Galerkin least squares finite element method for the obstacle problem ⋆

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Abstract

We construct a consistent multiplier free method for the finite element solution of the obstacle problem. The method is based on an augmented Lagrangian formulation in which we eliminate the multiplier by use of its definition in a discrete setting. We prove existence and uniqueness of discrete solutions and optimal order a priori error estimates for smooth exact solutions. Using a saturation assumption we also prove an a posteriori error estimate. Numerical examples show the performance of the method and of an adaptive algorithm for the control of the discretization error.

Key words: Obstacle problem, augmented Lagrangian method, a priori error estimate, a posteriori error estimate, adaptive method

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1 Introduction

Our aim in this paper is to design a simple consistent penalty method for contact problems that avoids the solution of variational inequalities. We elim-

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inate the need for Lagrange multipliers to enforce the contact conditions by using its definition in a discrete setting, following an idea of Chouly and Hild [7] used for elastic contact.

1.1 The model problem

We consider the obstacle problem of finding the displacement u of a membrane constrained to stay above an obstacle given by $\psi = \psi(x, y)$ (with $\psi \leq 0$ at $\partial\Omega$):

$$\begin{aligned} -\Delta u - f &\geq 0 \text{ in } \Omega \subset \mathbb{R}^2 \\ u &\geq \psi \text{ in } \Omega \\ (u-\psi)(f+\Delta u) &= 0 \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

where Ω is a convex polygon. It is well known that this problem admits a unique solutions $u \in H^1(\Omega)$. This follows from the theory of Lions and Stampacchia applied to the corresponding variational inequality [14]. It is also known that if $\psi \in C^{1,1}(\Omega)$ then u also has this regularity in the contact zone [4], i.e. the second derivatives are bounded. We assume this regularity in the following.

1.2 The finite element method

There exists a large body of literature treating finite element methods for unilateral problems in general and obstacle problems in particular, including the following papers [13,16,9,12,11,5,20,24,3,22]. Discretization of (1) is usually performed directly starting from the variational inequality or using a penalty method. For a discussion and a analysis of the first approach we refer to [8]. The latter approach was first considered in [16].

An alternative is to use the augmented Lagrangian method. We introduce the Lagrange multiplier λ such that

$$\begin{aligned} -\Delta u + \lambda &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned} \tag{2}$$

under the Kuhn-Tucker side conditions

$$
\begin{aligned}
\psi - u &\leq 0 \text{ in } \Omega \\
\lambda &\leq 0 \text{ in } \Omega \\
(\psi - u)\,\lambda &= 0 \text{ in } \Omega.
\end{aligned}
\tag{3}
$$

Using the standard trick of rewriting the Kuhn-Tucker conditions as

$$
\lambda = -\frac{1}{\gamma}\left[\psi - u - \gamma\lambda\right]_+ \tag{4}
$$

where $[x]_+ = \max(0, x)$ and $\gamma \in \mathbb{R}^+$, cf., e.g., Chouly and Hild [7], we can formulate the augmented Lagrangian problem of finding (u, λ) that are stationary points to the functional

$$
\begin{aligned}
\mathfrak{F}(u,\lambda) := \frac{1}{2}\int_\Omega |\nabla u|^2\, d\Omega &+ \int_\Omega \frac{1}{2\gamma}\left[\psi - u - \gamma\lambda\right]_+^2\, d\Omega \\
&- \frac{1}{2}\int_\Omega \gamma\lambda^2\, d\Omega - \int_\Omega f u\, d\Omega,
\end{aligned}
\tag{5}
$$

cf. Alart and Curnier [1], leading to seeking $(u, \lambda) \in H^1_0(\Omega) \times L_2(\Omega)$ such that

$$
\int_\Omega \nabla u \cdot \nabla v\, d\Omega - \int_\Omega \frac{1}{\gamma}\left[\psi - u - \gamma\lambda\right]_+ v\, d\Omega = \int_\Omega f v\, d\Omega \quad \forall v \in H^1_0(\Omega) \tag{6}
$$

and

$$
\int_\Omega \frac{1}{\gamma}\left[\psi - u - \gamma\lambda\right]_+ \mu\, d\Omega + \int_\Omega \lambda\mu\, d\Omega = 0 \quad \forall \mu \in L_2(\Omega). \tag{7}
$$

For our discrete method, we assume that $\{\mathcal{T}\}_h$ is a family of conforming shape regular meshes on Ω, consisting of triangles $\mathcal{T} = \{T\}$ and we denote the set of interior faces of $\mathcal{T}$ by $\mathcal{F}$ Then we define V_h as the space of H^1–conforming piecewise polynomial functions on $\mathcal{T}$, satisfying the homogeneous boundary condition of Γ_D.

$$
V_h := \{v_h \in H^1_0(\Omega) : v_h|_T \in \mathbb{P}_k(T),\, \forall T \in \mathcal{T}\}, \qquad \text{for } k \geq 1.
$$

To obtain a discrete minimization problem, without multiplier, we formally replace λ element–wise by $\Delta u_h + f$: seek $u_h \in V_h$ such that

$$
u_h = \arg\min_{v \in V_h} \mathfrak{F}_h(v) \tag{8}
$$

where

$$\mathfrak{F}_h(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 \, d\Omega + \sum_{T\in\mathcal{T}} \int_T \frac{1}{2\gamma} \left[\psi - v - \gamma(\Delta v + f)\right]_+^2 \, d\Omega$$
$$- \frac{1}{2}\sum_{T\in\mathcal{T}} \int_T \gamma(\Delta v + f)^2 \, d\Omega - \int_\Omega f v \, d\Omega. \tag{9}$$

