Reconstruction and stability in Gel'fand's inverse interior spectral problem

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Abstract

Assume that M is a compact Riemannian manifold of bounded geometry given by restrictions on its diameter, Ricci curvature and injectivity radius. Assume we are given, with some error, the first eigenvalues of the Laplacian Δ on M as well as the corresponding eigenfunctions restricted on an open set in M. We then construct a stable approximation to the manifold (M, g). Namely, we construct a metric space and a Riemannian manifold which differ, in a proper sense, just a little from M when the above data are given with a small error. We give an explicit logarithmic stability estimate on how the constructed manifold and the metric on it depend on the errors in the given data. Moreover a similar stability estimate is derived for the Gel'fand's inverse problem. The proof is based on methods from geometric convergence, a quantitative stability estimate for the unique continuation and a new version of the geometric Boundary Control method.

1 Introduction

1.1 Inverse interior spectral data

Let (M, g, p) be a pointed compact Riemannian manifold, that is, (M, g) is a compact Riemannian manifold without boundary and $p \in M$ is a point of the manifold. We denote by $\mathrm{inj}_M(p)$ the injectivity radius of (M, g) at p. Also, by $\mathbf{M}^a_{n,p}$, where $n \in \mathbb{Z}_+$, $n \geq 2$, $a > 0$, we denote the collection of all pointed compact manifolds of dimension n such that $\mathrm{inj}_M(p) \geq a$.

Let Δ_g be the Laplace operator on (M, g), where we use the sign convention where $-\Delta_g$ is a non-negative definite operator.

Let φ_j, $j = 0, 1, 2, \ldots$ be the complete sequence of $L^2(M)$-orthonormal eigenfunctions of $-\Delta_g$ and let λ_j be the corresponding eigenvalues. The eigenvalues and the eigenfunctions are enumerated so that $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ Note that $\varphi_0(x) = \mathrm{vol}\,(M)^{-1/2}$, where vol stands for the volume on (M, g).

For $r_0 > 0$, let $B(p, r_0) \subset M$ be an open ball of radius r_0 centered at p. When $r_0 < \mathrm{inj}_M(p)$, in the ball $B(p, r_0)$ there are Riemannian normal coordinates $X : B(p, r_0) \to B_e(0, r_0)$, where $B_e(r_0) = B_e(0, r_0) \subset \mathbb{R}^n$ is the Euclidean ball of $\mathbb{R}^n$ of radius r_0, centered at zero. When there is no danger of misunderstanding, we use the coordinate map X to identify the metric ball $B(p, r_0)$ on the manifold and the Euclidean ball $B_e(r_0)$ on the coordinate chart. Also, we identify the metric tensor g on $B(p, r_0) \subset M$ and the corresponding tensor $X_* g$ on $B_e(r_0) \subset \mathbb{R}^n$ and denote this metric tensor by $g(x)$ or $g_{ij}(x)$, $x \in B_e(r_0)$. Also we identify the restrictions of the eigenfunctions φ_j to $B(p, r_0) \subset M$ with $X_* \varphi_j$ on $B_e(r_0) \subset \mathbb{R}^n$. We say that a metric tensor $g_{jk}(x)$ on $B_e(r_0)$ is a Riemannian metric in normal coordinates, if $g_{jk}(x) x^j x^k = |x|^2 = \sum_{j=1}^n (x^j)^2$, $x \in \mathbb{R}^n$, and $g_{jk}(x) x^j \xi^k = 0$, if $x \cdot \xi = \sum_{j=1}^n x^j \xi^j = 0$.

Definition 1 *Let* $(M, g, p) \in \mathbf{M}^a_{n,p}$*. Then*

(i) The pair, consisting of the ball $(B_e(r_0), g|_{B_e(r_0)})$ *on the Riemannian manifold* M *and the sequence* $\{(\lambda_j,\, \varphi_j|_{B_e(r_0)});\ j = 0, 1, 2, \ldots\}$ *of eigenvalues and eigenfunctions, is called the interior spectral data (ISD) of* (M, g, p)*.*

(ii) The pair, consisting of the ball $(B_e(r_0), g|_{B_e(r_0)})$ *and a finite collection* $\{(\lambda_j,\, \varphi_j|_{B_e(r_0)}),\ j = 0, 1, 2, \ldots, J_0\}$ *of the* $J_0 + 1$ *first eigenvalues and eigenfunctions, is called the finite interior spectral data (FISD) of* (M, g, p)*.*

In this paper we consider the problem of an approximate reconstruction of (M, g, p) when we know only its FISD, namely, the first eigenvalues, $\lambda_j < \delta^{-1}$ with some small $\delta \in (0, 1)$ and the values of $\varphi_j|_{B(p, r_0)}$ of the corresponding eigenfunctions φ_j. Furthermore, we assume that we know all these objects with some error.

To formalise the above, let $\mathbf{B}$ be a collection of elements

$$D = (\,(B_e(r_0), h)\,,\ \{(\mu_j,\ \psi_j|_{B_e(r_0)})\}_{j=0}^{\infty}\,) \tag{1}$$

where $B_e(r_0)$ is a ball of $\mathbb{R}^n$ with centre 0 and radius r_0, h is a Riemannian metric on $B_e(r_0)$ in normal coordinates, and the pairs $(\mu_j,\, \psi_j|_{B_e(r_0)}) \in \mathbb{R} \times L^2(B_e(r_0))$ are such that $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \dots$. Also, for $J_0 \in \mathbb{N}$ we define a cut-off operator P_{J_0} that maps the data element D of the form (1) to

$$P_{J_0}(D) = \left((B_e(r_0), h)\, ,\, \{(\mu_j,\, \psi_j|_{B_e(r_0)})\}_{j=0}^{J_0} \right)$$

In particular, P_{J_0} maps the ISD of manifold (M, g) to the FISD of length J_0. Let $\mathbf{B}_{J_0} = P_{J_0}(\mathbf{B})$.

We start with introducing a proper topology on the set $\mathbf{B}$.

Definition 2 *(Interior spectral topology.) Let $\delta > 0$. For $i = 1, 2$, consider the collections*

$$\left((B_e(r_0), h_i)\, ,\, \{(\mu_j^i,\, \psi_j^i)\}_{j=0}^{J_0^i} \right) \in \mathbf{B}_{J_0^i},$$

where $J_0^i \in \mathbb{Z}_+ \cup \{\infty\}$.

We say that these two collection are δ-close if the following is valid: There are disjoint intervals

$$I_p = (a_p, b_p) \subset (-\delta,\ \delta^{-1} + \delta), \quad p = 0, 1, \dots, P,$$

such that

i) $b_p - a_p < \delta$.

ii) For any μ_j^i, $i = 1, 2$ with $|\mu_j^i| < \delta^{-1}$ there is p such that $\mu_j^i \in I_p$.

iii) For $p = 0$, $n_0^i = 1$. For any $p \geq 1$, the total number n_p^i of elements in sets $\mathcal{J}_p^i = \{j \in \mathbb{Z}_+;\ \mu_j^i \in I_p\}$ coincide, i.e. $n_p^1 = n_p^2\,(= n_p)$.

iv) There is an orthogonal matrix $O \in O(n)$, defining a transformation $O : B_e(r_0) \to B_e(r_0)$, such that the metrics $O_* h_1$ and h_2 are Lipschitz δ-close on $B_e(r_0)$, i.e., for any $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$, $\xi \neq 0$,

$$(1 + \delta)^{-1} \leq \frac{(O_* h_1)_{jk}(x)\, \xi^j \xi^k}{(h_2)_{jk}(x)\, \xi^j \xi^k} \leq 1 + \delta, \tag{2}$$

v) For any p there is a unitary matrix

$$A_p = \left[a_{jk}^{(p)} \right]_{j,k \in \mathcal{J}_p} \in U(n_p),$$

such that

$$\|A_p\cdot(O_*\Psi_p^1)-\Psi_p^2\|_{(L^2(B_e(r_0),h_2))^{n_p}}\leq\delta, \tag{3}$$
$$\|A_p^{-1}\cdot((O^{-1})_*\Psi_p^2)-\Psi_p^1\|_{(L^2(B_e(r_0),h_1))^{n_p}}\leq\delta. \tag{4}$$

Here, Ψ_p^i *is the vector-function* $\{\psi_j\}_{j\in\mathcal{J}_p^i}$.

Above, (3) can be written also as

$$\left(\sum_{\kappa=1}^{n_p}\int_{L^2(B_e(r_0))}\left|\sum_{\ell=1}^{n_p}a^{(p)}_{j(\kappa),j(\ell)}\psi^1_{j(\ell)}(O^{-1}x)-\psi^2_{j(\kappa)}(x)\right|^2(\det(h_2(x)))^{\frac{1}{2}}dx\right)^{\frac{1}{2}}\leq\delta.$$

Remark 1 An appearance of the matrix O is due to the fact that Riemannian normal coordinates associated with p are defined up to an orthogonal transformation.

The appearance of the unitary matrices A_p is due to the possibility of a non-continuity of the eigenfunctions with respect to small perturbations of an operator which has an eigenvalue of multiplicity higher than one. However, in one considers clusters of eigenvalues, the Riesz projection onto the span of eigenfunctions corresponding to all eigenvalues in a cluster of eigenvalues is stable in small perturbations.

A more detailed analysis to eigenvalues and eigenvectors is presented in [39]. In particular, consider a compact Riemannian manifold, an interval $I=[a,b]$ such that $a,b\notin\sigma(-\Delta_g)$ and metric tensors g' on the manifold M. Let $P_{I,g}$ be the orthogonal projector, in $L^2(M,g)$, onto the space spanned the eigenvectors of $-\Delta_g$ corresponding to the eigenvalues in the interval I. Then it follows from Theorems IV.3.16 and VI.5.12 of [39] that if $\|g'-g\|_{L^\infty(M)}$ goes to zero, then the eigenvalues of $-\Delta_{g'}$ converge to those of $-\Delta_{g'}$ and the eigenprojectors satisfy $\|P_{I,g'}-P_{I,g}\|_{L^2(M,g)\to L^2(M,g)}\to 0$. This implies that the ISD of (M,g') converges to the ISD of (M,g).

We note that in a more restricted context of Gelfand's inverse problem for a Schrödinger operator with simple spectrum in a domain in $\mathbb{R}^n$ a similar topology was introduced by Alessandrini in [1], [2] who studied stability of the corresponding inverse problem.

1.2 Manifolds of bounded geometry and the main result

In the future we consider the following class of pointed manifolds.

Definition 3 *(Riemannian manifolds of bounded geometry). For any $n \in \mathbb{Z}_+$ and $R > 0$, $D > 0$, $i_0 > 0$, $\mathbf{M}_{n,p}(R, D, i_0)$ consists of n-dimensional pointed compact Riemannian manifolds (M, g, p) such that*

$$\begin{aligned} &i) \sum_{j=0}^{3} \|\nabla^j Ric(M,g)\|_{L^\infty(M)} \leq R, \\ &ii) \; diam\,(M,g) \leq D, \\ &iii) \; inj(M,g) \geq i_0. \end{aligned} \tag{5}$$

Here $Ric(M,g)$ stands for the Ricci curvature of M, $diam\,(M,g)$ for the diameter of M, and $inj(M,g)$ for the injectivity radius of (M,g). At last, ∇ stands for the covariant derivative on (M,g).

Below we assume that $i_0 \geq a$ so that $\mathbf{M}_{n,p}(R, D, i_0) \subset \mathbf{M}^a_{n,p}$. Then we can compare ISD of any two manifolds $M^i \in \mathbf{M}_{n,p}(R, D, i_0)$ using Definition 2.

To proceed we recall the notion of the Gromov-Hausdorff distance.

Definition 4 *(GH-topology). Let (X^i, d^i, p^i), $i = 1, 2$ be pointed compact metric spaces. Then the pointed Gromov-Hausdorff distance $d_{GH}((X^1, d^1, p^1),\, (X^2, d^2, p^2))$ is the infimum of all $\varepsilon > 0$ such that there is a metric space (Z, d_Z) and isometric embeddings $\mathbf{i}_1 : X^1 \to Z$ and $\mathbf{i}_2 : X^2 \to Z$ which satisfy*

$$d_H(\mathbf{i}_1(X^1),\, \mathbf{i}_2(X^2)) < \varepsilon, \quad d_Z(\mathbf{i}_1(p^1),\, \mathbf{i}_2(p^2)) < \varepsilon.$$

Here d_H denotes the Hausdorff distance in Z.

Our main result states a stability estimate for an approximate reconstruction of a Riemannian manifold from its interior spectral data.

Theorem 1 *Let $n \in \mathbb{Z}_+$, R, D, i_0 and $r_0 \in \left(0, \min\left(\frac{i_0}{2}, \frac{\pi}{2\sqrt{K}}\right)\right)$ be given. Let $(M, g, p) \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$. Then there exists a constant $\delta^* = \delta^*(n, R, D, i_0, r_0)$ such that, for all δ with*

$$0 < \delta \leq \delta^*, \tag{6}$$

the following is true:
We get a constant $C_{50} = C_{50}(\delta, n, R, D, i_0, r_0)$ and assume that we are given a collection

$$\big((B_e(r_0), g^a)\,,\, \{(\mu_j,\, \varphi^a_j);\, j = 0, 1, 2, \ldots, J_0\} \big) \quad \text{with } J_0 \geq C_{50},$$

that is δ*-close to the interior spectral data* $((B(p,r_0),g)\,,\,\{(\lambda_j,\,\varphi_j);\,j=0,1,2,\dots\}),$ *of the operator* $-\Delta_g$ *on* (M,g,p)*.*
Then we can construct a metric space (M^*, d_M^*) *such that*

$$d_{GH}(M,\,M^*) \le \frac{C_{43}}{\left(\ln\left(\ln\frac{1}{\delta}\right)\right)^{1/(72nC_{46})}}. \tag{7}$$

The coefficients $\delta^*, C_{43}, C_{46}, C_{50}$ *are calculated in the proof.*

The above inequality (7) combined with the sectional curvature bound (15) below and the solution of the geometric Whitney problem [28, Thm. 1, Cor. 1.9] implies the following stable construction result for the manifold M in the Lipschitz topology. Recall that the for C^1-diffeomorphic manifolds M_1 and M_2 the Lipschitz distance $d_L(M_1,M_2)$ is

$$d_L(M_1,M_2) = \inf_{F:M_1\to M_2}\Bigg(\ln(\mathrm{Lip}(F)) + \ln(\mathrm{Lip}(F^{-1}))\Bigg)$$

where the infimum is taken over bi-Lipschitz maps $F: M_1 \to M_2$ and $\mathrm{Lip}(F)$ is the Lipschitz-constant of the map F, see [32].

Corollary 1 *There are uniform constants* $K, C_0 > 1$ *satisfying the following. Let* $(M,g,p) \in \overline{\mathbf{M}_{n,p}(R,D,i_0)}$*,* $\delta > 0$*, and the metric space* M^* *be as in Theorem 1. Using* M^* *one can construct a smooth Riemannian manifold* (N, g_N) *that approximates the original manifold* (M,g) *so that the following holds:*
The manifolds M *and* N *are diffeomorphic and their Lipschitz distance satisfies the inequality*

$$d_L(M,N) \le C_0K^{1/3}\sigma_0^{2/3}, \quad \sigma_0 = C_{43}\big(\ln\big(\ln\frac{1}{\delta}\big)\big)^{-1/(72nC_{46})},$$

where $|Sec(N)| \le C_0K$*.*
Moreover, the injectivity radius of the manifold N *is such that*

$$inj(N) \ge \min\{(C_0K)^{-1/2}, (1 - C_0K^{1/3}\sigma_0^{2/3})\,inj(M)\}.$$

Another consequence of (7) is the following stability estimate for the solutions of the interior spectral problem.

Corollary 2 *Assume that $M^{(1)}$ and $M^{(2)}$ are two manifolds with $M^{(i)} \in \mathbf{M}_{n,p}(R, D, i_0, r_0)$, $i = 1, 2$, and such that the ISD of $M^{(1)}$ and $M^{(2)}$ are $\delta-$close with $\delta \in (0, exp(-e)]$.*
Then,

$$d_{GH}(M^{(1)}, M^{(2)}) \leq \frac{C_{84}}{\left(\ln\left(\ln\frac{1}{\delta}\right)\right)^{1/(72nC_{46})}}. \tag{8}$$

Notations: Here K is the uniform bound for the sectional curvature on $\mathbf{M}_{n,p}(R, D, i_0)$, see (15).
Here and later we will use notations c, C, C_1, etc. for the constants that depend only on n, R, D, i_0, and the radius r_0. We call such constants uniform constants. When the constants depend also on other parameters we will indicate this dependence explicitly. Given a set A we denote by $\chi_A(\cdot)$ its characteristic function (except for the Appendix, where $\chi_A(\cdot)$ is a smooth cut-off).

1.3 Earlier results and outline of the paper

The Gel'fand inverse problem, formulated by I. M. Gel'fand in 50's [31], is the problem of determining the coefficients of a second order elliptic differential operator in a domain $\Omega \subset \mathbb{R}^n$ from the boundary spectral data, that is, the eigenvalues and the boundary values of the eigenfunction of the operator. In the geometric Gel'fand inverse problem, a Riemannian manifold with boundary and a metric tensor on it need to be constructed from similar data. For Neumann boundary value problem for the operator $-\Delta_g$ on manifold M, the boundary spectral data consists of the boundary ∂M, the eigenvalues λ_j and the boundary values of the eigenfunction, $\varphi_j|_{\partial M}$, $j = 1, 2, \ldots$ The uniqueness of the solution of the Gel'fand inverse problem has been considered in [9, 11, 44, 45, 58, 37, 56]. The boundary spectral data is equivalent to the Neumann-to-Dirichlet map for the wave equation or the heat equation on the manifold (M, g), see [38]. The Gel'fand inverse problem on manifolds is closely related to the geometric Calderon's inverse problem [19] of determining the manifold (M, g) when one is given the Neumann-to-Dirichlet map for the elliptic operator $-\Delta_g$. In general setting, this problem is open but partial results have been given e.g. in Euclidean domains in [7, 8, 42, 55, 59, 63, 68] and on manifolds in [26, 27, 33, 50, 51, 54].

To formulate properly stability of the inverse problems, let us consider first the Gel'fand inverse on a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$ and a conformally Euclidian metric $g_{jk}(x) = \rho(x)^{-2}\delta_{jk}$. Here, $\rho(x) > 0$ is a smooth real valued function. Then the problem has the form

$$\left(-\sum_{k=1}^{2} \rho(x)^2 \left(\frac{\partial}{\partial x^k}\right)^2 - \lambda_j \right) \varphi_j(x) = 0, \quad \text{in } \Omega, \tag{9}$$
$$\partial_\nu \varphi_j|_{\partial\Omega} = 0.$$

The problem of determining $\rho(x)$ from the boundary spectral data is ill-posed in sense of Hadamard: The map from the boundary data to the coefficient $\rho(x)$ is not continuous so that small change in the data can lead to huge errors in the reconstructed function $\rho(x)$. Clearly this results in serious instability when solving inverse problems numerically in various applications. One way out of this fundamental difficulty is to assume a priori higher regularity of coefficients, that is a widely used trend in inverse problems for isotropic equations, like (9). This type of results is called conditional stability results. For example, assuming in (9) that ρ_1 and ρ_2 are bounded in some norm, $\max(\|\rho_1\|_{C^m(\overline{\Omega})}, \ \|\rho_2\|_{C^m(\overline{\Omega})}) \leq c_0$ one can conclude that if the boundary spectral data for ρ_1 and ρ_2 are "close", then $\|\rho_1 - \rho_2\|_{C^k(\overline{\Omega})}$ is small with some $k < m$ (see e.g. [1, 3, 41, 65] etc).

For inverse problems for general metric this approach bears significant difficulties. The reason is that the usual C^k norm bounds of coefficients are not invariant and thus this condition does not suit the invariance of the problem with respect to diffeomorphisms. Moreover, if the structure of the manifold M is not known a priori, the traditional approach is useless. The way to overcome these difficulties is to impose a priori constrains in an invariant form and consider a class of manifolds that satisfy a priori bounds similar to (5), for instance for curvature, second fundamental form, radii of injectivity, etc. Under such kind of conditions, invariant stability results for inverse problems have be proven in [6, 28, 37, 52, 65]. In particular, for the Gel'fand inverse problem for manifolds with non-trivial topology, an abstract, i.e., a non-quantitative stability result was proven in [6]. There, it was shown that the convergence of the boundary spectral data implies the convergence of the manifolds with respect to the Gromov-Hausdorff convergence. However, this result was based on compactness arguments and it did not provide any estimates. In this paper our aim is to improve this result and to give explicit estimates for an analogous inverse problem.

In this paper we consider a Gel'fand inverse problem for manifolds without boundary. Then, as explained above, instead of assuming that the boundary and the boundary values of the eigenfunctions are known we assume that we are given a small open ball $U \subset M$ and the eigenfunctions φ_j are known on this set. Similar type of formulation of the problem with measurements on open sets have been considered in [22, 23, 24, 34, 46]. We show that the ISD, that is, an open set $U \subset M$, the eigenvalues λ_j and the restrictions of the eigenfunctions $\varphi_j|_U$ determine the whole manifold (M, g) in stable way. Also, we quantify this stability by giving explicit inequalities under a priori assumptions on the geometry of M. We emphasise that we assume that the eigenfunctions are known only on an open subset U of M that may be chosen to be arbitrarily small but still e.g. the topology of M is determined in a stable way. We note that in spectral geometry a similar stability problem was studied in [13, 35, 36]. Here the authors assumed that the knowledge of the heat kernel on the whole manifold, however, these data is equivalent to knowing the eigenvalues and the eigenfunctions and the eigenfunctions on the whole manifold.

We now turn to a brief description of the basic techniques utilised in this paper. The fundamental method underlying the reconstruction procedure is the Boundary Control (BC) method in its geometric version using the distance functions, see [9, 12, 37]. To consider the uniqueness of the inverse problem, one assumes that a complete set of interior spectral data [47] is given, see also [46, 34] and references therein. Using these data one can determine the image of the interior distance map

$$R_M : M \to L^\infty(U_0), \quad R_M(x) = r_{M,x},$$

where $U_0 = B(p, r_0/25) \subset U = B(p, r_0)$ and

$$r_{M,x}(z) = d_M(x, z), \quad \text{for } z \in U_0.$$

When $R_M(M)$ is equipped with the L^∞-distance, $R_M(M)$ is homeomorphic to M. However, in the general case it is not isometric to M. To make $R_M(M)$ isometric to M, we need to do additional constructions for construct an appropriate structure of Riemannian manifold on $R_M(M)$.

