Analysis of first order systems on manifolds without boundary:

A spectral theoretic approach

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This dissertation is submitted for the degree of
Doctor of Philosophy

June 2017

To my family – my Mom and Dad, my sisters, my wife Wen, and my son An. Without you I would go insane. This is for you . . .

Declaration

I, Yan-Long Fang, hereby declare that except where specific reference is made to the work of others, the contents presented in this dissertation are original.

I also claim that my dissertation is not the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further claim that no part of my dissertation has been submitted for any such degree, diploma or other qualification.

Yan-Long Fang
June 2017

Acknowledgements

I would like to express my sincere gratitude to my advisor, Dmitri Vassiliev, for his help and advice throughout my doctoral research. He introduced me to a wide range of mathematical research areas and had many stimulating conversations with me.

I would also like to thank Zhirayr Avetisyan, Michael Levitin and Nikolai Saveliev for the many discussions on subjects related to my research.

I have to mention another wonderful mathematician, Yuri Safarov, who has sadly passed away. He taught me the basic ideas of microlocal analysis, and I have benefited from this ever since.

Throughout my doctoral studies, I have been generously funded by UCL and the Department of Mathematics. Without their support, my work would be impossible.

Finally, I would like to thank my family for their constant encouragement and understanding of my true love – mathematics.

Abstract

In this thesis we study first order systems of partial differential equations on manifolds without boundary. The thesis is, in part, based upon my publications [22, 23, 5, 21].

The importance of analysing first order systems can be seen directly from the special case, namely, the study of Dirac-type operators.

In the first part, I assume my manifolds to be 4-dimensional and consider the simplest non-trivial first order linear differential operators on such manifolds. I give a systematic way of extracting the geometric content encoded within these operators. More importantly, a new concept called covariant subprincipal symbol is defined and further employed in the spectral analysis developed in the next chapter.

In the second part, by applying the hyperbolic equation method and using Fourier Tauberian theorems, I establish the relationship between the Weyl coefficients of an elliptic self-adjoint first order differential operator and the residues of the corresponding eta function, which can be easily generalised to the pseudo-differential case. The special case of this relationship is examined explicitly and it is combined with the analysis of the first part.

The third part of the thesis involves a detailed analysis of the massless Dirac operator, whereas the massive case is also investigated with the help of the abstract adjugation operation on operators.

In contrast to the asymptotics of large eigenvalues, there has not been a systematic and robust way of analysing the asymptotics of small eigenvalues. The last part of the thesis gives a rigorous perturbation analysis of the massless Dirac operator on a topological 3-sphere, which complements the results obtained in the second part. In particular, explicit asymptotic formulae for small eigenvalues are derived for general perturbations of the standard metric. These asymptotic formulae are tested on generalised Berger spheres for which we have explicit expressions for eigenvalues.

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Chapter 1

Introduction

In this chapter I will introduce some basic definitions and basic notation that will be used throughout this thesis.

First order linear systems

Let the dimension of the manifold is n and we will only deal with the case $n \geq 2$. In this section we consider a first order self-adjoint $m \times m$ linear pseudo-differential operator L which acts on a column of m (complex valued) half-densities on a closed connected manifold.

Definition 1.0.1. Let (Φ, U, x) and (Ψ, V, y) be two coordinate charts on M, then on the overlapping set $U \cap V \subset M$, the two representations of a half-density v are related by the following formula:

$$v_\Phi(x) = \Big| \det \frac{\partial y}{\partial x} \Big|^{\frac{1}{2}} v_\Psi(y(x)). \tag{1.1}$$

There are many advantages in working with half-densities, one is that we can define an inner product on half-densities without any additional geometric assumptions. This will allow us to introduce the notion of formal self-adjointness without the need for a prescribed density. Furthermore, one can always transform a problem on columns of scalar functions to an equivalent spectral problem on columns of half-densities. So we will now focus on operators on half-densities and denote the inner product on half-densities by

$$\langle v, w \rangle := \int_M w^* v \, \mathrm{d}x \,, \tag{1.2}$$

where v and w are m-columns of half-densities on M and the superscript * denotes Hermitian conjugation in $\mathbb{C}^m$. For a given linear pseudodifferential operator L on half-densities with C^∞ coefficients, its full symbol can be easily adapted from [29]

$$\sigma_L(f,g) \sim \sum_{j=0}^{\infty} L_j(f,g)\,, \tag{1.3}$$

where $f \in C_0^\infty(M)$, $g \in C^\infty(M)$, $\mathrm{supp}(f) \subset \mathrm{supp}(\mathrm{d}g)$ and the $L_j(f,g)$ are given by

$$e^{-i\lambda g}\, L\, \left(f\, e^{i\lambda g}\right) \sim \sum_{j=0}^{\infty} L_j(f,g)\lambda^{s_j} \qquad \text{as} \qquad \lambda \to +\infty\,, \tag{1.4}$$

where $\{s_j\}$ is a strictly decreasing sequence tending to minus infinity. The formal sum on the LHS of asymptotic expansion (1.3) is known as the full symbol of L and L_0 is called the principal symbol. Note that the expression (1.4) is coordinate independent and f plays the role of a coordinate cut-off function when we introduce a partition of unity on M. Once a partition is chosen, we can think of L as acting on a bounded open subset of $\mathbb{R}^n$ and assume that $f = 1$ in a small neighbourhood of a given point of our manifold. Furthermore, we can choose $g(x) = x \cdot p = x^\alpha p_\alpha$ (note that this definition is coordinate dependent), where the Einstein summation convention is used and will be used throughout this thesis unless there is an ambiguity about the summation. Thus, expressions (1.3) and (1.4) become

$$\sigma_L(x,p) \sim \sum_{j=0}^{\infty} L_j(x,p)\,, \tag{1.5}$$

and

$$e^{-i\lambda x\cdot p}\, L\, \left(e^{i\lambda x\cdot p}\right) \sim \sum_{j=0}^{\infty} L_j(x,p)\lambda^{s_j} \qquad \text{as} \qquad \lambda \to +\infty\,. \tag{1.6}$$

Here we used $(x,p) := (x^1,\ldots,x^n,p_1,\ldots,p_n)$ to emphasize the dependence of the symbol on $T^*M\backslash\{0\}$ rather than f and g as in expressions (1.3) and (1.4). Note that when the sequence $\{s_j\}$ is integer-valued, the operator is said to be a classical pseudo-differential operator. For the case of a first order differential operator acting on a column of m half-densities, we can write it in local coordinates as

$$L = L_0^\alpha(x)\partial_{x^\alpha} + L_1(x)\,, \tag{1.7}$$

where $\partial_{x^\alpha} := \frac{\partial}{\partial x^\alpha}$ and $\alpha = 1, \ldots, n$. Each L_0^α is a $m \times m$ matrix function and so is the L_1. Now, its symbol is given by

$$L(x,p) = iL_0^\alpha(x)p_\alpha + L_1(x)\,, \tag{1.8}$$

with the principal and subprincipal symbols given by

$$L_{\text{prin}}(x,p) := iL_0^\alpha(x)p_\alpha\,, \tag{1.9}$$

and

$$L_{\text{sub}}(x) := L_1(x) + \frac{i}{2}\left(L_{\text{prin}}\right)_{x^\alpha p_\alpha}(x)\,, \tag{1.10}$$

Here the subscripts $_{x^\alpha}$ and $_{p^\alpha}$ denote the partial derivatives with respect to x^α and p^α. Let us explain why the formula for the subprincipal symbol has the particular structure (1.10). Firstly, using formulae (1.8) and (1.9) we rewrite (1.10) as

$$L_{\text{sub}}(x) = L_1(x) - \frac{1}{2}(L_0^\alpha)_{x^\alpha}(x)\,. \tag{1.11}$$

Secondly, given a differential operator L, we define its formal adjoint L^* by means of the formal identity

$$\langle Lv, w\rangle = \langle v, L^*w\rangle. \tag{1.12}$$

Working with the full symbol is inconvenient because the full symbol of a formally self-adjoint operator is not necessarily Hermitian and because it is not invariant under changes of local coordinates. The standard way of addressing these issues is by dealing with L_{prin} and L_{sub} which are invariantly defined matrix-functions on T^*M and M respectively, see subsection 2.1.3 in [45] for details.

Here and further on in this paragraph we drop, for the sake of brevity, the dependence on x. The advantage of representing the subprincipal symbol in the form (1.11) is that the RHS is written explicitly in terms of the matrix-valued coefficients L_0^α and L_1 of the differential operator (1.7). Let us now substitute (1.7) into the LHS of (1.12), use the formula for our inner product (1.2) and perform integration by parts. We arrive at the expression for the adjoint operator in local coordinates

$$L^* = \widehat{L}_0^\alpha \frac{\partial}{\partial x^\alpha} + \widehat{L}_1, \tag{1.13}$$

where

$$\widehat{L}_0^\alpha = -(L_0^\alpha)^*, \qquad \widehat{L}_1 = L_1^* - [(L_0^\alpha)^*]_{x^\alpha}\,. \tag{1.14}$$

We then calculate the subprincipal symbol of L^* using formula (1.11) and replacing matrix-valued coefficients accordingly, compare formulae (1.7) and (1.13). We get

$$(L^*)_{\text{sub}} = \widehat{L}_1 - \frac{1}{2}(\widehat{L}_0^{\alpha})_{x^\alpha} \,. \tag{1.15}$$

Substitution of (1.14) into (1.15) gives us

$$(L^*)_{\text{sub}} = L_1^* - \frac{1}{2}[(L_0^{\alpha})^*]_{x^\alpha} \,. \tag{1.16}$$

Comparing formulae (1.11) and (1.16) we conclude that

$$(L^*)_{\text{sub}} = (L_{\text{sub}})^*. \tag{1.17}$$

Thus, the whole point of introducing the correction term in (1.10) (last term in the RHS) is to ensure that we get the identity (1.17). It is also easy to see that our subprincipal symbol is invariant under changes of local coordinates. Had we defined the subprincipal symbol as $L_{\text{sub}} := L_1$ we would not have the identity (1.17) and we would not have invariance under changes of local coordinates.

The definition of the subprincipal symbol (1.10) originates from the classical paper [20] of J.J. Duistermaat and L. Hörmander: see formula (5.2.8) in this paper. Unlike [20], we work with matrix-valued symbols, but this does not affect the formal definition of the subprincipal symbol. In the above text we explained in great detail the concept of subprincipal symbol because it is not widely known outside the analysis community.

For the principal symbol things are much easier and, obviously, we have an analogue of formula (1.17):

$$(L^*)_{\text{prin}} = (L_{\text{prin}})^*. \tag{1.18}$$

Also, it is easy to see that the principal symbol is invariant under changes of local coordinates. Examination of formulae (1.7)–(1.10) shows that L_{prin} and L_{sub} uniquely determine the first order differential operator L. Thus, the notions of principal symbol and subprincipal symbol provide an invariant way of describing a first order differential operator.

For the sake of clarity, we write down the differential operator L explicitly, in local coordinates, in terms of its principal and subprincipal symbols:

$$L = -\frac{i}{2}\left([(L_{\text{prin}})_{p_\alpha}(x)]\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}[(L_{\text{prin}})_{p_\alpha}(x)]\right) + L_{\text{sub}}(x). \tag{1.19}$$

Remark 1.0.1. In writing formula (1.19) we used the convention that both operators of partial differentiation $\frac{\partial}{\partial x^\alpha}$ act on all terms which come (as a product) to the right, including the m-column of complex-valued half-densities v which is present in (1.19) implicitly. Thus, a more explicit way of writing formula (1.19) is

$$Lv = -\frac{i(L_{\text{prin}})_{p_\alpha}}{2}\,\frac{\partial v}{\partial x^\alpha} - \frac{i}{2}\,\frac{\partial((L_{\text{prin}})_{p_\alpha} v)}{\partial x^\alpha} + L_{\text{sub}}\, v\,.$$

Formulae (1.17) and (1.18) tell us that a first order differential operator is formally self-adjoint if and only if its principal and subprincipal symbols are Hermitian matrix-functions.

Now, we would like to define two classes of operators via their principal symbols.

Definition 1.0.2. We say that a formally self-adjoint first order differential operator L is *elliptic* if

$$\det L_{\text{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}, \tag{1.20}$$

and *non-degenerate* if

$$L_{\text{prin}}(x,p) \neq 0, \qquad \forall (x,p) \in T^*M \setminus \{0\}. \tag{1.21}$$

The ellipticity condition (1.20) is a standard condition in the spectral theory of differential operators, see, for example, [14]. Our non-degeneracy condition (1.21) is less restrictive: the class of linear differential operators satisfying this condition includes hyperbolic operators.

Remark 1.0.2. Ellipticity and the fact that dimension n is greater than or equal to two imply that m is even. Indeed, let us fix an arbitrary point $x \in M$ and consider $L_{\text{prin}}(x,p)$ as a function of momentum $p \in T^*_x M$. Since L is formally self-adjoint, the matrix-function $L_{\text{prin}}(x,p)$ is Hermitian, and, hence, $\det L_{\text{prin}}(x,p)$ is real. For $n \geq 2$ the set $T^*_x M \setminus \{0\}$ is connected, so the ellipticity condition (1.20) implies that the polynomial $\det L_{\text{prin}}(x,p)$ preserves sign on $T^*_x M \setminus \{0\}$. But our $m \times m$ matrix-function $L_{\text{prin}}(x,p)$ is linear in p, so $\det L_{\text{prin}}(x,-p) = (-1)^m \det L_{\text{prin}}(x,p)$, therefore the sign of $\det L_{\text{prin}}(x,p)$ can only be preserved if m is even. Of course, the case $n = 1$ is special as in this case m, the number of half-densities in our system, can be odd. For instance, a one dimensional Dirac operator on $\mathbb{S}^1$ is well defined.

In order to highlight the difference between the ellipticity condition (1.20) and the non-degeneracy condition (1.21) we consider two special cases.

Special case 1: $n = 3$, $m = 2$ and $\operatorname{tr} L_{\text{prin}}(x, p) = 0$. In this case conditions (1.20) and (1.21) are equivalent.

Special case 2: $n = 4$ and $m = 2$. In the proof of Lemma 2.2.1 we will show that for each $x \in M$ there exists a $p \in T^*_x M \setminus \{0\}$ such that $\det L_{\text{prin}}(x, p) = 0$, so it is impossible to satisfy the ellipticity condition (1.20). However, it is possible to satisfy the non-degeneracy condition (1.21). In fact, we will derive in Lemma 2.1.1 the necessary and sufficient conditions for a manifold M to admit a non-degenerate operator L.

There is a crucial mathematical operation on scalar matrix-functions on T^*M, which can be generalized further, namely, the Poisson bracket. We will use curly brackets to denote the Poisson bracket on scalar matrix-functions, $\{P, R\} := P_{x^\alpha} R_{p_\alpha} - P_{p_\alpha} R_{x^\alpha}$. Then it can be further generalised to the following:

$$\{F, G, H\} := F_{x^\alpha} G H_{p_\alpha} - F_{p_\alpha} G H_{x^\alpha}, \tag{1.22}$$

where the subscripts x^α and p_α indicate partial derivatives and the repeated index α indicates summation over $\alpha = 1, \ldots, n$.

Chapter 2

Analysis as a source of geometry

2.1 Introduction

In this chapter we will work on a manifold of dimension four. In the next chapter we will examine what happens when the dimension is reduced to three. Recall also that throughout the thesis we assume that our manifold does not have a boundary.

We start with the analysis of simplest non-trivial systems, namely, systems in which the number of components of half-densities, m, is equal to two. In other words, in this chapter we will be looking at 2×2 formally self-adjoint first order linear differential operators.

We will see that our analytic conditions, the non-degeneracy condition (1.21) and the standard ellipticity condition (1.20), have far reaching geometric consequences. Furthermore, we will give a definition of metrics associated with a PDE. More importantly, the new concept of *covariant subprincipal symbol* will be introduced, which extends the usual concept of subprincipal symbol by making it covariant under a wider group of transformations.

The motivation for analysing such a class of PDEs is twofold. The first motivating factor is that this is an attempt at developing a relativistic field theory based on the concepts from the analysis of partial differential equations as opposed to geometric concepts. The long-term goal of the work presenting in this chapter is to recast quantum electrodynamics in curved spacetime in such "non-geometric" terms. The potential advantage of formulating a field theory in "analytic" terms is that there might be a chance of describing the interaction of different physical fields in a more consistent, and, hopefully, non-perturbative manner. The second motivating factor comes from spectral analysis: this issue will be explored in the next chapter.

Now, let M be a 4-manifold without boundary. It can be shown that not every 4-manifold admits a non-degenerate operator. Indeed, the following lemma establishes the appropriate criterion.

Lemma 2.1.1. *The manifold M admits a non-degenerate self-adjoint operator L if and only if it is parallelizable.*

Proof. Decomposing $L_{\text{prin}}(x,p)$ with respect to the standard basis

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad s^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad s^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1}$$

in the real vector space of 2×2 Hermitian matrices, we get

$$L_{\text{prin}}(x,p) = s^j {e_j}^\alpha(x)\, p_\alpha\,, \tag{2.2}$$

where the e_j, $j = 1,2,3,4$, are some real-valued vector fields. Here each vector $e_j(x)$ has coordinate components ${e_j}^\alpha(x)$, $\alpha = 1,2,3,4$. Now, let η be the standard Minkowski metric with signature $(+,+,+,-)$, then we can define another basis for the real vector space of 2×2 Hermitian matrices by

$$s_j := \eta_{jk} s^k\,. \tag{2.3}$$

Hence, one can define the following:

$$E_j := \frac{1}{2}\operatorname{tr}(s_j L_{\text{prin}}(x,p)) = {e_j}^\alpha(x)\, p_\alpha\,. \tag{2.4}$$

In such a way we obtain a linear map, with matrix coefficients ${e_j}^\alpha(x)$, from the vector space $\{p = (p_1,p_2,p_3,p_4) : p_j \in \mathbb{R}\}$ to the vector space $\{E = (E_1,E_2,E_3,E_4) : E_j \in \mathbb{R}\}$. Of course, the real vector space E is isomorphic to the real vector space of 2×2 Hermitian matrices, with the latter being expressed through the basis from formula (2.1). Therefore (1.21) implies that the linear map ${e_j}^\alpha$ has only trivial kernel and, hence, is bijective. In other words, formulae (2.2) and (2.4) allow us to rewrite the non-degeneracy condition (1.21) as

$$\det {e_j}^\alpha(x) \neq 0, \qquad \forall x \in M. \tag{2.5}$$

But condition (2.5) is the condition of linear independence of the vector fields e_j. □

Let L be a first order linear differential operator acting on 2-columns of complex-valued half-densities over M. The standard invariant analytic way of describing this operator is by means of its principal symbol $L_{\mathrm{prin}}(x,p)$ and subprincipal symbol $L_{\mathrm{sub}}(x)$, see previous chapter.

Further on we assume that our differential operator L is formally self-adjoint and satisfies the non-degeneracy condition (1.21).

We now take an arbitrary smooth matrix-function

$$Q : M \to \mathrm{GL}(2,\mathbb{C}) \tag{2.6}$$

and consider the transformation of our differential operator

$$L \mapsto Q^* L Q. \tag{2.7}$$

The motivation for looking at such transformations is as follows. Let us write down the action (variational functional) associated with our operator, $\int_M v^*(Lv)\,\mathrm{d}x$, and let us perform an invertible linear transformation

$$v \mapsto Qv$$

in the vector space $V := \{v : M \to \mathbb{C}^2\}$ of 2-columns of complex-valued half-densities. Then the action transforms as

$$\int_M v^*(Lv)\,\mathrm{d}x \;\mapsto\; \int_M v^*(Q^*LQv)\,\mathrm{d}x\,.$$

We see that the transformation (2.7) of our differential operator describes the transformation of the integrand in the formula for the action. We choose to interpret (2.7) as a gauge transformation.

The transformation (2.7) of the differential operator L induces the following transformations of its principal and subprincipal symbols:

$$L_{\mathrm{prin}} \mapsto Q^* L_{\mathrm{prin}} Q, \tag{2.8}$$

$$L_{\mathrm{sub}} \mapsto Q^* L_{\mathrm{sub}} Q + \frac{i}{2}\left(Q^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} Q - Q^*(L_{\mathrm{prin}})_{p_\alpha} Q_{x^\alpha}\right), \tag{2.9}$$

where the subscripts indicate partial derivatives. Here we made use of formula (9.3) from [14].

Comparing formulae (2.8) and (2.9) we see that, unlike the principal symbol, the subprincipal symbol does not transform in a covariant fashion due to the appearance of terms with the gradient of the matrix-function $Q(x)$. In order to identify the sources of this non-covariance we observe that any matrix-function (2.6) can be written as a product of three terms: a complex matrix-function of determinant one, a positive scalar function and a complex scalar function of modulus one. Hence, we examine the three gauge-theoretic actions separately.

Take an arbitrary scalar function

$$\psi : M \to \mathbb{R} \tag{2.10}$$

and consider the transformation of our differential operator

$$L \mapsto e^{\psi} L e^{\psi}. \tag{2.11}$$

The transformation (2.11) is a special case of the transformation (2.7) with $Q = e^{\psi} I$, where I is the 2×2 identity matrix. Substituting this Q into formula (2.9), we get

$$L_{\text{sub}} \mapsto e^{2\psi} L_{\text{sub}}, \tag{2.12}$$

so the subprincipal symbol transforms in a covariant fashion.

Now take an arbitrary scalar function

$$\phi : M \to \mathbb{R} \tag{2.13}$$

and consider the transformation of our differential operator

$$L \mapsto e^{-i\phi} L e^{i\phi}. \tag{2.14}$$

The transformation (2.14) is a special case of the transformation (2.7) with $Q = e^{i\phi} I$. Substituting this Q into formula (2.9), we get

$$L_{\text{sub}}(x) \mapsto L_{\text{sub}}(x) + L_{\text{prin}}(x, (\operatorname{grad} \phi)(x)), \tag{2.15}$$

so the subprincipal symbol does not transform in a covariant fashion. We do not take any action with regards to the non-covariance of (2.15).

Finally, take an arbitrary matrix-function

$$R : M \to \mathrm{SL}(2, \mathbb{C}) \tag{2.16}$$

and consider the transformation of our differential operator

$$L \mapsto R^* L R. \tag{2.17}$$

Of course, the transformation (2.17) is a special case of the transformation (2.7): we are looking at the case when $\det Q(x) = 1$. It turns out that it is possible to overcome the resulting non-covariance in (2.9) by introducing the *covariant subprincipal symbol* $L_{\mathrm{csub}}(x)$ in accordance with formula

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} - f(L_{\mathrm{prin}}), \tag{2.18}$$

where f is a function (more precisely, a nonlinear differential operator) mapping a 2×2 non-degenerate Hermitian principal symbol $L_{\mathrm{prin}}(x, p)$ to a 2×2 Hermitian matrix-function $(f(L_{\mathrm{prin}}))(x)$. The function f is chosen from the condition so that the transformation (2.17) of the differential operator induces the transformation

$$L_{\mathrm{csub}} \mapsto R^* L_{\mathrm{csub}} R \tag{2.19}$$

of its covariant subprincipal symbol and the condition

$$f(e^{2\psi} L_{\mathrm{prin}}) = e^{2\psi} f(L_{\mathrm{prin}}), \tag{2.20}$$

where ψ is an arbitrary scalar function (2.10).

