# Informed Seller with Taste Heterogeneity*

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**Abstract**

Consider a seller with a single indivisible good facing a buyer whose willingness to pay depends on his privately-known taste and on product characteristics privately known by the seller. What selling procedure can arise as an equilibrium of the game in which the seller strategically chooses mechanisms conditional on his information? We characterize the set of equilibrium outcomes and establish that ex-ante revenue-maximizing mechanisms are in this set. There is generally a continuum of revenue-ranked equilibrium outcomes. Focusing on the revenue-maximizing equilibrium, we show that the seller, in general, benefits from private information and does not benefit from committing to a disclosure or a certification technology. We also provide conditions under which the privacy of the seller's information does not affect revenue.

Keywords: Informed seller; consumer heterogeneity; product information disclosure; mechanism design; value of information.

JEL Classification: C72; D82.

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In this paper, we analyze an informed-seller problem in which the valuation of the buyer depends both on his privately-known taste and on product characteristics privately known to the seller. What is an equilibrium selling procedure if the seller chooses a mechanism to maximize revenue after he observes his information? When is the privacy of the seller's information valuable, detrimental or irrelevant to revenue? Does he use simple selling protocols, such as posted prices, in equilibrium? These questions are of practical importance since in many real-world trading problems, sellers have information relevant to buyers whose taste is heterogeneous. For example, think of a privatization agency facing a sole potential acquirer for an asset. It has superior information about the asset that is relevant to the acquirer's willingness to pay, which can also depend on privately known potential uses of the asset. Or consider an IT technician whose past experience dealing with operating systems and programming languages is private information. The client needs determine how this experience affects his valuation of the IT services, whereas the IT technician cares about his fee and has a fixed opportunity cost of time. More generally, our model fits situations in which the seller's private information is about a horizontal dimension of a good (flavor, character, location), whereas the consumer's privately-known taste determines how he values these different varieties.

We use the informed principal methodology to characterize the set of equilibrium outcomes of this informed-seller game. There is typically a continuum of revenue-ranked equilibrium outcomes that always include the ex-ante revenue-maximizing allocation–the best that the seller can achieve if he can choose any feasible selling procedure before knowing his type. This characterization is obtained directly by appealing to the inscrutability principle (Myerson, 1983) and by appropriately choosing off-path beliefs to block deviations to arbitrary mechanisms.

Focusing on the revenue-maximizing equilibrium, we then investigate whether or not the seller benefits from having private information. We do so by showing that for any mechanism, we can construct an equivalent one–in terms of interim payoffs for the buyer and ex-ante revenues–that is incentive-compatible for the seller. This result implies that the privacy of the seller's information is always valuable ex-ante, so he does not have any incentive to invest in a certification technology (even if it is free); nor does he benefit from being able to commit ex-ante to *any* information disclosure policy. At the interim stage, however, some seller types may be worse off. These results do not hold when the seller can only post prices. We provide sufficient conditions on the valuation function and the distribution of the buyer's types for information irrelevance.

What does the revenue-maximizing equilibrium mechanism look like? While it cannot, in general, take the form of a posted price, as it requires communication between the buyer and the seller, we illustrate in an example realistic procedures that can be near- or even fully optimal. In one such procedure, the seller posts a consulting fee in order to release his information to the buyer, who then decides whether or not to pay an acquisition fee to get the good. In another procedure, the seller posts a price, and after the buyer pays, the seller discloses his information and the buyer receives a rebate if he decides to return the good. Another procedure, which resembles "book building" (Boone and Mulherin, 2007, Quint and Hendricks, 2013), involves bilateral cheap talk between the buyer and the seller and price quotes conditional on the content of the conversation. These procedures motivate the use of contingent payments, which are prevalent in practice but cannot be optimal in the leading frameworks with private values or an uninformed seller, where simple posted prices are optimal. More generally, we characterize revenue-maximizing mechanisms when the buyer's valuation is a convex function of his type that depends on the seller's type.

**Related Literature.** The literature on the design of revenue-maximizing selling procedures has been quite influential but has focused primarily on cases in which the seller has *no* private information. In such a setup, Myerson (1981) and Riley and Zeckhauser (1983) show that when the seller faces one buyer, the revenue-maximizing procedure is a posted price. Yilankaya (1999) allows for bilateral private information, but the seller's information does not affect the buyer's willingness to pay. He shows that a posted price that depends on the seller's information is optimal and that the revenue is the same as when the seller's information is commonly known. Such information-irrelevance results have been established and generalized in different private-value settings by Maskin and Tirole (1990), Tan (1996), Yilankaya (1999), Skreta (2011) and Mylovanov and Tröger (2013, 2014).

Beyond the private-value setup, fewer than a handful of papers have been written on mechanism selection when the principal's information affects the agent's valuation. Myerson (1983) formulates the inscrutability principle and proposes various solution concepts for general informed principal problems. Maskin and Tirole (1992) characterize the set of equilibria in a class of informed principal problems, but their characterization does not apply to our environment,[1] and they do not provide conditions for information irrelevance. Balkenborg and Makris (2015) propose an equilibrium refinement (called assured equilibrium) for a class of informed principal problems with common values. Balestrieri and Izmalkov (2012) consider a symmetric horizontal differentiation problem, in which the buyer's valuation depends on the buyer's type and the seller's privately-known location and characterize ex-ante optimal mechanisms. In contrast, we characterize the entire set of equilibria for general buyer valuations and show that ex-ante optimal mechanisms are always in this set. The linear, but possibly asymmetric, version of Balestrieri and Izmalkov's (2012) model is analyzed as an illustration of the characterization of revenue-maximizing mechanisms (Proposition 4) in the online appendix.

Apart from the literature on informed principal, this paper is related to the literature where sellers design the buyers' information structures (Eső and Szentes, 2007; Ottaviani and Prat, 2001; Rayo and Segal, 2010). In Eső and Szentes (2007), the seller is uninformed. He designs

[1] In particular, their characterization *requires* that the so-called Rothschild-Stiglitz-Wilson allocation is interim efficient for some strictly positive beliefs, a property that may not be satisfied in our setup, as we illustrate in Example 1 (Remark 3).

a machine that discloses a signal orthogonal to the buyer's initial value estimate and chooses a multi-stage mechanism that depends on reports from the buyer. First, the buyer reports his type, and then he observes the signal (which the seller does not observe) and reports it. In our paper, the seller offers a mechanism once he is informed, and the reports in the mechanism come from both the seller and the buyer. Despite the differences, there are similarities: in both papers, there is some signal that affects the buyer's willingness to pay that is not contained in the buyer's initial information. Both papers establish that there is no cost in terms of ex-ante revenue to incentivize truth-telling by the party that reports this signal to the mechanism: in Eső and Szentes (2007), this party is the buyer (Theorem 1 in Eső and Szentes, 2007), and in this paper, it is the seller (our Lemma 1). In other words, in both setups, the seller is able to achieve the same ex-ante maximal revenue as when a mechanism can condition on the seller's true signal. Interestingly, in our setup, this upper bound is not only achievable after we provide the seller with incentives to reveal information truthfully, but it is also an equilibrium outcome of our informed seller mechanism-selection game.

This paper is also related to the literature on signalling seller's information through the choice of the selling procedure. Among others, Cai et al. (2007) and Jullien and Mariotti (2006) study how a revenue-maximizing seller can signal information through the choice of the reserve price. Finally, our work is related to the literature on disclosure of product information, which restricts attention to posted prices (Anderson and Renault, 2006; Johnson and Myatt, 2006; Sun, 2011; Koessler and Renault, 2012; Sun and Tyagi, 2014).

# 1 Motivating Example: Selling Distressed Assets

Consider the situation where a seller (e.g., a privatization agency or a bank liquidating foreclosed properties) wants to sell a bundle of assets (e.g., a bundle of real estate properties) with privately known characteristics $s_R$ (e.g., the bundle has value for residential use) or $s_C$ (e.g., the bundle has value for commercial use). The seller wants to maximize revenue from the sale, as is most often the case with banks liquidating assets, and sometimes with privatization agencies. The buyer's (the acquisition firm) needs are private information: he needs commercial properties, $t_c$ (and is willing to pay up to 30 for an $s_C$ bundle and 0 otherwise), or needs residential ones, $t_r$ (and is willing to pay up to 32 for such a bundle and 0 otherwise), or is indifferent, $t_i$ (and wants to pay 20, regardless of features). Thus, the buyer's valuation function $u(s,t)$ depends on his needs and the seller's information (see Table 1).[2] Except when stated otherwise, all examples in the paper have uniformly distributed types.

