Uniqueness of Lagrangian self-expanders

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Abstract

In mean curvature flow an important class of solutions are the self-expanders, which move simply by dilations under the flow and provide models for smoothing of singular configurations. In Kähler–Einstein manifolds mean curvature flow preserves Lagrangian submanifolds, providing the notion of Lagrangian mean curvature flow. I will describe joint work with Neves [12] showing that Lagrangian self-expanders in $\mathbb{C}^n$ asymptotic to pairs of planes are locally unique if $n > 2$ and unique if $n = 2$.

1 Lagrangian mean curvature flow

Definition. A one-parameter family of immersions $\mathbf{x}_t : L \to M$ of a manifold L in a Riemannian manifold M is said to satisfy *mean curvature flow* if

$$\dot{\mathbf{x}}_t = H(\mathbf{x}_t),$$

where $H(\mathbf{x}_t)$ denotes the mean curvature vector of the immersion $\mathbf{x}_t$.

Mean curvature flow is the negative gradient flow of the area functional and its stationary points are minimal submanifolds. In the case of hypersurfaces, and particularly when L is 2-dimensional, the theory of mean curvature flow has been quite well developed, but for submanifolds of general dimension and codimension little is known and the flow does not usually respect distinguished classes of submanifolds in M. However, Smoczyk [20] showed the following.

Theorem. *If M is Kähler–Einstein then mean curvature flow preserves* Lagrangian *submanifolds: i.e. if ω is the Kähler form on M, $\mathbf{x}_t : L^n \to M^{2n}$ satisfies mean curvature flow and $\mathbf{x}_0^*(\omega) = 0$, then $\mathbf{x}_t^*(\omega) = 0$ for all $t > 0$ for which the flow exists.*

This defines the notion of *Lagrangian mean curvature flow*. When M is Calabi–Yau (so M is Kähler and Ricci-flat) the stationary points of the flow are *special Lagrangian*. It is well-known that special Lagrangian submanifolds are area-minimizing in their homology class [4], so all stationary points for the flow are minima in this case.

The natural and important open question which Lagrangian mean curvature flow seeks to address is the following.

Question. Given a Lagrangian L in a Calabi–Yau manifold M, is there a special Lagrangian representative L' of the homology or Hamiltonian isotopy class of L? Furthermore, is L' unique?

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Instead of using the flow to study this problem, one can instead take the variational approach and seek to minimize area amongst Lagrangians in a given class. Schoen and Wolfson [19] showed that when L is compact and 2-dimensional (so the ambient manifold is a Calabi–Yau 2-fold), a Lagrangian minimizer L' in the class always exists which is smooth except for finitely many singular points, but L' may not be special Lagrangian. As an indication of the inherent difficulties in the variational method, Wolfson [23] exhibited a Lagrangian 2-sphere L in a K3 surface with $[L] \neq 0$ such that the Lagrangian minimizing area in $[L]$ exist but is not special Lagrangian and the surface minimizing area in $[L]$ exists but is not Lagrangian.

Considering the flow approach to the question, Thomas and Yau [21] proposed a notion of "stability" for compact Lagrangian submanifolds in Calabi–Yau manifolds which led them to the following conjecture.

Conjecture. *Lagrangian mean curvature flow in a Calabi–Yau manifold starting from a stable Lagrangian exists for all time and converges to the unique special Lagrangian representative in its Hamiltonian isotopy class.*

However, Neves [16] has shown that any embedded Lagrangian surface in a Calabi–Yau 2-fold can be perturbed within its Hamiltonian isotopy class to a Lagrangian L so that Lagrangian mean curvature flow starting at L develops a singularity in finite time. Thus one sees that finite-time singularities of Lagrangian mean curvature flow are, in some sense, unavoidable and therefore one needs a good understanding of these singularities in order to study the flow.

2 Lagrangian self-expanders

Definition. A Lagrangian $\mathbf{x} : L^n \to \mathbb{C}^n$ is a *self-expander* if $H = \mathbf{x}^\perp$, where H is the mean curvature vector and $\mathbf{x}^\perp$ denotes the normal projection of the position vector $\mathbf{x}$ on L. This implies that $\mathbf{x}_t = \sqrt{2t}\mathbf{x}$ is a solution to Lagrangian mean curvature flow for $t > 0$; i.e. L simply dilates under the flow.

