump of the gradient on its normal and tangential part and applying the inverse inequality

$$\|h^{\frac{1}{2}}[(I - n_F \otimes n_F) \nabla x_h]\|_F \leq C \|h^{-\frac{1}{2}}[x_h]\|_F$$

in the latter. Relating the right hand side to the quantities in $|||\cdot|||_V$ already controlled in (4.7), this leads to the upper bound

$$\|\nabla x_h\|_h \leq C h^{-1} (\|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} + \gamma_V^{-\frac{1}{2}} |x_h|_{s_V}).$$

and hence

$$h \gamma_V^{\frac{1}{2}} \|\nabla x_h\|_h \leq C (\gamma_V^{\frac{1}{2}} \|h^{\frac{1}{2}}[n \cdot \nabla x_h]\|_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} + |x_h|_{s_V}).$$

We may conclude that there exists a positive constant $c_0 > 0$ independent of γ_V, γ_W and h such that

$$c_0 |||(x_h, y_h)|||^2 \leq A_h[(x_h, y_h), (\alpha x_h, -\alpha y_h + \xi_h)].$$

To end the proof we need to prove the stability of ξ_h in the triple norm. By the triangle inequality

$$|||(\alpha x_h, -\alpha y_h + \xi_h)||| \leq \alpha |||(x_h, y_h)||| + |||(0, \xi_h)|||.$$

Using now an inverse inequality followed by the inequality (4.5) we arrive at

$$|||(0,\xi_h)||| = \|\xi_h\|_{s_W} \leq \gamma_W^{\frac{1}{2}} C_i \|h^{-1}\xi_h\|_\Omega \leq C_i c_{\mathcal{T}} (\gamma_W \gamma_V)^{\frac{1}{2}} |||x_h|||_V.$$

Collecting terms we see that

$$|||(\alpha x_h, -\alpha y_h + \xi_h)||| \leq (\alpha + C_i c_{\mathcal{T}}) |||(x_h, y_h)|||.$$

This conclude the proof with $c_s = c_0/(\alpha + C_i c_{\mathcal{T}})$. □

Remark 4.2. Observe that the above analysis is restricted to the method using (3.10). If the stabilization operator defined by equation (3.6) is used the relation (4.1) and the norm (4.4) only holds on the semi-norm defined by (3.6). A contribution $\|h\nabla z_h\|_h$ may then be included in (4.4) with stability shown using a similar argument as above. The control of the dual variable in this case is nevertheless weaker than that provided using (3.11). This will have consequences for the perturbation analysis below.

Corollary 4.3. *The formulation* (3.8) *admits a unique solution* (u_h, z_h).

Proof. The system matrix corresponding to (3.4) is a square matrix and we only need to show that there are no zero eigenvalues. Assume that $l(w_h) = 0$. It then follows by Theorem 4.1 that for any solution (u_h, z_h) there holds

$$c_s |||(u_h, z_h)||| \leq \sup_{(v_h, w_h) \in \mathcal{V}_h} \frac{A_h[(u_h, z_h), (v_h, w_h)]}{|||(v_h, w_h)|||} = 0,$$

implying that $u_h = 0$, $z_h = 0$ which shows that the solution is unique. □

5. Error estimates

Even though Theorem 4.1 provides us with a stability estimate for the formulation, the norm is not sufficiently strong to allow for a proof of convergence. Indeed the only notion of stability at our disposal that can allow us to prove error estimates are (2.4) and (2.5). We will follow the approach introduced in [9] and first prove that $|||(u - u_h, z_h)||| \leq Ch|u|_{H^2(\Omega)}$. This tells us that the stabilization terms must vanish at an optimal rate for smooth u and that $\|\nabla u_h\|_h + \|\nabla z_h\|_h$ is uniformly bounded as $h \to 0$. Using this a priori bound we may conclude that the H^1-conforming part of u_h is uniformly bounded in H^1. This allows us to write the error $u - u_h$ as $u - \tilde{u}_h + \tilde{u}_h - u_h = \tilde{e} + e_h$, where $\tilde{u}_h$ denotes the V-conforming part of u_h. We may then control the part $\tilde{e}$ using the conditional stability estimates (2.4) and (2.5), while e_h is shown to be bounded by the stabilization.

Before proving the main result we introduce two technical Lemmas that will be useful in the analysis. Using the regularity assumptions on the data in $l(\cdot)$ it is straightforward to show that the formulation satisfies the following weak consistency

Lemma 5.1. *(Weak consistency) Let* $u \in H^2(\Omega)$ *be the solution of* (2.1), *with* $f \in L^2(\Omega)$ *and* $\psi \in H^{\frac{1}{2}}(\Gamma_N)$ *and let* $(u_h, z_h) \in \mathcal{V}_h$ *be the solution of* (3.4) *then, for all* $w_h \in W_h$, *there holds,*

$$(5.1)\quad |a_h(u_h - u, w_h) - b_h(u_h - u, w_h) - s_W(z_h, w_h)| \leq \sum_{F \in \mathcal{F}_i} \inf_{\nu_h \in V_h} \int_F |n_F \cdot (\nabla u - \{\nabla \nu_h\})||[w_h]|\, ds.$$

Proof. Multiplying (2.1) with $w_h \in W_h$ and integrating by parts we have

$$\int_\Omega f w_h \,\mathrm{d}x = -\int_\Omega \Delta u w_h \,\mathrm{d}x = -\sum_{\kappa \in \mathcal{T}_h} \sum_{\substack{F \subset \partial\kappa \\ F \notin \mathcal{F}_{\Gamma_N}}} \int_F n_\kappa \cdot \nabla u \, w_h \,\mathrm{d}s + a_h(u, w_h) - \int_{\Gamma_N} \psi w_h \,\mathrm{d}s$$

or by rearranging terms and using that $u|_{\Gamma_D} = 0$,

$$a_h(u, w_h) - b_h(u, w_h) = l(w_h) + \sum_{\kappa \in \mathcal{T}_h} \sum_{\substack{F \subset \partial\kappa \\ F \notin \mathcal{F}_{\partial\Omega}}} \int_F n_\kappa \cdot \nabla u \, w_h \,\mathrm{d}s.$$

