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ON THE PERFORMANCE OF A NONLINEAR VIBRATION ISOLATOR CONSISTING OF AXIALLY LOADED CURVED BEAMS

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ABSTRACT

A desirable characteristic for nonlinear vibration isolators is a high static stiffness and a low dynamic stiffness. A curved beam is a possible candidate for this role provided that the amplitude of vibration about the static equilibrium position is sufficiently small. However, for large amplitude oscillations, the nonlinear dynamics may have a detrimental effect. This paper considers the force transmissibility of a single degree-of-freedom system where the stiffness element is a curved, axially loaded beam. The transmitted force is calculated by numerical time domain integration of the equations of motion. The exact force-deflection relation for the beam is used for the spring. By comparison, a frequency domain solution is sought using the Harmonic Balance (HB) method in which the system is modelled as a Duffing oscillator. It is shown that the HB and time domain solutions are in close agreement for small amplitudes of excitation and both predict advantageous performance of the nonlinear isolator compared with its equivalent linear counterpart. However, significant discrepancies occur between the two solutions for large excitation since the beam can no longer be approximated by a linear and a cubic stiffness. It is also strongly asymmetric – soft in compression but stiff in extreme extension– which gives rise to an impulse in the transmitted force in each fundamental period. This numerical problem is alleviated by inserting a linear spring in series with the beam isolator with a modest compromise in isolation performance at the excitation frequency.

1. INTRODUCTION

The effective bandwidth of a vibration isolator is dependent upon the fundamental natural frequency of the isolated mass moving rigidly on the stiffness of the isolator [1]. Since it is not usually desirable to alter the supported mass, the remaining option to achieve a lower frequency of isolation is to reduce the stiffness. However, reducing the stiffness of a linear spring causes a large static deflection which can be prohibitive. Nonlinear springs can overcome this problem by having a High Static but Low Dynamic Stiffness (HSLDS).

A candidate component with HSLDS characteristics is a vertically orientated straight Euler buckled beam (column). At the buckling point substantial bending of the beam and hence shortening of its span occurs when a small additional compressive load is applied. The post-buckled beam is stable but offers little resistance to additional deformation and so has low stiffness. Prior to buckling the deflection is due to axial deformation which results in a high stiffness. This 'kink' in the force-deflection curve gives rise to a high static load bearing capability in conjunction with a low 'dynamic stiffness' in the post-buckled region [2].

Winterflood et al. [3, 4] used a buckled beam in a vibration isolator to achieve low frequency isolation. The buckled beam was connected to a lever mechanism, and a mass was suspended on the lever. Mathematical analysis and experimental results of a constrained Euler spring in an isolator was presented in [5]. Virgin and Davis [2] studied a buckled beam used directly as a vibration isolator. They used an approximate second order solution to find the force-deflection characteristic and stiffness of a simply supported straight buckled beam. They modelled imperfection by an initial transverse deflection. An experimental rig was used to validate their results. The response in the time domain was considered and they measured the amplitude of vibration and plotted it as a transmissibility graph. They showed that a lower level of transmissibility is achievable by implementing a buckled beam as a vibration isolator.

Plaut et al. [6] considered a fixed-fixed buckled beam as a vibration isolator. They obtained non-dimensional governing differential equations and considered harmonic based excitation. They considered a solution comprised of a static equilibrium part and dynamic part, solving them numerically. Single harmonic and two frequency excitations were applied and displacement transmissibility was obtained for different static loads and beam dimensions. It was shown that a wide range of isolation is achievable by implementing buckled beams. Plaut et al. [7] further developed their model to a two degree-of-freedom system. They considered an asymmetric bar supported by two buckled columns or two pairs of pre-bent columns bonded by a viscoelastic filler. Jeffers et al. [8] continued the previous work by Plaut et al. and considered a three-dimensional model. They isolated a plate with four pairs of pre-bent columns bonded with viscoelastic material. El-Kafrawy et al. considered a two degree-of-freedom [9] and a three degree-of-freedom [10] isolator with only straight buckled columns.

While low dynamic stiffness is beneficial in extending the bandwidth of isolation, the singularity in the force-deflection curve due to buckling is undesirable. Other configurations, such as a curved beam, are suggested to achieve a smoother transition in stiffness. It is shown in reference [11] that curved beams can maintain high static stiffness whilst providing low dynamic stiffness. However, nonlinear dynamics of such isolators are not considered in the study.