With a posted price and no information transmission between the seller and the buyer, the seller's highest feasible expected revenue is 15, in which case every buyer type purchases at

[2]A version of this example appeared in Koessler and Renault (2012), who study equilibrium information disclosure for a seller restricted to posting prices.

| $u(s,t) =$ | $t_c$ | $t_r$ | $t_i$ |
|---|---|---|---|
| $s_C$ | 30 | 0 | 20 |
| $s_R$ | 0 | 32 | 20 |

Table 1: The buyer's valuation function.

price 15. Another procedure is to post a price of 20 after a fully revealing message from the seller to the buyer, yielding expected revenue of $20 \times 2/3 < 15$. Posting a price equal to 20 is actually the revenue-maximizing mechanism when the seller's type is common knowledge, which we henceforth refer to as *full-information allocation*. Can the seller do better and even extract the entire surplus of the buyer?

Consider the following allocation $(p, x)(s, t) \in [0, 1] \times \mathbb{R}$, which specifies as a function of type $s$ of the seller and type $t$ of the buyer the probability of trade, $p(s, t)$, and the price, $x(s, t)$:

| $(p,x)(s,t) =$ | $t_c$ | $t_r$ | $t_i$ |
|---|---|---|---|
| $s_C$ | $1, 30$ | $0, 0$ | $1, 20$ |
| $s_R$ | $0, 0$ | $1, 32$ | $1, 20$ |

This allocation is incentive-compatible for the buyer, satisfies his participation constraints, and extracts all his surplus. However, it is not incentive-compatible for the seller since type $s_C$ gets a higher expected revenue by claiming to be $s_R$ instead of $s_C$. We now construct a new allocation $(\hat{p}, \hat{x})$–obtained from $(p, x)$ by averaging payments across the seller's types–that is feasible (hence incentive-compatible both for the buyer and the seller), and extracts all buyer's surplus:[3]

| $(\hat{p},\hat{x})(s,t) =$ | $t_c$ | $t_r$ | $t_i$ |
|---|---|---|---|
| $s_C$ | $1, 15$ | $0, 16$ | $1, 20$ |
| $s_R$ | $0, 15$ | $1, 16$ | $1, 20$ |

The interim expected revenue from $(\hat{p}, \hat{x})$ is equal to $51/3$ for both seller types.

The structure of $(\hat{p}, \hat{x})$ suggests procedures that are almost optimal and resemble real-world and commonly used procedures, as we now describe:

*Procedure 1: Bilateral direct cheap talk and conditional price quotes.*[4] First, the seller asks the buyer whether or not his type is $t_i$. If the buyer reveals that his type is $t_i$, then the seller asks a price 20 without revealing any information, in which case the seller's revenue is 20. If the buyer reveals that his type is not $t_i$, then the seller asks a price of 30 and reveals his type. In this case, the seller receives 30 if the buyer's needs and the type of the asset match, and 0

[3]This transformation is later used in Lemma 1 to show that the incentive-compatibility constraints of the seller are irrelevant in terms of feasible ex-ante expected revenue.

[4]This procedure resembles "book building," a prevalent selling procedure in private company and other asset sales.

otherwise, yielding expected revenue of 50/3. It is easy to check that this scenario implements an allocation identical to $(p, x)$, with the only difference being that the outcome for $(s_R, t_r)$ is $(1, 30)$, instead of $(1, 32)$.

*Procedure 2: Consulting fee followed by acquisition fee.* The seller posts a consulting fee of 10 and an acquisition price of 10. If the buyer pays the consulting fee, the seller reveals his type, after which the buyer decides whether or not to acquire the asset by paying the acquisition fee. It is easy to see that all buyer types agree to pay the consulting fee, but buyer types $t_c$ and $t_r$ do not pay the acquisition price if the asset does not match their needs. Those types end up paying 10 even when not getting the good. Also, given that all buyer types pay the consulting fee, the seller does not learn anything about the buyer. This protocol yields interim expected revenue of 50/3 to both seller types and implements the following allocation.

$(\tilde{p}, \tilde{x})(s, t) =$

| | $t_c$ | $t_r$ | $t_i$ |
|---|---|---|---|
| $s_C$ | $1, 20$ | $0, 10$ | $1, 20$ |
| $s_R$ | $0, 10$ | $1, 20$ | $1, 20$ |

*Procedure 3: Posted price and rebate.* In this third procedure, the seller asks a price of 20, and after the buyer accepts and pays the price, the seller reveals his information and offers a rebate of 10 if the buyer decides to return the asset. It is immediate to see that this procedure implements $(\tilde{p}, \tilde{x})$.

All three aforementioned procedures would be *fully* optimal if the buyer's valuation function were symmetric–that is, if $u(s_R, t_r) = 30$ instead of 32.

Consider, now, the game in which the seller can choose any (possibly mediated) selling procedure after he has learned his type. Would any of these selling procedures implement equilibrium outcomes of this mechanism-selection game? Theorem 1 shows that *all* aforementioned selling procedures do so. More generally, it establishes that the set of interim equilibrium revenue vectors coincides with the line connecting the points $(40/3, 40/3)$ and $(51/3, 51/3)$.

# 2 Model

Consider a monopoly seller facing a single buyer with unit demand.[5] The seller has perfect and private information about the *product's characteristics*, denoted by $s \in S$ and also called the *type of the seller*. The buyer has perfect and private information about his *taste*, denoted by $t \in T$ and also called the *type of the buyer*. Type spaces $S$ and $T$ are compact metric spaces, and $s$ and $t$ are independently distributed according to full support probability distributions. Both the seller and the buyer are risk-neutral.

[5] Or, equivalently, a continuum of anonymous buyers of mass one.

The *valuation function* $u(s,t) \in \mathbb{R}$ describes the buyer's willingness to pay for the product, which depends on his own type and the type of the seller. The value of the object for the seller and the outside option for the buyer are type-independent and, therefore, normalized to 0.[6] We assume that $u(s,t)$ is absolutely continuous in $t$ for every $s$.

An *allocation*, denoted by $(p,x) : S \times T \to [0,1] \times \mathbb{R}$, determines the probability of sale (i.e., the probability that the buyer gets the good), $p(s,t)$, and the expected transfer from the buyer to the seller, $x(s,t)$, as a function of the seller's type $s$ and the buyer's type $t$.[7] The seller's expected payoff from $(p,x)$ given $s$ and $t$ is $x(s,t)$, while the buyer's is $U(s,t) = p(s,t)u(s,t) - x(s,t)$. The interim expected payoffs are denoted by $X(s) \equiv E_T[x(s,t)]$ and $U(t) \equiv E_S[U(s,t)]$, respectively.

We say that an allocation $(p,x)$ is *feasible* if it satisfies (interim) incentive-compatibility and participation constraints for the seller and the buyer: For every $s$ and $s'$ in $S$, and for every $t$ and $t'$ in $T$:

$$X(s) = X(s'), \tag{S-IC}$$

$$X(s) \geq 0, \tag{S-PC}$$

$$U(t) \geq U(t'|t) \equiv E_S\big[p(s,t')u(s,t) - x(s,t')\big], \tag{B-IC}$$

$$U(t) \geq 0. \tag{B-PC}$$

Incentive-compatibility for the seller is equivalent to

$$X(s) = X(s') = \bar{X} \equiv E_S[X(s)], \tag{1}$$

for all $s$ and $s'$. In other words, incentive-compatibility for the seller imposes that his interim expected revenue is the same whatever his type, so it is the same as his ex-ante expected revenue.

**Definition 1** An allocation is *ex-ante optimal* and is denoted by $(p^{EO}, x^{EO})$ if it maximizes ex-ante revenue $\bar{X} = E_S[X(s)]$ under the feasibility constraints.

