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# Software Tools for Geometrical Problem Solving:
# Potentials and Pitfalls

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## Abstract

Dynamic geometry software provides tools for students to construct and experiment with geometrical objects and relationships. On the basis of their experimentation, students make conjectures that can be tested with the tools available. In this paper, we explore the role of software tools in geometry problem solving and how these tools, in interaction with activities that embed the goals of teachers and students, mediate the problem solving process. Through analysis of successful student responses, we show how dynamic software tools can not only scaffold the solution process but also help students move from argumentation to logical deduction. However, by reference to the work of less successful students, we illustrate how software tools that cannot be programmed to fit the goals of the students may prevent them from expressing their (correct) mathematical ideas and thus impede their problem solution.

## Introduction

Dynamic geometry systems provide access to a variety of geometrical objects and relations with which users can interact in order to construct and manipulate new objects and relations. In this sense, microworlds based on dynamic geometry systems, including the software under discussion in this paper Cabri-Géomètre (henceforth called Cabri), are little different from other microworlds for exploring geometry. However, geometry microworlds do differ in the processes that govern construction and manipulation. In Cabri, users interact directly through the mouse with the tools provided by the system in order to build, manipulate and explore figures. As Laborde and Laborde (1995) have pointed out; "Cabri-Géomètre provides a 'real' model of the theoretical field of Euclidean geometry in which it is possible to handle in a physical sense the theoretical objects which appear as diagrams on the screen". The visual artefact produced by interaction with Cabri resembles the traditional representation of paper and pencil geometry. Yet, in contrast to paper and pencil representation, the visual output of Cabri is not a drawing of one instance of a geometry figure, but rather can be moved – or dragged – around the screen with its constructed properties preserved. Thus the system offers feedback that can be used to distinguish "diagrams drawn in an empirical way from diagrams resulting from the use of geometrical primitives" (Laborde and Laborde, 1995). This distinction between drawings that can be 'messed up' and figures whose geometrical properties are retained under dragging can be exploited as a window through which students can come to appreciate the theoretical aspects of geometry (for a discussion of messing up, see Healy, Höelzl, Hoyles & Noss, 1994, and of dragging more generally, see Höelzl, 1996).

The interactivity fostered by dynamic geometry systems is different from that in the best-known microworlds that preceded them. For example, in the original Turtle Geometry microworld, interactions were entirely mediated by a symbolic language[i]. In order to produce geometrical designs on the computer screen, users have to write symbolic procedures to organise and execute the primitive objects of the system. Guided by the visual feedback resulting from their programming activity, users can edit or debug their symbolic constructions until they are satisfied with their final product. Any variables that define a turtle's drawings are controlled symbolically through the use of inputs to the programming code.

The critical difference between programming environments and direct manipulation interfaces revolves around this emphasis in interaction on symbolic control in the former case, as opposed to visual control in the latter. For some, the necessary presence of the formal representation of a programming language was (and still is) a bonus (see diSessa, Hoyles & Noss, 1995 for an extended discussion). The programming language provides both a text on which to reflect and a means by which learners can communicate their problem solving strategies. Eisenberg (1995), for example, characterised a programming environment as "providing a rich linguistic medium in which students can develop their own domain-orientated "vocabularies" " (p.179). Proponents of programming environments (for example, diSessa, 2000; Hoyles and Sutherland, 1989; Hoyles and Noss, 1992; Hoyles, Sutherland & Healy, 1992) claim that it is through these vocabularies that students come to express their mathematical ideas. For others, formal expression limits accessibility. For example, Laborde and Laborde (1995) have argued that the effort needed to follow the syntax of a system can be so overwhelming that geometrical problems are reduced to problems of language: it is only through direct manipulation that users can handle and change not only objects but also the relations between them.

It is certainly true that with Cabri's drop down menus it is relatively easy to construct and explore complex geometrical figures. What is still open to question is the relationship of learners' problem-solving strategies in Cabri with those of more expert mathematicians. Goldenberg and Cuoco (1997) have stressed the importance of recognising that the interactions with computer-based microworlds of those who already know the mathematics involved are not necessarily the same as those still coming to grips with the relevant concepts and processes. (Our use in this paper of the term learners or students is meant to denote the second set of users). Will students' use of Cabri tools help them to make transitions to and from theoretical constructs and their empirical manifestations? Or will students simply bypass mathematics and come to rely solely on empirical feedback to make judgements and solve problems? Or more troubling still, will the use of tools lead students down into mathematical cul-de-sacs from which they can neither progress nor backtrack?

In this paper, we explore these questions by reference to students' use of software tools for geometrical problem solving. We take as a starting point a sociocultural position, which argues that the actions students perform together with the knowledge they construct are shaped by the tools used (see for example Cole and Wertsch, 1996; Wertsch 1998 and Bartolini Bussi; Mariotti, 1999). We also recognise the influence on learning of the activities within which the software tools are to be used, and the nature and content of the teacher interventions. Our aim in this paper is to examine in detail how student strategies for constructing, conjecturing about and proving geometrical relationships are mediated by Cabri tools. To achieve this aim, we present an analysis of students' work during one activity that formed part of a teaching sequence designed to help students, through Cabri use, to make connections between empirical and theoretical modes of working. To set the context for this analysis, we begin by describing the teaching sequence as a whole, its aims and an outline of its activities. We then present some of the potentials and pitfalls of using the software by reference to successful and less successful solution strategies in the final activity in the sequence. In theory, both successful and unsuccessful strategies made sense in terms of the mathematical demands of the task, and both involved students in attempting to capitalise on the possibilities afforded by the Cabri tools. In practice, however, only one strategy led to successful task completion. In the concluding sections, we discuss why this was the case and point to possible ways forward.

## Construction and deduction with dynamic geometry tools

We have noted in our studies of students interacting in a range of microworlds that while constructing on the computer, some (although not all) students spontaneously provide justifications of their actions and explanations of why their actions would produce the outcomes they require (see for example, Healy and Hoyles, 1999). We also found that students generally want to explain why certain phenomena are observed on the computer screen, especially if visual feedback is surprising. A requirement to explain can also be institutionalised as part of the mathematics classroom culture, in a whole-class setting or as part of group work. Our aim was to investigate the effects of trying to help the students make one further step; that is to devise classroom-based activities where students would not only be required explain the output of their activity, but also to use their explanations as bases for making logical deductions.

