INVERSE PROBLEMS FOR THE CONNECTION LAPLACIAN

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Abstract. We reconstruct a Riemannian manifold and a Hermitian vector bundle with compatible connection from the hyperbolic Dirichlet-to-Neumann operator associated with the wave equation of the connection Laplacian. The boundary data is local and the reconstruction is up to the natural gauge transformations of the problem. As a corollary we derive an elliptic analogue of the main result which solves a Calderón problem for connections on a cylinder.

1. Introduction

The purpose of the present paper is to show how to reconstruct a Riemannian metric and a Hermitian vector bundle with compatible connection from partial boundary measurements associated with the wave equation of the connection Laplacian (or rough Laplacian). The recovery is possible up to the natural gauges of the problem, and the proof uses techniques from the Boundary Control method [1].

There is considerable literature on the topic, and we shall review it in due course, but the strength of our results lies in the geometric generality involved: there are no restrictions on the Riemannian manifold, Hermitian vector bundle or connection. Our methods also include a transparent and direct proof in the case of the trivial vector bundle that avoids gluing of local reconstructions. The problem is motivated by the Aharonov-Bohm effect which asserts that different gauge equivalence classes of electromagnetic potentials have different physical effects that can be detected by experiments. The solution to the inverse problem presented in this paper shows in great generality that different gauge equivalence classes of Hermitian connections (e.g. Yang-Mills potentials) will have different boundary data and therefore are detectable by boundary measurements.

We proceed to state our results in more detail. Let (M, g) be a smooth, compact, connected Riemannian manifold of dimension m with non-empty boundary ∂M. Let $E \to M$ be a smooth Hermitian vector bundle of rank n, and let us denote by $\langle \cdot, \cdot \rangle_E$ the Hermitian inner product on each fiber. Let ∇ be a connection compatible with the Hermitian structure, that is, if we think of ∇ as operating on sections

$$\nabla : C^\infty(M; E) \to C^\infty(M; E \otimes T^*M)$$

then for any pair of sections u, v, we have

$$d\langle u, v \rangle_E = \langle \nabla u, v \rangle_E + \langle u, \nabla v \rangle_E.$$

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Using the Riemannian volume dx of (M,g) we can define a natural L^2-inner product of sections by setting

$$\langle u, v\rangle_{L^2(M;E)} = \int_M \langle u, v\rangle_E \, dx.$$

Similarly we get a natural L^2-inner product in $C^\infty(M; E \otimes T^*M)$. The elements in $C^\infty(M; E \otimes T^*M)$ can be thought of as 1-forms taking values in E. A pointwise product $\langle \alpha, \beta\rangle_E$ is a complex-valued 2-tensor on M which can be contracted with g to obtain a complex-valued function, and then integrated in M. In other words, if $\alpha = \alpha_i dx^i$ and $\beta = \beta_i dx^i$, then

$$\langle \alpha, \beta\rangle_{L^2(M;E\otimes T^*M)} = \int_M g^{ij} \langle \alpha_i, \beta_j\rangle_E \, dx.$$

We denote by ∇^* the adjoint of ∇ with respect to these L^2-inner products, and define the *connection Laplacian* as

$$P = \nabla^* \nabla.$$

We denote by $\mathrm{End}(E)$ the vector bundle whose fiber at $x \in M$ is the space of linear maps from the fiber E_x to itself, and say that a section $V \in C^\infty(M; \mathrm{End}(E))$ is a *potential* if it is symmetric in the sense that for any pair of sections u, v of E,

$$\langle u, Vv\rangle_E = \langle Vu, v\rangle_E.$$

Let V be a potential and consider the wave equation on sections,

$$\begin{aligned}
&(\partial_t^2 + P + V)u(t,x) = 0, && (0,\infty)\times M,\\
&u|_{(0,\infty)\times\partial M} = f, && (0,\infty)\times \partial M,\\
&u|_{t=0} = \partial_t u|_{t=0} = 0, && \text{in } M.
\end{aligned} \tag{1}$$

Let $T > 0$, let $\mathcal{S} \subset \partial M$ be open, and define the restricted Dirichlet-to-Neumann operator

$$\Lambda_{\mathcal{S}}^{2T} f = \nabla_\nu u|_{(0,2T)\times\mathcal{S}}, \quad f \in C_0^\infty((0,2T)\times\mathcal{S}; E),$$

where ν is the interior unit normal on ∂M and u is the solution of (1).

Our main result is that, for a sharp time $T > 0$, the Hermitian vector bundle $E|_{\mathcal{S}}$ and the restricted Dirichlet-to-Neumann operator $\Lambda_{\mathcal{S}}^{2T}$ determine the Riemannian manifold (M,g), the Hermitian vector bundle E, the connection ∇ and potential V. Here $E|_{\mathcal{S}}$ is the pullback bundle j^*E given by the inclusion map $j : \mathcal{S} \to M$.

Theorem 1.1. *Let $(M_i, g_i, E_i, \nabla_i, V_i)$, $i = 1,2$, be two smooth Hermitian vector bundles that are defined on smooth, compact and connected Riemannian manifolds with boundary, and that are equipped with smooth Hermitian connections and smooth potentials. Suppose that $T > 0$ and open $\mathcal{S}_i \subset \partial M_i$, $i = 1,2$, satisfy*

$$T > \max_{x\in M_i} d_{g_i}(x, \mathcal{S}_i), \quad i = 1,2,$$

where d_{g_i} is the distance function on (M_i, g_i). Suppose, furthermore, that there is a Hermitian vector bundle isomorphism $\phi : E_1|_{\mathcal{S}_1} \to E_2|_{\mathcal{S}_2}$ intertwining the Dirichlet-to-Neumann operators $\Lambda_{\mathcal{S}_1}^{2T}$ and $\Lambda_{\mathcal{S}_2}^{2T}$, that is, $\phi^*\Lambda_{\mathcal{S}_2}^{2T} = \Lambda_{\mathcal{S}_1}^{2T}\phi^*$. Then there is a Hermitian *vector bundle isomorphism* $\Phi : E_1 \to E_2$ *that covers an isometry between* (M_i, g_i), $i = 1, 2$, *and that satisfies* $\Phi^* \nabla_2 = \nabla_1$, $\Phi^* V_2 = V_1$ *and* $\Phi|_{E_1|_{\mathcal{S}_1}} = \phi$.

It is a simple exercise to check that if an isomorphism Φ as in Theorem 1.1 exists, then the restriction of Φ on $E_1|_{\mathcal{S}_1}$ intertwines the Dirichlet-to-Neumann operators. Hence Theorem 1.1 is optimal in terms of the gauge invariances.

We recall that a generalized Laplacian H on E is a differential operator such that its principal symbol is

$$|\xi|^2 = g^{ij}(x)\xi_i\xi_j, \quad (x, \xi) \in T^*M,$$

and we say that H is symmetric if

$$\langle u, Hv \rangle_{L^2(M;E)} = \langle Hu, v \rangle_{L^2(M;E)}, \quad u, v \in C_0^\infty(M; E).$$

A symmetric generalized Laplacian H on E can be written in the form $P + V$ for some Hermitian connection ∇ and potential V, see e.g. [3, Proposition 2.5], and wave equations for generalized Laplacians are the most general hyperbolic equations for which unique continuation is known to hold in the whole domain of influence, see Theorem 2.3 below. Such time sharp unique continuation, that goes back to the seminal paper [31], is crucial to our proof. Let us also point out that if the symmetry assumptions in Theorem 2.3 are weakened, then all the known uniqueness results in the scalar case require additional assumptions on the global geometry of (M, g), see [13, 23, 26].

As a corollary of Theorem 1.1, let us consider the case when (M, g) is known, E is the trivial bundle $M \times \mathbb{C}^n$ with its usual Hermitian inner product and $V = 0$. Then ∇ is of the form

$$d_A = d + A, \tag{2}$$

where $A = A_i dx^i$ and each $A_i(x)$, $x \in M$, is a skew-Hermitian $(n \times n)$-matrix. The Dirichlet-to-Neumann operator depends on A and we write $\Lambda^{2T}_{\partial M} = \Lambda^{2T}_{\partial M;A}$.

Corollary 1.2. *Let* d_A *and* d_B *be two Hermitian connections on the trivial bundle* $M \times \mathbb{C}^n$ *over a fixed Riemannian manifold* (M, g), *and suppose that* $\Lambda^T_{\partial M;A} = \Lambda^T_{\partial M;B}$ *for* $T > \max_{x\in M} d_g(x, \partial M)$. *Then, there exists a smooth* $U : M \to U(n)$ *such that* $U|_{\partial M} = Id$ *and*

$$B = U^{-1}dU + U^{-1}AU. \tag{3}$$

Note that if A and B satisfy (3), then $U^{-1}d_A U = d_B$ and hence $P_B = U^{-1}P_A U$, where $P_i = d_i^* d_i$, $i = A, B$. Thus if u solves the wave equation for P_B, then Uu solves it for P_A. Hence the above corollary can not be improved, that is, if $U : M \to U(n)$ satisfies $U|_{\partial M} = Id$ and (3) holds, then $\Lambda^T_{\partial M;A} = \Lambda^T_{\partial M;B}$ for any T. In the context of the gauges in Theorem 1.1, we have that ϕ is the identity and $\Phi(x, s) = (x, U(x)s)$, where $(x, s) \in M \times \mathbb{C}^n$.

The situation of the corollary is the one that appears in the literature. For the abelian case $n = 1$, the corollary in essentially proved in [22] via the Boundary

Control method. The Boundary Control method was pioneered for the isotropic wave equation on a domain in [1] and developed for manifolds in [2]. Note, however, that in [22] the boundary spectral data is used, and therefore the result does not give the sharp time T.

In [16], the corollary is proved under the further assumptions that M is a two dimensional domain, g is the Euclidean metric tensor and the connection is small in a suitable sense. The proof uses geometric optics solutions and reduces the problem to an injectivity result about the non-abelian Radon transform, which is of independent interest; see [10] for the case of the Euclidean metric and compactly supported connections. More recently, the injectivity result for the non-abelian Radon transform was extended to any simply connected surface with strictly convex boundary and no conjugate points [30] and to higher dimensions and negative curvature [17].

There is a result due to G. Eskin [11] that implies Corollary 1.2 under the assumption that M is a domain in Euclidean space with obstacles. Our proof seems however simpler. Eskin also proves a related theorem for the case of time-dependent Yang-Mills potentials in [14]. A survey on these results, including amended statements, is given in [12].

The proof of Corollary 1.2 follows directly from of our local reconstruction procedure, so the full power of Theorem 1.1 is not needed. As far as we are aware, there are no previous results for this problem when the bundle is not trivial; perhaps the closest in spirit is the result in [24] for the hyperbolic Dirac equation. However in this reference it is assumed that the data is given on the whole boundary for an infinite time interval, whereas our main result assumes only partial data and is sharp in terms of T. One of the main contributions of the present paper is to develop a new method to glue local reconstructions. The method allows us to reconstruct an isomorphic copy of the structure (g, E, ∇, V) on the interior of M given the data $\Lambda_{\mathcal{S}}^{2T}$ corresponding to a sharp time T.

As a final corollary, let us consider an elliptic analogue of Theorem 1.1. This application is very much in the spirit of [7, Theorem 1.5] where an elliptic scalar valued equation was considered.

Let (M_0, g_0) be a compact Riemannian manifold with boundary, and let $C = \mathbb{R} \times M_0$ be the infinite cylinder with the product metric $g = dt^2 + g_0$. Here dt^2 is the Euclidean metric on $\mathbb{R}$. We consider a Hermitian vector bundle $E_0 \to M_0$ with a Hermitian connection ∇_0, and define the operator $P_0 = \nabla_0^* \nabla_0$. Moreover, we have an induced Hermitian bundle E with connection ∇ on C, that is, $E = \pi^* E_0$ and $\nabla = \pi^* \nabla_0$, where $\pi : C \to M_0$ is the canonical projection.

Let us denote by $\lambda_1 \leq \lambda_2 \leq \dots$ the Dirichlet eigenvalues of the operator P_0. A point $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ is not in the continuous spectrum of the operator $\nabla^* \nabla = -\partial_t^2 + P_0$ and, for any $f \in C_0^\infty(\partial C; E)$, the equation

$$(-\partial_t^2 + P_0 - \lambda) u = 0 \text{ in } C, \qquad u|_{\partial C} = f,$$

has a unique bounded solution $u \in C^\infty(C; E)$. We define the elliptic Dirichlet-to-Neumann map

$$\Lambda(\lambda) f = \nabla_\nu u|_{\partial C}, \quad \Lambda(\lambda) : C_0^\infty(\partial C; E) \to C^\infty(\partial C; E).$$

Our application is the following recovery result:

Corollary 1.3. *The Hermitian vector bundle $E|_{\partial C}$ and the elliptic Dirichlet-to-Neumann map $\Lambda(\lambda)$ for a fixed $\lambda \in \mathbb{C}\backslash[\lambda_1, \infty)$ determine the structure $(M_0, g_0, E_0, \nabla_0)$.*

Here, the structure is determined up to the natural gauge invariances as in Theorem 1.1. It is possible to prove also a version of the corollary assuming that λ is in the continuous spectrum of $-\partial_t^2 + P_0$ as long as it avoids the eigenvalues λ_i. This extension can be carried out as in [7, Theorem 1.7] but we do not include it here.

This paper is organized as follows. Section 1 is the introduction and states the main results. In Section 2 we include preliminaries, mostly having to do with the direct problem, finite speed of propagation, unique continuation and approximate controllability. The results here are standard, but some details are provided to ensure the usual techniques fit our setting. Section 3 contains the local reconstruction procedure near the boundary. We first reconstruct the metric g and the core of the section is the reconstruction of the Hermitian bundle and the connection. The main local result is Theorem 3.10 and Corollary 1.2 is immediately derived from this theorem and well-known properties of the cut locus. Section 4 contains the global reconstruction procedure, expains in detail how to build up the structure from local data and finishes the proof of Theorem 1.1. In the final Section 5 we prove Corollary 1.3.