Although we deal with data on eigenvalues and eigenfunctions, the BC-method applies to hyperbolic equations. The BC-method is based on two tools: the Tataru-type Carleman estimates [66, 67], see also [16, 17, 53], for the wave equation and the Blagovestchenskii identity [15] that gives the Fourier coefficients of the waves generated by sources supported in $U \times \mathbb{R}_+$. A combination of these makes it possible to construct the set of the Fourier coefficients of all functions supported in metric balls of an arbitrary radius $R > 0$

with centres in $z \in U_0$, $B_M(z,R)$. Similarly, we can find the set $\mathcal{F}_S \subset \ell^2$ of the Fourier coefficients of functions in $L^2(S) = \{u \in L^2(M);\ \mathrm{supp}\ (u) \subset S\}$. Here S is a finite intersection of the *slices* $B(z_\ell, \alpha_\ell^+) \setminus B(z_\ell, \alpha_\ell^-)$, $\alpha_\ell^+ > \alpha_\ell^- \geq 0$. Here z_ℓ, $\ell = 1, \dots, O(\tau^{-n})$, see (36), form τ-net in U_0. In particular, the norm of the closest element in $\mathcal{F}_S$ to the element $(1,0,0,\dots) \in \ell^2$ is equal to the norm of the projection of the zeroth eigenfunction, $\varphi_0(x) = \mathrm{vol}^{-1/2}(M)$ to $L^2(S)$. By computing such norms we can evaluate the volume of S.
When the ISD or FIS is given with errors, we introduce in this paper a new slicing method in the reconstruction of a manifold that is robust in presence of errors. Moreover, we estimate the errors in the constructions by using a quantitative version of Tataru's unique continuation theorem. These give us a discrete approximation R_* to $R_M(M) \subset L^\infty(U_0)$. We then show that the Hausdorff distance $d_H(R_*, R_M(M))$ is small, with a quantitative estimate of this distance. At last, applying a properly modified result of [40], see also [29], we obtain Theorem 1.

2 Geometric preliminaries

2.1 Properties of the manifolds of bounded geometry

Below, we define the norm of the space $C^k(M)$ invariantly by

$$\|f\|_{C^k(M)} := \sum_{j=0}^{k} \max_{x \in M} \|\nabla^j f(x)\|_g. \tag{10}$$

We use the notation $C^k(M;E)$ for the space of C^k-smooth sections of a bundle $\pi : E \to M$ and often use the short hand notation $C^k(M;E) = C^k(M)$ for C^k-smooth sections of E or tensors fields when the bundle E is clear from the context. Below, we denote balls on manifold M by $B(x,r) = B_M(x,r)$.

Also, we define the Hölder spaces $C^{k,\beta}(M) = C_*^{k+\beta}(M)$ and the Zygmund spaces $C_*^k(M)$ by interpolation [14] of function spaces $C^k(M)$, that is, $C_*^s(M) = [C^{k_1}(M), C^{k_2}(M)]_\theta$, for $s = \theta k_1 + (1-\theta)k_2 \in \mathbb{R}_+$ and $\theta \in (0,1)$. Note that $C^{k-1,1}(M) \subset C_*^k(M) \subset C^{k-1,\beta}(M)$ for $0 < \beta < 1$.

We say that $\{x_j;\ j = 1,2,\dots,J\} \subset M$ is an ε-net if, for every $x \in M$, there is x_j such that $d(x_j, x) \leq \varepsilon$. Also, a set $\{y_j;\ j = 1,2,\dots,J\} \subset M$ is τ-separated if $d(y_j, y_k) \geq \tau$ when $j \neq k$. Note that a maximal τ-separated set is a τ-net. Using ε-nets one can define the Gromov-Hausdorff topology as follows:

Let $\varepsilon > 0$ and consider pointed compact Riemannian manifolds (M^i, g^i, p^i), $i = 1, 2$. Let $\widetilde{d}_{GH}((M^1, g^1, p^1), (M^2, g^2, p^2))$ be the infimum of those $\varepsilon > 0$ for which there are ε-nets $\{x^i_j;\ j = 1, 2, \dots, J(\varepsilon)\} \subset M^i$, such that $x^i_1 = p^i$, and

$$|d_1(x^1_j, x^1_k) - d_2(x^2_j, x^2_k)| \le \varepsilon, \quad j, k = 1, 2, \dots, J(\varepsilon).$$

Here $d_i(\cdot,\cdot)$ stands for the distance on (M^i, g^i).

The distance $\widetilde{d}_{GH}((M^1, g^1, p^1), (M^2, g^2, p^2))$ is Lipschitz equivalent to the distance $d_{GH}((M^1, g^1, p^1), (M^2, g^2, p^2))$.

In the future we need the following facts about the structure of the class $\mathbf{M}_{n,p}(R, D, i_0)$ with respect to the GH-topology. These results can be found in or immediately follow from [5, 21] with further improvements in [6]. Namely, the class $\mathbf{M}_{n,p}(R, D, i_0)$ is precompact in GH-topology. Its closure, $\overline{\mathbf{M}_{n,p}(R, D, i_0)}$ consists of pointed Riemannian manifolds (M, g, p) with $g \in C^5_*(M)$. Moreover, estimates (5) remain valid for $(M, g, p) \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$.

Note that by [32], the class $\overline{\mathbf{M}_{n,p}(R, D, i_0)}$ is compact also in Lipschitz topology, that is, for any sequence $(M_j, g_j, p_j) \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$, $j \in \mathbb{Z}_+$, there is $(M, g, p) \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$ and a subsequence $j_k \to \infty$ and bi-Lipschitz maps $F_{j_k} : (M_{j_k}, g_{j_k}, p_{j_k}) \to (M, g, p)$ such that $\mathrm{Lip}(F_{j_k}) \to 1$ and $\mathrm{Lip}(F^{-1}_{j_k}) \to 1$ as $k \to \infty$. Note this implies also a stronger convergence in the C^α-sense, see [5].

To achieve the desired smoothness of g, one needs to use some special coordinates, e.g. harmonic coordinates. Note that, for any $C_{18} > 1$, there is a uniform constant $r_H = r_H(C_{18}, R, D, i_0) \in (0, i_0)$ such that, for any $(M, g, p) \in \overline{\mathbf{M}_n(R, D, i_0)}$ and $x \in M$, there are harmonic coordinates in $B(x, r_H)$, that we denote by $X : B(x, r_H) \to \mathbb{R}^n$. Moreover, in these coordinates, the metric tensor $(g^{(H)}_{jk})^n_{j,k=1} = X_* g$, where the superindex H indicates that we are in harmonic coordinates, satisfies

$$C^{-1}_{18} I \le (g^{(H)}_{jk}(x))^n_{j,k=1} \le C_{18} I, \quad x \in X(B(x, r_H)), \tag{11}$$
$$\|g^{(H)}_{jk}\|_{C^5_*(X(B(x,r_H)))} \le C_{18}.$$

Therefore, using [5, 21], with the terminology described in [60, Sec. 10.3.2], we see that when $(M_k, g_k, p_k) \in \overline{\mathbf{M}_n(R, D, i_0)}$ and $(M_k, g_k, p_k) \to (M, g, p)$ in the Gromov-Hausdorff topology as $k \to \infty$, then for all $\beta \in (0, 1)$ there are $C^{5,\beta}$-smooth diffeomorphism $F_k : M_k \to M$ such

$$F_*(g_k) \to g \quad \text{in } C^{4,\beta}(M), \text{ as } k \to \infty. \tag{12}$$

Considering the ordinary differential equations for geodesics in harmonic coordinates, we see that, for all $\alpha \in (0,1)$, there is a uniform constant C_α such that the exponential map $\exp_x : B_{T_xM}(0,2D) \subset T_xM \to M$ satisfies

$$\|\exp_x\|_{C^{3,\alpha}(B_{T_xM}(0,2D);M)} \leq C_\alpha. \tag{13}$$

Here, $C^{3,\alpha}(B_{T_xM}(0,2D);M)$ is the space of $C^{3,\alpha}$-smooth functions from the ball $B_{T_xM}(0,2D) \subset T_xM$ to M, where the norm is defined using a suitable partition of unity and harmonic coordinate neighbourhoods on M.

The inequality (11) implies that the Riemannian curvature tensor, Riem_M, is uniformly bounded in $C^3_*(M) \subset C^{2,\alpha}(M)$,

$$\|\mathrm{Riem}_M\|_{C^3_*(M)} \leq C_{19}, \tag{14}$$

and, in particular, its sectional curvature, Sec_M satisfies

$$\|\mathrm{Sec}_M\|_{C^3_*(M)} \leq K, \tag{15}$$

where K is a uniform constant. Note that this inequality implies that there is a uniform constant $C_1 > 1$, such that for all $(M,g,p) \in \overline{\mathbf{M}_{n,p}(R,D,i_0)}$ and $x \in M$, we have

$$C_1^{-1} r^n \leq \mathrm{vol}(B(x,r)) \leq C_1 r^n, \quad 0 \leq r \leq D. \tag{16}$$

The class $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$ consists of a finite number of diffeomorphic classes of manifolds and there is $\sigma = \sigma(R,D,i_0)$ such that, if $d_{GH}(M^1,M^2) < \sigma$, then M^1 and M^2 are diffeomorphic. Note that using Riemannian normal coordinates decreases the smoothness of the metric tensor by 2, see e.g. [25]. Therefore, using [5] and [25], we see, that when $(M_k,g_k,p_k) \in \overline{\mathbf{M}_n(R,D,i_0)}$ and $(M_k,g_k,p_k) \to (M,g,p)$ in the Gromov-Hausdorff topology as $k \to \infty$, then in the Riemannian normal coordinates, centred at suitable points,

$$g_k^{(n)} \to g^{(n)} \quad \text{in } C^{2,\beta}(B_e(r)), \quad r < r_H. \tag{17}$$

Here we denote by $g^{(n)}$ the metric tensor in normal coordinates.

In the following, we assume that

$$0 < r_0 < \min\left(\frac{i_0}{2}, \frac{\pi}{2\sqrt{K}}\right), \tag{18}$$

where K is the constant in (15). Then any ball $B(x,r), r < r_0$ is geodesically convex, see e.g. [20, Thm. IX. 6.1] or [57].

Let $(M,g,p) \in \overline{\mathbf{M}_n(R,D,i_0)}$, and consider the normal coordinates $X : B(p,r_0) \to B_e(r_0) \subset \mathbb{R}^n$. Let $(g^{(n)}_{jk}(x))^n_{j,k=1} = X_g$, $x = (x^1,\dots,x^n)$. Then using estimates for metric tensor in the Riemannian normal coordinates, see [60], we see that there is a uniform constant $C_2 = C_2(\beta) > 1$ such that

$$\begin{aligned} &C_2^{-1} I \le (g^{(n)}_{jk}(x))^n_{j,k=1} \le C_2\, I, \quad \text{for } x \in B_e(r_0), \\ &\|(g^{(n)}_{jk})^n_{j,k=1}\|_{C^{2,\beta}(B_e(r_0))} \le C_2. \end{aligned} \tag{19}$$

Therefore, when dealing with $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$, without loss of generality we can take in (3) the L^2-norm defined with respect to the Euclidian metric rather than the metric $h = (g^{(n)}_{jk})^n_{j,k=1}$.

We turn now to the spectral properties of manifolds in $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$. First, using Courant's minmax principle, we see that, as $d_{GH}(M_k,M) \to 0$, then ISD of M_k tend to ISD of M in the sense of Definition 2. We note that this is valid even under less restrictive conditions then (5), in particular those that allow collapsing, as is shown in [30], [46].

Second, by the compactness of $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$, we can, for any $\sigma > 0$, cover it by a finite number of balls, in the GH-metric, of radius σ. Using Courant's minmax principle, it then follows from the metric convergence result (12) that there exists $C_3 > 1$ such that, for any $M \in \overline{\mathbf{M}_{n,p}(R,D,i_0)}$,

$$C_3^{-1} j^{2/n} \le \lambda_j(M) \le C_3 j^{2/n}, \quad j = 0,1,2,\dots. \tag{20}$$

Note that estimate (20) remains valid under a much weaker assumption that $\mathrm{Ric}(M,g)$ is bounded from below, see [13].

Third, instead of harmonic coordinates, we can use coordinates made of the eigenfunctions φ_j. It turns out, cf. [12, 6], that in a neighbourhood of any $x \in M$ there are $\varphi_{j(1;x)},\dots,\varphi_{j(n;x)}$ which form a C^6_*-smooth coordinate system. Moreover, by the compactness arguments, there are $N \in \mathbb{Z}_+$, $C > 1$, $r > 0$ such that, for any $M \in \overline{\mathbf{M}_{n,p}(R,D,i_0)}$, $x \in M$, we can take $j(\ell;x) \le N$, $\ell = 1,\dots,n$ and the metric tensor g in these coordinates satisfies (11).

Using the eigenpairs made of eigenvalues and the corresponding eigenfunctions, (λ_j,φ_j) of $-\Delta_g$, we introduce the Sobolev spaces $H^s(M)$, $s \in \mathbb{R}$,

$$f(x) = \sum_{j=0}^\infty f_j \varphi_j(x) \in H^s(M) \quad \text{if and only if } \|f\|^2_{H^s} := \sum_{j=0}^\infty \langle \lambda_j \rangle^s |f_j|^2 < \infty, \tag{21}$$

where $\langle \lambda \rangle = (1+\lambda^2)^{1/2}$.

Remark 2.1. Using the compactness and interpolation arguments, it follows from (19) that, instead of the $L^2(B(r_0))$-norm in condition iv) of definition 2, we can use a stronger, $C^{2,\beta}(B(r_0))$, $\beta < 1$, and $H^s(B(r_0))$, $s < 3$, norms in (3). However, we will not be using this fact.

Remark 2.2. Returning to C^6_*-coordinates, we see from (20) that the above definition of $H^s(M)$, $-6 < s < 6$, is equivalent to the standard definition that uses the covariant derivatives of functions, with the equivalence constants that are uniform for all (M, g, p) in $\overline{\mathbf{M}_{n,p}(R, D, i_0)}$.

2.2 Distance coordinates

Proposition 1 *For any $\beta < 1$, there are uniform constants $\tau_0, \rho_0 > 0$, $C_4, C_6 > 1$, $C_5 = C_5(\beta) > 1$ such that, for any $(M, g, p) \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$, the following holds true: For $\tau \in (0, \tau_0)$ there is a maximal τ-separated net in $B(p, r_0/4)$ with at most $L - 1$ points, where $L = L(\tau) \in \mathbb{Z}_+$. Let $\{z_1, \dots, z_{L-1}\} \subset B(p, r_0/4)$ be such a τ-net. Then,*

(i) For all $x \in M$, there are n points $z_{j(i)} \in Z$, $j(i) = j(i; x)$, $i = 1, 2, \dots, n$ such that the map $X : B(x, \rho_0) \to \mathbb{R}^n$,

$$X : y \mapsto (d(y, z_{j(1)}), d(y, z_{j(2)}), \dots, d(y, z_{j(n)}))$$

defines $C^{3,\beta}$-smooth coordinates in a ball $B(x, \rho_0)$, that is, $X : B(x, \rho_0) \to X(B(x, \rho_0))$ is a $C^{3,\beta}$-smooth diffeomorphism and

$$\|DX\|_{L^\infty(B(x,\rho_0))} + \|DX^{-1}\|_{L^\infty(X(B(x,\rho_0))))} \le C_4. \tag{22}$$

Moreover, $z_{j(i)}$ can be chosen so that $d(x, z_{j(i)}) > 2^{-8} r_0$ and the metric tensor $(g_{ij})_{i,j=1}^n = X_* g$ in these coordinates satisfies

$$\begin{aligned} &C_4^{-1} I \le (g_{ij}(z))_{i,j=1}^n \le C_4 I, \quad \text{for } z \in X(B(x, \rho_0)), \\ &\|g_{ij}\|_{C^{2,\beta}(X(B(x,\rho_0))} \le C_5. \end{aligned} \tag{23}$$

(ii) The map $H : M \to \mathbb{R}^{L-1}$ defined by $H(x) = (d(x, z_j))_{j=1}^{L-1}$ satisfies

$$L^{-1} \le \frac{d(x,y)}{|H(x) - H(y)|} \le C_6 \tag{24}$$

for all $x, y \in M$, $x \ne y$.

Note that condition (23) implies that $B(X(x), \rho_0/\sqrt{C_4}) \subset X(B(x, \rho_0))$ and thus, after shifting the origin by $X(x)$, we can speak about the covering of M by a finite system of distance coordinate with images are of the form $B_e(\rho_0/\sqrt{C_4})$.

Observe also that the parameters τ_0, ρ_0, C_5, C_4, and C_6 are uniform on $\overline{\mathbf{M}_{n,p}(R, D, i_0)}$. Later, we will fix $\beta = \frac{1}{2}$ and choose τ to depend on R, D, i_0, so that also parameters $L = L(\tau)$ and $C_5 = C_5(\beta)$ will depend on R, D, and i_0.

Proof. (i) Recall that $\text{inj}(M) \geq r_0$ and let $r_1 = 2^{-10} r_0$, where $r_0 < \pi/\sqrt{K}$. Below, we use the Sasaki metric on TM.

Consider a point $x_0 \in M$ and let $p_0 = p_0(x_0) \in B(p, \frac{1}{8} r_0)$ be such that $\ell = d(x_0, p_0) > 16 r_1$. Then $B(p_0, 9r_1) \subset B(p, \frac{1}{4} r_0)$.

Let $N = B(p_0, 9r_1) \setminus B(p_0, r_1)$ and $M_1 = M \setminus B(p, 9r_1)$. We can consider N as a layer that is glued to M_1 along the surface ∂M_1. Recall that the C^1-norm of the Ricci tensor Ric_M of M is bounded by a uniform constant and the sectional curvature is bounded by the uniform constant K. Also, because of the definition of r_1, the second fundamental form of the surface $\Sigma = \partial M_1$ is bounded by a uniform constant. These observations imply that we can below use the considerations in [40, Section 4], with minor modifications. In fact, these minor modifications are simplifications in the sense that in this paper the metric is smooth across Σ, and thus one does not need to consider the intersection angle of geodesics and the surface Σ as is done in [40].

Let $\xi \in S_{x_0} M$ be such that $\gamma_{x_0,\xi}(\ell) = p_0$ and $q_0 = \gamma_{x_0,\xi}(\ell - 5r_1)$. Then considering the shortest geodesics connecting x_0 to points in ball $B(q_0, r_1)$, and using the same arguments as in the proof of [40, Prop. 1], with $a = r_1$ so that $B(q_0, a) \subset N$, we see that there is a uniform constant $c_{220} > 0$ such that there are $v_0 \in S_{x_0} M$, $t_0 > 10 r_1$ that satisfy $y_0 = \gamma_{x_0,v_0}(t_0) \in B(p_0, 6r_1) \setminus B(p_0, 4r_1)$, $d(x_0, y_0) = t_0$, $|t_0 - (\ell - 5r_1)| < r_1$, and the differential of the map $\zeta \mapsto \exp_{x_0}(\zeta)$ at $\zeta_0 = t_0 v_0$ satisfies

$$\min_{\eta \in T_{x_0} M \setminus 0} \frac{\|d \exp_{x_0}|_{\zeta_0}(\eta)\|_g}{\|\eta\|_g} \geq c_{220}.$$

Let now $\widetilde{t}_0 = t_0 - r_1$ and $\widetilde{y}_0 = \gamma_{x_0,v_0}(\widetilde{t}_0) \in B(p_0, 7r_1) \setminus B(p_0, 3r_1)$. Also, let $\widetilde{\zeta}_0 = \widetilde{t}_0 v_0$ and note that the geodesic $\gamma_{x_0,v_0}([0, \widetilde{t}_0])$ from x_0 to $\widetilde{y}_0$ can be extended to a distance minimizing geodesic $\gamma_{x_0,v_0}([0, \widetilde{t}_0 + r_1])$ that does not intersect $B(p_0, 2r_1)$. Then the proof of [40, Lemma 4], with minor modifications, yield that is a uniform constant $c_{221} \in (0, r_1)$ such the following holds:

Let $\mathcal{B} := B_{TM}((x_0, \widetilde{\zeta}_0), c_{221}) \subset TM$ be a ball with center $(x_0, \widetilde{\zeta}_0) \in TM$ and radius c_{221} and let $N = B(p_0, 9r_1) \setminus B(p_0, r_1)$. Then for all $(x, tv) \in \mathcal{B}$, where $v \in S_xM$ and $t > 0$, we have $\gamma_{x,v}(t) \in N$, the geodesic $\gamma_{x,v}([0,t])$ is the unique shortest geodesics in M between its end points, and $\gamma_{x,v}([0,t]) \cap B(p_0, r_1) = \emptyset$. Moreover, there are uniform constants $C_{10}, C_{51} > 1$ such that the differential $D_wF(x,w) = d\exp_x|_w$ of the map

$$F : TM \to M \times M,$$
$$F(x,w) = (x, \exp_x(w))$$

satisfies

$$\|D_wF(x,w)\| \leq C_{10}, \quad \|(D_wF(x,w))^{-1}\| \leq C_{51}, \quad \text{for all } (x,w) \in \mathcal{B}. \quad (25)$$

Let $\kappa = \min(r_1, c_{221}/C_{51})$ and $\mathcal{V} = \{(x,w) \in \mathcal{B} : \ F(x,w) \in B(x_0, \kappa) \times B(\widetilde{y}_0, \kappa)\}$. Then $B(\widetilde{y}_0, \kappa) \subset B(p_0, 8r_1) \setminus B(p_0, r_1)$ and the above yields that the restriction of the map F in $\mathcal{V}$,

$$F|_{\mathcal{V}} : \mathcal{V} \to B(x_0, \kappa) \times B(\widetilde{y}_0, \kappa)$$

is a diffeomorphism.

The above yields that there is uniform constant $\delta_1 > 0$ such that there are $w_j \in T_{x_0}M$, $j = 1, 2, \dots, n$ so that $(x_0, w_j) \in \mathcal{V}$ and

$$4\delta_1 \leq \|w_j - w_k\|_g \leq 4n\delta_1, \quad \text{for } j \neq k, \quad (26)$$

and moreover, $B_{TM}((x_0, w_j), 8n\delta_1) \subset \mathcal{V}$ and

$$B_{TM}((x_0, w_j), 2\delta_1) \cap B_{TM}((x_0, w_k), 2\delta_1) = \emptyset, \quad \text{for } j \neq k. \quad (27)$$

Let $y_j = \exp_{x_0}(w_j) \in N$ and define $\widehat{\tau} = \delta_1/(2C_{51})$.Then,

$$B(x_0, \widehat{\tau}) \times B(y_j, \widehat{\tau}) \subset F(B_{TM}((x_0, w_j), 2\delta_1)). \quad (28)$$

Moreover, then (25), (27), (28) and the definition of $\widehat{\tau}$ yield that if $x' \in B(x_0, \widehat{\tau})$ and $y'_j \in B(y_j, \widehat{\tau})$, then

$$(x', y'_j) = F(x', \theta_j), \quad \text{where } \theta_j = s_j\eta_j,\ \eta_j \in S_{x'}M,\ s_j = |\theta_j|,$$

are such that $(x', \theta_j) \in B_{TM}((x_0, w_j), 2\delta_1) \subset \mathcal{V}$, $s_j > r_1$,

$$\|\theta_j - w_j\|_g < 2C_{51}\,\widehat{\tau} \leq \delta_1.$$

These and (26) yield that $\|\theta_j-\theta_k\|_g > 2\delta_1$ and $\|\eta_j-\eta_k\|_g > 2r_1^{-1}\delta_1$ for $j\neq k$. Since $\nabla d(\,\cdot\,,y_j')|_{x'})=\eta_j$, we have

$$\|\nabla d(\,\cdot\,,y_j')|_{x'}-\nabla d(\,\cdot\,,y_k')|_{x'}\|_g > 2r_1^{-1}\delta_1,\quad \text{for } j\neq k. \tag{29}$$

In particular, this yields that if $z_1,\dots,z_{L-1}\in B(p,r_0/4)$ is a $\widehat{\tau}$-net then there are $k(j,x')\in\mathbb{Z}_+$ for which $z_{k(j,x')}\in B(y_j,\widehat{\tau})$. For such points $y_j'=z_{k(j,x')}\in\{z_1,\dots,z_{L-1}\}$ the inequalities (29) are valid, This proves (22) with $\rho_0=\widehat{\tau}$ and and $\tau_0\le\widehat{\tau}$ with some suitable uniform constant C_5. Then, by using inverse function theorem and the $C^{3,\beta}$-smoothness of the exponential map and its inverse, see (13) and (25), to analyse the map $X(x')=(\|\exp_{y_j'}^{-1}(x')\|_g)_{j=1}^n$, we obtain the inequality (23) with some constants C_4 and $C_5=C_5(\beta)$. This proves claim (i) holds when $\tau_0\le\widehat{\tau}$.