The Euler–Lagrange equations corresponding to (9) take the form: Find $u_h \in V_h$ such that

$$a(u_h, v_h) + b(u_h, \psi, f; v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h \tag{10}$$

where $(\cdot,\cdot)_\Omega$ denotes the standard L^2-inner product, $a(u_h, v_h) := (\nabla u_h, \nabla v_h)_\Omega$ and

$$b(u_h, \psi, f; v_h) := \left\langle \gamma^{-1}[\psi - u_h - \gamma(\Delta u_h + f)]_+, -v_h - \gamma \Delta v_h \right\rangle_h$$
$$- \left\langle \gamma(\Delta u_h + f), \Delta v_h \right\rangle_h \tag{11}$$

where

$$\langle x_h, y_h \rangle_h := \sum_{T\in\mathcal{T}} \int_T x_h y_h \, \mathrm{d}x$$

and, for use below,

$$\|x_h\|_h := \langle x_h, x_h \rangle_h^{1/2}.$$

To simplify the notation below we introduce $P_\gamma(u_h) = \gamma \Delta u_h + u_h$ and

$$b(u_h, \psi, f; v_h) := \left\langle \gamma^{-1}[\psi - \gamma f - P_\gamma(u_h)]_+, -P_\gamma(v_h) \right\rangle_h - \left\langle \gamma(\Delta u_h + f), \Delta v_h \right\rangle_h.$$

We will also omit ψ and f from the argument of b below, and use the notation $\Psi := \psi - \gamma f$ so that

$$b(u_h; v_h) := \left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+, -P_\gamma(v_h) \right\rangle_h - \left\langle \gamma(\Delta u_h + f), \Delta v_h \right\rangle_h. \tag{12}$$

Note that the form b can be interpreted as a nonlinear consistent least squares penalty term for the imposition of the contact condition. A similar method was proposed in Stenberg et al. [10] in the framework of variational inequalities. Since for the case $k = 1$, the high order terms vanish in (10), this choice leads to a method that bears a strong ressemblence to that of [16], showing the relation between our method and the classical penalty method.

We will below alternatively use the compact notation

$$A_h(u_h; v_h) := a(u_h, v_h) + b(u_h; v_h)$$

and the associated formulation, find $u_h \in V_h$ such that

$$A_h(u_h; v_h) = (f, v_h)_\Omega, \text{ for all } v_h \in V_h. \tag{13}$$

1.3 Summary of main results and outline

In Section 2 we recall some technical results. In Section 3 we derive an existence result for the discrete solution using Brouwer's fixed point theorem and we prove uniqueness of the solution using monotonicity of the the nonlinearity. Section 4 is consecrated to error estimates and we prove an a priori error estimate an a posteriori error estimate. In the latter case the analysis relies on a saturation assumption. Finally in Section 5 we present numerical results confirming our theoretical results and illustrating the performance of an adaptive algorithm based on our a posteriori error estimate.

2 Technical results

Below we will use the notation $a \lesssim b$ for $a \leq Cb$ where C is a constant independent of h, but not of the local mesh geometry.

First we recall the following inverse inequality,

$$\|\nabla v_h\|_T \leq C_i h_T^{-1} \|v_h\|_T, \qquad T \in \mathcal{T} \tag{14}$$

see Thomée [19]. An immediate consequence of this is that $P_\gamma(v_h)$ satisfies the bound

$$\|P_\gamma(v_h)\|_h \leq \|v_h\|_\Omega + C_i \gamma h^{-1} \|\nabla v_h\|_\Omega. \tag{15}$$

We will use the Scott-Zhang interpolant preserving boundary conditions, denoted $i_h : H_0^1(\Omega) \mapsto V_h$. This operator is H^1-stable, $\|i_h u\|_{H^1(\Omega)} \lesssim \|u\|_{H^1(\Omega)}$ and the following interpolation error estimate is known to hold [17],

$$\|u - i_h u\|_\Omega + h\|u - i_h u\|_{H^1(\Omega)} + h^2 \|\Delta(u - i_h u)\|_h \lesssim h^{k+1} |u|_{H^{k+1}(\Omega)}. \tag{16}$$

The essential properties of the nonlinearity are collected in the following lemmas.

Lemma 1 *Let $a, b \in \mathbb{R}$ then there holds*

$$([a]_+ - [b]_+)^2 \leq ([a]_+ - [b]_+)(a - b),$$

$$|[a]_+ - [b]_+| \leq |a - b|.$$

PROOF. Developing the left hand side of the expression we have

$$[a]_+^2 + [b]_+^2 - 2[a]_+[b]_+ \leq [a]_+a + [b]_+b - a[b]_+ - [a]_+b = ([a]_+ - [b]_+)(a - b).$$

For the proof of the second claim, this is trivially true in case both a and b are positive or negative. If a is negative and b positive then

$$|[a]_+ - [b]_+| = |b| \leq |b - a|$$

and the case with b is negative and a positive follows by symmetry.

Lemma 2 *(Continuity of $b(\cdot;\cdot)$) For all $u_1, u_2, v \in V_h$, the form* (11) *satisfies*

$$|b(u_1; v) - b(u_2; v)| \lesssim \gamma^{-1}(\|(u_1 - u_2)\|_\Omega + \gamma h^{-1}\|\nabla(u_1 - u_2)\|_\Omega)(\|v\|_\Omega + \gamma h^{-1}\|\nabla v_h\|_\Omega). \tag{17}$$

PROOF.

$$\begin{aligned} b(u_1; v_h) - b(u_2; v_h) = \gamma^{-1} \left\langle [\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+, -P_\gamma(v_h) \right\rangle_h \\ - \left\langle \gamma \Delta(u_1 - u_2), \Delta v_h \right\rangle_h . \end{aligned}$$