2. Preliminaries

2.1. **Local trivializations.** The connection ∇ is of the form (2) on a local trivialization of E. Let us derive local expressions for d_A^* and $P = d_A^* d_A$. To this end, we consider a section $u : M \to E$ and a E-valued 1-form $\beta = \beta_i dx^i$ supported on a local trivialization. As A is skew-hermitian,

$$\langle Au, \beta \rangle_{L^2(M;E\otimes T^*M)} = \int_M g^{ij} \langle A_i u, \beta_j \rangle_E \, dx = -\int_M \langle u, g^{ij} A_i \beta_j \rangle_E \, dx.$$

We define $(A, \beta) = g^{ij} A_i \beta_j$ and see that $d_A^* = d^* - (A, \cdot)$. Thus

$$Pu = d^* du + d^*(Au) - (A, du) - (A, Au).$$

We recall that for a 1-form α in local coordinates

$$d^*\alpha = -|g|^{-1/2} \frac{\partial}{\partial x^i} \left(|g|^{1/2} g^{ij} \alpha_j \right),$$

hence $d^*(Au) = (d^*A)u - (A, du)$, and

$$Pu = d^* du - 2(A, du) + (d^*A)u - (A, Au). \tag{4}$$

This exposes the nature of P: the principal part is the usual Laplacian and the first order term given by $-2(A, du)$.

When working near the boundary ∂M, it is convenient to use boundary normal coordinates, that is, semigeodesic coordinates adapted to the boundary. Let $\Gamma \subset \partial M$ be open. Then the semigeodesic coordinates adapted to Γ are given by the map

$$(s, y) \mapsto \gamma(s; y, \nu), \quad y \in \Gamma, \ s \in [0, \sigma_\Gamma(y)), \tag{5}$$

where the cut distance $\sigma_\Gamma : \Gamma \to (0,\infty)$ is defined by

$$\sigma_\Gamma(y) = \max\{s \in (0, \tau_M(y)];\ d_g(\gamma(s; y, \nu), \Gamma) = s\},$$
$$\tau_M(y) = \sup\{s \in (0,\infty);\ \gamma(s; y, \nu) \in M^{\text{int}}\}. \tag{6}$$

Here $\gamma(\cdot; x, \xi)$ is the geodesic with the initial data $(x, \xi) \in TM$. We recall that ν is the interior unit normal on ∂M, and define

$$M_\Gamma = \{\gamma(s; y, \nu);\ y \in \Gamma,\ s \in [0, \sigma_\Gamma(y))\}. \tag{7}$$

Then a point $x \in M_\Gamma$ is represented in the coordinates (5) by (s, y), where s is the distance $d_g(x, \Gamma)$ and y is the unique closest point to x in Γ. Moreover, g has the form $ds^2 + h_{jk}(s, y)dy^j dy^k$ and the principal part of P is

$$-\partial_s^2 - h^{jk}(s, y)\partial_{y^j}\partial_{y^k}. \tag{8}$$

2.2. **The direct problem.** Let us consider the initial-boundary value problem

$$(\partial_t^2 + P + V)u(t, x) = F, \quad (0, T) \times M,$$
$$u|_{(0,\infty)\times\partial M} = f, \quad (0, T) \times \partial M,$$
$$u|_{t=0} = \psi,\ \partial_t u|_{t=0} = \phi, \quad \text{in } M, \tag{9}$$

where $T > 0$. When $f = 0$ we have the energy estimate

$$\|u(t)\|_{H_0^1(M;E)} + \|\partial_t u(t)\|_{L^2(M;E)}$$
$$\le C(\|\psi\|_{H_0^1(M;E)} + \|\phi\|_{L^2(M;E)} + \|F\|_{L^2((0,t)\times M;E)}), \tag{10}$$

for all $t \in (0, T)$. For a proof in the scalar valued case, we refer to [15, Section 7.2]. The proof is analogous in the vector valued case and we omit it. We have also higher regularity results under suitable compatibility conditions. In what follows, we need only the following estimate

$$\|u\|_{H^m((0,T)\times M;E)} \le C(\|\phi\|_{H^{m-1}(M;E)} + \|F\|_{H^{m-1}((0,T)\times M;E)}), \tag{11}$$

where $m \ge 1$, f and ψ vanish, F is compactly supported in the time interval $(0, T)$ (but not necessarily in space), and ϕ is compactly supported in M^{int}, see e.g. [15]. We can extend a function $f \in C_0^\infty((0,\infty) \times \partial M; E)$ as a smooth function on the whole domain $(0,\infty) \times M$ and substract it from u. By using (11) we see that the solution of (1) is smooth for such f.

We need a sharp regularity result for the Neumann trace. The result is due to Lasiecka, Lions and Triggiani in the scalar valued case [25]. The proof in the present setting is analogous but we give it for the convenience of the reader. We will use the following identity

$$\langle \nabla^* u, v\rangle_{L^2(M;E)} - \langle u, \nabla v\rangle_{L^2(M;E\otimes T^*M)}$$
$$= \langle d^* u, v\rangle_{L^2(M;E)} - \langle u, dv\rangle_{L^2(M;E\otimes T^*M)} = \int_{\partial M} \langle i_\nu u, v\rangle_E \, dS, \tag{12}$$

where $u \in C^\infty(M; E \otimes T^*M)$, $v \in C^\infty(M; E)$, and dS is the Riemannian volume of $(\partial M, g)$. This follows from [32, Prop. 2.9.1] since the principal symbol of ∇ coincides with the principal symbol of d.

Theorem 2.1. *Suppose that F, f and ψ vanish and let $\phi \in L^2(M; E)$. Then the solution u of (9) satisfies $\nabla_\nu u \in L^2((0,T) \times \partial M; E)$.*

Proof. We will first suppose that $\phi \in C_0^\infty(M; E)$. Then u is smooth by (11). We extend ν as a smooth vector field on the whole domain M, and denote this extension still by ν. We have

$$\begin{aligned} &\langle Pu, \nabla_\nu u \rangle_{L^2((0,T)\times M;E)} \\ &\quad = \langle \nabla u, \nabla \nabla_\nu u \rangle_{L^2((0,T)\times M;E\otimes T^*M)} + \int_0^T \int_{\partial M} |\nabla_\nu u|_E^2 \, dS. \end{aligned}$$

Here $|u|_E^2 = \langle u, u \rangle_E$. In local coordinates, the principal part of both

$$\langle \nabla_j u, \nabla_k \nabla_\nu u \rangle_E \, g^{jk} \quad \text{and} \quad \frac{1}{2} \nu(\langle \nabla_j u, \nabla_k u \rangle_E \, g^{jk})$$

is $\langle \partial_j u, \nu^p \partial_p \partial_k u \rangle_E \, g^{jk}$. Thus

$$\langle \nabla u, \nabla \nabla_\nu u \rangle_{L^2((0,T)\times M;E\otimes T^*M)} = \frac{1}{2} \int_0^T \int_M \nu(\langle \nabla_j u, \nabla_k u \rangle_E \, g^{jk}) dx + R,$$

where the remainder term R satisfies $|R| \le C \|u\|_{H^1((0,T)\times M;E)}^2$. Moreover,

$$\begin{aligned} \int_0^T \int_M \nu(\langle \nabla_j u, \nabla_k u \rangle_E \, g^{jk}) dx = &- \int_0^T \int_M (\operatorname{div} \nu) \langle \nabla_j u, \nabla_k u \rangle_E \, g^{jk} dx \\ &- \int_0^T \int_{\partial M} \langle \nabla_j u, \nabla_k u \rangle_E \, g^{jk} dS. \end{aligned}$$

As u vanishes on the boundary, we have in the boundary normal coordinates $(s, y) \in [0, \epsilon) \times \partial M$ that

$$\langle \nabla_j u, \nabla_k u \rangle_E \, g^{jk} = |\partial_s u|_E^2 = |\nabla_\nu u|_E^2.$$

Hence

$$\langle Pu, \nabla_\nu u \rangle_{L^2((0,T)\times M;E)} = \frac{1}{2} \|\nabla_\nu u\|_{L^2((0,T)\times \partial M;E)}^2 + R, \tag{13}$$

where the remainder term R satisfies $|R| \le C \|u\|_{H^1((0,T)\times M;E)}^2$.

Analogously

$$\begin{aligned} &\langle \partial_t^2 u, \nabla_\nu u \rangle_{L^2((0,T)\times M;E)} \\ &\quad = -\frac{1}{2} \int_0^T \int_M \nu \, \langle \partial_t u, \partial_t u \rangle_E \, dx + \left[\int_M \langle \partial_t u, \nabla_\nu u \rangle_E \, dx \right]_{t=0}^{t=T}, \end{aligned}$$

and

$$\int_0^T \int_M \nu \left\langle \partial_t u, \partial_t u \right\rangle_E dx = -\int_0^T \int_M (\operatorname{div} \nu) \left\langle \partial_t u, \partial_t u \right\rangle_E dx - \int_0^T \int_{\partial M} \left\langle \partial_t u, \partial_t u \right\rangle_E dx,$$

where the second term on the right-hand side is zero since $u = 0$ on ∂M. Hence

$$|\left\langle \partial_t^2 u, \nabla_\nu u \right\rangle_{L^2((0,T)\times M;E)}| \leq C \|u\|^2_{H^1((0,T)\times M;E))} + C \max_{t=0,T} (\|u(t)\|^2_{H^1_0(M;E)} + \|\partial_t u(t)\|^2_{L^2(M;E)}). \tag{14}$$

Clearly

$$|\left\langle Vu, \nabla_\nu u \right\rangle_{L^2((0,T)\times M;E)}| \leq C \|u\|^2_{H^1((0,T)\times M;E)}. \tag{15}$$

Combining (13)-(15) with the energy estimate (10), we get

$$\|\nabla_\nu u\|^2_{L^2((0,T)\times \partial M;E)} \leq C \|\phi\|^2_{L^2(M;E)}.$$

The claim follows since $C_0^\infty(M;E)$ is dense in $L^2(M;E)$. □

The estimate (11) implies, via a duality argument, that (9) has a unique solution $u \in H^{-m+1}((0,T)\times M;E)$ when f, ψ and ϕ vanish and $F \in H^{-m}((0,T)\times M;E)$ is compactly supported in $(0,T)\times M^{\text{int}}$. In the estimate (11), the norm on the left-hand side can be replaced with the norm of $C([0,T];H^m(M;E))$. From this it follows that (9) has a unique solution $u \in H^{-m+1}((0,T)\times M;E)$ when F, f and ψ vanish and $\phi \in H^{-m}(M;E)$ is compactly supported in M^{int}.

Suppose that $u \in H^{-m+1}((0,T)\times M;E)$ and that F vanishes near $(0,T)\times\partial M$. As the principal part of $P+V$ is of the form (8) in the boundary normal coordinates $(s,y) \in [0,\epsilon)\times\partial M$, we may repeat the proof of [19, Th. 4.3.1] without any changes in the present setting. By combining this with [19, Th. 2.5.6], we see that there is m' such that that the maps $s \mapsto u(s)$ and $s \mapsto \nabla_\nu u(s)$ are continuous with values in $H^{-m'}((0,T)\times\partial M;E)$. In particular, the traces $u|_{(0,T)\times\partial M}$ and $\nabla_\nu u|_{(0,T)\times\partial M}$ of such a solution are well defined.

2.3. **Finite speed of propagation, unique continuation and approximate controllability.**

The equation (1) has the following finite speed of propagation property:

Theorem 2.2. *Let $T > 0$, $U \subset M$ be open and define the cone*

$$\mathcal{C} = \{(t,x) \in (0,T)\times M;\ d_g(x,U) < T-t\}.$$

Suppose that $f \in C_0^\infty((0,T)\times\partial M;E)$ vanishes in the intersection

$$\mathcal{C} \cap ((0,T)\times\partial M).$$

Then the solution u of (1) vanishes in $\mathcal{C}$. In particular, if $\Gamma \subset \partial M$ is open, $r \in (0,T)$, and $\operatorname{supp}(f) \subset (T-r,T)\times\Gamma$, then $\operatorname{supp}(u(T))$ is contained in the domain of influence

$$M(\Gamma,r) = \{x \in M;\ d_g(x,\Gamma) \leq r\}.$$

We refer to [20, Lemma 4.1] for a proof in the scalar valued case. The proof in the present setting is analogous and we omit it.

The operator $P+V$ is of principally scalar form, and the local unique continuation result [8] can be applied. The local result implies the following result due to Eller and Toundykov [9] that is analogous to the semi-global Holmgren theorem.

Theorem 2.3. *Let $T > 0$ and let $\Gamma \subset \partial M$ be open. Let $s \in \mathbb{R}$, and suppose that $u \in H^s((0,2T)\times M;E)$ satisfies $(\partial_t^2 + P + V)u = 0$ and*

$$u|_{(0,2T)\times\Gamma} = 0, \quad \nabla_\nu u|_{(0,2T)\times\Gamma} = 0.$$

Then $u(T,x) = 0$ whenever $x \in M(\Gamma,T)^{\mathrm{int}}$.

Let us denote $Wf = u(T)$, where u is the solution of (1). The formal adjoint of W is $\phi \mapsto \nabla_\nu v|_{(0,T)\times\partial M}$, where v is the solution of

$$\begin{aligned} &(\partial_t^2 + P + V)v(t,x) = 0, \quad (0,T)\times M, \\ &v|_{(0,\infty)\times\partial M} = 0, \quad (0,T)\times\partial M, \\ &v|_{t=T} = 0,\ \partial_t v|_{t=T} = -\phi, \quad \text{in } M. \end{aligned} \tag{16}$$

Indeed,

$$\begin{aligned} 0 &= \left\langle(\partial_t^2 + P + V)u, v\right\rangle_{L^2((0,T)\times M;E)} - \left\langle u, (\partial_t^2 + P + V)v\right\rangle_{L^2((0,T)\times M;E)} \\ &= \Big[\langle\partial_t u, v\rangle_{L^2(M;E)} - \langle u, \partial_t v\rangle_{L^2(M;E)}\Big]_{t=0}^{t=T} \\ &\quad + \langle\nabla_\nu u, v\rangle_{L^2((0,T)\times\partial M;E)} - \langle u, \nabla_\nu v\rangle_{L^2((0,T)\times\partial M;E)} \\ &= \langle u(T), \phi\rangle_{L^2(M;E)} - \langle f, \nabla_\nu v\rangle_{L^2((0,T)\times\partial M;E)}\,. \end{aligned} \tag{17}$$

For an open set $S \subset M$, we consider $L^2(S;E)$ as the subspace of $L^2(M;E)$ consisting of the sections that vanish outside the set S. If $\Gamma \subset \partial M$ is open and nonempty and $r > 0$, then Theorem 2.3 implies that the map

$$\phi \mapsto \nabla_\nu v|_{(0,r)\times\Gamma} : L^2(M(\Gamma,r);E) \to L^2((0,r)\times\Gamma;E)$$

is injective. A duality argument implies that the wave equation (1) is approximately controllable in the sense of the lemma below. Note that to carry out the duality argument, we need the L^2-regularity of the Neumann trace, see Theorem 2.1.