Given that to construct on the computer requires explicit attention to the processes used, we hypothesised that explanations derived from computer interactions would be more readily formalisable into logical deductive chains of reasoning, than those arising from activities in which construction processes remain at an implicit level (for an earlier discussion of this approach, see Chazan and Yerushalmy, 1998). Our goal was therefore to design construction activities in which students had to attend to and make explicit the relationships they used in their constructions – the “given” properties and relationships – and distinguish these from properties that could be deduced as necessary, after observing their invariance on dragging in Cabri.

To achieve this goal, we organised the computer activities into a teaching sequence comprising four phases. In the first phase, students were required to construct a geometrical figure with Cabri, identify and describe the properties and relations they had used in their constructions, use the computer tools to generate and test conjectures about further properties that might also be true, and finally make informal explanations of why they might be true. During this phase of work, the role of the teacher was to facilitate and prompt rather than to instruct. In the second phase, the teacher played a more directive role, bringing the students together to introduce them to writing proofs. Students were helped to organise the explanations they had generated during the computer activity into logical deductive chains. The third phase was essentially a repetition of the first phase with the added requirement to write a proof of one conjecture. In the final phase (the core of this paper), the students were given a challenging problem, in which they were asked to construct a figure with Cabri using any of the tools with which they had become familiar, and identify, explain and prove any of its properties.

## The context of the teaching sequence

To be effective in the classroom, it is almost self-evident that teaching must fit the curriculum context. The students who followed our teaching sequence had been introduced to an approach to proving prescribed in the statutory mathematics National Curriculum for England and Wales, in which emphasis is placed on engaging in proving in a range of contexts rather than on learning proofs in for example Euclidean Geometry. Thus students were used to engaging in activities involving conjecturing, experimenting and explaining (for a more detailed picture of this curriculum approach, see Hoyles, 1997).

To examine the proof conceptions resulting from this approach, we conducted a nationwide survey of 2459 students (aged 14-15 years) of above-average attainment in mathematics[ii]. In brief, the results from this survey showed that the majority had developed considerable expertise in generating and organising data and in constructing and explaining conjectures, especially within the arithmetic/algebra domain. But, although most recognised that a proof required a general, analytic justification, only a few students could actually construct a valid deductive argument. The students also had considerable difficulties engaging with “formal proofs”[iii]: they found them hard to evaluate and difficult to construct[iv]. Having sketched the landscape of students’ approaches to proof, we designed a teaching sequence, which would exploit the students’ confidence in experimentation and informal explanation, and attempt to link these processes, through computer construction, with verification and logical argument.

The teaching sequence was undertaken by six groups of six students from 3 different schools. The schools were located in the same area of London, one was mixed-sex, one girl-only and one-boy only. The students were all aged between 14 and 15 years and identified by their teachers as high attainers (approximately representative of the top 20% in mathematics). None of the students had used Cabri before taking part in this work, nor did they have any prior experience of creating sequences of logically justified geometrical statements. The activities took place at their respective schools, over a period of six to eight weeks at the times allocated for mathematics lessons, but in a room where the six students were separated from the rest of the class. It is not our intention here to present a complete analysis of the experiments (a summary of the results appears in Hoyles and Healy, 1999), but rather to focus on students' solution strategies in the final computer task, *the angle bisectors construction*. However, in order to make sense of student responses to this activity, we first summarise the activities in the previous phases of the sequence, and describe the Cabri tools to which the students had been introduced.

All the computer-based activities were designed for use with Cabri I. Cabri II had been released, but we decided not to use it for three reasons: we were more familiar with the old version, not all the machines available for use within the schools could support Cabri II at that time, and most relevant here, the set of tools available in Cabri II for our purposes were significantly different from those of Cabri I. The first two issues we could have overcome, but the third ruled out the use of the newer version. Some of the Cabri I tools we intended to embed in our activities had changed radically and one had disappeared completely. Given that we were unable to modify or reprogram the new tools, we chose to structure our activities around the original Cabri I.[v]

### The construction and conjecturing phase

In the first phase of the teaching sequence, we introduced the students to a set of tasks involving congruent triangles. Our aim was that students would discover the conditions for congruency by finding out whether or not pairs of triangles, one given and one constructed according to certain conditions (for example with equal sides, or with two equal angles and one pair of equal sides), remained exactly the same, or congruent, under dragging. The students opened a file containing a general Cabri triangle, ABC. They were then asked to find minimum sets of properties required to construct a triangle congruent to ABC. To help them specify the properties, we added two construction tools, or macros, to the Cabri construction menu. The first macro, `compass,` allowed a user to construct line segments equal in length to a given segment; the second, `angle-carry,` enabled them to do the same for angles. Hence triangle DEF in Figure 1 was constructed by using the `compass` tool to copy the lengths AC and CB and the `angle-carry` tool to copy the angle $A\hat{C}B$.

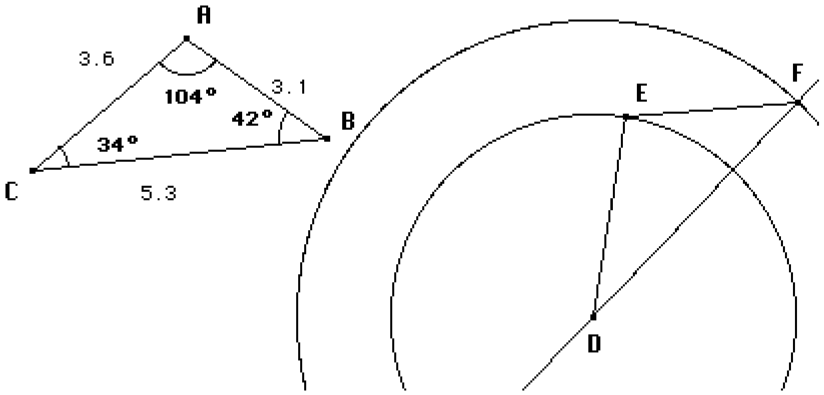
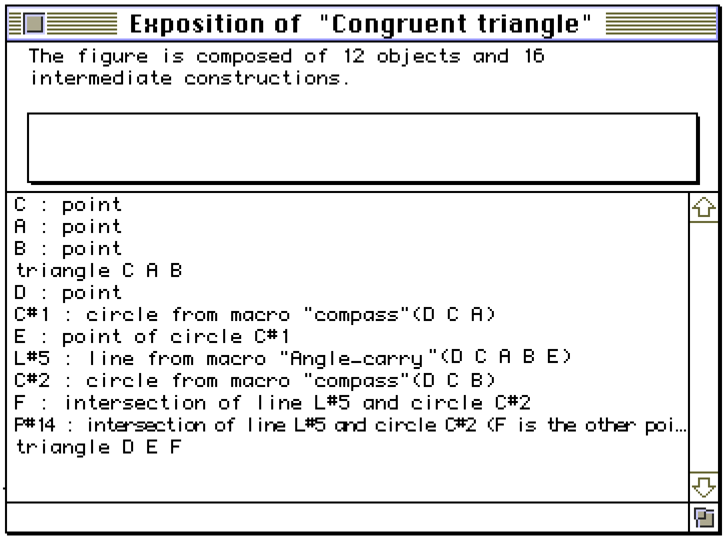