Using the second inequality of Lemma 1 we see that the nonlinearity satisfies

$$\begin{aligned} &\gamma^{-1} | \left\langle [\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+, -P_\gamma(v_h) \right\rangle_h | \\ &\qquad \leq \gamma^{-1} \|P_\gamma(u_1) + \Psi - P_\gamma(u_2) - \Psi\|_h \|P_\gamma(v_h)\|_h. \end{aligned} \tag{18}$$

By the inverse inequality (14) we have

$$\left\langle \gamma \Delta(u_1 - u_2), \Delta v_h \right\rangle_h \leq C_i^2 \gamma h^{-2} \|\nabla(u_1 - u_2)\|_\Omega \|\nabla v_h\|_\Omega. \tag{19}$$

Collecting (18),(19) and using (15) we have

$$\begin{aligned} &|b(u_1; v_h) - b(u_2; v_h)| \\ &\qquad \leq \gamma^{-1}(\|u_1 - u_2\|_\Omega + C_i \gamma h^{-1} \|\nabla(u_1 - u_2)\|_\Omega)(\|v_h\|_\Omega + C_i \gamma h^{-1} \|\nabla v_h\|_\Omega) \\ &\qquad\quad + C_i^2 \gamma h^{-2} \|\nabla(u_1 - u_2)\|_\Omega \|\nabla v_h\|_\Omega \end{aligned}$$

and the claim follows.

3 Existence of unique discrete solution

In the previous works on lateral contact problems using Nitsche's method [7] existence and uniqueness was proved by using the monotonicity and hemi-continuity of the operator. Here we propose a different approach where we use the Brouwer's fixed point theorem to establish existence and the monotonicity of the nonlinearity for uniqueness. We start by showing some positivity results and a priori bounds. Since we are interested in existence and uniqueness for a fixed mesh parameter h, we do not require that the bounds in this section are uniform in h.

Lemma 3 *Let $u_1, u_2 \in V_h$ and assume that*

$$\gamma < C_i^{-2} h^2/2. \tag{20}$$

Then there holds

$$\frac{\alpha}{2}\|u_1 - u_2\|^2_{H^1(\Omega)} + \gamma^{-1}\|[\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+\|_h^2 \\ \leq A_h(u_1; u_1 - u_2) - A_h(u_2; u_1 - u_2)$$

and

$$\frac{\alpha}{4}\|u_1\|^2_{H^1(\Omega)} \leq A_h(u_1; u_1) + C\alpha^{-1}\gamma^{-2}\|[\Psi]_+\|^2_\Omega.$$

PROOF. First we consider the form $b(\cdot;\cdot)$,

$$b(u_1; v_h) - b(u_2; v_h) \\ = \gamma^{-1}\left\langle [\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+, -v_h - \gamma\Delta v_h + \Psi - \Psi\right\rangle_h \\ - \left\langle \gamma\Delta(u_1 - u_2), \Delta v_h\right\rangle_h.$$

Using the monotonicity of Lemma 1 we obtain

$$b(u_1; u_1 - u_2) - b(u_2; u_1 - u_2) \geq \gamma^{-1}\|[\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+\|_h^2 \\ - \gamma\|\Delta(u_1 - u_2)\|_h^2.$$

Observe that using the inverse inequality (14) we have

$$\gamma\|\Delta(u_1 - u_2)\|_h^2 \leq C_i \gamma h^{-2}\|\nabla(u_1 - u_2)\|^2_\Omega.$$

We may then write

$$(1 - C_i^2 h^{-2}\gamma)\|\nabla(u_1 - u_2)\|^2_\Omega + \gamma^{-1}\|[\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+\|_h^2 \\ \leq A_h(u_1; u_1 - u_2) - A_h(u_2; u_1 - u_2).$$

It follows that choosing $\gamma < C_i^{-2} h^2/2$ and applying the Poincaré inequality

$$\alpha^{\frac{1}{2}}\|u\|_{H^1(\Omega)} \leq \|\nabla u\|_\Omega, \quad \forall u \in H^1_0(\Omega) \tag{21}$$

there holds

$$\frac{\alpha}{2}\|u_1 - u_2\|^2_{H^1(\Omega)} + \gamma^{-1}\|[\Psi - P_\gamma(u_1)]_+ - [\Psi - P_\gamma(u_2)]_+\|_h^2 \\ \leq A_h(u_1; u_1 - u_2) - A_h(u_2; u_1 - u_2).$$

The second inequality follows by taking $u_2 = 0$ above and noting that then

$$\begin{aligned} &\frac{\alpha}{2}\|u_1\|^2_{H^1(\Omega)} + \gamma^{-1}\|[\Psi - P_\gamma(u_1)]_+ - [\Psi]_+\|_h^2 \\ &\quad \leq A_h(u_1; u_1) - \gamma^{-1}\langle [\Psi]_+, -P_\gamma(u_1)\rangle_h \\ &\quad \leq A_h(u_1; u_1) + (1 + C_i \gamma h^{-1})\alpha^{-\frac{1}{2}}\gamma^{-1}\|[\Psi]_+\|_\Omega \alpha^{\frac{1}{2}}\|u_1\|_{H^1(\Omega)} \end{aligned}$$

where we used (15) in the last step. Considering the condition on γ and using an arithmetic-geometric inequality we conclude.

Proposition 4 *Assume that γ satisfy* (20)*. Then formulation* (13) *using the contact operator* (12)*, admits a unique solution.*

PROOF. The uniqueness is an immediate consequence of Lemma 3. If u_1 and u_2 both are solution to (13), then

$$A_h(u_1; u_1 - u_2) - A_h(u_2; u_1 - u_2) = (f, u_1 - u_2)_\Omega - (f, u_1 - u_2)_\Omega = 0$$

and we conclude that $\|u_1 - u_2\|_{H^1(\Omega)} = 0$ and hence $u_1 \equiv u_2$.