Lemma 2.4. *Let $\Gamma \subset \partial M$ be open and $r > 0$. Then*

$$\{Wf;\ f \in C_0^\infty((T-r,T)\times\Gamma;E)\} \tag{18}$$

is dense in $L^2(M(\Gamma,r);E)$.

For a proof in the scalar valued case we refer to [21]. The proof in the present setting is analogous and we omit it.

We need also a refined version of approximate controllability. In order to formulate this, let us define the modified domain of influence as follows. Let $\Gamma \subset \partial M$ and $h : \Gamma \to \mathbb{R}$. we define

$$M(\Gamma,h) = \{x \in M;\ \inf_{y\in\Gamma}(d_g(x,y) - h(y)) \le 0\},$$

and denote for $T > 0$

$$\mathcal{B}(\Gamma, h; T) = \{(t, y) \in (0, T) \times \Gamma;\ T - h(y) < t\}.$$

If $r > 0$ and $h(y) = r$, $y \in \Gamma$, then $M(\Gamma, h)$ coincides with our earlier definition of $M(\Gamma, r)$. We denote by 1_S the indicator function of a set $S \subset M$, that is, $1_S(x) = 1$ if $x \in S$ and $1_S(x) = 0$ otherwise.

For the convenience of the reader, we give a proof of the following lemma. An analogous lemma is stated in [26] without a proof.

Lemma 2.5. *Let $T > 0$ and suppose that $\Gamma \subset \partial M$ is open. Let $L \in \mathbb{N}$, let $\Gamma_\ell \subset \Gamma$ be open and let $h_\ell \in C(\bar{\Gamma}_\ell)$, $\ell = 1, \dots, L$. We define*

$$h = \sum_{\ell=1}^{L} h_\ell 1_{\Gamma_\ell}, \tag{19}$$

and suppose that $h \le T$ pointwise. Then

$$\{Wf;\ f \in C_0^\infty(\mathcal{B}(\Gamma, h; T); E)\} \tag{20}$$

is dense in $L^2(M(\Gamma, h); E)$.

Proof. Let $\epsilon > 0$. There is a simple function

$$h_\epsilon(y) = \sum_{j=1}^{J} T_j 1_{\Gamma_j}(y),$$

where $J \in \mathbb{N}$, $T_j \in (0, T)$ and $\Gamma_j \subset \Gamma$ are open and disjoint, such that $h < h_\epsilon + \epsilon$ almost everywhere on Γ and $h_\epsilon < h$ on $\bar{\Gamma}$, see e.g. [27, Lemma 4.2].

We show by induction on J that the density holds when $h = h_\epsilon$. The base case $J = 1$ follows from Lemma 2.4. We define $\tilde{h}_\epsilon = h_\epsilon - T_J 1_{\Gamma_J}$, and use the shorthand notation $M_0 = M(\Gamma, \tilde{h}_\epsilon)$ and $M_1 = M(\Gamma_J, T_J)$. Let $\psi \in L^2(M(\Gamma, h_\epsilon); E)$. Note that $M(\Gamma, h_\epsilon) = M_0 \cup M_1$. By the induction hypothesis there is a sequence of smooth functions $(f_k^0)_{k=1}^\infty$ supported in $\mathcal{B}(\Gamma, \tilde{h}_\epsilon; T))$ such that

$$Wf_k^0 \to 1_{M_0}\psi, \quad k \to \infty.$$

Moreover, by Lemma 2.4 there is a sequence of smooth functions $(f_k^1)_{k=1}^\infty$ supported in $\mathcal{B}(\Gamma_J, T_J; T))$ such that

$$Wf_k^1 \to 1_{M_1}(\psi - 1_{M_0}\psi), \quad k \to \infty.$$

Thus $W(f_k^0 + f_k^1) \to \psi$. This proves that the density holds for h_ϵ.

Suppose now that $\psi \in L^2(M(\Gamma, h); E)$. We have shown that there is a smooth function f supported in $\mathcal{B}(\Gamma, h_\epsilon; T)$ such that

$$\left\| 1_{M(\Gamma, h_\epsilon)}\psi - Wf \right\|_{L^2(M;E)}^2 < \epsilon.$$

Thus

$$\|\psi - Wf\|_{L^2(M;E)}^2 < \epsilon + \left(\int_{M(\Gamma,h)} |\psi|_E^2 dx - \int_{M(\Gamma,h_\epsilon)} |\psi|_E^2 dx \right).$$

The Riemannian volumes converge $|M(\Gamma, h_\epsilon)| \to |M(\Gamma, h)|$ as $\epsilon \to 0$, see [27, Lemma 4.3]. Thus the claimed density holds. $\square$

3. Local reconstruction near the boundary

3.1. **Inner products.** We begin by generalizing an integration by parts technique due to Blagovestchenskii in the $1+1$ dimensional scalar case [4]. For a multidimensional scalar case this was first used by Belishev [1]. We recall the notation $Wf = u(T)$, where u is the solution of (1).

Lemma 3.1. *Let $T > 0$, let $\mathcal{S} \subset \partial M$ be open, and let f and h be functions in $C_0^\infty((0,2T) \times \mathcal{S}; E)$. Then*

$$\langle Wf, Wh \rangle_{L^2(M;E)} = \left\langle f, J\Lambda_{\mathcal{S}}^{2T} h \right\rangle_{L^2((0,2T)\times\mathcal{S};E)} - \left\langle f, (\Lambda_{\mathcal{S}}^{2T})^* J h \right\rangle_{L^2((0,2T)\times\mathcal{S};E)},$$

where J is the integral operator in the time variable with the kernel $\operatorname{sgn}(t-s) 1_L(t,s)/4$. *Here $L = \{(s,t) \in \mathbb{R}^2 :\ 0 \le t+s \le 2T,\ t,s > 0\}$.*

Proof. We write $u^f = u$ for the solution of (1) and define $w(t,s) = \langle u^f(t), u^h(s)\rangle_{L^2(M;E)}$. We have

$$\begin{aligned}
(\partial_t^2 - \partial_s^2) w(t,s) &= \langle \partial_t^2 u^f(t), u^h(s)\rangle_{L^2(M;E)} - \langle u^f(t), \partial_s^2 u^h(s)\rangle_{L^2(M;E)} \\
&= -\langle \nabla^*\nabla u^f(t), u^h(s)\rangle_{L^2(M;E)} + \langle u^f(t), \nabla^*\nabla u^h(s)\rangle_{L^2(M;E)} \\
&= -\int_{\partial M} \langle \nabla_\nu u^f(t), u^h(s)\rangle_E \, dS + \int_{\partial M} \langle u^f(t), \nabla_\nu u^h(s)\rangle_E \, dS \\
&= \int_{\partial M} \langle f(t), \Lambda_{\mathcal{S}}^{2T} h(s)\rangle_E \, dS - \int_{\partial M} \langle \Lambda_{\mathcal{S}}^{2T} f(t), h(s)\rangle_E \, dS.
\end{aligned}$$

Since $w(0,s) = w(t,0) = \partial_t w(0,s) = \partial_s w(0,s) = 0$ and w solves the above $1+1$ dimensional wave equation, the result follows by considering $w(T,T)$. $\square$

Corollary 3.2. *Let $T > 0$, $\mathcal{S} \subset \partial M$ be open. Then $\Lambda_{\mathcal{S}}^{2T}$ determines the inner products*

$$\langle Wf, Wh \rangle_{L^2(M;E)}, \quad f, h \in C_0^\infty((0,2T) \times \mathcal{S}; E). \tag{21}$$

Moreover, $\Lambda_{\mathcal{S}}^{2T}$ determines, for all $(f_j)_{j=1}^\infty \subset C_0^\infty((0,2T)\times\mathcal{S};E)$, if the sequence $(Wf_j)_{j=1}^\infty$ converges, in the strong or weak sense, in $L^2(M;E)$.

Proof. We allow the metric tensor g to be *a priori* unknown on $\mathcal{S}$. However, $\Lambda_{\mathcal{S}}^{2T}$ determines the distances $d_g(x,y)$, $x, y \in \mathcal{S}$, see e.g. [6, Section 2.2], and these distances determine g on $\mathcal{S}$. Thus we can assume without loss of generality that the Riemannian volume measure dS of $(\mathcal{S}, g)$ is known, and Lemma 3.1 implies that $\Lambda_{\mathcal{S}}^{2T}$ determines the inner products (21).

For the second claim, we observe that the inner products (21) can be used to determine if $(Wf_j)_{j=1}^\infty$ is a Cauchy sequence in $L^2(M;E)$. This allows us to determine if $(Wf_j)_{j=1}^\infty$ converges in the strong sense. Moreover, using again (21) we can determine if $(Wf_j)_{j=1}^\infty$ is bounded in $L^2(M;E)$, and we may test the weak convergence analogously to [26, Lemma 3]. $\square$

3.2. **Reconstruction of the metric tensor.** Our reconstruction of the metric tensor is based on the proof in [26]. The following lemma is a variation of [26, Lemma 6]. We give a short proof for the convenience of the reader.

Lemma 3.3. *Let $T > 0$, $s \in (0, T]$, let $\Sigma, \Gamma \subset \partial M$ be open and let $h : \Gamma \to [0, T]$. Suppose that (20) is dense in $L^2(M(\Gamma, h); E)$. Then the following are equivalent:*

(i) $M(\Sigma, s) \subset M(\Gamma, h)$.
(ii) *For all $f_0 \in C_0^\infty(\mathcal{B}(\Sigma, s; T); E)$ there is a sequence $(f_j)_{j=1}^\infty$ in $C_0^\infty(\mathcal{B}(\Gamma, h; T); E)$ such that $(W(f_0 - f_j))_{j=1}^\infty$ tends to zero in $L^2(M; E)$.*

Proof. The implication from (i) to (ii) follows from the assumption that (20) is dense in $L^2(M(\Gamma, h); E)$. We will now show that (ii) implies (i). We denote

$$M_0 = M(\Sigma, s), \quad M_1 = M(\Gamma, h),$$
$$S_0 = \mathcal{B}(\Sigma, s; T), \quad S_1 = \mathcal{B}(\Gamma, h; T).$$

Let us assume that (i) does not hold. There is a nonempty open set $U \subset M_0$ such that $U \cap M_1 = \emptyset$, see [26, Lemma 6]. By Lemma 2.4 there is a smooth function f_0 supported in S_0 such that $\int_U W f_0 dx \neq 0$. However, by finite speed of propagation $W f|_U = 0$ for any f supported in S_1. Thus

$$\langle W(f_0 - f), 1_U \rangle_{L^2(M;E)} = \langle W f_0, 1_U \rangle_{L^2(M;E)} \neq 0,$$

for all f supported in S_1 and (ii) does not hold. □

Let $\Gamma \subset \partial M$ be open and let $T > 0$. We recall that the cut distance σ_Γ is defined by (6), and define

$$\sigma_\Gamma^T(y) = \min(\sigma_\Gamma(y), T), \quad y \in \Gamma, \tag{22}$$
$$M_\Gamma^T = \{\gamma(s; y, \nu);\ y \in \Gamma,\ s \in [0, \sigma_\Gamma^T(y))\}.$$

Theorem 3.4. *Let $T > 0$ and let $\Gamma \subset \partial M$ be open. Then the Riemannian manifold (Γ, g), the Hermitian vector bundle $E|_\Gamma$ and Λ_Γ^{2T} determine (M_Γ^T, g).*

Proof. By combining Corollary 3.2 and Lemmas 2.5 and 3.3 we can determine the relation

$$\{(\Sigma, s, h);\ M(\Sigma, s) \subset M(\Gamma, h)\}$$

for any open $\Sigma \subset \Gamma$, $s \in (0, T]$ and a function h of form (19). This relation determines σ_Γ^T and the Riemannian manifold (M_Γ^T, g) by using the purely geometric method of [26]. □

3.3. **Reconstruction of the connection.** Our reconstruction method is based on a use of sequences of sources $(f_j)_{j=1}^\infty$ such that $\mathrm{supp}(W f_j)$ converges to a point.

Lemma 3.5. *Let $\Gamma_1, \Gamma_2 \subset \partial M$ be open and $r_1, r_2 > 0$. Suppose that for a sequence $(f_j)_{j=1}^\infty \subset C_0^\infty((T - r_1, T) \times \Gamma_1; E)$ the sequence $(W f_j)_{j=1}^\infty$ converges weakly to a function $\phi \in L^2(M; E)$, and that*

$$\langle W f_j, W h \rangle_{L^2(M;E)} \to 0, \quad h \in C_0^\infty((T - r_2, T) \times \Gamma_2; E).$$

Then $\mathrm{supp}(\phi) \subset M(\Gamma_1, r_1) \setminus M(\Gamma_2, r_2)^{\mathrm{int}}$.

Proof. The lemma follows immediately from the density of the set (18). □

Lemma 3.6. *Let $T > 0$, $\Gamma \subset \partial M$ be open, and let $x \in \Gamma \cup M^{\text{int}}$ satisfy $d_g(x, \Gamma) < T$. Then there are functions $h_\ell \in C_0^\infty((0, 2T) \times \Gamma; E)$ such that $Wh_\ell(x)$, $\ell = 1, \dots, n$, form an orthonormal basis of the fiber E_x of E at x.*

Proof. If $x \in \Gamma$, then $Wh(x) = h(T, x)$ and the claim clearly holds in this case. Suppose now that $x \in M^{\text{int}}$. It is enough to show that the fiber E_x is spanned by the vectors

$$Wh(x), \quad h \in C_0^\infty((0,T) \times \Gamma; E).$$

In order to show this it is enough to show that if $e \in E_x$ and

$$\langle e, Wh(x) \rangle_E = 0, \quad h \in C_0^\infty((0,T) \times \Gamma; E), \tag{23}$$

then $e = 0$.

