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Learning in lectures: multiple representations

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Lectures remain the lynchpin of mathematics teaching at university even with advances in information technology and access to the internet. This paper examines the requirements for learning mathematics and shows how important it is for lecturers to be aware of the different modes of presentation they are using. We consider ways to assist students to make the connections between different representations with particular reference to students whose first language is not English.

Keywords*: University mathematics; Lectures; International students; Learning.*

AMS Subject Classification*:* 97B40; 97D40

1 Introduction

Lectures have long been established as the most common method of teaching in higher education. As lecturers of mathematics, we are interested fundamentally in the question: 'how does learning in mathematics take place as a result of attending a lecture?'. Good lectures can inspire and motivate learning, yet some lectures make students bored, confused, anxious, frustrated and even angry. And, in the middle lies the ordinary, day-to-day mathematics lecture that occurs in universities across the globe. It is this 'regular' lecture that we are considering here; the product of a professional university mathematician's good-practice that s/he plans for students to learn a branch of mathematics.

This paper investigates learning that comes about through multimodal representation of mathematics presented in a lecture. We claim that a central purpose of lectures is to be a natural venue for the links between different representations of mathematics experienced by learners; and having different representations available is central to building a 'concept-image' [1] and thus to learning mathematics at undergraduate level. Lectures are not just information delivery venues, nor are they classrooms: for in a classroom the prior learning and on-going attainment of individuals is important for teacher planning. This level of contact with the individual is not practical where there are hundreds of students, as in many undergraduate mathematics lectures. Though not central to the topic of this paper, we note that other aspects of the lecture experience are important, in particular, the social cohesion and identity formation that occurs just by being in that lecture hall together. Furthermore, given the centrality of multimodality, we consider in the discussion students for whom this aspect might, nevertheless, be disadvantageous; for example, students whose skills in the language of instruction do not match/align with the modalities of the lecture being offered.

2 On the nature of lectures

There are various ideas regarding what a lecture actually is and what lecturing actually involves. Traditionally, a university lecture was 50–55 minutes of largely uninterrupted monologue from a lecturer with student activity being focused on listening and note-taking [2, p. 9]. According to Edwards et al. [3], the word 'lecture' is also used to specify a time where students are taught in a designated space, a lecture theatre, in a group whose size can vary from 20 to 800 or more, and where one lecturer has sole responsibility for delivering content. In recent years, there have been challenges to the prevalence of this form of teaching. The development of information technologies has influenced the type of resources and learning materials that are now available to students. In many universities, the course material for subjects is available in several modes. For example, the overhead transparencies (OHTs) and lecture notes may be uploaded onto a webpage specifically designed for that unit. Unit outlines, assessments, assignment feedback and discussion boards are now commonly available in an online format for many university subjects. The prescribed textbook readings and recommended papers may also be made available for download.

It would appear that the prevalence of lecturing as the main method of university teaching may be challenged by the convenience and accessibility of material online and lecture recordings. However, a recent study at the University of Queensland, [4] found that despite the availability of online materials and textbooks, the majority of students attributed a large percentage of their learning in mathematics to lectures and tutorials. A similar finding was established by researchers at the University of New England [5]. Undergraduate students in agricultural science degrees were asked to rate which of a wide range of learning activities and resources provided for the unit had greatest overall value. These included: a study guide, WebCT online resources (discussion, boards, quizzes etc), practical classes, lectures and an interactive CD-ROM resource. Over 98% of the students rated lectures most highly. Cretchley [6] found that despite students having full and easy access to all course information and materials, online and in hardcopy, lecture attendance has not diminished. This suggests that for many students, human contact is still a very important part of their learning experience at university. Indeed, some lecturers exploit technology to enhance this human dimension within a lecture format [7]. So, despite, or even because of, the enormous advances in communication and information technology, lecturing continues to be a significant component of university education.

During the past forty years, there has been an increase in the number of students attending institutes of higher education. As a result, a number of guides and booklets have appeared aiming to facilitate the improvement of university teaching [2, 8]. Interestingly, research examining what students experience during a typical lecture is limited, though Bligh's classic critique of lectures is now in its 5th edition [9]. The research that does exist is anecdotal or personally reflective in nature and based on the lecturer's experience of the lecture [10, 11]. Furthermore, while some studies have examined student feedback, perceptions, and lecture attendance [5, 11, 12, 13], there has been little or no research examining components of a lecture. Examination of the different facets of a lecture (for any particular topic) may provide insight into the factors fundamental to successful teaching and learning at university.