To prove existence we use the Brouwer's fixed point Theorem, see for instance, Temam [18, Chapter 2, Lemma 1.4]. Let N_V denote the number of degrees of freedom of V_h.

Consider the mapping $G : \mathbb{R}^{N_V} \mapsto \mathbb{R}^{N_V}$ defined by

$$(G(U), V)_{\mathbb{R}^{N_V}} := A_h(u_h; v_h) - (f, v_h)_\Omega,$$

where $U = \{u_i\}_{i=1}^{N_V}$, $V = \{v_i\}_{i=1}^{N_V}$, where $\{u_i\}$ and $\{v_i\}$ denotes the vectors of unknowns associated to the basis functions of V_h.

By the second claim of Lemma 3, there holds

$$\frac{\alpha}{4}\|u_h\|^2_{H^1(\Omega)} - C\alpha^{-1}\gamma^{-2}\|[\Psi]_+\|^2_\Omega - (f, u_h)_\Omega \leq A_h(u_h; u_h) - (f, u_h)_\Omega \\ = (G(U), U)_{\mathbb{R}^{N_V}}.$$

Since

$$\frac{\alpha}{4}\|u_h\|^2_{H^1(\Omega)} - (f, u_h)_\Omega \geq \frac{\alpha}{8}\|u_h\|^2_{H^1(\Omega)} - C\frac{1}{\alpha}\|f\|^2_\Omega$$

we have that for any fixed h the following positivity holds for U sufficiently large

$$0 < \frac{\alpha}{8}\|u_h\|^2_{H^1(\Omega)} - \frac{C}{\alpha}(\gamma^{-2}\|[\Psi]_+\|^2_\Omega + \|f\|^2_\Omega) \leq (G(U), U)_{\mathbb{R}^{N_V}}. \qquad (22)$$

Assume that this positivity holds whenever $|U| \geq q \in \mathbb{R}_+$. Denote by B_q the (closed) ball in $\mathbb{R}^{N_V}$ with radius q and assume that there is no $U \in B_q$ such that $G(U) = 0$. Define the function

$$\phi(U) = -qG(U)/|G(U)|.$$

Then $\phi : B_q \mapsto B_q$, ϕ is continuous by Lemma 2 and the assumption that $|G(U)| > 0$ for all $U \in B_q$. Hence there exists a fixed point $X \in B_q$ such that

$$X = \phi(X).$$

It follows that

$$|X|^2 = q^2 = -q(G(X), X)/|G(X)|,$$

but this contradicts (22) and hence a solution exists.

4 Error estimates

In this section we prove optimal error estimates under the assumption that the exact solution is sufficiently regular for the formulation to be strongly consistent. First we prove a best approximation estimate, which then leads to optimal error estimates using interpolation. Then under a saturation assumption we prove the upper and lower bounds of an a posteriori error estimate.

Theorem 5 *(A priori error estimate) Assume that $u \in H^1_0(\Omega)$ with $\Delta u \in L^2(\Omega)$ is the solution of* (1) *and that u_h is the solution to* (10) *with* (11) *and $0 < \gamma = \gamma_0 h^2$, where $\gamma_0 \in \mathbb{R}$, $\gamma_0 < C_i^{-2}/2$. Then there holds for all $v_h \in V_h$*

$$\begin{aligned} \alpha\|u - u_h\|^2_{H^1(\Omega)} + \gamma^{-1}\|[\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+\|^2_h \\ \lesssim \frac{1}{\alpha}\|u - v_h\|^2_{H^1(\Omega)} + \|\gamma^{-\frac{1}{2}}(u - v_h)\|^2_\Omega + \|\gamma^{\frac{1}{2}}\Delta(u - v_h)\|^2_h. \end{aligned} \qquad (23)$$

If in addition $u \in H^{k+1}(\Omega)$ then there holds

$$\alpha\|u - u_h\|_{H^1(\Omega)} + \gamma^{-1/2}\|[\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+\|_h \lesssim h^k |u|_{H^{k+1}(\Omega)}. \qquad (24)$$

PROOF. Using the definition of $a(\cdot,\cdot)$ we write the decomposition

$$\begin{aligned}\|\nabla(u-u_h)\|_\Omega^2 &= a(u-u_h,u-u_h)\\ &= a(u-u_h,u-v_h)+a(u-u_h,v_h-u_h)\\ &\leq \frac{\alpha}{4}\|u-u_h\|_{H^1(\Omega)}^2+\frac{1}{\alpha}\|u-v_h\|_{H^1(\Omega)}^2+a(u-u_h,v_h-u_h).\end{aligned}$$

Observe that

$$\begin{aligned}a(u,v_h-u_h) &= (-\Delta u-f+f,v_h-u_h)_\Omega\\ &= (\gamma^{-1}[\Psi-P_\gamma(u)]_+,(v_h-u_h))_\Omega+(f,v_h-u_h)_\Omega. \end{aligned}\tag{25}$$

If $[\Psi-P_\gamma(u)]_+\in L^2(\Omega)$ the following equality also holds

$$\langle\Delta u+f,\gamma\Delta(v_h-u_h)\rangle_h+\left\langle\gamma^{-1}[\Psi-P_\gamma(u)]_+,\gamma\Delta(v_h-u_h)\right\rangle_h=0.$$

It follows that

$$\begin{aligned}a(u,v_h-u_h) &= (f,v_h-u_h)_\Omega-\left\langle\gamma^{-1}[\Psi-P_\gamma(u)]_+,-P_\gamma(v_h-u_h)\right\rangle_h\\ &\quad+\langle\Delta u+f,\gamma\Delta(v_h-u_h)\rangle_h\\ &= (f,v_h-u_h)_\Omega-b(u;v_h-u_h).\end{aligned}\tag{26}$$