Using (3.8) we obtain

$$a_h(u_h - u, w_h) - b_h(u_h - u, w_h) - s_W(z_h, w_h) = -\sum_{\kappa \in \mathcal{T}_h} \sum_{\substack{F \subset \partial\kappa \\ F \notin \mathcal{F}_{\partial\Omega}}} \int_F n_\kappa \cdot \nabla u \, w_h \,\mathrm{d}s.$$

By the definition of the finite element space X_h on $\mathcal{F}_i$ and since every internal face appears twice with different orientation of n_κ we have for all $\nu_h \in X_h$,

$$\sum_{\substack{F \subset \partial\kappa \\ F \notin \mathcal{F}_{\partial\Omega}}} \int_F n_\kappa \cdot \nabla u \, w_h \,\mathrm{d}s = \sum_{\substack{F \subset \partial\kappa \\ F \notin \mathcal{F}_{\partial\Omega}}} \int_F n_\kappa \cdot (\nabla u - \{\nabla \nu_h\}) \, w_h \,\mathrm{d}s.$$

We now observe that by replacing w_h with the jump $[w_h]$ we may write the sum over the faces of the mesh, replacing n_κ by n_F. The conclusion follows by taking absolute values on both sides and moving the absolute values under the integral sign creating the desired inequality. □

Lemma 5.2. *For any $v \in H^1(\Omega)$ and for all $w_h \in X_h$ there holds*

$$a_h(v - r_h v, w_h) = 0. \tag{5.2}$$

For any $v \in H^2(\Omega)$ and for all $w_h \in X_h$ there holds

$$b_h(v - r_h v, w_h) \leq C h |v|_{H^2(\Omega)} \|h^{-\frac{1}{2}} w_h\|_{\Gamma'_N}. \tag{5.3}$$

Proof. By integration by parts we have

$$a_h(v - r_h v, w_h) = \sum_{\kappa \in \mathcal{T}_h} \sum_{F \subset \partial\kappa} \int_F (v - r_h v) n_\kappa \cdot \nabla w_h \,\mathrm{d}s = 0,$$

where the last equality is a consequence of the definition of $r_h v$. The inequality (5.3) follows in a similar fashion observing that by the definition of $r_h v$ and the Cauchy-Schwarz inequality we have

$$\begin{aligned} b_h(v - r_h v, w_h) &= \sum_{F \in \mathcal{F}_{\Gamma'_N}} \int_F n \cdot \nabla(v - r_h v) \, w_h \,\mathrm{d}s \\ &\leq \|h^{\frac{1}{2}} n \cdot \nabla(v - r_h v)\|_{\mathcal{F}_{\Gamma'_N}} \|h^{-\frac{1}{2}} w_h\|_{\Gamma'_N}. \end{aligned}$$

We conclude by applying the second approximation estimate of equation (3.2). □

We now proceed by first showing that the error in the triple-norm must go to zero and then we use this result together with conditional stability to obtain error estimates in the L^2-norm.

Proposition 5.3. *Let $u \in H^2(\Omega)$ be the solution of* (2.1) *and $(u_h, z_h) \in \mathcal{V}_h$ the solution of* (3.4)*. Then*

$$|||(u - u_h, z_h)||| \leq C(\gamma_V^{\frac{1}{2}} + c_s^{-1}(\gamma_W^{-\frac{1}{2}} + \gamma_V^{\frac{1}{2}}))h\|u\|_{H^2(\Omega)} \tag{5.4}$$

and

$$\|\nabla u_h\|_h \leq C(1 + c_s^{-1}(\gamma_W^{-\frac{1}{2}}\gamma_V^{-\frac{1}{2}} + 1))\|u\|_{H^2(\Omega)}. \tag{5.5}$$

Proof. Using a triangle inequality and the approximation (4.3) it is sufficient to consider the discrete error $\mu_h = u_h - r_h u$. By Theorem 4.1 we have the stability

$$c_s|||(\mu_h, z_h)||| \leq \sup_{(v_h,w_h)\in\mathcal{V}_h} \frac{A_h[(\mu_h, z_h), (v_h, w_h)]}{|||(v_h, w_h)|||}. \tag{5.6}$$

By adding and subtracting u in the formulation we observe that

$$\begin{aligned} A_h[(\mu_h, z_h), (v_h, w_h)] = a_h(u_h - u, w_h) - b_h(u_h - u, w_h) - s_W(z_h, w_h) \\ + a_h(u - r_h u, w_h) - b_h(u - r_h u, w_h) - s_V(r_h u, v_h). \end{aligned}$$

Applying Lemma 5.1 and 5.2 to the right hand side with $\nu_h := r_h u$ we obtain

$$\begin{aligned} |A_h[(\mu_h, z_h), (v_h, w_h)]| \leq \sum_{F\in\mathcal{F}_i} \int_F |n_F \cdot (\nabla u - \{\nabla r_h u\})||[w_h]| \,\mathrm{d}s + |s_V(r_h u, v_h)| \\ + C\gamma_W^{-\frac{1}{2}} h\|u\|_{H^2(\Omega)}\|w_h\|_{s_W}. \end{aligned}$$

We proceed using the Cauchy-Schwarz inequality followed by element wise trace inequalities and the approximation (3.2) to obtain

$$\begin{aligned} &\sum_{F\in\mathcal{F}_i} \int_F |n_F \cdot (\nabla u - \{\nabla r_h u\})||[w_h]| \,\mathrm{d}s + |s_V(r_h u, v_h)| \\ &\quad \leq C\gamma_W^{-\frac{1}{2}}\|h^{\frac{1}{2}} n_F \cdot (\nabla u - \{\nabla r_h u\})\|_{\mathcal{F}_i}\|w_h\|_{s_W} + |u - r_h u|_{s_V}|v_h|_{s_V} \\ &\quad \leq C(\gamma_W^{-\frac{1}{2}} + \gamma_V^{\frac{1}{2}})h\|u\|_{H^2(\Omega)}|||(v_h, w_h)|||. \end{aligned}$$

Applying the above inequalities in (5.6) completes the proof of (5.4). The inequality (5.5) then is an immediate consequence of (5.4) and the H^1-stability of r_h.