(ii) To start the proof, we observe that because of triangular inequality, we have

$$|H(x)-H(y)|\le (L-1)d(x,y). \tag{30}$$

Next, we prove an opposite inequality to (30) with a uniform constant when τ_0 is sufficiently small. To show it, assume that the claim is not valid. Then for all $k\in\mathbb{Z}_+$ there are $\tau_k>0$, such that $\tau_k\le\frac{1}{k}$, manifolds $(M_k,g_k,p_k)\in\overline{\mathbf{M}_{n,p}(R,D,i_0)}$, τ_k-nets $\{z_j^k: j=1,2,\dots,L_k\}\subset B_{M_k}(p_k,r_0/4)$ that define functions $H_k:M_k\to\mathbb{R}^{L_k-1}$, $H_k(y)=(d_{M_k}(y,z_j^k))_{j=1}^{L-1}$, and points $x_k,y_k\in M_k$ for which

$$\lim_{k\to\infty}\frac{|H_k(x_k)-H_k(y_k)|}{d_{M_k}(x_k,y_k)}=0. \tag{31}$$

Then, by [32], the class $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$ is compact with respect to the Lipschitz topology and all elements in the class $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$ are compact manifolds. Thus we can assume, by choosing a subsequence if necessary, that there are $(M,g,p)\in\overline{\mathbf{M}_{n,p}(R,D,i_0)}$ and bi-Lipschitz maps $F_k:(M_k,g_k,p_k)\to(M,g,p)$ such that

$$d_M(F_k(p_k),p)\le\frac{1}{k},\quad \mathrm{Lip}(F_k)\le 1+\frac{1}{k},\quad \mathrm{Lip}(F_k^{-1})\le 1+\frac{1}{k}.$$

Let now $\widetilde{x}_k=F_k(x_k)$, $\widetilde{y}_k=F_k(y_k)$, and $\widetilde{z}_j^k=F_k(z_j^k)$. Again, by choosing a subsequence if necessary, we can assume that $\widetilde{x}_k\to\widetilde{x}$ and $\widetilde{y}_k\to\widetilde{y}$ as $k\to\infty$.

Also, by using Cantor's diagonalization argument, we can assume that for all $j \in \mathbb{Z}_+$ we have $\lim_{k\to\infty} \widetilde{z}_j^k = \widetilde{z}_j$.

First, we consider the case when $\widetilde{x} \neq \widetilde{y}$, so that

$$\lim_{k\to\infty} d_k(x_k, y_k) = d(\widetilde{x}, \widetilde{y}) \neq 0, \quad d_k(x_k, y_k) := d_{M_k}(x_k, y_k). \tag{32}$$

Then, for all $q \in B(p, r_0/4)$ and a sufficiently large k there is $q_k = F_k^{-1}(q) \in B_k(p_k, r_0/2) := B_{M_k}(p_k, r_0/2)$. Also, there is z_j^k for which $d_k(z_j^k, q_k) \leq \tau_k$, and hence

$$d(\widetilde{z}_j, q) \leq d_k(z_j^k, q_k) + \frac{1}{k} \leq \tau_k + \frac{1}{k} \leq \frac{2}{k}.$$

This shows that $S = \{\widetilde{z}_j : j \in \mathbb{Z}_+\}$ is $\widetilde{\tau}$-net in $B(p, r_0/4)$ for arbitrary $\widetilde{\tau} > 0$, that is, the set S is dense in $B(p, r_0/4)$. By (31), we have $\lim_{k\to\infty} H_k(x_k, y_k) = 0$. Hence for all $j \in \mathbb{Z}_+$, we have

$$d(\widetilde{x}, \widetilde{z}_j) - d(\widetilde{y}, \widetilde{z}_j) = \lim_{k\to\infty} d_k(x_k, z_j^k) - d_k(y_k, z_j^k) = 0,$$

and as the set S is dense in $B_M(p, r_0/4)$, we see that

$$d(\widetilde{x}, \widetilde{z}) = d(\widetilde{y}, \widetilde{z}), \quad \text{for all } \widetilde{z} \in B(p, r_0/4). \tag{33}$$

Let $\widetilde{p} = \widetilde{p}(\widetilde{x}, \widetilde{y}) \in B_M(p, 2^7 r_1) \subset B_M(p, \frac{1}{8} r_0)$ be such that $\ell_x = d(\widetilde{x}, \widetilde{p}) > 16 r_1$ and $\ell_y = d(\widetilde{y}, \widetilde{p}) > 16 r_1$. Then $B(\widetilde{p}, r_1) \subset B(p, \frac{1}{4} r_0)$. As in [6], we see that if $z \in \partial B(\widetilde{p}, r_1)$ is a closest point of $\partial B(\widetilde{p}, r_1)$ to $\widetilde{x}$, then the shortest curve from z to $\widetilde{x}$ is a normal geodesics. By (33), $z \in \partial B(\widetilde{p}, r_1)$ is also a closest point of $\partial B(\widetilde{p}, r_1)$ to $\widetilde{y}$ and the shortest curve from z to $\widetilde{y}$ is a normal geodesics. Moreover, then (33) implies that $\ell_x - r_1 = \ell_y - r_1$, and

$$\widetilde{x} = \gamma_{z,\nu(z)}(s) = \widetilde{y}, \tag{34}$$

where $s = \ell_x - r_1$ and $\nu(z)$ is the exterior normal vector of $\partial B(\widetilde{p}, r_1)$ at z. Now, equation (34) is in contradiction with (32). Hence, (32) is not possible.

Second, we consider the case when $\widetilde{x} = \widetilde{y}$, so that

$$\lim_{k\to\infty} d_k(x_k, y_k) = d(\widetilde{x}, \widetilde{y}) = 0. \tag{35}$$

Then for sufficiently large k_0 we have for all $k > k_0$ that $\tau_k < \widehat{\tau}$ and $d_k(x_k, y_k) \leq \rho_0$. Then the inequality (22) implies that (31) can not be valid. This proves (ii).

Finally we note that, if $z_1, \dots, z_{L-1}$ is a maximal τ-separated net in $B(p, r_0/4)$, then balls $B(z_j, \tau/2)$ are disjoint. Then, using (16), we see that

$$L \le L(\tau) \le C_7 \tau^{-n} + 1 \tag{36}$$

with some uniform constant $C_7 > 0$. This shows that $L(\tau)$ can be chosen to depend only on D, R, and i_0.

$\square$

The above considerations leading to (36) bring about the following result

Lemma 1 *There exist $C_{13} > 0$, $C_{14} \ge 1$ such that, the following hold true:*

(i) Let $\gamma > 0$. There exists a maximal γ separated set $x_1, \dots, x_{N(\gamma)}$ in M with

$$N(\gamma) \le \widetilde{N}(\gamma) = C_{13}\gamma^{-n}, \tag{37}$$

Moreover, the balls $B(x_k, 4\gamma)$ enjoy the finite intersection property with constant C_{14}, that is, the number of balls having a non-empty intersection is bounded by C_{14}.

(ii) Let $\gamma > 0$. There exist points $z_1, \dots, z_{N_1(\gamma)}$ which form a maximal γ-separated net in $B(p, r_0/4)$ with $N_1(\gamma) \le \widetilde{N}(\gamma)$, and the balls $B(z_k, 4\gamma)$ enjoy the finite intersection property with constant C_{14}.

Proof. It remains to prove the finite intersection property. It follows from (16) if we take into the account that $B(x_k, 4\gamma) \cap B(x_j, 4\gamma) = \emptyset$ if $d(x_k, x_j) \ge 9\gamma$ and $B(x_k, \gamma/2) \cap B(x_j, \gamma/2) = \emptyset$. $\square$

3 Wave equation: estimates and unique continuation results

Let $(M, g, p) \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$. On the manifold (M, g) we consider the initial-value problem for the wave equation

$$\begin{aligned} &\partial_t^2 w - \Delta_g w = 0 \text{ in } M \times \mathbb{R}, \\ &w|_{t=0} = v, \quad w_t|_{t=0} = 0, \end{aligned} \tag{38}$$

and denote its solution by $w = W(v)$.

Our main interest lies in the case when $v \in \mathcal{H}^s_{\Lambda_s(M)}$, $\Lambda_s > 0$,

$$\mathcal{H}^s_{\Lambda_s}(M) = \{v \in H^s(M) : \|v\|_{H^s(M)} \leq \Lambda_s\} \tag{39}$$

and we assume in the following that $3/2 < s < 2$.

Using the Fourier decomposition, we see that, if $v \in H^s(M)$, then

$$w \in C(\mathbb{R}; H^s(M)) \cap C^1(\mathbb{R}; H^{s-1}(M)) \cap C^2(\mathbb{R}; H^{s-2}(M)),$$

and

$$\|w\|_{C(\mathbb{R};H^s(M)) \cap C^1(\mathbb{R};H^{s-1}(M)) \cap C^2(\mathbb{R};H^{s-2}(M))} \leq 3\|v\|_{H^s(M)}.$$

Thus, if $T < 2D$, then

$$\|w\|_{H^s(M \times [-T,T])} \leq 6\sqrt{T}\|v\|_{H^s(M)} \leq C_D\|v\|_{H^s(M)} \tag{40}$$

where

$$C_D = 6\langle D\rangle, \quad \langle D\rangle = \sqrt{1 + D^2}. \tag{41}$$

3.1 Unique continuation

Associated to the wave operator are the double cones of influence. To define these, let $V \subset M$ be open, $T \in \mathbb{R}_+$. Denote by

$$\Gamma(V, T) := V \times (-T, T).$$

Then the double cone of influence is given by

$$\mathcal{D}(V, T) := \{(t, x); d(x, V) + |t| < T\}. \tag{42}$$

Note that, by Tataru's uniqueness theorem [66], [67], if u is a solution to the wave equation

$$\partial_t^2 u - \Delta_g u = 0, \quad \text{in } M \times (-T, T),$$

which satisfies $u = 0$ in $\Gamma(V, T)$, then $u = 0$ in $\mathcal{D}(V, T)$. However, for our purposes we need an explicit estimate which follows from Theorem 3.3 in [17]. To formulate the results we introduce, for

$$0 < \gamma < r_0/16, \quad r_0/4 < T < 2D, \tag{43}$$

and $z \in M$, the domains

$$\Gamma = \Gamma(z,T) = B(z,r_0/16) \times (-T + r_0/16, T - r_0/16), \tag{44}$$
$$\mathcal{D} = \mathcal{D}(z,\gamma,T) = \{(t,x) :\, (T - d(x,z))^2 - t^2 \geq \gamma^2,\, |t| < T - r_0/16\},$$
$$\Omega(T) = M \times (-T + r_0/16,\, T - r_0/16).$$

Theorem 2 *Let $(M,g) \in \overline{\mathbf{M}_{n,p}(R,D,i_0)}$. Let $P = P(x,D) = \partial_t^2 - \Delta_g$ be the wave operator associated with M. Assume that $w(t,x) = 0$ for all $(t,x) \in \Gamma$. Then, for any $\theta < 1$, there is $c_{206} \geq 1$, such that the following stability estimate holds true:*

$$\|w\|_{L^2(\mathcal{D}(z,\gamma,T))} \leq c_{206} \frac{\|w\|_{H^1(\Omega(T))}}{\left(\ln\left(1 + \frac{\|w\|_{H^1(\Omega(T))}}{\|Pw\|_{L^2(\Omega(T))}}\right)\right)^\theta}.$$

Therefore, for any $0 \leq m \leq 1$,

$$\|w\|_{H^{1-m}(\mathcal{D}(z,\gamma,T))} \leq c_{206}^m \frac{\|w\|_{H^1((\Omega(T))}}{\left(\ln\left(1 + \frac{\|w\|_{H^1((\Omega(T))}}{\|Pw\|_{L^2((\Omega(T))}}\right)\right)^{\theta m}}. \tag{45}$$

Moreover there is $c_{205} = c_{205}(n,R,D,i_0,r_0,T)$ such that

$$c_{206} = c_{205}\, exp(\gamma^{-c_{200}}), \quad c_{200} = 58(n+1) + 1. \tag{46}$$

Proof Theorem 2 follows from Theorem 3.3 in [17] with $\ell = r_0/16$ and $\mathcal{D}(z,\gamma,T) = S(z,r_0/16,T,\gamma)$. Using that $w = 0$ in Γ, the domain Λ in the final equation of Theorem 3.3 can be changed into $\mathcal{D}(z,\gamma,T)$. Moreover, for $\theta < 1$, the function $f_\theta(a,b)$, $a,b > 0$,

$$f_\theta(a,b) = \frac{a}{\left(\ln(1 + \frac{a}{b})\right)^\theta}, \tag{47}$$

increases when either a or b increases. Thus, we can change $\|w\|_{H^1(\Omega_1)}$ and $\|Pw\|_{L^2(\Omega_1)}$ in Theorem 3.3 to $\|w\|_{H^1(\Omega(T))}$ and $\|Pw\|_{L^2(\Omega(T))}$. Let us also note that, although the results in [17] are formulated for $M \subset \mathbb{R}^n$, they can be easily reformulated for an arbitrary compact Riemannian manifold. For the calculation of (46) see the Appendix. Moreover, recall that the constants in [17] explicitly depend on parameters $c_1, c_2, c_3 > 0$ such that

$$c_1|\xi|^2 \leq g^{jk}(x)\xi_j\xi_k \leq c_2|\xi|^2, \quad \|g^{jk}(x)\|_{C^{2,\beta}(M)} \leq c_3.$$

Moreover, due to (19), these constraints and the estimates in [17] are uniformly valid on $\overline{\mathbf{M}_{n,p}(R,D,i_0)}$. $\Box$

Our main interest will be an estimate for $v(\cdot) = w(0,\cdot)$ in (38) in the domain $B(z, T-2\gamma)$. To analyse it, we need the following trace-type theorem.

Proposition 2 *For any $\alpha > 1/2$ there exists $C_{11}(\alpha)$ such that the following holds true:*
1. Let $r \geq r_0/16$, $z \in M$ and $w \in H^\alpha((-\gamma,\gamma); L^2(B(z,r)))$. Then

$$\begin{aligned} \|w(\,\cdot\,,0)\|_{L^2(B(z,r))} &\leq C_{11}(\alpha)\,\gamma^{-\alpha}\|w\|_{H^\alpha((-\gamma,\gamma);L^2(B(z,r)))} \\ &\leq C_{11}(\alpha)\,\gamma^{-\alpha}\|w\|_{H^\alpha(B(z,r)\times(-\gamma,\gamma))}. \end{aligned} \tag{48}$$

2. Let $T - 2\gamma \geq r_0/16$, $z \in M$. Then, since $B(z, T-2\gamma)\times(-\gamma,\gamma) \subset \mathcal{D}(z,\gamma,T)$

$$\|w(\,\cdot\,,0)\|_{L^2(B(z,T-2\gamma))} \leq C_{11}(\alpha)\,\gamma^{-\alpha}\|w\|_{H^\alpha(\mathcal{D}(z,\gamma,T))}. \tag{49}$$

Corollary 3 *Assume* (43) *and let $\theta \in [1/2, 1)$, $\varepsilon_2 \in (0, \Lambda_s]$ and $v \in \mathcal{H}^s_{\Lambda_s}(M)$. Denote by $w = W(v)$ the solution to initial-value problem (38) and assume that,*

$$\|w\|_{L^2(B(z,r_0/16+\gamma)\times(-T+r_0/16,\,T-r_0/16))} \leq \varepsilon_2. \tag{50}$$

Then, calling $\beta = \theta^2/2$ and defining $\varepsilon_1 := \mathcal{E}_1(\varepsilon_2;\theta,\gamma,\Lambda_s)$, we get

$$\|v\|_{L^2(B(z,T-2\gamma))} \leq \varepsilon_1, \tag{51}$$

$$\text{where}\quad \mathcal{E}_1(\varepsilon_2;\theta,\gamma,\Lambda_s) = c_{202}\,\frac{\Lambda_s}{\gamma^{(2-\theta/2)}\left(\ln\left[1+\gamma\Lambda_s^{(s-1)/s}\varepsilon_2^{-(s-1)/s}\right]\right)^\beta}, \tag{52}$$

and with $C_{30} = C_{30}(\theta, n, R, D, i_0, r_0)$ such that

$$c_{202}(\theta,\gamma) = C_{30}(\theta)\,exp\big(\gamma^{-(c_{200}\,\theta/2)}\big). \tag{53}$$

Proof. Let the cut-off function $\eta(x) \in C^2_0(B(z, r_0/16 + \gamma/2))$ be equal to one in $B(z, r_0/16)$ and $\|\eta\|_{C^i(M)} \leq C\gamma^{-i}$, $i = 0, 1, 2$. Then $w_\eta(x,t) = (1-\eta(x))w(x,t)$ vanishes in Γ and we have $(\partial_t^2 - \Delta)w_\eta(x,t) = F$, where

$$\begin{aligned} F(x,t) &= \big(\Delta_g\eta(x)\big)\,w(x,t) + 2g(\nabla\eta(x), \nabla_x w(x,t)) \\ &= \big(\Delta_g\eta(x)\big)\,(\widetilde{\eta}(x)w(x,t)) + 2g(\nabla\eta(x), \nabla_x(\widetilde{\eta}(x)w(x,t)))\,. \end{aligned} \tag{54}$$

Here $\widetilde{\eta}(x) \in C_0^2(B(z, r_0/16 + \gamma))$ is equal to one in $B(z, r_0/16 + \gamma/2)$ and $\|\widetilde{\eta}\|_{C^i(M)} \leq C\gamma^{-i}$, $i = 0, 1, 2$.
The L^2-norm of the first term in the right hand side of (54) is bounded by $C\gamma^{-2}\varepsilon_2$. To estimate the norm of the second term, observe that,

$$\|\widetilde{\eta} w\|_{H^s(M\times(-T+\gamma,T-\gamma))} \leq C\gamma^{-s}\Lambda_s,$$

where we have also used (40). Since

$$\|\widetilde{\eta} w\|_{L^2(M\times(-T+r_0/16,T-r_0/16))} \leq \varepsilon_2, \tag{55}$$

by interpolation arguments, we see first that

$$\|\widetilde{\eta} w\|_{H^1(M\times(-T+r_0/16,T-r_0/16))} \leq C\gamma^{-1}\Lambda_s^{1/s}\varepsilon_2^{1-1/s} \tag{56}$$

Since $\mathrm{supp}(\nabla\eta)\cap\mathrm{supp}(\nabla\widetilde{\eta}) = \emptyset$, equations(55), (56) imply that the L^2-norm of the second term in right hand side of (54) is estimated by $C\gamma^{-1}\Lambda_s^{1/s}\varepsilon_2^{1-1/s}$. Since $\varepsilon_2 \leq \Lambda_s$, these yield

$$\|F\|_{L^2(M\times(-T+r_0/16,T-r_0/16))} \leq C\gamma^{-2}\Lambda_s^{1/s}\varepsilon_2^{1-1/s}.$$

As $s > 1$, we have

$$\|w_\eta\|_{H^1(M\times(-T+\gamma,T-\gamma))} \leq C\gamma^{-1}\Lambda_s.$$

Using growth properties of the function f_θ of form (47), it follows from Theorem 2 that

$$\|w_\eta\|_{H^{1-\theta/2}(\mathcal{D})} \leq C c_{206}^{\theta/2} \frac{\gamma^{-1}\Lambda_s}{\left(\ln\left[1+\gamma\Lambda_s^{(s-1)/s}\varepsilon_2^{-(s-1)/s}\right]\right)^\beta}. \tag{57}$$

It follows from (49) with $\alpha = 1-\theta/2$ and (57) that,

$$\|w_\eta(\cdot\,,0)\|_{L^2(B(z,T-2\gamma))} \leq CC_{11}(\alpha)\, \frac{c_{206}^{\theta/2}\Lambda_s}{\gamma^{2-\theta/2}\left(\ln\left[1+\gamma\Lambda_s^{(s-1)/s}\varepsilon_2^{-(s-1)/s}\right]\right)^\beta}. \tag{58}$$

Next define $\alpha = (1-\beta)s + \beta > 1/2$. Then by interpolation,

$$\|\eta w\|_{H^\alpha(B(z,r)\times(-\gamma,\gamma))} \leq c_{201}\,\|\eta w\|_{H^s(B(z,r)\times(-\gamma,\gamma))}^{\alpha/s}\,\|\eta w\|_{L^2(B(z,r)\times(-\gamma,\gamma))}^{(s-\alpha)/s}.$$

Using the fact that $\mathrm{supp}(\eta) \subset B(z, r_0/16+\gamma)$, we can apply (48) with $r = r_0/16+\gamma$, the previous inequality and (50), to obtain

$$\|\eta(\cdot)w(\cdot,0)\|_{L^2(B(z,T-2\gamma))} \leq C_{11}(\alpha)\gamma^{-\alpha}c_{201}(C_D\Lambda_s)^{\alpha/s}\epsilon_2^{\beta(s-1)/s}$$
$$\leq \frac{C_{11}(\alpha)\gamma^{\beta-\alpha}c_{201}C_D^{\alpha/s}\Lambda_s}{\left(\ln\left[1+\gamma\Lambda_s^{(s-1)/s}\varepsilon_2^{-(s-1)/s}\right]\right)^{\beta}}. \tag{59}$$