The purpose of this paper is to assess the performance of a nonlinear isolator comprised of a curved beam in isolating a harmonic force with due consideration of its nonlinear dynamics. This work is a continuation of previous work [11] which focussed mainly on the static behaviour of a curved beam isolator. The resulting force-deflection relations are now incorporated into a dynamic model for a rigid mass suspended by a curved beam. Numerical solution of the equations of motion is straightforward but a cubic polynomial fit to the force-deflection curve further enables the system to be modelled simply as a Duffing oscillator. The equations of motion and numerical and analytical methods of solution are outlined briefly in section 2. The force transmissibility results for low and high levels of excitation are presented in section 3. Section 4 considers the waveform of the steady state response and addresses an inherent limitation of the buckled beam isolator for large amplitude motion. Conclusions are drawn in section 5.

2. DYNAMIC MODEL OF A CURVED BEAM ISOLATOR

A schematic view of a curved beam isolator is shown in Figure 1(a) in which the weight of the isolated mass is acting in the x-direction. The configuration shown is a parallel arrangement of two spring combinations, each combination comprising a curved beam in series with a comparatively stiff linear spring k_c. The latter is included initially to improve numerical convergence when solving the dynamic equations of motion, but subsequently to adjust the force-deflection relation, as discussed in section 4. The fundamental mode of vibration of the system is assumed to be that of the rigid mass moving vertically on the axial stiffness of the isolator. This mode is represented by the single degree-of-freedom system shown in Figure 1(b) which assumes that the beams are massless. The stiffness of one beam can be computed from an analytical solution [11]. A dashpot is included to model damping.

Figure 2 shows the force deflection curve for one axially loaded beam with small to modest initial curvature (2˚, 5˚, and 15˚). The isolator is initially very stiff when loaded and softens considerably in the post-buckled region. The position at which the dynamic stiffness is at a minimum is assumed to be the optimal static equilibrium position for the purpose of vibration isolation and the origin of the force deflection curves shown has been transformed to this point. In these results, the linear spring in series with the beam is 100 times stiffer than the static stiffness of the beam at its equilibrium position, and this has the effect of slightly reducing the steepness of the curve in the pre-buckled region.

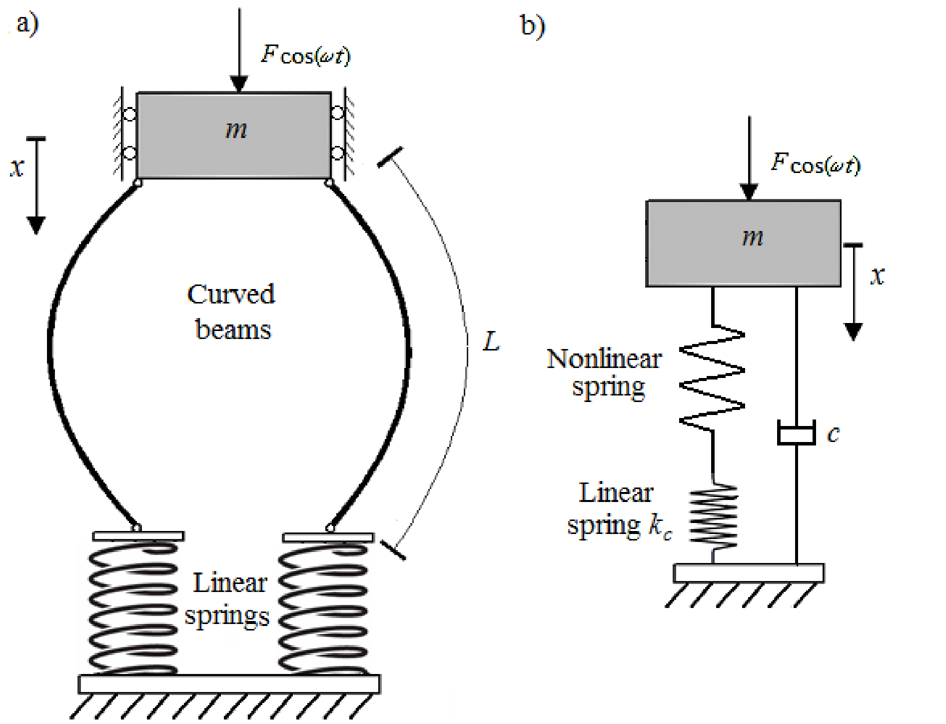

Figure 1: Curved beam isolator. a) Schematic view, b) Dynamic model with additional linear spring in series