3 Data for analysis

In this section, we will present components of a 'typical' large undergraduate lecture, which was video- and audio-taped (for research purposes), together with learning materials that the lecturer requires or recommends students to engage with outside the lecture. In this case, the data consists of four sets: (1) the spoken presentation by the lecturer (in English), (2) written-as-he-was-lecturing overhead transparencies, (3) lecture notes written by the lecturer (or colleagues) prior to the lecture and given to all students in booklet form and available to annotate as the lecturer is speaking, and (4) Mathematica® files on the topic (students are required to use Mathematica® for some assignments in the course). The topic of the lecture is de Moivre's Theorem.

Figures 1 and 2 show what the lecturer is saying and writing as he states that de Moivre's theorem is true for negative numbers. He does not draw the diagram or repeat the derivation given in the notes. It is a large class of 200 in a typical tiered lecture. It is difficult, even with this level of detail in the data, to ascertain whether the lecturer had eye contact with enough students to make them feel as an audience [12] and indeed, this important mode of communication (connecting with the audience) is likely to be significant in developing the students' motivation to engage. But what we can present are the extra materials that are expected to enhance the learning and are specific to this lecture.
Figures 1 and 2 here
Figure 3 shows the lecture notes in the course booklet that many students had with them. Students are trained in the use of the computer algebra system (CAS) Mathematica® and are required to use it for assignments. Figure 4 is the computer file that is returned when HELP is searched for de Moivre's theorem.
From figures 1 and 2, we see a feature that this multiple modes data gives us: students experience (nearly) the same words spoken and written (nearly) simultaneously. The example here being: "it turns out that this result also holds/is true for negative integers". This exemplifies a feature of the multimodality of lectures: a concept is reinforced because more than one sense is being fed at the same time. The lecturer missed the opportunity to make the connection with the diagram in the notes.
Figures 3 and 4 here
The lecture notes, figure 3, use the same notations as is used in the lecturer's live writing (done on the overhead transparency). The printed notes also use an image in the form of a complex-plane (Argand) diagram, another representation of the theorem.

$$re^{i\theta}$$

The Mathematica® files, figure 4, via HELP when 'de Moivre' is entered, do not mention the theorem specifically. Furthermore, the notations used by the CAS are different to those used in the other modes: for example, the CAS 'Abs[z]Exp[I Arg[z]]' is equivalent to the written . To be able to move between the different modes students would need to understand that:

$$z = x + iy = re^{i\theta} = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$$

= Re[z]+Im[z]=Abs[z]Exp[I Arg[z]].
The textbook and the written text use of *i*, representing the square root of -1, are textually similar to each other but different from the computer program's 'I'. Thus, the CAS constitutes a different representation in terms of interface and also textually.

4 Tools for analysis and interpretation

Discourse is a technical term used in both linguistics and social science that covers not only the language items, like vocabulary and syntax, but also how context and unconscious processes shape and are shaped by the language used. There is a unique mathematical vocabulary and language fundamental to learning mathematics which is central to the mathematical discourse. As well as the particular mathematical vocabulary, some of the characteristics of natural language also transfer into mathematical discourse, e.g. sentence word order, logical structures. Therefore mathematical discourse is a mixture of characteristics that are exclusive to mathematics as well as characteristics that are derived from the natural language being used [14]. Barton and Barton-Neville [15] discuss mathematics discourse in the specific context of undergraduate study:
[t]here is some evidence that there is a fundamental change in the nature of the discourse [at undergraduate, compared with school level]: not only do the normal features continue to get more complex, but also the use of mathematical discourse changes in several ways. Logical statements become the essence of mathematical

meaning, not just a way of describing mathematical relationships. The roles of definitions, axioms and theorems in mathematical argumentation are subtly indicated in their linguistic expression. General English is used in increasingly creative ways to describe the increasingly sophisticated nuances of mathematical concepts. Understanding mathematical discourse successfully at earlier levels may not be sufficient to understand it at advanced levels.

In a related vein, Duval [16] proposes that we can only gain access to mathematical information (e.g., concepts, skills and problem-details) through their representations. He states that "the characteristic feature of mathematical activity is the simultaneous mobilization of at least two registers of representation" [16, p. 3]. For example, the equation of the linear function $y = 2x$ and the 'graph' that represents this function are two different representations of the concept of doubling proportionality. Duval claims that learning mathematics is a translation between registers. The mathematical register in English is the distinct way in which mathematical meaning is expressed in that language. This view is compatible with the learning theories in 'advanced mathematical thinking' developed by Tall and Vinner [1].