The existence of a function f satisfying conditions (2.19) and (2.20) is a nontrivial fact, a feature specific to a system of two equations in dimension four. The explicit formula for the function f is formula (2.47).

Let us summarise the results of our gauge-theoretic analysis.

- Our first order differential operator L is completely determined by its principal symbol $L_{\mathrm{prin}}(x, p)$ and covariant subprincipal symbol $L_{\mathrm{csub}}(x)$.
- The transformation (2.7) of the differential operator induces the transformation 2.8 of its principal symbol.

- Transformations (2.11), (2.14) and (2.17) of the differential operator induce transformations

$$L_{\mathrm{csub}} \mapsto e^{2\psi} L_{\mathrm{csub}}\,, \tag{2.21}$$

$$L_{\mathrm{csub}}(x) \mapsto L_{\mathrm{csub}}(x) + L_{\mathrm{prin}}(x, (\operatorname{grad}\phi)(x)) \tag{2.22}$$

  and (2.19) of its covariant subprincipal symbol.

We use the notation

$$L = \mathrm{Op}(L_{\mathrm{prin}}, L_{\mathrm{csub}}) \tag{2.23}$$

to express the fact that our operator is completely determined by its principal symbol and covariant subprincipal symbol. The differential operator L can be written down explicitly, in local coordinates, via the principal symbol L_{prin} and covariant subprincipal symbol L_{csub} in accordance with the following formula

$$L = -\frac{i}{2}\left([(L_{\mathrm{prin}})_{p_\alpha}(x)]\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}[(L_{\mathrm{prin}})_{p_\alpha}(x)]\right) - \frac{i}{16}\left(g_{\alpha\beta}\{L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}\right)(x) + L_{\mathrm{csub}}(x).$$

See the paragraph above formula (2.64) for details. Indeed, formula (2.23) is shorthand the above expression. We call (2.23) the *covariant representation* of the differential operator L.

Recall now a definition from elementary linear algebra. The *adjugate* of a 2×2 matrix is defined as

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} =: \operatorname{adj} P. \tag{2.24}$$

Using the covariant representation (2.23) and matrix adjugation (2.24) we can define the adjugate of the differential operator L as

$$\operatorname{Adj} L := \mathrm{Op}(\operatorname{adj} L_{\mathrm{prin}}, \operatorname{adj} L_{\mathrm{csub}}). \tag{2.25}$$

Remark 2.1.1. Note that in the case when the principal symbol does not depend on the position variable x (this corresponds to Minkowski spacetime, which is the case most important for applications) the definition of the adjugate differential operator simplifies. In this case the subprincipal symbol coincides with the covariant subprincipal symbol and one can treat the differential operator L as if it were a matrix: formula (2.25)

becomes

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \mapsto \begin{pmatrix} L_{22} & -L_{12} \\ -L_{21} & L_{11} \end{pmatrix} = \operatorname{Adj} L. \tag{2.26}$$

Observe now that out of the 2×2 operator L we can further construct a 4×4 operator, denoted by D. In fact, we shall call the operator D the massive Dirac operator as this is the usual massive Dirac operator defined in an analytic (as opposed to the geometric) way. A detailed exposition will be provided in Chapter 4. To be more precise, we can define the operator D as the following differential operator

$$D := \begin{pmatrix} L & m_e\, I \\ m_e\, I & \operatorname{Adj} L \end{pmatrix} \tag{2.27}$$

acting on 4-columns v of complex-valued half-densities. Here I is the 2×2 identity matrix and the constant m_e is the electron mass. We claim that the system of four scalar equations

$$Dv = 0 \tag{2.28}$$

is equivalent to the massive Dirac equation in its traditional geometric formulation.

Before proving the above claim in Chapter 4, let us first see that what kind of geometric information we can extract from formula (2.27). Indeed, one can easily see that the examination of formula (2.27) raises the following questions.

- Where is the Lorentzian metric?
- Why do not we encounter topological obstructions?
- Where are the Pauli matrices?
- Where are the spinors?
- Where are the connection coefficients for spinor fields?
- Where is the electromagnetic covector potential?
- Where is Lorentz invariance?

These questions will be answered in Sections 2.2–2.8. In Section 4.6 we will collect together all the formulae from Sections 2.2–2.8 and show, by direct substitution, that our equation (2.28) is indeed the massive Dirac equation (4.39). This fact will be presented in the form of Theorem 4.6.1, one of the main results in Chapter 4.

2.2 Lorentzian metric

Observe that the determinant of the principal symbol is a quadratic form in the dual variable (momentum) p:

$$\det L_{\text{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta \,. \tag{2.29}$$

We interpret the real coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha, \beta = 1, 2, 3, 4$, appearing in formula (2.29) as components of a (contravariant) metric tensor.

Lemma 2.2.1. *Our metric is Lorentzian, i.e. it has three positive eigenvalues and one negative eigenvalue.*

Proof. Decomposing $L_{\text{prin}}(x,p)$ with respect to the standard basis (2.1) as in equation (2.2):

$$L_{\text{prin}}(x,p) = s^j e_j{}^\alpha(x)\, p_\alpha \,.$$

The quartet of real-valued vector fields e_j, $j = 1, 2, 3, 4$, is called the *frame*. Note that Lemma 2.1.1 ensures that the vector fields e_j are linearly independent at every point of our manifold M.

Substituting (2.1) into (2.2), we get

$$L_{\text{prin}}(x,p) = s^j e_j{}^\alpha(x)\, p_\alpha = \begin{pmatrix} e_4{}^\alpha p_\alpha + e_3{}^\alpha p_\alpha & e_1{}^\alpha p_\alpha - i e_2{}^\alpha p_\alpha \\ e_1{}^\alpha p_\alpha + i e_2{}^\alpha p_\alpha & e_4{}^\alpha p_\alpha - e_3{}^\alpha p_\alpha \end{pmatrix}. \tag{2.30}$$

Calculating the determinant of (2.30) and substituting the result into the LHS of (2.29), we get $g^{\alpha\beta}\, p_\alpha p_\beta = (e_1{}^\alpha p_\alpha)^2 + (e_2{}^\alpha p_\alpha)^2 + (e_3{}^\alpha p_\alpha)^2 - (e_4{}^\alpha p_\alpha)^2$

□

The proof of Lemma 2.2.1 warrants the following remark. We will not encounter topological obstructions when we construct the Dirac operator via our analytic approach (see Chapter 4), as condition (1.21) implies that our manifold is parallelizable.

It is also easy to see that our frame defined in accordance with formula (2.30) is orthonormal with respect to the metric (2.29):

$$g_{\alpha\beta}\, e_j{}^\alpha e_k{}^\beta = \begin{cases} 0 & \text{if} \quad j \neq k, \\ 1 & \text{if} \quad j = k \neq 4, \\ -1 & \text{if} \quad j = k = 4. \end{cases} \tag{2.31}$$

2.3 Geometric meaning of our transformations

In Section 2.1 we defined four transformations of a formally self-adjoint 2×2 first order linear differential operator:

- conjugation (2.11) by a positive scalar function,
- conjugation (2.14) by a complex scalar function of modulus one,
- conjugation (2.17) by an $\mathrm{SL}(2, \mathbb{C})$-valued matrix-function and
- adjugation (2.25).

In this section we establish the geometric meaning of the transformations (2.11), (2.17) and (2.25). We do this by looking at the resulting transformations of the principal symbol.

We choose to examine the three transformations listed above in reverse order: first (2.25), then (2.17) and, finally, (2.11).

We know that L_{prin} can be written in terms of the standard basis (2.1) and frame e_j as (2.30)). Similarly, $\operatorname{adj} L_{\mathrm{prin}}$ can be written as

$$\operatorname{adj} L_{\mathrm{prin}}(x,p) = s^j \tilde{e}_j{}^\alpha(x)\, p_\alpha = \begin{pmatrix} \tilde{e}_4{}^\alpha p_\alpha + \tilde{e}_3{}^\alpha p_\alpha & \tilde{e}_1{}^\alpha p_\alpha - i\tilde{e}_2{}^\alpha p_\alpha \\ \tilde{e}_1{}^\alpha p_\alpha + i\tilde{e}_2{}^\alpha p_\alpha & \tilde{e}_4{}^\alpha p_\alpha - \tilde{e}_3{}^\alpha p_\alpha \end{pmatrix}, \tag{2.32}$$

where $\tilde{e}_j$ is another frame. Examination of formulae (2.24), (2.30) and (2.32) shows that the two frames, e_j and $\tilde{e}_j$, differ by spatial inversion:

$$e_j \mapsto -e_j, \quad j = 1, 2, 3, \qquad e_4 \mapsto e_4. \tag{2.33}$$

The transformation (2.17) of the differential operator induces the following transformation of its principal symbol:

$$L_{\mathrm{prin}} \mapsto R^* L_{\mathrm{prin}} R. \tag{2.34}$$

If we recast the transformation (2.34) in terms of the frame e_j (see formula (2.30)), we will see that we are looking at a linear transformation of the frame,

$$e_j \mapsto \Lambda_j{}^k e_k\,, \tag{2.35}$$

with some real-valued coefficients $\Lambda_j{}^k(x)$. The transformation of the principal symbol (2.34) preserves the Lorentzian metric (2.29), so the linear transformation of the frame (2.35) is a Lorentz transformation.

Of course, the transformation (2.33) is also a Lorentz transformation and it can be written in the form (2.35) with $(\Lambda_j{}^k) = \text{Diagonal}(-1, -1, -1, +1)$. The difference between the two Lorentz transformations is that in the case of adjugation (2.25) we get $\det \Lambda_j{}^k = -1$, whereas in the case of conjugation (2.17) by an $\mathrm{SL}(2, \mathbb{C})$-valued matrix-function we get $\det \Lambda_j{}^k = +1$.

Finally, let us establish the geometric meaning of conjugation (2.11) by a positive scalar function. The transformation (2.11) of the differential operator induces the following transformation of its principal symbol:

$$L_{\text{prin}} \mapsto e^{2\psi} L_{\text{prin}}. \tag{2.36}$$

Comparing formulae (2.29) and (2.36) we see that we are looking at a conformal scaling of the metric,

$$g^{\alpha\beta} \mapsto e^{4\psi} g^{\alpha\beta}. \tag{2.37}$$

Remark 2.3.1. We did not examine in this section the geometric meaning of the transformation (2.14). We did not do it because this transformation does not affect the principal symbol: one has to look at the subprincipal symbol to understand the geometric meaning of the transformation (2.14). We will do this later, in Section 2.6: see formula (2.82).

2.4 Pauli matrices

Definition 2.4.1. We say that the 2×2 Hermitian matrix-functions $\sigma^\alpha(x)$ are *Pauli matrices* if these matrix-functions satisfy the identity

$$\sigma^\alpha \tilde{\sigma}^\beta + \sigma^\beta \tilde{\sigma}^\alpha = -2I g^{\alpha\beta}, \tag{2.38}$$

where I is the 2×2 identity matrix and the tilde indicates matrix adjugation.

Remark 2.4.1. The identity (2.38) is, of course, equivalent to

$$\tilde{\sigma}^\alpha \sigma^\beta + \tilde{\sigma}^\beta \sigma^\alpha = -2I g^{\alpha\beta}. \tag{2.39}$$

The principal symbol $L_{\text{prin}}(x,p)$ of our operator L is linear in the dual variable p, so it can be written as

$$L_{\text{prin}}(x,p) = \sigma^{\alpha}(x)\, p_{\alpha}\,. \tag{2.40}$$

The four matrix-functions $\sigma^{\alpha}(x)$, $\alpha = 1,2,3,4$, appearing in (2.40) are, by definition, our Pauli matrices associated with the operator L.

The adjugate of the principal symbol can be written as

$$\operatorname{adj} L_{\text{prin}}(x,p) = \tilde{\sigma}^{\alpha}(x)\, p_{\alpha}\,. \tag{2.41}$$

The matrices $\tilde{\sigma}^{\alpha}(x)$, $\alpha = 1,2,3,4$, appearing in formula (2.41) are the adjugates of those from (2.40)

We have

$$[L_{\text{prin}}(x,p)][\operatorname{adj} L_{\text{prin}}(x,p)] = [\operatorname{adj} L_{\text{prin}}(x,p)][L_{\text{prin}}(x,p)] = -Ig^{\alpha\beta}p_{\alpha}p_{\beta}, \tag{2.42}$$

where I is the 2×2 identity matrix and $g^{\alpha\beta}$ is the metric from formula (2.29). Formula (2.42) implies

$$[L_{\text{prin}}(x,p)][\operatorname{adj} L_{\text{prin}}(x,q)] + [L_{\text{prin}}(x,q)][\operatorname{adj} L_{\text{prin}}(x,p)] = -2Ig^{\alpha\beta}p_{\alpha}q_{\beta}\,,$$

$$[\operatorname{adj} L_{\text{prin}}(x,p)][L_{\text{prin}}(x,q)] + [\operatorname{adj} L_{\text{prin}}(x,q)][L_{\text{prin}}(x,p)] = -2Ig^{\alpha\beta}p_{\alpha}q_{\beta}\,.$$

Substituting (2.40) and (2.41) into the above formulae we arrive at (2.38) and (2.39). This means that our matrices $\sigma^{\alpha}(x)$ defined in accordance with formula (2.40) satisfy the abstract definition of Pauli matrices, Definition 2.4.1.

2.4.1 Additional properties of Pauli matrices

Lemma 2.4.1. *If P is a 2×2 matrix then*

$$\sigma_{\alpha}P\tilde{\sigma}^{\alpha} = -2(\operatorname{tr} P)I, \tag{2.43}$$

$$\sigma_{\alpha}P\sigma^{\alpha} = 2\operatorname{adj} P. \tag{2.44}$$

Proof. Formulae (2.30), (2.32), (2.40) and (2.41) imply

$$\sigma^{\alpha} = s^{j}e_{j}{}^{\alpha}, \qquad \tilde{\sigma}^{\alpha} = s^{j}\tilde{e}_{j}{}^{\alpha}, \tag{2.45}$$

where the matrices s^j are defined in accordance with (2.1). Substituting (2.45) into (2.43) and (2.44) and using the identities (2.31) and (2.33), we get

$$\sigma_\alpha P\tilde{\sigma}^\alpha = -s^1Ps^1 - s^2Ps^2 - s^3Ps^3 - s^4Ps^4,$$

$$\sigma_\alpha P\sigma^\alpha = s^1Ps^1 + s^2Ps^2 + s^3Ps^3 - s^4Ps^4.$$

The rest is a straightforward calculation. □

Note that an alternative way of proving formula (2.43) is by means of formula (1.2.27) from [11].

2.5 Covariant subprincipal symbol

Recall that we defined the covariant subprincipal symbol $L_{\rm csub}(x)$ in accordance with formula (2.18). We need now to determine the function f appearing in this formula.

Let $R(x)$ be as in (2.16). Formulae (2.9) and (2.18) imply that the transformation (2.17) of the differential operator induces the following transformation of the matrix-function $L_{\rm csub}(x)$:

$$L_{\rm csub} \mapsto R^*(L_{\rm csub} + f(L_{\rm prin}))R - f(R^*L_{\rm prin}R) + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\rm prin})_{p_\alpha}R - R^*(L_{\rm prin})_{p_\alpha}R_{x^\alpha}\right).$$

Comparing with (2.19) we see that our function f has to satisfy the condition

$$f(R^*L_{\rm prin}R) = R^*f(L_{\rm prin})R + \frac{i}{2}\left(R^*_{x^\alpha}(L_{\rm prin})_{p_\alpha}R - R^*(L_{\rm prin})_{p_\alpha}R_{x^\alpha}\right) \tag{2.46}$$

for any non-degenerate 2×2 Hermitian principal symbol $L_{\rm prin}(x,p)$ and any matrix-function (2.16). Thus, we are looking for a function f satisfying conditions (2.20) and (2.46).

Put

$$f(L_{\rm prin}) := -\frac{i}{16}\, g_{\alpha\beta}\{L_{\rm prin}, \operatorname{adj} L_{\rm prin}, L_{\rm prin}\}_{p_\alpha p_\beta}, \tag{2.47}$$

where subscripts p_α, p_β indicate partial derivatives and $\{F,G,H\}$ is the generalised Poisson bracket on matrix-functions defined in (1.22). Note that the matrix-function in the RHS of formula (2.47) is Hermitian.

Lemma 2.5.1. *The function (2.47) satisfies conditions (2.20) and (2.46).*

Proof. Substituting (2.36) into (2.47) we see that the terms with the gradient of the function $\psi(x)$ cancel out, which gives us (2.20).

To show that the function (2.47) satisfies the condition (2.46), we first note that formulae (1.22), (2.40) and (2.41) give us

$$\frac{1}{2}\, g_{\alpha\beta}\{L_{\text{prin}}, \operatorname{adj} L_{\text{prin}}, L_{\text{prin}}\}_{p_\alpha p_\beta} = (\sigma^\alpha)_{x^\gamma}\tilde{\sigma}_\alpha\sigma^\gamma - \sigma^\gamma\tilde{\sigma}_\alpha(\sigma^\alpha)_{x^\gamma}. \tag{2.48}$$

Note also that if we transform Pauli matrices σ^α as

$$\sigma^\alpha \mapsto R^*\sigma^\alpha R, \tag{2.49}$$

where $R(x)$ is as in (2.16), then the adjugate Pauli matrices $\tilde{\sigma}^\alpha$ transform as

$$\tilde{\sigma}^\alpha \mapsto R^{-1}\tilde{\sigma}^\alpha(R^{-1})^*, \tag{2.50}$$

see formula (2.83).

Substituting formulae (2.47), (2.40) and (2.48)–(2.50) into (2.46) we rewrite the latter as $Q + Q^* = 0$, where

$$Q := -\frac{i}{8}\left[R^*\sigma^\alpha R_{x^\gamma}R^{-1}\tilde{\sigma}_\alpha\sigma^\gamma R - R^*\sigma^\gamma\tilde{\sigma}_\alpha\sigma^\alpha R_{x^\gamma}\right] + \frac{i}{2}R^*\sigma^\alpha R_{x^\alpha}. \tag{2.51}$$

Hence, in order to prove (2.46) it is sufficient to prove

$$Q = 0. \tag{2.52}$$

Formula (2.39) implies that $\tilde{\sigma}_\alpha\sigma^\alpha = -4I$, so formula (2.51) becomes

$$Q = -\frac{i}{8}R^*\sigma^\alpha R_{x^\gamma}R^{-1}\tilde{\sigma}_\alpha\sigma^\gamma R. \tag{2.53}$$

The matrix-functions $R_{x^\gamma}R^{-1}$ are trace-free, so, by formula (2.43),

$$\sigma^\alpha R_{x^\gamma}R^{-1}\tilde{\sigma}_\alpha = 0. \tag{2.54}$$

Formulae (2.53) and (2.54) imply (2.52).

□

It is interesting that the generalised Poisson bracket on matrix-functions (1.22) was initially introduced for the purpose of abstract spectral analysis, see formula (1.17) in [14]. It has now come handy in formula (2.47).

We will see later, in Section 4.6, that the RHS of (2.47) is just a way of writing the usual, Levi-Civita, connection coefficients for spinor fields. More precisely, the RHS of (2.47) does not give each spinor connection coefficient separately, it rather gives their sum, the way they appear in the Dirac operator.

Remark 2.5.1. The function f is not a function in the usual sense, it is actually a nonlinear differential operator mapping a 2×2 non-degenerate Hermitian principal symbol $L_{\text{prin}}(x, p)$ to a 2×2 Hermitian matrix-function $(f(L_{\text{prin}}))(x)$. Moreover, the function (2.47) is not a unique solution of the system of equations (2.20) and (2.46): one can always add $L_{\text{prin}}(x, B(x))$, where B is an arbitrary prescribed real-valued covector field with components $B_\alpha(x)$, $\alpha = 1, 2, 3, 4$. Thus we arrive at the following proposition.

Proposition 2.5.1. The function defined in (2.47) satisfying conditions (2.20) and (2.46) is unique up to the transformation $f(L_{\text{prin}}) \mapsto f(L_{\text{prin}}) + L_{\text{prin}}(x, B(x))$, for some covector field B on M.

Proof. Let f_1 and f_2 both satisfy conditions (2.20) and (2.46) . Then their difference, denoted by $\tilde{f} = f_1 - f_2$, satisfies the following identity:

$$\tilde{f}(R^* L_{\text{prin}} R) = R^* \tilde{f}(L_{\text{prin}}) R \,. \tag{2.55}$$

Therefore, we only need to prove that the only solution to equations (2.20) and (2.55) is of the type $\tilde{f}(L_{\text{prin}}) = L_{\text{prin}}(x, B(x))$, for some covector B on T^*M.

Now, note that if we choose a frame, say $\hat{e}_j{}^\alpha(x)$, then we can define the following objects:

$$\hat{L}_{\text{prin}}(x, p) := s^j \hat{e}_j{}^\alpha(x)\, p_\alpha \tag{2.56}$$

and

$$\tilde{f}\left(\hat{L}_{\text{prin}}(x, p)\right) := s^j \hat{e}_j(x) \,. \tag{2.57}$$

In particular, there exists a covector, B on T^*M, such that

$$\hat{e}_j{}^\alpha(x) B_\alpha(x) = \hat{e}_j(B)(x) = \hat{e}_j(x) \,. \tag{2.58}$$

Now, in view of equation (2.2), for every 2×2 non-degenerate Hermitian principal symbol $L_{\mathrm{prin}}(x,p)$, there exists a unique section $\Lambda \in C^\infty(M, SO(3,1))$ such that

$$L_{\mathrm{prin}}(x,p) = s^j \Lambda_{jk}(x) \hat{e}_k{}^\alpha(x)\, p_\alpha \,. \tag{2.59}$$

Using Robert P. Geroch's work on the existence of spin structures on a 4-dimensional space-time [25] and Lemma 2.1.1, we conclude that for every continuous section $\Lambda \in C^\infty(M, SO(3,1))$, there exists a continuous section $R_\Lambda \in C^\infty(M, SL(2;\mathbb{C}))$ such that

$$L_{\mathrm{prin}}(x,p) = R_\Lambda^*(x) \hat{L}_{\mathrm{prin}}(x,p) R_\Lambda(x) \,. \tag{2.60}$$

Using equations (2.59) and (2.60), we have

$$\tilde{f}(L_{\mathrm{prin}}) = \tilde{f}\left(s^j \Lambda_{jk}(x) \hat{e}_k{}^\alpha(x)\, p_\alpha\right) = \tilde{f}\left(R_\Lambda^* \hat{L}_{\mathrm{prin}} R_\Lambda\right) . \tag{2.61}$$

On the other hand, applying equation (2.55), we also have

$$\tilde{f}\left(R_\Lambda^* \hat{L}_{\mathrm{prin}} R_\Lambda\right) = R_\Lambda^* \tilde{f}(\hat{L}_{\mathrm{prin}}) R_\Lambda \,. \tag{2.62}$$

Moreover, contracting (2.59) and (2.60) with the covector B found in (2.58), we thereby arrive at

$$R_\Lambda^* \tilde{f}(\hat{L}_{\mathrm{prin}}) R_\Lambda = s^j \Lambda_{jk}(x) \hat{e}_k = L_{\mathrm{prin}}(x, B(x)) \,. \tag{2.63}$$

Therefore, equations (2.61)–(2.63) yield the desired result. □

Remark 2.5.2. Proposition 2.5.1 implies that the function (2.47) is singled out amongst all solutions of the system of equations (2.20) and (2.46) by the property that it does not depend on any prescribed external fields.