The equation of motion for the dynamic system subject to a time-harmonic force is,

$$m\ddot{x} + c\dot{x} + F_k(x) = F\cos(\omega t) \quad (1)$$

where F_k is the restoring force due to the spring. Equation (1) can be written in terms of non-dimensional parameters and variables as,

$$y'' + 2\zeta y' + f_k(y) = f\cos(\Omega\tau) \tag{2}$$

where $(\bullet)'$ denotes differentiating with respect to non-dimensional time τ,

$$\tau = \omega_n t, \qquad \Omega = \omega/\omega_n, \quad y = x/x_s, \quad k_s = mg/x_s, \quad \omega_n = \sqrt{k_s/m},$$
$$\zeta = c/2m\omega_n\,, \qquad f = F/mg, \quad f_k = F_k/mg$$

and x_s is the static displacement of the isolator due to the weight of the isolated mass. Note that ω_n is the natural frequency of a comparable linear isolator which has the same static displacement x_s. The damping parameter $\zeta = c/2m\omega_n$ is therefore *not* the equivalent damping ratio of the system, which is dependent on the natural frequency of the nonlinear system when linearized about its equilibrium position. At low levels of dynamic response, i.e. $|y| \ll 1$, the restoring force can be adequately represented by a linear stiffness given by the tangent to the force-deflection curve, k_d, referred to here as the dynamic stiffness. The non-dimensional natural frequency of the linearized system is then given by $\Omega_n = \sqrt{k_d/k_s}$. A linear spring, by comparison, has a normalised natural frequency of unity and serves as a convenient benchmark for isolator performance. If y approaches -1 the curved beams move out of their preloaded state and become substantially more stiff, a problem which is encountered and discussed further in section 4. At modest levels of response, however, the force-deflection curve can be represented by a polynomial featuring both odd and even terms owing to asymmetry. A cubic polynomial is chosen here and equation (2) can be rewritten as,

$$y'' + 2\zeta y' + \beta y + \epsilon y^2 + \gamma y^3 = f\cos(\Omega\tau) \tag{3}$$

where β, ϵ and γ are the fit coefficients. The quadratic term can be circumvented by appropriate transformation of the axis ($z = y - \delta$ and $\delta = \epsilon/3\gamma$) [12] to yield the classic Duffing oscillator with the addition of a static preload *p*,

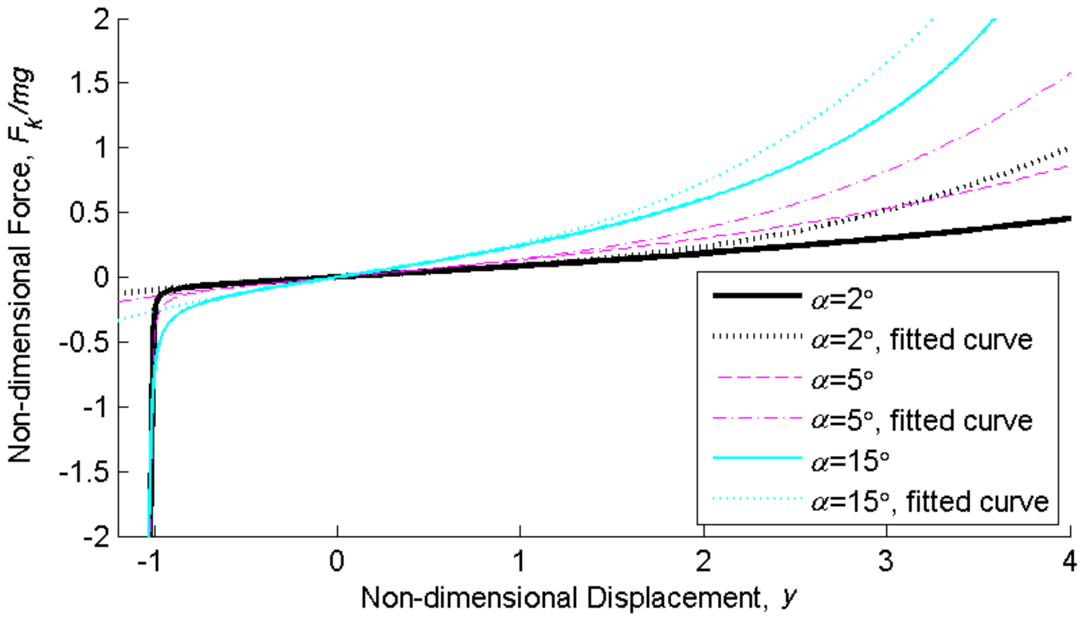

Figure 2: Non-dimensional force-deflection curve for a curved beam isolator for different initial angles of curvature.

$$z'' + 2\zeta z' + \kappa z + \gamma z^3 = p + f\cos(\Omega\tau) \qquad (4)$$

where $\kappa = \beta - \epsilon^2/3\gamma$, and $p = \beta\epsilon/3\gamma - 2\epsilon^3/27\gamma^2$. The polynomial coefficients for fits after transformation of axis are listed in Table 1. The fits are shown in Figure 2 for three different initial curvature angles. For modest excursions from the equilibrium position the fits are reasonable approximations to the true characteristic in compression but become radically different in extension. However, simplification of the system in this way is convenient for the purpose of obtaining an analytical solution for the dynamic response of the system using the Harmonic Balance method.