Here at the last step we use the fact that $X \geq \ln(1+X)$ for $X > 0$, with $X = \gamma\Lambda_s^{(s-1)/s}\varepsilon_2^{-(s-1)/s}$.
Recall that $v(x) = w_\eta(x,0) + \eta(x)w(x,0)$. Comparing (58) and (59), we obtain equation (52). The coefficient c_{202} defined in (53) fulfills the inequality

$$c_{202} \geq CC_{11}(\alpha)c_{206}^{\theta/2}\gamma^{\theta/2-2} + C_{11}(\alpha)c_{201}C_D^{(1-\beta)+\beta/s}\gamma^{(\beta-1)s},$$

by using (46) and a proper multiplicative coefficient C_{30} independent on γ. □

Corollary 4 *We define*

$$\mathcal{E}_2(\varepsilon_1;\,\theta,\gamma,\Lambda_s) := \Lambda_s\left[\frac{\gamma}{exp\left[\left(\frac{\Lambda_s}{(\varepsilon_1\gamma^{2-\theta/2})}C_{30}(\theta)\,exp(\gamma^{-c_{200}})\right)^{1/\beta}\right]}\right]^{s/(s-1)} \tag{60}$$

and observe that $\mathcal{E}_2(\varepsilon_1;\,\theta,\gamma,\Lambda_s) = \mathcal{E}_1^{-1}(\varepsilon_1)$ with $\mathcal{E}_1$ given in (52).
In the following we can assume in (51) that:

$$0 < \varepsilon_1 \leq \Lambda_s. \tag{61}$$

We then assume in (50) that

$$\varepsilon_2 \leq \mathcal{E}_2(\varepsilon_1;\,\theta,\gamma,\Lambda_s). \tag{62}$$

From the growth properties of $\mathcal{E}_2(\varepsilon_1)$ it follows that

$$\mathcal{E}_2(\varepsilon_1;\,\theta,\gamma,\Lambda_s) \leq \varepsilon_1, \quad \varepsilon_1 \in (0,\Lambda_s]. \tag{63}$$

4 Computation of the projection

4.1 Approximate projections

Our ultimate goal is to approximately construct the values of the distance functions from a variable point $x \in M$ to all z_ℓ, $\ell = 1, 2, \dots, \widetilde{N}(\sigma)$, defined in Lemma 1. The main step to achieve that is to approximately compute the Fourier coefficients of the functions of form $\chi_\Omega v$, where χ_Ω is the characteristic function of some special subdomains $\Omega \subset M$ and v has a finite Fourier expansion. These subdomains Ω are defined using distances to L points $\{z_1, \dots, z_{L-1}, z_i\}$, where $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ is arbitrary. For $i \in \{L, L+1, \dots, \widetilde{N}(\sigma)\}$, we denote

$$K_i = \{1, 2, \dots, L-1\} \cup \{i\}$$

and define $\mathcal{A}^{(i)}$ be the set of those $\alpha = (\alpha_\ell)_{\ell=1}^{\widetilde{N}(\sigma)} \in \mathbb{R}^{\widetilde{N}(\sigma)}$ that satisfy conditions

$$\begin{aligned} &\alpha_\ell = A_\ell \gamma, \text{ such that } A_\ell \in \mathbb{Z}_+ \text{ and } r_0/8 \le \alpha_\ell \le 2D, \text{ for } \ell \in K_i, \\ &\alpha_\ell = 0, \text{ for } \ell \notin K_i. \end{aligned} \tag{64}$$

Next, we consider for a while a fixed index $i \in \{L, \dots, \widetilde{N}(\sigma)\}$. To construct subdomains Ω, we start with more general observation sets $\Gamma(\alpha)$, $\alpha \in \mathcal{A}^{(i)}$,

$$\Gamma(\alpha) = \bigcup_{\ell \in K_i} \Gamma(z_\ell, \alpha_\ell - \gamma). \tag{65}$$

Then the corresponding double cone of influence is defined as

$$\mathcal{D}(\alpha) = \bigcup_{\ell \in K_i} \mathcal{D}(z_\ell, \gamma, \alpha_\ell - \gamma) \tag{66}$$

where $\Gamma(z_\ell, \alpha_\ell - \gamma)$, $\mathcal{D}(z_\ell, \gamma, \alpha_\ell - \gamma)$ are given by (44) with $T = \alpha_\ell - \gamma$. Let $h^{(i)} = (h_\ell^{(i)})_{\ell=1}^{\widetilde{N}(\sigma)} \in \mathbb{R}^{\widetilde{N}(\sigma)}$ be such that

$$h_\ell^{(i)} = \begin{cases} 1, & \text{for } \ell \in K_i, \\ 0, & \text{for } \ell \notin K_i. \end{cases}$$

At last, for $b \in \mathbb{R}$, we define

$$M(\alpha + b\gamma h^{(i)}) = \bigcup_{\ell \in K_i} B(z_\ell, \alpha_\ell + b\gamma). \tag{67}$$

We have the following volume estimate.

Lemma 2 *Let* $\alpha \in \mathcal{A}^{(i)}$, $i = L, \dots, N(\gamma)$ *and*

$$A = A(\alpha, \gamma) = \{x \in M : \ d(x, \partial M(\alpha + \gamma h^{(i)})) \leq 4\gamma\}, \tag{68}$$

where $\gamma \geq 0$*. There is a uniform* $C_{14} > 0$ *such that*

$$vol(A) \leq C_{14} L \gamma.$$

Proof. Let $d(x, \partial M(\alpha + \gamma h^{(i)})) \leq 4\gamma$. Then, for some $\ell \in \{1. \dots, L\}$,

$$\alpha_\ell - 3\gamma \leq d(x, z_\ell) \leq \alpha_\ell + 5\gamma. \tag{69}$$

Since $\|d \exp_{z_\ell}|_{\mathbf{v}}\|$ is uniformly bounded on $\overline{\mathbf{M}_{n,p}(R, D, i_0)}$ for $\mathbf{v} \in T_{z_\ell}M$, $|\mathbf{v}| \leq 2D$, the volume of the set of the points satisfying (69) is uniformly bounded by $C_{14}\gamma$. We obtain the claim by taking the union of these sets when the index ℓ runs from 1 to L. $\square$

Remark 2 *It follows from the proof that* ∂M_α *is a closed set with* $vol(\partial M_\alpha) = 0$*. Therefore, in the following we would not distinguish between* $vol(M_\alpha)$ *and* $vol(M_\alpha^{int})$ *and similar type of objects occurring later.*

Remark 3 *We define* $b(s)$ *as*

$$b(s) = 1/2 \text{ for } n = 2, 3 \text{ and } b(s) = s/n \text{ for } \tfrac{3}{2} < s < 2,\ n \geq 4. \tag{70}$$

By the Sobolev embedding $H^s(M) \to C(M)$*, for* $n \in \{2, 3\}$*, and* $H^s(M) \to L^q(M)$*,* $q = (2n)/(n - 2s)$ *for* $n \geq 4$*. Note that the norm of this map is a uniform constant as the embedding can be done in harmonic coordinates that are defined in balls having a uniform radius. This, together with the volume estimate*

$$vol\ (B(z, T + \gamma) \setminus B(z, T - 2\gamma)) \leq c\gamma, \quad \text{for } T \leq 2D, \tag{71}$$

of Lemma 2 and the generalized Hölder inequality with $\widetilde{q} > 1$ *such that* $q^{-1} + \widetilde{q}^{-1} = \frac{1}{2}$*, imply that, with some* $c_1(s) > 0$*,*

$$\begin{aligned} \|\chi_{B(z,T+\gamma)\setminus B(z,T-2\gamma)}\, v\|_{L^2(M)} &\leq \|\chi_{B(z,T+\gamma)\setminus B(z,T-2\gamma)}\|_{L^{\widetilde{q}}(M)}\, \|v\|_{L^q(M)} \\ &\leq c(s)\gamma^{b(s)} \|v\|_{H^s(M)}. \end{aligned} \tag{72}$$

Theorem 3 *Let $\varepsilon_0 \in (0, \Lambda_s)$. There are $\gamma_0(\varepsilon_0; s, \Lambda_s)$ and $j_0(\varepsilon_0; \gamma, s, \Lambda_s)$, with the following properties:*
Let $\gamma \leq \gamma_0(\varepsilon_0; s, \Lambda_s)$. Assume that

$$u(x) = \sum_{j=0}^{j_0} a_j \varphi_j(x) \in \mathcal{H}^s_{\Lambda_s}(M), \quad j_0 \geq j_0(\varepsilon_0; \gamma, s, \Lambda_s)$$

and the eigenvalues and restrictions of eigenfunctions of $-\Delta_g$,

$$\left(\lambda_j,\ \varphi_j|_{B(p,r_0)}\right), \quad j = 0, 1, 2, \ldots, j_0,$$

be given. Then, for any $i \in \{L, \ldots, \widetilde{N}(\sigma)\}$ and $\alpha \in \mathcal{A}^{(i)}$, it is possible to determine the Fourier coefficients $(d_j)_{j=0}^{j_0}$, $d_j = d_j(\alpha, i)$, such that

$$v(x) = \sum_{j=0}^{j_0} d_j \varphi_j(x)$$

satisfies

$$\|v - \chi_{M(\alpha - 2\gamma h^{(i)})} u\|_{L^2(M)} < \varepsilon_0. \tag{73}$$

Moreover $v \in \mathcal{H}^s_{(2C_{16}(s,\gamma)\Lambda_s)}(M)$, where

$$C_{16}(s, \gamma) = C_{17}(s) \gamma^{-s}. \tag{74}$$

Remark 4 *The critical values considered in Theorem 3, that is, the functions $\gamma_0(\varepsilon_0; s, \Lambda_s)$ and $j_0(\varepsilon_0; \gamma, s, \Lambda_s)$ are defined later. Namely,*

$$\gamma_0(\varepsilon_0; s, \Lambda_s) = \frac{1}{(8Lc(s))^{1/(2b(s))}} \left(\frac{\varepsilon_1}{\Lambda_s}\right)^{1/(b(s))}, \quad \varepsilon_1 = \frac{\varepsilon_0^2}{10\Lambda_s}, \tag{75}$$

cf. (100), where $c(s)$ is given in (72). As for j_0, we have

$$j_0(\varepsilon_0; \gamma, s, \Lambda_s) = C_{20} C_{16}(s, \gamma)^{n/s} \left(\frac{\Lambda_s}{\varepsilon_2}\right)^{n/s}, \tag{76}$$

cf. (86), where

$$\varepsilon_2 = \mathcal{E}_2\left(\frac{\varepsilon_1}{L};\ \theta, \gamma, C_{16}(s, \gamma)\Lambda_s\right),$$

cf. (83). Here $\mathcal{E}_2$ is defined in (60), C_{20} in Lemma 4 and C_{16} in Lemma 3 below.

The rest of this section is devoted to the proof of Theorem 3 which is divided into several steps.
In the subsections 4.2, 4.3 and 4.4 below, we keep the index $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ fixed and omit marking the index when defining some new notations that depend on i.

4.2 Cut-off estimates and finite dimensional projections

We start with a partition of unity associated with points $\{x_1, \dots, x_{N(\gamma)}\}$ defined in Lemma 1.

They determine a covering of M consisting of (2γ)-neighbourhoods of these points, $B(x_\ell, 2\gamma)$, $\ell = 1, 2, \dots, N(\gamma)$. We construct next a partition of unity using these neighbourhoods. To this end, using harmonic coordinates in $B(x_\ell, 2\gamma)$ and C^∞-smooth bump-functions in these coordinates, we construct the $C^6_*(M)$-smooth functions $\psi_\ell : M \to \overline{\mathbb{R}}_+$, such that

$$\begin{aligned} \|\psi_\ell\|_{C^{k,\beta}(M)} \ &\leq \ c_{ik,\beta}\gamma^{-(k+\beta)}, \quad k = 0, 1, 2, \quad 0 \leq \beta < 1; \\ &\text{supp } (\psi_\ell) \subset B(x_\ell, 2\gamma), \quad \sum_{\ell=1}^{N(\gamma)} \psi_\ell(x) = 1. \end{aligned} \tag{77}$$

Next we analyze a smooth cut-off of a function $u \in \mathcal{H}^s_{\Lambda_s}(M)$ to $M \setminus M(\alpha + \gamma h^{(i)})$.

Lemma 3 *There exists*

$$C_{16}(s, \gamma) = C_{17}(s)\gamma^{-s} \tag{78}$$

with the following property:
Let $u \in \mathcal{H}^s_{\Lambda_s}(M)$, $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ and $\alpha \in \mathcal{A}^{(i)}$. There exists $u_\alpha \in \mathcal{H}^s_{\frac{1}{2}C_{16}(s,\gamma)\Lambda_s}(M)$ which satisfies

$$\begin{aligned} u_\alpha(x) = u(x), \qquad &\textit{for } x \in M \setminus M(\alpha + 5\gamma h^{(i)}), \\ u_\alpha(x) = 0, \qquad &\textit{for } x \in M(\alpha + \gamma h^{(i)}). \end{aligned} \tag{79}$$

Proof. Define

$$u_\alpha(x) = \Psi(x)u(x), \quad \Psi(x) = \sum_{\text{supp } (\psi_\ell) \cap M(\alpha + \gamma h^{(i)}) = \emptyset} \psi_\ell(x). \tag{80}$$

Then, due to the finite intersection property of supports of ψ_ℓ, see Lemma 1 and (77), and calling $c_{230} = c_{230}(K, n)$ the finite number of intersections,

$$\|\Psi\|_{C^{k+\beta}(M)} \leq c_{230} c_{k,\beta} \gamma^{-(k+\beta)}, \quad k = 0, 1, 2, \quad 0 \leq \beta < 1.$$

Since $u \in \mathcal{H}^s_{\Lambda_s}(M)$, this implies the claimed properties of u_α with an appropriate $C_{17}(s)$. $\square$

We introduce sets of the finite-dimensional functions.

Definition 5 *Let $\underline{b} = (b_j)_{j=0}^{j_0} \in \mathbb{R}^{(j_0+1)}$ and $\mathcal{F}^*(\underline{b})$ be its Fourier coimage*

$$\mathcal{F}^*(\underline{b}) = \sum_{j=0}^{j_0} b_j \varphi_j \in L^2(M).$$

For $a > 0$ the class of Fourier coefficients $\mathcal{C}_{j_0,s}(a)$ is defined as

$$\mathcal{C}_{j_0,s}(a) \quad := \quad \{\underline{b} \in \mathbb{R}^{(j_0+1)};\ \sum_{j=0}^{j_0} (1+\lambda_j^2)^s |b_j|^2 \leq a^2\}. \tag{81}$$

Furthermore, if $w = W(v)$ is the solution to the initial-value problem (38), then, for $\underline{c} \in \mathbb{R}^{(j_0+1)}$, we denote

$$\mathcal{W}(\underline{c}) = W(\mathcal{F}^*(\underline{c})) \in L^2(M).$$

and, for any $\varepsilon_* > 0$, $\alpha \in \mathcal{A}^i$, we denote

$$\mathcal{C}_{j_0,s}(\varepsilon_*;\, a, \alpha) = \{\underline{c} \in \mathcal{C}_{j_0,s}(a) :\ \|W(\mathcal{F}^*(\underline{c}))\|_{L^2(\Gamma(z_\ell, \alpha_\ell - \gamma))} \leq \varepsilon_*,\ \forall \ell \in K_i\}.$$

Observe that $\underline{b} \in \mathcal{C}_{j_0,s}(a)$ if and only if $v = \mathcal{F}^*(\underline{b})$ satisfies

$$\mathcal{F}^*(\underline{b}) \in \mathcal{H}^s_a(M). \tag{82}$$

In the future, we always assume that

$$\varepsilon_2 \leq \mathcal{E}_2\left(\frac{\varepsilon_1}{L}; \theta, \gamma, C_{16}(s;\gamma)\Lambda_s\right), \tag{83}$$

see (60) for $\mathcal{E}_2$.

In particular, this implies that if $u \in \mathcal{H}^s_{C_{16}(s,\gamma)\Lambda_s}(M)$ and $w = W(u)$ satisfies

$$\|w\|_{L^2(\Gamma(z_\ell, \alpha_\ell - \gamma))} \leq \varepsilon_2 \leq \mathcal{E}_2(\frac{\varepsilon_1}{L}; \theta, \gamma, C_{16}(s,\gamma)\Lambda_s), \tag{84}$$

then, by Corollary 3 and (67), we have for $\ell \in K_i$

$$\|w(0,\cdot)\|_{L^2(B(z_\ell, \alpha_\ell - 2\gamma))} \leq \frac{\varepsilon_1}{L}, \quad \|w(0,\cdot)\|_{L^2(M(\alpha - 2\gamma h^{(i)}))} \leq \varepsilon_1. \tag{85}$$

Lemma 4 *Let P_{j_0} be the orthoprojection*

$$P_{j_0} v = \sum_{j=0}^{j_0} \langle v, \varphi_j \rangle_{L^2(M)} \, \varphi_j,$$

and $v \in \mathcal{H}^s_{(\frac{1}{2} C_{16}(s,\gamma)\Lambda_s)}(M)$. There is $C_{20}(s)$ such that, for any $\alpha \in \mathcal{A}^{(i)}$, if

$$j_0 \geq j_0(\varepsilon_2; \gamma, \Lambda_s) = C_{20} C_{16}(s,\gamma)^{n/s} \left(\frac{\Lambda_s}{\varepsilon_2} \right)^{n/s}, \tag{86}$$

then

$$\| P_{j_0} v - v \|_{L^2(M)} \leq \frac{\varepsilon_2}{4 \langle 2D \rangle^{1/2}}. \tag{87}$$

Proof. For $v = \sum_{j=0}^{\infty} b_j \varphi_j$, we have

$$\| P_{j_0} v - v \|^2_{L^2(M)} = \sum_{j > j_0} |b_j|^2 \leq \lambda_{j_0}^{-s} \left(\sum_{j=0}^{\infty} |\lambda_j|^s |b_j|^2 \right) \leq |\lambda_{j_0}|^{-s} C_{16}(s;\gamma)^2 \Lambda_s^2.$$

Using (20) this implies that

$$\| P_{j_0} v - v \|^2_{L^2(M)} \leq C_3^s j_0^{-(2s/n)} C_{16}(s,\gamma)^2 \Lambda_s^2.$$

This inequality implies (87), if j_0 satisfies (86) with

$$C_{20} = C_3^{n/2} 4^{n/s} \langle 2D \rangle^{n/2s}. \tag{88}$$

□

Observe, that the condition $\| W(\mathcal{F}^*(\underline{c})) \|_{L^2(\Gamma(z_\ell, \alpha_\ell - \gamma))} \leq \varepsilon_*$ is equivalent to

$$\left\| \Big(\sum_{j=0}^{j_0} c_j \cos(\sqrt{\lambda_j t}) \, \varphi_j(x) \Big) \Big|_{\Gamma(z_\ell, \alpha_\ell - \gamma)} \right\|_{L^2(\Gamma(z_\ell, \alpha_\ell - \gamma))} \leq \varepsilon_*, \;\; \ell \in K_i. \tag{89}$$

Note, that if we know $\{(\lambda_j, \, \varphi_j|_{B(p,r_0)})\}_{j=0}^{j_0}$, condition (89) can be directly verified.

Lemma 5 *Let $u \in \mathcal{H}^s_{\Lambda_s}(M)$ and u_α be the function given by (80). Assume that j_0 satisfies (86). Then,*

$$v_\alpha = P_{j_0} u_\alpha \in \mathcal{F}^* \left(\mathcal{C}_{j_0,s}(\frac{1}{4}\varepsilon_2;\ \frac{1}{2} C_{16}(s;\gamma)\Lambda_s, \alpha) \right). \tag{90}$$

Proof. Since $u_\alpha|_{M(\alpha+\gamma h^{(i)})} = 0$, then, by the finite speed of the wave propagation, we have $W(u_\alpha)|_{\Gamma(z_\ell,\alpha_\ell-\gamma)} = 0$ for all $\ell \in K_i$. In addition, by Lemma 4,

$$\|v_\alpha\|_{L^2(M(\alpha+\gamma h^{(i)}))} \le \|v_\alpha - u_\alpha\|_{L^2(M)} \le \frac{\varepsilon_2}{4\langle 2D\rangle^{1/2}}. \tag{91}$$

Therefore, for any $\ell \in K_i$,

$$\|W(v_\alpha)|_{\Gamma(z_\ell,\alpha_\ell-\gamma)}\|_{L^2(\Gamma(z_\ell,\alpha_\ell-\gamma))} \le \frac{\sqrt{2D}\varepsilon_2}{4\langle 2D\rangle^{1/2}} \le \frac{1}{4}\varepsilon_2. \tag{92}$$

Moreover, since for any $\widetilde{s}$, $\|v_\alpha\|_{H^{\widetilde{s}}(M)} = \|P_{j_0} u_\alpha\|_{H^{\widetilde{s}}(M)} \le \|u_\alpha\|_{H^{\widetilde{s}}(M)}$, it follows from Lemma 3 that $v_\alpha \in \mathcal{H}^s_{(\frac{1}{2}C_{16}(s;\gamma)\Lambda_s)}(M)$. □

4.3 Minimisation algorithm

Assume that we are given $\underline{a} = (a_j)_{j=1}^{j_0} \in \mathbb{R}^{(j_0+1)}$ so that

$$u = \mathcal{F}^*(\underline{a}) = \sum_{j=0}^{j_0} a_j \varphi_j \in \mathcal{H}^s_{\Lambda_s}(M).$$

Our next goal is to use FISD to find a vector $\underline{b} \in \mathcal{C}_{j_0,s}(C_{16}(s,\gamma)\Lambda_s)$ such that $\mathcal{F}^*(\underline{b})$ is close to $\chi_{M(\alpha)}\mathcal{F}^*(\underline{a})$. To achieve this goal we will use a minimisation method. Let ε_1 satisfy

$$0 < \varepsilon_1 \le \frac{\varepsilon_0^2}{10\Lambda_s}, \quad \text{with } \varepsilon_0 < \frac{1}{10}\Lambda_s. \tag{93}$$

Let ε_2 satisfy (83) and let

$$\mathcal{U} := \mathcal{F}^*\left(\mathcal{C}^*\right), \quad \text{where } \ \mathcal{C}^* = \mathcal{C}_{j_0,s}(\varepsilon_2;\ C_{16}(s,\gamma)\Lambda_s, \alpha). \tag{94}$$

Definition 6 *(i) A function $v \in \mathcal{U}$ is called an ε_1-minimizer of the minimization problem*

$$\min_{h\in\mathcal{U}} \mathcal{L}_u(h), \quad \textit{where } \mathcal{L}_u(h) = \|h-u\|^2_{L^2(M)}, \tag{95}$$

if v satisfies

$$\|v-u\|_{L^2(M)} \le J_{min} + 5\Lambda_s\varepsilon_1, \quad J_{min} := \inf_{h\in\mathcal{U}} \|h-u\|_{L^2(M)}. \tag{96}$$

(ii) A vector $\underline{b} = (b_j)_{j=0}^{j_0} \in \mathcal{C}^*$ is an ε_1-minimizer of the minimization problem

$$\min_{\underline{c}\in\mathcal{C}^*} \mathcal{L}_{\underline{a}}(\underline{c}), \quad \textit{where } \mathcal{L}_{\underline{a}}(\underline{c}) = \|\underline{c}-\underline{a}\|^2_{\mathbb{R}^{(j_0+1)}}, \tag{97}$$

if

$$\|\underline{b}-\underline{a}\|^2_{\mathbb{R}^{(j_0+1)}} \le J_{min} + 5\Lambda_s\varepsilon_1, \quad J_{min} := \inf_{\underline{c}\in\mathcal{C}^*} \|\underline{c}-\underline{a}\|^2_{\mathbb{R}^{(j_0+1)}}. \tag{98}$$

Remark 5 *Then $\underline{b} \in \mathcal{C}^*$ satisfies (98) if and only if $v = \sum_{j=0}^{j_0} b_j\varphi_j$ satisfies (96). Note that $\mathcal{L}_{\underline{a}}(\underline{c}) = \mathcal{L}_u(\mathcal{F}^*(\underline{c}))$.*

The problem of finding $\underline{b} \in \mathcal{C}^*$ satisfying (98) can be solved using the data given in Theorem 3 with j_0 satisfying (86). Indeed, for given $\underline{c} \in \mathcal{C}^*$ and

$$h = \mathcal{F}^*(\underline{c}) = \sum_{j=0}^{j_0} c_j\varphi_j(x),$$

we have

$$Wh|_{B(p,r_0)\times\mathbb{R}} = \sum_{j=0}^{j_0} c_j \cos(\sqrt{\lambda_j}\, t)\, \varphi_j|_{B(p,r_0)}. \tag{99}$$

Therefore, given $\{(\lambda_j, \varphi_j|_{B(p,r_0)})\}_{j=0}^{j_0}$, we can evaluate $\|Wh\|_{L^2(\Gamma(z_\ell,\alpha_\ell-\gamma))}$ for all $\ell \in K_i$. Hence, $\underline{c} \in \mathcal{C}^*$ if and only if it satisfies (81), with $a = C_{16}(s,\gamma)\Lambda_s$, and (89) with $\varepsilon_* = \varepsilon_2$.
Summarizing the above, we can find $\underline{b} \in \mathcal{C}^*$ satisfying (98) by solving a minimization problem for the functional $\|\underline{c}-\underline{a}\|^2_{\mathbb{R}^{(j_0+1)}}$ with $\underline{c} \in \mathcal{C}^*$.