For the sake of clarity, we write down the differential operator L explicitly, in local coordinates, it terms of its principal symbol L_{prin} and covariant subprincipal symbol L_{csub}. Combining formulae (1.19), (2.18) and (2.47), we get

$$\begin{aligned} L = -\frac{i}{2}\left([(L_{\mathrm{prin}})_{p_\alpha}(x)]\frac{\partial}{\partial x^\alpha} + \frac{\partial}{\partial x^\alpha}[(L_{\mathrm{prin}})_{p_\alpha}(x)]\right) \\ - \frac{i}{16}\left(g_{\alpha\beta}\{L_{\mathrm{prin}}, \mathrm{adj}\, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}\right)(x) + L_{\mathrm{csub}}(x). \end{aligned} \tag{2.64}$$

Here the covariant symmetric tensor $g_{\alpha\beta}(x)$ is the inverse of the contravariant symmetric tensor $g^{\alpha\beta}(x)$ defined by formula (2.29), $\{\,\cdot\,,\,\cdot\,,\,\cdot\,\}$ is the generalised Poisson bracket on

matrix-functions defined by formula (1.22) and adj is the operator of matrix adjugation (2.24). See also Remark 1.0.1 which explains how to read formula (2.64) correctly.

2.5.1 The correction function f and axial torsion

We would like to examine the quantity $f(L_{\text{prin}})$ in this subsection. Before doing this, let us first introduce a geometric object that can be constructed out from the quartet of real-valued vector fields e_j, $j = 1, 2, 3, 4$ (presented in equation (2.2)). We will follow the conventions used in the paper [15], adapting them to the 4-dimensional Lorentzian setting.

Definition 2.5.1 (The teleparallel connection). For a given frame e_j on a parallelizable manifold M there exists a metric compatible affine connection induced by the frame, which is called the teleparallel connection associated with the frame e_j. Its connection coefficients are given by

$$\Gamma^{\alpha}{}_{\beta\gamma} = e_j{}^{\alpha} \frac{\partial}{\partial x^{\beta}} e^{j}{}_{\gamma} \tag{2.65}$$

in local coordinates. By ∇^{T} we denote the corresponding covariant derivative on TM.

Remark 2.5.3.

- The teleparallel connection is defined via satisfying the conditions

$$\nabla^{\mathrm{T}} e_j = 0\,.$$

- Although the teleparallel connection has zero curvature, it has, in general, non-trivial torsion. Its torsion is given in local coordinates by the formula

$$T^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} - \Gamma^{\alpha}{}_{\gamma\beta}\,. \tag{2.66}$$

Definition 2.5.2 (Axial torsion). The axial torsion of $\{e_j\}_{j=1}^4$ is the totally antisymmetric part of $T_{\alpha\beta\gamma}$.

Definition 2.5.3 (Hodge dual). For a Lorentzian 4-manifold M with a metric $g_{\mu\nu}$, there is a standard (differential) volume form on M, namely $\omega = \sqrt{|\det g_{\mu\nu}|}\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$ in local coordinates. The Hodge dual of a degree r differential form, R, is defined as

$$(*_{\omega} R)_{\mu_{r+1}\ldots\mu_4} := \frac{1}{r!}\sqrt{|\det g_{\mu\nu}|}\, R^{\mu_1\ldots\mu_r} \varepsilon_{\mu_1\ldots\mu_4}\,, \tag{2.67}$$

where $\varepsilon_{\mu_1\ldots\mu_4}$ is the totally antisymmetric quantity with $\varepsilon_{1234} = 1$.

Remark 2.5.4. The subscript ω in the star symbol, $*_\omega$, is to indicate the dependence of the Hodge dual on the volume form ω. When the volume form is the standard volume form as shown in Definition 2.5.3, we will omit the subscript ω.

Proposition 2.5.2. The correction function $f(L_{\text{prin}})$ and axial torsion are related as follows:

$$f(L_{\text{prin}}) = \frac{3}{4}(\mathbf{c} * T^{\text{ax}})_\alpha \sigma^\alpha \,, \tag{2.68}$$

where $*$ is the Hodge dual associated with the metric $g_{\mu\nu}$ defined in formula (2.29) and $\mathbf{c} := \text{sgn}(\det(e_i{}^\alpha))$, with the frame e_j given by formula (2.2).

Proof. Let the indices $\{m, n, p, q\}$ run through $\{1, 2, 3, 4\}$ and the indices $\{i, j, k, l\}$ run through $\{1, 2, 3\}$. Define s^m and s_n as in formulae (2.1) and (2.3). Then we have

$$\begin{aligned} s^m s^n = \sum_{p=1}^{4} \Big[& (\delta^m{}_4 \eta^{np} + \eta^{mp} \delta^n{}_4 - \delta^m{}_4 \delta^n{}_4 \eta^{p4}) \\ & + (1 - \delta^m{}_4 - \delta^n{}_4 + \delta^m{}_4 \delta^n{}_4) \left(i(1 - \delta^p{}_4) \varepsilon^{mnp} + \delta^{mn} \eta^{p4} \right) \Big] s_p \,, \end{aligned} \tag{2.69}$$

where $(\delta^m{}_n) = (\delta^{mn}) = (\delta_{mn}) = \text{Diagonal}(1, 1, 1, 1)$ is the 4×4 identity matrix and $(\eta^{mn}) = (\eta_{mn}) = \text{Diagonal}(1, 1, 1, -1)$. Note that the transformation (2.33) gives the relation between e_j and $\tilde{e}_j$. Therefore, together with equation (2.69), we obtain

$$\tilde{\sigma}^\alpha \sigma^\beta = s^m s^n \tilde{e}_m{}^\alpha e_n{}^\beta = (e_4{}^\alpha e_i{}^\beta - e_i{}^\alpha e_4{}^\beta) s^i + e_4{}^\alpha e_4{}^\beta s^4 - (i\varepsilon^{ijk} s_k + \delta^{ij} s^4) e_i{}^\alpha e_j{}^\beta \,.$$

After some calculations (see Appendix A.1), we arrive at

$$\begin{aligned} \sigma^\alpha \tilde{\sigma}^\beta \sigma^\gamma = & \Big(e_4{}^\alpha e_4{}^\beta e_m{}^\gamma s^m - e_4{}^\alpha e_m{}^\beta s^m e_4{}^\gamma - \delta^{ij} e_m{}^\alpha s^m e_i{}^\beta e_j{}^\gamma \\ & + \delta^{ij} e_i{}^\alpha e_m{}^\beta s^m e_j{}^\gamma - \delta^{ij} e_i{}^\alpha e_j{}^\beta e_m{}^\gamma s^m + e_m^\alpha s^m e_4{}^\beta e_4^\gamma \Big) \\ & + i\varepsilon^{ijk} \Big(e_i{}^\alpha e_j{}^\beta e_k{}^\gamma s_4 - e_4{}^\alpha e_i{}^\beta e_j{}^\gamma s_k + e_i{}^\alpha e_4{}^\beta e_j{}^\gamma s_k - e_i{}^\alpha e_j{}^\beta e_4{}^\gamma s_k \Big) \,, \end{aligned} \tag{2.70}$$

which further implies (see Appendix A.1)

$$\begin{aligned} & (\sigma^\alpha)_{x^\gamma} \tilde{\sigma}_\alpha \sigma^\gamma - \sigma^\gamma \tilde{\sigma}_\alpha (\sigma^\alpha)_{x^\gamma} \\ & \quad = 2i\varepsilon^{ijk} \Big((e_i{}^\alpha)_{x^\gamma} e_{j\alpha} e_k{}^\gamma s_4 - (e_4{}^\alpha)_{x^\gamma} e_{i\alpha} e_j{}^\gamma s_k + (e_i{}^\alpha)_{x^\gamma} e_{4\alpha} e_j{}^\gamma s_k - (e_i{}^\alpha)_{x^\gamma} e_{j\alpha} e_4{}^\gamma s_k \Big) \,. \end{aligned} \tag{2.71}$$

In order to write equation (2.71) in an elegant way, we introduce the exterior algebra associated with the vector space of Hermitian matrices $V = \mathrm{Span}\{s^1, s^2, s^3, s^4\} = \mathrm{Span}\{s_1, s_2, s_3, s_4\}$. Also, we equip our vector space with an inner product $(\cdot,\cdot)_V$ defined as

$$(s^m, s^n)_V := \eta^{mn}\,.$$

Now, we have the algebraic volume form ω_V on the vector space V defined as

$$\omega_V = s_1 \wedge s_2 \wedge s_3 \wedge s_4 = -s^1 \wedge s^2 \wedge s^3 \wedge s^4\,. \tag{2.72}$$

Observe that for a given frame ${e_m}^\alpha(x)$ on M we also have another basis for V, namely $\{\sigma^\alpha(x) = {e_m}^\alpha(x)s^m; \alpha = 1,2,3,4\} = \{\sigma^1(x), \sigma^2(x), \sigma^3(x), \sigma^4(x)\}$. Moreover, the algebraic volume form (2.72) and the new algebraic form are related by

$$\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4 = {e_{i_1}}^1 s^{i_1} \wedge {e_{i_2}}^2 s^{i_2} \wedge {e_{i_3}}^3 s^{i_3} \wedge {e_{i_4}}^4 s^{i_4} = -\det({e_i}^\alpha)\,\omega_V\,, \tag{2.73}$$

where we omit the dependence on x. Now, let $\mathbf{c} = \mathrm{sgn}(\det({e_i}^\alpha))$ and $g_{\rho\lambda}$ as given in formula (2.31). Then we define a differential volume form on M as

$$\omega_{-\mathbf{c}} := -\frac{1}{\det({e_i}^\alpha)}\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4 = -\mathbf{c}\sqrt{|\det(g_{\rho\lambda})|}\sigma^1 \wedge \sigma^2 \wedge \sigma^3 \wedge \sigma^4\,. \tag{2.74}$$

In fact, the differential volume form (2.74) is the same as viewing the quantity (2.72) as a differential volume form on M. Furthermore, if we identify the exterior algebra generated by $\{\sigma^\alpha\}_{\alpha=1}^4$ with the one generated by $\{\mathrm{d}x^\alpha\}_{\alpha=1}^4$, then we have

$$\omega_{-\mathbf{c}} = -\mathbf{c}\,\omega\,. \tag{2.75}$$

Hence, equation (2.71) becomes

$$\begin{aligned}(\sigma^\alpha)_{x^\gamma}\tilde{\sigma}_\alpha\sigma^\gamma - \sigma^\gamma\tilde{\sigma}_\alpha(\sigma^\alpha)_{x^\gamma} &= 2i\Big(*_{\omega_{-\mathbf{c}}}[({e_m}^\alpha)_{x^\gamma}s^m \wedge e_{n\alpha}s^n \wedge {e_p}^\gamma s^p], \sigma_\lambda\Big)_V\sigma^\lambda \\ &= 2i\Big(*_{\omega_{-\mathbf{c}}}[({e_m}^\rho)_{x^\gamma}{e^m}_\alpha\, g_{\rho\beta}\,\sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma], \sigma_\lambda\Big)_V\sigma^\lambda\,.\end{aligned} \tag{2.76}$$

Now, observe that

$$\begin{aligned}g(\nabla^{\mathrm{T}}_{e_p}e_m, e_n) &= g\Big([{e_p}^\gamma({e_m}^\alpha)_{x^\gamma} + {e_p}^\lambda {e_m}^\rho\Gamma^\alpha_{\lambda\rho}]\,\frac{\partial}{\partial x^\alpha}, {e_n}^\beta\frac{\partial}{\partial x^\beta}\Big) \\ &= ({e_m}^\alpha)_{x^\gamma}e_{n\alpha}{e_p}^\gamma + {e_m}^\rho{e_n}^\beta{e_p}^\lambda g_{\alpha\beta}\Gamma^\alpha_{\lambda\rho}\,.\end{aligned} \tag{2.77}$$

Multiplying both sides of above equation (2.77) by $s^m \wedge s^n \wedge s^p$ and sum over all repeated indices, one obtains (see Appendix A.1)

$$g(\nabla^{\mathrm{T}}_{e_p} e_m, e_n) s^m \wedge s^n \wedge s^p = (e_m{}^\rho)_{x^\gamma} e^m{}_\alpha \, g_{\rho\beta} \, \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma + 3 \times \frac{1}{3!} T^{\mathrm{ax}}_{\alpha\beta\gamma} \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma \,. \quad (2.78)$$

Since $\nabla^{\mathrm{T}}_{e_p} e_m = 0$, we have, from equations (2.75)–(2.78),

$$6i(\mathbf{c} * T^{\mathrm{ax}})_\alpha \sigma^\alpha = (\sigma^\alpha)_{x^\gamma} \tilde{\sigma}_\alpha \sigma^\gamma - \sigma^\gamma \tilde{\sigma}_\alpha (\sigma^\alpha)_{x^\gamma} \,. \quad (2.79)$$

Formulae (2.47), (2.48) and (2.79) yield the required result. □

2.6 Electromagnetic covector potential

The non-degeneracy condition (1.21) implies that for each $x \in M$ the matrices $(L_{\mathrm{prin}})_{p_\alpha}(x)$, $\alpha = 1, 2, 3, 4$, form a basis in the real vector space of 2×2 Hermitian matrices. Here and throughout this section the subscript p_α indicates partial differentiation.

Decomposing the covariant subprincipal symbol $L_{\mathrm{csub}}(x)$ with respect to this basis, we get

$$L_{\mathrm{csub}}(x) = (L_{\mathrm{prin}})_{p_\alpha}(x) \, A_\alpha(x) \quad (2.80)$$

with some real coefficients $A_\alpha(x)$, $\alpha = 1, 2, 3, 4$.

Formula (2.80) can be rewritten in more compact form as

$$L_{\mathrm{csub}}(x) = L_{\mathrm{prin}}(x, A(x)), \quad (2.81)$$

where A is a covector field with components $A_\alpha(x)$, $\alpha = 1, 2, 3, 4$. Formula (2.81) tells us that the covariant subprincipal symbol L_{csub} is equivalent to a real-valued covector field A, the electromagnetic covector potential.

It is easy to see that our electromagnetic covector potential A is invariant under Lorentz transformations (2.17) and conformal scalings of the metric (2.11), whereas formulae (2.22) and (2.81) imply that the transformation (2.14) of the differential operator induces the transformation

$$A \mapsto A + \operatorname{grad} \phi. \quad (2.82)$$

2.7 Properties of the adjugate operator

In this section we list gauge-theoretic properties of operator adjugation (2.25).

Matrix adjugation (2.24) has the property

$$\operatorname{adj}(R^*PR) = R^{-1}(\operatorname{adj} P)(R^{-1})^* \tag{2.83}$$

for any matrix $R \in \mathrm{SL}(2,\mathbb{C})$. It is easy to see that operator adjugation (2.25) has a property similar to (2.83):

$$\operatorname{Adj}(R^*LR) = R^{-1}(\operatorname{Adj} L)(R^{-1})^* \tag{2.84}$$

for any matrix-function (2.16).

It is also easy to see that operator adjugation (2.25) commutes with the transformations (2.11) and (2.14):

$$\operatorname{Adj}(e^{\psi}Le^{\psi}) = e^{\psi}(\operatorname{Adj} L)e^{\psi}, \qquad \operatorname{Adj}(e^{-i\phi}Le^{i\phi}) = e^{-i\phi}(\operatorname{Adj} L)e^{i\phi}.$$

Finally, let us observe that the map (2.47) anticommutes with matrix adjugation (2.24),

$$\operatorname{adj} f(L_{\mathrm{prin}}) = -f(\operatorname{adj} L_{\mathrm{prin}}).$$

This implies that the full symbol of the operator $\operatorname{Adj} L$ is not necessarily the matrix adjugate of the full symbol of the operator L.

In the special case when the principal symbol does not depend on the position variable x we get $f(L_{\mathrm{prin}}) = f(\operatorname{adj} L_{\mathrm{prin}}) = 0$, so in this case the full symbol of the operator $\operatorname{Adj} L$ is the matrix adjugate of the full symbol of the operator L. The definition of the adjugate operator then simplifies and becomes (2.26).

2.8 Lorentz invariance of the operator (2.27)

In this section we show that our Dirac operator (2.27) is Lorentz invariant. Recall that this operator acts on 4-columns of complex-valued half-densities.

Let $R(x)$ be as in (2.16). Define the 4×4 matrix-function

$$S := \begin{pmatrix} R & 0 \\ 0 & (R^{-1})^* \end{pmatrix}.$$

Then

$$S^* D S = \begin{pmatrix} R^* L R & m_e\, I \\ m_e\, I & R^{-1}(\mathrm{Adj}\, L)(R^{-1})^* \end{pmatrix}. \tag{2.85}$$

The operator identity (2.84) tells us that the diagonal terms in (2.85) are adjugates of each other, so formula (2.85) can be rewritten as

$$S^* D S = \begin{pmatrix} R^* L R & m_e\, I \\ m_e\, I & \mathrm{Adj}(R^* L R) \end{pmatrix}. \tag{2.86}$$

We see that the operator (2.86) has the same structure as (2.27), which proves Lorentz invariance.

Chapter 3

Spectral asymptotics for first order systems

The work of this chapter is inspired by the famous Weyl type asymptotic formulae. The first one of this type was given by Hermann Weyl in 1911 for the counting function of the Laplace operator with Dirichlet boundary condition on a bounded domain $\Omega \subset \mathbb{R}^n$. This formula reads

$$N(\lambda) = \left[\frac{\mathrm{Vol}(B_n)}{(2\pi)^n} \times \mathrm{Vol}(\Omega) \right] \lambda^{\frac{n}{2}} + o(\lambda^{\frac{n}{2}}), \quad \text{as} \quad \lambda \to +\infty. \tag{3.1}$$

Here B_n is the unit ball in $\mathbb{R}^n$. Later on, Weyl conjectured that the remainder estimate can be improved. Since the conjecture was made, a number of distinguished mathematicians had and have been working on it, including Richard Courant, Lars Gårding, T. Carleman, V. G. Avakumovič, Boris Levitan, Lars Hörmander, Robert T. Seeley, Hans Duistermaat, Victor Guillemin, Richard B. Melrose and Victor Ivrii. In particular, Lars Gårding generalized Weyl's law for elliptic systems with constant coefficients. Later on, Avakumovič and Levitan independently proved a sharp remainder term estimate, $O(\lambda^{\frac{n-1}{2}})$, for the counting function of the Laplace–Beltrami operator on a closed Riemannian manifold. In the same period, inspired by Levitan's and Seeley's works, Hörmander obtained a sharp remainder term estimate for a general scalar elliptic pseudo-differential operator on a closed manifold.

Weyl also conjectured that the basic asymptotic formula for $N(\lambda)$ can be improved by singling out a second asymptotic term. Namely, for the Laplacian in a domain in $\Omega \subset \mathbb{R}^n$ he conjectured that this second asymptotic term should be proportional to $\lambda^{\frac{n-1}{2}}$ and the $(n-1)$-dimensional measure of the smooth boundary $\partial\Omega$. This became known as *Weyl's conjecture*. In 1980 Victor Ivrii proved [31] Weyl's conjecture for the

Laplace–Beltrami operator acting on a Riemannian manifold with smooth boundary, under the assumption that the starting points of periodic geodesic billiard trajectories have measure zero on T^*M. Ivrii also conjectured that the above geometric assumption is always fulfilled for the Laplace operator in a domain $\Omega \subset \mathbb{R}^n$, but this has not been proven to date.

In 1983–1984 D. Vassiliev established [49, 50] a two-term asymptotic formula for the counting function of a general even order self-adjoint scalar elliptic differential operator acting on a compact manifold with smooth boundary. A detailed exposition of this result was later provided in [45].

In 1989, Yu. Safarov performed a comprehensive analysis [44] of "non-classical" two-term asymptotic formulae for the counting function, i.e. asymptotic formulae which are not purely polynomial. Such situations occur when there are many periodic trajectories (measure of their starting points is nonzero),

In 2013 [14], O. Chervova, R. J. Downes and D. Vassiliev derived a two-term asymptotic formula for first order elliptic systems on closed manifolds, with the second asymptotic coefficient written out explicitly. Note that the formula for the second Weyl coefficient of a system is fundamentally different from the well-known scalar formula of Duistermaat and Guillemin [19].

In this chapter we will give a brief review of a typical methodology used in some proofs of Weyl-type asymptotic formulae, which is known as the *hyperbolic equation method* and was first introduced by Boris Levitan.

3.1 The hyperbolic equation method for the local counting function

In this section we give an overview of the hyperbolic equation method, which is also known as the wave equation method. Following Levitan [39], Seeley [46] and Hörmander [30],we start our analysis with the local counting function. We will be using Levitan's hyperbolic equation method which is the only physically meaningful way of introducing a time coordinate when dealing with an operator which is not semi-bounded.