Initial curvature angle (°)	Polynomial coefficients of fit, $\kappa z + \gamma z^3 = p$			Coordinate shift
	γ	κ	p	$\delta = y - z$
2	0.0121	0.0803	-0.0142	-0.176
5	0.0178	0.1269	-0.0114	-0.089
15	0.0380	0.2221	-0.0082	-0.037

Table 1: coefficients of the fit to the force-deflection curve in the region of the minimum dynamic stiffness (static equilibrium) position

3. FORCE TRANSMISSIBILITY OF A CURVED BEAM ISOLATOR

The transmitted force to the base of the isolator modelled in Figure 1(b) is given by,

$$f_t = 2\zeta y' + f_k(y) \qquad (5)$$

The forced response is obtained by two independent means (although these intermediate results are not presented here for brevity).

(i) The Harmonic Balance method is applied to obtain the amplitude A_1 and the phase of the assumed time-harmonic response due to a time harmonic input force. A bias term A_0 arises from the asymmetry in the force-deflection curve. Higher order harmonics are not included. The isolator is represented by a cubic polynomial fit to the force-deflection curve, as described in section 2. The reader is referred to references [13, 14] for further details. The transmissibility can then be determined from,

$$T = \sqrt{1 + (A_1^2/f^2)\Omega^2(6\gamma A_0^2 + (3/2)A_1^2\gamma - \Omega^2)} \qquad (6)$$

(ii) The equation of motion for forced response (equation (2)) is solved directly by numerical integration in the time domain using the 'ode45' function in Matlab. The isolator is represented by the exact force-deflection of the curved beam isolator, as given by Figure 2. Equation (5) is then used to determine the transmitted force. The amplitude of the transmitted force at the excitation frequency is obtained by applying the Fast Fourier Transform to the transmitted force once steady state is achieved. Whilst strictly defined only for linear systems, force transmissibility is taken here as the ratio of the transmitted force at the excitation frequency to the applied force.

The force transmissibility obtained by both methods is shown in Figure 3 for three different initial curvature angles and a non-dimensional dynamic force amplitude of 0.01, i.e. 1% of the weight of the isolated mass. There is very good agreement at all frequencies. For such small oscillations about the equilibrium position the force deflection curve of the isolator is adequately modelled as a cubic relationship. The curve resembles that of the system linearised about its equilibrium position with a natural frequency of $\Omega_n = \sqrt{k_d/k_s}$. For all values of curvature presented the transmissibility is considerably reduced when compared to the curve one would expect for a comparable linear isolator for which $\Omega_n = 1$. A small curvature is preferable in this instance due to its more dramatic stiffness reduction. The height of the resonance peak also reduces with a smaller curvature which is a consequence of a reduced effective critical damping coefficient. A more practical scenario might be to vary the damping coefficient so as to maintain a constant acceptable value for the peak transmissibility. This would have the advantage of preventing the curves in Figure 3 from converging to the performance of the comparable linear isolator far beyond the frequency range shown.