**Figure 1: A triangle DEF constructed to preserve the lengths of two sides and one angle of triangle ABC**

Students could verify whether or not the second triangle, DEF, was always the same as the first under dragging and so could convince themselves that the two triangles were congruent or not. In addition to the dragging test, students could make use of various other tools during their verification activities: they could ask for measures of any segments and angles to check invariant relationships, or they could use the Cabri tool, `Check-property`. We encouraged students to use this latter tool, as it required them to declare explicitly their conjectured relationships between objects and to test whether visually-apparent properties (such as equality) were true in general. Additionally, we thought that by providing a counter-example to illustrate when conjectures were not true, the use of this tool would assist in the students in 'seeing' important relationships. The `Check-property` tool is one of the tools that has changed with the advent of Cabri II; the counterexample feature has gone and properties are declared true only on the basis of specific cases.

As well as verifying their constructions, we required students to identify explicitly the properties they had used to build their second triangle, and so wished to take advantage of the tools Cabri provided to facilitate communication. We conjectured on the basis of our Logo experience, that tools that described the construction process in linguistic terms would take on special significance when construction was achieved by a set of physical manipulations. Two tools were available in Cabri I, but not in Cabri II. First, a user could replay the entire construction process using the `History` tool, which included a step-by-step description of the macros used. For the congruent triangles tasks, this level of detail of steps in the macros turned out to obscure the pertinent triangle properties. Second, with the `exposition` tool, the user could display a symbolic description of the menu selections and mouse clicks used. Since this output closely matched what we required, we encouraged the students to use it as a means to help them to reflect on their constructions and decide on the properties necessary and sufficient for congruency, (see Figure 2 for the description of the figure that appeared in Figure 1, given by the `exposition` tool).



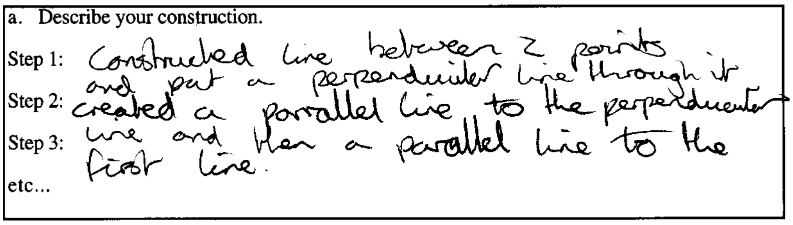
**Exposition of "Congruent triangle"**

The figure is composed of 12 objects and 16 intermediate constructions.

```
C : point
A : point
B : point
triangle C A B
D : point
C#1 : circle from macro "compass"(D C A)
E : point of circle C#1
L#5 : line from macro "Angle-carry"(D C A B E)
C#2 : circle from macro "compass"(D C B)
F : intersection of line L#5 and circle C#2
P#14 : intersection of line L#5 and circle C#2 (F is the other poi…
triangle D E F
```

**Figure 2: The procedure  used to create the triangles ABC and DEF**

In the second phase of the teaching sequence, the teacher (one of the authors) introduced the students to writing formal proofs. The students were brought together in a group away from the computers, and asked to share their explanations of what had happened on the computer. They were then shown how to organise these explanations into logical deductive chains of argument. The students then experimented with more Cabri construction tools, including parallel and perpendicular lines and angle bisectors, before working on the next set of constructing and conjecturing activities in phase three. These activities involved the construction, exploration and deduction of properties of familiar quadrilaterals (for example, parallelograms, rectangles, squares and rhombi). The students were asked to construct each quadrilateral and identify the properties that they had chosen to use in order to define it (what we called, the *givens*). Figure 3 shows how one student, Theo, described the construction of a rectangle and pulled out a subset of these constructions to identify its given properties[vi].



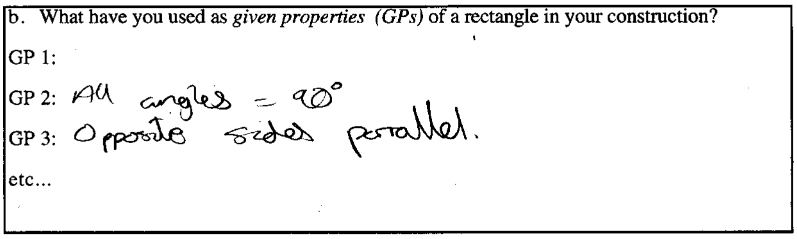
a. Describe your construction.

Step 1: Constructed line between 2 points and put a perpendicular line through it

Step 2: created a parrallel line to the perpendicular line and then a parallel line to the

Step 3: first line.

etc...



b. What have you used as *given properties (GPs)* of a rectangle in your construction?