We recall that the adjoint of W is given by $\phi \mapsto \nabla_\nu v|_{(0,T)\times\partial M}$ where v is the solution of (16). We choose $\phi = e\delta_x$. Then the traces $v|_{(0,T)\times\Gamma}$ and $\nabla_\nu v|_{(0,T)\times\Gamma}$ are well defined as explained at the end of Section 2.2. Moreover, the former trace vanishes by the boundary condition in (16) and the latter by (23) and (17). We extend v on the time interval $(0, 2T)$ by the odd reflection with respect to $t = T$, and denote the extension still by v. The extension satisfies $(\partial_t^2 + P + V)v = 0$ on $(0, 2T) \times M$. Theorem 2.3 implies that $e = 0$. □

Lemma 3.7. *Let $\Gamma \subset \partial M$ be open, let $T > 0$, and let $e : M \to E$ be a section of E. Let $U \subset M^{\text{int}} \cup \Gamma$ be open in M and suppose that $U \subset M(\Gamma, T)$. Suppose, furthermore, that $x \mapsto \langle e(x), Wh(x) \rangle_E$ is smooth on U for all $h \in C_0^\infty((0, 2T) \times \Gamma; E)$. Then e is smooth on U.*

Proof. Let $x \in U$, and let us choose h_ℓ, $\ell = 1, \dots, n$, as in Lemma 3.6. Then the functions Wh_ℓ form a smooth frame near x, and the representation of e in this frame is smooth. □

We recall that $|X|$ denotes the Riemannian volume of a measurable set $X \subset M$, and that the set M_Γ is defined by (7).

Lemma 3.8. *Let $\Gamma \subset \partial M$ be open. Let $x = \gamma(s; y, \nu) \in M_\Gamma$, and define $s_k = s + 1/k$,*

$$Y_k = \{\tilde{y} \in \Gamma;\ d_g(\tilde{y}, y) < 1/k\}, \quad X_k = M(Y_k, s_k) \setminus M(\Gamma, s).$$

Suppose that a double sequence $\Phi = (f_{jk})_{j,k=1}^\infty$ of functions in $C_0^\infty((T - s_k, T) \times Y_k; E)$ satisfies the following

(i) *For each $k = 1, 2, \dots,$ the sequence $(Wf_{jk})_{j=1}^\infty$ converges weakly in $L^2(M; E)$ to a function supported in X_k.*

(ii) *There is $C > 0$ such that*

$$\|Wf_{jk}\|_{L^2(M;E)} \le C|X_k|^{-1/2}, \quad j, k = 1, 2, \dots.$$

(iii) *The limit $\lim_{k\to\infty} \lim_{j\to\infty} \langle Wf_{jk}, Wh \rangle_{L^2(M;E)}$ exists for any function h in the space $C_0^\infty((0, 2T) \times \Gamma; E)$.*

Then there is a vector $e(x;\Phi) \in E_x$ that depends on x and Φ such that

$$\lim_{k\to\infty}\lim_{j\to\infty} \langle Wf_{jk}, \phi\rangle_{L^2(M;E)} = \langle e(x;\Phi), \phi(x)\rangle_E, \quad \phi \in C^\infty(M;E). \tag{24}$$

Note that we allow here the case $x \in \Gamma$, i.e. $s = 0$.

Proof. By Lemma 3.6 there are h_ℓ such that $Wh_\ell(x)$, $\ell = 1,\dots,n$, form an orthonormal basis of E_x. Let write $b_\ell = Wh_\ell$ and denote weak limit of $(Wf_{jk})_{j=1}^\infty$ by u_k. We choose local coordinates $\tilde{x}$ in a neighborhood $U \subset M$ of x, and suppose that k is large enough so that $X_k \subset U$ and that the sections $b_\ell(\tilde{x})$ form a basis in $E_{\tilde{x}}$ for all $\tilde{x} \in X_k$. Let $\phi \in C^\infty(M;E)$ and write $\phi(\tilde{x}) = c^\ell b_\ell(\tilde{x}) + (x^p - \tilde{x}^p)\psi_p(\tilde{x})$, where $c^\ell \in \mathbb{C}$ and $\psi_p \in C^\infty(U;E)$, $p = 1,\dots,m$. Then

$$\langle u_k, \phi\rangle_{L^2(M;E)} = \overline{c^\ell}\,\langle u_k, b_\ell\rangle_{L^2(M;E)} + R_k, \tag{25}$$

where the remainder term satisfies

$$\begin{aligned}|R_k| &\le m \max_{p=1,\dots,m} \|\psi_p\|_{C(U)} \operatorname{diam}(X_k) \int_{X_k} |u_k(\tilde{x})|_E d\tilde{x} \\ &\le m \max_{p=1,\dots,m} \|\psi_p\|_{C(U)} \operatorname{diam}(X_k)\, \|u_k\|_{L^2(M;E)} |X_k|^{1/2}.\end{aligned}$$

Note that $\operatorname{diam}(X_k) \to 0$ since $X_k \supset X_{k+1}$ and $X_k \to x$ as $k \to \infty$. Thus (ii) implies that $R_k \to 0$. By (iii) the limits

$$a^\ell = \lim_{k\to\infty} \langle u_k, b_\ell\rangle_{L^2(M;E)}, \quad \ell = 1,\dots,n,$$

exist. We set $e = a^\ell b_\ell(x)$. Then

$$\lim_{k\to\infty} \langle u_k, \phi\rangle_{L^2(M;E)} = \overline{c^\ell} \lim_{k\to\infty} \langle u_k, b_\ell\rangle_{L^2(M;E)} = \sum_{\ell=1}^n a^\ell \overline{c^\ell} = \langle e, \phi(x)\rangle_E.$$

□

Lemma 3.9. *Let $\Gamma \subset \partial M$ be open. Let $x \in M_\Gamma$ and let $e \in E_x$. Then there is a double sequence $\Phi = (f_{jk})_{j,k=1}^\infty$ that satisfy the conditions of Lemma 3.8, and furthermore, $e(x;\Phi) = e$ where $e(x;\Phi)$ is as in (24).*

Proof. Let $\tilde{e} \in C^\infty(M;E)$ satisfy $\tilde{e}(x) = e$. By Lemma 2.4 there is a double sequence $\Phi = (f_{jk})_{j,k=1}^\infty$ of functions in $C_0^\infty((T - s_k, T) \times Y_k; E)$ such that $(Wf_{jk})_{j=1}^\infty$ converges to the function $u_k = |X_k|^{-1} 1_{X_k} \tilde{e}$. We recall that 1_{X_k} is the indicator function of the set X_k and $|X_k|$ is its volume. Moreover, u_k satisfies $\|u_k\|_{L^2(M;E)} \le |X_k|^{-1/2} \|\tilde{e}\|_{L^\infty(M;E)}$ and, for a function $\phi \in C^\infty(M;E)$,

$$\langle u_k, \phi\rangle_{L^2(M;E)} = \frac{1}{|X_k|} \int_{X_k} \langle \tilde{e}(\tilde{x}), \phi(\tilde{x})\rangle_E\, d\tilde{x} \to \langle e, \phi(x)\rangle_E,$$

where $\tilde{x}$ are local coordinates on X_k. □

Theorem 3.10. *Let $T > 0$, let $\Gamma \subset \partial M$ be open and suppose that the vector bundle $E|_\Gamma$ is trivial. Then the Riemannian manifold (M_Γ^T, g), where M_Γ^T is defined in (22), the Hermitian vector bundle $E|_\Gamma$ and the restricted Dirichlet-to-Neumann map Λ_Γ^{2T}*

determine the Hermitian vector bundle $E|_{M_\Gamma^T}$, the connection ∇ and the potential V on $E|_{M_\Gamma^T}$.

Proof. We choose for each $x \in M_\Gamma^T$ a double sequence $\Phi^x = (f_{jk}^x)_{j,k=1}^\infty$ satifying conditions (i)–(iii) of Lemma 3.8. Observe that, by combining Corollary 3.2 and Lemma 3.5, we can determine if condition (i) of Lemma 3.8 is valid, while conditions (ii) and (iii) can be verified by using Lemma 3.1 alone. We use Lemma 3.1 once again to compute the inner products $\langle e(x;\Phi^x), Wh(x)\rangle_E$ for $h \in C_0^\infty((0,2T)\times\Gamma;E)$. Next we will impose some further conditions on the choice of the sequences Φ^x.

First, we choose the sequences Φ^x, $x \in M_\Gamma^T$ so that the functions

$$x \mapsto \langle e(x;\Phi^x), Wh(x)\rangle_E\,, \quad h \in C_0^\infty((0,2T)\times\Gamma;E), \tag{26}$$

are smooth in M_Γ^T. Then Lemma 3.7 implies that $e(x) = e(x;\Phi^x)$ is a smooth section of the vector bundle $E|_{M_\Gamma^T}$.

Second, we pick an orthonormal frame $\mathcal{B} = (b_\ell)_{\ell=1}^n$ of $E|_\Gamma$ and choose double sequences $\Phi_\ell^x = (f_{jk,\ell}^x)_{j,k=1}^\infty$, $\ell = 1,\dots,n$, so that the corresponding smooth sections $e_\ell(x) = e(x;\Phi_\ell^x)$ satisfy,

$$\langle e_\ell(x), Wh(x)\rangle_E = \langle b_\ell(x), h(T,x)\rangle_E\,, \quad x \in \Gamma,\ h \in C_0^\infty((0,2T)\times\Gamma;E).$$

This condition implies that $e_\ell = b_\ell$ on Γ.

Our next goal is to choose Φ_ℓ^x so that the corresponding sections e_ℓ form an orthonormal frame also on the set $M_0 = M_\Gamma^T \cap M^{\text{int}}$. To this end, we observe that the vector bundle $E|_{M_\Gamma^T}$ is trivial. This follows from [18, Th. 4.2.4], since the identity map on M_Γ^T is smoothly homotopic with the map $(s,y)\mapsto(0,y)$ in coordinates (5).

Let $x \in M_0$, and choose a cut off function $\chi \in C_0^\infty(M_0)$ such that $\chi(x) = 1$. As the functions (26) and the geometry (M_Γ^T, g) are known, we can compute the limits

$$\lim_{k\to\infty}\lim_{j\to\infty}\left\langle \chi e_\kappa, W f_{jk,\ell}^x\right\rangle_{L^2(M;E)} = \langle e_\kappa(x), e_\ell(x)\rangle_E\,, \quad \kappa,\ell = 1,\dots,n, \tag{27}$$

where the equality follows from Lemma 3.8. Hence, by modifying Φ_ℓ^x, we can enforce $\mathcal{E} = (e_\ell)_{\ell=1}^n$ to form an orthonormal frame on M_0.

Now $(x,a)\mapsto a^\ell e_\ell(x)$, where $a = (a^\ell)_{\ell=1}^n \in \mathbb{C}^n$ and $x \in M_\Gamma^T$, is a trivialization of $E|_{M_\Gamma^T}$, and the Hermitian inner product is given by

$$\left\langle a^\ell e_\ell(x), c^\kappa e_\kappa(x)\right\rangle_E = \sum_{\ell=1}^n a^\ell \overline{c^\ell}, \quad a,c \in \mathbb{C}^n,\ x \in M_\Gamma^T,$$

on this trivialization.

Let us write $u^h = u$ for the solution of (1) with $f = h$. The functions (26) determine the representation of

$$Wh(x) = u^h(t,x), \quad t = T,\ x \in M_0,\ h \in C_0^\infty((0,2T)\times\Gamma;E), \tag{28}$$

in the frame $\mathcal{E}$. To avoid cumbersome notation, we will not make explicit distinction between the functions (28) and their representation until Section 4.2.

As the wave equation (1) is translation invariant in time, the functions (28) are determined also for $t \in (0,T)$. We differentiate twice in time and obtain the functions

$$(P+V)u^h(t,x), \quad t \in (0,T),\ x \in M_0,\ h \in C_0^\infty((0,2T)\times\Gamma;E).$$

Let $\phi \in C_0^\infty(M_0;E)$. We can compute the inner products

$$\left\langle (P+V)u^h(T),\phi\right\rangle_{L^2(M;E)} = \langle Wh,(P+V)\phi\rangle_{L^2(M;E)},\ h \in C_0^\infty((0,2T)\times\Gamma;E).$$

As the functions (28) are known and dense in $L^2(M_0;E)$, we can determine $(P+V)\phi$ on M_0.

Let $x \in M_0$, $\ell = 1,\dots,n$ and $k = 1,\dots,m$. We choose $\phi = \phi_\ell^k$ such that $\phi(x) = 0$ and $\partial_j\phi(x) = \delta_j^k e_\ell$ for $j = 1,\dots,m$. As the metric tensor is known near x, we can compute $d^*d\phi$ at x. Thus we can recover the first order term in $(P+V)\phi$ at x. By (4), this is

$$-2(A,d\phi)(x) = -2g^{ik}(x)A_i e_\ell(x),$$

and therefore A can be determined. Finally, A and g determine P, and we can determine V by $V = P+V-P$. $\square$

3.4. **Reconstruction of ∇ when (M,g) is known and E is trivial.** We will show next that Corollary 1.2 follows from the above local reconstruction step, that is, from the proof of Theorem 3.10.

Corollary 3.11. *Suppose that (M,g) is known, E is the trivial bundle $M\times\mathbb{C}^n$, and that $T > \max_{x\in M} d_g(x,\partial M)$. Let d_A be a Hermitian connection on E. Then the Dirichlet-to-Neumann map $\Lambda_{\partial M;A}^{2T}$ determines the orbit*

$$\mathcal{O}(A) = \{U^{-1}AU + U^{-1}dU;\ U : M \to U(n),\ U|_{\partial M} = Id\}.$$

Proof. Let $b_1,\dots,b_n$ be the standard basis of $\mathbb{C}^n$ and let $\mathcal{B}$ be the corresponding constant frame of E. Let $\mathcal{E}$ be the orthonormal frame of $E|_{M_{\partial M}}$ chosen in the proof of Theorem 3.10. We recall that $\mathcal{E}$ can be enforced to satisfy $\mathcal{E} = \mathcal{B}$ on ∂M.

We have $M_{\partial M} = M \setminus N$ where the cut locus N is of measure zero, see e.g. [5]. In particular, $M_{\partial M}$ is dense in M. We know the representation of the functions Wh, $h \in C_0^\infty((0,2T)\times\partial M;E)$, in the frame $\mathcal{E}$, see (28) above. Let us impose the further condition on the choice of Φ_ℓ^x in the proof of Theorem 3.10 that the representation of $Wh(x)$ in the frame $\mathcal{E}$ is smooth in $M = \overline{M_{\partial M}}$ for all $h \in C_0^\infty((0,2T)\times\partial M;E)$. Then Lemma 3.7 implies that $\mathcal{E}$ gives a smooth frame for the whole vector bundle E.