$$\begin{aligned} &\|\nabla u_h\|_h \leq \|\nabla \mu_h\|_h + \|\nabla r_h u\|_h \\ &\quad \leq C(\gamma_V^{-\frac{1}{2}} h^{-1}|||(\mu_h, z_h)||| + \|u\|_{H^1(\Omega)}) \leq C(1 + c_s^{-1}(\gamma_W^{-\frac{1}{2}} + \gamma_V^{\frac{1}{2}})\gamma_V^{-\frac{1}{2}})\|u\|_{H^2(\Omega)}. \end{aligned}$$

□

Theorem 5.4. *Let $u \in H^2(\Omega)$ be the solution of* (2.1) *and $(u_h, z_h) \in \mathcal{V}_h$ the solution of* (3.4)*. Then, with $j(\cdot)$ and $\Xi(\cdot)$ defined in* (2.4) *or* (2.5)*, there exists $h_0 > 0$ and a constant $C > 0$ independent of h such that for all $h < h_0$*

$$|j(u - u_h)| \leq \Xi(\eta(h, l, u_h, z_h)) + C\gamma_V^{-\frac{1}{2}} h|u_h|_{s_V}$$

where

$$\eta(h, l, u_h, z_h) = C(h\|f\|_\Omega + \gamma_V^{-\frac{1}{2}}|u_h|_{s_V} + \gamma_W^{\frac{1}{2}}\|z_h\|_{s_W}) + C\left(\sum_{F\in\mathcal{F}_{\Gamma_N}} h \inf_{\alpha_F\in\mathbb{R}} \|\psi - \alpha_F\|_F^2\right)^{\frac{1}{2}}.$$

In addition the following a priori bound holds

$$\eta(h,l,u_h,z_h) + |u_h|_{s_V} \le Ch(\|f\|_\Omega + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_N)} + \|u\|_{H^2(\Omega)}),$$

where the constant includes that of (5.4).

Proof. By the definition $j(\cdot)$ is an L^2-norm and therefore well defined for functions in $V+V_h$. We then consider the decomposition of $u-u_h$ into one V-conforming part and its residual. To this end introduce a function $\tilde{u}_h \in V \cap V_h$. To get a V-conforming approximation we define the values of $\tilde{u}_h$ in the vertices x_i of the tessellation $\mathcal{T}_h$ by $\tilde{u}_h|_{\bar{\Gamma}_D} = 0$ and,

$$\tilde{u}_h(x_i) = \mathfrak{C}_{x_i}^{-1} \sum_{\kappa: x\in\kappa} u_h(x_i)|_\kappa, \quad x_i \notin \bar{\Gamma}_D, \tag{5.7}$$

where $\mathfrak{C}_{x_i} := \operatorname{card}(\{\kappa \in \mathcal{T}_h : x_i \in \kappa\})$. With this definition it holds that $\tilde{u}_h \in V \cap V_h$. For the discrete error $e_h := u_h - \tilde{u}_h$ it is well known that the following estimate holds (see [1, 22])

$$\|e_h\| + h\|\nabla e_h\|_h \le Ch\gamma_V^{-\frac{1}{2}} |u_h|_{s_V}. \tag{5.8}$$

We may then construct the H^1-conforming part of the error as $\tilde{e} := u - \tilde{u}_h \in V$, making it a valid function to use in the conditional stability (2.3). For any $w \in W$ there holds

$$a(\tilde{e}, w) = l(w) - a(\tilde{u}_h, w) =: \langle r, w\rangle_{W',W}$$

where we have identified $r \in W'$. To apply (2.3) we need to upper bound $\|r\|_{W'}$. To this end we write

$$\sup_{\substack{w\in W\\ \|w\|_W=1}} \langle r, w\rangle_{W',W} = \sup_{\substack{w\in W\\ \|w\|_W=1}} (l(w - r_h w) + a_h(e_h, w) \\ - s_W(z_h, r_h w) - b_h(u_h, r_h w)) \tag{5.9}$$

where we have used the symmetry of $a_h(\cdot,\cdot)$ and the relation (5.2). By the definition of $l(\cdot)$ we see that the first term on the right hand side may be bounded by

$$l(w - r_h w) = (f, w - r_h w)_\Omega + \sum_{F\in\mathcal{F}_{\Gamma_N}} (\psi - \alpha_F, w - r_h w)_F \\ \le Ch\|f\|_\Omega + C\left(\sum_{F\in\mathcal{F}_{\Gamma_N}} h \inf_{\alpha_F\in\mathbb{R}} \|\psi - \alpha_F\|_F^2\right)^{\frac{1}{2}}.$$

To bound the second term in the right hand side of (5.9) we use the Cauchy-Schwarz inequality and the discrete interpolation result (5.8) to write

$$a_h(e_h, w) \le \|e_h\|_{1,h}\|w\|_{H^1(\Omega)} \le C\gamma_V^{-\frac{1}{2}} |u_h|_{s_V}.$$

For the second to last term in the right hand side of (5.9) there holds by the Cauchy-Schwarz inequality (followed by a trace inequality and approximation if (3.6) is used) and the H^1-stability of r_h,

$$|s_W(z_h, r_h w)| \le C\|z_h\|_{s_W} \gamma_W^{\frac{1}{2}} \|\nabla r_h w\|_h \le C\gamma_W^{\frac{1}{2}} \|z_h\|_{s_W}.$$

For the last term in the right hand side we similarly obtain using a trace inequality (recalling that $r_h w \in W_h$)

$$|b_h(u_h, r_h w)| = |\sum_{F \in \Gamma_D} \int_F n \cdot \nabla r_h w \, u_h \, \mathrm{d}s| \leq C C_t \gamma_V^{-\frac{1}{2}} |u_h|_{s_V}.$$

It follows from (5.4) and standard approximation that for h small enough, $\tilde{e}$ satisfies the assumptions of the conditional stability (2.3). However note that in order to apply (2.4) or (2.5) to $\tilde{e}$ we must show that there exists $E > 0$ such that the bound $\|\tilde{e}\|_{H^1(\Omega)} \leq E < \infty$ holds uniformly in h, since otherwise the constants in the estimates may blow up. This a priori bound is a consequence of a triangle inequality, the bound (5.5) and the estimate (5.8) as follows

$$\begin{aligned} \|\tilde{e}\|_{H^1(\Omega)} &\leq \|u\|_{H^1(\Omega)} + \|u_h\|_{1,h} + \|e_h\|_{1,h} \\ &\leq \|u\|_{H^1(\Omega)} + \|u_h\|_{1,h} + C\gamma_V^{-\frac{1}{2}} |u_h|_{s_V} \leq C(1+h)\|u\|_{H^2(\Omega)}. \end{aligned} \tag{5.10}$$

