I thought I knew all about square roots

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Following on from my observations of the inconsistencies and misuse of the radical symbol amongst pupils, undergraduates, teachers and some authors of school textbooks, I became interested in those decisions that teachers take when confronted with inaccurate or ambiguous representations of the square root concept and its associated symbol notation. The impact that the ambiguous treatment of this mathematical concept and its associated symbol notation has on a number of PGCE students' conceptual understanding and pedagogical affinity will be discussed.

Keywords: square roots, ambiguous definition, textbooks

How it all started

My interest with this particular mathematical concept started a number of years ago, just as I was embarking on teaching my Year 8 pupils about square roots. It was my first year of teaching mathematics at secondary school level after having taught various pure mathematics courses at university level for over ten years. I remember glancing at the textbook the pupils were using and as I did so I was very surprised to find a new symbol which I was not familiar with. The textbook introduced the symbol $\sqrt[\pm]{\ }$, according to which the notation $\sqrt[\pm]{16}$ was understood to stand for the positive and negative square root of 16. As I expected, my pupils found this new notation confusing, especially after having studied the square root the previous year when the textbook simply and clearly stated that "A square root is represented by the symbol $\sqrt{\ }$. For example, $\sqrt{16}=4$ and -4" (Evans et al. 2008) (note *and* not *or* in the definition above, introducing or indicating a further ambiguity about yet another mathematical symbol, namely $\pm$).

As a mathematician, I felt uncomfortable with the situation. The square root symbol $\sqrt{\ }$, referred to as 'the radical symbol' is assigned to the positive square root of any non-negative real number, since $\sqrt{x^2}=|x|$ for any real number *x* and thus its value is always a non-negative real number. While I did not expect this level of rigour in defining new concepts or symbols to Year 8 pupils (nor did I think that was desirable at this level of pupils' mathematical education), I was worried by the textbook's incorrect definition and use of a mathematical symbol together with the lack of consistency and rigour in treating a mathematical concept..

In Crisan (2008, 2012) I identified the widespread misuse of the radical symbol amongst the authors of a large number of school textbooks. Most of the many teachers I talked to about the square root of a number did not seem to question the textbook definition; but used it according to how it was introduced by the class textbooks. It was not unusual for teachers to report to me that they taught pupils that $\sqrt{9}=\pm 3$ at KS3 and KS4 foundation level, while teaching pupils that $\sqrt{9}=3$ at KS4 higher level and KS5. Just a handful of teachers said that they were very keen to point out the textbook inaccuracies to their pupils, teaching them to use the symbol $\sqrt{\ }$ for the positive square root value of a number only. They did so despite running into difficulties at times, such as when confronted with examination marking schemes that awarded marks for the negative values of a square root.

The study

While Ball and Phelps (2008) argue that teachers need to be able to make judgments about the mathematical quality of instructional materials and modify them as necessary, can we rest assured that users (teachers) of these resources are able to identify inaccuracies and ambiguities and know what to do about 'putting them right' given for example, the constraints of the departmental practices or exam board syllabus specifications?

For this reason I decided to carry out a small study involving prospective teachers, students on a Post Graduate Certificate of Education (PGCE) course, and present them with a number of mathematics questions to solve involving the square root. The aim of this study was to explore the participants' knowledge about the square root and its associated symbol notation and to the decisions they take in the planning for teaching when confronted with inaccurate definitions or ambiguous representations of this concept held by other participants or present in the instructional materials consulted. I was also interested to find their sources of conviction when adopting a particular 'definition' of the concept and how they justify their choices.

In this study the eight secondary mathematics PGCE volunteers were engaged in a number of mathematics and pedagogically specific tasks with the aim of gaining access to their knowledge, views, beliefs and intended practices. The participants were split into two groups according to their availabilities for group discussion (group I – pseudonyms: Jan, Jemma, Jack and Joan; group II – pseudonyms: Billy, Barry, Ben and Bea).

Data Collection

Participants were first given a piece of homework consisting of questions where the concept of square root was likely to be employed. The mathematics questions were designed so that they would bring to the surface the ambiguities and inconsistencies of this concept and its associated symbol. The participants were then invited to talk to each other about how they solved/answered the questions set. During the discussion, implications for teaching about square roots arose naturally, either through the participants' reflection on how they had been taught the topic or how they would teach the topic themselves. Immersion of the participants' mathematical work in the pedagogical space was taken further through another task, namely fictional pupils' scenarios. The participants were asked to give written feedback to three fictional pupils' responses (Emma-KS3, Peter-KS4 and Lucy-KS5) characterised by a subtle mathematical error in a question involving the square root, throwing further light on the choices the participants made about treating this concept.

Discussion and findings

In the following I will report on some aspects of the participants' approaches to solving some of the questions set as homework, supporting their written and oral explanations with data collected during the group discussions and some of their written feedback to the fictional pupils' scenarios where relevant.