There is a smooth transition function $U : M \to U(n)$ between the two frames $\mathcal{E}$ and $\mathcal{B}$, and $U = Id$ on ∂M. Moreover, we can reconstruct the representation of d_A in the frame $\mathcal{E}$. Let us denote the representation by $d_{\widetilde{A}}$. Then

$$\widetilde{A} = U^{-1}AU + U^{-1}dU,$$

and hence we can determine the orbit $\mathcal{O}(\widetilde{A}) = \mathcal{O}(A)$. $\square$

Suppose now that d_A and d_B are two Hermitian connections on E, and that the assumptions of Corollary 1.2 are satisfied. Then the above corollary implies that $\mathcal{O}(A) = \mathcal{O}(B)$, and we have shown Corollary 1.2.

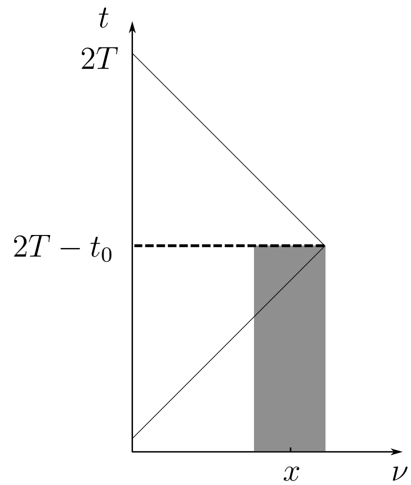

FIGURE 1. A schematic of the unique continuation argument in the proof of Lemma 4.1. The origin represents the set Γ, and the gray area is the cylinder $(0, 2T - t_0) \times B$. In order to recover u on $\{2T - t_0\} \times B$, data $(u, \nabla_\nu u)$ is needed on the cylinder $I \times \Gamma$ where $I = (2T - 2t_0, 2T)$. We may translate the interval I to cover the whole gray cylinder.

4. Global reconstruction

In this section we show how to iterate the local reconstruction step. The iteration is based on continuation of the data Λ_S^{2T} inside the region that we have already reconstructed.

4.1. **Continuation of the data.** For $T > 0$ and open sets $B \subset M$ and $\Gamma \subset \partial M$, we define the map

$$L_{\Gamma,B}^T f = u|_{(0,T)\times B}, \quad f \in C_0^\infty((0,T) \times \Gamma; E),$$

where u is the solution of (1). Moreover, for open $B \subset M^{\text{int}}$, we define the map

$$L_B^T F = u|_{(0,T)\times B}, \quad F \in C_0^\infty((0,T) \times B; E),$$

where u is the solution of

$$(\partial_t^2 + P + V)u(t,x) = F, \quad (0,\infty) \times M, \tag{29}$$
$$u|_{(0,\infty)\times\partial M} = 0,$$
$$u|_{t=0} = \partial_t u|_{t=0} = 0.$$

We write $B(x,\epsilon) = \{y \in M;\ d_g(y,x) < \epsilon\}$ for $x \in M$ and $\epsilon > 0$.

Lemma 4.1. *Let $T > 0$, $\Gamma \subset \partial M$ be open and let $x \in M_\Gamma^T$. Define $s = d_g(x,\Gamma)$, let $\epsilon \in (0, T - s)$ and define*

$$B = B(x,\epsilon), \quad t_0 = s + \epsilon.$$

Then Λ_Γ^{2T} and the structure (g, E, ∇, V) on M_Γ^T determine the map $L_{\Gamma,B\cap M_\Gamma^T}^{2T-t_0}$. Furthermore, if $B \subset M_\Gamma^T \cap M^{\text{int}}$ then they determine also the map $L_B^{2(T-t_0)}$.

Proof. Let $f \in C_0^\infty((0,2T) \times \Gamma; E)$. As $P+V$ is known on M_Γ^T, we can use unique continuation $\Lambda_\Gamma^{2T} f$ to determine $L_{\Gamma, B \cap M_\Gamma^T}^{2T-t_0} f$. Indeed, let us first extend the solution u of (1) by 0 to $(-\infty, 0) \times M$. We denote the distance function of (M_Γ^T, g) by $\tilde{d}_g$ and observe that

$$\tilde{d}_g(x,\Gamma) = d_g(x,\Gamma), \quad x \in M_\Gamma^T,$$

by the definition of M_Γ, see (7). Let $\tilde{u}$ be a solution of

$$(\partial_t^2 + P + V)\tilde{u} = 0, \quad (0,\infty) \times M_\Gamma^T,$$

satisfying the boundary conditions $\tilde{u} = f$ and $\nabla_\nu \tilde{u} = \Lambda_\Gamma^{2T} f$ on $(-\infty, 2T) \times \Gamma$. As $P+V$ is known on M_Γ^T, we can determine all such functions $\tilde{u}$. We apply Theorem 2.3 on the function $w = \tilde{u} - u$ with M replaced by M_Γ^T and with suitable translations in the time variable, see Figure 1. This implies that $\tilde{u} = u$ on $(0, 2T - t_0) \times (B \cap M_\Gamma^T)$, and we have shown the first claim.

Let us now assume that $B \subset M_\Gamma^T \cap M^{\text{int}}$. We will reconstruct the map $L_B^{2(T-t_0)}$ in two steps that we outline before giving a detailed proof. Note that $L_{\Gamma,B}^{2T-t_0}$ can be interpreted as data with sources on Γ and receivers on B. We will first transpose $L_{\Gamma,B}^{2T-t_0}$ and obtain data with sources on B and receivers on Γ. Then we will use unique continuation to obtain data with both sources and receivers on B, that is, the map $L_B^{2(T-t_0)}$.

By taking the adjoint of $L_{\Gamma,B}^{2T-t_0}$ and conjugating it with the operator reversing the time on the interval $(0, 2T-t_0)$, we get the map

$$F \mapsto \nabla_\nu u : C_0^\infty((0,2T-t_0) \times B; E) \to C^\infty((0,2T-t_0) \times \Gamma; E), \tag{30}$$

where u is the solution of (29). We extend u by 0 to $(-\infty, 0) \times M$, and let $\tilde{u}$ be a solution of

$$(\partial_t^2 + P + V)\tilde{u} = F, \quad (0,\infty) \times M_\Gamma^T,$$

satisfying the boundary conditions $\tilde{u} = 0$ and $\nabla_\nu \tilde{u} = \nabla_\nu u$ on $(-\infty, 2T - t_0) \times \Gamma$. Then $w = \tilde{u} - u$ satisfies conditions of Theorem 2.3 with M again replaced by M_Γ^T, and therefore $\tilde{u} = u$ on $(0, 2T - t_0 - t_0) \times B$. This implies the second claim. □

We denote by SM the unit sphere bundle of M. Similarly to σ_Γ and σ_Γ^T, see (6) and (22), we define for $x \in M^{\text{int}}$, $\xi \in S_x M$ and $T > 0$,

$$\sigma_x(\xi) = \sup\{t \in (0, \tau_x(\xi)];\ d_g(\gamma(t;x,\xi), x) = t\},$$
$$\tau_x(\xi) = \sup\{t \in (0,\infty);\ \gamma(t;x,\xi) \in M^{\text{int}}\},$$

and $\sigma_x^T(\xi) = \min(\sigma_x(\xi), T)$. Moreover, we define

$$M_x^T = \{\gamma(t;x,\xi);\, \xi \in S_x M, t \in [0, \sigma_x^T(\xi))\}.$$

Note that the injectivity radius inj_x at a point $x \in M^{\text{int}}$ satisfies

$$\text{inj}_x = \min_{\xi \in S_x M} \sigma_x(\xi).$$

Lemma 4.2. *Let $T > 0$, $x \in M^{\mathrm{int}}$, $\epsilon \in (0, inj_x)$, and define $B = B(x, \epsilon)$. Then L_B^{2T} and the structure (g, E, ∇, V) on B determine the structure (g, E, ∇, V) on $M_x^{T+\epsilon}$.*

Proof. We define $\tilde{M} = M \setminus B$ and consider the wave equation

$$
\begin{aligned}
&(\partial_t^2 + P + V)\tilde{u} = 0, && (0,\infty) \times \tilde{M}, \\
&\tilde{u}|_{(0,\infty)\times\partial B} = f, \quad \tilde{u}|_{(0,\infty)\times\partial M} = 0, \\
&\tilde{u}|_{t=0} = \partial_t \tilde{u}|_{t=0} = 0.
\end{aligned} \tag{31}
$$

We will show that L_B^{2T} determines the restricted Dirichlet-to-Neumann map $\Lambda_{\partial B}^{2T}$ of $\tilde{M}$, that is, the map

$$\Lambda_{\partial B}^{2T} f = \nabla_\nu \tilde{u}|_{(0,2T)\times\partial B}, \quad f \in C_0^\infty((0,2T)\times\partial B; E),$$

where $\tilde{u}$ is the solution of (31). Let $f \in C_0^\infty((0,2T)\times\partial B; E)$ and extend the solution of (31) smoothly into $(0,\infty)\times B$ keeping the notation $\tilde{u}$ for the extension. Then $\tilde{u}$ satisfies (29) with $\tilde{F} = (\partial_t^2 + P + V)\tilde{u}$, and $\tilde{F}$ belongs to

$$\dot{C}^\infty((0,\infty)\times\overline{B}; E) = \{F \in C^\infty((0,\infty)\times M; E);\ \mathrm{supp}(\tilde{F}) \subset (0,\infty] \times \overline{B}\}. \tag{32}$$

By closing L_B^{2T} in $L^2((0,2T)\times B; E)$, we can compute $L_B^{2T} F$ for F in (32), and therefore we can find all $F \in \dot{C}^\infty((0,\infty)\times\overline{B}; E)$ such that the corresponding solution u of (29) satisfies $u|_{(0,2T)\times\partial B} = f$. Using again $L_B^{2T} F$ we can determine $\nabla_\nu u|_{(0,2T)\times\partial B}$. Since the solution of (31) is unique, we have

$$\nabla_\nu u|_{(0,2T)\times\partial B} = \nabla_\nu \tilde{u}|_{(0,2T)\times\partial B}.$$

We shown that the map L_B^{2T} determines the map $\Lambda_{\partial B}^{2T}$.

We denote by $\sigma_{\partial B}$ the cut distance on the manifold $\tilde{M}$ defined analogously to (6) and define $\sigma_{\partial B}^T(y) = \max(\sigma_{\partial B}(y), T)$, $y \in \partial B$. Note that the vector bundle $E|_{\partial B}$ is trivial, in fact, E is trivial over M_x^T due to its contractibility via the radial geodesics emanating from x. We apply Theorems 3.4 and 3.10 with $M = \tilde{M}$ and $\Gamma = \partial B$. This gives us the structure (g, E, ∇, V) on

$$\tilde{M}_{\partial B}^T = \{\gamma(s; y, \nu);\ y \in \partial B,\ s \in [0, \sigma_{\partial B}^T(y))\}.$$

Note that $\sigma_x(\xi) = \sigma_{\partial B}(y) + \epsilon$, where $y = \gamma(\epsilon; x, \xi)$, and therefore $M_x^{T+\epsilon} = B \cup \tilde{M}_{\partial B}^T$. □

Lemma 4.3. *Let $T_0, \epsilon_0 > 0$, $x_0 \in M^{\mathrm{int}}$, and define $B_0 = B(x_0, \epsilon_0)$ and $M_0 = M_{x_0}^{T_0}$. Let $x \in M_0 \setminus B_0$ and define $s = d_g(x, x_0)$. Let $T > 0$ and let $\epsilon \in (0, inj_x)$ satisfy*

$$\epsilon < d_g(x, \partial M_0), \quad \epsilon < T - s + \epsilon_0.$$

Define $B_1 = B(x, \epsilon)$ and $t_1 = s + \epsilon - \epsilon_0$. Then $L_{B_0}^{2T}$ and the structure (g, E, ∇, V) on M_0 determine the map $L_{B_1}^{2(T-t_1)}$. Furthermore, for open $\Gamma \subset \partial M$, L_{Γ, B_0}^{2T} and the structure (g, E, ∇, V) on M_0 determine the map $L_{\Gamma, B_1}^{2T-t_1}$.

Proof. By the proof of Lemma 4.2, $L_{B_0}^{2T}$ determines $\Lambda_{\partial B_0}^{2T}$. As $d_g(x, \partial B_0) = s - \epsilon_0$, Lemma 4.1 shows that $\Lambda_{\partial B}^{2T}$ and the structure (g, E, ∇, V) on M_0 determine $L_{B_1}^{2(T-t_1)}$. Finally, L_{Γ, B_0}^{2T} determines $L_{\Gamma, B_1}^{2T-t_1}$ by a unique continuation argument similar to that in the proof of Lemma 4.1. □

4.2. **Gluing local reconstructions in the interior.** In this section we show the following theorem:

Theorem 4.4. *Let $\mathcal{S} \subset \partial M$ be open and suppose that*

$$T > \max_{x \in M} d_g(x, \mathcal{S}). \tag{33}$$

Then the Hermitian vector bundle $E|_{\mathcal{S}}$ and the restricted Dirichlet-to-Neumann operator $\Lambda_{\mathcal{S}}^{2T}$ determine the smooth manifold M^{int} and the structure (g, E, ∇, V) on M^{int}.

Up to this point we have avoided writing all the isomorphisms explicitly, but in this section the distinction between different representations is crucial. Let us choose an open cover $\mathcal{G}_{\mathcal{S}}$ of $\mathcal{S}$ consisting of small enough sets $\Gamma \subset \mathcal{S}$ so that each Γ is a coordinate neighborhood in ∂M and that the vector bundle $E|_\Gamma$ is trivial. Then we may choose an open set $Y_\Gamma \subset \mathbb{R}^{m-1}$ and a unitary trivialization

$$\begin{array}{ccc} E & \xrightarrow{\phi_\Gamma} & Y_\Gamma \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow[\psi_\Gamma]{} & Y_\Gamma \end{array} \tag{34}$$

By a unitary trivialization we mean that the diagram (34) commutes, ϕ_Γ is a smooth bijection that is linear in fibers, and that the Hermitian structure is preserved, that is, $\phi_\Gamma^* \langle \cdot, \cdot \rangle_{\mathbb{C}^n} = \langle \cdot, \cdot \rangle_E$.