Let L be an $m \times m$ matrix first order self-adjoint elliptic pseudo-differential operator acting on half-densities over a closed manifold. It is known that L has a discrete spectrum but this spectrum is not necessarily semi-bounded. Let λ_k and $v_k(x) = (v_{k1}(x) \cdots v_{km}(x))^T$ be the eigenvalues and orthonormal (in the sense of the inner product (1.1)) eigenfunctions of the operator. The particular choice of enumeration of

these eigenvalues (with account of multiplicities) is irrelevant. Consider the expression $\|v_k(x)\|$, the Euclidean norm of the m-column v_k evaluated at the point $x \in M$. Of course, $\|v_k(x)\|^2$ is a real-valued density. Hence, we can define the positive local counting function (which is sometimes called the positive spectral function) as

$$N_+(L; x, \lambda) := \sum_{\lambda_k(L)>0} \|v_k(x)\|^2 H(\lambda - \lambda_k(L))\,, \tag{3.2}$$

where $H(\lambda)$ is the Heaviside step function with $H(0) = 0$. Integration of formula (3.2) yields the usual (global) positive counting function:

$$N_+(L; \lambda) := \int_M N_+(L; x, \lambda)\,\mathrm{d}x = \sum_{0<\lambda_k<\lambda} 1\,. \tag{3.3}$$

Let us now define the local negative counting function $N_-(L; x, \lambda)$ and the global negative counting function $N_-(L; \lambda)$. Following [33, 5], we define them as

$$N_-(L; x, \lambda) := \sum_{\lambda_k(L)<0} \|v_k(x)\|^2 H(\lambda + \lambda_k(L)) \tag{3.4}$$

and

$$N_-(L; \lambda) := \int_M N_-(L; x, \lambda)\,\mathrm{d}x\,. \tag{3.5}$$

In fact, it is enough to derive an asymptotic formula for the positive counting function (3.3), as we can describe the negative one by the following relationship.

$$N_-(L; \lambda) = N_+(-L; \lambda)\,. \tag{3.6}$$

Therefore, from now on, we will drop the dependence on L, since we are only focusing on calculations associated with L rather than with $-L$. Thus

$$N_\pm(x, \lambda) = N_\pm(L; x, \lambda) \qquad \text{and} \qquad N_\pm(\lambda) = N_\pm(L; \lambda)\,. \tag{3.7}$$

It is known that the positive local counting function can be recovered from a more general mathematical object known as the (wave) propagator associated with operator L. The *propagator* (see Subsection 3.1.1 for details) is the one-parameter family of unitary operators

$$U(t) := e^{-itL} = \sum_k e^{-it\lambda_k}\, v_k(x) \int_M [v_k(y)]^*(\,\cdot\,)\,\mathrm{d}y\,, \tag{3.8}$$

where $t \in \mathbb{R}$ is the time variable. Its integral kernel is

$$u(x,t,y) := \sum_k e^{-it\lambda_k} v_k(x) [v_k(y)]^* . \tag{3.9}$$

By taking the (matrix) trace of the integral kernel, evaluating it at $y = x$ and then applying the inverse Fourier transform[1] from t to λ, we get

$$\mathcal{F}^{-1}_{t\to\lambda} [\mathrm{tr}\,(u(x,t,x))] = \sum_k \|v_k(x)\|^2 \delta(\lambda - \lambda_k) , \tag{3.10}$$

where tr stands for the matrix trace. Combing the above formula with the definitions of local counting functions, one can easily obtain

$$\mathcal{F}^{-1}_{t\to\lambda} [\mathrm{tr}\,(u(x,t,x))] = N'_+(x,\lambda) + N'_-(x,-\lambda) + \sum_{\lambda_k=0} \|v_k(x)\|^2 \delta(\lambda) , \tag{3.11}$$

where the $N'_\pm(x,\lambda)$ are the (distributional) derivatives of the functions $N_\pm(x,\lambda)$ with respect to λ. Now, let ϵ to be a positive number which is strictly less than the first positive eigenvalue. Then integrating from ϵ to some positive λ, we get, from equations (3.2), (3.4), (3.7) and (3.10), the following:

$$\int_\epsilon^\lambda \mathcal{F}^{-1}_{t\to\mu} [\mathrm{tr}\,(u(x,t,x))]\, \mathrm{d}\mu = N_+(x,\lambda) . \tag{3.12}$$

Equations (3.2) and (3.12) suggest that the asymptotic behaviour of $N_+(\lambda)$ as λ going to positive infinity boils down to understanding the behaviour of singularities of $\mathrm{tr}\,(u(x,t,x))$. A rigorous exposition involving Fourier Tauberian theorems is presented in [45].

Thus, we only need to reconstruct the singularities of $\mathrm{tr}\,(u(x,t,x))$ and the smooth contributions are irrelevant. Since oscillatory integrals (also known as Fourier integral operators, which are abbreviated as *FIOs*) are used to study singularities of distributions, we only need to find a FIO approximating to $\mathrm{tr}\,(u(x,t,x))$ modulo a C^∞ function. In fact, we can do even better by constructing a FIO approximating $u(x,t,y)$, which will be explained further in this chapter.

[1] $\mathcal{F}_{\lambda\to t}[f(\lambda)] = \hat{f}(t) = \int e^{-it\lambda} f(\lambda)\,\mathrm{d}\lambda$ and $\mathcal{F}^{-1}_{t\to\lambda}[\hat{f}(t)] = f(\lambda) = \frac{1}{2\pi}\int e^{it\lambda}\hat{f}(t)\,\mathrm{d}t$.

3.1.1 The propagator

From now on we assume that our first order $m \times m$ matrix operator L is differential (as opposed to pseudo-differential). This implies, in particular, that m is even: see Remark 1.0.2.

Let $h^{(j)}(x,p)$ be the eigenvalues of the matrix-function $L_{\text{prin}}(x,p)$. Throughout this subsection we assume that these are simple for all $(x,p) \in T^*M \setminus \{0\}$. We enumerate the $h^{(j)}(x,p)$ in increasing order, using a negative index $j = -m/2, \dots, -1$ for negative $h^{(j)}(x,p)$ and a positive index $j = 1, \dots, m/2$ for positive $h^{(j)}(x,p)$. By $v^{(j)}(x,p)$ we denote the corresponding normalised eigenvectors. Note that as our operator L is first order and differential (as opposed to pseudo-differential) we have the following symmetry:

$$h^{(-j)}(x,p) = -h^{(j)}(x,-p), \quad v^{(-j)}(x,p) = v^{(j)}(x,-p), \quad j = 1, \dots, \frac{m}{2}. \tag{3.13}$$

Now, let $x^{n+1} \in \mathbb{R}$ be the additional 'time' coordinate. Consider the Cauchy problem

$$w\big|_{x^{n+1}=0} = v \tag{3.14}$$

for the hyperbolic system

$$(-i\partial_{x^{n+1}} + L)w = 0 \tag{3.15}$$

on $M \times \mathbb{R}$. The m-column of half-densities $v = v(x^1, \dots, x^n)$ is given and the m-column of half-densities $w = w(x^1, \dots, x^n, x^{n+1})$ is to be found. The solution of the Cauchy problem (3.14), (3.15) can be written as $w = U(x^{n+1})\, v$, where

$$U(x^{n+1}) := e^{-ix^{n+1}L} = \sum_k e^{-ix^{n+1}\lambda_k}\, v_k(x^1, \dots, x^n) \int_M [v_k(y^1, \dots, y^n)]^*(\,\cdot\,)\, \mathrm{d}y^1 \dots \mathrm{d}y^n \tag{3.16}$$

is the propagator associated with operator L defined in formula (3.8) with t replaced by x^{n+1}.

Remark 3.1.1. We chose to denote the 'time' coordinate by x^{n+1} rather than by t because some constructions presented in the current chapter work in the relativistic setting, i.e. when there is no distinguished time direction and the coordinates $x^1, \dots, x^n$ and x^{n+1} are 'mixed up'. Such an approach was pursued in [22] and, to a certain extent, in [23].

It was shown by Safarov [44] that the propagator can be constructed explicitly, modulo C^∞, as a sum of m oscillatory integrals (Fourier integral operators)

$$U(x^{n+1}) \overset{\mathrm{mod}\, C^\infty}{=} \sum_j U^{(j)}(x^{n+1}), \tag{3.17}$$

where the phase function of each oscillatory integral $U^{(j)}(x^{n+1})$ is associated with the corresponding Hamiltonian $h^{(j)}(x^1, \dots, x^n, p_1, \dots, p_n)$ and summation is carried out over all nonzero integers j from $-m/2$ to $+m/2$. The notion of a phase function associated with a Hamiltonian is defined in Section 2 of [14] and Section 2.4 of [45]. Safarov's initial exposition [44] of the construction leading up to (3.17) was quite concise. A more detailed exposition was later given in [14].

We will now state the two main results regarding the properties of the oscillatory integrals $U^{(j)}(x^{n+1})$ appearing in the RHS of formula (3.17). From this point till the end of the subsection we assume the index j to be fixed.

The first result concerns the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$. The notion of the principal symbol of an oscillatory integral is defined in accordance with Definition 2.7.12 from [45]. The principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ is a complex-valued $m \times m$ matrix-function on $M \times \mathbb{R} \times (T^*M \setminus \{0\})$. We denote the arguments of this principal symbol by $x^1, \dots, x^n$ (local coordinates on M), x^{n+1} ('time' coordinate on $\mathbb{R}$), $y^1, \dots, y^n$ (local coordinates on M) and $q_1, \dots, q_n$ (variable dual to $y^1, \dots, y^n$).

Further on in this subsection and the next subsection we use x, y, p and q as shorthand for $x^1, \dots, x^n$, $y^1, \dots, y^n$, $p_1, \dots, p_n$ and $q_1, \dots, q_n$ respectively. The additional 'time' coordinate x^{n+1} will always be written separately.

In order to write down the principal symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ we need to introduce some notation.

We define the scalar function $f^{(j)} : T^*M \setminus \{0\} \to \mathbb{R}$ in accordance with the formula

$$f^{(j)} := [v^{(j)}]^* L_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, L_{\mathrm{prin}} - h^{(j)}, v^{(j)}\} - i[v^{(j)}]^*\{v^{(j)}, h^{(j)}\}.$$

By $(x^{(j)}(x^{n+1}; y, q), p^{(j)}(x^{n+1}; y, q))$ we denote the Hamiltonian trajectory originating from the point (y, q), i.e. solution of the system of ordinary differential equations (the dot denotes differentiation in x^{n+1})

$$\dot{x}^{(j)} = h^{(j)}_p(x^{(j)}, p^{(j)}), \qquad \dot{p}^{(j)} = -h^{(j)}_x(x^{(j)}, p^{(j)})$$

subject to the initial condition $(x^{(j)}, p^{(j)})\big|_{x^{n+1}=0} = (y, q)$.

Theorem 3.1.2. *The formula for the principal symbol of the oscillatory integral* $U^{(j)}(x^{n+1})$ *reads as follows:*

$$[v^{(j)}(x^{(j)}(x^{n+1}; y, q), p^{(j)}(x^{n+1}; y, q))]\,[v^{(j)}(y, q)]^* \times \exp\left(-i\int_0^{x^{n+1}} f^{(j)}(x^{(j)}(\tau; y, q), p^{(j)}(\tau; y, q))\,\mathrm{d}\tau\right).$$

This principal symbol is positively homogeneous in momentum q *of degree zero.*

Theorem 3.1.2 is due to Safarov [44]. It was later confirmed by the more detailed analysis carried out in [14].

Theorem 3.1.2 is insufficient for the determination of the second term in spectral asymptotics because one needs information about the lower order terms of the symbol of the oscillatory integral $U^{(j)}(x^{n+1})$. Namely, one needs information about terms positively homogeneous in momentum q of degree -1. The algorithm described in Section 2 of [14] provides a recursive procedure for the calculation of all lower order terms, of any degree of homogeneity in momentum q. However, there are two issues here. Firstly, calculations become very complicated. Secondly, describing these lower order terms in an invariant way is problematic. It was pointed out by Safarov, a few months before his untimely death, the concept of subprincipal symbol has never been defined for time-dependent oscillatory integrals (Fourier integral operators).

We overcome the problem of invariant description of lower order terms of the symbol of the oscillatory integral $U^{(j)}(x^{n+1})$ by restricting our analysis to $U^{(j)}(0)$. It turns out that knowing the properties of the lower order terms of the symbol of $U^{(j)}(0)$ is sufficient for the derivation of two-term spectral asymptotics. And $U^{(j)}(0)$ is a pseudo-differential operator, so one can use here the standard notion of subprincipal symbol of a pseudo-differential operator.

The following result was recently established in [14].

Theorem 3.1.3. *We have*

$$\operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}} = -i\{[v^{(j)}]^*, v^{(j)}\}, \tag{3.18}$$

where tr *stands for the matrix trace.*

It is interesting that the RHS of formula (3.18) admits a geometric interpretation: it can be interpreted as the scalar curvature of a U(1) connection on $T^*M \setminus \{0\}$, see

Section 5 of [14] for details. This connection is to do with gauge transformations of the normalised eigenvector $v^{(j)}(x,p)$ of the principal symbol $L_{\mathrm{prin}}(x,p)$ corresponding to the eigenvalue $h^{(j)}(x,p)$. Namely, observe that if $v^{(j)}(x,p)$ is an eigenvector and $\phi^{(j)}(x,p)$ is an arbitrary smooth real-valued function, then $e^{i\phi^{(j)}(x,p)}v^{(j)}(x,p)$ is also an eigenvector. Careful analysis of the gauge transformation

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)} \tag{3.19}$$

leads to the appearance of a curvature term.

3.1.2 Mollified spectral asymptotics

In this subsection we will study the integral kernel of the propagator, which is defined by formula (3.9). Denote by

$$u(x, x^{n+1}, y) := \sum_k e^{-ix^{n+1}\lambda_k} v_k(x)[v_k(y)]^* \tag{3.20}$$

the integral kernel of the propagator (3.16). The quantity (3.20) can be understood as a matrix-valued distribution in the variable $x^{n+1} \in \mathbb{R}$ depending on the parameters $x, y \in M$. Further on in this subsection we will be studying the quantity

$$\hat{f}(x, x^{n+1}) := \operatorname{tr} u(x, x^{n+1}, x) = \sum_k \|v_k(x)\|^2 e^{-ix^{n+1}\lambda_k}\,. \tag{3.21}$$

In order to understand the reason for our interest in (3.21), put

$$f(x,\lambda) := \sum_k \|v_k(x)\|^2 \delta(\lambda - \lambda_k), \tag{3.22}$$

which is essentially the object presented in equation (3.10). Then (3.21) and (3.22) are related as $\hat{f} = \mathcal{F}_{\lambda \to x^{n+1}}[f\,]$ and $f = \mathcal{F}^{-1}_{x^{n+1}\to\lambda}[\hat{f}\,]$, where the one-dimensional Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are the same as the previous definitions. The quantity (3.22) contains all the information on the spectrum of our operator L and it also contains some information on the eigenfunctions.

Let $\hat{\rho} : \mathbb{R} \to \mathbb{C}$ be a smooth function such that $\hat{\rho}(0) = 1$, $\hat{\rho}'(0) = 0$ and the support of $\hat{\rho}$ is sufficiently small. Here 'sufficiently small' means that $\operatorname{supp}\hat{\rho} \subset (-\mathbf{T}, \mathbf{T})$, where $\mathbf{T}$ is the infimum of the lengths of all possible loops. See Section 6 in [14] for details. Denote also $\rho(\lambda) := \mathcal{F}^{-1}_{x^{n+1}\to\lambda}[\hat{\rho}(x^{n+1})]$.

We mollify the distributions (3.21) and (3.22) by switching to $\hat{\rho}(x^{n+1})\hat{f}(x, x^{n+1})$ and $(\rho * f)(x, \lambda)$, where the star indicates convolution in the variable λ. It was shown in [14] that Theorems 3.1.2 and 3.1.3 imply the following result.

Theorem 3.1.4. *We have, uniformly over $x \in M$,*

$$(\rho * f)(x, \lambda) = n\, a(x)\, \lambda^{n-1} + (n-1)\, b(x)\, \lambda^{n-2} + O(\lambda^{n-3}) \quad as \quad \lambda \to +\infty.$$

Here the densities $a(x)$ and $b(x)$ are given by formulae

$$a(x) = (2\pi)^{-n} \sum_{j=1}^{m/2} \int\limits_{h^{(j)}(x,p)<1} \mathrm{d}p\,, \tag{3.23}$$

$$\begin{aligned} b(x) = -n(2\pi)^{-n} \sum_{j=1}^{m/2} \int\limits_{h^{(j)}(x,p)<1} \Big(& [v^{(j)}]^* L_{\mathrm{sub}} v^{(j)} \\ & - \frac{i}{2}\{[v^{(j)}]^*, L_{\mathrm{prin}} - h^{(j)}, v^{(j)}\} + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big)(x,p)\, \mathrm{d}p\,, \end{aligned} \tag{3.24}$$

where $\mathrm{d}p = \mathrm{d}p_1 \ldots \mathrm{d}p_n$.

Theorem 3.1.4 warrants the following remarks.

Remark 3.1.5. It is easy to see that the RHS of formula (3.24) is invariant under gauge transformations (3.19) of the eigenvectors of the principal symbol.

Remark 3.1.6. Let $R : M \to \mathrm{U}(m)$ be an arbitrary smooth unitary matrix-function. As one would expect, the RHS of formula (3.24) is invariant under gauge transformations $L \mapsto R^* L R$ of our operator, but establishing this is not that easy. The corresponding calculations are presented in Section 9 of [14].

Let us now leave in (3.22) only terms with positive λ_k and define the quantity

$$f_+(x, \lambda) := \sum_{\lambda_k > 0} \|v_k(x)\|^2 \delta(\lambda - \lambda_k), \tag{3.25}$$

Note that formula (3.25) is the same as the (distributional) derivative with respect to λ of local positive counting function described in (3.2). Theorem 3.1.4 implies the following Corollary.

Corollary 3.1.1. *We have, uniformly over $x \in M$, the following two results:*

$$(\rho * f_+)(x, \lambda) = n\, a(x)\, \lambda^{n-1} + (n-1)\, b(x)\, \lambda^{n-2} + O(\lambda^{n-3}) \quad as \quad \lambda \to +\infty$$

and $(\rho * f_+)(x,\lambda)$ *vanishes faster than any negative power of* $|\lambda|$ *as* $\lambda \to -\infty$.

Recall the definitions of local counting functions in (3.2) and (3.4), which can also be written as

$$N_\pm(x,\lambda) := \begin{cases} 0 & \text{if} \quad \lambda \le 0, \\ \sum_{0<\pm\lambda_k<\lambda} \|v_k(x)\|^2 & \text{if} \quad \lambda > 0. \end{cases} \tag{3.26}$$

The function $N_+(x,\lambda)$ counts the eigenvalues λ_k between zero and λ, whereas the function $N_-(x,\lambda)$ counts the eigenvalues λ_k between $-\lambda$ and zero. In both cases counting eigenvalues involves assigning them weights $\|v_k(x)\|^2$.

We have $(\rho * N_+)(x,\lambda) = \int_{-\infty}^{\lambda} (\rho * f_+)(x,\mu)\,\mathrm{d}\mu$, so Corollary 3.1.1 implies

$$(\rho * N_+)(x,\lambda) \;=\; a(x)\,\lambda^n + b(x)\,\lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if} \quad n \ge 3, \\ O(\ln\lambda) & \text{if} \quad n = 2, \end{cases} \quad \text{as} \quad \lambda \to +\infty. \tag{3.27}$$

The asymptotics for $(\rho * N_-)(x,\lambda)$ is obtained by applying the above result to the operator $-L$ and using the symmetries (3.13). This gives

$$(\rho * N_-)(x,\lambda) \;=\; a(x)\,\lambda^n - b(x)\,\lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if} \quad n \ge 3, \\ O(\ln\lambda) & \text{if} \quad n = 2, \end{cases} \quad \text{as} \quad \lambda \to +\infty. \tag{3.28}$$

Note that the second terms in the asymptotic formulae (3.27) and (3.28) have opposite signs and that the remainders are uniform in $x \in M$.

Finally, recall the two global counting functions

$$N_\pm(\lambda) := \begin{cases} 0 & \text{if} \quad \lambda \le 0, \\ \sum_{0<\pm\lambda_k<\lambda} 1 & \text{if} \quad \lambda > 0. \end{cases} \tag{3.29}$$

We have $N_\pm(\lambda) = \int_M N_\pm(x,\lambda)\,\mathrm{d}x$, where $\mathrm{d}x = \mathrm{d}x^1 \ldots \mathrm{d}x^n$. Therefore, formulae (3.27) and (3.28) imply

$$(\rho * N_\pm)(\lambda) = a\,\lambda^n \pm b\,\lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if} \quad n \ge 3, \\ O(\ln\lambda) & \text{if} \quad n = 2, \end{cases} \quad \text{as} \quad \lambda \to +\infty, \tag{3.30}$$

where

$$a = \int_M a(x)\,\mathrm{d}x\,, \qquad b = \int_M b(x)\,\mathrm{d}x\,. \tag{3.31}$$

3.1.3 Unmollified spectral asymptotics

In this subsection we write down asymptotic formulae for the local and global counting functions without mollification. These can be obtained from the mollified asymptotic formulae (3.27), (3.28) and (3.30) by applying appropriate Fourier Tauberian theorems, see Section 8 of [14].

Theorem 3.1.7. *We have, uniformly over $x \in M$,*

$$N_\pm(x,\lambda) = a(x)\,\lambda^n + O(\lambda^{n-1}) \quad as \quad \lambda \to +\infty. \tag{3.32}$$

Corollary 3.1.2. *We have*

$$N_\pm(\lambda) = a\lambda^n + O(\lambda^{n-1}) \quad as \quad \lambda \to +\infty. \tag{3.33}$$

In particular, applying Corollary 3.1.2 and formula (3.23) to the square root of the Laplace-Beltrami operator $\sqrt{-\Delta}$, we have

Corollary 3.1.3.

$$N_+(-\Delta;\lambda) = \left[\frac{\mathrm{Vol}(B_n)}{(2\pi)^n} \times \mathrm{Vol}(\Omega)\right] \lambda^{\frac{n}{2}} + O(\lambda^{\frac{n-1}{2}}) \quad as \quad \lambda \to +\infty. \tag{3.34}$$

This is the sharp remainder term estimate of Weyl's conjecture as stated in formula (3.1).

Theorem 3.1.8. *If the point $x \in M$ is nonfocal then*

$$N_\pm(x,\lambda) = a(x)\,\lambda^n \pm b(x)\,\lambda^{n-1} + o(\lambda^{n-1}) \quad as \quad \lambda \to +\infty. \tag{3.35}$$

Here we call a point x nonfocal if the measure of momenta which serve as starting directions for loops originating from x and generated by each Hamiltonian $h^{(j)}$ is zero on $T^*_x M \setminus \{0\}$. See [14, Section 8.2] for precise definition.

Theorem 3.1.9. *If the nonperiodicity condition is fulfilled then*

$$N_\pm(\lambda) = a\lambda^n \pm b\lambda^{n-1} + o(\lambda^{n-1}) \quad as \quad \lambda \to +\infty. \tag{3.36}$$

Here we say that the nonperiodicity condition is fulfilled if the measure of starting points of periodic trajectories generated by each Hamiltonian $h^{(j)}$ is zero on $T^*M \setminus \{0\}$. See [14, Subsection 8.2] for details.