Therefore, under our regularity assumption on the exact solution, the H^1-norm of the conforming part of the error is uniformly bounded for all h. For the case of (2.4) or (2.5) we note that for all $\omega \subset \Omega$ there holds

$$\|u - u_h\|_{L^2(\omega)} \leq \|\tilde{e}\|_{L^2(\omega)} + \|e_h\|_{L^2(\omega)} \leq \Xi(\eta(h, l, u_h, z_h)) + C\gamma_V^{-\frac{1}{2}} h |u_h|_{s_V}$$

where $\Xi(\cdot)$ is defined by (2.4) or (2.5) depending on the choice of ω. The upper bounds on $\eta(h, l, u_h, z_h)$ and $|u_h|_{s_V}$ are immediate consequences of Proposition 5.3 and the approximation properties of piecewise constant functions. First note that since $u \in H^1(\Omega)$, by consistency

$$|u_h|_{s_V} = |u - u_h|_{s_V} \leq |||(u - u_h), z_h|||$$

and we may use the upper bound (5.4). For the bound on η we use the approximation $\inf_{\alpha \in \mathbb{R}} \|\psi - \alpha\|_F \leq C h^{\frac{1}{2}} \|\psi\|_{H^{\frac{1}{2}}(F)}$ and the consistency of $s_V(\cdot, \cdot)$ to obtain

$$\eta(h, l, u_h, z_h) \leq Ch(\|f\|_\Omega + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_N)}) + (\gamma_V^{-\frac{1}{2}} + \gamma_W^{\frac{1}{2}})|||(u - u_h, z_h)|||$$

and conclude once again using (5.4). □

5.1. **The case of perturbed data.** In this section we consider the realistic case that we have at our disposal only measurements of the fluxes $\psi + \delta\psi$ on the boundary part Γ_C. In practice these measurements are always polluted by measurement errors, $\delta\psi$. It is then of interest to study how fine it is reasonable to make the mesh, knowing that the perturbed data might not be in the range of the operator. We assume that $\delta f \in L^2(\Omega)$ and $\delta\psi \in L^2(\Gamma_N)$. The perturbed problem may be written, find $u_\delta \in V$ such that

$$a(u_\delta, v) = l_\delta(w) := l(w) + \delta l(w) \tag{5.11}$$

where

$$\delta l(w) := \int_\Omega \delta f w \, \mathrm{d}x + \int_{\Gamma_N} \delta\psi \, w \, \mathrm{d}s.$$

We introduce the h-weighted dual norm,

$$\|(\delta f, \delta\psi)\|_{h,W'} := h\|\delta f\|_\Omega + \|\delta f\|_{W'} + h^{\frac{1}{2}} \|\delta\psi\|_{\Gamma_N} + \|\delta\psi\|_{H^{-\frac{1}{2}}(\Gamma_N)}.$$

This norm will be used to measure the perturbation induced by errors in measurements. The reason for the combination of strong and weak norms is the following boundedness results.

Lemma 5.5. *Let* $s_W(\cdot,\cdot)$ *be defined by* (3.5). *Then*

$$\sup_{\substack{w_h\in W_h\\ \|w_h\|_{s_W}=1}} |l(w_h)-l_\delta(w_h)| \le C\gamma_W^{-\frac{1}{2}}\|(\delta f,\delta\psi)\|_{h,W'}. \tag{5.12}$$

$$\sup_{\substack{w\in W\\ \|w\|_{W}=1}} |l(r_h w)-l_\delta(r_h w)| \le C\|(\delta f,\delta\psi)\|_{h,W'}. \tag{5.13}$$

Proof. By definition $\delta l(w_h) = l(w_h)-l_\delta(w_h)$ and by the linearity of the operator

$$|\delta l(w_h)| \le |\delta l(\tilde w_h)| + |\delta l(w_h-\tilde w_h)|,$$

where $\tilde w_h \in W_h$ is the H^1-conforming part of w_h defined similarly as in (5.7), but with $\tilde w_h|_{\Gamma_N'}=0$. We may then use an estimate similar to (5.8), but with $\|\cdot\|_{s_W}$, to obtain the bounds

$$\begin{aligned}|\delta l(\tilde w_h)| &= |\,\langle \delta f,\tilde w_h\rangle_{W',W} + \langle\delta\psi,\tilde w_h\rangle_{H^{-\frac12},H^{\frac12}}\,|\\ &\le C(\|\delta f\|_{W'}+\|\delta\psi\|_{H^{-\frac12}})\|\tilde w_h\|_{H^1(\Omega)}\\ &\le C(\|\delta f\|_{W'}+\|\delta\psi\|_{H^{-\frac12}})(\|\tilde w_h-w_h\|_{1,h}+\|w_h\|_{1,h})\\ &\le C(\|\delta f\|_{W'}+\|\delta\psi\|_{H^{-\frac12}})\gamma_W^{-\frac12}\|w_h\|_{s_W}\end{aligned}$$

and,

$$\begin{aligned}|\delta l(w_h-\tilde w_h)| &\le \|\delta f\|_\Omega\|w_h-\tilde w_h\|_\Omega + \|\delta\psi\|_{\Gamma_N}\|w_h-\tilde w_h\|_{\Gamma_N}\\ &\le C(h\|\delta f\|_\Omega + h^{\frac12}\|\delta\psi\|_{\Gamma_N})\gamma_W^{-\frac12}\|w_h\|_{s_W}.\end{aligned}$$

Similarly the bound on $|\delta l(r_h w)|$ is obtained by

$$|\delta l(r_h w)| = |\delta l(r_h w - w)+\delta l(w)| \le C\|(\delta f,\delta\psi)\|_{h,W'}$$

where we used the approximation (3.2) with $t=1$ and the duality pairing $\delta l(w) = \langle\delta f,w\rangle_{W',W} + \langle\delta\psi,w\rangle_{H^{-\frac12},H^{\frac12}}$. □

Remark 5.6. The inequality (5.12) of Lemma 5.5 only holds when the stabilization of (3.5) is used in (3.8). If instead (3.6) is used, one may only obtain control of $\|h\nabla w_h\|_h$ in the triple norm (see Remark 4.2), leading to an additional factor h^{-1} in the right hand side of (5.12) above.