Starting from the representation of Λ_Γ^{2T} on the trivialization (34), the local reconstruction method in Section 3 gives the function $\sigma_\Gamma^T : \Gamma \to (0, T)$, a metric tensor g_Γ on X_Γ^T, and a connection ∇_Γ and potential V_Γ on $X_\Gamma^T \times \mathbb{C}^n$, such that there is a unitary trivialization

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\Phi}_\Gamma} & X_\Gamma^T \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ M_\Gamma^T & \xrightarrow[\tilde{\Psi}_\Gamma]{} & X_\Gamma^T \end{array} \tag{35}$$

satisfying $g = \tilde{\Psi}_\Gamma^* g_\Gamma$, $\nabla = \tilde{\Phi}_\Gamma^* \nabla_\Gamma$ and $V = \tilde{\Phi}_\Gamma^* V_\Gamma$. Here

$$X_\Gamma^T = \{(s, y) \in \mathbb{R}^m;\ s \in [0, \sigma_\Gamma^T \circ \psi_\Gamma^{-1}(y)),\ y \in Y_\Gamma\}$$

is the representation of M_Γ^T in boundary normal coordinates, and the restriction of $\tilde{\Phi}_\Gamma$ on the vector bundle $E|_\Gamma$ coincides with ϕ_Γ.

We will next iterate the procedure in Section 4.1. The initial step is the following:

1. Given Λ_Γ^{2T} and a representation of the structure (g, E, ∇, V) on M_Γ^T, that is, g_Γ, $X_\Gamma^T \times \mathbb{C}^n$, ∇_Γ and V_Γ, we choose $(s_0, y_0) \in X_\Gamma^T$ and $\epsilon_0 > 0$ such that

$$B_0 = B(z_0, \epsilon_0) \subset M_\Gamma^T \cap M^{\mathrm{int}}, \tag{36}$$

where $z_0 = \tilde{\Psi}_\Gamma^{-1}(s_0, y_0) \in M_\Gamma^T$.

We invoke Lemma 4.1 to reconstruct the representations of $L_{\Gamma,B_0}^{2T-t_0}$ and $L_{B_0}^{2(T-t_0)}$ on the trivialization (35). Here

$$t_0 = s_0 + \epsilon_0, \tag{37}$$

and we emphasize that we do not know the point $z_0 \in M$, only its representation (s_0, y_0) in the boundary normal coordinates.

We iterate Lemmas 4.2 and 4.3 as follows:

2. Given a representation of $L_{B_j}^{2(T-t_j)}$, where $B_j = B(z_j, \epsilon_j)$, we reconstruct a representation of the structure (g, E, ∇, V) on $M_j = M_{z_j}^{T-t_j+\epsilon_j}$.
3. We choose $s_{j+1} > 0$, $\xi_{j+1} \in S_{z_j}M$ and $\epsilon_{j+1} > 0$ such that

   $$B_{j+1} = B(z_{j+1}, \epsilon_{j+1}) \subset M_j,$$

   where $z_{j+1} = \gamma(s_{j+1}; z_j, \xi_{j+1})$. Again, we do not know z_{j+1}, only its representation (s_{j+1}, ξ_{j+1}) in normal coordinates at z_j. Given representations of $L_{B_j}^{2(T-t_j)}$ and $L_{\Gamma,B_j}^{2T-t_j}$, we reconstruct representations of $L_{B_{j+1}}^{2(T-t_{j+1})}$ and $L_{\Gamma,B_j}^{2T-t_{j+1}}$, where

$$t_{j+1} = t_j + s_{j+1} + \epsilon_{j+1} - \epsilon_j. \tag{38}$$

We terminate the iteration after repeating the steps 2 and 3 a finite number of times denoted by $N = 0, 1, 2, \dots$. Note that we must satisfy the condition $t_j < T$ in each step of the iteration.

If $N = 0$ then we do not need to satisfy the constraint (36). That is, we can use Lemma 4.1 to reconstruct a representation of $L_{\Gamma,B_0\cap M_\Gamma^T}^{2T-t_0}$ where $B_0 = B(z_0, \epsilon_0)$, $z_0 \in M_\Gamma^T$ and $\epsilon_0 \in (0, T - s_0)$. In particular, for $y_0 \in \Gamma$ and for small enough $\epsilon_0 > 0$ we can reconstruct a representation of $L_{\Gamma,C_0}^{2T-\epsilon_0}$ where C_0 is the cylinder

$$C_0 = \{\gamma(s; y, \nu);\ s \in (0, \epsilon_0),\ y \in B_\partial(y_0, \epsilon_0)\}, \tag{39}$$

and $B_\partial(y_0, \epsilon_0) = \{y \in \partial M;\ d_g(y, y_0) < \epsilon_0\}$.

There are is a lot freedom in our iteration process. Namely, we can choose N, the points z_j and the radii ϵ_j freely within the constraints of the iteration. Let A_Γ denote the set of all choices that are allowed within the constraints of iteration when starting from $\Gamma \in \mathcal{G}_\mathcal{S}$. We define also the disjoint union $A = \bigsqcup_{\Gamma\in\mathcal{G}_\mathcal{S}} A_\Gamma$.

We denote by $B_\alpha = B_{N(\alpha)}$ the set chosen in the last invocation of step 3 in the iteration process $\alpha \in A_\Gamma$, and use analogous notation for other chosen quantities. The iteration gives us a metric tensor g_α, a connection ∇_α and a potential V_α such that there is a unitary trivialization

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\Phi}_\alpha} & X_\alpha \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ B_\alpha & \xrightarrow[\tilde{\Psi}_\alpha]{} & X_\alpha \end{array} \tag{40}$$

satisfying $g = \tilde{\Psi}_\alpha^* g_\alpha$, $\nabla = \tilde{\Phi}_\alpha^* \nabla_\alpha$ and $V = \tilde{\Phi}_\alpha^* V_\alpha$. Here X_α is the open ball of radius $\epsilon_{N(\alpha)}$ in $\mathbb{R}^m$ with center at the origin, and $\tilde{\Psi}_\alpha$ gives normal coordinates at $z_{N(\alpha)}$. The iteration gives also the representation L_α of $L_{\Gamma,B_\alpha}^{2T-t_{N(\alpha)}}$ on the trivialization (40).

If the iteration is terminated immediately after the initial step (that is, $N(\alpha) = 0$) we allow B_α to be also of the form (39).

Let us show that the balls B_α, $\alpha \in A_\Gamma$, cover $M(\Gamma, T)^{\mathrm{int}}$ and that they separate points:

(G1) For all distinct $z, z' \in M(\Gamma, T)^{\mathrm{int}}$ there are $\alpha, \beta \in A_\Gamma$ such that $z \in B_\alpha$, $z' \in B_\beta$ and $B_\alpha \cap B_\beta = \emptyset$.

Proof. Let $z \in M(\Gamma, T)^{\mathrm{int}}$. Then there is a shortest path γ from $\bar{\Gamma}$ to z having length strictly less than T. The path γ can be perturbed to get a broken geodesic $\tilde{\gamma}$ from $y \in \Gamma$ to z having length strictly less than T. Moreover, $\tilde{\gamma}$ can be chosen so that it intersects ∂M only at its starting point y. Then the points z_j, $j = 1, \dots, N$, can be chosen along $\tilde{\gamma}$. Moreover, when z_0 is close to Γ and the radius ϵ_N is chosen small enough, we have $t_N < T$. Indeed, by (37) and (38),

$$t_N = \epsilon_N + s_0 + \sum_{j=1}^{N} s_j,$$

where $s_0 = d_g(z_0, \Gamma)$ and $s_j = d_g(z_j, z_{j-1})$. In particular, the balls B_α, $\alpha \in A$, form an open cover of $M(\Gamma, T)^{\mathrm{int}}$.

Let $z' \in M(\Gamma, T)^{\mathrm{int}}$ and suppose that $z' \neq z$. We may choose the radius ϵ_N small enough so that $\epsilon_N < d_g(z, z')/2$, and perform an analogous construction for z'. This gives us disjoint balls as claimed. □

Let us point out that the assumption (33) does not imply that $M(\Gamma, T) = M$ since Γ might be smaller than $\mathcal{S}$. However, it implies that the sets $M(\Gamma, T)^{\mathrm{int}}$, $\Gamma \in \mathcal{G}_{\mathcal{S}}$, form an open cover of M^{int}, and therefore the sets B_α, $\alpha \in A$, form an open cover of M^{int} by (G1). We will show next how to glue together the local representations of (g, E, ∇, V) on the sets B_α, $\alpha \in A$.

Lemma 4.5. *Let $T > 0$, $\Gamma \subset \partial M$ be open, and suppose that $B \subset M^{\mathrm{int}}$ is open and satisfies $B \subset M(\Gamma, T)$. Let $h \in C_0^\infty(B; E)$ and $s \in (0, T)$. Then the maps Λ_Γ^{2T} and $L_{\Gamma,B}^{2T-s}$ together with the structure (g, E) on B determine the non-empty set*

$$\{(f_j)_{j=1}^\infty \subset C_0^\infty((0, 2T) \times \Gamma; E);\ \lim_{j\to\infty} W f_j = h \text{ in } L^2(M; E)\}. \tag{41}$$

Proof. We expand the squared norm

$$\|W f_j - h\|_{L^2(M;E)}^2 = \langle W f_j, W f_j \rangle_{L^2(M;E)} - 2\,Re\,\langle W f_j, h \rangle_{L^2(M;E)} + \langle h, h \rangle_{L^2(M;E)},$$

and observe that Λ_Γ^{2T} determines the first term on the right-hand side by Corollary 3.2, $L_{\Gamma,B}^{2T-s}$ and (g, E) on B determine the second term, and (g, E) on B determines the third term. To conclude we observe that Lemma 2.4 implies that the set (41) is non-empty. □

Lemma 4.6. *Suppose that open $\mathcal{S} \subset \partial M$ and $T > 0$ satisfy (33). Let $x_1, x_2 \in M^{\text{int}}$. We have $x_1 = x_2$ if and only if for all sufficiently small $\epsilon > 0$ and any $h_1 \in C_0^\infty(B(x_1, \epsilon); E)$ there is $h_2 \in C_0^\infty(B(x_2, \epsilon); E)$ such that*

$$\langle h_1 - h_2, Wf \rangle_{L^2(M;E)} = 0, \quad f \in C_0^\infty((0, 2T) \times \mathcal{S}; E). \tag{42}$$

Proof. Let us suppose that $x_1 \neq x_2$. We choose small enough $\epsilon > 0$ so that the balls $B(x_j, \epsilon)$, $j = 1, 2$, are disjoint. We choose non-zero $h_1 \in C_0^\infty(B(x_1, \epsilon); E)$ and let $h_2 \in C_0^\infty(B(x_2, \epsilon); E)$ be arbitrary. Then $h_1 \neq h_2$ and Lemma 2.4 implies that there is $f \in C_0^\infty((0, 2T) \times \mathcal{S}; E)$ satisfying

$$\langle h_1 - h_2, Wf \rangle_{L^2(M;E)} \neq 0.$$

The other implication is trivial. □

Lemmas 4.5 and 4.6 allow us to determine if two points $x_i \in X_{\alpha_i}$, $\alpha_i \in A_{\Gamma_i}$, $\Gamma_i \in \mathcal{G}_\mathcal{S}$, $i = 1, 2$, satisfy

$$\tilde{\Psi}_{\alpha_1}^{-1}(x_1) = \tilde{\Psi}_{\alpha_2}^{-1}(x_2). \tag{43}$$

Indeed, let $\epsilon > 0$ be small, let $\tilde{B}_i$, $i = 1, 2$, be the geodesic ball in $(X_{\alpha_i}, g_{\alpha_i})$ with center x_i and radius ϵ, and let $\tilde{h}_i \in C_0^\infty(\tilde{B}_i; E)$. Then using Lemma 4.5, we can find sequences $(f_j^i)_{j=1}^\infty \subset C_0^\infty((0, 2T) \times \Gamma_i; E)$ such that $\lim_{j \to \infty} Wf_j^i = h_i$ where $h_i = \Phi_{\alpha_i}^* \tilde{h}_i$. Note that in order to apply Lemma 4.5 it is enough to know Λ_Γ^{2T} and the representations L_{α_i} and g_{α_i}, $i = 1, 2$. By Corollary 3.2, we can compute

$$\lim_{j \to \infty} \langle Wf_j^1 - Wf_j^2, Wf \rangle_{L^2(M;E)} = \langle h_1 - h_2, Wf \rangle_{L^2(M;E)}, \tag{44}$$

for all $f \in C_0^\infty((0, 2T) \times \mathcal{S}; E)$. Hence we can use (42) to determine if (43) holds.

The equation (43) gives an equivalence relation on the disjoint union $\tilde{\mathcal{X}} = \bigsqcup_{\alpha \in A} X_\alpha$ and we denote by $\mathcal{X}$ and $q : \tilde{\mathcal{X}} \to \mathcal{X}$ the corresponding quotient space and the canonical map. Moreover, we define $\mathcal{U}_\alpha = q(X_\alpha) \subset \mathcal{X}$ and $q_\alpha = q|_{X_\alpha}$, $\alpha \in A$. We will show that $\mathcal{X}$ is a smooth manifold:

(G2) The maps $q_\alpha : X_\alpha \to \mathcal{U}_\alpha$ are bijective, and there is a unique Hausdorff topology and a complete atlas on $\mathcal{X}$ such that each q_α^{-1} is a coordinate system.

As we can determine if x and x' are equivalent given the data Λ_Γ^{2T}, we see that the smooth structure of $\mathcal{X}$ is determined. Let us show (G2) simultaneously with the following:

(G3) Let us define a map $\Psi : M^{\text{int}} \to \mathcal{X}$ by $\Psi(z) = q \circ \tilde{\Psi}_\alpha(z)$ when $z \in B_\alpha$. Then Ψ is a well-defined diffeomorphism.

Proof of (G2) and (G3). Let $z \in M^{\text{int}}$. Then (G1) implies that there is $\alpha \in A$ such that $z \in B_\alpha$. If $z \in B_\beta$ also for $\beta \in A$, then $q(x) = q(x')$ where $x = \tilde{\Psi}_\alpha(z)$ and $x' = \tilde{\Psi}_\beta(z)$. Thus Ψ is well-defined.

Note that the sets $\mathcal{U}_\alpha$ cover $\mathcal{X}$ since the sets $X_\alpha = \tilde{\Psi}_\alpha(B_\alpha)$ cover $\tilde{\mathcal{X}}$. This implies that Ψ is surjective. Suppose that $\Psi(z) = \Psi(z')$ for some $z \in B_\alpha$ and $z' \in B_\beta$. Then $q(x) = q(x')$ where $x = \tilde{\Psi}_\alpha(z)$ and $x' = \tilde{\Psi}_\beta(z)$. Thus $z = z'$ by the definition of q, and we have shown that Ψ is injective.