Next we assume that, in addition to ε_2 satisfying (83), γ satisfies

$$\gamma \le \frac{1}{(2Lc(s))^{1/(2b(s))}} \left(\frac{\varepsilon_1}{\Lambda_s}\right)^{1/(b(s))}, \tag{100}$$

where $b(s)$, $c(s)$ are defined in (70), (72).

Now we are ready to consider the properties of the ε_1-minimizers.

Lemma 6 *Let* $u \in \mathcal{H}^s_{\Lambda_s}(M)$ *and assume that* ε_2 *satisfies (83),* j_0 *satisfies (86) and* γ *satisfies (100). Let* $\mathcal{U}$ *be defined in (94).*

(i) For all $h \in \mathcal{U}$*, we have*

$$\mathcal{L}_u(h) \geq \|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} - 2\Lambda_s\varepsilon_1 + \|h-u\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))}. \tag{101}$$

(ii) The function v_α *defined by (90), (80) satisfies* $v_\alpha \in \mathcal{U}$ *and*

$$\begin{aligned}\mathcal{L}_u(v_\alpha) &\leq \|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + 2\Lambda_s\varepsilon_2 + 2\varepsilon_2^2 + \varepsilon_1^2 \\ &\leq \|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + 2\Lambda_s\varepsilon_1 + 3\varepsilon_1^2.\end{aligned} \tag{102}$$

(iii) Moreover, v_α *is an* ε_1-*minimiser,*

$$\mathcal{L}_u(v_\alpha) \leq J_{min} + 5\Lambda_s\varepsilon_1. \tag{103}$$

Proof. (i) We have, for $h \in \mathcal{U}$,

$$\begin{aligned}\|h-u\|^2_{L^2(M)} &= \|h-u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + \|h-u\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} \\ &\geq (\|u\|_{L^2(M(\alpha-2\gamma h^{(i)}))} - \|h\|_{L^2(M(\alpha-2\gamma h^{(i)}))})^2 + \|h-u\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))}.\end{aligned}$$

Due to definition (82), (94) and the condition (83) it follows from (84), (85) that $\|h\|_{L^2(M(\alpha-2\gamma h^{(i)}))} \leq \varepsilon_1$. Thus,

$$\|h-u\|^2_{L^2(M)} \geq \|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} - 2\Lambda_s\varepsilon_1 + \varepsilon_1^2 + \|h-u\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))},$$

which proves (101).

(ii) Recall that we consider $u \in \mathcal{H}^s_{\Lambda_s}(M)$. We have

$$v_\alpha = (u-u_\alpha) + (u_\alpha - v_\alpha),$$

where u_α is defined by (80) and v_α by (90). Then, by (90), $v_\alpha \in \mathcal{U}$. Moreover,

$$\begin{aligned}&\|u-v_\alpha\|^2_{L^2(M)} = \|u-v_\alpha\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + \|u-v_\alpha\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} \qquad (104)\\ &\leq (\|u\|_{L^2(M(\alpha-2\gamma h^{(i)}))} + \|v_\alpha\|_{L^2(M(\alpha-2\gamma h^{(i)}))})^2 + \|u-v_\alpha\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} \\ &\leq \|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + 2\Lambda_s\frac{\varepsilon_2}{\langle 2D\rangle^{1/2}} + \left(\frac{\varepsilon_2}{\langle 2D\rangle^{1/2}}\right)^2 \\ &\quad + 2\|u-u_\alpha\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} + 2\|u_\alpha - v_\alpha\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))},\end{aligned}$$

where we use the fact that $\|u\|_{L^2(M)} \leq \Lambda_s$ and by (91), $\|v_\alpha\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} \leq \left(\frac{\varepsilon_2}{\langle 2D\rangle^{1/2}}\right)^2$. Observe, that by (79),

$$\begin{aligned} \|u-u_\alpha\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} &\leq \|u\|^2_{L^2(M(\alpha+5\gamma h^{(i)})\setminus M(\alpha-2\gamma h^{(i)}))} \\ &\leq c(s)\Lambda_s^2 L^2 \gamma^{2b(s)} \leq \varepsilon_1^2. \end{aligned} \tag{105}$$

Here the last inequality follows from the volume estimate of Lemma 2, the Sobolev embedding estimate (72) and (100). Using (91), we see that

$$\|u_\alpha - v_\alpha\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} \leq \left(\frac{\varepsilon_2}{\langle 2D\rangle^{1/2}}\right)^2.$$

Since $\langle 2D\rangle \geq 1$, this inequality together with (104) and (105) yield the first inequality in (102). To obtain the second inequality we use $\varepsilon_2 \leq \varepsilon_1$, see (63).

(iii) The claims (i) and (ii) yield that

$$\begin{aligned} &\mathcal{L}_u(v_\alpha) - J_{min} = \mathcal{L}_u(v_\alpha) - \min_{h\in\mathcal{U}} \mathcal{L}_u(h) \\ \leq\ &\left(\|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + 2\Lambda_s\varepsilon_1 + 3\varepsilon_1^2\right) - \left(\|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} - 2\Lambda_s\varepsilon_1\right) \\ \leq\ &5\Lambda_s\varepsilon_1, \end{aligned}$$

where the last inequality follows from (93). These yield (103). As $v_\alpha \in \mathcal{U}$ by (ii), the claim follows.

□

Lemma 7 *Let $u \in \mathcal{H}^s_{\Lambda_s}(M)$, ε_1 satisfies (93), ε_2 satisfies (83), j_0 satisfies (86) and γ satisfies (100).*
Let $v^* = \sum_{j=0}^{j_0} b_j\varphi_j$ be any ε_1-minimizer of the minimization problem (95), with $\underline{b} \in C_{j_0,s}(\varepsilon_2; C_{16}(\gamma,s)\Lambda_s,\alpha)$,. Then

$$\|v^* - \chi_{(M\setminus M(\alpha-2\gamma h^{(i)}))}u\|^2_{L^2(M)} \leq \varepsilon_0^2. \tag{106}$$

Proof. Let $v^* \in \mathcal{U}$ be any ε_1-minimizer of the minimization problem (95), i.e., we have $\|v^*-u\|^2_{L^2(M)} \leq J_{min} + 5\Lambda_s\varepsilon_1$. By Lemma 6, the function $v_\alpha \in \mathcal{U}$ and satisfies (103)). Thus an ε_1-minimizer satisfies

$$\begin{aligned} \|v^*-u\|^2_{L^2(M)} &\leq \|v_\alpha - u\|^2_{L^2(M)} + 5\Lambda_s\varepsilon_1 \\ &\leq \|u\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} + 7\Lambda_s\varepsilon_1 + 3\varepsilon_1^2. \end{aligned} \tag{107}$$

On the other hand, since $v^* - u$ satisfies (101), (107) implies that

$$\|v^* - u\|^2_{L^2(M\setminus M(\alpha-2\gamma h^{(i)}))} \le 9\Lambda_s\varepsilon_1 + 3\varepsilon_1^2. \tag{108}$$

In addition, since $v^* \in \mathcal{U}$, see (94), $w^* = W(v^*)$ satisfies estimate (89). As ε_2 satisfies (83), it follows from (84), (85) that

$$\|v^*|_{M(\alpha-2\gamma h^{(i)})}\|^2_{L^2(M(\alpha-2\gamma h^{(i)}))} \le \varepsilon_1^2.$$

Due to (93), this inequality together with (108), implies (106). □

Proof of Theorem 3. Assume that $\underline{a} := (a_j)_{j=0}^{j_0}$ satisfies the hypothesis. First we determine $(b_j)_{j=0}^{j_0}$ so that $v^* = \sum_{j=0}^{j_0} b_j\varphi_j(x)$ is an ε_1-minimizer of (95) and $v^* \in \mathcal{U}$. Then we see from (106) that

$$\|\chi_{M(\Gamma(\alpha-2\gamma h^{(i)})}u - \sum_{j=0}^{j_0}(a_j - b_j)\varphi_j\|_{L^2(M)} < \varepsilon_0.$$

Thus, by setting $d_j = a_j - b_j$, the function $v(x) = \sum_{j=0}^{j_0} d_j\varphi_j(x)$ satisfies equation (73).
Finally, since $u \in \mathcal{H}^s_{\Lambda_s}(M)$ and $v^* \in \mathcal{H}^s_{(C_{16}(s,\gamma)\Lambda_s)}(M)$, we see that $v = u - v^* \in \mathcal{H}^s_{(2C_{16}(s,\gamma)\Lambda_s)}(M)$. This proves Theorem 3. □

4.4 Finite interior spectral data with error

Above we have obtained the necessary conditions on the bounds $\gamma(\varepsilon_0; s, \Lambda_s)$ and $j_0(\varepsilon_0; \gamma, s, \Lambda_s)$, see (100), (93), (86) and (83) for evaluation of parameters γ and j_0, in the case when there are no errors in FISD. Let us next consider an approximate construction for the case when there is δ-error, in the sense of Definition 2, in FISD. So, let $\delta > 0$ and assume that we are given $(B_e(r_0), g^a)$ and a collection $\{(\lambda_j^a,\, \varphi_j^a|_{B_e(r_0)});\, j = 0, 1, 2, \dots, J_0\}$ that is δ-close to FISD, that is, to $(B_e(r_0), g)$ and $\{(\lambda_j,\, \varphi_j|_{B_e(r_0)});\, j = 0, 1, 2, \dots\}$, where $J_0 \in \mathbb{Z}^+$.

We recall that in this subsection we keep $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ fixed and omit marking the index when defining some new notations that depend on i.

First observe that, due to Weyl's asymptotics (20), in order to achieve (86), we would require

$$\delta < C_3^{-1} j_0^{-2/n}. \tag{109}$$

Next recall that, by Definition 2, there are intervals $I_p \subset \mathbb{R}$, $p = 0, \dots, P$ such that each interval I_p contains n_p eigenvalues λ_j and n_p approximate eigenvalues λ_j^a. Denote

$$J_0 + 1 = \sum_{p=0}^{P} n_p, \quad j_0(\varepsilon_0; \gamma, s, \Lambda_s) \le J_0 \le C_3\, j_0(\varepsilon_0; \gamma, s, \Lambda_s). \tag{110}$$

Let $\widetilde{\varphi}_j$, $j = 0, 1, \dots, J_0$, be the orthonormal set

$$\widetilde{\varphi}_j(x) = \sum_{\lambda_k \in I_p} a_{jk}^p \varphi_k(x), \quad \text{if } \lambda_j \in I_p, \tag{111}$$

where $A_p \in O(n_p)$ is the matrix defined in Definition 2, item (v). Let $E_p = A_p^{-1} = A_p^*$. We use below the matrix $E = \text{diag}\,(E_1, E_2, \dots, E_P) \in O(J_0 + 1)$

$$E = [e_{jk}]_{j,k=0}^{J_0}, \quad e_{jk} = \langle \widetilde{\varphi}_k, \varphi_j \rangle_{L^2(M)} \tag{112}$$

and note that $e_{jk} = 0$ if λ_j, λ_k do not lie in the same I_p.

Let $\underline{b} = (b_0, b_1, \dots, b_{J_0}) \in \mathbb{R}^{J_0+1}$ then, for $\widetilde{\underline{b}} = E(\underline{b})$ we have

$$\sum_{j=0}^{J_0} \widetilde{b}_j \widetilde{\varphi}_j(x) = \sum_{j=0}^{J_0} b_j \varphi_j(x),$$

and $\widetilde{\varphi}_j$ is almost an eigenfunction, namely,

$$\|\Delta_g \widetilde{\varphi}_j + \lambda_j \widetilde{\varphi}_j\|_{L^2(M)} \le \delta. \tag{113}$$

As $\delta < 1$, we see easily that

$$\begin{aligned} &\sum_{j=0}^{J_0} \langle \lambda_j^a + \delta \rangle^s |a_j|^2 \le \Lambda_s^2 \quad \text{implies} \quad \sum_{j=0}^{J_0} \langle \lambda_j \rangle^s |a_j|^2 \le \Lambda_s^2; \\ &\sum_{j=0}^{J_0} \langle \lambda_j \rangle^s |a_j|^2 \le (\frac{1}{2}\Lambda_s)^2 \quad \text{implies} \quad \sum_{j=0}^{J_0} \langle \lambda_j^a + \delta \rangle^s |a_j|^2 \le \Lambda_s^2. \end{aligned} \tag{114}$$

Theorem 4 *Let $0 < \varepsilon_0 < \Lambda_s/10$. Let also ε_1 satisfy (93), γ satisfy* (100), *ε_2 satisfy (83) and j_0 satisfy (86). There is $C_{26} > 0$ such that, if*

$$\delta < \delta_0(\varepsilon_2, \gamma, j_0, \Lambda_s) = C_{26}\, j_0^{-1/2}(\varepsilon_2; \gamma, s, \Lambda_s)\, \frac{\varepsilon_2}{C_{16}(\gamma, s)\, \Lambda_s}, \tag{115}$$

then the following holds true:
Assume that $\underline{\widetilde{a}} = (\widetilde{a}_0, \widetilde{a}_1, \dots, \widetilde{a}_{j_0})$

$$\sum_{j=0}^{j_0} \langle \lambda_j^a + \delta \rangle^s |\widetilde{a}_j|^2 \leq \Lambda_s^2, \quad u(x) = \sum_{j=0}^{j_0} \widetilde{a}_j \widetilde{\varphi}_j(x), \tag{116}$$

and $z_1, \dots, z_{\widetilde{N}(\sigma)} \in B(p, r_0/4)$ *is a* $\sigma-$*net. Let* $g^a|_{B_e(r_0)}$ *and* $(\lambda_j^a, \varphi_j^a|_{B_e(r_0)})$*, be* δ*-close, in the sense of Definition 2, to FISD* $g|_{B_e(r_0)}$ *and* $(\lambda_j, \varphi_j|_{B_e(r_0)})$ *of a manifold* $(M, g, p) \in \overline{\mathbf{M}_{n,p}(R, D, i)}$*.*
Then, for any $\Gamma(\alpha)$ *of form (65) with* $\alpha \in \mathcal{A}^{(i)}$*, it is possible to determine* $\widetilde{c} = (\widetilde{c}_j)_{j=0}^{J_0}$*,* $c_j = c_j(\alpha)$*, such that*

$$v(x) = \sum_{j=0}^{J_0} \widetilde{c}_j \widetilde{\varphi}_j(x), \quad \widetilde{c}_j = \langle v, \widetilde{\varphi}_j \rangle_{L^2(M)} \tag{117}$$

satisfies

$$\|v - \chi_{M(\alpha - 2\gamma h^{(i)})} u\|_{L^2(M)} < \varepsilon_0. \tag{118}$$

Proof. By (110) we have $J_0 \geq j_0(\varepsilon_2; \gamma, s, \Lambda_s)$, and, taking $\widetilde{a}_j = 0$ for $j > J_0$, we will assume further that $\underline{\widetilde{a}} \in \mathbb{R}^{J_0+1}$.

For $\underline{\widetilde{b}} \in \mathbb{R}^{J_0+1}$, denote

$$w^a(x,t) = \mathcal{W}^a(\underline{\widetilde{b}}) := \sum_{j=0}^{J_0} \widetilde{b}_j \cos(\sqrt{\lambda_j^a}\, t)\, \varphi_j^a(x), \quad x \in B_e(r_0). \tag{119}$$

Next, similar to (81), (82), we introduce

$$\mathcal{C}^a_{J_0,s}(a) = \{\underline{\widetilde{b}} \in \mathbb{R}^{(J_0+1)} : \ \sum_{j=0}^{J_0} \langle \lambda_j^a + \delta \rangle^s |\widetilde{b}_j|^2 \leq a^2\}; \tag{120}$$
$$\mathcal{C}^a_{J_0,s}(\varepsilon_*; a, \alpha) = \{\underline{\widetilde{b}} \in \mathcal{C}^a_{J_0,s}(a);\ \|\mathcal{W}^a(\underline{\widetilde{b}})\|_{L^2(\Gamma(z_\ell, \alpha_\ell - \gamma))} \leq \varepsilon_*,\ \ell \in K_i\}.$$

Let $\underline{\widetilde{b}} \in \mathcal{C}^a_{J_0,s}(C_{16}(\gamma, s)\Lambda_s)$ and $w(x,t)$ be as follows

$$w(x,t) = \widetilde{\mathcal{W}}(\underline{\widetilde{b}}) := (Wh)(x,t), \quad x \in M; \quad h(x) = \sum_{j=0}^{J_0} \widetilde{b}_j \widetilde{\varphi}_j(x). \tag{121}$$

Lemma 8 *Let w^a and w be defined by (119) and (121) with $\underline{\widetilde{b}} \in \mathcal{C}^a_{J_0,s}(C_{16}(s,\gamma)\Lambda_s)$. There is C_{27} such that, if δ satisfies (115) with*

$$C_{26} = \frac{1}{8C_{27}^{1/2}\langle 2D\rangle^{3/2}}, \tag{122}$$

then

$$\|w - w^a\|_{L^2(B^g_e(r_0)\times(-2D,2D))} \le \frac{\varepsilon_2}{8\langle 2D\rangle}. \tag{123}$$

Proof. Recall that $I_p = (a_p, b_p)$ and let ω_p be

$$\omega_p = \frac{1}{2}(a_p + b_p),\ p \ge 1; \quad \omega_0 = 0, \tag{124}$$

and $\mathcal{J}_p = \{j:\ \lambda_j \in I_p\}$. Due to (20), for $j \in \mathcal{J}_p$,

$$\big|\sqrt{\lambda_j} - \sqrt{\omega_p}\big| \le c\delta, \quad |\sqrt{\lambda^a_j} - \sqrt{\omega_p}| \le c\delta, \tag{125}$$

so that, for $|t| \le 2D$,

$$|\cos(\sqrt{\omega_p}\,t) - \cos(\sqrt{\lambda^a_j}\,t)| \le c2D\,\delta. \tag{126}$$

Consider next the functions

$$\widetilde{w}^a(x,t) = \sum_{p=1}^{P}\sum_{j\in\mathcal{J}_p} \widetilde{b}_j \cos\left(\sqrt{\omega_p}\,t\right)\ \varphi^a_j(x); \quad x \in B(p,r_0); \tag{127}$$

$$\widetilde{w}(x,t) = \sum_{p=1}^{P}\sum_{j\in\mathcal{J}_p} \widetilde{b}_j \cos\left(\sqrt{\omega_p}\,t\right)\ \widetilde{\varphi}_j(x), \quad x \in M.$$

As $\delta < 1$, we have $\|\varphi^a_j\|_{L^2(B^g_e(r_0))} \le 2$. Thus, using (126), we obtain for $|t| \le 2D$,

$$\begin{aligned}\|\widetilde{w}^a(\cdot,t) - w^a(\cdot,t)\|^2_{L^2(B^g_e(r_0))} &\le C\,J_0\left(\sum_{j=0}^{J_0}\widetilde{b}_j^2\right)\delta^2 D^2 \\ &\le C\langle 2D\rangle^2 J_0\,C_{16}^2(\gamma,s)\Lambda_s^2\,\delta^2. \end{aligned} \tag{128}$$

Similarly, since $(W\widetilde{\varphi}_j)(x,t) = \sum_{k=0}^{J_0} a_{kj}\varphi_k(x)\cos(\sqrt{\lambda_k}\, t)$, we have

$$\|\widetilde{w}(\cdot,t) - w(\cdot,t)\|^2_{L^2(B^g_e(r_0))} \le \|\widetilde{w}(\cdot,t) - w(\cdot,t)\|^2_{L^2(M)} \tag{129}$$
$$\le C\delta(2D)^2 \left(\sum_{j=0}^{J_0} \widetilde{b}_j^2\right) \le C\langle 2D\rangle^2 C_{16}(\gamma,s)^2\Lambda_s^2\delta^2.$$

Next, since by (3) $\|\widetilde{\varphi}_j - \varphi_j^a\|_{L^2(B(p,r_0))} \le \delta$, we have

$$\begin{aligned}\|\widetilde{w}(\cdot,t) - \widetilde{w}^a(\cdot,t)\|^2_{L^2(B(p,r_0))} &= \|\sum_{p=1}^{P}\sum_{k\in\mathcal{J}_p} \cos(\sqrt{\omega_p}t)\,\widetilde{b}_j\left(\varphi_j^a - \widetilde{\varphi}_j\right)\|^2_{L^2(B(p,r_0))} \\ &\le C J_0\delta^2\, C_{16}^2(\gamma,s)\,\Lambda_s^2. \end{aligned}\tag{130}$$

Using (128)–(130) together with (110), we see that there is $C_{27} > 0$ such that

$$\|w - w^a\|^2_{L^2(B^g_e(r_0)\times[-T,T])} \le C_{27}\, j_0(\varepsilon_0,\gamma,s,\Lambda_s)\,\delta^2\langle 2D\rangle^3\, C_{16}^2(\gamma,s)\,\Lambda_s^2, \tag{131}$$

where at the last step we use that $\delta < 1$ and (110). Due to (115), equation (123) follows from (131) taking into the account (122).