A full-information optimal allocation[8] maximizes the seller's interim revenue, whatever his type, when the seller's type is commonly known:

**Definition 2** An allocation is *full-information optimal* and is denoted by $(p^{FI}, x^{FI})$ if, for every

[6] Note that our analysis and results also hold when the seller has a reservation value $v(t)$ that depends on the buyer's type, $t$. In this case, his expected profit would be $x(s,t) - p(s,t)v(t)$, and incentive-compatibility for the seller would impose that his interim profit is the same whatever his type. However, the assumption that the seller's valuation does not depend on his own type $s$ is crucial for our results.

[7] Hence, we assume implicitly that transfers are balanced ex-post. Without loss of generality, we directly consider expected transfers because both players are assumed to be risk-neutral.

[8] This terminology is used in Maskin and Tirole (1990, 1992). Note that it refers only to the seller's information about $s$ being commonly known; the agent's information about $t$ is still private.

$s \in S$, it maximizes $X(s)$ under the following ex-post incentive-compatibility and participation constraints of the buyer:

$$U(s,t) \geq p(s,t')u(s,t) - x(s,t'), \text{ for every } t,t' \in T; \tag{2}$$

$$U(s,t) \geq 0, \text{ for every } t \in T. \tag{3}$$

We respectively denote by $X^{EO}$ and $X^{FI}(s)$ the ex-ante optimal and full-information revenue of type $s$.

# 3 Mechanism-Selection Game and Equilibria

Now we consider the mechanism-selection game in which the seller strategically chooses the mechanism after he has learned his type. Therefore, we are dealing with an *informed principal* problem (Myerson, 1983), where the principal is the seller, and the agent is the buyer. The timing of the mechanism-selection game is as follows:

1. Nature selects the types of the seller and the buyer, $(s,t) \in S \times T$, according to the probability distributions $\sigma \in \Delta(S)$ and $\tau \in \Delta(T)$. The seller is privately informed about $s \in S$, and the buyer is privately informed about $t \in T$.

2. The seller proposes a mechanism, that consists of (finite) sets of messages $M_S$ for the seller and $M_T$ for the buyer, and a function

$$m : M_S \times M_T \to [0,1] \times \mathbb{R},$$

   mapping a message profile $(m_S, m_T)$ to a probability of trade and a selling price.

3. The buyer observes the mechanism $m$ selected by the seller and chooses a message $m_T \in M_T$ and whether or not to participate. Simultaneously, the seller chooses a message $m_S \in M_S$.

The allocation is determined by $m$ and the messaging and participation decisions. If the buyer does not participate, then there is no trade and both players get zero.

An allocation $(p,x)$ is an *(expectational) equilibrium*[9] outcome of the mechanism-selection game iff: (i) it is feasible; and (ii) for every mechanism $\tilde{m}$, there exists a belief $\pi \in \Delta(S)$ for the buyer, a messaging and participation strategy profile that is a continuation Nash equilibrium of $\tilde{m}$ given $\pi$, with associated allocation $(\tilde{p}, \tilde{x})$, such that $X(s) \geq E_T[\tilde{x}(s,t)] \equiv \tilde{X}(s)$, for every $s \in S$.

[9]For full details, see Myerson (1983) pages 1779–1780; An expectational equilibrium is equivalent to the strong version of perfect Bayesian equilibrium used, for instance, in Maskin and Tirole (1990).

In every continuation Nash equilibrium, we assume that the buyer chooses to participate whenever he is indifferent between participating or not. We further assume that if the seller proposes a direct mechanism $m : S \times T \to [0,1] \times \mathbb{R}$, then the buyer and seller always report their true type if they have a weak incentive to do so. These tie-breaking assumptions are used only to prove the necessary part of the following characterization of expectational equilibrium interim revenues.[10]

**Theorem 1** *A profile of interim revenues $(X(s))_{s \in S}$, is a profile of expectational equilibrium interim revenues if and only if*

$$\min_{s \in S} X^{FI}(s) \leq X(s) = X(s') \leq X^{EO}, \text{ for every } s, s' \in S. \tag{4}$$

*Proof.* First, observe that the property that interim expected revenues are equal across all seller types holds for every continuation Nash equilibrium outcome $(\tilde{p}, \tilde{x})$ (given some belief $\pi$) of every mechanism $\tilde{m}$: $\tilde{X}(s) = \tilde{X}(s')$, for every $s$ and $s'$.

($\Leftarrow$) We show that if (4) is satisfied, then $(X(s))_{s \in S}$ is a profile of interim equilibrium revenues. We first show that (4) implies that $(X(s))_{s \in S}$ is feasible. Consider an ex-ante optimal allocation $(p^{EO}, x^{EO})$, which is feasible by definition. Consider the mechanism $(p^{EO}, x)$ such that

$$x(s,t) = x^{EO}(s,t) - [X^{EO} - X(s)], \text{ for every } s \text{ and } t.$$

By construction, $(p^{EO}, x)$ is feasible and generates the interim revenues $(X(s))_{s \in S}$. We next show that for every deviation of the seller from the direct mechanism $(p^{EO}, x)$ to a mechanism $\tilde{m}$, there exists a belief $\pi$ for the buyer such that the deviation is not profitable for the seller, whatever his type. Consider the degenerate belief $\pi$ that assigns probability one on some type,

$$\underline{s} \in \arg\min_{s \in S} X^{FI}(s),$$

and let $(\tilde{p}, \tilde{x})$ be a continuation Nash equilibrium of $\tilde{m}$ given this belief. By definition of the full-information allocation, we have

$$\tilde{X}(\underline{s}) \leq X^{FI}(\underline{s}).$$

Now, from the equilibrium property that interim expected revenues are equal across all seller types, we get $\tilde{X}(s) \leq X^{FI}(\underline{s}) \leq X(s)$ for every $s$. Hence, the seller does not benefit from the

[10]Precisely, these assumptions are used to show that expectational equilibrium revenues are necessarily higher than the full information revenues of the "worst" seller type, $\arg\min_{s \in S} X^{FI}(s)$. They guarantee that the seller is able to select his preferred continuation equilibrium given the buyer's belief (but, of course, he is not able to select the buyer's belief). The assumption that the buyer always participates when he is indifferent is easy to dispense with because the seller can always slightly modify the mechanism by reducing all transfers by an arbitrary small amount.

deviation.

($\Rightarrow$) We show that if $(X(s))_{s\in S}$ is a profile of interim equilibrium revenues, then it satisfies (4). By feasibility of $(X(s))_{s\in S}$, we immediately get $X(s) = X(s') \leq X^{EO}$ for every $s$ and $s'$. To show that $X(s) \geq X^{FI}(\underline{s})$, we show that the seller can always deviate to a mechanism $\tilde{m}$ inducing a continuation Nash equilibrium revenue higher than or equal to $X^{FI}(\underline{s})$ whatever his type $s$ and whatever the belief $\pi$ of the buyer. To construct such a deviation, consider a full-information allocation $(p^{FI}, x^{FI})$ and let $\tilde{m} = (\tilde{p}, \tilde{x})$ be the direct mechanism such that $\tilde{p} = p^{FI}$ and

$$\tilde{x}(s,t) = x^{FI}(s,t) - [X^{FI}(s) - X^{FI}(\underline{s})], \text{ for every } s \text{ and } t.$$

This direct mechanism is incentive-compatible for the seller since $\tilde{X}(s) = X^{FI}(\underline{s})$ for every $s$. The interim utility of the buyer in the allocation $(\tilde{p}, \tilde{x})$ is equal to the interim utility of the buyer in the allocation $(p^{FI}, x^{FI})$ plus the constant $E_S[X^{FI}(s) - X^{FI}(\underline{s})] \geq 0$. The full-information allocation $(p^{FI}, x^{FI})$ is incentive-compatible for the buyer, whatever his belief $\pi$, because it satisfies the buyer's ex-post incentive and participation constraints (it is "safe" according to the terminology of Myerson, 1983). Therefore, $(\tilde{p}, \tilde{x})$ is also incentive-compatible for the buyer, whatever his belief. ■

**Remark 1** The proof that every interim revenue vector satisfying (4) is an expectational equilibrium is constructed using off-path beliefs that assign probability one to the "worst" seller type (the type with the smallest full-information revenue). It can be shown that the ex-ante optimal allocation can also be supported as an expectational equilibrium with passive beliefs.[11]

Compared to private-value setups (Maskin and Tirole, 1990), equilibrium is typically not unique and many equilibria are dominated. The unique undominated equilibrium allocation is the ex-ante optimal allocation.[12] As an illustration, consider the introductory example in Section 1. The full-information interim revenue is equal to 40/3, whatever the seller's type, while the ex-ante optimal revenue is 51/3. From Theorem 1, the set of expectational equilibrium allocations is the set of all feasible allocations giving a revenue between 40/3 and 51/3. In particular, the allocation implemented by the optimal posted price with no information transmission, and the allocations implemented by procedures involving consulting and acquisition fees, acquisition fees and rebates, and cheap-talk communication with conditional price offers are all expectational equilibrium allocations.