We define $\Psi_\alpha : B_\alpha \to \mathcal{U}_\alpha$ as the restiction $\Psi_\alpha = \Psi|_{B_\alpha}$. It is clearly bijective. Now $\Psi_\alpha = q_\alpha \circ \tilde{\Psi}_\alpha$ implies that $q_\alpha = \Psi_\alpha \circ \tilde{\Psi}_\alpha^{-1}$. Hence the maps q_α are bijective. Moreover, if $\mathcal{U} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \emptyset$ then we have on $q_\alpha^{-1}(\mathcal{U})$ that

$$q_\beta^{-1} \circ q_\alpha = \tilde{\Psi}_\beta \circ \Psi^{-1} \circ \Psi \circ \tilde{\Psi}_\alpha^{-1} = \tilde{\Psi}_\beta \circ \tilde{\Psi}_\alpha^{-1},$$

and we see that $q_\beta^{-1} \circ q_\alpha$ is smooth on the open set $q_\alpha^{-1}(\mathcal{U}) = \tilde{\Psi}_\alpha(B_\alpha \cap B_\beta)$. We have shown that the conditions (1) and (2) of [28, Prop. 1.42] hold. To finish the proof of (G2) we need only to verify the separation condition (3) in [28, Prop. 1.42].

Let $p, p' \in \mathcal{X}$ be distinct. Then $z \neq z'$ where $z = \Psi^{-1}(p)$ and $z' = \Psi^{-1}(p')$. Let $\alpha, \beta \in A$ be as in (G1). Then $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are disjoint sets containing p and p' respectively, since $\mathcal{U}_\alpha = \Psi(B_\alpha)$ and $\mathcal{U}_\beta = \Psi(B_\beta)$. Now (G2) follows from [28, Prop. 1.42].

To show that Ψ is smooth, it is enough to show that each $q_\alpha^{-1} \circ \Psi \circ \tilde{\Psi}_\alpha^{-1}$ is smooth. But this is simply the identity map on X_α. □

Let us show that the metric tensors g_α can be glued together:

(G4) We have $(q_\alpha^{-1})^* g_\alpha = (\Psi^{-1})^* g$ on each $\mathcal{U}_\alpha$.

Proof. We recall that $g = \tilde{\Psi}_\alpha^* g_\alpha$ on B_α. Thus we have on $\mathcal{U}_\alpha$ that

$$(\Psi^{-1})^* g = (\tilde{\Psi}_\alpha \circ \Psi^{-1})^* g_\alpha = (\tilde{\Psi}_\alpha \circ \Psi_\alpha^{-1})^* g_\alpha = (q_\alpha^{-1})^* g_\alpha.$$

□

Let us now turn to gluing of the vector bundles $X_\alpha \times \mathbb{C}^n$. Let $\mathcal{E}^\alpha = (e_\ell^\alpha)_{\ell=1}^n$ be the constant frame on $X_\alpha \times \mathbb{C}^n$ corresponding to the standard basis of $\mathbb{C}^n$. Suppose that $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ intersect for some $\alpha, \beta \in A$, and write

$$X_{\alpha\beta} = q_\alpha^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta), \quad X_{\beta\alpha} = q_\beta^{-1}(\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

We define functions $h_1 = \Phi_\alpha^* \tilde{h}_1$ and $h_2 = \Phi_\beta^* \tilde{h}_2$, where

$$\tilde{h}_1 = 1_{X_{\alpha\beta}} e_\ell^\alpha \in L^2(X_\alpha; \mathbb{C}^n), \quad \tilde{h}_2 = 1_{X_{\beta\alpha}} a_\ell^\kappa e_\kappa^\beta \in L^2(X_\beta; \mathbb{C}^n).$$

Here $\ell, \kappa = 1, \dots, n$ and $a_\ell^\kappa \in C^\infty(X_{\beta\alpha})$. Analogously to the considerations preceeding (44), we can choose two sequences of sources $(f_j^i)_{j=1}^\infty$, $i = 1, 2$, such that $(W f_j^i)_{j=1}^\infty$ converges to h_i, and determine if (42) holds. Suppose now that we have chosen $a_\ell^\kappa \in C^\infty(X_{\beta\alpha})$, $\kappa = 1, \dots, n$, so that (42) holds. We define $U_{\beta\alpha} = (a_\ell^\kappa)_{\kappa,\ell=1}^n$ on $X_{\beta\alpha}$. Moreover, we define an equivalence relation on $\tilde{\mathcal{X}} \times \mathbb{C}^n$ by

$$q(x) = q(x'), \quad \xi' = U_{\beta\alpha}(x')\xi, \tag{45}$$

where $x \in X_\alpha$, $x' \in X_\beta$ and $\xi, \xi' \in \mathbb{C}^n$. We have:

(G5) The equations (45) hold if and only if $\tilde{\Phi}_\alpha^{-1}(x, \xi) = \tilde{\Phi}_\beta^{-1}(x', \xi')$.

Proof. Note that $x \in X_\alpha$, $x' \in X_\beta$ and $q(x) = q(x')$ imply that $x' \in X_{\beta\alpha}$. Therefore, the second equation in (45) is well-defined whenever the first one holds.

We write $B = B_\alpha \cap B_\beta$. Let $Z \in \pi_E^{-1}(B)$ where $\pi_E : E|_{M^{\text{int}}} \to M^{\text{int}}$ is the bundle projection, and take $z = \pi_E(Z)$. Moreover, denote by $Z_p = (Z_p^\ell)_{\ell=1}^n$ the representation of Z in the frame $\tilde\Phi_p^* e_\ell^p$, $p = \alpha, \beta$. Then, since h_1 and h_2 are smooth in B and satisfy (42), Lemma 2.4 implies that

$$Z = Z_\alpha^\ell \tilde\Phi_\alpha^* e_\ell^\alpha|_z = Z_\alpha^\ell \tilde\Phi_\beta^*(a_\ell^\kappa e_\kappa^\beta)|_z = Z_\alpha^\ell a_\ell^\kappa(\tilde\Psi_\beta(z))\tilde\Phi_\beta^* e_\kappa^\beta|_z.$$

Hence $Z_\beta = U_{\beta\alpha}(\tilde\Psi_\beta(z))Z_\alpha$.

Suppose that (45) holds, and define $Z = \tilde\Phi_\alpha^{-1}(x,\xi)$. Then $Z \in \pi_E^{-1}(B)$ and we have, using the above notation $z = \pi_E(Z)$ and $Z_p = (Z_p^\ell)_{\ell=1}^n$, $p = \alpha,\beta$, that $\tilde\Psi_\alpha(z) = x$ and $Z_\alpha = \xi$. Moreover, $\tilde\Phi_\beta(Z) = (\tilde\Psi_\beta(z), Z_\beta)$ where $\tilde\Psi_\beta(z) = x'$ as $q(x) = q(x')$, and

$$Z_\beta = U_{\beta\alpha}(\tilde\Psi_\beta(z))Z_\alpha = U_{\beta\alpha}(x')\xi = \xi'.$$

On the other hand, if $Z = \tilde\Phi_\alpha^{-1}(x,\xi) = \tilde\Phi_\beta^{-1}(x',\xi')$, then $q(x) = q(x')$ and

$$\xi' = Z_\beta = U_{\beta\alpha}(\tilde\Psi_\beta(z))Z_\alpha = U_{\beta\alpha}(x')\xi.$$

□

We denote by F the quotient space with respect to the equivalence (45) and by $Q : \tilde{\mathcal{X}} \times \mathbb{C}^n \to F$ the corresponding canonical map. Moreover, we define

$$\pi_F : F \to \mathcal{X} : \pi_F(Q(x,\xi)) = q(x), \quad (x,\xi) \in \tilde X \times \mathbb{C}^n, \tag{46}$$

and Q_α as the restriction of Q on $X_\alpha \times \mathbb{C}^n$, $\alpha \in A$. These maps define a smooth vector bundle structure:

(G6) The map π_F is a well-defined surjection and the maps $Q_\alpha : X_\alpha \times \mathbb{C}^n \to \pi_F^{-1}(\mathcal{U}_\alpha)$ are bijective. There is a unique Hausdorff topology and a complete atlas on F such that each Q_α^{-1} is a coordinate system. The maps $\xi \mapsto Q_\alpha(x,\xi)$ are bijective from $\mathbb{C}^n$ to $\pi_F^{-1}(\{q(x)\})$ for $x \in X_\alpha$ and $\alpha \in A$, and, if the fibers $\pi_F^{-1}(\{p\})$, $p \in \mathcal{X}$, are equipped with the vector space structure that is pulled back from $\mathbb{C}^n$ via the inverses of these maps, then $\pi_F : F \to \mathcal{X}$ is a smooth vector bundle that is trivial on each $\mathcal{U}_\alpha$.

Let us show (G6) simultaneously with the following:

(G7) Let us define a map $\Phi : E|_{M^{\text{int}}} \to F$ by $\Phi(Z) = Q \circ \tilde\Phi_\alpha(Z)$ when $Z \in \pi_E^{-1}(B_\alpha)$. Here π_E is the bundle projection $E|_{M^{\text{int}}} \to M^{\text{int}}$. Then Φ is a well-defined vector bundle isomorphism that covers Ψ.

Proof of (G6) and (G7). Clearly π_F is a well-defined surjection. A proof that Φ is a well-defined bijection is essentially identical with the above proof that Ψ is a well-defined bijection, and we omit it.

Let $\alpha \in A$, $x \in X_\alpha$, and consider the map $Q_\alpha^x(\xi) = Q_\alpha(x,\xi)$. The definition of π_F implies that $Q_\alpha^x : \mathbb{C}^n \to F^x$ where $F^x = \pi_F^{-1}(\{q(x)\})$. Let us show that Q_α^x is surjective. Let $\beta \in A$ and $x' \in X_\beta$ satisfy $q(x') = q(x)$ and let $\xi' \in \mathbb{C}^n$. Then, if we choose $\xi = U_{\beta\alpha}(x')^{-1}\xi'$, we have $Q_\beta^{x'}(\xi') = Q_\alpha^x(\xi)$ due to (45). Thus Q_α^x is surjective. The surjectivity implies that

$$Q(X_\alpha \times \mathbb{C}^n) = \bigcup_{x \in X_\alpha} Q_\alpha^x(\mathbb{C}^n) = \pi_F^{-1}(q(X_\alpha)) = \pi_F^{-1}(\mathcal{U}_\alpha).$$

We write $E_\alpha = \pi_E^{-1}(B_\alpha)$, $F_\alpha = \pi_F^{-1}(\mathcal{U}_\alpha)$, and define $\Phi_\alpha = \Phi|_{E_\alpha}$. The sets

$$\Phi(E_\alpha) = Q(X_\alpha \times \mathbb{C}^n) = F_\alpha, \quad \alpha \in A,$$

cover F, and $\Phi_\alpha : E_\alpha \to F_\alpha$ is bijective. The factorization $\Phi_\alpha = Q_\alpha \circ \tilde{\Phi}_\alpha$ implies that Q_α is bijective, and $Q_\beta^{-1} \circ Q_\alpha = \tilde{\Phi}_\beta \circ \tilde{\Phi}_\alpha^{-1}$ is smooth on the open set $Q_\alpha^{-1}(F_\alpha \cap F_\beta) = \tilde{\Phi}_\alpha(E_\alpha \cap E_\beta)$.

Let $p, p' \in F$, and define $z = \pi_E \circ \Phi^{-1}(p)$ and $z' = \pi_E \circ \Phi^{-1}(p')$. If $z \neq z'$ then we may choose $\alpha, \beta \in A$ as in (G1). Then E_α and E_β are disjoint, whence F_α and F_β are disjoint sets containing p and p' respectively. On the other hand, if $z = z'$ then there is $\alpha \in A$ such that $p, p' \in F_\alpha$. Now [28, Prop. 1.42] implies that F has a unique smooth manifold structure.

To show that π_F is smooth, it is enough to show that each $q_\alpha^{-1} \circ \pi_F \circ Q_\alpha$ is smooth. But this is simply the map $\pi_\alpha : X_\alpha \times \mathbb{C}^n \to X_\alpha$, $\pi_\alpha(x, \xi) = x$. A proof that Φ is smooth is essentially identical with the above proof that Ψ is smooth, and we omit it.

We define a vector space structure on F^x by pulling back the addition and scalar multiplication via $(Q_\alpha^x)^{-1} : F^x \to \mathbb{C}^n$. That is,

$$Q_\alpha^x(\xi) + cQ_\alpha^x(\eta) = Q_\alpha^x(\xi + c\eta), \quad \xi, \eta \in \mathbb{C}^n \, c \in \mathbb{C}.$$

Let us show that this does not depend on the choice of $x' \in q^{-1}(\{x\})$. Suppose that $F^x = F^{x'}$ for some $\beta \in A$ and $x' \in X_\beta$, and let $\xi', \eta' \in \mathbb{C}^n$. We choose $\xi = U_{\beta\alpha}(x')^{-1}\xi'$ and $\eta = U_{\beta\alpha}(x')^{-1}\eta'$. Then $Q_\beta^{x'}(\xi') = Q_\alpha^x(\xi)$, $Q_\beta^{x'}(\eta') = Q_\alpha^x(\eta)$ and $Q_\beta^{x'}(\xi' + c\eta') = Q_\alpha^x(\xi + c\eta)$ for all $c \in C$.

Next let us construct local trivializations for F. We define $\rho : \tilde{\mathcal{X}} \times \mathbb{C}^n \to \mathcal{X} \times \mathbb{C}^n$ by $\rho = q \otimes id$, that is, $\rho(x, \xi) = (q(x), \xi)$, and set $\rho_\alpha = \rho \circ Q_\alpha^{-1}$. Then $\rho_\alpha : F_\alpha \to \mathcal{U}_\alpha \times \mathbb{C}^n$ is a smooth bijection since $(q_\alpha^{-1} \otimes id) \circ \rho_\alpha \circ Q_\alpha$ is the identity on $X_\alpha \times \mathbb{C}^n$. Moreover, $\pi_F \circ \rho_\alpha^{-1}$ is the identity on $\mathcal{U}_\alpha$, and, for $x \in X_\alpha$, the map $\xi \mapsto \rho_\alpha^{-1}(q(x), \xi)$ is Q_α^x. Thus the maps ρ_α^{-1}, $\alpha \in A$, give local trivializations for F, and $\pi_F : F \to \mathcal{X}$ is a smooth vector bundle.