Accounting for the perturbed data introduces a minor modification of the weak consistency that holds for the formulation (3.8), when the right hand side is substituted for the perturbed functional $l_\delta(w_h)$.

Lemma 5.7. *(Weak consistency with perturbed data) Let* u *be the solution of* (2.1), *with* $f\in L^2(\Omega)$ *and* $\psi\in H^{\frac12}(\Gamma_N)$ *and let* $(u_h,z_h)\in\mathcal{V}_h$ *be the solution of* (3.8), *with the right hand side given by* $l_\delta(w_h)$. *Then, for all* $w_h\in W_h$, *there holds,*

$$\begin{aligned}&|a_h(u_h-u,w_h)-b_h(u_h-u,w_h)-s_W(z_h,w_h)|\\ &\qquad\le \sum_{F\in\mathcal{F}_i}\inf_{\nu_h\in V_h}\int_F |n_F\cdot(\nabla u-\{\nabla\nu_h\})||[w_h]|\;ds + |\delta l(w_h)|.\end{aligned} \tag{5.14}$$

Proof. Following the proof of Lemma 5.1 we now find that

$$a_h(u_h-u,w_h)-b_h(u_h-u,w_h)-s_W(z_h,w_h) = -\sum_{\kappa\in\mathcal{T}_h}\sum_{\substack{F\subset\partial\kappa\\ F\not\in\mathcal{F}_{\Gamma_N}}}\int_F n_\kappa\cdot\nabla u\, w_h\,\mathrm{d}s+\delta l(w_h).$$

We conclude as in Lemma 5.1. □

It is then straightforward to derive modified versions of Proposition 5.3 and Theorem 5.4. We give the results for the perturbed case below, detailing only the parts of the proofs that are modified by the perturbed right hand side in (3.8). Observe that if the problem (5.11) admits a solution $u_\delta \in H^2(\Omega)$, then the Proposition 5.3 still holds if u is exchanged with u_δ. If on the other hand (5.11) does not have a solution, or $\|u_\delta\|_{H^2(\Omega)}$ is very large, the perturbation can be included in the following way.

Proposition 5.8. *Let $u \in H^2(\Omega)$ be the solution of* (2.1) *and $(u_h,z_h)\in\mathcal{V}_h$ the solution of* (3.8) *using* (3.10) *and with the perturbed right hand side $l_\delta(w_h)$. Then*
(5.15)
$$|||(u-u_h,z_h)||| \le C((\gamma_V^{\frac12}+c_s^{-1}(\gamma_W^{-\frac12}+\gamma_V^{\frac12}))h\|u\|_{H^2(\Omega)}+c_s^{-1}\gamma_W^{-\frac12}\|(\delta f,\delta\psi)\|_{h,W'})$$

and

$$\|\nabla u_h\|_h \le C((1+c_s^{-1}(\gamma_W^{-\frac12}\gamma_V^{-\frac12}+1))\|u\|_{H^2(\Omega)}+c_s^{-1}\gamma_W^{-\frac12}h^{-1}\|(\delta f,\delta\psi)\|_{h,W'}). \tag{5.16}$$

Proof. The proof follows the arguments of the proof of Proposition 5.3, but this time we use the modified weak consistency of Lemma 5.7

$$\begin{aligned}|A_h[(\mu_h,z_h),(v_h,w_h)]| \le \sum_{F\in\mathcal{F}_i}\int_F |n_F\cdot(\nabla u-\{\nabla r_h u\})||[w_h]|\,\mathrm{d}s+|\delta l(w_h)|\\ +|s_V(r_h u,v_h)|+C\gamma_W^{-\frac12}h\|u\|_{H^2(\Omega)}\|w_h\|_{s_W}.\end{aligned} \tag{5.17}$$

The second term of the right hand side is then bounded using inequality (5.12). The bound (5.16) follows as before using the definition of the norm $|||\cdot|||_V$ and the estimate (5.15). □

We observe that the uniform H^1-bound on u_h no longer holds. Indeed since it can not be assumed that the solution u_δ of the perturbed problem (5.11) exists the method can fail to converge in the limit $h\to 0$. Assuming that the contribution from the discretization error dominates the upper bound (5.15) an error estimate in the spirit of Theorem 5.4 can nevertheless be derived.

Theorem 5.9. *Let $u \in H^2(\Omega)$ be the solution of* (2.1) *and $(u_h,z_h)\in\mathcal{V}_h$ the solution of* (3.8) *using* (3.10) *and with the perturbed right hand side $l_\delta(w_h)$. Assume that there exists $h_1>h_0>0$ such that*

$$\max(1,\gamma_W^{-\frac12})\|(\delta f,\delta\psi)\|_{h,W'}\le h_0\|u\|_{H^2(\Omega)} \tag{5.18}$$

and for $h\le h_1$

$$\begin{aligned}\eta_\delta(h,l,u_h,z_h) := C(h\|f\|_{L^2(\Omega)}+|u_h|_{s_V}+\|z_h\|_{s_W}\\ +\left(\sum_{F\in\mathcal{F}_{\Gamma_N}} h\inf_{\alpha_F\in\mathbb{R}}\|\psi-\alpha_F\|_F^2\right)^{\frac12}+\|(\delta f,\delta\psi)\|_{h,W'})<1.\end{aligned}$$

Then, with $j(\cdot)$ and $\Xi(\cdot)$ defined in (2.4) *or* (2.5) *we have for $h_0 \le h \le h_1$*

$$|j(u-u_h)| \le \Xi(\eta_\delta(h,l,u_h,z_h)) + Ch|u_h|_{s_V}. \tag{5.19}$$

In addition the following a priori bound holds

$$\eta_\delta(h,l,u_h,z_h) + |u_h|_{s_V} \le Ch(\|f\| + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_N)}) + Ch_0\|u\|_{H^2(\Omega)},$$

where the constant includes that of (5.15).