The following example is another illustration of Theorem 1, which also shows that, even if the valuation function is monotonic both in the seller's and in the buyer's type, every posted price equilibrium mechanism can be dominated by the ex-ante optimal equilibrium.

[11]The proof appears in a previous version of the paper, which is available from the authors upon request.

[12]Note that, according to the terminology of Myerson (1983), the ex-ante optimal allocation is not a strong solution because it is not safe in general. It can be shown, however, that the set of ex-ante optimal allocations coincides exactly with the set of core mechanisms as defined in Myerson (1983).

**Example 1** Let $s_1$ and $s_2$ be the seller's types and $t_1$ and $t_2$ the buyer's types, with uniform prior probability distributions. The buyer's valuation is given by:

$u(s,t) =$

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | 6 | 7 |
| $s_2$ | 0 | 1 |

If the seller reveals his type to the buyer and posts a price, then, since the seller-incentive-compatibility constraints imply that both types must have the same expected revenue, the highest equilibrium revenue is 1/2, with a price of 1/2 for seller type $s_1$ (which is accepted by both buyer types) and a price of 1 for seller type $s_2$ (which is accepted only by buyer type $t_2$). Instead, posting a price of 3 without information revelation yields a revenue of 3, whatever the type of the seller (both buyer types buy at this price). Both of these selling procedures are dominated by the following allocation (which can be shown to be ex-ante optimal), which yields 13/4 to both types of the seller:

$(p,x)(s,t) =$

| | $t_1$ | $t_2$ |
|---|---|---|
| $s_1$ | $1,3$ | $1,7/2$ |
| $s_2$ | $0,3$ | $1,7/2$ |

From the previous theorem, equilibrium revenues are exactly those in between $\min_s X^{FI}(s) = X^{FI}(s_2) = 1/2$ and $X^{EO} = 13/4$. ◇

**Remark 2** In the previous example, even though private information for the seller is ex-ante irrelevant, the optimal allocation is *not* a posted price. This is different from the private-value case in which, when utilities are quasi-linear, information is irrelevant and posted prices are optimal.

**Remark 3** Maskin and Tirole (1992) provide an equilibrium characterization for common-value informed-principal problems. However, their characterization requires that the Rothschild-Stiglitz-Wilson allocation is interim efficient for some strictly positive beliefs. In the previous example, the Rothschild-Stiglitz-Wilson allocation gives 1/2 to each seller type, which is not interim efficient, whatever the (strictly positive) belief of the buyer.[13]

# 4 Value of Information

A natural question to ask is whether a privately informed seller fares better than a seller whose information is common knowledge, or than a seller who can commit to disclose some information

[13]For example, posting a price equal to $6\pi(s_1) + 1$ gives revenue $(6\pi(s_1) + 1)/2 > 1/2$ to every seller type.

to the buyer. In other words, we would like to assess whether the seller benefits from having access to a disclosure or certification technology.

A central step towards addressing these questions is to investigate the impact of the seller's incentive-compatibility constraints on the set of feasible allocations. Interestingly, we establish that requiring an allocation to be incentive-compatible for the seller does not affect the set of feasible interim utilities for the buyer and the set of feasible ex-ante revenues:

**Lemma 1** *For every allocation $(p, x)$ that gives the buyer interim utilities $U(t'|t)$, $t, t' \in T$, there exists an allocation $(\tilde{p}, \tilde{x})$ that satisfies the seller's incentive constraint, generates the same ex-ante revenue, and gives the buyer the same interim utilities: that is, $\tilde{U}(t'|t) = U(t'|t)$, for all $t, t' \in T$.*

*Proof.* For every allocation $(p, x)$ it suffices to consider the allocation $(\tilde{p}, \tilde{x})$ such that $\tilde{x}(s, t) = E_S[x(s, t)]$ and $\tilde{p}(s, t) = p(s, t)$, for all $(s, t) \in S \times T$. ■

Lemma 1 has two consequences:

**Proposition 1** The seller benefits from private information. *The ex-ante optimal allocation generates weakly higher ex-ante expected revenue than the full-information allocation.*

*Proof.* The full-information allocation may not be incentive-compatible for the seller. However, following Lemma 1, we can construct an equivalent allocation in terms of ex-ante expected revenue and interim utilities of the buyer that is incentive-compatible for the seller. Hence, an allocation that generates the same ex-ante revenue is feasible. The conclusion immediately follows. ■

**Proposition 2** Access to certification and disclosure rules does not benefit the seller. *The ability of the seller to certify his information or to commit to some information disclosure rule ex-ante does not lead to a higher ex-ante expected revenue.*

Indeed, any additional disclosure about the seller's type would just make the incentive-compatibility and participation constraints for the buyer harder to satisfy. It is worth noticing that the seller can strictly benefit from commitment to a disclosure rule if one restricts to posted prices mechanisms. In Example 1, the best posted price mechanism yields a revenue of 3, whatever the type of the seller, but if the seller can commit to complete information disclosure, then he gets his full information revenue, 6 for $s_1$ and 0.5 for $s_2$, yielding a higher ex-ante profit. This example illustrates that the seller's ability to employ general selling procedures renders certification and disclosure rules unnecessary if one focuses on revenue.

We already know from the previous examples that the seller sometimes strictly benefits from private information. In Section 5.2, we investigate the conditions on the valuation function and the distribution of the buyer's types for information irrelevance.

# 5 Convex Environments

We have seen that we can obtain one equilibrium of the mechanism-selection game for the seller by solving for the ex-ante optimal allocation subject to (S-IC), (S-PC), (B-IC) and (B-PC). Lemma 1 implies that we can ignore (S-IC). We now investigate the implications of (B-IC) and (B-PC) for feasible and ex-ante optimal allocations by imposing more structure. We assume that $t \in T = [\underline{t}, \bar{t}]$ is distributed according to a continuous density function $f$ with c.d.f. $F$, where $0 \leq \underline{t} < \bar{t} < +\infty$. No specific assumption is required on the seller's set of types. We also assume that for every $s \in S$, $u(s,t)$ is convex in $t$ and has a unique minimizer $t^{min}(s) \in \arg\min_t u(s,t)$ for every $s$.

## 5.1 Characterization

The following derivations are relatively standard and follow from arguments analogous to the ones in Krishna and Maenner (2001) or Figueroa and Skreta (2011). Let $t^* \in T$ be any type of the buyer and[14]

$$P(t) \equiv E_S \left[ p(s,t) \frac{\partial u(s,t)}{\partial t} \right]. \tag{5}$$

An allocation $(p,x)$ is incentive-compatible for the buyer iff

$$P(t') \geq P(t) \qquad \text{for all } t' \geq t; \tag{6}$$

$$U(t) = U(t^*) + \int_{t^*}^{t} P(\tau) d\tau \quad \text{for all } t \in T. \tag{7}$$

Condition (7) is satisfied if and only if, for every $t \in T$:

$$E_S[x(s,t)] = E_S \left[ p(s,t)u(s,t) - \int_{t^*}^{t} p(s,\tau) \frac{\partial u(s,\tau)}{\partial \tau} d\tau \right] - U(t^*). \tag{8}$$

Letting

$$J(s,t;t^*) \equiv \begin{cases} J^L(s,t) \equiv u(s,t) + \frac{F(t)}{f(t)} \frac{\partial u(s,t)}{\partial t} & \text{if } t < t^* \\ J^R(s,t) \equiv u(s,t) - \frac{1-F(t)}{f(t)} \frac{\partial u(s,t)}{\partial t} & \text{if } t > t^*, \end{cases}$$

we can express the seller's revenue at an incentive-compatible allocation as a function of the allocation rule and the virtual valuation $J$:

$$\bar{X} = \int_{\underline{t}}^{\bar{t}} E_S[p(s,t)J(s,t;t^*)] f(t) dt - U(t^*). \tag{9}$$

[14]By absolute continuity of $u(s,t)$ in $t$, its derivative with respect to $t$ exists almost everywhere, but we omit the qualification henceforth.