The participants' knowledge and understanding of the square root of a number

When discussing the answer to the question asking them to solve the equation $x^2 = 16$, all the participants were in agreement that the solutions were $x = \pm 4$. The solutions were reached either by solving the equation by factorisation (one participant), by using the graphical approach (two participants) or, in the most popular approach, by 'taking the square root' of both sides, the latter giving $\sqrt{x^2} = \sqrt{16}$ hence $x = \pm 4$ since $\sqrt{16}$ equals ± 4 (five participants). Group I were happy with this above explanation when given by Jan. A similar solution was put forward by Billy in group II, but he changed his mind very soon after offering his explanation. He then quickly said: *Actually, strictly speaking that is not right, is it? Looking at it now, I would amend it to say that* $x = \pm\sqrt{16}$ *since* $\sqrt{x^2} = \pm x$ *and* $\sqrt{16}$ *equals* 4. After this contribution, the participants debated whether the answer when 'taking the square root' was either positive or negative. *Sometimes it could be +, sometimes it could be –*, said Barry, while Ben attempted to clarify this point by saying: *It depends how you want to define the root function.* Billy interrupted abruptly to say: *The root function is defined as two numbers multiplied together to give the original number and so* $\sqrt{x^2} = \pm x$. However, he then changed his mind to say that $\sqrt{16}$ should equal ± 4, and so the equation $x^2 = 16$ reduces to solving $\pm x = \pm 4$, an equation in a format unfamiliar to all participants in group II.

The explanations put forward by Bill, Barry and Ben illustrate the two facets of this 'elementary procept' (Gray and Tall 1994), an amalgam of a *process* (the inverse of the square function) which produces a mathematical *object* (the square root of a number) and a *symbol* which is used to represent either process or object (the radical symbol notation). The radical symbol $\sqrt{}$ is used for both a *process* and a *concept*, giving thus rise to ambiguity.

Indeed, such ambiguity gave rise to a further interesting debate which took place when solving another question asking them to give the answer to $\sqrt{9^2}$. The following solutions were put forward:

- $\sqrt{9^2} = \sqrt{81} = \pm 9$;
- $\sqrt{9^2} = (9^2)^{\frac{1}{2}}$, which can then be taken forward by using the order of the operations (brackets first) as $(81)^{\frac{1}{2}} = \sqrt{81} = \pm 9$;
- $\sqrt{9^2} = (9^2)^{\frac{1}{2}}$, which if using the order of the operations (laws of indices) can be taken further as $9^{2\times\frac{1}{2}} = 9^1 = 9$ and finally,
- $\sqrt{9^2} = 9$ since the square and square root cancel each other (given that the square root and square functions are inverse of each other)

Despite the obvious equality $\sqrt{9^2} = \sqrt{81}$, all four explanations were regarded as being valid and the participants in group I did not seem to be able to find any 'fault' in the reasoning approaches presented above, as all explanations seemed to have a logical, firm foundation. *This is not an identity, but they can be equal*, Jan then said. The participants understood that this was ambiguous, and tried to 'get to the bottom' of this ambiguity. While doing so, they had a lengthy discussion about the differences between a mapping and a function. Jack concluded that *it is all to do with the fiddling things like … between … functions and mappings, which I cannot quite put my finger on why.*

The participants in group II had a similar debate when comparing each other's answers to another question asking them to simplify $\sqrt{25y^2}$. In the light of the earlier discussion about solving the equation $x^2 = 16$, they settled for the following convention: for a variable $\sqrt{y^2} = y$, while for a number $\sqrt{25} = \pm 5$ and so the solution of the equation was $\pm 5y$, which 'worked' when these values were substituted back into the equation. At this point, Ben summarised that perhaps in different contexts, the square root could mean different things. He went on to say that if working in the context of graphs and functions at KS5, one can consider only the positive value, whereas when finding the square root of numbers, one could consider the + or –. Both Billy and Barry illustrated this aspect with the formula for calculating the roots of a quadratic equation, namely $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, justifying the presence of the $\pm$ as the result of calculating the square root of a number (the numerical value of $b^2 - 4ac$). When prompted to consider more carefully the quadratic formula, the participants realized that in fact the $\pm$ becomes redundant in the formula.

Sources of conviction

During the group discussion, if conflicting or non-equivalent views of how to work with the square root were encountered, the participants were invited to discuss, debate and reach a consensus. Most of the participants' sources of conviction, which they used in order to justify their answers, were external in nature. The participants relied on what they remembered from school or what they learned from the instructional materials they consulted when doing the mathematics homework.

While consulting the materials available to them (textbooks, dictionary, mathematics glossary, examination papers with marking schemes, web sites), the participants commented on the inconsistencies in how the square root was presented. For example, while browsing an A-level textbook (Pledger et. al., 2004), the participants realised that according to the chapter on surds, $\sqrt{25} = 5$ with no mention of the $\pm$, while the following chapter on quadratic functions draws pupils attention to the fact that $\sqrt{25} = +5 \text{ or } -5$. The other instructional materials reviewed suggested that $\sqrt{16} = 4 \textit{ or } -4$, that $\sqrt{16} = 4 \textit{ and } -4$, that $\sqrt{16} = \pm 4$, introduced the new notation $\sqrt[\pm]{16}$ standing for the positive and negative square root of 16, or gave pupils a choice, namely that $\sqrt{16}$ is 4 most of the time, but that it could also be -4, depending on the context of the problem to be solved. Quite annoyed by this, Billy thought that this was *abuse of language and notation at A-level* and that mathematics should not be about free choices. Billy went on to say that in his view this was the result of simplifying things for the sake of our pupils. He explained how taking an easy route with Year 7 pupils when introduced to the positive and negative square root of 25 without a clear distinction about the symbols in use is similar to the difficulties pupils have with the incorrect (but widely accepted) way of reading -7-12 as 'minus 7 minus 12', leading to difficulties in understanding the operation that needs to be performed.