□

It now follows from (114) and (123) that

$$\begin{aligned} E\,\mathcal{C}_{J_0,s}\left(\frac{1}{2}C_{16}(\gamma,s)\,\Lambda_s\right) &\subset \mathcal{C}^a_{J_0,s}(C_{16}(\gamma,s)\,\Lambda_s) \\ &\subset E\,\mathcal{C}_{J_0,s}(C_{16}(\gamma,s)\,\Lambda_s), \\ E\,\mathcal{C}_{J_0,s}\left(\frac{1}{4}\varepsilon_2;\frac{1}{2}C_{16}(\gamma,s)\Lambda_s,\alpha\right) &\subset \mathcal{C}^a_{J_0,s}\left(\frac{1}{2}\varepsilon_2;C_{16}(\gamma,s)\Lambda_s,\alpha\right) \\ &\subset E\,\mathcal{C}_{J_0,s}(\varepsilon_2;C_{16}(\gamma,s)\Lambda_s,\alpha). \end{aligned}\tag{132}$$

Consider the quadratic function $\mathcal{L}_{\underline{a}} : \mathbb{R}^{J_0+1} \to \mathbb{R}$,

$$\mathcal{L}_{\underline{\widetilde{a}}}(\underline{\widetilde{c}}) = \sum_{j=0}^{J_0} |\widetilde{c}_j - \widetilde{a}_j|^2, \tag{133}$$

cf. (97). As E is an orthogonal matrix, we have for $\underline{\widetilde{a}} = E\underline{a}$ and $\underline{\widetilde{c}} = E\underline{c}$

$$\mathcal{L}_{\underline{\widetilde{a}}}(\underline{\widetilde{c}}) = \mathcal{L}_{\underline{a}}(\underline{c}). \tag{134}$$

Denote $\mathcal{C}^a = \mathcal{C}^a_{J_0,s}(\frac{1}{2}\varepsilon_2; \Lambda_s, \alpha)$ and let $\underline{\widetilde{b}} \in \mathcal{C}^a$ be a minimizer of $\mathcal{L}_{\underline{\widetilde{a}}}$ in $\mathcal{C}^a$. Note that now we do not anymore consider ε_1-minimizers, but exact minimizers of $\mathcal{L}_{\underline{\widetilde{a}}}$. Since $\mathcal{C}^a$ is a bounded and closed set in $\mathbb{R}^{J_0+1}$ such minimizer exists. This means that

$$\mathcal{L}_{\underline{\widetilde{a}}}(\underline{\widetilde{b}}) = J^a_{min} := \min_{\underline{\widetilde{c}} \in \mathcal{C}^a} \sum_{j=0}^{J_0} |\widetilde{c}_j - \widetilde{a}_j|^2. \tag{135}$$

Let $\underline{b} = E^{-1}\underline{\widetilde{b}}$ and denote

$$v^*(x) = \sum_{j=0}^{J_0} \widetilde{b}_j \widetilde{\varphi}_j(x) = \sum_{j=0}^{J_0} b_j \varphi_j(x).$$

Next we follow the same steps as in the proof of Theorem 3.

First, we define $\underline{a} = (a_0, \dots, a_{J_0}) = E^{-1}\underline{\widetilde{a}}$ and consider the function

$$u(x) = \sum_{j=0}^{J_0} a_j \varphi_j(x) = \sum_{j=0}^{J_0} \widetilde{a}_j \widetilde{\varphi}_j(x)$$

Next we define the function $u_\alpha(x) = \Psi(x)u(x)$ where Ψ is given in (80). Then, let $\underline{c} = (c_j)_{j=0}^{J_0}$, $c_j = \langle u_\alpha, \varphi_j \rangle_{L^2(M)}$, and by Lemmas 4 and 5, we have

$$\underline{\widetilde{c}} = E\underline{c} \in E\,\mathcal{C}_{J_0,s}\left(\frac{1}{4}\varepsilon_2,\, \frac{1}{2}\Lambda_s, \alpha\right) \subset \mathcal{C}^a_{J_0,s}\left(\frac{1}{2}\varepsilon_2, \Lambda_s, \alpha\right) = \mathcal{C}^a.$$

Using Lemma 6 (iii), we see then that

$$J^a_{min} \;\leq\; \mathcal{L}_{\underline{\widetilde{a}}}(\underline{\widetilde{c}}) = \mathcal{L}_{\underline{a}}(\underline{c}) \leq J_{min} + 5\Lambda_s\varepsilon_1. \tag{136}$$

However, $\underline{\widetilde{b}}$ is a minimizer of $\mathcal{L}_{\underline{\widetilde{a}}}$ in $\mathcal{C}^a$. Thus,

$$\mathcal{L}_{\underline{a}}(\underline{b}) = \mathcal{L}_{\underline{\widetilde{a}}}(\underline{\widetilde{b}}) = J^a_{min} \leq J_{min} + 5\Lambda_s\varepsilon_1.$$

At last, since $\underline{b} \in E^{-1}\mathcal{C}^a_{J_0,s}(\frac{1}{2}\varepsilon_2,\, \Lambda_s, \alpha) \subset \mathcal{C}_{J_0,s}(\varepsilon_2; \Lambda_s, \alpha)$, the above shows that $\underline{b} = E^{-1}\underline{\widetilde{b}}$ is an ε_1-minimizer of $\mathcal{L}_{\underline{a}}$ in $\mathcal{C}_{J_0,s}(\varepsilon_2; \Lambda_s, \alpha)$. By Lemma 7 this implies that v^* satisfies (106). Then, choosing $d_j = a_j - b_j$, $j = 0, 1, \dots, J_0$, we see that $v = \sum_{j=0}^{J_0} d_j \widetilde{\varphi}_j$ satisfies (118). This proves Theorem 4. $\square$

5 Construction of the approximate interior distance maps.

Below we consider several indexes $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ and consider i as a varying parameter.

5.1 Volume estimates

Our next goal is to approximately evaluate the volume of $M(\alpha)$. We have the following result:

Lemma 9 *There are uniform constants $\varepsilon_0(n, R, D, i_0, r_0) > 0$, $C_{33} > 0$, with the following properties:*

Let $\varepsilon_0 \leq \varepsilon_0(n, R, D, i_0, r_0)$, $\gamma < \gamma_0(\varepsilon_0, s)$, where γ_0 is defined by (75) with $\Lambda_s = 1$, and $\delta \leq \delta_0(\varepsilon_0, \gamma, s)$, where δ_0 is defined by (115) with j_0 given by the rhs of (76) with $\Lambda_s = 1$ and ε_2 given by (83). Assume that we are given $\left(g^a|_{B_e(r_0)}; \{\lambda_j^a, \varphi_j|_{B_e(r_0)}\}_{j=0}^{J_0}\right)$, with J_0 given by (110), which are $\delta-$close to FISD of $M \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$.

Let also $i = L, \dots, \widetilde{N}(\sigma)$, $\alpha \in \mathcal{A}^{(i)}$, see (37) for $\widetilde{N}(\sigma)$ and (64) and considerations thereafter for $\mathcal{A}^{(i)}$.

Then we can compute an approximate volume, $vol^a(M(\alpha))$, of the set $M(\alpha)$ so that

$$|vol^a(M(\alpha)) - vol(M(\alpha))| \leq C_{33}\varepsilon_0. \tag{137}$$

Here $M(\alpha)$ is defined in (67) with $b = 0$.

Proof. Recall that

$$\varphi_0(x) = \mathrm{vol}\,(M)^{-1/2}, \quad \mathcal{F}(\varphi_0) = (1, 0, 0, \dots), \quad \|\varphi_0\|_s = 1 \text{ for } s > 0. \tag{138}$$

By our assumption the interval $I_0 = (a_0, b_0)$ in Definition 2 contains only the eigenvalue $\lambda_0 = 0$. Thus $\varphi_0^a|_{B_e(r_0)}$ is a δ-approximation of $\varphi_0|_{B_e(r_0)}$. It then follows from (16), (138) and Definition 2 (iii)–(v) that

$$\left|\varphi_0^2 - \frac{\int_{B_e(r_0)} |\varphi_0^a(x)|^2 dV_{g^a}}{\int_{B_e(r_0)} dV_{g^a}}\right| \leq c\delta^2. \tag{139}$$

Recall that the eigenfunction $\varphi_0(x)$ corresponding to the eigenvalue $\lambda_0 = 0$ is the constant function $\varphi_0(x) = \mathrm{vol}\,(M)^{-1/2}$. Using Theorem 4 we evaluate the Fourier coefficients $(\widetilde{c}_j)_{j=0}^{j_0}$ of $v(x)$ which satisfies (118) with $u = \varphi_0$. Let

$$\mathrm{vol}^{\,a}(M(\alpha)) = \mathrm{vol}^{\,a}(M) \left(\sum_{j=0}^{j_0} \widetilde{c}_j^{\,2} \right)^{1/2} \tag{140}$$

Then, again using Theorem 4 together with (139), and the volume estimate of M due to (16) we see that, with some uniform $C > 0$,

$$|\mathrm{vol}^{\,a}(M(\alpha)) - \mathrm{vol}\,(M(\alpha - 2\gamma h^{(i)}))| \le C(\varepsilon_0 + \delta).$$

Since $\mathrm{vol}\,(M(\alpha)) - \mathrm{vol}\,(M(\alpha - 2\gamma h^{(i)})) < C\gamma$ (cf. Lemma 2), the above inequality implies estimate (137), if $\varepsilon_0 \le \varepsilon_0(n, R, D, i_0, r_0)$ with a uniform constant C_{33}. Here we use the fact that $\delta < \varepsilon_0$, $\gamma < \varepsilon_0$ for small ε_0, see (115), (75) where we take into account that $2/b(s) \ge 1$.

□

Next we show how to use FISD with errors to approximately find the distances from various points $x \in M$ to points $z \in B(p, r_0/4)$. The principal tool to achieve this goal is to approximately find the volumes of subdomains of M obtained by the slicing procedure. We use slicing related to a small parameter $0 < \sigma < \tau/2$ which will be chosen later so that if satisfies $\sigma = p_0\gamma$ with $p_0 \in \mathbb{Z}_+$.

For $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ and $\beta \in \mathcal{A}^{(i)} \cap (\sigma\mathbb{N})^L$ we use the notation $M(\beta)$ for domains defined in (67) with α replaced by β. We consider the intersection of slices,

$$M^*_{(i)}(\beta) = \{x \in M;\, d(x, z_l) \in ((\beta_\ell - 2\sigma), (\beta_\ell + 2\sigma)),\ \ell \in K_i\}, \tag{141}$$

that can be written as

$$M^*_{(i)}(\beta) = \bigcap_{\ell \in K_i} (B(z_\ell, \beta_\ell + 2\sigma) \setminus B(z_\ell, \beta_\ell - 2\sigma)).$$

To compute approximately $\mathrm{vol}\,(M^*_{(i)}(\beta))$, we write it as

$$\begin{aligned} M^*_{(i)}(\beta) &= \left(\bigcap_{\ell \in K_i} B(z_\ell, \beta_\ell + 2\sigma) \right) \cap \left(\bigcap_{\ell \in K_i} B(z_\ell, \beta_\ell - 2\sigma)^c \right) \\ &= \left(\bigcap_{\ell \in K_i} B(z_\ell, \beta_\ell + 2\sigma) \right) \cap \left(\bigcup_{\ell \in K_i} B(z_\ell, \beta_\ell - 2\sigma) \right)^c, \end{aligned}$$

where for $\Omega \subset M$, $\Omega^c = M \setminus \Omega$.

Recall that $\sigma = p_0\gamma$, where $p_0 \in \mathbb{Z}_+$. Thus for $\ell \in K_i$ we have $\beta_\ell \pm 2\sigma = A_\ell^\pm \gamma$, where $A_\ell^\pm = A_\ell \pm 2p_0$. Let us denote $a_\ell^\pm = a_\ell \pm 2p_0\gamma h^{(i)}$. Then

$$M^*_{(i)}(\beta) = \left(\bigcap_{\ell\in K_i} B(z_\ell, \alpha_\ell^+)\right) \cap \left(\bigcup_{\ell\in K_i} B(z_\ell, \alpha_\ell^-)\right)^c. \tag{142}$$

To proceed, observe that, for any Ω, $\widetilde{\Omega} \subset M$,

$$\begin{aligned}
&\mathrm{vol}\,(\Omega \cap \widetilde{\Omega}^c) = \mathrm{vol}\,(\Omega \cup \widetilde{\Omega}) - \mathrm{vol}\,(\widetilde{\Omega}),\\
&\mathrm{vol}\,(\Omega \cap \widetilde{\Omega}) = \mathrm{vol}\,(\Omega) + \mathrm{vol}\,(\widetilde{\Omega}) - \mathrm{vol}\,(\Omega \cup \widetilde{\Omega}),\\
&\mathrm{vol}\,\left((\Omega_1 \cap \Omega_2) \cup \widetilde{\Omega}\right) = \mathrm{vol}\,(\Omega_1 \cup \widetilde{\Omega}) + \mathrm{vol}\,(\Omega_2 \cup \widetilde{\Omega}) - \mathrm{vol}\,(\Omega_1 \cup \Omega_2 \cup \widetilde{\Omega}).
\end{aligned}$$

Thus, by induction,

$$\begin{aligned}
&\mathrm{vol}\,\left((\bigcap_{\ell\in K_i} \Omega_\ell) \bigcap \widetilde{\Omega}^c\right) = \sum_{\ell\in K_i} \mathrm{vol}\,(\Omega_\ell \cup \widetilde{\Omega}) \\
&- \sum_{\ell\neq\ell'=1}^{n} \mathrm{vol}\,(\Omega_\ell \cup \Omega_{\ell'} \cup \widetilde{\Omega}) + \dots + (-1)^{(L+1)} \mathrm{vol}\,\left((\bigcup_{\ell\in K_i} \Omega_\ell) \cup \widetilde{\Omega}\right) - \mathrm{vol}\,(\widetilde{\Omega}).
\end{aligned} \tag{143}$$

Returning to (142), we see that $M^*_{(i)}(\beta)$ has form (143) with $\Omega_\ell = B(z_\ell, \alpha_\ell^+)$, $\widetilde{\Omega} = \bigcup_{\ell\in K_i} B(z_\ell, \alpha_\ell^-)$.

Since, for any $\alpha_1, \alpha_2 \in \mathcal{A}^{(i)}$ we have

$$M(\alpha_1) \cup M(\alpha_2) = M(\alpha_m), \quad \text{where } (\alpha_m)_\ell = \max((\alpha_1)_\ell, (\alpha_2)_\ell),$$

it follows that all terms in (143) are of form $\mathrm{vol}\,(M(\alpha))$ for some $\alpha \in \mathcal{A}^{(i)}$. Thus, using Lemma 9, we can approximately compute each term of (143) with error $C_{33}\varepsilon_0$. Since there are $2^L + 1$ terms in (143), we obtain the following result.

Lemma 10 *There exists $\varepsilon_4(n, R, D, i_0, r_0) > 0$ with the following property:*

Let $0 < \varepsilon_4 < \varepsilon_4(n, R, D, i_0, r_0)$ and ε_0 satisfies

$$\varepsilon_0 \le C_{54}\varepsilon_4, \quad \textit{where} \quad C_{54} = \frac{1}{C_{33}(2^L + 1)}. \tag{144}$$

Let $\delta > 0$ satisfy conditions of Lemma 9 and $((B_e(r_0), g^a), \{(\lambda_j^a, \varphi_j^a|_{B_e(r_0)});\ j = 0, 1, \dots, J_0\})$ be δ-close, in the sense of Definition 2, to the FISD $((B_e(r_0), g), \{(\lambda_j, \varphi_j|_{B_e(r_0)});\ j = 0, 1, \dots\})$ of some $M \in \overline{\mathbf{M}_{n,p}(R, D, i_0)}$.

Let $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ and $\beta \in \mathcal{A}^{(i)} \cap (\sigma\mathbb{N})^L$. Then, for any $\sigma = p_0\gamma > 0$, it is possible to evaluate approximate volumes, $vol^a(M^*_{(i)}(\beta))$, of the sets $M^*_{(i)}(\beta)$ of form (141). Moreover,

$$\left| vol^a(M^*_{(i)}(\beta)) - vol(M^*_{(i)}(\beta)) \right| \le \varepsilon_4. \tag{145}$$

5.2 Distance functions approximation

We say that a function $r_M \in C^0(B(p, r_0/4))$ is an *interior distance function* if there is $x \in M$ such that

$$r_M(z) = d_M(x, z), \quad \text{for any} \quad z \in B(p, r_0/4).$$

Notice that from now on we denote the distance on M by d_M. Denoting such $r_M(\cdot) = r_{M,x}(\cdot)$, we have $r_{M,x} \in C^{0,1}(B(p, r_0/4)) \subset L^\infty(B(p, r_0/4))$.

The interior distance functions determine the interior distance map

$$R_M :\ (M, g) \to L^\infty(B(p, r_0/4)), \quad R_M(x) = r_{M,x}(\cdot).$$

We note that the map R_M or, more precisely, its image

$$R_M(M) := \{r_{M,x}(\cdot),\ x \in M\} \subset L^\infty(B(p, r_0/4)), \tag{146}$$

may be used to reconstruct (M, g). Namely, in [43], [37] it was shown how to reconstruct $(N,\ g|_N)$, where

$$N := M \setminus B(p, r_0/25). \tag{147}$$

To this end we define

$$R^{\partial N}(N) = \{r_x^{\partial N} \in L^\infty(\partial N) : x \in N\}, \tag{148}$$

where

$$r_x^{\partial N}(z) = d_N(x, z), \quad \text{for } z \in \partial B(p, r_0/25)$$

and d_N is the distance in N. Later, in section 6.1 we show that $R_M(M)$ or, more precisely, its approximation R^*, determines an approximation to $R^{\partial N}(N)$.

Note that, in general, the metric d_M induced on $R_M(M)$ from (M, g) is different from the metric d_∞ induced from $L^\infty(B(p, r_0/4))$.

Our next goal is to construct a finite approximation R^* to the set $R_M(M)$. To this end, we use the volume approximations described in the previous subsection.

First, let us define for $z, z' \in B(p, r_0/2)$ an approximative distance $d^a(z, z')$ to be the infimum of the lengths, with respect to the metric g^a, of all piecewise smooth paths $\mu : [0, 1] \to B(p, r_0)$ connecting z and z'. Assume that

$$\delta < C_{44}\sigma. \tag{149}$$

Then Definition 3 (iv) implies that

$$|d^a(z, z') - d_M(z, z')| \le C_{45}\sigma. \tag{150}$$

Let

$$\{z_1, \dots, z_{\widetilde{N}(\sigma)}\} \subset B(p, r_0/4), \quad \widetilde{N}(\sigma) \le C_{13}\sigma^{-n}$$

be a maximal σ-separated net in $B(p, r_0/4)$ with respect to the distance function d_a, see (37). Moreover, assuming $\tau \ge \sigma$, let $z_1, \dots, z_{L-1}$ form a τ-net in $B(p, r_0/4)$ with respect to metric d_a, see Lemma 1.

For any $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ and $\beta \in \mathbb{R}^{\widetilde{N}(\sigma)}$ we define a truncation operation $\mathcal{T}^{(i)} : \mathbb{R}^{\widetilde{N}(\sigma)} \to \mathcal{A}^{(i)}$, by setting

$$\mathcal{T}^{(i)}\beta = \widetilde{\beta} = (\widetilde{\beta}_\ell)_{\ell=1}^{\widetilde{N}(\sigma)},$$

where $\widetilde{\beta}_1 = \beta_1$, $\widetilde{\beta}_2 = \beta_2, \dots, \widetilde{\beta}_{L-1} = \beta_{L-1}$ and $\widetilde{\beta}_i = \beta_i$, and finally, $\widetilde{\beta}_\ell^{(i)} = 0$ for $\ell \notin K_i$. To proceed, observe that for any $x \in M \setminus B(p, 3r_0/8 + \sigma)$ and any $\ell = 1, \dots, \widetilde{N}(\sigma)$ there is $\beta_\ell(x) \in \mathbb{Z}$ such that

$$\beta_\ell(x) - \sigma \le d_M(x, z_\ell) \le \beta_\ell(x) + \sigma.$$

Then, $\beta_\ell = \beta_\ell(x)$ satisfies

$$B(x, \sigma) \subset B(z_\ell, \beta_\ell + 2\sigma) \setminus B(z_\ell, \beta_\ell - 2\sigma).$$

Note that by (16), we have

$$\varepsilon_4 \le \frac{1}{4} C_1^{-1}\sigma^n. \tag{151}$$

Let $\beta(x) = (\beta_\ell(x))_{\ell=1}^{\widetilde{N}(\sigma)} \in \mathbb{R}^{\widetilde{N}(\sigma)}$. Then, for any $i \in \{L, \dots, \widetilde{N}(\sigma)\}$, the truncated element $\beta^{(i)}(x) = \mathcal{T}^{(i)}(\beta(x)) \in \mathcal{A}^{(i)}$ defines $M^*_{(i)}(\beta^{(i)}(x))$ such that

$$\mathrm{vol}\,(M^*_{(i)}(\beta^{(i)}(x))) \geq 4\varepsilon_4, \quad \text{so that} \quad \mathrm{vol}^{\,a}(M^*_{(i)}(\beta^{(i)}(x))) \geq 3\varepsilon_4. \tag{152}$$

These suggest the following

Definition 7 *Let $\beta = (\beta_\ell)_{\ell=1}^{\widetilde{N}(\sigma)}$, $\beta_\ell \in \sigma\mathbb{Z}_+$ and $\beta_\ell \geq r_0/8$. We say that such β is* admissible, *if for all $i \in \{L, L+1, \dots, \widetilde{N}(\sigma)\}$ the truncated element $\beta^{(i)} = \mathcal{T}^{(i)}(\beta) \in \mathcal{A}^{(i)}$ satisfies*

$$vol^a(M^*_{(i)}(\beta^{(i)})) \geq 3\varepsilon_4. \tag{153}$$

We define the set $\mathcal{B} = \{\beta \in \sigma\mathbb{Z}_+^{\widetilde{N}(\sigma)};\ \beta$ is admissible$\}$.

Lemma 11 *For any $x \in M \setminus B(p, 3r_0/8 + \sigma)$, there exists an admissible $\beta = (\beta_1, \dots, \beta_{\widetilde{N}(\sigma)}) \in \sigma\mathbb{N}^{\widetilde{N}(\sigma)}$ such that*

$$|d_M(x, z_\ell) - \beta_\ell| \leq 2\sigma, \quad \ell \in \{1, 2, \dots, \widetilde{N}(\sigma)\}.$$

Conversely, there is $C_{28} > 0$ such that, if $\beta = (\beta_1, \dots, \beta_{\widetilde{N}(\sigma)})$ is admissible, then there is $x \in M \setminus B(p, 3r_0/8 - C_{28}\sigma)$ such that, for all $\ell \in \{1, 2, \dots, \widetilde{N}(\sigma)\}$,

$$|\beta_\ell - d_M(x, z_\ell)| \leq C_{28}\sigma. \tag{154}$$

Proof. The first statement follows from considerations before Definition 7.