Furthermore, using standard arguments (see, e.g., Milgrom and Segal, 2002), we can show that at an ex-ante optimal allocation $U(t^*) = 0$ for some $t^* \in T$. Hence:

**Proposition 3** *The ex-ante optimal allocation is fully characterized by a type $t^*$ such that $U(t^*) = 0$ and an assignment rule $p$ that maximize*

$$\bar{X} = \int_{\underline{t}}^{\bar{t}} E_S \left[p(s,t)J(s,t;t^*)\right] f(t)dt, \tag{10}$$

*subject to $P(\cdot)$ is increasing and $P(t) \leq 0$ for $t < t^*$ and $P(t) \geq 0$ for $t > t^*$.*

*Proof.* Feasibility follows from (6) and (7) and the following observations. The constraints that $P(\cdot)$ is increasing and $\int_{t^*}^{t} P(\tau)d\tau \geq 0$ together are equivalent to: $P(\cdot)$ is increasing and $P(t) \leq 0$ for $t < t^*$ and $P(t) \geq 0$ for $t > t^*$. That is, under (B-IC), the participation constraint for the buyer is simply $P(t) \leq 0$ for $t < t^*$ and $P(t) \geq 0$ for $t > t^*$. The payment rule $x$ can be obtained from $p$ via (8). ■

We now consider a case in which we can tell in advance what the optimal $t^*$ referred to in Proposition 3 will be:

**Proposition 4** *Suppose that $t^{min}(s) \equiv t^{min}$ for every $s$, and consider the allocation $(p, x)$ where,*

$$p(s,t) \equiv \begin{cases} 1 & \text{if } J(s,t;t^{min}) > 0 \\ 0 & \text{if } J(s,t;t^{min}) \leq 0, \end{cases}$$

*and*

$$x(s,t) = E_S \left[ p(s,t)u(s,t) - \int\limits_{t^{min}}^{t} p(s,\tau)\frac{\partial u(s,\tau)}{\partial \tau}d\tau \right]. \tag{11}$$

*If*

$$P(t) \equiv E_S \left[ p(s,t)\frac{\partial u(s,t)}{\partial t} \right], \tag{12}$$

*is increasing in $t$, then $(p, x)$ is an ex-ante optimal allocation for the seller.*

*Proof.* We first show that if $t^{min} \in \arg\min_t u(s,t)$ for every $s$, then at an ex-ante optimal allocation, the buyer's participation constraint binds at $t^{min}$, i.e., $U(t^{min}) = 0$. To prove this, it suffices to observe that if $t^{min}$ minimizes $u(s,t)$ in $t$ for all $s$, then at every (B-IC) allocation (and, therefore, at every feasible allocation) $U(t)$ is minimized at $t = t^{min}$. Indeed, we have:

$$E_S[U(s,t)] \geq E_S \left[p(t^{min}, s)u(t,s) - x(t^{min}, s)\right] \geq E_S \left[U(s, t^{min})\right],$$

where the first inequality follows from (B-IC), and the second follows from the fact that $t^{min}$ is a minimizer of $u$ for each $s$. The allocation is ex-ante optimal because $p$ maximizes $\bar{X}$ (described in (10)) pointwise, and the payment rule sets $U(t^{min}) = 0$, as required. We now establish that the allocation is feasible. Incentive-compatibility constraints follow trivially since the proposition requires that $P(t) \equiv E_S\left[p(s,t)\frac{\partial u(s,t)}{\partial t}\right]$ is increasing in $t$. Participation constraints follow immediately from (7) and the earlier considerations. The incentive constraints for the seller are satisfied because $x(s,t)$ does not depend on $s$. ■

## 5.2 Information Irrelevance

Proposition 4 also characterizes the full-information allocations if we simply remove the expectation operators from Equations (11) and (12). When the seller's type is commonly known, the incentive and participation constraints for the buyer are the hardest to satisfy, since they have to hold for every realization of the seller's type $s$, rather than in expectation. The seller benefits from having private information because, in general, uncertainty about his type relaxes (B-IC) or (B-PC) or both.

As we established in Proposition 4, if $t^{min}(s) \equiv t^{min}$ is the same for all the seller's types, then (B-PC) always binds at $t^{min}$ at an ex-ante optimal allocation, regardless of how much the buyer knows about the seller.[15] Then, uncertainty about $s$ does not relax (B-PC). We now show that this is not enough for the seller's information to be irrelevant. We do so in an example in which $u(s,t)$ is strictly increasing *both* in $s$ and $t$:

**Example 2** Let $s \in \{s_1, s_2\}$, where the prior probability of $s_1$ is $\sigma \in (0,1)$, and let $t \in [0,1]$ with a uniform prior distribution. Assume that the valuation function is

$$u(s_1,t) = t^2 + 1/4,$$
$$u(s_2,t) = t + 2/3.$$

Note that it is strictly increasing in $s$ and $t$ and convex in $t$ for every $s$. The virtual valuations are $J(s_1,t) = 3t^2 - 2t + 1/4$ and $J(s_2,t) = 2t - 1/3$. Maximizing pointwise, we obtain the following assignment rule:

$$p(s_1,t) = \begin{cases} 0 & \text{if } t \in (1/6, 1/2) \\ 1 & \text{otherwise,} \end{cases} \qquad p(s_2,t) = \begin{cases} 0 & \text{if } t < 1/6 \\ 1 & \text{if } t > 1/6. \end{cases}$$

This allocation is not incentive-compatible for the buyer when he knows the seller's type because $p(s_1,t)\frac{\partial u(s_1,t)}{\partial t} = \begin{cases} 0 & \text{if } t \in (1/6,1/2) \\ 2t & \text{otherwise,} \end{cases}$ is not monotonic in $t$. The full-information interim revenue is $X^{FI}(s_1) = 1/4$ and $X^{FI}(s_2) = 25/36$, whereas the corresponding ex-ante revenue is

[15] And this is also a necessary condition. Details are available from the authors.

$\bar{X}^{FI} = \frac{25-16\sigma}{36}$. The solution of the pointwise maximization is, however, incentive-compatible when the seller's type is private information if

$$P(t) = \sigma p(s_1,t)\frac{\partial u(s_1,t)}{\partial t} + (1-\sigma)p(s_2,t)\frac{\partial u(s_2,t)}{\partial t} = \begin{cases} \sigma 2t & \text{if } t < 1/6 \\ 1-\sigma & \text{if } t \in (1/6, 1/2) \\ 1+\sigma(2t-1) & \text{if } t > 1/2, \end{cases}$$

is increasing in $t$–i.e., if $\sigma \leq 3/4$. The corresponding ex-ante expected revenue is given by

$$X^{EO} = \bar{X}^{FI} + \int_0^{1/6} J(s_1,t)\,dt = \frac{29}{108}\sigma + \frac{25}{36}(1-\sigma).$$

Information is, therefore, ex-ante valuable for the seller when $\sigma < 3/4$. For type $s_1$, however, information is not interim valuable because when $\sigma < 3/4$, $X^{EO} < X^{FI}(s_1)$ . ◇

We now proceed to add conditions under which uncertainty about the seller's type does not relax the incentive constraints (B-IC), and we establish that the seller's information is then irrelevant.

**Definition 3** (*Single-Crossing*) A function $f : X \to \mathcal{R}$ satisfies *single crossing* if for every $x \leq x'$, $f(x) \geq (>)\ 0$ implies that $f(x') \geq (>)\ 0$.

**Assumption SC** (Single Crossing). Both $J^R$ and $-J^L$ satisfy *single-crossing* in $t$ for *every* $s$.

Observe that $J(s_1,t)$ in Example 2 violates Assumption SC.