One can also define a self-expander by $H = \kappa\mathbf{x}^\perp$ for constant $\kappa > 0$, but choosing $\kappa = 1$ just amounts to a re-scaling. Lagrangian self-expanders are in some sense analogues of special Lagrangians because they are stationary points for a weighted version of the area functional.

2.1 Singularities

Self-expanders are examples of self-similar solutions to mean curvature flow, which are known to provide local models for the behaviour of the flow near singularities. Moreover, Neves and Tian [17] show that blow-downs of eternal solutions to Lagrangian mean curvature flow are self-expanders for positive time, which indicates the particular importance of self-expanders in understanding the singularities of the flow.

Mean curvature flow is essentially a heat flow so it has regularizing properties, and thus typically makes singular configurations instantaneously smooth. If we set $L_t = \sqrt{2t}L$ then, as $t \to 0$, the limit L_0 is a cone. Hence we see that self-expanders give a simple model for how the flow regularizes the singular cone L_0. This observation has potential applications for defining a notion of surgery for Lagrangian mean curvature flow.

2.2 Examples

Anciaux [1] began the systematic study of examples of Lagrangian self-expanders by classifying the $\mathrm{SO}(n)$-invariant examples in $\mathbb{C}^n$. The examples, which are topologically $\mathbb{R} \times \mathcal{S}^{n-1}$, are asymptotic to an $\mathrm{SO}(n)$-invariant pair of transverse Lagrangian planes in the following sense.

Let $P_1, P_2 \subseteq \mathbb{C}^n$ be two Lagrangian planes intersecting transversely, denote the space of bounded smooth functions with compact support by $C_0^\infty(\mathbb{C}^n)$ and let $\mathcal{H}^n$ be n-dimensional Hausdorff measure. Let $P_1 + P_2$ denote the varifold whose support is given by the union $P_1 \cup P_2$.

Definition. A self-expander L is *asymptotic* to $L_0 = P_1 + P_2$ if

$$\lim_{t\to 0} \int_{\sqrt{2t}L} \phi \,\mathrm{d}\mathcal{H}^n = \int_{L_0} \phi \,\mathrm{d}\mathcal{H}^n$$

for all $\phi \in C_0^\infty(\mathbb{C}^n)$.

An alternative approach which provides examples in [10, 11] is to search for Lagrangian self-expanders which are Hamiltonian stationary (so minimal under Hamiltonian variations). All of the aforementioned examples were generalized by Joyce, Lee and Tsui [8], yielding an array of new Lagrangian self-expanders including the following.

Theorem. *Given any pair of transverse Lagrangian planes P_1, P_2 in $\mathbb{C}^n$ such that $P_1 + P_2$ and $P_1 - P_2$ are not area-minimizing, there exists a Lagrangian self-expander $L \cong \mathbb{R} \times \mathcal{S}^{n-1}$ asymptotic to $P_1 + P_2$.*

Remark. If $P_1 + P_2$ or $P_1 - P_2$ is area-minimizing then there is a special Lagrangian called a "Lawlor neck" [9], rather than a self-expander, asymptotic to the pair of planes.

Further examples are given by Castro and Lerma [2] in their classification of Hamiltonian stationary Lagrangian self-expanders in $\mathbb{C}^2$. In addition, Chau, Chen and He [3] exhibit a one-to-one correspondence between Lagrangian cones which are graphs over a real plane and Lagrangian self-expanders which are graphs over the same plane and asymptotic to the cone, when the Hessian of the potential functions for the graphs defining the cone and the self-expander both have eigenvalues uniformly bounded in the interval $(-1, 1)$.

3 The uniqueness result

As we have seen, Lagrangian self-expanders are asymptotic to cones and model how Lagrangian mean curvature flow may smooth out the cone. It is thus natural to speculate whether, given a Lagrangian cone, one can describe the Lagrangian self-expanders asymptotic to the cone (if any). A simple example of a cone is a pair of planes, which then motivates the following question.

Question. Can we classify Lagrangian self-expanders with two planar ends?

In the minimal case, Schoen [18] gave a well-known uniqueness result for the catenoid amongst minimal hypersurfaces with two planar ends. More pertinently, special Lagrangian surfaces in $\mathbb{C}^2$ can be realised as complex surfaces after a hyperkähler rotation of the complex structure, and thus it is possible to classify the special Lagrangians in $\mathbb{C}^2$ with two planar ends. However in $\mathbb{C}^n$ for $n > 2$ a classification of special Lagrangians with two planar ends is still unknown.