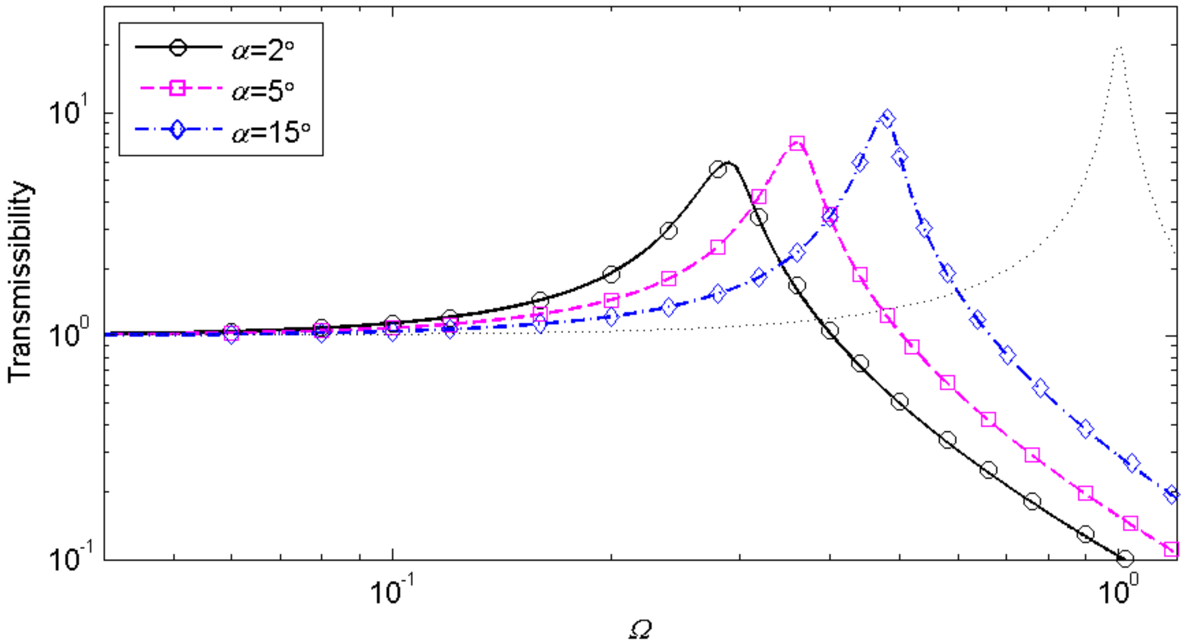

Figure 3: Modulus of the force transmissibility as a function of non-dimensional frequency Ω for the curved beam isolator with three different initial angles of curvature α, for ζ=0.025 and f =0.01, k_c/k_s =100, lines: approximate solution (coefficients for Duffing oscillator model as listed in Table 1), Markers: results of exact numerical model, dotted line: comparable linear isolator

Figure 4 shows the force transmissibility under the same conditions as Figure 3 except that the dynamic force is increased ten-fold to 10% of the weight of the isolated mass. The HB and numerical time integration approaches give dissimilar results mainly around resonance where the response is the largest. The discrepancies are due not to the method of solution but largely due to the inadequacy of the Duffing model to represent the force deflection curve over the operational range of dynamic displacements. In particular, the ‘jump-down’ frequency, at which the response jumps from one stable branch to another with increasing frequency, is not well predicted by the HB method. The isolation frequency, i.e. the frequency above which the force is attenuated, is underestimated. The numerically obtained results illustrate that the isolation region can be severely curtailed by the jump-down phenomenon when large dynamic forces are applied to such a strongly nonlinear system with light damping. However, once in the isolation region the benefits of the low dynamic stiffness are fully realised.

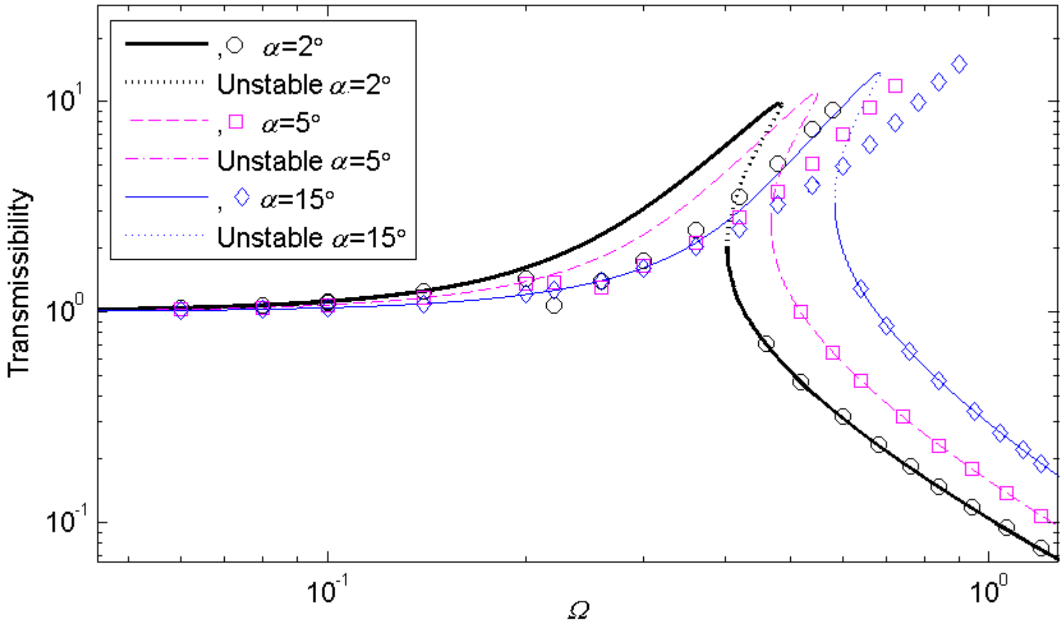

Figure 4: Force transmissibility as a function of non-dimensional frequency Ω for the curved beam isolator with three different initial angles α, for ζ=0.025, $f = 0.1$, and k_c/k_s =100, lines: approximate solution stable branch and unstable branch, Markers: results of numerical time integration using the exact force-deflection relation.