This example of the De Moivre's theorem lecture illustrates the way students are expected to change between different representations and learning modes of the same mathematical object. The reading, listening and writing skills required are complex. From the data samples it is evident that the lecturer, by working in a number of modes – oral language, written language, mathematical notations, visual diagrams – organises the students' attention to each mode of representation through verbal and non-verbal cues. In this way, the students' experience switching representations modelled for them by a person whom they respect (at least with respect to his/her mathematical knowledge!). This modelling includes nuanced use of the English language as it is used within mathematics. For example, from the lecture note extract we have: "the geometric interpretation of multiplication is: multiply moduli and add arguments". The English is tricky, the emphasis in the first part of the quotation is on 'geometric', it is a 'representation' rather than an 'interpretation' and to describe multiplying as in part adding, is surely confusing – unless it is already known. Indeed many students would not have ever thought of multiplication geometrically.

This claim that the multimodality, essential to learning mathematics, is experienced in a lecture is borne out by data from another set of lecture observations. This second set of observations was of a third-year option module on homotopy and surfaces. Earlier, we observed discernible differences between the oral, live-writing, text notes and CAS representations. This shows how important it is to have mediation between the modalities so that the learner can model fluent switching between representations, as this is a key to good concept-image development and mathematical understanding. And this mediation is what students expect from a lecture; some of them even holding a belief that this may occur subliminally:

Sometimes you read it and you think, well I think I understand that but then you hear it and it might even be the same words as are in the notes but somehow you hear it differently to what you read and sometimes it just clicks that way.

An example of this mediation was given by the lecturer on explaining the 'syllable' as used in polyhedral surfaces '$a\bar{a}$': he drew a diagram of two ray segments emanating from a point (looking a bit like an undone zip) and said "you zip them up so they essentially disappear". Here, the lecturer is using an everyday item as a metaphor in a humorous manner, and this will help some students not only relate the meaning of $a\bar{a}$ with the topological transformation but also offer them a way of remembering it too. However, some will miss out on this opportunity for learning. For example, a student with English as an additional language who has learnt 'zipper' rather than 'zip' might miss that mediation and the memory-enhancing visual image that helps switch representations.

5 Language background of students

Another important factor in the effectiveness of lectures is the large number of students with English as an Additional Language (EAL) background now attending universities in English-speaking countries. (Our discussion is also pertinent for any students who study in other than their home language, such as English-background students studying in Spanish.) In some, and maybe the majority of cases, this is associated with difficulties in learning from lectures. However language is confounded with other variables, such as previous learning experiences, culture and class which may have a stronger effect but are harder to measure.

For example, Huang [17] has shown that EAL students experience difficulties in English academic listening at American universities. Research in New Zealand has found that their EAL university mathematics students, experience a 10% disadvantage in overall performance through lack of textural understanding [15]. It is likely that EAL students experience similar challenges in other countries, although the most able bilingual students may in fact derive an advantage from their languages [18]. Mathematics is popular with students who are under the impression that they will not be disadvantaged, as they perceive that mathematics is relatively language free.

A conclusion from this viewpoint is that EAL students may experience greater difficulties in understanding a lecture: they may have different perceptions regarding the lecture, and the way in which the material is presented may have a particular impact on their learning of the relevant topics. EAL mathematics students experience greater difficulties with technical rather than general English and have an unjustified reliance on symbolic modes of working.

6 Discussion

This analysis of lectures draws attention to the importance of the transitions between representations as well as the different modes of representation themselves. We see that transitions between representations are sometimes made by a lecturer in a relaxed, humorous fashion. Students who are culturally aligned with the lecturer, both culturally and linguistically, like this and can make learning gains from transitional gambits that are fun or provide an image or metaphor to 'hook' representations together. Lecturers do have opportunities to make alignments with their students in the representations they choose. But for the transitions between representations, which we have argued are central for learning in lectures, the main challenge is that student bodies are very diverse and appealing to one cultural or linguistic group might exclude another. (For example, in times hopefully past, in some lectures, female students have felt excluded from male-culture banter that engaged male students.)

What we have shown here is that lecturers can take opportunities to assist students to make links between various representations of mathematical concepts. These links between representations form the basis for deep learning and fluency in working with mathematical ideas.