GP 1:

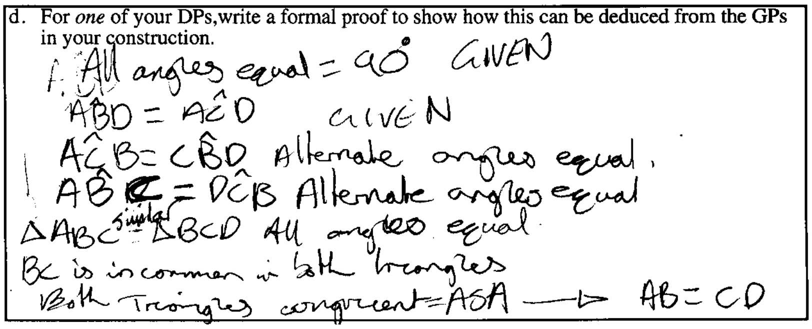
GP 2: All angles = 90°

GP 3: Opposite sides parallel.

etc...

**Figure 3: Choosing the given properties of a rectangle construction**

Next, the students explored their Cabri quadrilateral by dragging, in order to isolate further properties that always appeared to hold true. Theo, for example, noted that the opposite sides of his rectangle were always of equal length. Finally, the students had to select one of the deduced properties (DPs) from their list, and write a proof to show how it could be deduced from the properties they had declared as their givens. Figure 4 presents Theo's argument in which he used his givens to prove congruency and then deduce the equality of opposite sides.



d. For *one* of your DPs,write a formal proof to show how this can be deduced from the GPs in your construction.

All angles equal = 90° GIVEN

$A\hat{B}D = A\hat{C}D$ GIVEN

$A\hat{C}B = C\hat{B}D$ Alternate angles equal.

$A\hat{B}C = D\hat{C}B$ Alternate angles equal

$\triangle ABC \overset{similar}{=} \triangle BCD$ All angles equal

BC is in common in both triangles

Both Triangles congruent = ASA ⟶ AB = CD

**Figure 4: A proof to deduce that the opposites sides of a rectangle are of equal length**

After the first three phases of work, the students were comfortable using Cabri and were becoming accustomed to the requirements of the activities; that is to experiment, conjecture, explain and finally transform explanations into logical deductions. They then moved to the final problem solving phase, where we hoped they would bring together all their knowledge of the tools of Cabri to solve a new challenge.

## The problem-solving phase: the angle bisectors construction

In the final activity in the teaching sequence, the angle bisectors construction, the students were again asked to construct a quadrilateral, but this time, the quadrilateral was one with which they were not familiar. Students were given a property of this unknown quadrilateral, namely that the angle bisectors of two adjacent angles crossed at right angles, and asked to construct it. Their challenge was to discover other properties that also had to be true if the quadrilateral was to satisfy its initial conditions, by experimentation with any Cabri tools they wished, and then to put together a logical argument starting with the initial conditions and deducing these new properties.

In the following sections, we present some of the students' strategies in response to this task to illustrate how interactions with the Cabri tools on some occasions facilitated transitions to and from conjectures to proofs (in the sense proposed by Garuti, Boero and Lemut, 1999), while on others served as obstacles to the development of promising ideas.

### *Successful student responses: smooth transitions between creation, construction, conjecture and deduction*

We begin with a description of how two students successfully used the software to solve the challenge. This pair exploited a mixture of creation and construction tools to produce and explore the quadrilateral required, conjecture about the necessary geometrical relationships involved, verify their conjectures in particular cases, explain why they must be true in an informal way, and finally write a deductive proof based on their experimentation.

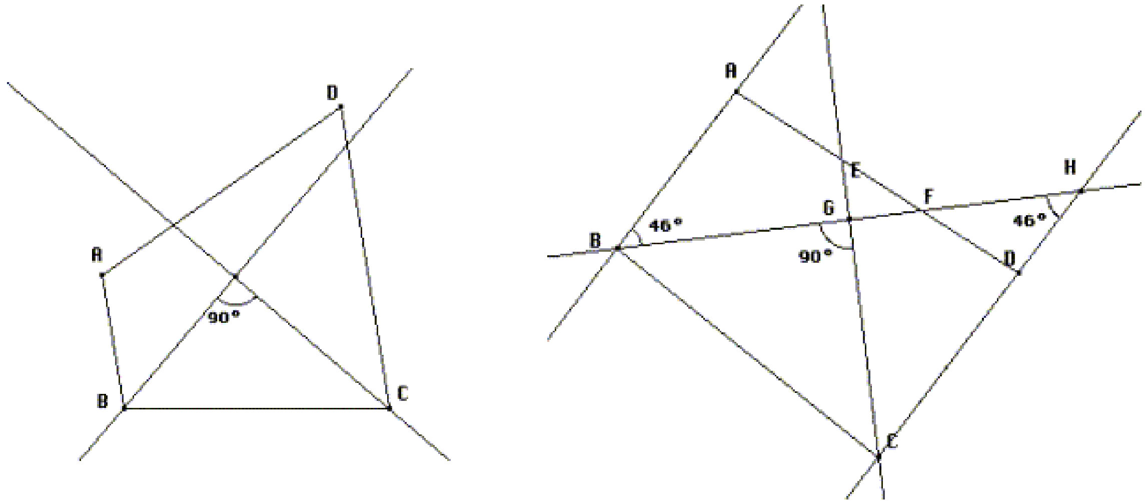
Tim and Richard began by creating a quadrilateral ABCD consisting of 4 line segments arranged in no particular configuration. After labelling the four vertices, they added the angle bisectors of angles $A\hat{B}C$ and BCD, and measured the angle where these two lines crossed. They then carefully dragged the vertices of the quadrilateral until this angle measured 90° (see Figure 5a). Thus the constraints of the required quadrilateral were not constructed but simply created “by eye”. At the moment at which the angle between the two bisectors measured 90°, the boys noticed that BA was parallel to CD, and they almost immediately conjectured, on the basis of this one example, that whenever the two angle bisectors were at right angles, BA must be parallel to CD. We might call this conjecture an *abduction*. Certainly, wanting to hypothesise a theorem to account for the figures on the screen guided the boys' activity, prior to their conjecture and also subsequent to it (see Arzarello, Micheletti, Olivero, & Robutti, 1998 for a discussion of how abduction is generally used at the conjecturing stage with Cabri). To investigate further, Tim and Richard dragged the quadrilateral purposefully in ways that both did and did not preserve – *visually* – the conjectured parallel property, noting the effect on the angle between the angle bisectors.