4. TRANSMITTED FORCE - TIME DOMAIN

The frequency domain representation of the transmitted force presented in the previous section considers only the component of the response at the excitation frequency. However, nonlinear systems will invoke response at other frequencies such as harmonics which cannot be ignored. A dramatic example of a near-periodic but non-time-harmonic response is shown in Figure 5. This is the time history of the transmitted force for the curved beam isolator with an initial curvature angle of 2° and an excitation non-dimensional frequency of 0.58, which is the jump-down frequency. There is a large impulsive force once per fundamental period which occurs when the isolator is at its maximum extent. The isolator becomes orders of magnitude more stiff when it is further extended, i.e. $y < -1$ and acts almost as a rigid end stop. The component of the response at the excitation frequency is small by comparison and is not a true reflection of the performance of the isolator. Higher order harmonic components should be quantified by, for example, evaluation of the total harmonic distortion.

The dramatic increase in isolator stiffness as the curved beam moves into extension can be mitigated to some extent by reducing the stiffness of the linear spring inserted in series. A value of k_c/k_s =2 is chosen for the following analysis in contrast to the value of 100 adopted previously. The effect on the force-deflection curve of softening the linear spring can be seen clearly in Figure 6. The isolator stiffness in extension is now governed by the linear spring which also has the adverse effect of increasing slightly the dynamic stiffness in the vicinity of the equilibrium position.

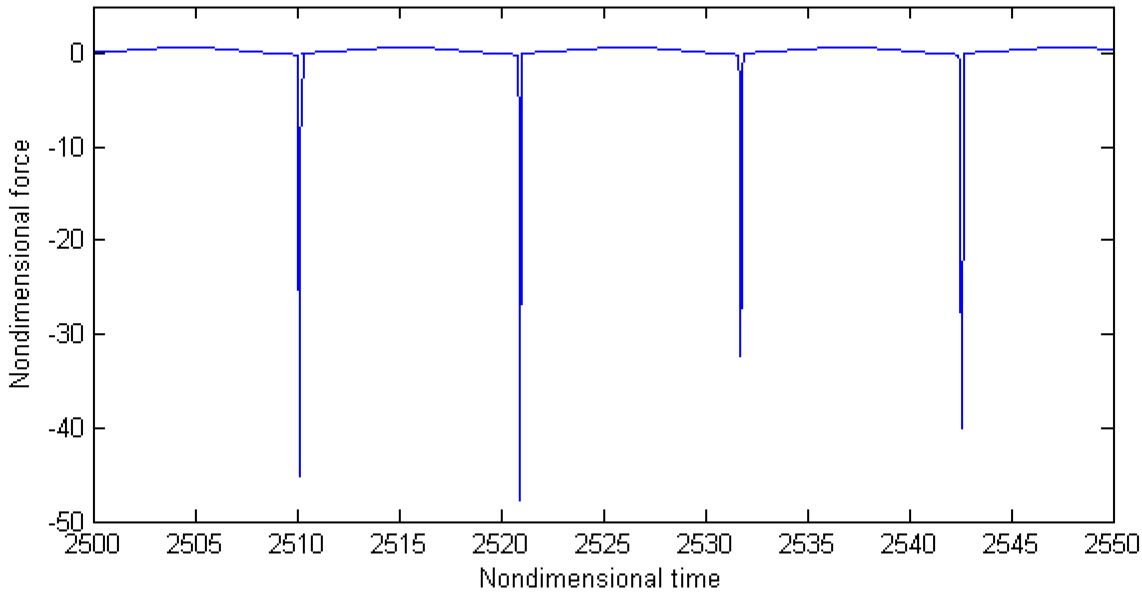

Figure 5: Time history of transmitted force through the curved beam isolator for an excitation frequency of 0.58 for α=2˚, f =0.1, ζ=0.025, and k_c/k_s=100