A lecturer personifies the content of his/her lecture. In mathematics, abstraction is central, so the lecturer's personal mediation helps learners connect to the abstract nature of the subject through its multiple representations. The emotional connection between lecturer and student body is important and not only the lecturer's representations but also the transitions play an important role in learning in lectures. The transitions can be fleeting, like eye contact, significant pauses, arm gestures (such as vectors) and personalised stories, images, models and metaphors. The lecturer's presentation of mathematics makes mathematics human, quirky and interesting; makes it worth doing.

Figures 1 and 2. Spoken text and accompanying hand written text

... the implicit assumption here is that n is actually a positive integer and in fact I should make that explicit (WRITES) "n a positive integer" but it turns out in fact that it holds for all n and it's fairly easy to prove for negative integers and it's quite obviously true for n zero n a fraction is a little more difficult and in fact there's a sort of way round that which you'll deal with shortly it turns out and I don't I won't prove (…) I'll just state the result so it turns out that this result also holds for negative (WRITING) "negative integers"	Consider z^2 with $z = (\cos\theta + i\sin\theta)$ $z^2 = \cos(\theta+\theta) + i\sin(\theta+\theta)$ $= \cos 2\theta + i\sin 2\theta$ $(\text{cis}\,\theta)^2 = \text{cis}(2\theta)$ In general we find $\boxed{(\text{cis}\,\theta)^n = \text{cis}(n\theta)}$ (n is +ve integer) de MOIVRE'S THEOREM. It turns out that this result is also true for negative integers.

Figure 3. Lecture notes

Suppose now that z_1 and z_2 are complex numbers with polar forms

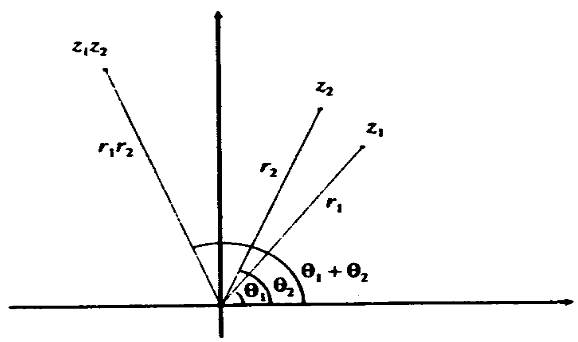
$$z_1 = r_1\text{cis}\theta_1, \quad z_2 = r_2\text{cis}\theta_2$$

Then

$$\begin{aligned} z_1z_2 &= r_1(\cos\theta_1 + i\sin\theta_1) \,.\, r_2(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)) \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \\ &= r_1r_2\text{cis}(\theta_1+\theta_2) \end{aligned}$$

Thus the geometric interpretation of multiplication is:

multiply moduli and add arguments.



There is an important special case of this result. Let $z = r\text{ cis }\theta$. Then

$$z^2 = r^2\text{cis }2\theta.$$

Further

$$\begin{aligned} z^3 &= z^2z \\ &= r^3\text{cis }3\theta, \end{aligned}$$

and in general, for any positive integer n,

$$z^n = r^n\text{ cis }n\theta.$$

It can be shown that this result is also true if n is a negative integer or zero. The result is known as *de Moivre's Theorem* and has many uses, of which we shall consider two. The first is an example of the way in which complex numbers can be used to derive results which are only about real numbers.

Figure 4. Mathematica® 5.2 files for de Moivre's theorem (abridged)

3.3.8 Expressions Involving Complex Variables

Mathematica usually pays no attention to whether variables like x stand for real or complex numbers. Sometimes, however, you may want to make transformations which are appropriate only if particular variables are assumed to be either real or complex.
The function ComplexExpand expands out algebraic and trigonometric expressions, making definite assumptions about the variables that appear.

ComplexExpand[*expr*]	expand *expr* assuming that all variables are real
ComplexExpand[*expr*, {x_1, x_2, ... }]	expand *expr* assuming that the x_i are complex

This expands the expression, assuming that x and y are both real.

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ComplexExpand[Tan[x + I y]]
```

$$\frac{\mathrm{Sin}[2\,x]}{\mathrm{Cos}[2\,x]+\mathrm{Cosh}[2\,y]}+\frac{i\,\mathrm{Sinh}[2\,y]}{\mathrm{Cos}[2\,x]+\mathrm{Cosh}[2\,y]}$$

There are several ways to write a complex variable z in terms of real parameters. As above, for example, z can be written in the "Cartesian form" Re[z] + I Im[z]. But it can equally well be written in the "polar form" Abs[z] Exp[I Arg[z]].

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