Proof. Under the assumption (5.18) the proof is analogous to that of Theorem 5.4, since by (5.18) equations (5.15) and (5.16) take the same form as (5.4) and (5.5). This means that $\|\tilde{e}\|_{H^1(\Omega)}$ is uniformly bounded in h under the condition (5.18) and $h_0 \le h$ and therefore the constants in (2.4) and (2.5) remain bounded. The only difference in the proof appears in the estimation of the residual term $r \in W'$, here

$$\sup_{\substack{w\in W\\ \|w\|_{H^1(\Omega)}}} \langle r, w\rangle_{W',W} = \sup_{\substack{w\in W\\ \|w\|_{H^1(\Omega)}}} (l(w-r_hw) - \underbrace{\delta l(r_hw)}_{perturbation} + a_h(e_h,w) \\ - s_W(z_h,r_hw) - b_h(u_h,r_hw).$$

The new contribution is the second term of the right hand side due to the perturbed data. This term is upper bounded using (5.13) and the result follows. □

We see that the estimate only is valid when $\|(\delta f,\delta\psi)\|_{h,W'}$ is small compared to $h\|u\|_{H^2(\Omega)}$. This is not a very useful condition in practice since $\|u\|_{H^2(\Omega)}$ is unknown. However, assuming that $\|(\delta f,\delta\psi)\|_{h,W'}$ is known, the quantities that form the upper bound (5.19) are all computable, leading to an a posteriori bound that allows to monitor the computation adaptively, requiring only the minimal regularity assumption $u \in H^1(\Omega)$ (for the conditional stability). Indeed $\eta_\delta(h,l,u_h,z_h)$ can be computed and the bound (5.18) is necessary only to ensure that the H^1-norm of $\tilde{e}$ stays bounded. However this last quantity can also be controlled a posteriori using (5.10),

$$\|\tilde{e}\|_{H^1(\Omega)} \le \|u\|_{H^1(\Omega)} + \|u_h\|_{1,h} + Ch\gamma_V^{-\frac{1}{2}}|u_h|_{s_V}.$$

Therefore it follows from Theorem 5.9 that mesh refinement will improve the solution as long as the following three criteria are satisfied

(1) $\|\nabla u_h\|_h$ stays bounded. This is necessary to ensure the uniformity of the E-dependent constants of the conditional stability estimate through (5.10).
(2) $|u_h|_{s_V} + \|z_h\|_{s_W}$ decreases. This is necessary for the reduction of the a posteriori quantity $\eta_\delta(h,l,u_h,z_h)$, as well as for the uniformity of the constants in the conditional stability, through (5.10).
(3) The perturbation error, measured in the discrete dual norm $\|(\delta f,\delta\psi)\|_{h,W'}$ is dominated by the discretization error:

$$\|(\delta f,\delta\psi)\|_{h,W'} < h\|f\|_{L^2(\Omega)} + |u_h|_{s_V} + \|z_h\|_{s_W} \\ + \left(\sum_{F\in\mathcal{F}_{\Gamma_N}} h \inf_{\alpha_F\in\mathbb{R}} \|\psi-\alpha_F\|_F^2\right)^{\frac{1}{2}}.$$

This is to ensure that $\eta_\delta(h,l,u_h,z_h)$ decreases significantly under mesh refinement. When the perturbation dominates the residual the error can no longer be expected to decrease.

If one or more of the above criteria fail we can expect the error to remain constant, or even grow under mesh refinement.

6. Numerical example

As a numerical illustration of the theory we consider the original Cauchy problem discussed by Hadamard. In (2.1) let $\Omega := (0,\pi) \times (0,1)$, $\Gamma_N := \{x \in (0,\pi); y = 0\}$, $\Gamma_D := \Gamma_N \cup \{x \in \{0,\pi\}; y \in (0,1)\}$ and

$$\psi := -A_n \sin(nx). \tag{6.1}$$

It is then straightforward to verify that

$$u_n = A_n n^{-1} \sin(nx) \sinh(ny) \tag{6.2}$$

solves (2.1). One may easily show show that the choice $A_n = n^{-p}$, $p > 0$ leads to $\psi \to 0$ uniformly as $n \to \infty$, whereas, for any $y > 0$, $u_n(x,y)$ blows up. Stability can only be obtained conditionally, under the assumption that $\|u_n\|_{H^1(\Omega)} < E$ for some $E > 0$, leading to the relations (2.4) and (2.5) (see [2] for detailed proofs and further discussion of (2.3), (2.4), (2.5).)

We choose $A_n := 1$ in (6.1) and study the error in the relative L^2-norms,

$$\frac{\|u - u_h\|_{\Omega_\zeta}}{\|u\|_{\Omega_\zeta}}, \text{ where } \Omega_\zeta := (0,\pi) \times (0,\zeta), \quad \zeta \in \{1/4,\ 1/2,\ 1\}. \tag{6.3}$$

Recall that for $\zeta < 1$ the stability (2.3), holds with (2.4) and for $\zeta = 1$ (2.3) with (2.5) holds. All computations below were performed using formulation (3.8) in the package FreeFEM++ [21].

6.1. **Tuning of penalty parameters.** To tune the parameters we set $\gamma_V = \gamma_{V,bc} = \gamma_{W,bc} = 1$ and then varied the parameter for the adjoint stabilization in the bulk γ_W in the interval $[10^{-8}, 1]$. Computations were performed with $n = 1$ on a coarse mesh with $h = 0.1$. In Figure 1 we report the results for the method (3.8) using (3.11) in the left plot and using (3.10) in the right plot. The filled line represents the error in the global L^2-norm plotted against γ_W for unperturbed data and the dotted line represents the same quantity, but for data perturbed by random noise at the level of 1%.

For both methods we see that γ_W can vary over several orders of magnitude while keeping the error below 2%, both for perturbed and unperturbed data. From Figure 1 we then chose the parameters for the rest of the study below as $\gamma_W = 5 \cdot 10^{-4}$ when the operator (3.11) is used and $\gamma_W = 5 \cdot 10^{-5}$ when (3.10) is used.

Remark 6.1. We do not claim that this choice of parameters results in the smallest error, but rather that it is one valid choice among many. Indeed for the method (3.11) numerical experiments indicated that all the penalty parameters could be chosen to the same value and tuned as one parameter. This was not the case for (3.10). We chose to use the above simple strategy in both cases to make the results for the two methods comparable and show that one set of parameters gives good results for all the numerical experiments considered.