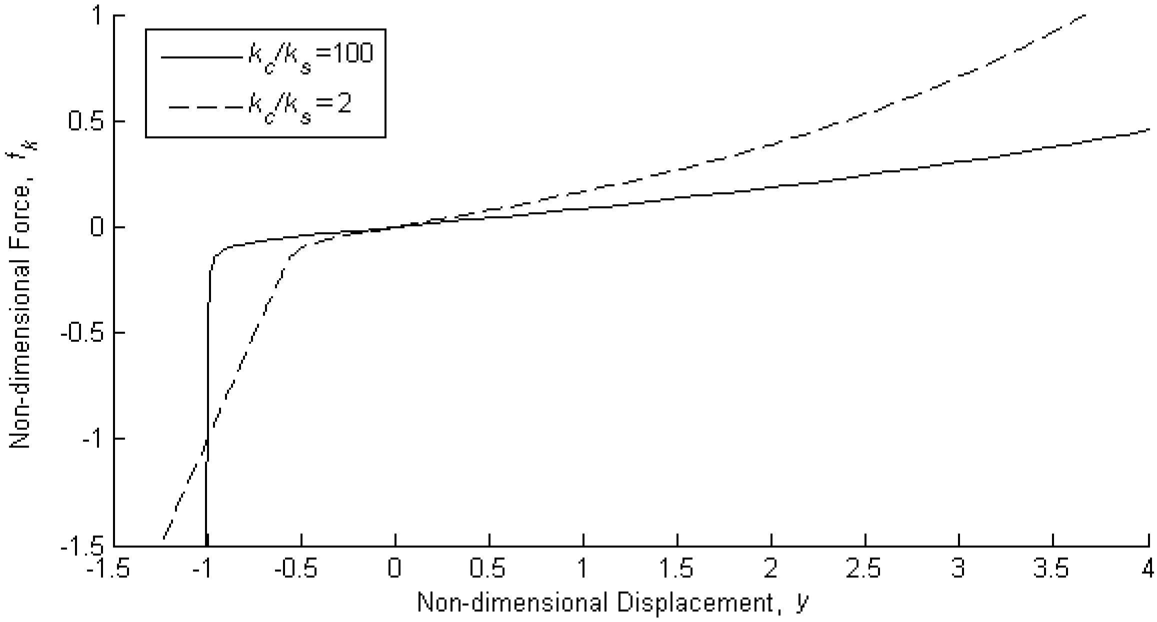

Figure 6: Non-dimensional force-deflection curve of the curved beam isolator for an initial curvature angle of α=2˚

Figure 7 shows the transmissibility curve for the curved beam isolator for each of the two linear spring stiffness ratios, k_c/k_s=2 and 100. The corresponding result is also shown for a linear isolator with the same static deflection due to the weight of the isolated mass. The effect of reducing the linear spring stiffness is to increase the jump-down frequency, and hence the non-dimensional frequency above which isolation occurs, from about 0.58 to 0.68. A significant benefit is still apparent compared to the linear isolator for which isolation occurs for $\Omega > \sqrt{2}$.

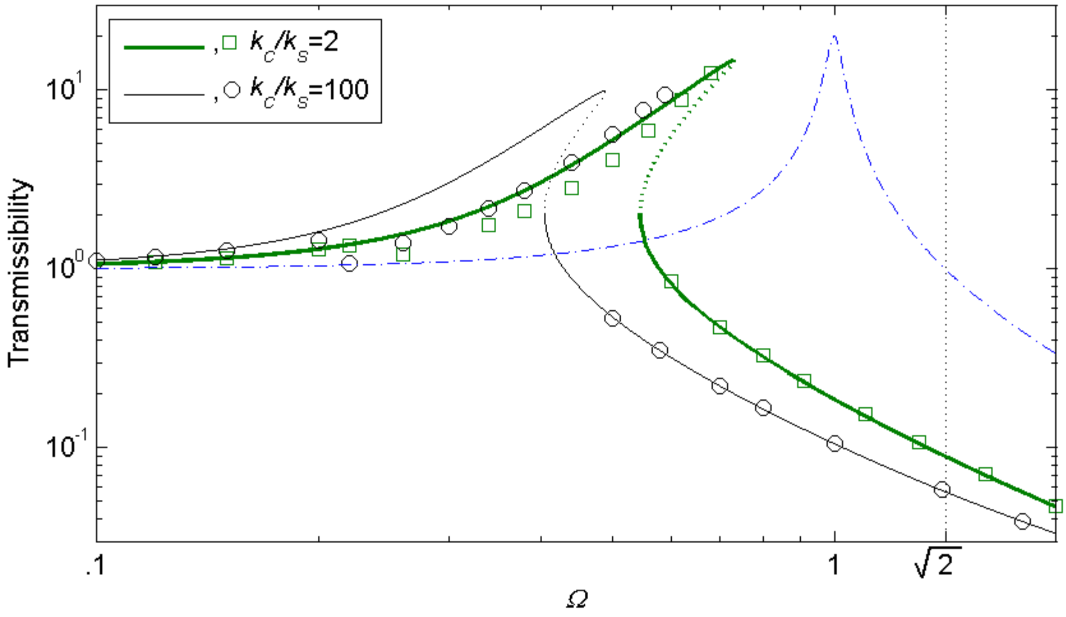

Figure 7: Force transmissibility as a function of non-dimensional frequency Ω for the curved beam isolator with two different linear springs stiffness, ζ=0.025 and f =0.1, solid line: approximate solution stable branch, dotted line: approximate solution unstable branch, markers: results of numerical time integration using the exact force-deflection relation, dotted-dashed line: linear isolator with the same static deflection due to the weight of the isolated mass

The steady state time history for the isolator with k_c/k_s=2 is shown in Figure 8. The non-dimensional excitation frequency is again chosen to be the jump down frequency, $\Omega = 0.68$ in this case. There is still a 'dip' caused by the curved beams 'locking up' in extension but this behaviour is much less severe than seen previously in Figure 5 due to the compliance of the linear spring placed in series.