Let us show that Φ is a vector bundle homomorphism. We recall that $q_\alpha = \Psi \circ \tilde{\Psi}_\alpha^{-1}$, $q_\alpha^{-1} \circ \pi_F \circ Q_\alpha = \pi_\alpha$ and $Q_\alpha = \Phi_\alpha \circ \widetilde{\Phi}_\alpha^{-1}$, where π_α is the projection on right in (40). thus, we have

$$q_\alpha^{-1} \circ \pi_F \circ \Phi \circ \tilde{\Phi}_\alpha^{-1} = q_\alpha^{-1} \circ \pi_F \circ Q_\alpha = \pi_\alpha, \tag{47}$$

and, as the diagram (40) commutes, we have also

$$q_\alpha^{-1} \circ \Psi \circ \pi_E \circ \tilde{\Phi}_\alpha^{-1} = q_\alpha^{-1} \circ \Psi \circ \tilde{\Psi}_\alpha^{-1} \circ \pi_\alpha = \pi_\alpha. \tag{48}$$

Thus $\pi_F \circ \Phi = \Psi \circ \pi_E$. Let $\alpha \in A$, $z \in B_\alpha$. Then Φ is linear from the fiber $\pi_E^{-1}(\{z\})$ to the fiber $\pi_F^{-1}(\{\Psi(z)\})$, since $(Q_\alpha^x)^{-1}$, $\Theta(\xi)^x = \tilde{\Phi}_\alpha^{-1}(x, \xi)$ and $(Q_\alpha^x)^{-1} \circ \Phi \circ \Theta = id$ are linear where $x = \tilde{\Psi}_\alpha(z)$ and the last equation follows from (47) and (48). Hence Φ is a vector bundle homomorphism. As it is bijective, it is a vector bundle isomorphism. □

The connections ∇_α, potentials V_α and the Hermitian structures can be glued together:

(G8) $(Q_\alpha^{-1})^*\nabla_\alpha = (\Phi^{-1})^*\nabla$, $(Q_\alpha^{-1})^* V_\alpha = (\Phi^{-1})^* V$ and $(Q_\alpha^{-1})^* \langle\cdot,\cdot\rangle_{\mathbb{C}^n} = (\Phi^{-1})^* \langle\cdot,\cdot\rangle_E$ on each $\pi_F^{-1}(\mathcal{U}_\alpha)$.

A proof is essentially identical with the proof of (G4) and we omit it.

To summarize, we have shown that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\Phi} & F \\ \downarrow & & \downarrow \\ M^{\text{int}} & \xrightarrow[\Psi]{} & \mathcal{X} \end{array}$$

gives an isomorphism of the structure (g, E, ∇, V) on M^{int} when $\mathcal{X}$ is equipped with the metric tensor given by the gluing (G4) and F is equipped with the connection, the potential and the Hermitian structure given by the gluing (G8). This concludes the proof of Theorem 4.4.

Let us show that Φ extends to the accessible part $\mathcal{S}$ of the boundary. If $\alpha \in A_\Gamma$, $\Gamma \in \mathcal{G}_\mathcal{S}$, corresponds to an iteration that is terminated immediately after the initial step, then we can use $B_\alpha = C_0$, where C_0 is of the form (39) and $\tilde{\tilde{\Phi}}_\alpha = \tilde{\Phi}_\Gamma|_{C_0}$. Thus $Q_\alpha^{-1} \circ \Phi|_{B_\alpha} = \tilde{\Phi}_\Gamma|_{C_0}$ extends to $C_0 \cup B_\partial(y_0, \epsilon_0)$ and

$$Q_\alpha^{-1} \circ \Phi = \phi_\Gamma, \quad \text{on } B_\partial(y_0, \epsilon_0). \tag{49}$$

4.3. **Extension to the inaccessible part of boundary.** We will give a non-constructive proof that the structure (g, E, ∇, V) is determined up to the boundary, and this will conclude the proof of Theorem 1.1. To this end, let $(M_i, g_i, E_i, \nabla_i, V_i)$, $i = 1, 2$, be two structures as in Theorem 1.1. Let $\mathcal{S}_i \subset \partial M_i$ be open and nonempty, and suppose that there is an isomorphism between the induced Hermitian vector bundles on $\mathcal{S}_i$, $i = 1, 2$,

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi} & E_2 \\ \downarrow & & \downarrow \\ \mathcal{S}_1 & \xrightarrow[\psi]{} & \mathcal{S}_2 \end{array}$$

Note that we do not assume *a priori* that ψ is an isometry.

Let us choose an open cover $\mathcal{G}_{\mathcal{S}_1}$ of $\mathcal{S}_1$ as in the proof of Theorem 4.4. Then for each $\Gamma_1 \in \mathcal{G}_{\mathcal{S}_1}$ there is a unitary trivialization

$$\begin{array}{ccc} E_1 & \xrightarrow{\phi_{\Gamma_1}} & Y_{\Gamma_1} \times \mathbb{C}^n \\ \downarrow & & \downarrow \\ \Gamma_1 & \xrightarrow[\psi_{\Gamma_1}]{} & Y_{\Gamma_1} \end{array}$$

We define $\Gamma_2 = \psi(\Gamma_1)$ and $\phi_{\Gamma_2} = \phi_{\Gamma_1} \circ \phi^{-1}$. Then $\phi_{\Gamma_2} : E_2|_{\Gamma_2} \to Y_{\Gamma_1} \times \mathbb{C}^n$ is a unitary trivialization, and, if ϕ intertwines the maps $\Lambda^{2T}_{\mathcal{S}_1}$ and $\Lambda^{2T}_{\mathcal{S}_2}$, then their representations on the respective trivializations coincide.

Theorem 4.4 implies that there is a Hermitian vector bundle $F \to \mathcal{X}$, that is equipped with a Hermitian connection $\tilde{\nabla}$ and a potential $\tilde{V}$, and whose base manifold $\mathcal{X}$ is equipped with a Riemannian metric $\tilde{g}$, such that, for both $i = 1, 2$, there is an Hermitian vector bundle isomorphism $\Phi_i : E_i|_{M_i^{\rm int}} \to F$, covering an isometry $\Psi_i : M_i^{\rm int} \to \mathcal{X}$, such that $\nabla_i = \Phi_i^* \tilde{\nabla}$ and $V_i = \Phi_i^* \tilde{V}$. Hence $\Phi_2^{-1} \circ \Phi_1$ gives an isomorphims between the structures $(g_i, E_i, \nabla_i, V_i)$ on $M_i^{\rm int}$, $i = 1, 2$.

It follows from [29] that $\Psi = \Psi_2^{-1} \circ \Psi_1$ extends smoothly to the boundary ∂M_1 and (M_i, g_i), $i = 1, 2$, are isometric via the extended Ψ. By considering the pullback bundle $\Psi^* E_2$, we can assume without loss of generality that $M_1 = M_2$. Thus the following proposition implies that also the bundle isomorphism $\Phi = \Phi_2^{-1} \circ \Phi_1$ extends smoothly to the boundary.

Proposition 4.7. *Let $E_i \to M$, $i = 1, 2$, be two Hermitian vector bundles over a smooth manifold with boundary ∂M, and let ∇_i be a Hermitian connection on E_i, $i = 1, 2$. Suppose that the exists a Hermitian vector bundle isomorphism Φ between $E_1|_{M^{\rm int}}$ and $E_2|_{M^{\rm int}}$ such that it covers the identity and that $\Phi^* \nabla_2 = \nabla_1$ on $M^{\rm int}$. Then Φ extends smoothly to ∂E_1 and the bundles and connections are isomorphic on M via the extended Φ.*

Proof. Fix a point $x \in \partial M$ and introduce coordinates

$$(x^1, \dots, x^m) \in W := [0, \varepsilon) \times (-\varepsilon, \varepsilon)^{m-1}$$

around x such that the boundary of M is given by $x^1 = 0$. Without loss of generality we may assume that the bundles E_1 and E_2 are trivial over these coordinates and that $\nabla_1 = d + A$, $\nabla_2 = d + B$. The bundle isomorphism Φ can be represented by a smooth $U(n)$-valued function $u(x^1, \dots, x^m)$ defined for $x^1 > 0$ and such that

$$B = u^{-1} du + u^{-1} A u.$$

Consider the smooth map $u_A : W \to U(n)$ uniquely defined by solving the following parallel transport equation along the curves $x^1 \mapsto (x^1, \dots, x^m)$:

$$\begin{aligned} &\frac{du_A}{dx^1} + A_{(x^1, \dots, x^m)}(\partial_{x^1}) u_A = 0, \\ &u_A(0, x^2, \dots, x^m) = Id. \end{aligned}$$

Consider a similar map $u_B : W \to U(n)$ associated to B. These two maps are convenient because, if we set

$$\tilde{A} = u_A^{-1} du_A + u_A^{-1} A u_A, \quad \tilde{B} = u_B^{-1} du_B + u_B^{-1} B u_B,$$

then $\tilde{A}(\partial_{x^1}) = \tilde{B}(\partial_{x^1}) = 0$. For $x^1 > 0$ define $v = u_A^{-1} u u_B$. Then, a simple calculation shows that

$$\tilde{B} = v^{-1} dv + v^{-1} \tilde{A} v, \quad x^1 > 0.$$

This implies $dv(\partial_{x^1}) = 0$ and the map v is independent of x^1. Hence v smoothly extends to $x^1 = 0$ and, since $u = u_A v u_B^{-1}$, u is also smooth up to the boundary $x_1 = 0$. □

In order to finish the proof of Theorem 1.1 we still need to show that $\Phi|_{\mathcal{S}_1} = \phi$. Using the coordinate systems Q_α^{-1} on F corresponding to choices α as in (49), we see that $\Phi_i = \phi_{\Gamma_i}$ on Γ_i. Thus

$$\Phi = \Phi_2^{-1} \circ \Phi_1 = \phi_{\Gamma_2}^{-1} \circ \phi_{\Gamma_1} = \phi$$

on each $\Gamma_i \in \mathcal{G}_{\mathcal{S}_1}$. This concludes the proof of Theorem 1.1.

5. Calderón problem for connections on a cylinder

The proof of Corollary 1.3 is based on a simple relation between the Dirichlet-to-Neumann map $\Lambda(\lambda)$ of the operator $-\partial_t^2 + P_0 - \lambda$ and that of the transversal operator P_0 defined analogously to $\Lambda(\lambda)$. That is, if $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ then we define

$$\Lambda_0(\lambda)h = (\nabla_0)_\nu u|_{\partial M_0}, \quad h \in C^\infty(\partial M_0; E_0),$$

where u is the solution of the equation

$$(P_0 - \lambda)u = 0 \text{ in } C, \qquad u|_{\partial C} = f.$$

We consider an L^2-space with a weight in the Euclidean direction,

$$L^2_\delta(C; E) = \{f \in L^2_{loc}(C; E);\ (1 + t^2)^{\delta/2} f \in L^2(C; E)\}, \quad \delta \in \mathbb{R},$$

and define the corresponding Sobolev spaces H^s_δ analogously to [7, Section 5]. Now we can formulate a relation between $\Lambda(\lambda)$ and $\Lambda_0(\lambda)$.

Proposition 5.1. *Let $\lambda \in \mathbb{C} \setminus [\lambda_1, \infty)$ and $\delta \in \mathbb{R}$. Then $\Lambda(\lambda)$ extends as a bounded linear map $\Lambda(\lambda) : H^{3/2}_\delta(\partial C; E) \to H^{1/2}_\delta(\partial C; E)$. Moreover, if $k \in \mathbb{R}$, then*

$$\Lambda_0(\lambda - k^2)h = e^{-kit}\Lambda(\lambda)(e^{ikt}h).$$

Note that if $h \in H^{3/2}(\partial M_0; E_0)$, then $e^{ikt}h \in H^{3/2}_\delta(\partial C; E)$ for any $\delta < -1/2$.

Proof. The proof that $\Lambda(\lambda)$ extends as claimed is analogous to the scalar case [7, Proposition 5.1] and we omit it. Let $h \in H^{3/2}(\partial M_0; E_0)$ and let $v_h \in H^2(M_0; E_0)$ solve

$$(P_0 - (\lambda - k^2))v_h = 0 \text{ in } M_0, \qquad v_h|_{\partial M_0} = h.$$

Since $\lambda \notin [\lambda_1, \infty)$, the number $\lambda - k^2$ is not a Dirichlet eigenvalue of P_0 and there is a unique solution v_h. Set $f(t, x) = e^{ikt}h(x)$ and $u(t, x) = e^{ikt}v_h(x)$. The function u is in $H^2_\delta(C; E)$ for any $\delta < -1/2$, and solves

$$(-\partial_t^2 + P_0 - \lambda)u = 0 \text{ in } C, \qquad u|_{\partial C} = f.$$

Note that $-\partial_t^2 + P_0 = \nabla^*\nabla$, where $\nabla = \pi^*\nabla_0$ and $\pi : C \to M_0$ is the canonical projection. It follows that

$$\Lambda(\lambda)f = \nabla_\nu u|_{\partial C} = e^{ikt}(\nabla_0)_\nu v_h|_{\partial M_0} = e^{ikt}\Lambda_0(\lambda - k^2)h,$$

and the proposition is proved. □

Proof of Corollary 1.3. Using that $C_0^\infty(\partial C; E)$ is dense in $H_\delta^{3/2}(\partial C; E)$ for all δ together with Proposition 5.1, we can determine the map

$$\Lambda_0(\lambda - k^2) : H^{3/2}(\partial M_0; E_0) \to H^{1/2}(\partial M_0; E_0)$$

for all $k \in \mathbb{R}$. Since $\mu \mapsto \Lambda_0(\mu)$ is a meromorphic map whose poles are contanied in $\{\lambda_1, \lambda_2, \dots\}$, see e.g. [21, Lemma 4.5], we can recover $\Lambda_0(\mu)$ for all $\mu \in \mathbb{C}$. This is equivalent to knowing the Dirichlet-to-Neumann map $\Lambda_{\partial M_0}^T$ for the wave operator $\partial_t^2 + P_0$ for any $T > 0$ [21, Chapter 4]. Thus Theorem 1.1 implies that we can recover the structure $(M_0, g_0, E_0, \nabla_0)$ as claimed. □

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