**Proposition 5 (Irrelevance of the Seller's Information)** *If Assumption SC holds and $t^{min}(s) = t^{min}$ for every $s$, then the ex-ante optimal revenue is equal to the ex-ante revenue of the full-information allocation.*

*Proof.* Under these conditions, the allocation described in Proposition 4 satisfies the requirement that $p(s,t)\frac{\partial u(s,t)}{\partial t}$ is increasing in $t$. The fact that $-J^L$ and $J^R$ satisfy single-crossing implies that the pointwise optimum $p$ is either always 1, or it is zero for $t$'s around $t^{min}$ and 1 for low $t$'s, and 1 again for high $t$'s. By the convexity of $u$ and the definition of $t^{min}$, $\frac{\partial u(s,t)}{\partial t}$ is negative for $t \leq t^{min}$ and positive for $t \geq t^{min}$, establishing (6), while (7) follows immediately since $E_S[U(s,t)] = E_S\left[\int_{t^{min}}^{t} p(s,\tau)\frac{\partial u(s,\tau)}{\partial \tau}d\tau + U(s,t^{min})\right]$. ■

The following proposition provides sufficient conditions for information irrelevance without relying on the specific assumptions imposed in this section.

**Proposition 6** *The ex-ante optimal revenue is equal to the ex-ante revenue of the full-information allocation if (i) at all feasible allocations, $U(t)$ is minimal at the same type $t^{min}$; and (ii) for* *every ex-ante optimal allocation, there exists an equivalent (in terms of interim payoffs for the buyer and the seller) feasible allocation satisfying ex-post incentive-compatibility for the buyer.*[16]

*Proof.* Proposition 1 established that the ex-ante optimal allocation $(p^{EO}, x^{EO})$ generates at least the ex-ante revenue of the full-information allocation $(p^{FI}, x^{FI})$–i.e., $X^{EO} \geq \bar{X}^{FI}$. We establish by contradiction that under conditions (i) and (ii), we have that $X^{EO} = \bar{X}^{FI}$. Suppose that $X^{EO} > \bar{X}^{FI}$. Condition (ii) implies that there exists an allocation $(\tilde{p}, \tilde{x})$ that is ex-post incentive-compatible for the buyer and generates the same interim utilities as $(p^{EO}, x^{EO})$; in particular, $U^{EO}(t^{min}) = \tilde{U}(t^{min}) = 0$. By condition (i), $t^{min}$ is the worst type both at $(p^{EO}, x^{EO})$ and $(\tilde{p}, \tilde{x})$. Hence, $(\tilde{p}, \tilde{x})$ satisfies (interim) participation constraints but may not satisfy ex-post participation constraints. Now, consider the modified allocation $(\tilde{p}, \hat{x})$ such that $\hat{x}(s, t) = \tilde{x}(s, t) + \tilde{U}(s, t^{min})$ for every $s$ and $t$. This allocation is still ex-post incentive-compatible for the buyer since for every $s$, $\hat{U}(s, t)$ is obtained from $\tilde{U}(s, t)$ by adding a constant: $\hat{U}(s, t) = \tilde{U}(s, t) - \tilde{U}(s, t^{min})$. Hence, by condition (i), we have $\hat{U}(s, t) \geq \hat{U}(s, t^{min})$ for every $s$ and $t$–i.e., $\hat{U}(s, t) \geq 0$ for every $s$ and $t$. Thus, $(\tilde{p}, \hat{x})$ satisfies ex-post participation. Since $\tilde{U}(t^{min}) = 0$, the ex-ante expected revenue at the allocation $(\tilde{p}, \hat{x})$ is the same as with the allocation $(\tilde{p}, \tilde{x})$, and is, therefore, $X^{EO}$. Now, since we assumed that $X^{EO} > \bar{X}^{FI}$, we conclude that there is some type $s$ for the seller that obtains a strictly higher interim revenue at the allocation $(\tilde{p}, \hat{x})$ than $(p^{FI}, x^{FI})$, contradicting that $(p^{FI}, x^{FI})$ is the full-information optimum. ■

If either (i) or (ii) fails, then it is possible that the seller strictly benefits from having private information. To see that, consider the example in the online appendix when $1 < V_0 = V_1 = V < 2$. This example violates condition (i) but satisfies condition (ii) since at the ex-ante optimal allocation the seller posts a price of $\bar{X} = V - \frac{1}{2}$ and the buyer always accepts. This allocation is ex-post incentive-compatible for the buyer. However, in this example, we have seen that the seller generates strictly more revenue when his information is private. Next, consider Example 2. This example satisfies condition (i) but fails condition (ii) since the ex-ante optimal allocation is not ex-post incentive-compatible for the buyer.

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[16]See Gershkov et al. (2013) for sufficient conditions for (ii) to hold. Their conditions are different from ours in Proposition 5.

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# Online Appendix

# A Application: Horizontal and Vertical Differentiation

There are two possible types of products in $S = \{0,1\}$. The buyer's taste parameter is in $T = [0,1]$. All types are uniformly distributed. The buyer's valuation function is:

$$u(0,t) = V_0 - t$$
$$u(1,t) = V_1 - 1 + t.$$

Without loss of generality, we assume that, ex ante, the type 1 product is more desirable for the buyer–i.e., $V_1 \geq V_0$. To exclude uninteresting extreme configurations, we also assume that $-1 \leq V_0 \leq V_1 \leq 2$. First, note that $u(s,t)$ is linear and, therefore, convex in $t$. Since $t^{min}(0) = 1$, while $t^{min}(1) = 0$, it is not a priori obvious at which type $t^*$ (B-PC) binds at the ex-ante optimal allocation. For every $t^* \in [0,1]$ we can write:

$$J(0,t;t^*) = \begin{cases} V_0 - 2t & \text{if } t < t^* \\ V_0 - 2t + 1 & \text{if } t > t^*, \end{cases} \qquad J(1,t;t^*) = \begin{cases} V_1 + 2t - 1 & \text{if } t < t^* \\ V_1 + 2t - 2 & \text{if } t > t^*. \end{cases}$$

Notice that $J(0,t;t^*) + J(1,t;t^*) = V_0 + V_1 - 1$ for every $t$ and $t^*$, and $2P(t) = p(1,t) - p(0,t)$ for every $t \in [0,1]$.

**Full-Information Allocation:** We first analyze the case in which the seller's type is common knowledge. Pointwise optimization yields:

$$p^{FI}(0,t) = \begin{cases} 1 & \text{if } t < \frac{V_0}{2} \\ 0 & \text{otherwise}, \end{cases} \qquad p^{FI}(1,t) = \begin{cases} 1 & \text{if } t > \frac{2-V_1}{2} \\ 0 & \text{otherwise}. \end{cases}$$

Since $p(s,t)\frac{\partial u(s,t)}{\partial t}$ is increasing in $t$ for every $s$, the pointwise optimum is a revenue-maximizing assignment rule and the corresponding payment rule is

$$x^{FI}(0,t) = p(0,t)u(0,t) - \int_1^t p(0,\tau)\frac{\partial u(0,\tau)}{\partial \tau}d\tau = \begin{cases} \frac{V_0}{2} & \text{if } t \leq \frac{V_0}{2} \\ 0 & \text{if } t > \frac{V_0}{2}, \end{cases}$$

and

$$x^{FI}(1,t) = p(1,t)u(1,t) - \int_0^t p(1,\tau)\frac{\partial u(1,\tau)}{\partial \tau}d\tau = \begin{cases} \frac{V_1}{2} & \text{if } t \geq \frac{2-V_1}{2} \\ 0 & \text{if } t < \frac{2-V_1}{2}. \end{cases}$$

When $V_s < 0$, the interim revenue of seller type $s$ is zero. Otherwise, when $V_0, V_1 \geq 0$, the full-information interim revenue is

$$X^{FI}(s) = \int_T x^{FI}(s,t)dt = \begin{cases} \frac{V_0^2}{4} & \text{if } s = 0 \\ \frac{V_1^2}{4} & \text{if } s = 1, \end{cases}$$

and the full-information ex-ante expected revenue is $\bar{X}^{FI} = \frac{1}{8}(V_0^2 + V_1^2)$.