On the other hand, assume that $\beta = (\beta_\ell)_{\ell=1}^{\widetilde{N}(\sigma)} \in \mathcal{B}$. Then equations (145) and (153) guarantee that, for any $i \in \{L, \dots, \widetilde{N}(\sigma)\}$ the truncated element $\beta^{(i)} = \mathcal{T}^{(i)}(\beta)$ satisfies

$$\mathrm{vol}\,(M^*_{(i)}(\beta^{(i)})) \geq 2\varepsilon_4, \tag{155}$$

so that there is $x_i \in M^*_{(i)}(\beta^{(i)})$, $i \in \{L, \dots, \widetilde{N}(\sigma)\}$. Thus, $|d_M(x_i, z_i) - \beta_i| \leq 2\sigma$ and

$$|d_M(x_i, z_\ell) - \beta_\ell| \leq 2\sigma, \quad \text{for all } \ell = 1, 2, \dots, L.$$

Moreover, in view of (24), for $j, k \in \{L, \dots, \widetilde{N}(\sigma)\}$,

$$d_M(x_j, x_k) \le C_6 |H(x_i j - H(x_k)| \le 4C_6\sqrt{L}\sigma.$$

The above two equations prove (154) for any $x = x_i$ with

$$C_{28} = 4C_6\sqrt{L} + 3. \tag{156}$$

Moreover, since $\{z_\ell\}_{\ell=1}^{\widetilde{N}(\sigma)}$ form a $\sigma-$net in $B(p, r_0/4)$, condition $\beta_\ell \ge 3r_0/8$, $\ell = 1, \dots, \widetilde{N}(\sigma)$, implies that $x \in M \setminus B(p, 3r_0/8 - C_{28}\sigma)$.

□

For any $\ell = 1, \dots, \widetilde{N}(\sigma)$, let $V_\ell \subset B(p, r_0/4)$ be the corresponding Voronoy region, i.e. the set of points $z \in B(p, r_0/4)$ for which z_ℓ is a closest point among z_k, $k = 1, \dots, \widetilde{N}(\sigma)$. Note that as z_k, $k = 1, \dots, \widetilde{N}(\sigma)$, form a $\sigma-$net,

$$V_\ell \subset B(z_\ell, \sigma). \tag{157}$$

For any admissible $\beta \in \mathcal{B}$ we associate a piecewise constant function $r_\beta \in L^\infty(B(p, r_0/4))$

$$r_\beta(z) = \beta_\ell, \quad \text{for } z \in V_\ell, \quad \ell = 1, \dots, \widetilde{N}(\sigma). \tag{158}$$

Let

$$R^*_> = \{r_\beta(\cdot) : \ \beta \in \mathcal{B}\ \} \subset L^\infty(B(p, r_0/4)).$$

Choose a $\sigma-$net $\{x_1, \dots, x_{N'(\sigma)}\} \subset B(p, r_0/2)$ by adding to $z_1, \dots, z_{\widetilde{N}(\sigma)}$ a $\sigma-$net in $B(p, r_0/2) \setminus B(p, r_0/4)$. Next we define

$$r_k(z) = d^a(x_k, z_\ell), \quad \text{for } z \in V_\ell, \quad k = 1, \dots, N'(\sigma),\ \ell = 1, \dots, \widetilde{N}(\sigma),$$

where d^a is defined before (150). Let

$$R^*_< = \{r_k(\cdot) : \ k = 1, \dots, N'(\sigma)\} \subset L^\infty(B(p, r_0/4)), \quad \text{and}$$
$$R^* = R^*_> \cup R^*_<. \tag{159}$$

Lemma 12 *We have*

$$d_H(R_M(M), R^*) \le C_{29}\sigma, \quad \textit{where } C_{29} = 2C_{28} + 2C_{45} + 1, \tag{160}$$

where d_H stands for the Hausdorff distance in $L^\infty(B(p, r_0/4))$, C_{28} is defined in (156) and C_{45} is determined in (150).

Proof. By Lemma 11 and equation (157),

$$d_H(R^*_>,\ R_M(M \setminus B(p, 3r_0/8))) \le (2C_{28}+1)\sigma.$$

On the other hand, it follows from the definition (159) and (150) that

$$d_H(R^*_<,\ R_M(B(p, r_0/2))) \le (2C_{45}+1)\sigma.$$

The above two inequalities imply (160).

□

6 Proof of Theorem 1 and Corollary 2

6.1 From interior distance functions to boundary distance functions

By standard estimates for the differential of the exponential map, see [60, Ch. 6, Cor. 2.4] the diameter of the sphere $\partial B(p,r)$, $r < r_0$, is bounded

$$\text{diam}\,(\partial B(p,r)) \le \pi r \cdot \frac{\sinh(\sqrt{K} r)}{\sqrt{K} r} \le \pi r\, \cosh(\frac{\pi}{2}) \le 10\, r, \tag{161}$$

where we use condition (18). Let $N = M \setminus B(p, r_0/25)$.

Lemma 13 *Let $x \in M \setminus B(p, r_0/4)$ and $y \in \partial N$ and $z \in \partial B(p, r_0/4)$, let*

$$\begin{aligned} f(y,x,z) &= d_N(y,z) + d_M(z,x), \\ f(y,x) &= \min_{z_1 \in \partial B(p,r_0/4)} f(x,y;z_1), \end{aligned} \tag{162}$$

where d_N and d_M are the distances in N and M, respectively. Then,

$$d_N(y,x) = f(y,x) \tag{163}$$

Proof. Clearly, as $d_M(z,x) \le d_N(z,x)$ and a shortest curve in N from y to x intersects the sphere $\partial B(p, r_0/4)$, we see that $d_N(y,x) \ge f(y,x)$.

On the other hand let $z' =\ \text{argmin}_z(f(y,x;z))$ and $\mu([0, f(y,x)]$ be the corresponding union of the distance minimizing paths from y to z' and from z' to x for which the minimum in (162) is achieved. Denote $s_1 = d_N(y,z')$ and consider $\mu([s_1, f(y,x)]$. We show next that $\mu([s_1, f(y,x)] \subset N$. If this is not the case, there would exists $s_1 < s_2 < s_3 < f(y,x)$ such that $\mu(s_1), \mu(s_3) \in \partial B(p, r_0/4)$, $\mu(s_2) \in \partial B(r_0/25)$ and $\mu[s_3, f(y,x)] \subset M \setminus B(p, r_0/4)$. Then,

$$s_1 \geq r_0 \left(\frac{1}{4} - \frac{1}{25}\right), \; s_2 - s_1 \geq r_0 \left(\frac{1}{4} - \frac{1}{25}\right), \; s_3 - s_2 \geq r_0 \left(\frac{1}{4} - \frac{1}{25}\right). \quad (164)$$

On the other hand, consider a curve $\mu'([0,l])$ which is parametrised by the arclength and consists of the radial path from $\mu(s_3)$ to $y' \in \partial B(r_0/25)$ followed by a shortest path along $\partial B(r_0/25)$ from y' to y. Due to (161) and (164),

$$l \leq r_0 \left(\frac{10}{25} + \frac{1}{4} - \frac{1}{25}\right) < 3r_0 \left(\frac{1}{4} - \frac{1}{25}\right) \leq s_3.$$

Taking the union of the path $\mu'([0,l])$, connecting $\mu(s_3)$ to y', and the path $\mu(s_3, f(y,x))$, connecting y' to x, we get a contradiction to definition (163). Thus, $\mu([s_1, f(y,x)]) \subset N$, i.e., $d_N(y,x) \leq f(y,x)$.

□

Next, using the already constructed set R^*, see (159) together with Lemmata 12 and 13, we construct a set $R^*(N) \subset L^\infty(\partial N)$ which approximates $R^{\partial N}(N)$ defined (148).

Lemma 14 *Let R^* be the set given in (159), which satisfies (160) be given. Then it defines a set $R^*(N) \subset L^\infty(\partial N)$ such that*

$$d_H(R^{\partial N}(N),\, R^*(N)) \leq C_{35}\sigma, \quad C_{35} = 2C_{29} + C_{45}. \quad (165)$$

Here C_{29} is defined in (160) and C_{45} is defined in (150).

Note that here we assume that δ satisfies (115), σ satisfies (151) with the related equations for $\varepsilon_4, \varepsilon_0$, etc.

Proof The proof is based on the construction of $R^*(N)$ which satisfies (165).

Observe first that it follows from the proof of Lemma 13 that, if $x, y \in B(p, r_0/4) \setminus B(p, r_0/25) \subset N$, then

$$d_N(x,y) \leq \frac{r_0}{2} + \frac{8r_0}{25},$$

so that a shortest path in N connecting x and y lies in $B(p, r_0)$. Thus it is possible, using (2), to construct an approximation $\widetilde{r}_x^{\partial N} : \partial N \to \mathbb{R}$ that satisfies

$$\|r_x^{\partial N} - \widetilde{r}_x^{\partial N}\|_{L^\infty(\partial N)} \leq C_{45}\sigma, \quad (166)$$

cf. (150). Denote $R^*_<(N) = \{\widetilde{r}^{\partial N}_x;\, x \in B(p, r_0/4) \setminus B(p, r_0/25)\}$, then

$$d_H(R^{\partial N}(B(p, r_0/4)),\, R^*_<(N)) \leq C_{45}\sigma, \tag{167}$$

for $\delta < \delta_0$, cf. construction of $R^*_<$ in subsection 5.2.

Next, let

$$R^*_c = \{r \in R^* :\, \min_{z \in \partial N}(r(z)) \geq \frac{r_0}{8}\}$$

For $y, z \in B(p, r_0/4) \setminus B(p, r_0/25)$ denote by $d^a_N(y, z)$ the distance between y and z in the metric g^a along the curves lying in $B(p, r_0/2) \setminus B(p, r_0/25)$. For each $r \in R^*_c$ we define

$$\begin{aligned} &\widetilde{r}^{\partial N} \in L^\infty(\partial N) :\, \widetilde{r}^{\partial N}(y) = \inf_{z \in \partial B(p, r_0/4)} \left(d^a_N(z, y) + r(z)\right); \qquad (168)\\ &R^*_>(N) = \{\widetilde{r}^{\partial N}(\cdot) :\, r \in R^*_c\}. \end{aligned}$$

Then, with $R^*(N) = R^*_<(N) \cup R^*_>(N)$, we have that

$$d_H(R^{\partial N}(N),\, R^*(N)) \leq (C_{45} + 2C_{29})\sigma.$$

Here $C_{45}\sigma-$error comes from an approximation of $d_N(y, z)$, $z \in \partial B(p, r_0/4)$, see (150), and $2C_{29}\sigma-$error comes from approximating $d_M(z, x)$ and $d_N(y, z)$ in formula (163), see (160). At last, we use again that δ satisfies the uniformly bound (115). □

Recall that the metric tensor g on $B(p, r_0)$ is a representation of a metric in Riemannian normal coordinates and the $C^{2,\alpha}$-norm of the metric is uniformly bounded. Using the fundamental equations of the Riemannian geometry, [60, Ch. 2, Prop. 4.1 (3)], we have that the shape operator S of the surface $\partial B(p, r)$, $r < r_0$, can be given in the Riemannian normal coordinates centered at p in terms of the metric tensor as $S = g^{-1}\partial_\nu g$, where ν is the unit normal vector of $\partial B(p, r)$. Taking $r = r_0/25$, we see that the $C^{1,\alpha}$-norm of the shape operator S of ∂N is uniformly bounded. Also, by (18), the boundary injectivity radius of $(N, g|_N)$ is bounded below by $\frac{24}{25}i_0$. As the sectional curvature of M and the second fundamental form (that is equivalent to the shape operator) of its submanifold ∂N are bounded, the Gauss-Codazzi equations imply that the sectional curvature of ∂N is bounded. As the metric tensor of M is bounded in normal coordinates in $B(p, r_0)$, we see that the $(n-1)$-dimensional volume of $\partial N = \partial B(p, r_0/25)$ is bounded from below by a uniform constant. Thus by Cheeger's theorem, see [60, Ch. 10, Cor. 4.4], the injectivity radius of ∂N is bounded from below by a uniform constant.

Summarising the above, the Ricci curvature of $(N, g|_N)$ is uniformly bounded in C^α, the second fundamental form of ∂N is uniformly bounded in $C^{1,\alpha}$, and the diameter and injectivity radii of N and ∂N, and the boundary injectivity radius of $(N, \partial N)$ are uniformly bounded. By [37], using the knowledge of the set, $R^*(N)$ of approximate boundary distance functions, which are $C_{35}\sigma-$Hausdorff close to the set, $R^{\partial N}(N)$ of the boundary distance functions of manifold $(N, g|_N)$, one can construct on the set $R^*(N)$ a new distance function $d_N^* : R^*(N) \times R^*(N) \to \mathbb{R}_+$, such that

$$d_{GH}((N, d_N), (R^*(N), d_N^*)) \leq C_{36}(C_{35}\sigma)^{1/36}, \tag{169}$$

with a uniform $C_{36} > 0$.

Having constructed $(R^*(N), d_N^*)$ we can now construct an approximate metric space (M^*, d_M^*) which is $C_{36}(C_{35}\sigma)^{1/36}-$ close to (M, d_M). Indeed, let $x, y \in N$ and $\mu[0, l]$, $l = d_M(x, y)$ be a shortest between x and y. If $\mu[0, l] \subset N$ then $d_M(x, y)) = d_N(x, y)$. If, however, $\mu[0, l]$ intersects with $B(p, r_0/25)$ then, due to the convexity of $B(p, r_0/25)$, there are $0 < s_1 < s_2 < l$ such that

$$\mu[0, s_1] \subset N, \quad \mu[s_1, s_2] \subset B(p, r_0/25), \quad \mu[s_2, l] \subset N.$$

Therefore, similar to Lemma 13, we obtain

Corollary 5 *Let $x, y \in N$. Then*

$$d_M(x, y) = \tag{170}$$
$$\min\left(d_N(x, y), \min_{z_1, z_2 \in \partial B(p, r_0/25)} [d_N(x, z_1) + d_M(z_1, z_2) + d_N(z_2, y)]\right).$$

Next define, for $\widetilde{r}_1^{\partial N}, \widetilde{r}_2^{\partial N} \in R^*(N)$,

$$d_M^*(\widetilde{r}_1^{\partial N}, \widetilde{r}_2^{\partial N}) = \tag{171}$$
$$\min\left(d_N^*(\widetilde{r}_1^{\partial N}, \widetilde{r}_2^{\partial N}), \min_{z_1, z_2 \in \partial B(p, r_0/25)} \left[\widetilde{r}_1^{\partial N}(z_1) + d^a(z_1, z_2) + \widetilde{r}_2^{\partial N}(z_2)\right]\right)$$

Using (170) together with (150), (169) and (2), we see that

$$d_{GH}((N, d_M), (R^*(N), d_M^*) \leq (2C_{36} + 1)(C_{35}\sigma)^{1/36} \quad \text{if } C_{45}\sigma \leq (C_{35}\sigma)^{1/36} \tag{172}$$

Here (N, d_M) is the manifold N with the distance function inherited from M and $\delta < \delta_0$, cf. (167).
Let us define the disjoint union $M^* = R^*(N) \cup B(p, r_0/25)$. Next we define a metric d_M^* on this set. To this end, consider first $\widetilde{r}^{\partial N} \in R^*(N)$, $y \in B(p, r_0/25)$. Recall, see the proof of Lemma 14, that the set $R^*(N)$ is bijective with $R_c^* \cup (B(p, r_0/4) \setminus B(p, r_0/25))$. In the case when $\widetilde{r}^{\partial N}$ is obtained from $r \in R_c^*$, we define $d_M^*(\widetilde{r}^{\partial N}, y) = r(y)$. Moreover, in the case when $\widetilde{r}^{\partial N}$ is obtained from $x \in B(p, r_0/4) \setminus B(p, r_0/25)$, we define $d_M^*(\widetilde{r}^{\partial N}, y) = d^a(x, y)$. At last, if $x, y \in B(p, r_0/25)$, we take $d_M^*(x, y) = d^a(x, y)$.

It follows from (172) together with equations (150), (2), (160) and considerations preceding Lemma 12 that

$$d_{GH}((M^*, d_M^*), (M, d_M)) \leq (2C_{36} + 1)(C_{35}\sigma)^{1/36}. \tag{173}$$

Summarizing, we obtain

Lemma 15 *Let R^* satisfy (160) and $M^* = R^*(N) \cup B(p, r_0/25)$ with metric d_M^*. Then,*

$$d_{GH}((M, d_M), (M^*, d_M^*)) \leq C_{47}\sigma^{1/36}, \quad C_{47} = (2C_{36} + 1)C_{35}^{1/36}. \tag{174}$$

6.2 Proof of Theorem 1

To prove the statement of the Theorem, we collect all the previous estimates. The aim is to find the relation between the final error ε (i.e. $d_{GH}((M, d_M), (M^*, d_M^*)) \leq \varepsilon$) and the initial error δ. We proceed by following the chain of relations:

$$\varepsilon \mapsto \sigma \mapsto \varepsilon_4 \mapsto \varepsilon_0 \mapsto \varepsilon_1 \mapsto \gamma \mapsto \varepsilon_2 \mapsto j_0 \mapsto \delta. \tag{175}$$

By (174) we determine $\sigma = \left(\frac{\varepsilon}{C_{47}}\right)^{36}$ and use it in (151), (144) and (93) with $\Lambda_s = 1$ to calculate ε_0 and ε_1 so that

$$\varepsilon_0 \leq \frac{C_{54}}{4C_1 C_{47}^{36n}}\varepsilon^{36n}, \quad \varepsilon_1 \leq C_{40}\varepsilon^{72n} \quad \text{with} \quad C_{40} = \frac{C_{54}^2}{160C_1^2 C_{47}^{72n}}. \tag{176}$$

To proceed with γ, it follows from (100) that

$$\gamma \leq C_{41}\varepsilon_1^{1/b(s)}, \quad C_{41} = (2Lc(s))^{-1/(2b(s))}. \tag{177}$$

Now use (115), (83), (86), (60) and (177) to get

$$\delta \le \frac{C_{80}\varepsilon_1^{\frac{1}{b(s)}\left(s+\frac{n}{2}+\frac{s}{s-1}(1+\frac{n}{2s})\right)}}{\exp\left[\frac{s}{s-1}\left(1+\frac{n}{2s}\right)\left(\frac{C_{30}}{C_{41}^{2-\theta/2}}\right)^{1/\beta}\frac{L^{1/\beta}}{\varepsilon_1^{(1+(2-\theta/2)/b(s))/\beta}}\exp\left(\frac{C_{41}^{-c_{200}}}{\beta}\varepsilon_1^{-c_{200}/b(s)}\right)\right]}, \tag{178}$$

i.e.

$$\delta \le \frac{C_{80}\,\varepsilon_1^{C_{39}}}{\exp\left[C_{81}\,\varepsilon_1^{-C_{48}}\exp(C_{82}\,\varepsilon_1^{-C_{49}})\right]}, \tag{179}$$

with

$$C_{39} = \frac{1}{b(s)}\left(s + \frac{s}{s-1}\left(1+\frac{n}{2s}\right)+\frac{n}{2}\right), \quad C_{81} = \frac{sL^{1/\beta}}{s-1}\left(1+\frac{n}{2s}\right)\left(\frac{C_{30}}{C_{41}^{2-\frac{\theta}{2}}}\right)^{1/\beta},$$

$$C_{82} = \frac{C_{41}^{-c_{200}}}{\beta}, \quad C_{49} = \frac{c_{200}}{b(s)}, \quad C_{48} = \frac{1}{\beta}\left(1+\frac{2-\frac{\theta}{2}}{b(s)}\right), \quad C_{80} = \frac{C_{26}C_{41}^{s+\frac{n}{2}+\frac{s}{s-1}(1+\frac{n}{2s})}}{C_{20}^{1/2}C_{17}^{1+\frac{n}{2s}}}.$$

We use the inequality $x \le \exp(x)$ to bound from below the right hand side of the estimate above to obtain, by calling $C_{46} = \max(C_{39}, C_{48}, C_{49}, 1/(2n))$,

$$\delta \le \frac{1}{\exp\left[\exp((C_{80}^{-1}+C_{81}+C_{82})\varepsilon_1^{-C_{46}})\right]}, \tag{180}$$

Notice that (149) and (109) are also satisfied, by substituting to C_{80} the quantity $C_{85} = \min(C_3^{-1}C_{80},\, C_{44}/(C_{47}^{36}C_{40}^{1/(2n)}))$. Assuming $0 < \delta \le \exp(-e)$, we get

$$\frac{(C_{85}^{-1}+C_{81}+C_{82})}{\ln\left(\ln\frac{1}{\delta}\right)} \le \varepsilon_1^{C_{46}}, \tag{181}$$

According to (93) with $\Lambda_s = 1$, the condition $\varepsilon_1 \le 1/1000$ implies

$$\delta \le C_{42}, \;\; \text{with } C_{42} = \min\left(\exp(-e),\; 1/\exp[\exp[1000^{C_{46}}(C_{85}^{-1}+C_{81}+C_{82})]]\right).$$

Finally by using (176) and defining

$$C_{43} = \frac{(C_{85}^{-1}+C_{81}+C_{82})^{1/(72nC_{46})}}{C_{40}^{1/(72n)}}$$

we obtain (7). We also define $\delta^* = C_{42}$. By (110) we define $C_{50} = [C_3 j_0(\varepsilon_0; \gamma, s, 1)]$, where [] denotes the integer part. Using (76) we get

$$C_3 j_0(\varepsilon_0; \gamma, s, 1) = C_3 C_{20} \frac{C_{17}^{n/s}}{C_{41}^{n+\frac{n}{s-1}}} \varepsilon_1^{\frac{-1}{b(s)}(n+\frac{n}{s-1})} \exp\left[\frac{nC_{81}}{sL^{1/\beta}(1+\frac{n}{2s})} \varepsilon_1^{-C_{48}} \exp\Big(C_{82} \varepsilon_1^{-C_{49}} \Big) \right] \quad (182)$$

and then we substitute the ε_1 obtained from (181) (with equality sign) in order to get the δ dependency. This completes the proof of Theorem 1. □

6.3 Proof of Corollary 2

Let $\delta \leq \delta^* = C_{42}$ and let the ISD of $M^{(i)}$, $i = 1, 2$ be $\delta-$close.
Call $\varepsilon =: \varepsilon(\delta) = C_{43} / \big(\ln \big(\ln \frac{1}{\delta} \big) \big)^{1/(72nC_{46})}$. Now define the set

$$\mathcal{D} = \Big((B^e(r_0), g^{(1)}),\ \{\lambda_j^{(1)},\ \varphi_j^{(1)}\}_{j=0}^{J_0} \Big)$$

where the index (1) is related to the IDS of $M^{(1)}$. For J_0 sufficiently large and by construction we see that the data $\mathcal{D}$ are δ-close to the ISD of both $M^{(1)}$ and $M^{(2)}$. By Theorem 1 the metric space (M^*, d_M^*) constructed with these data is $\varepsilon-$close to both $(M^{(i)}, d^{(i)}), i = 1, 2$, i.e.