As a consequence we have the following property reminiscent of Galerkin orthogonality,

$$\begin{aligned}&a(u-u_h,v_h-u_h)\\ &\quad= b(u_h;v_h-u_h)-b(u;v_h-u_h)\\ &\quad= \left\langle\gamma^{-1}[\Psi-P_\gamma(u_h)]_+-\gamma^{-1}[\Psi-P_\gamma(u)]_+,-P_\gamma(v_h-u_h)\right\rangle_h\\ &\qquad-\gamma\left\langle\Delta(u_h-u),\Delta(v_h-u_h)\right\rangle_h\end{aligned}\tag{27}$$

First observe that

$$\begin{aligned}&\gamma\left\langle\Delta(u_h-u),\Delta(v_h-u_h)\right\rangle_h\\ &\quad\leq\|\gamma^{\frac12}\Delta(u_h-v_h)\|_h^2+\|\gamma^{\frac12}\Delta(u-v_h)\|_h\|\gamma^{\frac12}\Delta(u_h-v_h)\|_h\\ &\quad\leq\frac12 C_i^2h^{-2}\gamma\|\nabla(u_h-v_h)\|_\Omega^2+\|\gamma^{\frac12}\Delta(u-v_h)\|_h^2\\ &\quad\leq C_i^2h^{-2}\gamma\|\nabla(u_h-u)\|_\Omega^2+C_i^2h^{-2}\gamma\|\nabla(v_h-u)\|_\Omega^2+\|\gamma^{\frac12}\Delta(u-v_h)\|_h^2.\end{aligned}$$

Considering the first term in the right hand side of equation (27) we see that

$$
\begin{aligned}
&\left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+ - \gamma^{-1}[\Psi - P_\gamma(u)]_+, -P_\gamma(v_h - u_h)\right\rangle_h \\
&\quad = \underbrace{\left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+ - \gamma^{-1}[\Psi - P_\gamma(u)]_+, -P_\gamma(v_h - u)\right\rangle_h}_{I} \\
&\qquad + \underbrace{\left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+ - \gamma^{-1}[\Psi - P_\gamma(u)]_+, -P_\gamma(u - u_h)\right\rangle_h}_{II} \\
&\quad = I + II.
\end{aligned}
$$

The term I may be bounded using the Cauchy-Schwarz inequality followed by the arithmetic geometric inequality

$$
I \leq \epsilon\gamma^{-1}\|[\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+\|_h^2 + \frac{1}{4\epsilon}\|\gamma^{-\frac{1}{2}}P_\gamma(v_h - u)\|_h^2.
$$

For the term II we use the monotonicity property of Lemma 1, with $a = \Psi - P_\gamma(u_h)$ and $b = \Psi - P_\gamma(u)$ so that

$$
([a]_+ - [b]_+)(b-a) = ([\Psi - P_\gamma(u_h)]_+ - \gamma^{-1}[\Psi - P_\gamma(u)]_+)(\Psi - P_\gamma(u) - \Psi + P_\gamma(u_h))
$$

to deduce that

$$
II \leq -\gamma^{-1}\|[\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+\|_h^2.
$$

Collecting the above bounds and using the Poincaré inequality (21) we find,

$$
\begin{aligned}
\alpha\left(\frac{3}{4} - C_i^2 h^{-2}\gamma\right)&\|u - u_h\|_{H^1(\Omega)}^2 \\
&+ (1-\epsilon)\,\gamma^{-1}\|[\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+\|_h^2 \\
&\leq \frac{1}{\alpha}\|u - v_h\|_{H^1(\Omega)}^2 + \frac{1}{4\epsilon}\|\gamma^{-\frac{1}{2}}P_\gamma(u - v_h)\|_h^2 + \|\gamma^{\frac{1}{2}}\Delta(u - v_h)\|_h^2 \qquad (28)
\end{aligned}
$$

Fixing $\epsilon = \frac{1}{2}$, and choosing γ sufficiently small so that $C_i^2 h^{-2}\gamma \leq \alpha/4$ then there holds

$$
\begin{aligned}
\alpha\|u - u_h\|_{H^1(\Omega)}^2 &+ \gamma^{-1}\|[\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+\|_h^2 \\
&\leq \frac{1}{\alpha}\|u - v_h\|_{H^1(\Omega)}^2 + \|\gamma^{-\frac{1}{2}}(u - v_h)\|_h^2 + \|\gamma^{\frac{1}{2}}\Delta(u - v_h)\|_h^2. \quad (29)
\end{aligned}
$$

This concludes the proof of (23). The error estimate (24) then follows by choosing v_h to be the interpolant $i_h u$, applying the approximation error estimate

(16) on the form

$$\begin{aligned}\|u - i_h u\|_{H^1(\Omega)} + \|\gamma^{-\frac{1}{2}}(u - i_h u)\|_h + \|\gamma^{\frac{1}{2}}\Delta(u - i_h u)\|_h \\ \lesssim (h^k + \gamma^{-1/2}h^{k+1} + \gamma^{1/2}h^{k-1})|u|_{H^{k+1}(\Omega)},\end{aligned}$$

and using the bound on γ.

Observe that the regularity requirement $\Delta u \in L^2(\Omega)$ in general is too strong in practice, i.e. for less regular obstacles than assumed here. We make it here as a technical assumption in order to prove optimal convergence using strong consistency. For cases where the exact solution is less regular an alternative approach based on weak consistency could be attempted. Such a finer analysis however is beyond the scope of the present work. Staying with the strong regularity assumption we instead proceed to prove an a posteriori error estimate for the method in the spirit of [12], where the penalty method was considered. To extend this analysis to the more general case presented here we need to handle the second order consistency term. To this end we introduce a saturation assumption. A similar approach has frequently been used in previous work on domain decomposition or contact problems. For instance for Nitsche's method in domain decomposition [2] or unilateral contact [6] and for work on mortar finite element methods see [23]. We now proceed to state the saturation assumption that we need and unsing this we prove the a posteriori error estimate.