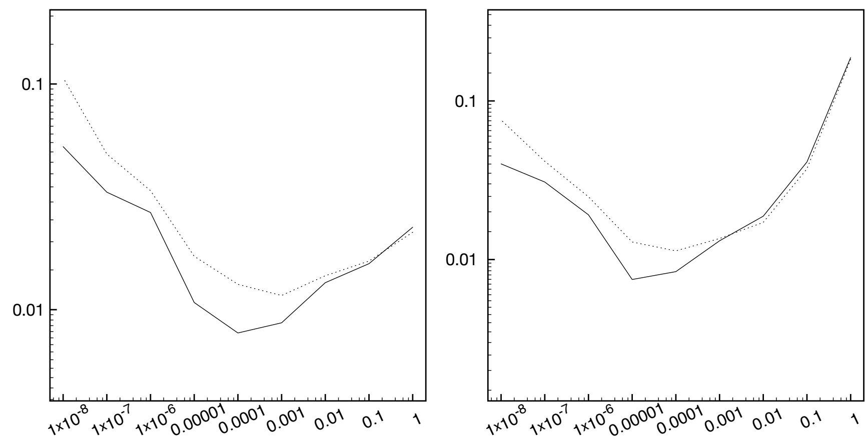

FIGURE 1. Study of the global relative L^2-error against penalty parameter γ_W ($\gamma_V = \gamma_{V,bc} = \gamma_{W,bc} = 1$) for the methods using (3.11) (left) and (3.10) (right). Dotted lines represent results obtained with 1% random perturbation in data.

6.2. **Convergence studies.** We performed computations varying n in $\{1, 3, 5\}$ on a series of unstructured meshes with approximate mesh sizes in the set,

$$\{0.1,\ 0.05,\ 0.025,\ 0.0125,\ 0.008333\}.$$

Elevations of the reconstructions using the stabilization (3.10) for $n = 5$ are given in Figure 2 with mesh size $h = 0.1$ (left plot) and $h = 0.0125$ (middle plot) with the exact solution interpolated on the finer mesh presented for comparison (right plot). Some spurious oscillations are present in the coarse mesh computation, but are completely eliminated on the finer mesh. The convergence results are given in the graphs of Figure 3. We have studied the relative L^2-norms for the three different values of ζ given in (6.3). Each value of ζ is represented by a different symbol according to: $\zeta = 1$, symbol: $\circ$; $\zeta = 1/2$, symbol: $\square$; $\zeta = 1/4$, symbol: $\diamond$. As we increase the value of n the H^1-norm of the exact solution, denoted E, increases and is given in the captions of the figures. We see that the error level increases with increasing E. The full lines refer to the method using (3.10) and the dashed lines refer to the method using (3.11). The dotted lines are reference curves defined as follows.

- Left plot, $n = 1$, from top to bottom: $y = -0.02(\log(x))^{-1}$; $y = 0.004x^{\frac{2}{5}}$; $y = 0.25x^2$.
- Middle plot, $n = 3$, from top to bottom: $y = -0.02(\log(x))^{-1}$; $y = 0.024x^{\frac{2}{5}}$; $y = 0.75x^2$.
- Right plot, $n = 5$, from top to bottom: $y = -0.5(\log(x))^{-1}$; $y = 6x$; $y = 5x^2$.

In all three cases we observe the logarithmic convergence given by the stability (2.5) for the global error. For the local errors we observe the convergence of type h^ς obtained from the stability (2.4). In all cases we see that the adjoint consistent

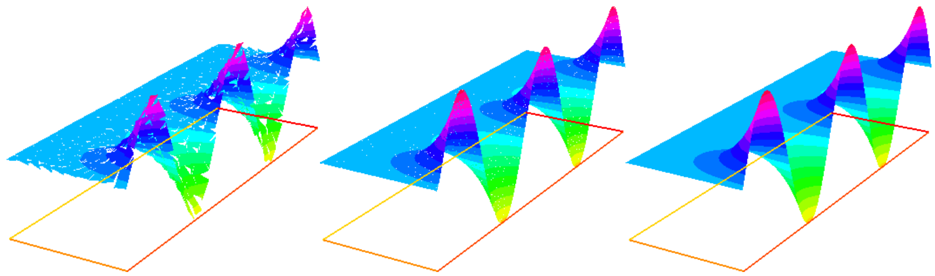

FIGURE 2. Comparison of reconstructions using (3.5) for $n = 5$ with $h = 0.1$ (left), $h = 0.0125$ (middle) and interpolated exact solution (right)

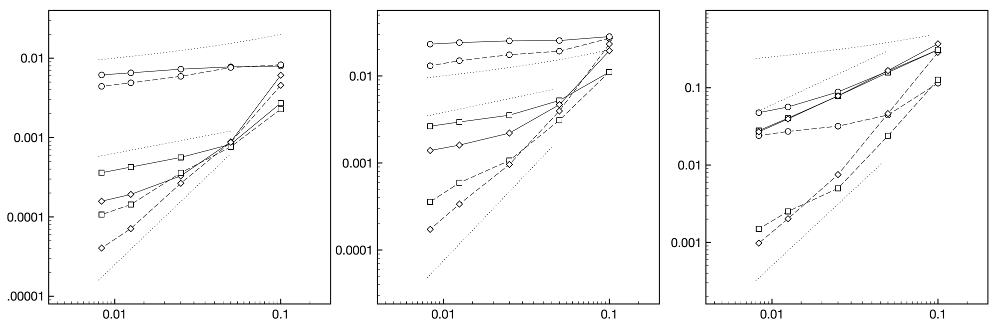

FIGURE 3. Relative L^2-error against mesh-size, $\zeta = 1$, $\zeta = \frac{1}{2}$ and $\zeta = \frac{1}{4}$. Left: $n = 1$ and $E = 1.68$. Middle: $n = 3$ and $E = 7.27$. Right: $n = 5$ and $E = 41.6$

method has superior convergence properties for the local errors, typically $\varsigma \in [0.5, 1]$ for the method using (3.10) and $\varsigma \in [1, 2]$ for (3.11).