**Ex-ante optimal allocation:** From Proposition 3, we have to find $t^*$ and $p(s,t)$ that maximize

$$\bar{X} = \frac{1}{2}\int_0^1 \Big[p(0,t)J(0,t;t^*) + p(1,t)J(1,t;t^*)\Big] f(t)dt, \tag{13}$$

subject to $P(\cdot)$ is increasing and $P(t) \leq 0$ for $t < t^*$ and $P(t) \geq 0$ for $t > t^*$. The last equation can be rewritten as

$$\bar{X} = \int_0^1 \left[-P(t)J(0,t;t^*) + p(1,t)\left(\frac{V_0 + V_1 - 1}{2}\right)\right] f(t)dt \tag{14}$$

$$= \int_0^1 p(1,t)\frac{V_0 + V_1 - 1}{2} f(t)dt - \int_0^{t^*} P(t)(V_0 - 2t)f(t)dt - \int_{t^*}^1 P(t)(V_0 - 2t + 1)f(t)dt, \tag{15}$$

and incentive and participation constraints for the buyer reduce to the conditions that $2P(t) = p(1,t) - p(0,t)$ is increasing, negative for $t < t^*$, and positive for $t > t^*$. The allocation rule that maximizes the revenue pointwise for some presumed $t^* \in [0,1]$ is:

$$p(0,t) = \begin{cases} 1 & \text{if } t < \frac{V_0}{2} \text{ and } t < t^* \\ 1 & \text{if } t^* < t < \frac{V_0+1}{2} \\ 0 & \text{otherwise,} \end{cases} \qquad p(1,t) = \begin{cases} 1 & \text{if } \frac{1-V_1}{2} < t < t^* \\ 1 & \text{if } t > \frac{2-V_1}{2} \text{ and } t > t^* \\ 0 & \text{otherwise.} \end{cases}$$

We start with two preliminary observations: If $V_0$ and $V_1$ are below zero, then the buyer's valuation is always negative, so he is never assigned the good and the ex-ante optimal and full-information allocations coincide. If $V_0 \geq 1$ and $V_1 \geq 1$, the buyer is always assigned the good, and the seller extracts the entire surplus from the buyer by selling the good at a price equal to $\frac{V_0+V_1-1}{2}$. In this case, we have full pooling. The ex-ante optimal expected revenue is strictly higher than the full-information ex-ante revenue. The following proposition characterizes the ex-ante optimal allocation for the more interesting intermediate values of $V_s$.

**Proposition 7** *In the horizontal and vertical differentiation example, the ex-ante optimal allocation has the following properties:*

- *Partial Pooling: If $V_0 + V_1 > 1$, then*

$$(p(0,t), p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{1-V_1}{2} \\ (1,1) & \text{if } \frac{1-V_1}{2} < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}. \end{cases} \qquad E_S\left[x(s,t)\right] = \begin{cases} \frac{2V_0+V_1-1}{4} & \text{if } t \leq \frac{1-V_1}{2} \\ \frac{V_0+V_1-1}{2} & \text{if } \frac{1-V_1}{2} \leq t \leq \frac{V_0+1}{2} \\ \frac{V_0+2V_1-1}{4} & \text{if } t \geq \frac{V_0+1}{2}, \end{cases}$$

  *and the ex-ante expected revenue is*

$$X^{EO} = \begin{cases} \frac{1}{8}(2V_0 + 2V_1 + V_0^2 + V_1^2 - 2) & \text{if } V_1 \leq 1 \\ \frac{1}{8}(4V_0 + 2V_1 + V_0^2 - 3) & \text{if } V_1 \geq 1, \end{cases}$$

  *which is strictly higher than the full-information ex-ante revenue.*

- *Full Separation: If $V_0 + V_1 \leq 1$ , then the allocation rule and ex-ante expected revenue are* *the same as in the full-information allocation:*

$$(p(0,t),p(1,t)) = \begin{cases} (1,0) & \textit{if } t < \frac{V_0}{2} \\ (0,0) & \textit{if } \frac{V_0}{2} < t < \frac{2-V_1}{2} \\ (0,1) & \textit{if } t > \frac{2-V_1}{2}. \end{cases} \qquad E_S\left[x(s,t)\right] = \begin{cases} \frac{V_0}{4} & \textit{if } t \le \frac{V_0}{2} \\ 0 & \textit{if } \frac{V_0}{2} \le t \le \frac{2-V_1}{2} \\ \frac{V_1}{4} & \textit{if } t \ge \frac{2-V_1}{2}, \end{cases}$$

$$X^{EO} = \bar{X}^{FI} = \frac{1}{8}\left((\max\{0,V_0\})^2 + (\max\{0,V_1\})^2\right).$$

*Proof.*

*Possibility 1: $V_0 \le 1 \le V_1$ and $V_0 + V_1 \ge 2$.* In this case, we have $\frac{1-V_1}{2} \le 0 \le \frac{2-V_1}{2} \le \frac{V_0}{2} < \frac{V_0+1}{2} \le 1$.

(1a) Let $\frac{2-V_1}{2} \le t^* \le \frac{V_0}{2}$. Pointwise maximization yields

$$(p(0,t),p(1,t)) = \begin{cases} (1,1) & \text{if } t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}, \end{cases}$$

so $2P(t) = 0$ for $t < \frac{V_0+1}{2}$ and $2P(t) = 1$ for $t > \frac{V_0+1}{2}$, which is feasible. Equation (13) simplifies to $\bar{X} = \frac{1}{8}(4V_1 + 2V_0 + V_0^2 - 3)$. The interim utilities and payments of the buyer are given, respectively, by

$$U(t) = \int_{t^*}^{t} P(\tau)d\tau = \begin{cases} 0 & \text{if } t < \frac{V_0+1}{2} \\ \frac{1}{2}(t - \frac{V_0+1}{2}) & \text{if } t > \frac{V_0+1}{2}, \end{cases}$$

and

$$E_S\left[x(s,t)\right] = E_S\left[p(s,t)u(s,t)\right] - U(t) = \begin{cases} \frac{V_0+V_1-1}{2} & \text{if } t < \frac{V_0+1}{2} \\ \frac{V_0+2V_1-1}{4} & \text{if } t > \frac{V_0+1}{2}, \end{cases} \tag{16}$$

and the solution is consistent for every $t^*$ such that $\frac{2-V_1}{2} \le t^* \le \frac{V_0}{2}$.

(1b) Let $0 \le t^* < \frac{2-V_1}{2} \le \frac{V_0}{2}$. Pointwise maximization yields

$$(p(0,t),p(1,t)) = \begin{cases} (1,1) & \text{if } t < t^* \\ (1,0) & \text{if } t^* < t < \frac{2-V_1}{2} \\ (1,1) & \text{if } \frac{2-V_1}{2} < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}, \end{cases}$$

so $2P(t) = 0$ for $t < t^*$, $2P(t) = -1$ for $t^* < t < \frac{2-V_1}{2}$, $2P(t) = 0$ for $\frac{2-V_1}{2} < t < \frac{V_0+1}{2}$ and $2P(t) = 1$ for $t > \frac{V_0+1}{2}$, which is not feasible. Given the constraints, we have to set $p(1,t) = p(0,t) = 1$ for $t^* < t < \frac{2-V_1}{2}$, and, therefore, we get the same allocation as in case (1a), and the solution is consistent for every $t^* < \frac{2-V_1}{2}$.

(1c) Let $\frac{V_0}{2} \le t^* \le \frac{V_0+1}{2}$. This gives exactly the same solution as in case (1a).

(1d) Let $\frac{V_0+1}{2} \le t^* \le 1$. Pointwise maximization yields

$$(p(0,t),p(1,t)) = \begin{cases} (1,1) & \text{if } t < \frac{V_0}{2} \\ (0,1) & \text{if } t > \frac{V_0}{2}, \end{cases}$$

which is not feasible because $2P(t) = 1 > 0$ for $\frac{V_0}{2} < t < t^*$, so we have to set $p(1,t) = p(0,t) = 1$ on this interval, so that

$$(p(0,t), p(1,t)) = \begin{cases} (1,1) & \text{if } t < t^* \\ (0,1) & \text{if } t > t^*. \end{cases}$$

The transfers are then obtained as in (16):

$$E_S[x(s,t)] = E_S[p(s,t)u(s,t)] - U(t) = \begin{cases} \frac{V_0+V_1-1}{2} & \text{if } t < t^* \\ \frac{V_0+2V_1-1}{4} & \text{if } t > t^*. \end{cases}$$

Since $\frac{V_0+2V_1-1}{4} > \frac{V_0+V_1-1}{2} \iff t > \frac{V_0+1}{2}$, it is optimal to set $t^* = \frac{V_0+1}{2}$ and we get again the same allocation as in case (1a).