The results presented in this subsection were first obtained by Victor Ivrii [33, 34] but without an explicit formula for the second asymptotic coefficient.

Asymptotic formulae of the type (3.32)–(3.36) are called *Weyl-type formulae* and the coefficients in such formulae are often referred to as *Weyl coefficients*.

3.2 The eta function

Let L be an elliptic self-adjoint first order differential operator (as opposed to pseudo-differential defined in Chapter 1). In this section, we will give a picture of the relationship between the *Weyl coefficients* of the counting functions and the residues of the *eta function* associated with operator L.

As shown in Section 3.1, we have both positive and negative counting functions associated with operator L. Therefore, we should also have positive and negative Weyl coefficients correspondingly. To define the Weyl coefficients without imposing geometric condition on the Hamiltonian flows we take a convolution of our counting functions with a 'nice' function $\rho(\lambda)$. Namely, let $\hat{\rho} : \mathbb{R} \to \mathbb{C}$ be a smooth function with support in a sufficiently small neighbourhood, $\mathbf{O}_1$, of the point $t = 0$. Moreover, suppose that we have $\hat{\rho}(t) = 1$ in a smaller neighbourhood, $\mathbf{O}_2 \subset \mathbf{O}_1$, including the point $t = 0$. Note that we can choose $\mathbf{O}_1$ so small that the only singularity of the propagator contained in $\mathbf{O}_1$ is at the point $t = 0$. Let the functions $N_\pm(\lambda)$ be as defined in the previous section and let $\rho(\lambda)$ denote, as usual, the inverse Fourier transform of $\hat{\rho}(t)$. In line with Subsection 3.1.2 and Theorem 4.1.2 of [45], we define the Weyl coefficients of the positive and negative counting function as follows.

Definition 3.2.1 (Weyl coefficients). Let ρ be as described above. Then

$$(\rho * N'_\pm)(\lambda) = n(a_\pm)_0\,\lambda^{n-1} + (n-1)(a_\pm)_1\,\lambda^{n-2} + \cdots + (a_\pm)_{n-1} + (a_\pm)_n\,\lambda^{-1} + \cdots + (n-k)(a_\pm)_k\,\lambda^{n-k-1} + \cdots \quad \text{as} \quad \lambda \to +\infty\,. \tag{3.37}$$

We call the coefficients $(a_+)_k$ and $(a_-)_k$ the k-th positive and negative Weyl coefficients respectively.

The validity of formula (3.37) is guaranteed by Theorem 4.1.2 from [45]. For $k \neq n$, the reason for putting factors in front of $(a_\pm)_k$ is to have agreement with coefficients $(a_\pm)_k$ appearing in actual (unmollified positive and negative respectively) counting

functions, see Section 3.1.3 for more details. For $k = n$, this corresponds to the logarithmic contribution to the unmollified counting functions. To simplify further mathematical expressions in this section, we introduce the rescaled Weyl coefficients as follows:

$$\begin{cases} (b_\pm)_k = (n-k)(a_\pm)_k \quad \text{for} \quad k \neq n\,, \\ (b_\pm)_n = (a_\pm)_n\,. \end{cases} \tag{3.38}$$

The second main object examined in this section is the (global) eta function of our operator L, which is defined as follows.

Definition 3.2.2 (Eta function).

$$\eta(s) := \sum_{\lambda_k \neq 0} \frac{\operatorname{sgn} \lambda_k}{|\lambda_k|^s} = \int_0^{+\infty} \lambda^{-s} \left(N'_+(\lambda) - N'_-(\lambda)\right) \mathrm{d}\lambda\,, \tag{3.39}$$

where summation is carried out over all nonzero eigenvalues λ_k of L and $s \in \mathbb{C}$ is the independent variable.

According to formula (3.1.2), the series (3.39) converges absolutely for $\operatorname{Re} s > n$ and defines a holomorphic function in this half-plane. Moreover, it is known [4] that it extends meromorphically to the whole s-plane with simple poles which can only occur at real integer values of s. The eta function is the accepted way of describing the asymmetry of the spectrum. Before proving the main theorem in this section, we first prove the following lemma.

Lemma 3.2.1. *Let r be a natural number, s and c be real positive numbers, and suppose that $s > -r-1$. Let $\hat{\rho}$ be a smooth compactly supported cut-off function satisfying*

$$\begin{cases} \hat{\rho}(0) = 1\,, \\ \hat{\rho}^{(k)}(0) = 0 \qquad \textit{for} \qquad 1 \leq k \leq r\,. \end{cases} \tag{3.40}$$

Then

$$\int_c^{+\infty} \lambda^{-s} \left[\delta(\lambda - \nu) - \rho(\lambda - \nu)\right] \mathrm{d}\lambda = O(\nu^{-s-r-1}) \quad \textit{as} \quad \nu \to +\infty. \tag{3.41}$$

Proof. We have

$$\begin{aligned} \int_c^{+\infty} \lambda^{-s} \left[\delta(\lambda - \nu) - \rho(\lambda - \nu)\right] \mathrm{d}\lambda &= \int_{c-\nu}^{+\infty} (\nu + \mu)^{-s} \left[\delta(\mu) - \rho(\mu)\right] \mathrm{d}\mu \\ &= \nu^{-s} - \int_{c-\nu}^{+\infty} (\nu + \mu)^{-s}\, \rho(\mu)\, \mathrm{d}\mu. \end{aligned}$$

But Taylor's formula gives us

$$(\nu+\mu)^{-s} = \nu^{-s} - s\nu^{-s-1}\mu + \cdots + \frac{(-1)^r}{(r)!}[s\cdots(s+r-1)]\nu^{-s-r}\mu^r + \frac{(-1)^{r+1}}{(r+1)!}[s\cdots(s+r)](\nu+\xi)^{-s-r-1}\mu^{r+1}$$

for some ξ strictly between 0 and μ. Thus, in order to prove (3.41) it is sufficient to prove

$$\int_{-\infty}^{c-\nu} \rho(\mu)\,\mathrm{d}\mu = O(\nu^{-\infty}), \tag{3.42}$$

$$\int_{-\infty}^{c-\nu} \mu^k\,\rho(\mu)\,\mathrm{d}\mu = O(\nu^{-\infty}) \quad \text{for} \quad 1 \le k \le r, \tag{3.43}$$

$$\int_{c-\nu}^{+\infty} (\nu+\xi)^{-s-r-1}\mu^{r+1}\,\rho(\mu)\,\mathrm{d}\mu = O(\nu^{-s-r-1}). \tag{3.44}$$

Formulae (3.42) and (3.43) follow immediately from the fact that our function ρ is from Schwartz space and formula (3.40). In order to prove formula (3.44), it is sufficient to prove

$$\int_{c-\nu}^{-\nu/2} (\nu+\xi)^{-s-r-1}\mu^{r+1}\,\rho(\mu)\,\mathrm{d}\mu = O(\nu^{-\infty}), \tag{3.45}$$

$$\int_{-\nu/2}^{+\infty} (\nu+\xi)^{-s-r-1}\mu^{r+1}\,\rho(\mu)\,\mathrm{d}\mu = O(\nu^{-s-r-1}). \tag{3.46}$$

Now, in order to prove (3.45) and (3.46) it is sufficient to prove

$$\int_{c-\nu}^{-\nu/2} c^{-s-r-1}|\mu|^{r+1}\,|\rho(\mu)|\,\mathrm{d}\mu = O(\nu^{-\infty}), \tag{3.47}$$

$$\int_{-\nu/2}^{+\infty} \Big(\frac{\nu}{2}\Big)^{-s-r-1}|\mu|^{r+1}\,|\rho(\mu)|\,\mathrm{d}\mu = O(\nu^{-s-r-1}). \tag{3.48}$$

Again, formula (3.47) follows immediately from the fact that our ρ is from Schwartz space. For formula (3.48), we see that it follows from the inequality

$$\int_{-\nu/2}^{+\infty} |\mu|^{r+1}\,|\rho(\mu)|\,\mathrm{d}\mu \le \int_{-\infty}^{+\infty} |\mu|^{r+1}\,|\rho(\mu)|\,\mathrm{d}\mu.$$

□

Remark 3.2.1. It is easy to see that the remainder term in formula (3.41) is uniform in s when s takes values from a bounded interval.

We are now ready to prove the following theorem.

Theorem 3.2.2 (Weyl coefficients and residues of the eta function)**.** *Let the Weyl coefficients* $(a_\pm)$ *be as defined in Definition 3.2.1. Then the residues of the eta function associated with operator* L *are related to the Weyl coefficients as follows:*

$$\operatorname{Res}(\eta\,, s = n-k) = (b_+)_k - (b_-)_k\,. \tag{3.49}$$

Proof. First of all, we will find a meromorphic function, denoted by $\tilde{\eta}_+(s)$, in $\mathbb{C}$ that differs from the positive part of the eta function by a function, denoted by $F_+(s)$ with the following property: we want $F_+(s)$ to be bounded by a holomorphic function on the real line. Then finding the poles and corresponding residues on the real line of the positive eta function reduces to finding those of the meromorphic function $\tilde{\eta}_+(s)$. Thereafter, we will apply the same method to the negative eta function.

Let $(\lambda_+)_k$ be the positive eigenvalues enumerated in increasing order with account of multiplicity. Choose a number $c > 0$ such that all these eigenvalues are greater than c. Then, we define the following two objects. The first one is the positive part of the eta function, which is defined as

$$\eta_+(s) := \sum_{k=1}^{+\infty} (\lambda_+)_k^{-s}\,. \tag{3.50}$$

The second one is the (positive) meromorphic function as given below:

$$\tilde{\eta}_+(s) := \sum_{k=1}^{+\infty} \int_c^{+\infty} \lambda^{-s} \rho\left(\lambda - (\lambda_+)_k\right) \mathrm{d}\lambda, \tag{3.51}$$

where $\rho(\lambda)$ is the same as defined at the beginning of this section.

Now, we consider the difference of the two quantities defined above, namely

$$F_+(s) := \eta_+(s) - \tilde{\eta}_+(s)\,. \tag{3.52}$$

Further on we fix a positive number r as in Lemma 3.2.1 and assume that $s \in [n-r-1+\delta, n+r+1] \subset (-r-1, +\infty)$ for some small positive number δ. In view of Lemma 3.2.1, we have

$$|F_+(s)| \leq c_1 \sum_{k=1}^{+\infty} (\lambda_+)_k^{-s-r-1}$$

for some constant c_1. From Corollary 3.1.2, we know that

$$N_+(\lambda) = a\lambda^n + O(\lambda^{n-1}) \quad \text{as} \quad \lambda \to +\infty,$$

which implies

$$(\lambda_+)_k = \left(\frac{k}{a}\right)^{1/n} + O(1) \quad \text{as} \quad k \to +\infty.$$

Thus, we have $(\lambda_+)_k \geq c_2 k^{1/n}$, so

$$|F_+(s)| \leq c_1 c_2 \sum_{k=1}^{+\infty} k^{-(s+r+1)/n}. \tag{3.53}$$

The series (3.53) converges if $\frac{s+r+1}{n} > 1$, that is, if $s > n - r - 1$. In other words, $F(s)$ is holomorphic in s for $\Re(s) > n - r - 1$.

Therefore, we have established the fact that the positive eta function $\eta_+(s)$ and the function $\tilde{\eta}_+(s)$ have the same real poles and residues on the half-line $s > n - r - 1$. Moreover, note that

$$\begin{aligned}\int_c^{+\infty} \lambda^{-s} \,\mathrm{d}(\rho * N_+)(\lambda) \\ = \int_c^{+\infty} \lambda^{-s} \,(\rho * N'_+)(\lambda)\,\mathrm{d}\lambda = \sum_{k=1}^{+\infty} \int_c^{+\infty} \lambda^{-s} \rho\,(\lambda - (\lambda_+)_k)\,\mathrm{d}\lambda = \tilde{\eta}_+(s)\,.\end{aligned} \tag{3.54}$$

Examining formulae (3.50), (3.51) and (3.54) we see that if we are looking at the function (3.50) and are interested in finding its real poles and residues on the interval $s \in (n - r - 1, +\infty)$, then we can instead work with the function

$$\int_c^{+\infty} \lambda^{-s} \,(\rho * N'_+)(\lambda)\,\mathrm{d}\lambda\,. \tag{3.55}$$

Similarly, we can define the negative part of the eta function

$$\eta_-(s) := \sum_{k=1}^{+\infty} (\lambda_-)_k^{-s}\,, \tag{3.56}$$

where $(\lambda_-)_k$ is the absolute value of the negative eigenvalues enumerated in increasing order with account of multiplicity.
Also, we have the negative meromorphic function as given below:

$$\tilde{\eta}_-(s) := \sum_{k=1}^{+\infty} \int_c^{+\infty} \lambda^{-s} \rho\,(\lambda - (\lambda_-)_k)\,\mathrm{d}\lambda\,. \tag{3.57}$$

According to formula (3.4) and the way we define the quantities $(\lambda_-)_k$, we have an analogue of formula (3.54), which is given as follows:

$$\int_c^{+\infty} \lambda^{-s} \, (\rho * N'_-)(\lambda) \, \mathrm{d}\lambda = \tilde{\eta}_-(s) \, . \tag{3.58}$$

Moreover, we also have

$$F_-(s) := \eta_-(s) - \tilde{\eta}_-(s) \, , \tag{3.59}$$

which can be bounded analogously as in equation (3.53).

Finally, the total eta function, as defined in formula (3.39), is given by

$$\eta(s) = \eta_+(s) - \eta_-(s) \, . \tag{3.60}$$

Combining equations (3.52), (3.54), (3.58), (3.59) and (3.60), we have

$$\eta(s) = \int_c^{+\infty} \lambda^{-s} \, \Big(\rho * (N'_+ - N'_-) \Big) \, (\lambda) \, \mathrm{d}\lambda + F(s) \, , \tag{3.61}$$

where c is less than the absolute values of all non-zero eigenvalues and the correction term $F(s) = F_+(s) - F_-(s)$ is holomorphic for $\Re(s) > n - r - 1$ (see equation (3.53) for details). Now, formulae (3.37) and (3.38) yield

$$(\rho * N'_\pm)(\lambda) = (b_\pm)_0 \, \lambda^{n-1} + \cdots + (b_\pm)_r \, \lambda^{n-r-1} + r(\lambda) \quad \text{as} \quad \lambda \to +\infty \, ,$$

where $r(\lambda) = O(\lambda^{n-r-2})$. Using equation (3.61), we derive that, for $\Re(s) > n$,

$$\begin{aligned} \eta(s) - F(s) &= \int_c^{+\infty} \lambda^{-s} \, \Big(\rho * (N'_+ - N'_-) \Big) \, (\lambda) \, \mathrm{d}\lambda \\ &= \sum_{k=0}^{r} \int_c^{+\infty} \left((b_+)_k - (b_-)_k \right) \lambda^{n-s-k-1} \, \mathrm{d}\lambda + R(s) \\ &= \sum_{k=0}^{r} \frac{c^{n-s-k}}{s-(n-k)} \left((b_+)_k - (b_-)_k \right) + R(s) \, , \end{aligned} \tag{3.62}$$

where $R(s) = \int_c^{+\infty} \lambda^{-s} \, r(\lambda) \, \mathrm{d}\lambda$ can be extended to a holomorphic function in s for $\Re(s) > n - r - 1$. In fact, from equations (3.62) we obtain a meromorphic extension of the eta function $\eta(s)$ to the half-plane $\Re(s) > n - r - 1$. More importantly, its residues are given by

$$\mathrm{Res}(\eta \, , s = n - k) = (b_+)_k - (b_-)_k \qquad \text{for} \qquad k \leq r \, .$$

Finally, with the help of the uniqueness of analytic extensions, we see that the extension is independent of constants c and r, which ends the proof. □

Theorem 3.2.2 warrants the following remarks.

Remark 3.2.3.

- Theorem 3.2.2 can be easily extended to the case of pseudo-differential operator L.
- Let $u(x,t,y)$ be the integral kernel of the propagator defined in accordance with formula (3.20). It is easy to see that the residues of the eta function are determined by the singularities of $\operatorname{tr} u(x,t,x)$ at $t=0$. The singularities of $\operatorname{tr} u(x,t,x)$ at $t\neq 0$ do not contribute to the residues of the eta function. This fact follows from the analysis performed in Subsection 3.1.2 and the definition of our cut-off function $\hat{\rho}(t)$.
- A full understanding of the mollified positive counting function (consequently the negative one) gives a complete picture of the residues of the eta function. Moreover, formula (3.49) explains how the residues of the eta function reflect the spectral asymmetry of the operator L.

Now, by setting $k=0,1$ and following equations (3.27), (3.28) and Theorem 3.2.2, we immediately arrive at the following corollary.

Corollary 3.2.1. *The first (from the right) non-trivial residue of the eta function is given by*

$$\operatorname{Res}(\eta\,, s=n-1)=2(n-1)b\,, \tag{3.63}$$

where b is the coefficient from (3.30).

Thus, for a generic differential operator L the first pole of the eta function is at $s=n-1$ and formulae (3.24) (3.31) and (3.63) give us an explicit expression for the residue. It is known [4, 26] that the eta function does not have a pole at $s=0$. The real number $\eta(0)$ is called the *eta invariant*. It can be interpreted as the number of positive eigenvalues minus the number of negative eigenvalues. This interpretation is based on the observation that if we were dealing with an Hermitian matrix L, then $\eta(0)$ would indeed be the number of positive eigenvalues minus the number of negative eigenvalues. Our differential operator L has infinitely many positive eigenvalues and infinitely many negative eigenvalues, and the concept of the eta function allows us to

regularise the expression 'number of positive eigenvalues minus the number of negative eigenvalues'.

The eta function may have poles at

$$s = n - 1, \dots, 1, -1, -2, \dots. \tag{3.64}$$

However, a more careful analysis [10, 40] shows that poles may occur only at values of s of the form

$$s = n - 1 - 2l, \qquad l = 0, 1, \dots. \tag{3.65}$$

The authors of [40] call values of s from the intersection of the sets (3.64) and (3.65) *admissible*. It was shown in [40] that residues of the eta function at positive admissible integers are generically nonzero. This agrees with our explicit calculation of the residue at $s = n - 1$.

3.3 Systems of two equations

From now on we assume that

$$m = 2 \tag{3.66}$$

and that

$$\operatorname{tr} L_{\text{prin}}(x, p) = 0. \tag{3.67}$$

In other words, we now restrict our analysis to 2×2 operators with trace-free principal symbols. The logic behind the assumptions (3.66) and (3.67) is that they single out the simplest class of first order systems and we expect to extract more geometry out of our asymptotic analysis and simplify the results.

It is easy to see that formulae (1.20), (3.66) and (3.67) imply that the dimension n of our manifold M is less than or equal to three. Further on we assume that

$$n = 3. \tag{3.68}$$

Remark 3.3.1. It was shown in [15] that a 3-manifold is parallelizable if and only if there exists a self-adjoint elliptic first order linear differential operator with trace-free principal symbol acting on 2-columns of complex-valued half-densities over this manifold. This means that once we restricted our analysis to the special case (3.66)–(3.68) we are working on a parallelizable manifold.

Remark 3.3.2. It is well known that a 3-manifold is orientable if and only if it is parallelizable, see Theorem 1 in Chapter VII of [36].

Use the same observation as in equation (2.29), we see that under the assumption (3.66) the determinant of the principal symbol is a quadratic form in the dual variable (momentum) p:

$$\det L_{\mathrm{prin}}(x,p) = -g^{\alpha\beta}(x)\, p_\alpha p_\beta\,. \tag{3.69}$$

Furthermore, the ellipticity condition (1.20) and (3.67) imply that the quadratic form $g^{\alpha\beta}(x)\, p_\alpha p_\beta$ is positive definite. We interpret the real coefficients $g^{\alpha\beta}(x) = g^{\beta\alpha}(x)$, $\alpha,\beta = 1,2,3$, as components of a (contravariant) metric tensor. Thus, 2×2 operators with trace-free principal symbols are special in that the concept of a Riemannian metric is encoded within such operators. This opens the way to the geometric interpretation of our analytic results.

Note also that under the assumptions (3.66) and (3.67) the principal symbol of the operator L^2 is automatically proportional to the 2×2 identity matrix I:

$$(L^2)_{\mathrm{prin}}(x,p) = (L_{\mathrm{prin}})^2(x,p) = \Big(g^{\alpha\beta}(x)\, p_\alpha p_\beta\Big)\, I\,. \tag{3.70}$$

Operators possessing the property (3.70) are called *Dirac-type operators*.

Now take an arbitrary smooth matrix-function

$$R : M \to \mathrm{SU}(2) \tag{3.71}$$

and consider the transformation of our 2×2 differential operator

$$L \mapsto R^* L R\,. \tag{3.72}$$

We interpret (3.72) as a gauge transformation because it does not affect our counting functions (3.26), (3.29) and the eta function (3.39). Note also that the transformation (3.72) preserves the condition (3.67).

The transformation (3.72) of the differential operator L induces the following transformations of its principal and subprincipal symbols:

$$L_{\mathrm{prin}} \mapsto R^* L_{\mathrm{prin}} R\,, \tag{3.73}$$

$$L_{\mathrm{sub}} \mapsto R^* L_{\mathrm{sub}} R + \frac{i}{2}\,(R^*_{x^\alpha}(L_{\mathrm{prin}})_{p_\alpha} R - R^*(L_{\mathrm{prin}})_{p_\alpha} R_{x^\alpha})\,. \tag{3.74}$$

Comparing formulae (3.73) and (3.74) we see that, unlike the principal symbol, the subprincipal symbol does not transform in a covariant fashion due to the appearance of terms with the gradient of the matrix-function $R(x)$.

Following the gauge theoretical analysis of Chapter 2 and using the concept of the covariant subprincipal symbol introduced in Section 2.5, we see that we can overcome the non-covariance in (3.74) by introducing the *covariant subprincipal symbol in the 3-dimensional setting*, which we shall also denote by $L_{\mathrm{csub}}(x)$. That is,

$$L_{\mathrm{csub}} := L_{\mathrm{sub}} - \frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta}, \tag{3.75}$$

where subscripts p_α and p_β indicate partial derivatives and curly brackets denote the generalised Poisson bracket on matrix-functions (1.22).