6.3. **Perturbed data.** In this section we present some preliminary results using random perturbations of the Cauchy data ψ. We consider

$$\delta\psi := \varrho \times rand$$

where $\varrho \in \mathbb{R}_+$ is the relative strength of the perturbation and *rand* is a random vector taking values in $[0, 1]$, generated in FreeFEM++ using the command `randreal1()`. We first study the effect of varying the strength of the perturbation by generating one random vector for each mesh and then using the same vector in each computation for the three different cases $n = 1$ (symbol $\triangle$), $n = 3$ (symbol: ∇), $n = 5$ (symbol: $\Diamond$) for the methods using stabilization (3.10) (full line) or (3.11) (dashed line). The value of ϱ is then varied in the interval $[0.01, 0.1]$ and the mesh size was fixed to $h = 0.025$. In the left plot of Figure 4 we see the dependence of the relative global L^2-norm error on the strength of the perturbation ϱ. The dependence varies for the different cases, for $n = 1$ we see a linear dependence indicating that the perturbation dominates the residual. For $n = 5$ the error is relatively independent of the perturbation, showing that the discretization error dominates the residual. In the intermediate case $n = 3$, finally, the behavior appears to change

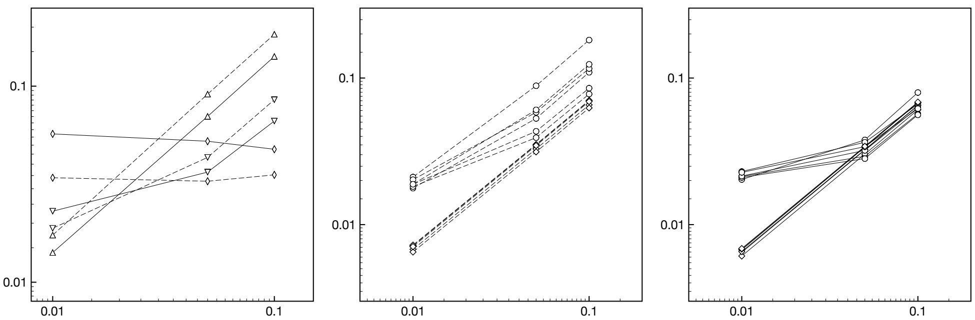

FIGURE 4. Left: Relative global L^2-error against size of the random perturbation ϱ, $h = 0.025$, $n = 1$, $n = 3$ and $n = 5$. Middle: six realizations of perturbation study using different random seeds, $h = 0.025$, $n = 3$, stabilization (3.11), $\zeta = 1$ and $\zeta = \frac{1}{4}$ in (6.3). Right: six realizations of perturbation study using different random seeds, $h = 0.025$, $n = 3$, stabilization (3.10) $\zeta = 1$ and $\zeta = \frac{1}{4}$ in (6.3).

depending on the size of ϱ, for large values of ϱ the behavior is similar as for $n = 1$, indicating that the perturbation is dominating. In the left plot of Figure 5 we report the relative global L^2-error against the mesh-size h for three different values of ϱ. Onset of stagnation of the errors at $h \approx \varrho$ is clearly visible. When (3.11) was used we also observe the growth of the error under mesh refinement when $h < \varrho$ predicted in Remark 5.6.

To assess the robustness for different random data we consider the same computations as above for a sample of six random vectors *rand* generated by varying the random seed (command `randinit(`*seed*`)`, with *seed* $\in \{2, 8, 76, 123, 2749, 31313\}$). The results are reported in the middle (stabilization (3.11)) and right (stabilization (3.10)) plots of Figures 4 and 5. The exact parameters of the different computations are given in the captions. Once again the superior stability for perturbed data of the method using (3.10) is clearly visible in the global error. For the local error the performance of the two methods is very similar.

7. CONCLUDING REMARKS

We have proposed a nonconforming stabilized finite element method for the approximation of elliptic Cauchy problems. Two different stabilization operators were studied and boundary conditions could either be set in the approximation spaces or introduced weakly in the variational formulation. We proved a posteriori and a priori error estimates for both approaches under the assumption of conditional stability. The method using (3.10) showed better stability for perturbed data, both theoretically and numerically, but needed separate tuning of the parameters γ_V and γ_W, whereas they could be chosen equal for the method using (3.11). The stabilization operator (3.11), which is adjoint consistent, on the other hand produced more accurate approximations when the data was not perturbed. Numerical tests using both unperturbed and perturbed data corroborated the theoretical results and showed the good stability properties of the proposed methods.

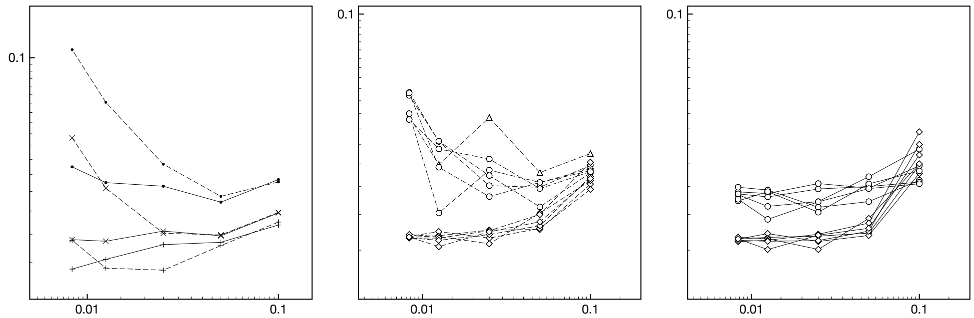

FIGURE 5. Left: relative global L^2-error plotted against h for $n = 3$ and (top to bottom) $\varrho = 0.05(\cdot), 0.025, (\times), 0.01(+)$. Middle: six realizations of convergence study using different random seeds, $n = 3$, $\varrho = 0.025$, stabilization (3.11), $\zeta = 1$ and $\zeta = \frac{1}{4}$ in (6.3). Right: six realizations of convergence study using different random seeds, $n = 3$, $\varrho = 0.025$ stabilization (3.10) $\zeta = 1$ and $\zeta = \frac{1}{4}$ in (6.3).

Finally it should be observed that it is straightforward to extend the present analysis to the interior penalty discontinuous Galerkin method using piecewise affine polynomials in the spirit of [20].

Acknowledgement

The author wishes to thank the anonymous reviewers whose comments helped improve this manuscript.

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