The `Check property` tool, with which the boys were familiar from their earlier work, was then used in an attempt to obtain a computer verification for their conjecture. They made the angle bisectors cross at right angles by making the measure equal to 90°, and then used the `Check-property` tool to test if BA and CD were parallel. The pair were disappointed and puzzled that the property that the two lines were parallel was declared *not* to be true in general. But the Cabri tool provided a justification for its judgement, by producing a counterexample that helped the boys to make sense of the feedback. In the counter-examples presented, the angle bisectors clearly did *not* cross at right angles, leading the boys to realise that the parallel property had been declared not true in general, because the given condition of 90° between the angle bisectors had not been kept invariant – as it had not been constructed.

After this surprising incident, the boys became more confident that the two properties of perpendicularity of angle bisectors and parallelism of one pair of parallel sides mutually depended upon each other.[vii] The final strategy used by this pair to verify their conjecture, was to construct a line parallel to CD passing through B. They found that whenever point A was dragged onto this parallel line, the two angle bisectors crossed at right angles, but if A was moved away from this line, the angle bisectors were no longer perpendicular.

By this time the boys were convinced that the quadrilateral satisfying the given requirements had to be a trapezium and they stopped searching for confirming instances. They had formulated a conjecture, that, when in its written form, was phrased in the following conditional terms: whenever the lines are parallel the angle bisectors are perpendicular. In this formulation, the property to be deduced appears in the first part of the conditional statement, and the given property in the second part. However, with the properties under their physical control, the boys were able to appreciate the necessary and sufficient relationship between the two properties: they could drag their figure to make the angle bisectors perpendicular, or to make the sides parallel, thus investigating at the same time both their conjecture and its converse.

Tim and Richard knew that they had now to come up with a proof of their conjecture. They returned to manipulate the quadrilateral, but now with A always on the line parallel to CD. They measured several angles, and watched out for angles that stayed equal as they dragged the vertices. Thus, they were using measurement as a tool to help them identify a relationship, a property, that could be used as a link in the logical chain, between the given right angle between the angle bisectors, and the end point of parallel lines. The boys noticed that the angles where the angle bisector of angle $A\hat{B}C$ cut the segments BA and CD were always the same and realised that if they could prove this, they would be able to conclude that the two line segments must be parallel (they were aware of alternate angles and parallel lines). In fact, this turned out to be more complicated than the boys first imagined as the angle bisector did not always cut the *segment* CD. They were unable to make more progress until the teacher intervened to suggest that they create a *line* through CD, which would always cut the angle bisector at some point, say H in Figure 5b. Tim and Richard made this addition to their construction and measured the relevant angles, $H\hat{B}A$ and $B\hat{H}C$ as shown in Figure 5b.

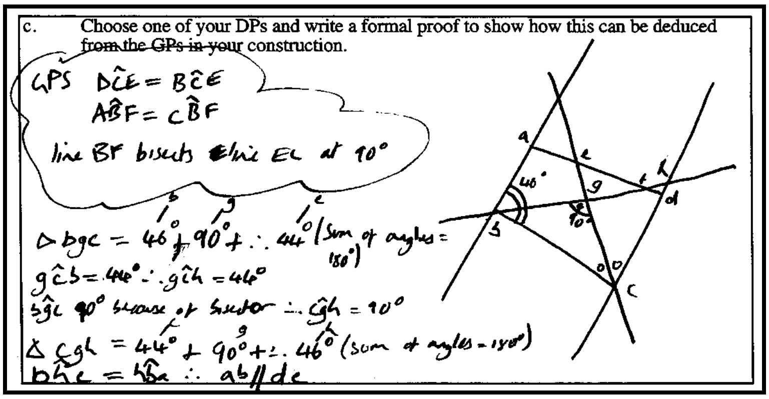


a: The Cabri figure in its original configuration

b: The extended Cabri figure with two more angles measured

**Figure 5: Using measurement to help to identify an invariant to use in a proof**

The Cabri figure was subjected to no further manipulations while the pair concentrated on building an explanation as to *why* the two angles $H\hat{B}A$ and $B\hat{H}C$ were always equal. They used the figure they had generated with its specific measurements as a generic example, to help them to formulate a general argument using their knowledge of the sum of the interior angles of a triangle, the equal angles generated by the angle bisectors and the perpendicularity of the angle bisectors. Richard's attempt to organise this argument into a deductive proof is presented in Figure 6. His written work shows how he used the scaffolding of *particular* numbers to formulate his proof, although the equality of the two numbers was used to identify a property that he recognised was generally true.


c. Choose one of your DPs and write a formal proof to show how this can be deduced ~~from the GPs in your~~ construction.

GPS $D\hat{C}E = B\hat{C}E$

$A\hat{B}F = C\hat{B}F$

line BF bisects line EC at 90°

$\triangle bgc = 46^\circ + 90^\circ + \therefore 44^\circ$ (sum of angles = 180°)

$g\hat{c}b = 44^\circ \therefore g\hat{c}h = 44^\circ$

$b\hat{g}c$ 90° because of bisector $\therefore c\hat{g}h = 90^\circ$

$\triangle cgh = 44^\circ + 90^\circ + \therefore 46^\circ$ (sum of angles = 180°)

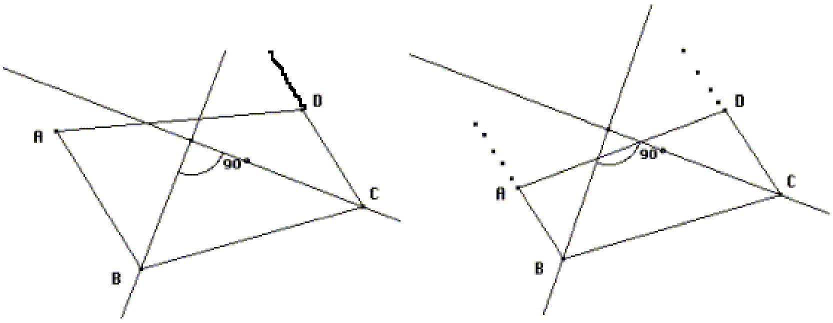
$b\hat{h}c = h\hat{b}a \therefore ab // dc$

**Figure 6: Richards' argument that side BA is parallel to side CD**

### *Tools to assist in "seeing" the necessary relationship*

Tim and Richard's strategy of mixing creation and construction tools to come up with a quadrilateral ABCD that could be dragged while satisfying the properties that we had specified was adopted by a number of other students. Not all of these students however were so quick to *see* that the sides BA and CD were parallel when the angle bisectors were at right angles. We suggested to these students that they use Cabri tools to keep track of the locations of the vertices A and D that produced the required perpendicularity in the figure, to help them notice the required relationship.