For self-expanders in $\mathbb{C}$, which are trivially Lagrangian, one has a uniqueness result given a pair of asymptotic lines: this is essentially because in this case the self-expanders are curves which may be viewed as geodesics with respect to a metric with non-positive curvature. It is clear for the classification question one should restrict to smooth self-expanders since Nakahara [13] gave examples of families of singular Lagrangian self-expanders with the same two planar ends.

To achieve a classification we restrict our attention to a distinguished class of Lagrangian submanifolds, for which we require a definition.

Definition. Let $\Omega = \mathrm{d}z_1 \wedge \ldots \wedge \mathrm{d}z_n$ be the standard holomorphic volume form on $\mathbb{C}^n$. For any Lagrangian L we have that

$$\Omega|_L = e^{i\theta}\mathrm{vol}_L$$

for some function θ, which is called the *Lagrangian angle* of L. We say that L has *zero-Maslov class* if θ is a single-valued function; i.e. if the closed form $\mathrm{d}\theta$ is exact.

There is a well-known relationship between the mean curvature vector and the Lagrangian angle given by $H = J\nabla\theta$ [21, Lemma 2.1], where J is the complex structure on $\mathbb{C}^n$. This shows that special Lagrangians are Lagrangians with constant Lagrangian angle. We can also characterize Hamiltonian stationary Lagrangians as those with harmonic Lagrangian angle, so $\Delta\theta = 0$.

We can now state the main results from [12] as the following theorem.

Theorem. *Let P_1, P_2 be a pair of transverse Lagrangian planes in $\mathbb{C}^n$ such that $P_1 + P_2$ and $P_1 - P_2$ are not area-minimizing. Zero-Maslov class Lagrangian self-expanders asymptotic to $P_1 + P_2$ are*

- *locally unique if $n > 2$;*
- *unique if $n = 2$.*

By the local uniqueness we mean that given a zero-Maslov class Lagrangian self-expander L asymptotic to P_1+P_2 there is $R > 0$ and $\varepsilon > 0$ so that any smooth zero-Maslov class Lagrangian self-expander which is asymptotic to $P_1 + P_2$ and ε-close in C^2 to L in B_R coincides with L.

Remark. Currently no corresponding local uniqueness statement is known for special Lagrangians with two planar ends in $\mathbb{C}^n$ for $n > 2$.

Since the asymptotically planar Lagrangian self-expanders in [8] have zero-Maslov class our result shows that these are the only zero-Maslov class Lagrangian self-expanders with two planar ends in $\mathbb{C}^2$.

Remark. As we already observed, we obtain a uniqueness result for special Lagrangians in $\mathbb{C}^2$ with two planar ends by viewing them as complex surfaces after a hyperkähler rotation. No such characterization is known for Lagrangian self-expanders so new ideas are required to prove our result in [12].

4 Sketch proof

The details of the proof of the main result are provided in [12], but we give an outline of the argument here.

4.1 Strategy

We shall focus attention on the case of zero-Maslov class Lagrangian self-expanders L in $\mathbb{C}^2$ asymptotic to a transverse pair of Lagrangian planes, which we denote by L_0. We begin with the following important observation.

Lemma. *If L_0 is $\mathrm{SO}(2)$-invariant then L is $\mathrm{SO}(2)$-invariant, and hence L is the unique zero-Maslov class Lagrangian self-expander asymptotic to L_0.*

Proof. If we let $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ be complex coordinates on $\mathbb{C}^2$ then L is $\mathrm{SO}(2)$-invariant if and only if the moment map $\mu = x_1y_2 - x_2y_1$ for the $\mathrm{SO}(2)$ action vanishes on L (see [16, Lemma 7.1] for example).

We have the following evolution equation for μ^2 along $L_t = \sqrt{2t}L$ [15, Lemma 3.3]:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu^2 = \Delta\mu^2 - 2|\nabla\mu|^2.$$

From Huisken's monotonicity formula [5] it essentially follows that μ^2 is decreasing in t (more precisely, the integral over L_t of μ^2 multiplied by the backwards heat kernel is decreasing). However, $\mu^2 = 0$ on L_0 as it is SO(2)-invariant and hence $\mu^2 = 0$ on L_t for all $t > 0$.

The uniqueness now follows from the classification of SO(2)-invariant zero-Maslov class Lagrangian self-expanders [1]. □

We can now describe our strategy for the proof of the uniqueness result.