Assumption: *(Saturation) We assume that there exists a constant C_s such that*

$$\|h\Delta(u - u_h)\|_h \le C_s\|\nabla(u - u_h)\|_\Omega. \tag{30}$$

Theorem 6 *(A posteriori error estimate) Assume that $u \in H^1_0(\Omega)$ with $\Delta u \in L^2(\Omega)$ is the solution of* (1) *and u_h the solution of* (10) *satisfying* (30) *and with the parameter γ defined element–wise and satisfying $\gamma|_T \le \frac{1}{2}C_s^{-2}h_T^2$, then*

$$\alpha\|u - u_h\|_{H^1(\Omega)} + \|\gamma^{-1/2}([\Psi - P_\gamma(u_h)]_+ - [\Psi - P_\gamma(u)]_+)\|_h \lesssim \left(\sum_{T\in\mathcal{T}}\eta_T^2\right)^{\frac{1}{2}}, \tag{31}$$

where

$$\eta_T := h_T\|f + \Delta u_h + \gamma^{-1}[\Psi - P_\gamma(u_h)]_+\|_T + \frac{1}{2}\sum_{F\in\partial T}\|h_T^{\frac{1}{2}}[\![\partial_n u_h]\!]\|_F.$$

PROOF. Let $e = u - u_h$ then, under the assumption (30) and using (3) we obtain the bound

$$
\begin{aligned}
&\frac{1}{2}\|\nabla e\|_\Omega^2 + \|([\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+)\|_h^2 \\
&\qquad \leq (\nabla(u-u_h), \nabla e)_\Omega \\
&\qquad\qquad + \left\langle \gamma^{-1}[\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+, -P_\gamma(e)\right\rangle_h \\
&\qquad\qquad - \langle \gamma\Delta(u-u_h), \Delta e\rangle_h
\end{aligned}
\tag{32}
$$

Now, using similar arguments as in Theorem 5 we deduce

$$
(\nabla u, \nabla e)_\Omega - \left\langle \gamma^{-1}[\Psi - P_\gamma(u)]_+, P_\gamma(e)\right\rangle_h - \langle \gamma\Delta u, \Delta e\rangle_h = \langle f, P_\gamma(e)\rangle_h .
$$

Using this relation to eliminate the exact solution u in the left slots of the right hand side of (35) we obtain

$$
\begin{aligned}
&\frac{1}{2}\|\nabla e\|_\Omega^2 + \|([\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+)\|_h^2 \\
&\qquad \leq \langle f, P_\gamma(e)\rangle_h - (\nabla u_h, \nabla e)_\Omega \\
&\qquad\qquad + \left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+, P_\gamma(e)\right\rangle_h + \langle \gamma\Delta u_h, \Delta e\rangle_h
\end{aligned}
\tag{33}
$$

Since by the formulation (10) there holds for all $v_h \in V_h$,

$$
\langle f, P_\gamma(v_h)\rangle_h - (\nabla u_h, \nabla v_h)_\Omega + \left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+, P_\gamma(v_h)\right\rangle_h + \langle \gamma\Delta u_h, \Delta v_h\rangle_h = 0,
$$

we may subtract $i_h e$ in the right hand slot to get

$$
\begin{aligned}
&\frac{1}{2}\|\nabla e\|_\Omega^2 + \|\gamma^{-1/2}([\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+)\|_h^2 \\
&\qquad \leq \langle f, P_\gamma(e - i_h e)\rangle_h - (\nabla u_h, \nabla(e - i_h e))_\Omega \\
&\qquad\qquad + \left\langle \gamma^{-1}[\Psi - P_\gamma(u_h)]_+, P_\gamma(e - i_h e)\right\rangle_h + \langle \gamma\Delta u_h, \Delta(e - i_h e)\rangle_h
\end{aligned}
\tag{34}
$$

Proceeding by integration by parts in the second term of the right hand side of this expression we see that

$$
\begin{aligned}
&\frac{1}{2}\|\nabla e\|_\Omega^2 + \|\gamma^{-1/2}([\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+)\|_h^2 \\
&\qquad \leq \left\langle f + \Delta u_h + \gamma^{-1}[\Psi - P_\gamma(u_h)]_+, (I + \gamma\Delta)(e - i_h e)\right\rangle_h \\
&\qquad\qquad + \langle [\![\partial_n u_h]\!], e - i_h e\rangle_{\mathcal{F}}
\end{aligned}
\tag{35}
$$

Therefore by applying the inverse inequality $\gamma\|\Delta i_h e\|_h \lesssim \gamma h^{-1}\|\nabla i_h e\|_\Omega$, the H^1 stability of i_h and standard interpolation results for i_h we may write, with

$\mathfrak{h}(h,\gamma)|_T = (h_T + \gamma|_T h_T^{-1} + \gamma|_T^{\frac{1}{2}})$

$$
\begin{aligned}
&\frac{1}{2}\|\nabla e\|_\Omega^2 + \gamma^{-1}\|[\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+\|_h^2 \\
&\quad \le C(\|\mathfrak{h}(h,\gamma)(f + \Delta u_h - \gamma^{-1}[\Psi - P_\gamma(u_h)]_+)\|_h \\
&\qquad \times (\|\nabla e\|_\Omega + \|\gamma^{\frac{1}{2}}\Delta e\|_h) \\
&\qquad + C\|h^{\frac{1}{2}}[\![\partial_n u_h]\!]\|_{\mathcal{F}}\|\nabla e\|_\Omega.
\end{aligned}
$$