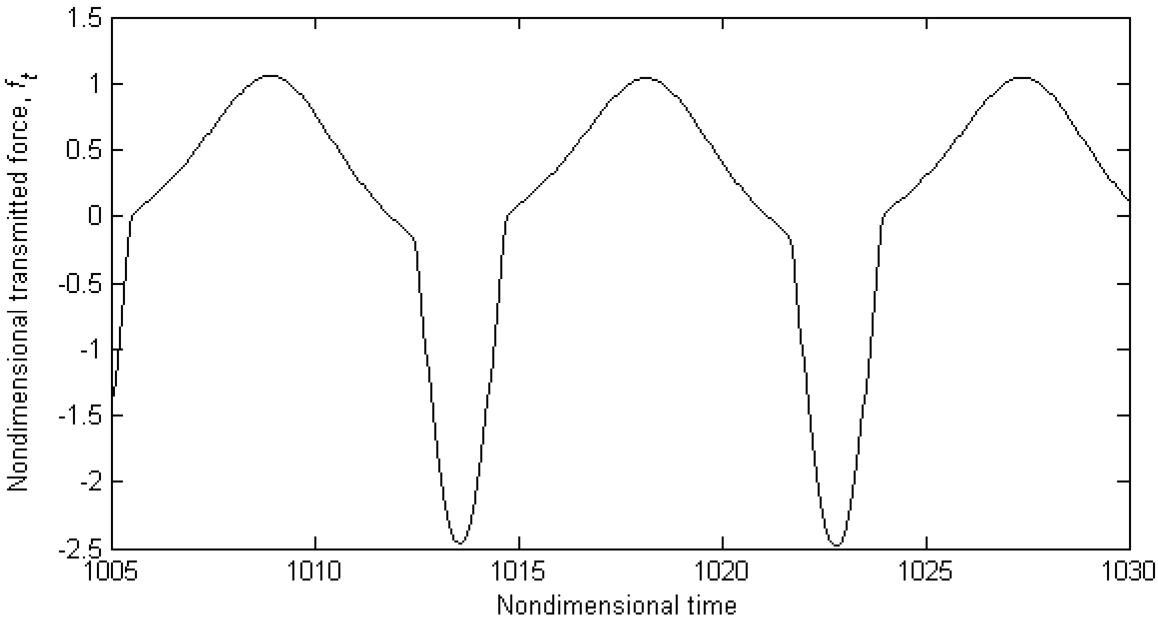

Figure 8: Steady state time history of the transmitted force of the curved beam isolator at a non-dimensional frequency Ω = 0.68 for k_c/k_s = 2, ζ = 0.025 and f = 0.1

5. CONCLUSIONS

The force-deflection curve for an axially loaded curved beam has been presented which shows that in its post-buckled region a low dynamic stiffness can be achieved whilst maintaining a large static load bearing capability, especially for small angles of curvature. This is a potentially desirable characteristic of a vibration isolator. A position of minimum dynamic stiffness can be identified numerically and is chosen as a desirable equilibrium position about which to consider dynamic motion. The force-deflection relation is fitted with a cubic function about this point so as to represent the beam with an attached rigid mass by a Duffing oscillator thus making solution by the Harmonic Balance method more convenient. Comparison with numerically obtained results using the exact force-deflection relation shows that the approximation is adequate except around resonance if the force excitation is large. The nonlinearity is of a hardening type in each direction from the equilibrium position which causes the resonance in the forced response curve to bend to the right. A jump-down is observed for large amplitudes of motion which has the undesirable effect of shifting the isolation region higher in frequency. However, the benefit for low levels of response is an order of magnitude reduction in dynamic stiffness and a corresponding three-fold reduction in the frequency above which isolation occurs.

Initial analysis is conducted in the frequency domain and considers only the component of the response at the excitation frequency. For large amplitudes of motion the response is significantly non-time-harmonic due to the beam acting almost rigidly in extension, the most striking manifestation of this being undesirable impulsive behaviour in the steady state time histories. These can be almost eliminated by introducing significant compliance in series with the curved beam although some compromise to the benefits of the nonlinear isolator results.

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