Some students used the `locus` tool so they could record a trace of the movement of the vertex as it was dragged – painstakingly – in a way that kept constant the 90° angle where the angle bisectors crossed. (Figure 7a). Although this strategy helped the students to determine the solution path for the chosen vertex, many still did not see the *relationship* between the sides of the resulting quadrilateral. They noticed that the locus of solutions was a line extending from CD, but *not* that this line was parallel to the opposite side, AB. Some students were able to find the path of vertices that produced perpendicularity by simply placing basic points at all the locations where this property was satisfied. This strategy had the advantage that multiple paths could be recorded and hence the relationship between the paths was more apparent (see Figure 7b). (The `trace` tool available in Cabri II that shows the locus of more than one point simultaneously would certainly have been helpful here).



a: Creating a trace with the locus tool

b: Dropping basic points to trace two paths

**Figure 7: Two ways to help visualise the relationship between BA and CD**

Up to this point, we have considered only one way of setting up the original quadrilateral ABCD, where the given property of perpendicularity of angle bisectors was satisfied visually, rather than by construction. As we have indicated, this was the most common approach adopted by our students and is ideally suited to problem-solving with Cabri. In fact, all of the students who were successful in this task adopted this strategy of loosening a constraint, referred to as "letting go" by Love (1996), or "relaxing" by Goldenberg (1999). But not all the students opted for this approach, which was not altogether surprising since, after all, we had asked the students to *construct* the quadrilateral ABCD, and the meaning for construction that we had stressed earlier in the teaching sequence, was to ensure that a given property was invariant under dragging. We now turn to describe the work of students who did start from the construction of perpendicular angle bisectors, but in so doing came up against unexpected difficulties.

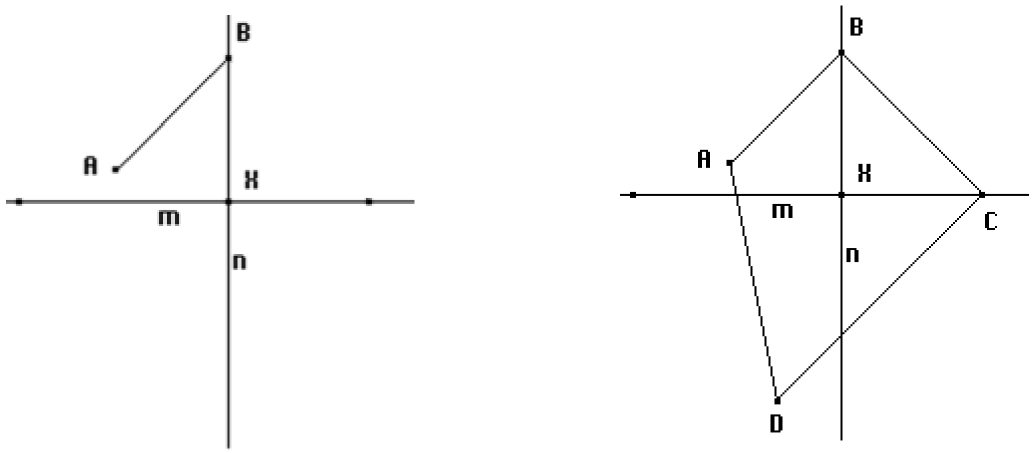
### *Less successful student responses: constructing the givens and reaching an impasse*

Among the students who started by constructing the perpendicularity of the angle bisectors rather than creating it, we observed two different strategies. The first strategy was to conjecture that the given properties would be satisfied by a *particular* quadrilateral, say a parallelogram, and then to construct this quadrilateral. This seemed a promising approach, but in practice resulted in students experiencing two difficulties that were hard to overcome with the tools available. First, there was some confusion over which properties were given and which were to be deduced, since  students used the parallel property in their construction.

Second, it was hard to convince students who managed to finalise a construction of a particular quadrilateral, that they had not in fact found a complete solution. The students had yet to encounter the aesthetic of geometry, which specified that one has to find the most general solution to any problem in order to pinpoint a critical property in a proof. We tried to convince them that they should find the property of the parallelogram, which specifically related to the perpendicularity of the angle bisectors, by relaxing some its properties in turn using the tool, `redefine an object`. This turned out to be of rather little help, first as the functionality of the tool was not transparent and, second, since the students, in trying to use it, lost control of the problem and became demotivated.

The second strategy followed closely how the problem was set, and involved using Cabri tools to construct the quadrilateral in such a way that the properties specified in the presentation of the problem were preserved under dragging. As the following story illustrates, this strategy came up against difficulties that the students could not predict.

Emily and Carrie wanted to construct the quadrilateral as requested. They created a line (m), dragged it until it was horizontal and constructed a point X on the line. They added a second line (n) perpendicular to m and passing through point X, intending that these lines would serve as the angle bisectors in the construction. Next, they constructed a point (B) on line n as one vertex of the quadrilateral and created a line segment joining B to new free point A to serve as another vertex (see Figure 8a). For the second side of the quadrilateral, they wanted to construct a segment BC such that the angle $X\hat{B}C$ would be equal to angle $A\hat{B}X$ and C would be on line m. Emily remembered that they had already used an `angle-carry` tool to construct congruent angles in the first set of activities The figure in front of Emily and Carrie at this time contained 5 points, 2 lines and a line-segment, as displayed in Figure 8a.



a: The two angle bisectors and two sides

b: An apparently correct construction of quadrilateral ABCD

**Figure 8: Constructing the quadrilateral around the angle bisectors**

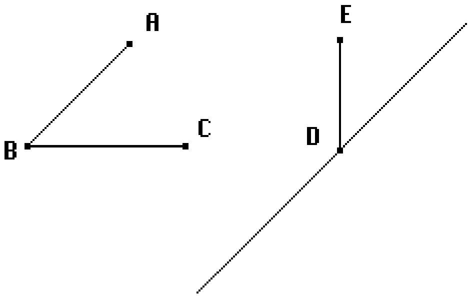
Before selecting the `angle-carry` tool from the construction menu, Emily traced out the line with her finger on the screen that they were expecting to produce. Carrie activated the tool and clicked on A, B and X, but could not complete the construction since the macro did not allow her to click on the points X and B again. It is not possible to reselect the same point when executing a Cabri macro. We had predicted this would be a problem when we were devising the activities and had provided a second macro, `angle-doubler`, that would copy an angle in the way that was required for this problem. We had decided that this second macro should be introduced exactly at this point, when it was needed. What we had not anticipated was the reaction of the students.