- Choose a path $L_0(s)$ of transverse Lagrangian planes for $s \in [0,1]$ such that $L_0(0) = L_0$, $L_0(1)$ is SO(2)-invariant and the difference of the Lagrangian angles of the planes in $L_0(s)$ remains constant. This is possible for any L_0 and ensures that $L_0(s)$ is not area-minimizing.
- Let $\mathcal{S}$ denote the set of zero-Maslov class Lagrangian self-expanders asymptotic to $L_0(s)$ for any $s \in [0,1]$. We show that $\mathcal{S}$ is *compact*.
- We study the *deformation theory* of zero-Maslov class Lagrangian self-expanders with two planar ends to give our local uniqueness result. We deduce that we have a local diffeomorphism $\pi : \mathcal{S} \to [0,1]$ which maps a self-expander L to s where $L_0(s)$ is the pair of asymptotic planes of L.
- Since $\mathcal{S}$ and $[0,1]$ are compact, π is a covering map. Moreover, $\pi^{-1}(1)$ consists of one element by our lemma above because $L_0(1)$ is SO(2)-invariant. Hence π is a diffeomorphism and the uniqueness result is proved.

4.2 Local uniqueness

On $\mathbb{C}^n$ with coordinates $z_j = x_j + iy_j$ for $j = 1,\ldots,n$ we have the Liouville form

$$\lambda = \sum_{j=1}^{n} x_j \mathrm{d}y_j - y_j \mathrm{d}x_j$$

which satisfies $\mathrm{d}\lambda = 2\omega$. Since ω vanishes on Lagrangians, the Liouville form is closed when restricted to Lagrangian submanifolds. Examples of closed forms are given by exact forms, which motivates the following definition.

Definition. A Lagrangian L in $\mathbb{C}^n$ is *exact* if $\lambda|_L = \mathrm{d}\beta$ for some function β.

Lemma. *Zero-Maslov class Lagrangian self-expanders are exact.*

Proof. Since $H = J\nabla\theta$ and $H = \mathbf{x}^\perp$ we see that $\nabla\theta = -J\mathbf{x}^\perp = -(J\mathbf{x})^{\mathrm{T}}$, where $(J\mathbf{x})^{\mathrm{T}}$ denotes the tangential projection of $J\mathbf{x}$. This is equivalent to $\lambda|_L = -\mathrm{d}\theta$ so the result follows by choosing $\beta = -\theta$ as the self-expander has zero-Maslov class. □

We now give a rough outline of the proof of the local uniqueness result for Lagrangian self-expanders L with two planar ends.

- Using a generalization of the Lagrangian neighbourhood theorem, we can define a one-to-one correspondence between exact zero-Maslov class Lagrangians near L and graphs L_u of $J\nabla u$, where u is a function on L. All zero-Maslov class Lagrangian self-expanders near L are necessarily of the form L_u for some u by the above lemma.

- We can determine whether L_u is a self-expander by the condition that $F(u) = \beta_u + \theta_u$ is constant, where $\mathrm{d}\beta_u = \lambda|_{L_u}$ and θ_u is the Lagrangian angle of L_u.
- The linearization of F at 0 is given by:

$$\mathrm{d}F|_0(u) = \Delta u + \langle \mathbf{x}, \nabla u \rangle - 2u.$$

  We show that $\mathrm{d}F|_0$ is an isomorphism between suitable Banach spaces. Heuristically, this is because Δ is a non-positive operator, so $\Delta - 2$ is strictly negative and dominates the term $\langle \mathbf{x}, \nabla u \rangle$ because we show that L converges exponentially to L_0.
- Applying the Inverse Function Theorem shows that F is invertible near 0, and we deduce our local uniqueness result.

Remark. We show that Lagrangian self-expanders converge exponentially to their asymptotic planes so as to deduce the local uniqueness result. This is not true of special Lagrangian Lawlor necks in $\mathbb{C}^n$, which only converge with order $O(r^{1-n})$ to their asymptotic planes.

4.3 Compactness

This is the heart of the argument giving our uniqueness result and contains the key technical work. Recall that $\mathcal{S}$ is the set of zero-Maslov class Lagrangian self-expanders asymptotic to some $L_0(s)$ in a path of transverse pairs of Lagrangian planes. We want to show that $\mathcal{S}$ is compact.