During the group discussion Bea expressed her frustration with the fact that her group were not making much progress in checking the rest of the homework questions due to confusion over the definition of the square root. She shared with the group that she was taught at school that the square root of a number is always a non-negative number and as a result her answers to this question (and other similar to this one) were non-negative numbers. In fact, she was confused by the polemic surrounded the + or -: *I cannot see what the problem is?* $\sqrt{x^2} = |x|$ *for any x real number; this is the definition of the square root, so why not use it*? Bea explained that taking the square root of both sides yields $|x| = \sqrt{16}$, hence $\pm x = 4$, resulting in $x = \pm 4$. The definition presented by Bea created some uneasiness amongst the other participants as they did not think it would be of much use since the square root is introduced to pupils much earlier than the concept of modulus function, or function for that matter. However, the participants in group II liked the clarity of this definition and adhered to it. For example, Barry in group II gives the following feedback to one fictional pupil scenario (Emma - KS3): *However, by convention, we usually take* $\sqrt{4}$ *to just mean the positive root, i.e. 2,* and he is consistent in the feedback to the pupils.

In group I, the discussion led to the participants making a clear distinction between the square root of a number and the square root of a square number written in index form and evidence collected through their feedback to fictional pupils' scenarios indicated that the participants were prepared to work with these two facets of the square root concept even if it led to conflicting pedagogical decisions. In her feedback, Jan tells Emma, the KS3 fictional pupil that $\sqrt{25-9} = \sqrt{16} = \pm 4$ *so when you see* $\sqrt{\ }$ *you must consider both the positive and the negative roots.* However, in her feedback to Peter, the KS5 pupil she explains that $\sqrt{7^2}$ can only equal 7, as this is about the square root being the inverse process to squaring,

Discussion and some findings

The participants brought to the group discussion different knowledge and understanding about the concept of square root of a number.

Strong held beliefs

With one exception, all the participants identified $+4$ and -4 as the square roots of 16 and their written answers revealed that they used the radical symbol to denote any of these square roots, i.e. $\sqrt{16} = \pm 4$. *This is how we were taught since very little*, said Jan and this explains why the participants (especially those in group I) invested a lot of energy in defending this knowledge. The participants' sources of conviction were external in nature in most cases, recalling and reproducing definitions they remembered from school or textbooks, while not claiming any ownership of the square root concept. Initially, when encountering ambiguities in the questions they were solving, the participants worked on the premise that their knowledge of square roots was correct, i.e. $\sqrt{16} = \pm 4$, as most of the participants were taught, hence they looked elsewhere for resolving any issues they encountered instead of revisiting their knowledge and understanding of the concept.

Competing Claims

However, the discomfort amongst the participants in group II caused by the logical inconsistencies ($\sqrt{9^2} \neq \sqrt{81}$) motivated the participants to reconsider their knowledge of this concept. They felt ready to alter it and to adhere to Bea's definition of the concept as it clearly was free of any ambiguities. Despite realizing that they were not going to be able to introduce this definition at KS3 and 4 levels, the participants were happy to present the use of the radical symbol to younger pupils as a 'convention' for positive square roots only, confident that they had a firm mathematical foundation for this argument.

The participants in group I however could not reach a consensus and as a result they accepted both facets of the square root. They were still not clear about the underlying mathematics of the concept, but made some pedagogical decisions: teaching pupils that $\sqrt{9} = \pm 3$ at KS3 and KS4 foundation level, while $\sqrt{9} = 3$ at KS4 higher level and KS5, complying with the textbooks they consulted. Both definitions were seen as valid and the participants' feedback to pupils' responses suggested that the square root symbol was used differently for different year groups.

The use of instructional materials

It was important to expose the prospective teachers to situations where textbooks give different but not equivalent or even ambiguous definitions of a mathematical concept. Good textbooks providing accurate information are needed. This does not necessarily mean that formal definitions should be introduced to the pupils, but authors of such textbooks have to be very careful when less formal definitions are introduced, without careful considerations for the implications for further learning

This study highlighted the need for prospective teachers to revisit their subject knowledge and develop an appreciation of mathematics as a coherent discipline, where different areas of mathematics are related and interconnected (square root definition, functions, mappings, relationships, identities, symbol use were aspects considered by the participants). It is this view and understanding of mathematics that enable teachers to scrutinise the available instructional resources and to decide for themselves on the appropriate pedagogical approaches and not rely on how they were taught when at school or on the authority of textbooks or examination boards.

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