The factor $\|\gamma^{\frac{1}{2}}\Delta e\|_h$ in the right hand side is now bounded using the saturation assumption leading to

$$
\begin{aligned}
&\frac{1}{2}\|\nabla e\|_\Omega^2 + \|\gamma^{-1/2}([\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+)\|_h^2 \\
&\quad \le C(\|\mathfrak{h}(h,\gamma)(f + \Delta u_h - \gamma^{-1}[\Psi - P_\gamma(u_h)]_+)\|_h^2 + \|h^{\frac{1}{2}}[\![\partial_n u_h]\!]\|_{\mathcal{F}}^2)^{\frac{1}{2}} \\
&\qquad \times (1 + C_s^{-1})\|\nabla e\|_\Omega
\end{aligned}
$$

Applying the assumption on γ we obtain the bound

$$
\begin{aligned}
&\|\nabla e\|_\Omega + \|\gamma^{-1/2}([\Psi - P_\gamma(u)]_+ - [\Psi - P_\gamma(u_h)]_+)\|_h \\
&\quad \le C(\|h(f + \Delta u_h - \gamma^{-1}[\Psi - P_\gamma(u_h)]_+)\|_h^2 + C\|h^{\frac{1}{2}}[\![\partial_n u_h]\!]\|_{\mathcal{F}}^2)^{\frac{1}{2}}
\end{aligned}
$$

We conclude the proof using the Poincaré inequality (21) and by decomposing the integrals on the right hand side into element–wise contributions.

Proposition 7 *Assume that $u \in H_0^1(\Omega)$ with $\Delta u \in L^2(\Omega)$ is the solution of* (1) *and u_h the solution of* (10)*, then the following lower bound holds*

$$\eta_T \lesssim \|\nabla(u - u_h)\|_{\Delta_T} + h_T \inf_{v_h \in Z_h} \|f - v_h\|_{\Delta_T}$$

where $\Delta_T := \{T' \in \mathcal{T} : \exists F \subset \partial T' \text{ s.t. } T \cap T' = F\}$ and $Z_h := \{v_h \in L^2(\Omega); \forall T \in \mathcal{T}, v_h|_T \in P_k(\Omega)\}$.

PROOF. This bound is obtained using standard arguments following [21]. For the part in the interior of the element we must first observe that under the regularity assumptions we may use the use the equality

$$f = -\Delta u - [\Psi - P_\gamma(u)]_+.$$

Remark 8 *This a posteriori error estimate has the disadvantage of the saturation assumption and also that the parameter γ depends on the constant in the saturation assumption. However as we shall see below it appears to give a very good representation of the H^1-error and can be used to drive adaptive refinement.*

5 Numerical examples

We will study the performance of the method on two model problems, one with smooth exact solution and one with reduced regularity. In both cases we will consider the case of piecewise quadratic finnite elements, $k = 2$. The nonlinearity is handled by fixed point iterations, using the solution from the previous iteration as a test for the contact criteria. For the adaptivity we use an error equilibrating criterion so that, given an element error indicator e_K, we refine the element K if $e_K > \text{TOL}/\sqrt{\text{NELE}}$, where TOL is a given tolerance and NELE denotes the current number of elements in the mesh.

5.1 A smooth rotational symmetric exact solution

This example, from [15], is posed on the square $\Omega = (-1,1) \times (-1,1)$ with $\psi = 0$ and

$$f = \begin{cases} -8r_0^2(1-(r^2-r_0^2)) & \text{if } r \leq r_0, \\ -8(r^2+(r^2-r_0^2)) & \text{if } r > r_0, \end{cases}$$

where $r = \sqrt{x^2+y^2}$ and $r_0 = 1/4$, and with Dirichlet boundary conditions taken from the corresponding exact solution

$$u = [r^2 - r_0^2]_+^2.$$

We choose $\gamma = \gamma_0 h^2$ with $\gamma_0 = 1/100$ and show the convergence in the L_2– and H^1–norms in Figure 1 together with the error indicator (with unknown constant chosen so that the indicator lies close to the H^1 error for convenience). We remark that the smoothness of the solution precludes mesh zoning for this example, but that the indicator has the same asymptotic behaviour as the H^1 error. An elevation of the computed solution on one of the meshes in a sequence is given in Fig. 2. We note the optimal convergence of $O(h^3)$ in L_2 and $O(h^2)$ in H^1. Here we use $h = 1/\sqrt{\text{NNO}}$, where NNO denotes the number of nodes in a uniformly refined mesh.

5.2 A non-smooth exact solution

This example, from [3], is posed on the L-shaped domain $\Omega = (-2,2)\times(-2,2)\setminus [0,2) \times (-2,0]$ with $\psi = 0$ and

$$f(r,\varphi) = -r^{2/3}\sin(2\varphi/3)(\gamma'(r)/r + \gamma''(r)) - \frac{4}{3}r^{-1/3}\gamma'(r)\sin(2\varphi/3) - \gamma_2(r)$$

(note the sign error in [3]), where, with $\hat{r} = 2(r - 1/4)$,

$$\gamma_1(r) = \begin{cases} 1, & \hat{r} < 0 \\ -6\hat{r}^5 + 15\hat{r}^4 - 10\hat{r}^3 + 1, & 0 \leq \hat{r} < 1 \\ 0, & \hat{r} \geq 1, \end{cases}$$

$$\gamma_2(r) = \begin{cases} 0, \; r \leq 5/4, \\ 1 \quad \text{elsewhere.} \end{cases}$$

with Dirichlet boundary conditions taken from the corresponding exact solution

$$u(r, \varphi) = r^{2/3}\gamma_1(r)\sin(2\varphi/3)$$

which belongs to $H^{5/3-\varepsilon}(\Omega)$ for arbitrary $\varepsilon > 0$.