- Let L^j be a sequence of zero-Maslov class Lagrangian self-expanders in $\mathcal{S}$ asymptotic to pairs of planes L_0^j.
- Using a result of Ilmanen [6, Theorem 7.1] allows us to deduce the existence of a subsequence, which we also denote by L^j, converging to a integral varifold L which is a Lagrangian self-expander.
- Moreover, possibly after taking a further subsequence, the pairs of planes L_0^j converge to a pair of transverse planes L_0 and L is asymptotic to L_0.
- It therefore suffices to show that L is a smooth self-expander to deduce that $\mathcal{S}$ is compact.

The first issue is that the limit L might be minimal, so we rule this out.

Lemma. *L is not a stationary varifold.*

Proof. Suppose for a contradiction that $H = 0$. Since $H = \mathbf{x}^\perp = 0$, we see that L is a cone. Moreover, L is asymptotic to L_0, so $L = L_0$.

Results by Neves [14] imply the existence of a constant β so that, essentially, $\beta^j \to \beta$ as $j \to \infty$ where $\lambda|_{L^j} = \mathrm{d}\beta^j$. Since $\theta^j = -\beta^j \to -\beta$ constant, we deduce that $L = L_0$ has constant Lagrangian angle and thus must be special Lagrangian.

However, we assumed that L_0^j was not area-minimizing for all j so L_0 is not area-minimizing, which gives our contradiction. □

Definition. We define the *Gaussian density*

$$\Theta(y, l) = \int_L (4\pi l)^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{4l}\right) \mathrm{d}\mathcal{H}^n$$

for $y \in \mathbb{C}^n$ and $l > 0$.

If L is a multiplicity one plane then $\Theta(y,l) \leq 1$ and equals 1 if and only if $y \in L$. Hence, $\lim_{l\to 0} \Theta(y,l) = 1$ if y is a smooth point on L. However, if L is a pair of transverse planes we have $\Theta(y,l) \leq 2$ with equality if and only if y is the intersection point of the planes.

Thus, intuitively, if we can make $\Theta(y,l)$ close to 1 for all y and all sufficiently small l then L must be "close" to being smooth. We can then apply a regularity result of White [22] to deduce that L is in fact smooth. This is what we do in $\mathbb{C}^2$.

Lemma. *Given $\varepsilon > 0$ there exists $\delta > 0$ such that $\Theta(y,l) < 1+\varepsilon$ for all $y \in \mathbb{C}^2$ and $l \leq \delta$.*

Proof. We break up the proof into two steps.

First step. Since L is asymptotic to the transverse pair of planes L_0, the monotonicity formula for Brakke flows [7, Lemma 7] implies that $\Theta(y,l) \leq 2$.

If $\Theta(y_j,l_j) \to 2$ for some sequence y_j, l_j then it follows again from the monotonicity formula that L must be a self-shrinker, i.e. $H = -\kappa \mathbf{x}^\perp$ for some $\kappa > 0$. Since L is a self-expander, we deduce that $H = 0$, which contradicts our previous lemma that L is not stationary.

Hence there exists $\varepsilon_0 > 0$ such that $\Theta(y,l) \leq 2 - 2\varepsilon_0$ for all y, l.

Second step. Suppose for a contradiction that there exists a sequence of $y_k \in \mathbb{C}^2$ and $\delta_k \to 0$ such that $\Theta(y_k,\delta_k) \geq 1+\varepsilon$.

Using the points y_k and the scales δ_k we can define a blow-up sequence which converges to a stationary varifold $\tilde{L}$, which cannot be a multiplicity one plane by the assumption on $\Theta(y_k,\delta_k)$. The blow-down C of $\tilde{L}$ is a special Lagrangian cone with density strictly greater than 1 at the origin, which thus means that C cannot be a multiplicity one plane either.

However, in $\mathbb{C}^2$, the only special Lagrangian cones are unions of planes and hence the density of C at the origin must be at least 2. (By contrast, in $\mathbb{C}^n$ for $n > 2$ there exist special Lagrangian cones with density at the origin in the interval $(1,2)$.) We deduce the existence of l such that $\Theta(y_k, l\delta_k) \to 2 - \varepsilon_0$ as $k \to \infty$. This contradicts the first step and the lemma follows. ☐

Using this lemma we can apply White's Regularity Theorem [22] to deduce that L is a smooth self-expander and the compactness of $\mathcal{S}$ is proved.

Remark. We see that it is in the compactness result that the assumption that the Lagrangian self-expander is a surface is crucial in order to prove uniqueness.

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