Lemma 3.3.1. *The transformation* (3.72) *of the differential operator induces the transformation* $L_{\mathrm{csub}} \mapsto R^* L_{\mathrm{csub}} R$ *of its covariant subprincipal symbol.*

Proof. Consider $\tilde{M} = M \times \mathbb{R}$ and a non-degenerate operator $\tilde{L} := L - i\partial_4$ acting on a column of $m = 2$ half-densities over $\tilde{M}$. In this case, equations (2.48) and (3.67) yield the desired formula (3.75). □

In our 3-dimensional Riemannian setting the correction term in the RHS of (3.75) turns out to be proportional to the 2×2 identity matrix I:

$$\frac{i}{16}\, g_{\alpha\beta}\{L_{\mathrm{prin}}, L_{\mathrm{prin}}, L_{\mathrm{prin}}\}_{p_\alpha p_\beta} = \frac{3}{4}(\mathbf{c} * T^{\mathrm{ax}})I\,, \tag{3.76}$$

where $\mathbf{c}$ and T^{ax} are the 3-dimensional analogues of those from equation (2.79) and Definition 2.5.2 respectively. In fact, $\mathbf{c}$ is the same as the quantity that will be defined in formula (3.81).

Theorem 3.3.3. *In the special case* (3.66)–(3.68) *formulae* (3.23) *and* (3.24) *read*

$$a(x) = \frac{1}{6\pi^2}\sqrt{\det g_{\alpha\beta}(x)}\,, \tag{3.77}$$

$$b(x) = -\frac{1}{4\pi^2}\Big((\operatorname{tr} L_{\mathrm{csub}})\sqrt{\det g_{\alpha\beta}}\Big)(x)\,. \tag{3.78}$$

Theorem (3.3.3) was established in [15], though the density $b(x)$ was written in [15] in a slightly different way. The use of the concept of covariant subprincipal symbol introduced in [22] allows us to replace formula (1.19) from [15] by the more compact expression (3.78).

Formula (3.77) tells us that the first local Weyl coefficient is proportional to the standard Riemannian density. The first global Weyl coefficient is obtained from the local one by integration, see formula (3.31), and is proportional to the Riemannian volume V of our manifold M: $a = \frac{1}{6\pi^2} V$.

In order to understand the geometric meaning of formula (3.78) we observe that the covariant subprincipal symbol can be uniquely represented in the form

$$L_{\mathrm{csub}}(x) = L_{\mathrm{prin}}(x, A(x)) + I A_4(x), \tag{3.79}$$

where $A = (A_1, A_2, A_3)$ is some real-valued covector field, A_4 is some real-valued scalar field, $x = (x^1, x^2, x^3)$ are local coordinates on M (we are working in the nonrelativistic setting) and I is the 2×2 identity matrix. Applying the results of [22] to the relativistic operator appearing in the LHS of (3.15) we conclude that $A = (A_1, A_2, A_3)$ is the magnetic covector potential and A_4 is the electric potential. Note that Lemma 3.3.1 and formulae (3.73) and (3.79) tell us that the magnetic covector potential and electric potential are invariant under gauge transformations (3.72).

Substituting (3.79) into (3.78) and making use of (3.67) we get

$$b(x) = -\frac{1}{2\pi^2} \left(A_4 \sqrt{\det g_{\alpha\beta}} \right)(x) . \tag{3.80}$$

Thus, the second Weyl coefficient is proportional to the electric potential and does not depend on the magnetic covector potential.

A number of researchers have studied the effect of the electromagnetic field on the spectrum of the first order differential operator L under the assumptions (3.66)–(3.68) and our formula (3.80) is a further contribution to this line of research. However, we believe that such results do not have a physical meaning because our 2×2 first order differential operator L describes a massless particle and no known massless particle has an electric charge. In the absence of an electric charge the particle cannot interact with the electromagnetic field.

The electron is an example of a charged massive particle but it is described by a 4×4 first order differential operator. Also, in the case of the electron it is more natural to do asymptotic analysis in a different setting, with Planck's constant tending to zero. Spectral problems for the electron in 3-dimensional Euclidean space in the presence of magnetic and electric potentials were extensively studied by Ivrii [33, 34]. An analytic (i.e. based on the concepts of principal symbol and covariant subprincipal symbol) representation of the massive Dirac equation in curved 4-dimensional Lorentzian spacetime was given in [22].

3.4 Spin structure from the operator theoretic perspective

Let M be a connected closed oriented Riemannian 3-manifold. Let us consider all possible self-adjoint elliptic first order 2×2 linear differential operators L with trace-free principal symbols corresponding, in the sense of formula (3.69), to the prescribed metric. See also Remarks 3.3.1 and 3.3.2. In this section our aim is to classify all such operators L.

We define the topological charge as

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \mathrm{tr}\Big((L_{\mathrm{prin}})_{p_1}(L_{\mathrm{prin}})_{p_2}(L_{\mathrm{prin}})_{p_3}\Big), \tag{3.81}$$

with the subscripts p_1, p_2, p_3 indicating partial derivatives with respect to the components of momentum $p = (p_1, p_2, p_3)$. It was shown in Section 3 of [15] that the number $\mathbf{c}$ defined by formula (3.81) is equivalent to the sign of the determinant of ${e_j}^{\alpha}$. That is

$$\mathbf{c} = \mathrm{sgn}\det {e_j}^{\alpha}\,, \tag{3.82}$$

which can take only two values, $+1$ or -1, and describes the orientation of the principal symbol relative to the chosen orientation of local coordinates $x = (x^1, x^2, x^3)$. Of course, the transformation $L \mapsto -L$ inverts the topological charge.

Further on we work with operators whose topological charge is $+1$.

We say that the operators L and $\tilde{L}$ are equivalent if there exists a smooth matrix-function (3.71) such that $\tilde{L}_{\mathrm{prin}} = R^* L_{\mathrm{prin}} R$. The equivalence classes of operators obtained in this way are called *spin structures*.

An example of non-equivalent operators L and $\tilde{L}$ on the 3-torus was given in Appendix A of [15]. Furthermore, using the above definition of spin structure one can show that there are eight distinct spin structures on the 3-torus whereas the spin structure on the 3-sphere is unique.

We see that an operator L is uniquely determined, modulo a gauge transformation (3.72), by the metric, topological charge, spin structure, magnetic covector potential and electric potential.

We claim that in dimension three our analytic definition of spin structure is equivalent to the traditional topological definition. We will provide a rigorous proof of this claim in a separate paper.

3.5 The massless Dirac operator

In this section we continue dealing with the special case (3.66)–(3.68) but make the additional assumption that the magnetic covector potential and electric potential vanish. The resulting operator L is called the *massless Dirac operator on half-densities*. It is uniquely determined, modulo a gauge transformation (3.72), by the metric, topological charge and spin structure.

In practice most researchers work with the massless Dirac operator which acts on 2-columns of complex-valued scalar fields (components of a Weyl spinor) rather than on 2-columns of complex-valued half-densities. As we have a Riemannian metric encoded in the principal symbol of our operator, scalar fields can be identified with half-densities: it is just a matter of multiplying or dividing by $(\det g_{\alpha\beta})^{1/4}$. Hence, the 'traditional' massless Dirac operator and the massless Dirac operator on half-densities are related by a simple formula, see formula (A.19) in [15], and their spectra are the same. For spectral theoretic purposes it is more convenient to work with half-densities because in this case the inner product does not depend on the metric.

The massless Dirac operator describes the massless neutrino. We are looking at a single neutrino living in a closed 3-dimensional Riemannian universe. The eigenvalues are the energy levels of the particle. The tradition is to associate positive eigenvalues with the energy levels of the neutrino and negative eigenvalues with the energy levels of the antineutrino.

Formula (3.80) tells us that the second Weyl coefficient for the massless Dirac operator is zero, both locally and globally. Formula (3.63), in turn, tells us that the eta function of the massless Dirac operator does not have a pole at $s = 2$.

The natural question is where is the first pole of the eta function? It was shown in [9] that the eta function of the massless Dirac operator is holomorphic in the half-plane $\operatorname{Re} s > -2$. This agrees with formulae (3.64) and (3.65).

Furthermore, Branson and Gilkey [10] have shown that generically the eta function of the massless Dirac operator has a pole at $s = -2$ and calculated the residue. Consider the covariant rank three tensor $(\nabla_\alpha Ric_{\beta\nu})Ric_\gamma{}^\nu$, where ∇ stands for the covariant derivative and Ric for Ricci curvature (both are understood in terms of the Levi-Civita connection), and antisymmetrize it. This gives a totally antisymmetric covariant rank three tensor which is equivalent to a 3-form. According to [10], the integral of this 3-form over the 3-manifold M gives, up to a particular nonzero constant factor, the residue of the eta function of the massless Dirac operator at $s = -2$.

The fact that the first pole of the eta function of the massless Dirac operator is at $s = -2$ indicates that with a very high accuracy the large (in terms of modulus)

positive and negative eigenvalues are distributed in the same way. This, in turn, means that the massless Dirac operator is special and has hidden symmetries encoded in it.

We end this section by highlighting one particular symmetry of the massless Dirac operator. Consider the following antilinear operator acting on 2-columns of complex-valued half-densities:

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} =: \mathrm{C}(v). \tag{3.83}$$

The operator C defined by formula (3.83) is called the *charge conjugation operator*. It is known, see Appendix A in [15], that the linear massless Dirac operator on half-densities L and the antilinear charge conjugation operator C commute:

$$\mathrm{C}(Lv) = L\mathrm{C}(v). \tag{3.84}$$

Formula (3.84) implies, in particular, that all eigenvalues of the massless Dirac operator have even multiplicity.

The addition of an electric potential preserves the symmetry (3.84), but the addition of a magnetic covector potential destroys it. This follows easily from formulae (3.79) and (3.83).

3.6 Small eigenvalues

Up till now we dealt with large, in terms of modulus, eigenvalues. In this section we will deal with small eigenvalues of the massless Dirac operator.

Suppose that we are working on a connected closed oriented Riemannian 3-manifold and let $\lambda^{(0)}$ be a double eigenvalue of the massless Dirac operator. As explained in the end of the previous section, multiplicity two is the lowest possible. We now perturb the metric, i.e. consider an arbitrary metric $g_{\alpha\beta}(x;\epsilon)$ the components of which are smooth functions of local coordinates x^α, $\alpha = 1, 2, 3$, and small real parameter ϵ; here we assume that for $\epsilon = 0$ we get the original metric. In this case one can expand the eigenvalue into an asymptotic series in powers of the small parameter ϵ: $\lambda(\epsilon) = \lambda^{(0)} + \lambda^{(1)}\epsilon + \lambda^{(2)}\epsilon^2 + \dots$ with some constants $\lambda^{(1)}, \lambda^{(2)}, \dots$. This asymptotic construction was described in Sections 3–5 of [17]. The construction is somewhat nontrivial because we are dealing with a double eigenvalue that cannot split.

We now consider two special cases. In both cases the unperturbed spectrum is symmetric but symmetry is broken under generic perturbations of the metric.

3.6.1 The 3-torus with standard spin structure

Here the unperturbed metric is assumed to be Euclidean and standard spin structure means that our unperturbed massless Dirac operator can be written as an operator with constant coefficients in the natural 2π-periodic cyclic coordinates parameterizing the 3-torus, see formula (1.1) in [17].

The spectrum of the unperturbed operator is known, see, for example, Appendix B in [15] or Section 1 in [17]. The smallest eigenvalue is the double eigenvalue $\lambda^{(0)} = 0$. It was shown in [17] that

$$\lambda(\epsilon) = \lambda^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0 \tag{3.85}$$

with an explicit expression for the constant $\lambda^{(2)}$. Examination of this explicit expression shows that under a generic perturbation of the metric we get $\lambda^{(2)} \neq 0$ which is an indication of spectral asymmetry.

Furthermore, two special families of metrics were identified in [17] for which the eigenvalue closest to zero, $\lambda(\epsilon)$, can be evaluated explicitly. Formula (3.85) was tested against explicit results for these two families of metrics.

3.6.2 The 3-sphere

Here the unperturbed metric is obtained by restricting the Euclidean metric from $\mathbb{R}^4$ to $\mathbb{S}^3$. There is no issue with spin structure because for the 3-sphere the spin structure is unique.

The spectrum of the unperturbed operator is known, see, for example, Appendix B in [15]. The smallest, in terms of modulus, eigenvalues are the double eigenvalues $\lambda_+^{(0)} = +\frac{3}{2}$ and $\lambda_-^{(0)} = -\frac{3}{2}$. We get

$$\lambda_\pm(\epsilon) = \pm\frac{3}{2} + \lambda_\pm^{(1)}\epsilon + \lambda_\pm^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0. \tag{3.86}$$

In order to write down the coefficients $\lambda_\pm^{(1)}$ we consider the Riemannian volume $V(\epsilon)$ of our manifold M and expand it in powers of ϵ:

$$V(\epsilon) = V^{(0)} + V^{(1)}\epsilon + O(\epsilon^2) \qquad \text{as} \qquad \epsilon \to 0, \tag{3.87}$$

where $V^{(0)} = 2\pi^2$ is the volume of the unperturbed 3-sphere. It turns out that

$$\lambda_\pm^{(1)} = \mp\frac{1}{4\pi^2}V^{(1)}. \tag{3.88}$$

Formulae (3.86)–(3.88) tell us that in the first approximation in ϵ spectral symmetry is preserved and the increments of the two eigenvalues closest to zero are determined by the increment of volume. If the volume increases then the moduli of the two eigenvalues closest to zero decrease and in the first approximation in ϵ they decrease in the same way.

Arguing along the lines of [17] one can write down explicit expressions for the constants $\lambda_{\pm}^{(2)}$. Examination of these explicit expressions shows that under a generic perturbation of the metric we get spectral asymmetry in the ϵ^2 terms: $\lambda_{+}^{(2)} + \lambda_{-}^{(2)} \neq 0$. A detailed exposition will be provided in Chapter 5. Note that there is a family of metrics for which the two eigenvalues closest to zero, $\lambda_{+}(\epsilon)$ and $\lambda_{-}(\epsilon)$, can be evaluated explicitly. These are the so-called generalized Berger metrics: see Proposition 3.1 in [28] or Definition 4 in [27].

3.7 Issue with eigenvalues of the principal symbol

Throughout this chapter we assumed that the eigenvalues of the matrix-function $L_{\mathrm{prin}}(x,p)$, the principal symbol of our operator L, are simple for all $(x,p) \in T^*M \setminus \{0\}$. In this section we briefly examine the issues that arise if one drops this assumption.

Ivrii showed that Theorem 3.1.7 holds without any assumptions on the eigenvalues of the principal symbol, see Theorem 0.1 in [32] or Theorem 0.1 in [33]. However, establishing analogues of Theorems 3.1.8 and 3.1.9 without the assumption that the eigenvalues of the principal symbol are simple is not, by any means, straightforward and there are two issues that have to be addressed. These are highlighted in the following two subsections.

3.7.1 Geometric conditions for the existence of two-term spectral asymptotics

If the multiplicity of eigenvalues of the principal symbol varies as a function of $(x,p) \in T^*M \setminus \{0\}$ then the expectation is that one needs to consider 'generalised' Hamiltonian trajectories, with branching occurring at points in the cotangent bundle where multiplicities of eigenvalues of the principal symbol change.

Ivrii [32, 33] dealt with the issue of variable multiplicities of eigenvalues of the principal symbol by assuming that the set of Hamiltonian trajectories encountering points where multiplicities of eigenvalues of the principal symbol change is, in some sense, small.

G.V. Rozenblyum [43] and later I. Kamotski and M. Ruzhansky [35] considered 'generalised' Hamiltonian trajectories with branching assuming that the principal symbol of the operator is well behaved at points where multiplicities of eigenvalues of the principal symbol change. Here good behaviour is understood as smooth microlocal diagonalisability of the principal symbol plus some conditions on the Poisson brackets of eigenvalues.

3.7.2 Explicit formulae for the second Weyl coefficient

In the case when the eigenvalues of the principal symbol are not simple explicit formulae for the second Weyl coefficient are not known.

A good starting point for the derivation of such formulae would be the analysis of the case when eigenvalues of the principal symbol have constant multiplicities for all $(x,p) \in T^*M \setminus \{0\}$. Let $L_{\text{prin}}(x,p)$, be our $m \times m$ principal symbol and let l_j, $j = 1, \dots, k$, be the multiplicities of its positive eigenvalues, so that $l_1 + \dots + l_k = m/2$. Then one can, by analogy with Section 5 of [14], introduce a $\mathrm{U}(l_j)$ connection associated with the jth positive eigenspace of the principal symbol. It is natural to conjecture that the curvature of this $\mathrm{U}(l_j)$ connection will appear in the explicit formula for the second Weyl coefficient.

Chapter 4

The massless and massive Dirac operators

In this chapter we will study both the massless Dirac operator and the corresponding massive one. In the first half of the chapter we will give two ways of defining the massless Dirac operator, or abbreviated as the Dirac operator. One approach is widely known within the geometry community [24, 38] and the other one has been introduced and promoted by Vassiliev [41, 51, 12, 16]. Both have their own advantages and the main difference between the two constructions reduces to the definition of connections (covariant derivatives) on the spin bundle. The geometric approach requires more abstract tools such as connections on a principal bundle of manifolds and its lifting to a double covering principal bundle, but this works without any requirement on the representations of the Clifford algebra and it works for all dimensions and any Clifford module bundle on a manifold. Although the geometric approach is relatively abstract, it gives an insight into the subsequent definition of the spin connections by Vassiliev. Despite the fact that the latter approach only works for a parallelizable 3- or 4-dimensional manifold, it gives an explicit local expression for the covariant derivatives on spinor fields. This gives substantial advantages when we study the spectral problem for the Dirac operator, which includes calculating spectra and the corresponding perturbation theory. More importantly, the latter approach consequently leads to a new definition of spin structure on the tangent bundle of the manifold. The new definition of spin structures is a joint work with Zhirayr Avetisyan, Nikolai Saveliev and Dmitri Vassiliev, and detailed proofs will be published in a separate paper.

4.1 Motivation and introduction

The first study of the massless Dirac operators dates back to the study of square roots of the Laplace operator by P. A. M. Dirac. Although the manifold considered by Dirac was the Minkowski spacetime (the most well-known 4-dimensional Lorentzian manifold), we will instead focus on the Riemannian setting. As a toy model, let us first work on $\mathbb{R}^n$ (Euclidean space). Now, our Laplacian becomes a vector Laplacian and it has the following form in local coordinates $\{x^\alpha\}$,

$$\Delta = \sum_{\alpha=1}^{n} I_n \frac{\partial^2}{(\partial x^\alpha)^2} = \delta^{\alpha\beta} I_n \frac{\partial^2}{\partial x^\alpha \partial x^\beta} \,.$$

The simplest form of a square root of the Laplace operator is a constant first order linear differential operator, which can be written locally as

$$W = \gamma^\alpha \frac{\partial}{\partial x^\alpha} \,, \tag{4.1}$$

where $\{\gamma^\alpha\}_{\alpha=1}^n$ are matrices. If $W^2 = \Delta$, we would have

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\delta^{\alpha\beta} I_n \,, \tag{4.2}$$

which implies

$$(\gamma^\alpha)^2 = I_n \,. \tag{4.3}$$

Equations (4.2) and (4.3) indicate that the γ^α's satisfy the Clifford algebra conditions in $\mathbb{R}^n$, which brings us to a more abstract setting of the Clifford algebra bundle on manifolds.

4.1.1 Clifford algebras

Let Q be a quadratic form on vector space V over the field $\mathbb{K}$.

Definition 4.1.1. A pair $\Big(Cl(V,Q), C\Big)$ is called a *Clifford algebra* for (V,Q) if

1. $Cl(V,Q) := T(V)/I(Q)$ is an associative $\mathbb{K}$-algebra with unity $1_{Cl(V,Q)}$, where $T(V) = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \cdots$ is the tensor algebra of V and $I(Q)$ is the two-sided ideal generated by $\{v \otimes v - Q(v) \cdot 1_{T(V)} : v \in V\}$.

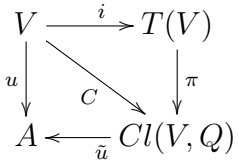
2. $C : V \to Cl(V, Q)$ is a linear map, called Clifford map satisfying

$$C(v)^2 = -Q(v) \cdot 1_{Cl(V,Q)}$$

for all $v \in V$.

3. if A is another $\mathbb{K}$-algebra with unity 1_A and $u : V \to A$ is a linear map satisfying $u(v)^2 = -Q(v) \cdot 1_A$, then there exists a unique algebra homomorphism $\tilde{u} : Cl(V, Q) \to A$ such that $u = \tilde{u} \circ C$.

The three properties can be described by the following commuting diagram.



where π is the quotient projection of $Cl(V, Q)$ from $T(V)$.

Example 4.1.1. *Let us work in $\mathbb{R}^2$ with standard basis $\{e_1, e_2\}$ and quadratic form given by*

$$Q(a^1 e_1 + a^2 e_2) = -(a^1)^2 - (a^2)^2.$$

Then $Cl(\mathbb{R}^2, Q)$ is the algebra of two by two real matrices, i.e. $\mathbb{R}^{2\times 2} =: \mathbb{R}(2)$.

Proof. $Cl(\mathbb{R}^2, Q)$ is generated by $\{1_{Cl(\mathbb{R}^2,Q)}, C(e_1), C(e_2), C(e_1) \cdot C(e_2)\}$ such that

$$C(e_1)^2 = C(e_2)^2 = -\Big(C(e_1) \cdot C(e_2)\Big)^2 = 1_{Cl(\mathbb{R}^2,Q)}\,.$$

Let

$$1_{Cl(\mathbb{R}^2,Q)} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad C(e_1) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$C(e_2) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C(e_1) \cdot C(e_2) \to \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that the above matrices form a basis for $\mathbb{R}^{2\times 2}$. □

Example 4.1.2. *Let us work in $\mathbb{R}^2$ with standard basis as above but now the quadratic form is given by*

$$Q'(a^1 e_1 + a^2 e_2) = (a^1)^2 + (a^2)^2.$$

Then $Cl(\mathbb{R}^2, Q')$ is the algebra of quaternions, denoted by $\mathbb{H}$.

Let $Cl_{r,s} := Cl(\mathbb{R}^{r+s}, q_{r,s})$ be the Clifford algebras associated with the quadratic forms

$$q_{r,s}(v) = (v_1)^2 + \cdots + (v_r)^2 - (v_{r+1})^2 - \cdots - (v_{r+s})^2 \,. \tag{4.4}$$

It is easy to see that

$$\begin{aligned} Cl_{1,0} &= \mathbb{C} \qquad & Cl_{0,1} &= \mathbb{R} \oplus \mathbb{R} \\ Cl_{2,0} &= \mathbb{H} \qquad & Cl_{0,2} &= Cl_{1,1} = \mathbb{R}(2) \end{aligned} \tag{4.5}$$

where the isomorphisms of $Cl_{0,2}$ and $Cl_{2,0}$ are given in the Example 4.1.1 and Example 4.1.2 respectively. The following theorem is given in [38].