Emily did not understand why it was not possible to use a tool with which they were familiar and was frustrated that they were unable to reselect points. While Emily was bemoaning the limitation of macros with one of us, Carrie wanted to get on with the problem. But she also rejected the new `angle-doubler` tool, and chose to abandon their attempted construction. She decided instead to use an existing point that she labelled C (one of the two that had been used to create the line m) as the third vertex, perhaps motivated by the fact that the segment BC more or less followed the visual trajectory suggested earlier by her partner. She created a fourth line segment to complete the quadrilateral by connecting C to a point D, again so that angle $X\hat{C}D$ looked more or less the same as the angle $B\hat{C}X$ (see Figure 8b). Since this figure *resembled* a trapezium, we (and, we suppose, Emily when she rejoined her partner) were not aware that the given condition that the two lines m and n should be angle bisectors had not been constructed. It is a didactical problem to keep track of what students are doing when engaged in activities where tools can be direct manipulated, without careful attention at every step. The pair carried on exploring Carrie's construction oblivious to the fact that the equal angle property they had originally wanted to construct, was not satisfied.

We were particularly concerned by this incident, as a promising approach ended in confusion, leaving us all feeling dissatisfied. Could this have been avoided? Would it have been possible for us, as experienced teachers, to write the macro in such a way that the same points could have been selected more than once? In order to consider more closely the relationship between tool design and student activity, we re-examine the macro construction in some detail.

### *Redefining a macro*

The `angle-carry` macro that we had used was one of a collection of macros that came with the application when it was purchased. We had not constructed it ourselves, so our first step was to discover exactly how it had been built. We created the simple figure in which the angle $A\hat{B}C$ was copied to DE creating a congruent angle, used the `history` tool to replay the construction[viii] and noted, in the form of a Logo-like program, the intermediate constructions on which the macro was based (see Figure 9). Rewriting the procedure provided a means through which we could talk about the objects of the macro, and expressing initial objects as variables helped us to describe the different configurations of the actions we performed.



```
ANGLE-CARRY P1 P2 P3 P4 P5
MIDPOINT  P5  P1    =>(P6)
SYMPOINT  P2 (P6)   =>(P7)
MIDPOINT  P5  P3    =>(P8)
SYMPOINT  P2 (P8)   =>(P9)
BISECTOR (P9) P5 P4 =>(L1)
SYMPOINT (P7)(L1)   =>(P10)
LINE P5 (P10)
```

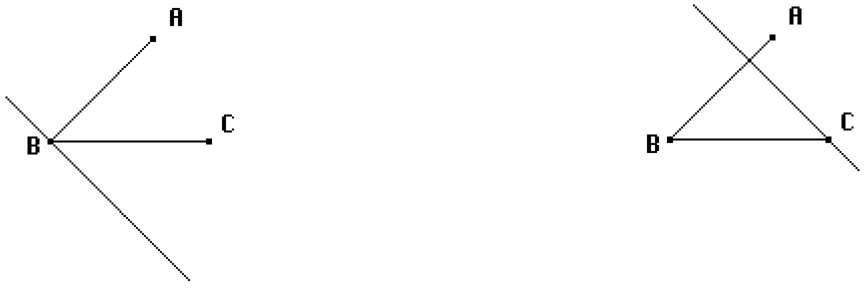
Copying an angle using angle-carry (`ANGLE-CARRY A B C E D`)

A Logo-like procedure representing the macro (brackets indicate intermediate constructions)

**Figure 9: The angle-carry macro**

The original angle-carry macro was defined using five points as initial objects and a line as the final object – or in Logo terms, it needed five inputs and output a line. Now, although it is not possible to click more than once on the same point when running a macro, it is possible that a macro will accept a point twice if one is creative about the choice of initial objects. We created a second macro, `angle-carry2`, in which the initial objects were three points and a line segment, but the final object was still the required line. The intermediate constructions in history produced by this new macro were identical to that of the first version, but with the new macro it was now possible to create an angle using only three points, as long as at least two of them were joined by a line segment.

One result of running this macro is shown in Figure 10a, where the angle $A\hat{B}C$ is to be copied on CB. It was executed by first clicking on A, B and C, in that order, and then clicking on the line segment BC. Emily's problem appeared to have been solved. But sadly not necessarily. Figure 10b shows another result of using the same macro, with the objects selected in the exactly same order: A, B, C and then the line segment BC. Why is there a difference? The reason is that the order in which a line segment is *originally* created determines the order in which the points at its ends will be interpreted in the macro. In Figure 10a, point C was created before point B, and this meant that when the macro was used it is equivalent to angle-carry A B C C B. In Figure 10b, B was created first so the functioning of the macro can be represented as angle-carry A B C B C[ix].



a: `ANGLE-CARRY A B C C B` b: `ANGLE-CARRY A B C B C`

**Figure 10: In the angle-carry2 macro the order of points is defined by the order in which the points of the line segment are created**

## Cabri and the future

Interacting with Cabri can help learners to explore, conjecture, construct and explain geometrical relationships, and can even provide them with a basis from which to build deductive proofs, as we have illustrated in the work of our successful students. Many of the strategies students used and the experimental stance that they adopted to challenging problems were unimaginable before the advent of dynamic geometry systems. We have illustrated how the Cabri tools mediated students’ problem-solving activities and shaped their developing conceptions. For example, as a result of their Cabri interactions, the students’ understanding of given properties and their place in the proving process became closely interconnected with their appreciation of geometrical construction and the use of properties in any construction to define a Cabri figure.  We also showed how two students combined geometrically-based constructions with visual methods to come up with conjectures, to ‘see’ how they might be explained, and finally, with the help of measuring tools, begin to formulate a proof.