For this example we plot, in Fig. 3 the error on consecutive adaptively refined meshes, using the minimum meshsize as a measure of h. We note the suboptimal convergence and that the indicator still approximately follows the H^1 error asymptotically. In Fig. 4 we show the corresponding sequence of refined meshes, and in Fig. 5 we show an elevation of the approximate solution on the final mesh in the sequence.

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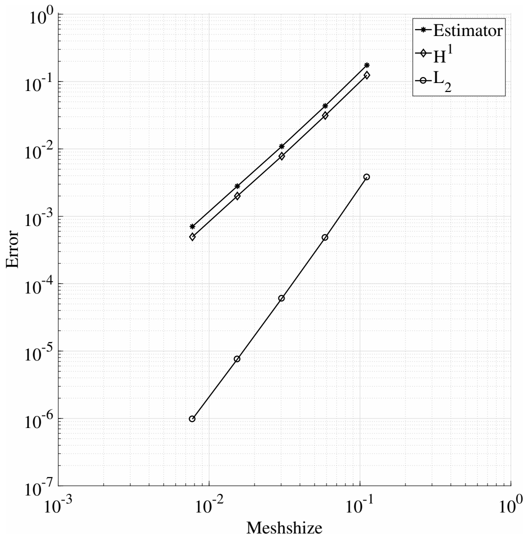

Figure 1. Convergence for the smooth case.

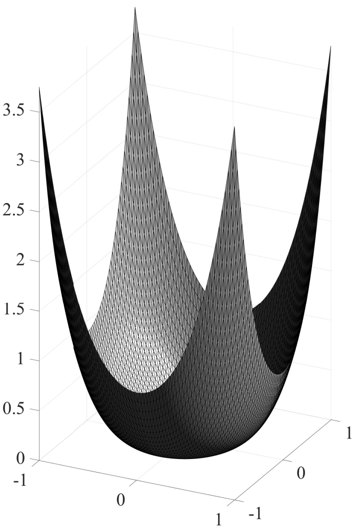

Figure 2. Elevation of the discrete solution, smooth case.

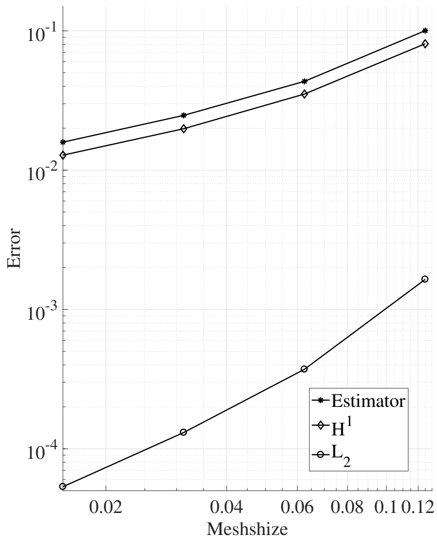

Figure 3. Convergence for the nonsmooth case.

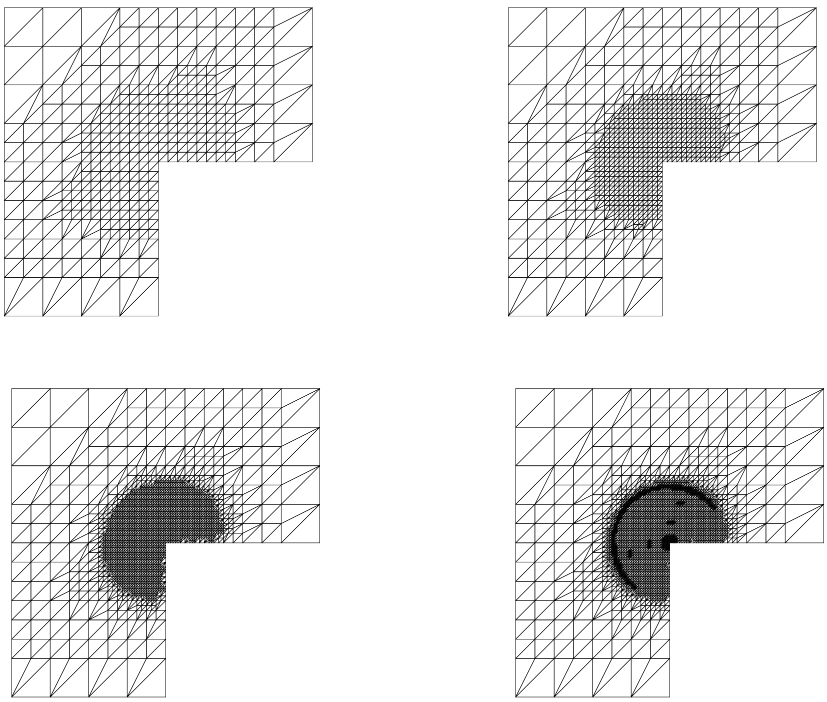

Figure 4. Sequence of refined meshes.

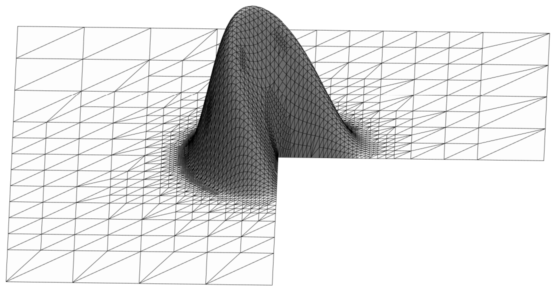

Figure 5. Elevation of the discrete solution, nonsmooth case.