$$d_{GH}((M^*, d_M^*),\ (M^{(i)}, d^{(i)})) \leq \varepsilon, \quad (183)$$

where ε is given by the right hand side of (7). We then conclude by triangular inequality, see [18, Prop. 3.7.16], for any $0 < \delta \leq C_{42}$,

$$d_{GH}((M^{(1)}, d^{(1)}),\ (M^{(2)}, d^{(2)})) \leq 2\varepsilon = 2C_{43} / \big(\ln \big(\ln \frac{1}{\delta} \big) \big)^{1/(72nC_{46})}. \quad (184)$$

We now extend this estimate to the case $\delta \in (0, \exp(-e)]$, when $C_{42} < \exp(-e)$. To this end, observe that the definition of the GH-topology and (5) imply that, independently on δ,

$$d_{GH}((M^{(1)}, d^{(1)}),\ (M^{(2)}, d^{(2)})) \leq D. \quad (185)$$

By combining (184) and (185), and comparing them in $\delta = C_{42}$, we obtain the inequality (8) with

$$C_{84} \;=\; 2 \max \Big(1,\ \frac{D}{2\varepsilon(\delta)|_{\delta = C_{42}}} \Big) C_{43} = \max \Big(2C_{43}, D \big(\ln \big(\ln \frac{1}{C_{42}} \big) \big)^{1/(72nC_{46})} \Big)$$

$\square$

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7 Appendix

7.1 Calculation of c_{206} in Theorem 2

In the following section we will not follow the notations of the article, since all is derived from calculations done in [17].
Let $\alpha \in [1/3, 1)$, and T, ℓ, γ be defined as in Assumption A5, [17]. We define the following Gevrey function as the smooth cut-off (see [61], Ex 1.4.9 for definition):

$$\chi_1(t) = \chi(1+t)\chi(1-t), \quad \text{with } \chi(s) = \exp(-s^{\frac{\alpha}{\alpha-1}}) \text{ for s} > 0,\ \chi(s) = 0 \text{ for s} \leq 0.$$

One can slightly modify the definition such that $\chi_1 = 1$ in a ball $B_1 \subset \mathbb{R}$, $\chi_1 = 0$ outside the ball B_2, and $0 \leq \chi_1 \leq 1$. Observe that $\chi_1 \in G_0^{1/\alpha}(\mathbb{R})$ since:

$$|D^\kappa \chi_1(v)| \leq c_{0X} c_{1X}^{|\kappa|} |\kappa|^{|\kappa|/\alpha}, \quad \text{with } c_{0X} = O(1),\, c_{1X} = O\Big(\frac{1}{1-\alpha}\Big). \tag{186}$$

Furthermore, define $\chi_\delta(v) := \chi_1(v/\delta)$, $v \in \mathbb{R}^M$. Hence, $\mathcal{F}_{v\to\zeta}\chi_\delta(v) = \delta^M \mathcal{F}_{v\to\delta\zeta}\chi_1(v)$ for $\zeta \in \mathcal{C}$, and calling $c_{2X} = 1/(eMc_{1X})^\alpha$ we get

$$|\mathcal{F}_{v\to\zeta}\chi_\delta(v)| \;\leq\; \delta^M c_{0X} \exp(\delta H_{B_2}(\mathrm{Im}\zeta) - c_{2X}\delta^\alpha |\mathrm{Re}\zeta|^\alpha) \cdot \mathrm{Vol}(\mathrm{supp}(\chi_1), dv). \tag{187}$$

Product: For $v \in B_2(\mathbb{R}^M)$,

$$|D^\kappa \chi_1(v)\chi_2(v)| \;\leq\; c_{0X,1}c_{0X,1} \max\{c_{1X,1}, c_{1X,2}\}(\max\{c_{1X,1}, c_{1X,2}\})^{|\kappa|}|\kappa|^{|\kappa|/\alpha}.$$

We also recall and improve the coefficients in Lemma 2.1, [17], for L^2 and H^m:

$$c_{107} = c_3\Big(\frac{8}{\beta_1}\Gamma\Big(\frac{1}{\alpha}\Big)\frac{1}{\alpha(c_{117})^{\frac{1}{\alpha}}}\Big)^{\frac{1}{2}}\frac{1}{(\alpha c_{106})^{\frac{1}{\alpha}}}, \tag{188}$$

$$c_{108} = c_{107}(1 + |D_x^m f|_{C^0}) + c_{107}\frac{(1+m)^{\frac{(m+1)}{\alpha}}}{(\alpha c_{106})^{\frac{m}{\alpha}}}$$

$$\|A(\beta_1 D_0/\mu) f(1 - A(D_0/\mu))v\|_1 \leq c_{108} e^{-c_{106}\mu^\alpha} \|v_{\mathrm{supp}(f)}\|_m\,.$$

In our application we are interested in the γ and $(1-\alpha)$ dependency of the geometric parameters, since both quantities tend to zero. We link these coefficients with the following assumption:

$$\alpha^N = \frac{1}{2} \text{ that implies } \alpha^{1/r^{(n+1)}} = \frac{1}{2} \Rightarrow (1-\alpha) \sim r^{n+1} \text{ as } \alpha \to 1, \qquad (189)$$

with $N = c_{170}$ and r defined in the following Table 1. From now on $\sim$ means "up to a coefficient independent of γ or $(1-\alpha)$". Consequently, for χ_1 and c_{1X} defined above, we get

$$c_{1X} \sim \frac{1}{1-\alpha} \sim \frac{1}{\gamma^{58(n+1)}}, \qquad |\chi_1'|_{C^0(\Omega_0)} \sim c_{1X}, \qquad |\chi_1''|_{C^0(\Omega_0)} \sim c_{1X}^2.$$

We now consider the hyperbolic surfaces defined in Remark A.1. of [17] and we calculate the corresponding Table A.3. in [17] (see also Table in [16]). The lower indexes of the coefficients and the formulas in brecket correspond to the ones used in [17, 16].

Table 1

Name		Order w.r.t. γ and $(1-\alpha)$
C_l	$\sim$	γ (4.7)-[17]
p_1	$\sim$	γ^2 (4.7)-[17]
$dist\{\partial\Omega_0, \Omega_a\}$	$\sim$	γ^2 (4.12)-[17]
$\lvert\psi'\rvert_{C^k}$	$\sim$	1 (4.8)-[17]
$d_g(x,z)$	$\in$	$[\ell, T-\gamma]$ (in $\Gamma\setminus$ cylinder)
$\lvert\partial_k d_g\rvert$	$\sim$	1 (4.6)-[17]
C_3	$\geq$	1
M_P	$\leq$	1
M_1	$\geq$	$\frac{1}{(p_1)^2} = \frac{1}{\gamma^4}$
M_2	$\geq$	$M_1 = \frac{1}{\gamma^4}$
λ	$\geq$	$\max\{M_1, e, \frac{1}{C_l^2}\} = \frac{1}{\gamma^4}$
ϕ_0	$\geq$	e^{-1}
ϕ_M	$\leq$	e
R_1	$\leq$	$\min\{1, \gamma^2, \frac{1}{\lambda}\} = \gamma^4$
c_T	$\sim$	$c_{T1} + c_{T2} = \lambda^3 = \frac{1}{\gamma^{12}}$
c_{100}	$\geq$	1
ϵ_0	$\leq$	$\frac{1}{(\lambda(1+\lambda)+c_T)} = \frac{1}{\lambda^3} = \gamma^{12}$
R_2	$\leq$	$\min\Big\{R_1, \frac{C_l}{(1+\lambda+c_T/\lambda)}, \frac{\lambda^2 C_l^2}{c_T}, \Big(\frac{1}{c_T^2 M_1(1+\lambda^2)}\Big)^{\frac{1}{4}}, \frac{\epsilon_0}{\sqrt{2M_2}}, \frac{\lambda}{c_T\big(1+\lambda^2+\lambda^2(1+\lambda)\big)}\Big\},$ $= \min\{\gamma^4, \gamma^9, \gamma^6, \gamma^9, \gamma^{14}, \gamma^{20}\} = \gamma^{20}$
σ	$\geq$	$c_T R_2 = \gamma^8$
τ_0	$\geq$	$M_1\Big(\big(\lambda^2 + c_T R_2\big)^2 + \lvert h\rvert^2_{C^0(\Omega_0)}(1 + \big(\lambda + c_T R_2^2\big)^2) + \lvert q\rvert^2_{C^0(\Omega_0)}\Big) = \frac{1}{\gamma^{20}}$
R	$\leq$	$R_2 = \gamma^{20}$
δ	$\leq$	$c_T R_2^3 = \gamma^{48}$
r	$\leq$	$\frac{\lambda^2 C_l^2 R_2^3}{\big(\lambda + c_T R_2^2\big)} = \gamma^{58}$
$c_{1,T}$	$\geq$	$\sqrt{\Big(\frac{M_1}{\tau_0} + \frac{1}{\lambda}\Big)} = \gamma^2$
$c_{2,T}$	$\geq$	$\sqrt{M_2}(1 + \frac{\lvert\chi_1'\rvert_{C^0(\Omega_0)}}{\tau_0 R}) + \frac{c_{1,T}}{\sqrt{\tau_0}} c_{133} = \frac{1}{\gamma^2} + \frac{1}{\gamma^8}(\lvert\chi_1''\rvert_{C^0(\Omega_0)} + \frac{\lvert\chi_1'\rvert_{C^0(\Omega_0)}}{\gamma^4}) \sim \frac{c_{1X}^2}{\gamma^8}$
c_{133}	$\geq$	$\frac{\lvert\chi_1''\rvert_{C^0(\Omega_0)}}{\tau_0 R^2} + \frac{\lvert\chi_1'\rvert_{C^0(\Omega_0)}}{R}(1 + \lambda + c_T R_2^2 + \frac{\lvert h\rvert_{L^\infty(\Omega_0)}}{\tau_0}) = \frac{1}{\gamma^{20}}(\lvert\chi_1''\rvert_{C^0(\Omega_0)} +$ $\frac{\lvert\chi_1'\rvert_{C^0(\Omega_0)}}{\gamma^4})$

We are now ready to calculate the coefficients used in the proof of Th.1.2 (resp. Th.3.3) in [17] (see the next Table 2 and the calculations below). First we split each smooth localizer in time and space (see Remark 2.8 (4) in [17]):

$$b(\frac{y-y_0}{R}) = b(\frac{t-t_0}{R})b(\frac{x-x_0}{R}),$$

with $b(t)=\chi_1(t)\in G_0^{1/\alpha}(\mathbb{R})$ and $b(x)\in C_0^2(\mathbb{R}^n)$. Consequently the functions $f_1(y), f_2(y), f_3(y)$ (see (2.21) in [17]) can be written as: $f(y)=f(t)f(x)$, with $f(t)=D_0^2 b_{j-1}(t)+D_0 b_{j-1}(t)+b_{j-1}(t)$ and $f(x)=D_rD_sb_{j-1}(x)+D_rb_{j-1}(x)+b_{j-1}(x)$, for $b_{j-1}(t):=b(2(t-t_{j-1})/r)$. Let $v=b((y-y_{j-1})/r)u_{j-1}$, then

$$\begin{aligned}\|A\Big(\frac{3D_0}{\nu}\Big)f(t)\Big(1-A\Big(\frac{D_0}{\nu}\Big)\Big)f(x)v\|_1 &\le \|A\Big(\frac{3D_0}{\nu}\Big)(D_0f(t))\Big(1-A\Big(\frac{D_0}{\nu}\Big)\Big)f(x)v\|_0\\ +\|A\Big(\frac{3D_0}{\nu}\Big)f(t)\Big(1-A\Big(\frac{D_0}{\nu}\Big)\Big)(D_0+D_x+1)(f(x)v)\|_0 &\le c_{108}c_{152}\exp(-c_{106}\nu^\alpha)\end{aligned}$$

with c_{108} calculated as in (188) with $\beta_1=3, m=3, c_3=(r/2)c_{0X}$. Moreover,

$$\begin{aligned}
c_{162,j} &= 2c_{162,j-1}+c_{153}c_{164}+c_{155,j-1}|-P_2b_{j-1}+h^s(x)D_{x_s}b_{j-1}|_{C^0}\\
&\quad +c_{107}c_{152}\big(1+n^2|g^{kr}|_{C^0}+|h^s|_{C^0}\big)+c_{155,j-1}|2D_0b_{j-1}|_{C^1}+c_{152}c_{108}\\
&\quad +c_{155,j-1}|D_0(2D_0b_{j-1})|_{C^0}+c_{152}c_{107}\\
&\quad +c_{155,j-1}|2ng^{kr}D_kb_{j-1}|_{C^1}+c_{152}c_{108}n^2|g^{kr}|_{C^1}\\
&\quad +c_{155,j-1}|D_r(2g^{kr}D_kb_{j-1})|_{C^0}+c_{107}c_{152}n^2|g^{kr}|_{C^1}\\
&\sim 2c_{162,j-1}+\Big(\frac{N^2c_{1X}^2}{r^2}\Big)\frac{c_{1X}^{3/2}}{r^{1/2}}+c_{155,j-1}(1+|g^{kr}|_{C^1}+|h^s|_{C^0})\Big(\frac{|b'|_0}{r}+\frac{|b''|_0}{r^2}+\frac{|b'|_0^2}{r^2}\Big)+\\
&\quad +\Big(\frac{Nc_{1X}}{r}\Big)c_{108}(1+|g^{kr}|_{C^1}+|h^s|_{C^0}) \sim c_{162,j-1}+c_{155,j-1}\frac{c_{1X}^2}{r^2}\\
c_{154,j} &= c_{162,j}+c_{153}\widetilde{c}_{107}\sim c_{162,j}+\frac{N^2c_{1X}^3}{r^2}R^n\sim c_{162,j}\sim c_{155,j-1}\frac{c_{1X}^2}{r^2}\\
c_{116} &\sim \gamma^4c_{154,j}^2\Big(\frac{Nc_{1X}}{\gamma^{48}}\Big)^4.
\end{aligned}$$

By applying Lemma 2.6 in [17] with $c_U=c_{152}$, $c_P=c_{153}$, $c_A=c_{154,j}$, one obtains:

$$c_{155,j}=c_{150}(c_{152},c_{153},c_{154,j})\sim c_{1X}^3\frac{\sqrt{c_{116}}}{\gamma^{48}}\sim\frac{N^2c_{1X}^5}{\gamma^{46+58\cdot 2}}c_{155,j-1},$$

$$c_{156}=\min\Big(\frac{1}{18\beta c_{131}}, c_{132}^{1/\alpha}, c_{165}^{1/\alpha}, \frac{c_{106}^{1/\alpha}}{3c_{131}}\Big)=\frac{c_{106}^{1/\alpha}}{3c_{131}}\sim\gamma^{56\alpha+58(n+1)(\alpha+1)+28}.$$

Recalling that $\alpha^N = \theta = 1/2$, we get $c_{159} = c_{156}^{-\frac{1}{\alpha^{N-1}(1-\alpha)}} > 1$ and c_{160}:

$$c_{159} \sim \Big(\frac{1}{\gamma^{56\alpha+58(n+1)(\alpha+1)+28}}\Big)^{\frac{1}{2\gamma^{58(n+1)}}} = \exp\Big(\frac{-[56\alpha + 58(n+1)(\alpha+1)+28]}{2\gamma^{58(n+1)}}\ln(\gamma)\Big),$$

$$c_{158} = Nc_{155,N} + 3Nc_{131}c_{152}\Big(1 + \frac{|b'|_{C^0}}{r}\Big)c_{156}^{-\alpha/(1-\alpha)} \sim Nc_{131}c_{152}c_{159}^{1/2}$$

$$c_{160} = \big(\ln(1+e^{c_{159}})\big)^{1/2} + 2^{1/2}c_{158} \sim c_{158} \le \exp\Big(\frac{1}{\gamma^{c_{200}}}\Big), \quad c_{200} = 58(n+1)+2$$

To obtain c_{206} we proceed as in Remark 3.8. of [17], by repeating the previous calculations for different sets of translated hyperbolic surfaces. We get $c_{206} \sim c_{160}$, up to a multiple of i_0, the lower bound of the injectivity radius. We call c_{205} the multiplicative constant that includes all the geometric parameters T, i_0, D, r_0, R, n.

Table 2

Name	Value	Name	Value
c_{2X}	$= c_{102} = \frac{1}{(ec_{1X})^\alpha}$	c_{119}	$\delta c_{1X} \sim \gamma^{48} c_{1X}$
c_{118}	$1+\lvert\phi'\rvert_0(1+R_2)+5n\lvert\phi''\rvert_{0,\rho}R_2^{\rho+1}+\lvert\phi''\rvert_0(1+R_2^2)+\sigma(2+R_2^2) \sim \frac{1}{\gamma^8}$	c_{114}	$c_{1,T}^2\lvert g\rvert_{C^1}^2\lvert\chi_1\rvert_{C^2}^2(1 + \lvert\varphi'\rvert_{C^0}^4/\delta^4 + \lvert\varphi''\rvert_{C^0}^2/\delta^2) \sim \frac{c_{1X}^4}{\gamma^{12+48\cdot 4}}$
c_{115}	$c_{2,T}^2(\lvert\varphi'\rvert_{C^0}^2 + 1)(3^3e^{-3}/\delta^3)(1 + \lvert\chi_1'\rvert_{C^0}^2/\delta^2) \sim \frac{c_{1X}^6}{\gamma^{8\cdot 3+48\cdot 5}}$	c_{121}	$\frac{c_{1X}}{\delta}$
c_{122}	$\frac{c_{1X}^2}{\gamma^{44}}$	c_{123}	$\sim \frac{\gamma^{56\cdot\alpha}}{c_{1X}^\alpha}$
c_{128}	$\frac{1}{3^\alpha 2}c_{123} \sim c_{123}$	c_{110}	$c_{122}\left(\frac{8\Gamma(1/\alpha)}{3[\alpha c_{123}^{1/\alpha}(\alpha c_{128})^{1/\alpha}]}\right)^{1/2} \sim \frac{c_{1X}^3}{\gamma^{44+56}}$
c_{109}	$\min(\sqrt{\epsilon\,\delta/36}, c_{128}/2, 1) \sim \frac{\gamma^{56\cdot\alpha}}{c_{1X}^\alpha}$	c_{130}	$\frac{3c_{109}}{4\delta}\left(\frac{1}{16}\right)^5 \sim \frac{\gamma^{56\cdot\alpha-48}}{c_{1X}^\alpha}$
c_{131}	$\max(16^6\sqrt{2}, \frac{16^6 3^{\alpha-1}\sqrt{2\epsilon_0\delta}}{c_{123}}, \frac{16^6\sqrt{\epsilon_0\delta}}{3\sqrt{2}}) \sim \frac{c_{1X}^\alpha}{\gamma^{56\cdot\alpha-30}}$	c_{135}	$r^\alpha c_{2X}\frac{1}{4\cdot 3^\alpha} \sim \frac{\gamma^{58\cdot\alpha}}{c_{1X}^\alpha}$
c_{137}	$\min(\frac{1}{2}\left(c_{102}\delta^\alpha\frac{(c_{130})^\alpha}{(\sqrt{2})^\alpha} + \delta\frac{c_{130}}{2\sqrt{2}}\right), \frac{1}{2}c_{102}\delta^\alpha(\frac{1}{2\sqrt{2}}c_{130})^\alpha) \sim \frac{\gamma^{48\cdot\alpha}}{c_{1X}^\alpha}c_{130}^\alpha$	c_{132}	$\min(c_{135}, c_{137}) \sim \frac{\gamma^{56\cdot\alpha\cdot\alpha}}{c_{1X}^{\alpha\cdot\alpha}}\frac{1}{c_{1X}^\alpha}$
c_{170}	$N \sim \frac{1}{\gamma^{58(n+1)}}$		
$\widetilde{c}_{117}$	$c_{2X}R^\alpha = (ec_{1X})^{-\alpha}R^\alpha$	β	$2 + (\frac{4}{\widetilde{c}_{117}})^{1/\alpha} \sim \frac{c_{1X}}{R}$ (2.11)
$\widetilde{c}_{106}$	$\frac{1}{\beta^\alpha} \sim \frac{R^\alpha}{c_{1X}^\alpha}$	$\widetilde{c}_{107}$	$R^{n+1}c_{0X}\left(\frac{8}{\beta}\Gamma\left(\frac{1}{\alpha}\right)\frac{1}{\alpha(\widetilde{c}_{117})^{1/\alpha}}\right)^{1/2}\frac{1}{(\alpha\widetilde{c}_{106})^{\frac{1}{\alpha}}} \sim R^n c_{1X}$
$c_{154,1}$	$1 + \widetilde{c}_{107} \sim R^n c_{1X}$	$c_{155,1}$	$\max(c_{134}, c_{136}) = \max(c_{1X}^{2.5}\gamma^{58(n-\frac{3}{2})}, \frac{c_{1X}^6}{\gamma^{180}}) = \frac{c_{1X}^6}{\gamma^{180}}$
c_{153}	$1 + 2N\left(1 + n^2\lvert g^{kr}\rvert_{C^0} + \lvert h^s\rvert_{C^0}\right)\left(\frac{\lvert b'\rvert_{C^0}}{r} + \frac{\lvert b''\rvert_{C^0}}{r^2} + (N-1)\frac{\lvert b'\rvert_{C^0}^2}{r^2}\right) \sim \frac{N^2c_{1X}^2}{r^2}$	c_{152}	$2\left(1 + N\frac{\lvert b'\rvert_{C^0}}{r}\right) \sim \frac{Nc_{1X}}{r}$
$c_{162,1}$	1	c_{117}	$(r/2)^\alpha\frac{1}{(ec_{1X})^\alpha} \sim \frac{r^\alpha}{c_{1X}^\alpha}$
c_{165}	$c_{117}\beta^\alpha/(3^\alpha 4) \sim \frac{r^\alpha}{R^\alpha} \sim \gamma^{38\alpha}$	c_{164}	$\frac{r}{2}c_{0X}\left(\frac{8}{3}\Gamma\left(\frac{1}{\alpha}\right)\frac{ec_{1X}}{\alpha^{1/\alpha}(r/2)}\right)^{1/2}\frac{ec_{1X}(3^\alpha 4)^{\frac{1}{\alpha}}}{(\alpha^{\frac{1}{\alpha}}(r/2))} \sim \frac{c_{1X}^{3/2}}{r^{1/2}}$
c_{107}	$c_{164} \sim \frac{c_{1X}^{3/2}}{r^{1/2}}$	c_{108}	$\left(c_{107} + c_{107}\frac{4^{4/\alpha}}{(\alpha c_{106})^{3/\alpha}}\right)\left(1 + \frac{\lvert b'\rvert_0}{r} + \frac{\lvert b''\rvert_0}{r^2} + \frac{\lvert b'''\rvert_0}{r^3}\right)\left(1 + \frac{\lvert b'\rvert_0}{r}\right) \sim \frac{c_{1X}^{\frac{17}{2}}}{r^{\frac{15}{2}}}$

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