*Possibility 2:* $1 \leq V_0 + V_1 \leq 2$. In this case, we have $\frac{1-V_1}{2} \leq \frac{V_0}{2} \leq \frac{2-V_1}{2} \leq \frac{V_0+1}{2}$.

(2a) $\frac{V_0}{2} \leq t^* \leq \frac{2-V_1}{2}$. Pointwise maximization yields

$$(p(0,t), p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{1-V_1}{2} \\ (1,1) & \text{if } \frac{1-V_1}{2} < t < \frac{V_0}{2} \\ (0,1) & \text{if } \frac{V_0}{2} < t < t^* \\ (1,0) & \text{if } t^* < t < \frac{2-V_1}{2} \\ (1,1) & \text{if } \frac{2-V_1}{2} < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}. \end{cases}$$

This is not feasible because $P(t) > 0$ for $t \in [\frac{V_0}{2}, t^*]$ and $P(t) < 0$ for $t \in [t^*, \frac{2-V_1}{2}]$. Hence, we have to set $P(t) = 0$ on these intervals. Since $\frac{V_0+V_1-1}{2} > 0$, to maximize $\bar{X}$ given by (15), we should set $p(1,t) = p(0,t) = 1$ on that range. Then, the ex-ante optimal allocation in this case is:

$$(p(0,t), p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{1-V_1}{2} \\ (1,1) & \text{if } \frac{1-V_1}{2} < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}. \end{cases}$$

Equation (13) simplifies to

$$\bar{X} = \begin{cases} \frac{1}{8}(2V_0 + 2V_1 + V_0^2 + V_1^2 - 2) & \text{if } V_1 \leq 1 \\ \frac{1}{8}(4V_0 + 2V_1 + V_0^2 - 3) & \text{if } V_1 \geq 1, \end{cases}$$

which is strictly higher compared to the full-information revenue $\frac{1}{8}(V_0^2 + V_1^2)$. The interim utilities and payments of the buyer are given, respectively, by

$$U(t) = \int_{t^*}^{t} P(\tau)d\tau = \begin{cases} -\frac{1}{2}(t - \frac{1-V_1}{2}) & \text{if } t < \frac{1-V_1}{2} \\ 0 & \text{if } \frac{1-V_1}{2} < t < \frac{V_0+1}{2} \\ \frac{1}{2}(t - \frac{V_0+1}{2}) & \text{if } t > \frac{V_0+1}{2}, \end{cases}$$

and

$$E_S\left[x(s,t)\right] = E_S\left[p(s,t)u(s,t)\right] - U\left(t\right) = \begin{cases} \frac{2V_0+V_1-1}{4} & \text{if } t < \frac{1-V_1}{2} \\ \frac{V_0+V_1-1}{2} & \text{if } \frac{1-V_1}{2} < t < \frac{V_0+1}{2} \\ \frac{V_0+2V_1-1}{4} & \text{if } t > \frac{V_0+1}{2}, \end{cases} \tag{17}$$

and the solution is consistent for every $t^*$ such that $\frac{1-V_1}{2} < t^* < \frac{V_0+1}{2}$.

(2b) $0 \leq t^* \leq \frac{1-V_1}{2}$. Pointwise maximization yields

$$(p(0,t),p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{2-V_1}{2} \\ (1,1) & \text{if } \frac{2-V_1}{2} < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}. \end{cases}$$

This is not feasible since $t^* < \frac{2-V_1}{2}$, so we should set $p(1,t) = p(0,t) = 1$ for $t^* < t < \frac{2-V_1}{2}$ and we get

$$(p(0,t),p(1,t)) = \begin{cases} (1,0) & \text{if } t < t^* \\ (1,1) & \text{if } t^* < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}. \end{cases}$$

Equation (13) yields $\bar{X} = \frac{1}{8}(2V_0 + V_0^2 - 3 + 4V_1 + 4t^* - 4(t^*)^2 - 4t^*V_1)$, which is maximized for $t^* = \frac{1-V_1}{2}$ and, again, we get the same allocation as in case (2a).

(2c) $\max\{\frac{1-V_1}{2}, 0\} \leq t^* \leq \frac{V_0}{2}$. Pointwise maximization yields

$$(p(0,t),p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{1-V_1}{2} \\ (1,1) & \text{if } \frac{1-V_1}{2} < t < t^* \\ (1,0) & \text{if } t^* < t < \frac{2-V_1}{2} \\ (1,1) & \text{if } \frac{2-V_1}{2} < t < \frac{V_0+1}{2} \\ (0,1) & \text{if } t > \frac{V_0+1}{2}. \end{cases}$$

This is not feasible because $P(t) < 0$ for $t \in [t^*, \frac{2-V_1}{2}]$, so, as before, we should set $p(1,t) = p(0,t) = 1$ on that range and we get exactly the same allocation as in case (2a).

(2d) $\frac{V_0+1}{2} \leq t^* \leq 1$. This situation is similar to situation (2b) and yields the same ex-ante optimal allocation as in (2a).

(2e) $\frac{2-V_1}{2} \leq t^* \leq \frac{V_0+1}{2}$. This situation is similar to situation (2c) and yields the same ex-ante optimal allocation as in (2a).

*Possibility 3:* $V_0 + V_1 \leq 1$. Let $\frac{1-V_1}{2} \leq t^* \leq \frac{V_0+1}{2}$. Pointwise maximization yields

$$(p(0,t),p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{V_0}{2} \\ (0,0) & \text{if } \frac{V_0}{2} < t < \frac{1-V_1}{2} \\ (0,1) & \text{if } \frac{1-V_1}{2} < t < t^* \\ (1,0) & \text{if } t^* < t < \frac{V_0+1}{2} \\ (0,0) & \text{if } \frac{V_0+1}{2} < t < \frac{2-V_1}{2} \\ (0,1) & \text{if } t > \frac{2-V_1}{2}. \end{cases}$$

This is not feasible because $P(t) > 0$ for $t \in [\frac{1-V_1}{2}, t^*]$ and $P(t) < 0$ for $t \in [t^*, \frac{V_0+1}{2}]$. Hence, we have to set $P(t) = 0$ on these intervals. Since $\frac{V_0+V_1-1}{2} \leq 0$, to maximize $\bar{X}$ given by (15) we should set $p(1,t) = p(0,t) = 0$ on that range. Then, the allocation rule is the same as in the full-information allocation:

$$(p(0,t), p(1,t)) = \begin{cases} (1,0) & \text{if } t < \frac{V_0}{2} \\ (0,0) & \text{if } \frac{V_0}{2} < t < \frac{2-V_1}{2} \\ (0,1) & \text{if } t > \frac{2-V_1}{2}. \end{cases}$$

The same logic applies for $t^* \notin \left(\frac{1-V_1}{2}, \frac{V_0+1}{2}\right)$ and leads to the full-information allocation. ■

This proposition generalizes the observations made in the introductory example. In general, a simple posted price is suboptimal, and the seller strictly gains from the fact that the buyer does not know the product characteristics. When $V_0 = V_1 = V \in [1/2, 1]$, the optimum can be implemented, for example, via bilateral cheap talk followed by contingent prices: the buyer first reveals whether or not his type $t$ belongs to the interval $[\frac{1-V}{2}, \frac{V+1}{2}]$. If $t$ belongs to that interval, then the seller posts the price $\frac{2V-1}{2}$ without revealing product information. Otherwise, the seller posts the price $\frac{3V-1}{4}$ and reveals product information to the buyer. Balestrieri and Izmalkov (2012) examine a symmetric version of the same setup and use a different solution approach that leverages symmetry. In addition to the case of linear transportation costs considered here, they also analyze convex and concave costs.