Theorem 4.1.1. *For all $n, r, s \geq 0$, we have isomorphisms*

$$\begin{aligned} Cl_{n,0} \otimes Cl_{0,2} &\approx Cl_{0,n+2} \,, \\ Cl_{0,n} \otimes Cl_{2,0} &\approx Cl_{n+2,0} \,, \\ Cl_{r,s} \otimes Cl_{1,1} &\approx Cl_{r+1,s+1} \,. \end{aligned} \tag{4.6}$$

Formula (4.5) and Theorem 4.1.1 immediately imply that the Clifford algebras $Cl_{r,s}$ have either simple or semi-simple algebraic structures. Or, more precisely, the algebras $Cl_{r,s}$ have the forms of either $\mathbb{K}(p)$ or $\mathbb{K}(q) \oplus \mathbb{K}(q)$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and the numbers p and q depend on r and s only.

In fact, it is more convenient to work with complex Clifford algebras, which are given by

$$\mathbb{C}l_n := Cl(\mathbb{R}^n \otimes \mathbb{C}, q_{n,0} \otimes \mathbb{C}) \approx Cl_{n,0} \otimes \mathbb{C} \,, \tag{4.7}$$

where the quadratic form $q_{n,0}$ is given by formula (4.4). Moreover, Theorem 4.1.1 and formulae (4.5) and (4.7) give us

$$\mathbb{C}l_n \approx \mathbb{C}l_{n-2} \otimes_{\mathbb{C}} \mathbb{C}l_2 \approx \mathbb{C}l_{n-2} \otimes_{\mathbb{C}} \mathbb{C}(2) \,, \tag{4.8}$$

which can be further simplified as

$$\mathbb{C}l_n \approx \begin{cases} (\otimes_{\mathbb{C}})^k \mathbb{C}(2) \\ (\otimes_{\mathbb{C}})^k \mathbb{C}(2) \oplus (\otimes_{\mathbb{C}})^k \mathbb{C}(2) \end{cases} \approx \begin{cases} \mathbb{C}(2^k) & \text{for} \quad n = 2k \\ \mathbb{C}(2^k) \oplus \mathbb{C}(2^k) & \text{for} \quad n = 2k+1 \end{cases} \tag{4.9}$$

where $k \in \mathbb{N}$. Now, let $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$ be local coordinates for the tangent space of $\mathbb{R}^n$, which is denoted by $T\mathbb{R}^n$. Moreover, we choose an orthonormal basis $\{e_1, \cdots, e_n\}$ for $T\mathbb{R}^n$. Let us then equip $T\mathbb{R}^n$ with the standard quadratic form $\delta_{\alpha\beta}$, i.e. the Euclidean

metric. One thereby has a dual basis $\{e^1, \cdots, e^n\}$ of $T^*\mathbb{R}^n$ with respect to the basis $\{e_1, \cdots, e_n\}$. For the standard Euclidean metric one also has the Levi-Civita connection on $T\mathbb{R}^n$ and $T^*\mathbb{R}^n$. In this case it is trivial since taking covariant derivatives is the same as taking partial derivatives. Furthermore, we assume the 'induced' connection on $Cl(T^*\mathbb{R}^n, \delta)$ to be also trivial; we will discuss later how to construct such an 'induced' connection. Hence, by choosing a representation, equation (4.1) can be rewritten as

$$W = \rho\left[C(e_j)\right]\nabla_{e_j}\,, \tag{4.10}$$

where ρ is a $\mathbb{C}^m\backslash\mathbb{R}^m$-representation of $Cl(T^*\mathbb{R}^n, \delta)$ and maps $C(\mathrm{d}x^\alpha)$ into matrices $\{\gamma^\alpha\}_{\alpha=1}^n$, satisfying formulae (4.2) and (4.3). In fact, there is a fundamental representation for Clifford algebras, which is known as the spin representation.

Definition 4.1.2 (Spin representations). Let ρ be a complex irreducible representation: $\mathbb{C}l_n \to \mathrm{Hom}_{\mathbb{C}}(S, S)$. Then the corresponding complex spin representation of $Spin(n)$ is given by $\Delta_n := \rho|_{Spin(n)}$, where we have $Spin(n) \subset Cl_n \subset \mathbb{C}l_n$.

Remark 4.1.2.

- From the isomorphisms (4.9), we know that the complex irreducible module of the Clifford algebra $\mathbb{C}l_n$ is isomorphic to the vector space $\mathbb{C}^{2^k}$, and it is called the vector space of complex n-spinors. The elements of $\mathbb{C}^{2^k}$ are called complex spinors.
- Here and in what follows we will mainly focus on the complex representation of spin groups, whereas the treatments for the real spin representation can be found in [38].

4.2 Geometric construction of massless Dirac operators

Let M be a n-dimensional Riemann manifold with metric g. We can construct a Clifford algebra bundle associated with TM as follows.

- For a given point x on the manifold, we have the corresponding tangent bundle T_xM and quadratic form g_x.
- Construct the Clifford algebra associated with the vector space (T_xM, g_x) and denote this Clifford algebra by $Cl(T_xM, g_x)$.
- For each point $x \in M$ we have a fibre of the algebra $Cl(T_xM)$ on top of it. Glue all the algebras together over the manifold to get the Clifford algebra bundle

associated with (TM, g) and denote the algebra bundle by $Cl(TM, g)$. Thus, $Cl(TM, g) = \sqcup_{x\in M} Cl(T_xM, g_x)$.

Note that the map $Cl(T_xM, g_x)$ preserves the linear structure of T_xM. For simplicity we denote $Cl(TM, g)$ by $Cl(TM)$ if the metric is given. Once a $\mathbb{C}^n\backslash\mathbb{R}^n$-representation of a Clifford algebra is given, we can work with the $\mathbb{C}^n\backslash\mathbb{R}^n$-vector bundle induced by the representation. Say, the representation from Example 4.1.1 gives a 2-dimensional vector bundle structure on a 2-dimensional manifold. Or, in modern terminology, we shall call this a bundle of (left) Clifford modules.

Let us denote by S a bundle of Clifford modules, which means that at every point $x \in M$ the fibre S_x is a left module over $Cl(T_xM) \otimes \mathbb{C} =: \mathbb{C}l(T_xM)$. In order to take derivatives on S, we need a connection. Here are some compatibility assumptions defined in [42].

Definition 4.2.1 (Clifford bundle). A bundle of Clifford modules, S, is called a *Clifford bundle* if it is equipped with a Hermitian inner product $(\cdot\,,\cdot)$ and compatible connection such that

- The Clifford action of each vector $v \in T_xM$ on S_x is *skew-adjoint*:

$$(C(v)s_1, s_2) = -(s_1, C(v)s_2) \qquad \text{for} \qquad \forall s_1, s_2 \in S_x\,.$$

- The connection on S is *compatible* with the Levi-Civita connection on M:

$$\nabla_X(C(Y)s) = (\nabla_X C(Y))s + C(Y)\nabla_X s\,,$$

  for all $X, Y \in C^\infty(TM)$ and all $s \in C^\infty(S)$.

In what follows, we will produce a natural Clifford bundle constructed via the language of the principal G bundle.

4.2.1 Connection on the Clifford algebra bundle

Let M be orientable. Since we have the Levi-Civita connection on the tangent bundle, we also have an induced connection on the associated frame bundle, which is a principal $SO(n)$ bundle and is given (locally) by the following formula

$$Z^s = \frac{1}{2}\sum_{i,j=1}^{n} g(\nabla s_i, s_j)E_{ij}\,. \tag{4.11}$$

Here $s = (s_1, \ldots, s_n)$ is a local section of the $SO(n)$ bundle and E_{ij} is the standard basis for the Lie algebra of $SO(n)$, denoted by $\mathfrak{so}(n)$. From now on, the summation symbol will be dropped and the Einstein summation convention will apply. Recall that the standard two-fold group homomorphism $\lambda : Spin(n) \to SO(n)$ is induced by the map $\lambda(x) : \mathbb{R}^n \to \mathbb{R}^n$

$$\Big[\lambda(x)\Big](v) = C^{-1}\Big(x\, C(v)\, \gamma(x)\Big)\,,$$

where $x \in Spin(n)$, $v \in \mathbb{R}^n$ and γ is the anti-involution of the Clifford algebra. Hence one has $\lambda_* : \mathfrak{spin}(n) \to \mathfrak{so}(n)$ with the following property, see [24],

$$\lambda_*\Big(C(e_i)C(e_j)\Big) = 2E_{ij}\,.$$

Thus, we also have an induced connection on the $Spin(n)$ bundle given by

$$\tilde{Z}^s = \frac{1}{4} g(\nabla s_i, s_j) C(s_i) C(s_j)\,.$$

Furthermore, if the principal $SO(n)$ bundle admits a global section, say, e, then the connection on the $SO(n)$ bundle and the $Spin(n)$ bundle can be written respectively, without indicating e, as

$$\begin{cases} Z = \dfrac{1}{2} g(\nabla e_i, e_j) E_{ij}\,, \\ \tilde{Z} = \dfrac{1}{4} g(\nabla e_i, e_j) C(e_i) C(e_j)\,. \end{cases} \tag{4.12}$$

Once a connection on a principal G bundle and a representation are given, a connection on the associated vector bundle is also given. If we choose the fundamental representation for $SO(n)$ and the spin representation for $Spin(n)$, we can easily derive connections on the corresponding vector bundles. Let V be a section on the vector bundle associated with the fundamental $SO(n)$ representation and ξ be a section on the vector (spin) bundle associated with spin representation for $Spin(n)$. One can compute their covariant derivatives (both denoted by ∇) induced by (4.12) as follows:

$$\begin{cases} \nabla V = \mathrm{d}V + \dfrac{1}{2} g(e_i, \nabla e_j) E_{ij} V\,, \\ \nabla \xi = \mathrm{d}\xi + \dfrac{1}{4} g(e_i, \nabla e_j) C(e_i) C(e_j) \xi\,. \end{cases} \tag{4.13}$$

Remark 4.2.1.

- The $\{E_{ij}\}_{1\leq i<j\leq n}$ in the first equation is the standard basis for the Lie algebra $\mathfrak{so}(n)$ and it reads as
$$(E_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\,.$$
- The first covariant derivatives for vector fields should be understood in the anholonomic sense, which means that E_{ij} acts on the projection of V onto the frame $\{e_j\}_{j=1}^n$.
- The $C(e_i)$ in equation (4.12) is in the abstract Clifford algebra and the $C(e_i)$ in equation (4.13) should be understood as $\rho_{\text{spin}}[C(e_i)]$, where ρ_{spin} is the spin representation. We will continue using $C(e_i)$ for $\rho_{\text{spin}}[C(e_i)]$ when considering the Dirac operator on spin bundles.

The geometric way of defining the Dirac operator, denoted by W, is as follows. It acts on the spin bundle according to the formula

$$W := C(e_i)\nabla_{e_i}\,, \tag{4.14}$$

which is different from equation (4.10) as now we are working on a closed 3-manifold and the bundle W acts on is not any Clifford module bundle but the irreducible one corresponding to the spin representation.

4.3 The frame construction

Now suppose we are working on a 3-dimensional Riemannian manifold equipped with metric g. We also assume that the manifold is orientable, which implies that it is parallelizable (special feature of 3-manifolds). Then $Cl(T_xM, g_x)$ is isomorphic to $\mathbb{H}\oplus\mathbb{H}$ and the corresponding spin group is $Spin(3)$. Hence, by previous analysis, the spin representations of $Cl(T_xM, g_x)$ and $Spin(3)$ are the isomorphism representations of $\mathbb{C}^{2\times 2}$ and $SU(2)$ respectively. Let $\{e_1, e_2, e_3\}$ be an orthonormal frame for the tangent bundle of the manifold.

Definition 4.3.1. The Clifford map for the orthonormal frame $\{e_1, e_2, e_3\}$ is given by

$$C(e_j) = -i(S_j)_{ab}\,, \tag{4.15}$$

where the S_j's are the standard constant Pauli matrices defined as

$$S_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad S_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.16}$$

Remark 4.3.1.

- The minus sign in equation (4.15) is a convention to match the geometric approach with Vassiliev's approach.
- The reason for the imaginary number i in the equation is to make the Clifford action of a tangent vector on M to be skew-adjoint, see [42] for details.
- Choosing different constant Pauli matrices or, equivalently, $\{-iS_j\}_{j=1}^3$ will only result in a rigid rotation of the original frames. This is because that by requiring the skew-adjoint Clifford action and orthonormal conditions, $\{-iS_j\}_{j=1}^3$ can only be a base for the Lie algebra $\mathfrak{su}(2)$. In fact, we should view definition (4.15) as a map from a set of orthonormal bases in TM to a set of orthonormal bases in the Lie algebra $\mathfrak{su}(2)$.

The induced connection on spinors

Now we want to show that the second equation of (4.13) in this representation reduces to the covariant derivatives introduced by Vassiliev. Let $\{x^1, x^2, x^3\}$ be local coordinates on our 3-manifold. Then $e_j = e_j{}^\alpha \partial_\alpha$. Since the Clifford map is linear on the tangent space at x, one has

$$C(e_j) = C(e_j{}^\alpha \partial_\alpha) = e_j{}^\alpha C(\partial_\alpha),$$

which implies

$$\begin{aligned} \sigma_\alpha &:= iC(\partial_\alpha) = ie_{j\alpha} C(e_j) = e_{j\alpha}\, S_j, \\ \sigma^\alpha &:= g^{\alpha\beta}\sigma_\beta = e_j{}^\alpha\, S_j. \end{aligned} \tag{4.17}$$

Note that the σ's appearing in equations (4.17) are the (generalised) Pauli matrices defined by Vassiliev. Plugging (4.17) into the second equation (4.13) and contracting with respect to the vector field that generates x^μ, we get

$$\begin{aligned} \nabla_\mu \xi &= \partial_\mu \xi + \frac{1}{4} e_{i\alpha} \nabla_\mu e_j{}^\alpha C(e_i) C(e_j) \xi \\ &= \partial_\mu \xi + \frac{1}{4} \sigma_\alpha (\partial_\mu e_j{}^\alpha + \{{}^{\alpha}_{\mu\nu}\} e_j{}^\nu) C(e_j) \xi \\ &= \partial_\mu \xi + \frac{1}{4} \sigma_\alpha (\partial_\mu \sigma^\alpha + \{{}^{\alpha}_{\mu\nu}\} \sigma^\nu) \xi, \end{aligned} \tag{4.18}$$

where the $\{^{\alpha}_{\mu\nu}\}$ are the Christoffel symbols, and this coincides with the covariant derivative of a spinor field defined by Vassiliev. Thereby, according to equations (4.14) and (4.17), we have a more explicit definition of the Dirac operator.

Definition 4.3.2 (A frame construction for the Dirac operator)*.*

$$W\xi = -i\sigma^{\mu}\left[\partial_{\mu} + \frac{1}{4}\sigma_{\alpha}(\partial_{\mu}\sigma^{\alpha} + \{^{\alpha}_{\mu\nu}\}\sigma^{\nu})\right]\xi\,, \tag{4.19}$$

where the matrix-functions σ^{μ} are given by formula (4.17). The operator W is called the Dirac operator via frames in Vassiliev's work.

Now, a natural question to ask is that what will happen if we choose a different frame, say $\tilde{e}_j$, for the Dirac operator defined above starting from formula (4.15). Let us denote by $\tilde{W}$ the Dirac operator associated with the frame $\tilde{e}_j$. Firstly, note that for every two frames on TM, for instance, $\{e_j\}$ and $\{\tilde{e}_j\}$, we have a section of the $SO(3)$ bundle on M such that

$$\tilde{e}_i = O_{ij}e_j\,, \qquad\qquad O \in \Gamma(SO(3))\,. \tag{4.20}$$

It was shown in [15] that the two Dirac operators W and $\tilde{W}$ are related by

$$\tilde{W} = R\,W\,R^{*}\,, \tag{4.21}$$

where the matrix-function (not necessary a global section of the $SU(2)$ bundle) $R : M \to SU(2)$ is the lift of the section O.

Remark 4.3.2*.* If we are working in local coordinates, the matrix-function R will be a local section of the $SU(2)$ bundle. This explains why it does not matter which frame is chosen locally for the Dirac operator, as they are all locally unitary equivalent. What prevents R being global is exactly the spin structure.

Note that the fundamental group of $SO(3)$, denoted by $\pi_1(SO(3))$, is isomorphic to $\mathbb{Z}_2$, which classifies two classes of loops in $SO(3)$. It is not hard to see from formula (4.20) that this is equivalent to classifying frames in $\mathbb{R}^3$ into two classes. We will see in the next section how this leads to a new definition of spin structures.

4.3.1 The massless Dirac operator on half-densities

Before discussing spin structures, we will introduce the *massless Dirac operator on half-densities* corresponding to the given metric g and denote it by $W_{1/2}$. The massless Dirac operator on half-densities is a particular 2×2 matrix first order linear differential

operator acting on 2-columns of complex-valued half-densities. It is defined by the following four conditions:

$$\operatorname{tr}(W_{1/2})_{\mathrm{prin}} = 0\,, \tag{4.22}$$

$$\det(W_{1/2})_{\mathrm{prin}}(y,p) = -g^{\alpha\beta}(y)\,p_\alpha p_\beta\,, \tag{4.23}$$

$$(W_{1/2})_{\mathrm{sub}} = \frac{i}{16}\,g_{\alpha\beta}\{(W_{1/2})_{\mathrm{prin}},(W_{1/2})_{\mathrm{prin}},(W_{1/2})_{\mathrm{prin}}\}_{p_\alpha p_\beta}, \tag{4.24}$$

$$-\,i\,\operatorname{tr}\Big[((W_{1/2})_{\mathrm{prin}})_{p_1}((W_{1/2})_{\mathrm{prin}})_{p_2}((W_{1/2})_{\mathrm{prin}})_{p_3}\Big] > 0\,. \tag{4.25}$$

Here $y = (y^1, y^2, y^3)$ denotes local coordinates, $p = (p_1, p_2, p_3)$ denotes the dual variable (momentum), $(W_{1/2})_{\mathrm{prin}}(y,p)$ is the principal symbol, $(W_{1/2})_{\mathrm{sub}}(y)$ is the subprincipal symbol, curly brackets denote the generalized Poisson bracket on matrix-functions, see formula (1.22) for details.

The *massless Dirac operator*, W, is defined as

$$W := (\det g_{\kappa\lambda})^{-1/4}\,W_{1/2}\,(\det g_{\mu\nu})^{1/4}. \tag{4.26}$$

It acts on 2-columns of complex-valued scalar fields.

The analytic definition of the massless Dirac operator given in this subsection originates from [5, Section 8].

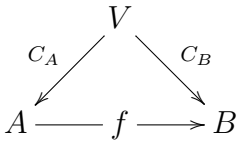
4.4 Spin structures via classifications of frames

The constructions of the massless Dirac operator from Section 4.3 work for any connected oriented Riemannian 3-manifold. Note, however, that they do not define the massless Dirac operator uniquely. Namely, let W be a massless Dirac operator and let $R(y)$ be an arbitrary smooth 2×2 special unitary matrix-function (3.71). One can check that then R^*WR is also a massless Dirac operator.

Let us now look at the issue of non-uniqueness of the massless Dirac operator the other way round. Suppose that W and $\tilde{W}$ are two massless Dirac operators. Does there exist a smooth matrix-function (5.3) such that $\tilde{W} = R^*WR$? If the operators W and $\tilde{W}$ are in a certain sense 'close' then the answer is yes, but in general there are topological obstructions and the answer is no. This motivates the introduction of the concept of spin structure, see [5, Section 7] for details.

Now, we assume that we are working on an n-dimensional parallelizable manifold. From Definition 4.1.1 of the Clifford algebra associated with a given quadratic form, one can derive the following property, see [24] for the proof.

Proposition 4.4.1. Let (A, C_A) and (B, C_B) be two Clifford algebras associated with (V, Q). Then there exists a unique algebra isomorphism $f : A \to B$



where A and B are both isomorphic to $Cl(V, Q)$.

The uniqueness of f comes from property (3) in Definition 4.1.1. Now let $V = T_xM$ and $Q = g_x$. According to Definition 4.3.1 we have a pointwise Clifford map $C_e|_x$ which is obtained by restricting the Clifford map, C in formula (4.15), to x. Here we use the subscript $_e$ to emphasise the dependence on the chosen frame e. Therefore, for a different frame, say, $\tilde{e}$, we also have another Clifford map at x, $C_{\tilde{e}}|_x$. In fact, they are related for all x via a special orthogonal matrix-function $O_{ij}(x)$ and the relationship is given by

$$C_e(e_j) = O_{jk} C_{\tilde{e}}(e_k)\,. \tag{4.27}$$

Now let C_A and C_B in (4.4.1) be $C_e|_x$ and $C_{\tilde{e}}|_x$ respectively. Then, according to Proposition 4.4.1, we have a unique algebra isomorphism $f|_x := \mathrm{Ad}_O(x)$ between $C_e|_x$ and $C_{\tilde{e}}|_x$,

$$\mathrm{Ad}_O(x)\,[C_e|_x(e_j)] = \lambda_O(x)\;[C_{\tilde{e}}|_x(e_j)]\;[\lambda_O(x)]^{-1}\,, \tag{4.28}$$

where $\lambda_O(x)$ is the lift of $O_{ij}(x)$ to its double cover $Spin(n)$ and $\mathrm{Ad}_O(x)$ is independent of the lift.

Remark 4.4.1.

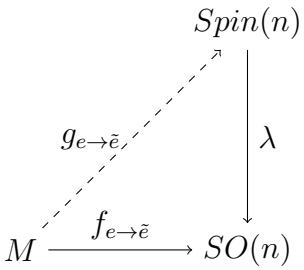
- Equation (4.28) is also know as the *adjoint representation* of $O_{ij}(x)$.
- The spin group $Spin(n)$ can be viewed as a subgroup of $SU\,(m\,(n))$, where the number $m(n)$ is the dimension of spinors in the spin representation of the Clifford algebra and the value of $m(n)$ is given by

$$\begin{aligned} m(n) &= 2^k\,, & n &= 2k+1\,,\\ m(n) &= 2^k\,, & n &= k\,. \end{aligned}$$

  Therefore, Hermitian conjugation can be used to replace the inverse in equation (4.28). This will be useful when we work in low dimensions. In particular, we know that $Spin(3) \approx SU(2)$.

- Initially one has two choices when lifting the value of O_{ij} at the point x_0 to $Spin(n)$. Once an initial value is chosen, say, p_0, $\lambda_O(\cdot)$ becomes a unique continuous path on the $Spin(n)$ bundle over M. Note that every closed loop at x_0 on M induces a path on the $Spin(n)$ bundle starting from p_0 and ending at p_1. More importantly, both p_0 and p_1 are in the fibre on top of x_0. Due to the double covering property of $Spin(n)$ over $SO(n)$, $p_1 = \pm p_0$. Thus, in this way, we construct a map from $\pi_1(M)$ to $\mathbb{Z}_2$ depending on the sign of p_1 with respect to p_0.