From all our observations of students’ interaction with Cabri, we have become increasingly aware that the mediation of students’ activities by the software is not necessarily positive for their engagement and for their learning (see also Jones, 1998). Students will not always follow the directions planned for them, however carefully we try to structure their activity. This is inevitable given the fact that, if we encourage experimentation, students will experiment and notice events not necessarily on the teacher's agenda. We have called this phenomenon the *play paradox* (see Noss & Hoyles, 1996). But this is not the only problem. As we have shown in the work of our less successful students, learners can also find themselves in a position where they are unable to use the tools they have in mind, even if they are convinced that their use would make sense mathematically, and they are familiar with how the tools should work.

This points to the importance of the decisions made by those who design software tools and interfaces. We need to recognise how difficult it is for students to come to understand and accept the syntactic limitations of menu-driven macros, and that although they can add or remove tools from the menus in Cabri, it is not possible to modify existing tools according to their needs. This can be a hard lesson to learn when you are a learner yourself and unsure when faced with a problem if it is your knowledge that is at fault or the way the tools work (for a discussion of the same issue from  a Logo perspective, see Hoyles, 1996).  When using Cabri, students have to learn that only tools based on action can be added to the environment, as well as master more general heuristics of successful Cabri use. Even expert teachers cannot (or at least *we* do not know how to) build new tools that support different modes of verifying constructions or that offer new ways of communicating our physical actions upon the objects of the system. Nor can expert teachers choose to interact with a symbolic language rather than through direct manipulation when it might be useful – as in the case of the `angle-carry` macro. Clearly most users of Cabri will not wish to take advantage of this opportunity. But some will and, as we have hoped to show in this paper, for expert teachers (and students) it might help to avoid the frustration we have described.

Cabri I offers considerable potential for exploring geometry. So does Cabri II and so will all the new versions that will undoubtedly come on stream[x]. But it is only natural, given any computer application, that users will *always* want just the tool that is not quite available, or become disconcerted by modifications to tools to which they have become accustomed or upset by the disappearance of tools they particularly valued. One way around this problem may be for Cabri, and any other dynamic geometry system, to have some programmable functionality; that is to integrate the complementary strengths of its direct manipulation interface with a programming capability, in the way envisaged by Eisenberg (1995), or instantiated in the vision of a computational medium as elaborated by diSessa (2000). What this might look like is clearly a matter for research, but perhaps there are already some clues to be found. In thinking about the problems associated with the use of the `angle-carry` macro, we found it useful to create for ourselves, by using the `history` tool, a text which made explicit the Cabri manipulations that were involved as the steps of the macro construction. This description was not unlike the feedback from the `exposition` tool available in some versions of Cabri I. We found it invaluable not only in understanding the various constructions that defined its outcome, but also in illuminating why the macro did not always produce what we wanted on the screen. Had we been able to interact with a text like this, then we could have done more than understand the angle replication problem, we also could have resolved it by typing in the relevant inputs. Of course, we would not want to lose the existing ways of physically executing construction tools through direct manipulation of screen objects and menu items. What we think might be useful is to have an additional means of communicating with the system, so that, especially when physical actions do not have the results expected, students have access to a language which they can use to reflect upon what happened and why, to communicate their thinking and to initiate further actions.

Returning to the particular problem of replicating angles encountered by our students, a question that remains is whether with access to different ways of building the equal angle property, success on the angle bisectors task would no longer be limited to those who used a mixture of properties constructed with software tools and created by eye. Additionally, we do not know whether the transition to proof could have been made. As it was, it seems almost paradoxical that of the two strategies, the one that might be described as less mathematical was associated with better results. Students had more success, on this particular challenge and with the particular tools they had available for use, when they found a way to avoid the need to build robust constructions. This goes to show, and this is our point, that solution strategies cannot be judged independently from the tool-set with which they are expressed. It is also why we continue to be attracted by the vision of programmable software in which the tool-set allows the user flexibility in how to express their developing understandings.

In the dynamic geometry we have in mind, rather than decreasing the emphasis on symbolic description that characterised Cabri I's transformation into Cabri II, a programmable Cabri might offer greater versatility in terms of the medium of interaction through which users could choose to communicate with the system and, at the same time, a wider opening of the window onto how users try to shape the system and the tools available for their purposes. This, we believe, could support a new web of connections between different representations of geometrical structures, and with these new resources, possibly new routes for learners to begin to make sense of the process of proving in mathematics.

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[i] The past tense is used since newer versions of the Logo language and the microworlds constructed with them include aspects of direct manipulation.

[ii] Both this survey and the teaching experiment described later were conducted as part of the research project, Justifying and Proving in School Mathematics, funded by the Economic and Social Research Council (ESRC), Grant Number R000236178.

[iii] The term "formal proof" is intended to carry the same sense as suggested by Grenier (2000), that is to represent and communicate an argument as a succession of sentences, using conventional mathematical formulations, showing how starting from assumptions (regarded as true) one can arrive logically at a conclusion.

[iv] For a complete description of the survey and its analysis, see Healy and Hoyles 1998; for a discussion of the results in geometry, see Hoyles and Healy (in press), and for the results in algebra, see Healy and Hoyles (2000).

[v] Clearly this is no longer an option. Cabri I is now very 'out of date' and the benefits of some of the older tools are outweighed by the abundance of new tools.

[vi] Strictly speaking, Theo used only one 90° angle as a given property, the other three right angles resulted from other aspects of the construction process. He overlooked the need to deduce the perpendicularity of the other three.

[vii] We note that had Tim and Richard been interacting with Cabri II, the line segments would have been declared parallel if they happened to be horizontal or vertical when the check was applied.

[viii] This investigation is possible with Cabri I, but would have been very difficult with Cabri II. From our point of view, it is a limitation of the new version that its `history` tool no longer records the procedure underlying a macro. So if you have not constructed a macro for yourself it is difficult to identify exactly how it was built.

[ix] Fortunately, this story has a happy ending. In Cabri II the angle-carry tool we wanted can be defined by using a new object in the place of the line segment – a ray which emanates from a given point in a given direction. It is still impossible to simply select the same object twice when applying a macro.

[x] Cabri III was announced in CabriWorld (Laborde, 1999)