Let $f_{e\to\tilde{e}}$ denote that map from M to $SO(n)$ relating e and $\tilde{e}$. Then the map constructed above is nothing but $(f_{e\to\tilde{e}})_* : \pi_1(M) \to \pi_1(SO(n))$, as $\pi_1(SO(n)) \approx \mathbb{Z}_2$. Since $\pi_1(Spin(n))$ is trivial, $\lambda_*[\pi_1(Spin(n))]$ is trivial as well. Therefore, by the homotopy lifting criterion, $f_{e\to\tilde{e}}$ can be lifted to a continuous function on $Spin(n)$ if and only if $(f_{e\to\tilde{e}})_*[\pi_1(M)]$ is trivial. Thus the following diagram commutes if and only if $(f_{e\to\tilde{e}})_*[\pi_1(M)] \approx 1$:



Here $g_{e\to\tilde{e}}$ is a continuous map from M to $Spin(n)$. Now we are ready to give a new definition of spin structure on a parallelizable manifold.

Definition 4.4.1 (Equivalent frames). Two frames e and $\tilde{e}$ on M are called *equivalent* if $(f_{e\to\tilde{e}})_*[\pi_1(M)] \approx 1$.

This leads us to a new definition of spin structures on parallelizable manifolds.

Theorem 4.4.2. *For a parallelizable manifold equivalence classes of frames are in one-to-one correspondence with traditional spin structures.*

Remark 4.4.3. An immediate consequence of this theorem is that if M is also simply connected, i.e. if $\pi_1(M) \approx 1$, then all frames are equivalent. Hence, for a simply connected parallelizable manifold the spin structure is unique.

4.5 The massive Dirac equation in its traditional form

The traditional way of writing the massive Dirac equation is as follows. We equip our manifold M with a prescribed Lorentzian metric and a prescribed electromagnetic covector potential, and write the Dirac equation using the rules of spinor calculus, see the text later in this section. In the process of doing this one may encounter topological obstructions: not every 4-manifold admits a Lorentzian metric and, even if it admits one, it may still not admit a spin structure.

We give now an analytic representation of the massive Dirac equation which, for parallelizable manifolds, turns out to be equivalent to the traditional geometric representation.

For the sake of clarity, prior to describing our analytic construction let us explain why we will not encounter topological obstructions related to the second Stiefel–Whitney class. We will work with operators satisfying the non-degeneracy condition (1.21) which is very natural from the analytic point of view as it is a generalisation (weaker version) of the standard ellipticity condition (1.20). It turns out that the imposition of the non-degeneracy condition (1.21) has far reaching geometric consequences: it implies that our manifold M is parallelizable. Thus, in our construction we deal only with parallelizable manifolds, but we do not state the parallelizability condition explicitly because it is automatically encoded in the analytic non-degeneracy condition (1.21).

Before writing down the massive Dirac equation in its traditional form, let us make several general remarks on the notation that we will be using.

- The notation in this section originates from [8, 11]. Covariant derivatives of spinor fields are defined in accordance with formulae (24) and (25) from [12], which is re-established in formulae (4.18). The difference with [8, 11, 12] is that in the current section we enumerate local coordinates with indices $1, 2, 3, 4$ rather than $0, 1, 2, 3$. Also, the difference with [8, 12] is that in the current section we use opposite Lorentzian signature.

- The construction in this section is a generalisation of that from Appendix A of [15]: in [15] authors dealt with the massless Dirac operator in dimension three.

- We will write the massive Dirac equation in its *spinor representation* as opposed to its *standard representation*, see Appendix B in [13] for details. The spinors ξ^a and $\eta_{\dot{b}}$ that we will be using will be Weyl spinors, i.e. left-handed and right-handed spinors. Let us note straight away that the 4×4 matrix differential operator in

the LHS of formula (B6) from [13] appears to have a structure different from (2.27). However, it is easy to see that the representation (B6) from [13] reduces to (2.27) if one multiplies by the constant 4×4 matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ from the left.

The construction presented below is local, i.e. we work in a neighbourhood of a given point of a 4-manifold M without boundary. We have a prescribed Lorentzian metric $g_{\alpha\beta}(x)$, $\alpha, \beta = 1, 2, 3, 4$, and a prescribed electromagnetic covector potential $A_\alpha(x)$, $\alpha = 1, 2, 3, 4$. The metric tensor is assumed to have three positive eigenvalues and one negative eigenvalue.

Consider a quartet of 2×2 Hermitian matrix-functions $\sigma^\alpha{}_{\dot{a}b}(x)$. Here the Greek index $\alpha = 1, 2, 3, 4$ enumerates the matrices, whereas the Latin indices $\dot{a} = \dot{1}, \dot{2}$ and $b = 1, 2$ enumerate elements of a matrix. Here and throughout the section the first spinor index always enumerates rows and the second columns. We assume that under changes of local coordinates our quartet of matrix-functions transforms as the four components of a vector. Throughout this section we use Greek letters for tensor indices and we raise and lower tensor indices by means of the metric.

Define the "metric spinor"

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{4.29}$$

We will use the rank two spinor (4.29) for raising and lowering spinor indices. Namely, given a quartet of 2×2 Hermitian matrix-functions $\sigma^\alpha{}_{\dot{a}b}(x)$ we define the quartet of 2×2 Hermitian matrix-functions $\tilde{\sigma}^{\alpha a\dot{b}}(x)$ as

$$\tilde{\sigma}^{\alpha a\dot{b}} := -\epsilon^{ab}\, \epsilon^{\dot{a}\dot{b}}\, \sigma^\alpha{}_{\dot{a}b}\,. \tag{4.30}$$

Note the order of spinor indices in the matrix-functions $\tilde{\sigma}^{\alpha a\dot{b}}(x)$: we choose it to be opposite to that in [12] but in agreement with that in [11].

Examination of formulae (4.29) and (4.30) shows that the 2×2 matrices $\sigma^\alpha{}_{\dot{a}b}$ and $\tilde{\sigma}^{\alpha a\dot{b}}$ are adjugates of one another, see formula (2.24) for definition of matrix adjugation. Hence, we could have avoided the use of the "metric spinor" in our construction of the Dirac equation, using the mathematically more sensible concept of matrix adjugation instead. The only reason we introduced the "metric spinor" is to relate the notation of the current section to that of [8, 11, 12].

Further on in this section we use matrix notation. This means that we hide spinor indices and write the matrix-functions $\sigma^{\alpha}{}_{\dot{a}b}(x)$ and $\tilde{\sigma}^{\alpha a\dot{b}}(x)$ as $\sigma^{\alpha}(x)$ and $\tilde{\sigma}^{\alpha}(x)$ respectively.

Further on we assume that our $\sigma^{\alpha}(x)$ are Pauli matrices.

Consider a pair of spinor fields which we shall write as 2-columns,

$$\xi = \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}, \qquad \eta = \begin{pmatrix} \eta_{\dot{1}} \\ \eta_{\dot{2}} \end{pmatrix}. \tag{4.31}$$

Using matrix notation, we define the covariant derivatives of these spinor fields as (see formula (4.18) for details)

$$\nabla_{\alpha}\xi := \frac{\partial \xi}{\partial x^{\alpha}} - \frac{1}{4}\tilde{\sigma}_{\beta}\left((\sigma^{\beta})_{x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^{\gamma}\right)\xi\,, \tag{4.32}$$

$$\tilde{\nabla}_{\alpha}\eta := \frac{\partial \eta}{\partial x^{\alpha}} - \frac{1}{4}\sigma_{\beta}\left((\tilde{\sigma}^{\beta})_{x^{\alpha}} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \tilde{\sigma}^{\gamma}\right)\eta \tag{4.33}$$

respectively, where

$$\left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} := \frac{1}{2}g^{\beta\delta}\left(\frac{\partial g_{\gamma\delta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\delta}}\right) \tag{4.34}$$

are the Christoffel symbols.

Formulae (4.32) and (4.33) warrant the following remarks.

- The sign in front of the $\frac{1}{4}$ in formula (4.32) is the opposite of that in formula (24) of [12]. This is because in the current section we use opposite Lorentzian signature.
- The RHS of formula (4.32) is a generalization of the expression appearing in the RHS of formula (A.3) from [15]. This follows from the observation that the adjugate of a trace-free 2×2 matrix σ_{β} is $-\sigma_{\beta}$.
- If we multiply formula (4.32) from the left by the "metric spinor" (4.29), apply complex conjugation and denote $\epsilon\bar{\xi}$ by η, this gives us (4.33).

The massive Dirac equation reads

$$\sigma^{\alpha}\left(-i\nabla + A\right)_{\alpha}\xi + m\eta = 0\,, \tag{4.35}$$

$$\tilde{\sigma}^{\alpha}\left(-i\tilde{\nabla} + A\right)_{\alpha}\eta + m\xi = 0\,, \tag{4.36}$$

see formulae (B1) and (B2) from [13] or formulae (20.2) and (20.5) from [8].

We define the Dirac operator written in traditional geometric form as

$$D_{\text{trad}} := \begin{pmatrix} \sigma^\alpha \left(-i\nabla + A\right)_\alpha & m_e I \\ m_e I & \tilde{\sigma}^\alpha \left(-i\tilde{\nabla} + A\right)_\alpha \end{pmatrix} \tag{4.37}$$

and the bispinor field as the 4-column

$$\psi := \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{4.38}$$

Formulae (4.35) and (4.36) can then be rewritten as

$$D_{\text{trad}}\, \psi = 0\,. \tag{4.39}$$

4.6 The massive Dirac equation in its analytic form

Formulae (2.64), (2.40), (2.29), (1.22) (2.81), (2.24) and (2.25) allow us to rewrite our Dirac operator (2.27) in geometric notation — in terms of Lorentzian metric, Pauli matrices and electromagnetic covector potential. This raises the obvious question: what is the relation between our Dirac operator (2.27) and the traditional Dirac operator (4.37)? The answer is given by the following theorem.

Theorem 4.6.1. *Our Dirac operator (2.27) and the traditional Dirac operator (4.37) are related by the formula*

$$D = |\det g_{\kappa\lambda}|^{1/4}\, D_{\text{trad}}\, |\det g_{\mu\nu}|^{-1/4}\,, \tag{4.40}$$

where the electromagnetic covector field A in the traditional Dirac operator is given by formula (2.80)*.*

Here, of course, $\det g_{\kappa\lambda} = \det g_{\mu\nu}$. We used different subscripts to avoid confusion because tensor notation involves summation over repeated indices.

Before proving this theorem let us point out that the operators D and D_{trad} appear to be very different. In particular, formula (2.27) does not contain any Pauli matrices, covariant derivatives or electromagnetic covector portential (at least in explicit form).

Proof of Theorem 4.6.1 Proving the 4×4 operator identity (4.40) reduces to proving the following two separate 2×2 operator identities:

$$L = |\det g_{\kappa\lambda}|^{1/4}\, \sigma^\alpha\, (-i\nabla + A)_\alpha\, |\det g_{\mu\nu}|^{-1/4}\,, \tag{4.41}$$

$$(\operatorname{Adj} L) = |\det g_{\kappa\lambda}|^{1/4}\, \tilde\sigma^\alpha\, (-i\tilde\nabla + A)_\alpha\, |\det g_{\mu\nu}|^{-1/4}\,. \tag{4.42}$$

Here σ^α are Pauli matrices (2.40), $\tilde\sigma^\alpha$ are their adjugates, and ∇_α and $\tilde\nabla_\alpha$ are covariant derivatives defined in accordance with formulae (4.32) and (4.33).

We shall prove the operator identity (4.41). The operator identity (4.42) is proved in a similar fashion.

In the remainder of the proof we work in some local coordinate system. The full symbols of the left- and right-hand sides of (4.41) read

$$(L_{\text{prin}})_{p_\alpha} p_\alpha - \frac{i}{2}(L_{\text{prin}})_{x^\alpha p_\alpha} - \frac{i}{16}\, g_{\alpha\beta}\{L_{\text{prin}}, \operatorname{adj} L_{\text{prin}}, L_{\text{prin}}\}_{p_\alpha p_\beta} + (L_{\text{prin}})_{p_\alpha} A_\alpha$$

and

$$\sigma^\alpha p_\alpha + \frac{i}{4}\sigma^\alpha (\ln|\det g_{\mu\nu}|)_{x^\alpha} + \frac{i}{4}\sigma^\alpha\tilde\sigma_\beta \left((\sigma^\beta)_{x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) + \sigma^\alpha A_\alpha$$

respectively, where $\left\{ \begin{smallmatrix} \beta \\ \alpha\gamma \end{smallmatrix} \right\}$ denotes Christoffel symbols (4.34); see also formulae (1.7) and (1.8) for the definition of the full symbol of a differential operator. Comparing these with account of the fact that $(L_{\text{prin}})_{p_\alpha} = \sigma^\alpha$, we see that the proof of the identity (4.41) reduces to the proof of the identity

$$\begin{aligned} -\frac{i}{2}(\sigma^\alpha)_{x^\alpha} - \frac{i}{16}\, g_{\alpha\beta}\{L_{\text{prin}}, \operatorname{adj} L_{\text{prin}}, L_{\text{prin}}\}_{p_\alpha p_\beta} \\ = \frac{i}{4}\sigma^\alpha (\ln|\det g_{\mu\nu}|)_{x^\alpha} + \frac{i}{4}\sigma^\alpha\tilde\sigma_\beta \left((\sigma^\beta)_{x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right). \end{aligned} \tag{4.43}$$

Using the formula (4.34), we have

$$2\left\{ \begin{matrix} \beta \\ \alpha\beta \end{matrix} \right\} = g^{\beta\gamma}\partial_{x^\alpha} g_{\beta\gamma} = \partial_{x^\alpha} \operatorname{tr}\bigl(\ln|g_{\mu\nu}|\bigr) = \partial_{x^\alpha}(\ln|\det g_{\mu\nu}|),$$

which means $(\ln|\det g_{\mu\nu}|)_{x^\alpha} = 2\left\{ \begin{smallmatrix} \beta \\ \alpha\beta \end{smallmatrix} \right\}$. Then we rewrite (4.43) as

$$\frac{1}{2}\, g_{\alpha\beta}\{L_{\text{prin}}, \operatorname{adj} L_{\text{prin}}, L_{\text{prin}}\}_{p_\alpha p_\beta} = -2\,(2I g^\alpha{}_\beta + \sigma^\alpha\tilde\sigma_\beta) \left((\sigma^\beta)_{x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right). \tag{4.44}$$

Finally, using formula (2.48) we rewrite (4.44) as

$$(\sigma^\alpha)_{x^\gamma}\tilde{\sigma}_\alpha\sigma^\gamma - \sigma^\gamma\tilde{\sigma}_\alpha(\sigma^\alpha)_{x^\gamma} = -2\,(2Ig^\alpha{}_\beta + \sigma^\alpha\tilde{\sigma}_\beta)\left((\sigma^\beta)_{x^\alpha} + \left\{\begin{array}{c}\beta\\ \alpha\gamma\end{array}\right\}\sigma^\gamma\right). \qquad (4.45)$$

Thus, we have reduced the proof of the operator identity (4.41) to the proof of the identity (4.45) for Pauli matrices. Calculations proving (4.45) are performed in Subsection 4.6.1. □

It remains only to note that formula (4.40) implies

$$D = |\det g_{\kappa\lambda}|^{1/4}\, D_{\text{trad}}\, |\det g_{\mu\nu}|^{-1/4}\,. \qquad (4.46)$$

We identify a 4-column of complex-valued half-densities v with a bispinor field ψ by means of the formula

$$v = |\det g_{\alpha\beta}|^{1/4}\,\psi\,. \qquad (4.47)$$

Substituting (4.46) and (4.47) into (2.28) we get

$$|\det g_{\kappa\lambda}|^{1/4}\, D_{\text{trad}}\,\psi = 0\,. \qquad (4.48)$$

Clearly, equation (4.48) is equivalent to equation (4.39).

4.6.1 Technical calculations

In this subsection we prove the identity (4.45).

Let us fix an arbitrary point $P \in M$ and prove the identity (4.45) at this point. As the left- and right-hand sides of (4.45) are invariant under changes of local coordinates x, it is sufficient to prove the identity (4.45) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the point P, the metric at $x = 0$ is Minkowski and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the identity we are proving involves only first partial derivatives, we may assume, without loss of generality, that the metric is Minkowski for all x in some neighbourhood of the origin.

Further on we assume that the metric is Minkowski. We need to prove

$$Q = 0\,, \qquad (4.49)$$

where

$$Q := (\sigma^\alpha)_{x^\gamma}\tilde{\sigma}_\alpha\sigma^\gamma - \sigma^\gamma\tilde{\sigma}_\alpha(\sigma^\alpha)_{x^\gamma} + 2\,(2Ig^\alpha{}_\beta + \sigma^\alpha\tilde{\sigma}_\beta)\,(\sigma^\beta)_{x^\alpha}. \qquad (4.50)$$

Formula (4.50) can be rewritten in more compact symmetric form

$$Q = (\sigma^\alpha)_{x^\gamma}\tilde{\sigma}_\alpha\sigma^\gamma + \sigma^\gamma\tilde{\sigma}_\alpha(\sigma^\alpha)_{x^\gamma} + 4(\sigma^\alpha)_{x^\alpha}. \tag{4.51}$$

Using formulae (2.38), (2.39) and the fact that the metric is Minkowski we can now rewrite (4.51) as

$$\begin{aligned} Q &= (\sigma^\alpha)_{x^\gamma}(-2g_\alpha{}^\gamma - \tilde{\sigma}^\gamma\sigma_\alpha) + (-2g^\gamma{}_\alpha - \sigma_\alpha\tilde{\sigma}^\gamma)(\sigma^\alpha)_{x^\gamma} + 4(\sigma^\alpha)_{x^\alpha} \\ &= -(\sigma^\alpha)_{x^\gamma}\tilde{\sigma}^\gamma\sigma_\alpha - \sigma_\alpha\tilde{\sigma}^\gamma(\sigma^\alpha)_{x^\gamma} = -(\sigma^\alpha)_{x^\gamma}\tilde{\sigma}^\gamma\sigma_\alpha - \sigma^\alpha\tilde{\sigma}^\gamma(\sigma_\alpha)_{x^\gamma} \\ &= \sigma^\alpha(\tilde{\sigma}^\gamma)_{x^\gamma}\sigma_\alpha - (\sigma^\alpha\tilde{\sigma}^\gamma\sigma_\alpha)_{x^\gamma}. \end{aligned} \tag{4.52}$$

Formula (2.44) allows us to rewrite formula (4.52) in the form

$$Q = 2\left[\operatorname{adj}\left((\tilde{\sigma}^\gamma)_{x^\gamma}\right) - (\operatorname{adj}\tilde{\sigma}^\gamma)_{x^\gamma}\right].$$

As the operations of matrix adjugation (2.24) and partial differentiation commute, we arrive at (4.49).

Chapter 5

Spectral analysis of the Dirac operator on a 3-sphere

5.1 Introduction

In this chapter we study the spectrum of the (massless) Dirac operator on a 3-sphere, $\mathbb{S}^3$, equipped with Riemannian metric.

By y^α, $\alpha = 1, 2, 3$, we denote local coordinates. We specify an orientation, see Appendix B.1, and use only local coordinates with positive orientation.

We will use the following conventions. Sometimes it will be convenient for us to view the 3-sphere as the hypersurface (B.1) in Euclidean space $\mathbb{R}^4$, in which case Cartesian coordinates in $\mathbb{R}^4$ will be denoted by $\mathbf{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = 1, 2, 3, 4$. Hereinafter we will use bold script for 4-dimensional objects and normal script for 3-dimensional objects. We will use Latin letters for *anholonomic (frame)* indices and Greek letters for *holonomic (tensor)* indices. We will use the convention of summation over repeated indices; this will apply both to Greek and to Latin indices. Also, we will heavily use the analytic concepts of principal and subprincipal symbol of a differential operator; see definitions in [45, subsection 2.1.3] for the case of a scalar operator acting on a single half-density and, more relevantly, [23, Section 1] and [18, Appendix A] for the case of a matrix operator acting on a column of half-densities.

We equip $\mathbb{S}^3$ with a Riemannian metric tensor $g_{\alpha\beta}(y)$, $\alpha, \beta = 1, 2, 3$, and study the corresponding (massless) Dirac operator W. The Dirac operator is a particular first order elliptic linear differential operator acting on 2-columns of complex-valued scalar fields (components of a Weyl spinor). It is written down explicitly in Chapter 4; note that the definition depends on the choice of orientation, see formula (4.25) or formula (B.3). It is known that the Dirac operator W is a self-adjoint operator in $L^2(\mathbb{S}^3; \mathbb{C}^2)$

whose domain is the Sobolev space $H^1(\mathbb{S}^3;\mathbb{C}^2)$, and that the spectrum of W is discrete, accumulating to $+\infty$ and to $-\infty$. Here the inner product in $L^2(\mathbb{S}^3;\mathbb{C}^2)$ is defined as

$$\langle v, w\rangle := \int_{\mathbb{S}^3} \left(w^* v \sqrt{\det g_{\alpha\beta}} \right) \mathrm{d}y\,, \tag{5.1}$$

where the star stands for Hermitian conjugation and $\mathrm{d}y = \mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3$. Furthermore, it is known that all eigenvalues have even multiplicity because the linear Dirac operator commutes with the antilinear operator of charge conjugation

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \mapsto \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix} =: \mathrm{C}(v),$$

see [15, Appendix A] for details.

The Dirac operator W describes the massless neutrino. We are looking at a single neutrino living in a closed 3-dimensional Riemannian universe. The eigenvalues are the energy levels of the particle. The tradition is to associate positive eigenvalues with the energy levels of the neutrino and negative eigenvalues with the energy levels of the antineutrino. In theoretical physics literature the (massless) Dirac operator is often referred to as the Weyl operator which explains our notation.

The Dirac operator is uniquely defined by the metric modulo the gauge transformation

$$W \mapsto R^* W R, \tag{5.2}$$

where

$$R : \mathbb{S}^3 \to \mathrm{SU}(2) \tag{5.3}$$

is an arbitrary smooth special unitary matrix-function. Obviously, the transformation (5.2), (5.3) does not affect the spectrum.

Fortunately, for the purposes of this chapter the issue of spin structure is irrelevant because it is known [7, Section 5], that the 3-sphere admits a unique spin structure. In other words, when we work on $\mathbb{S}^3$ equipped with a Riemannian metric g the constructions from Section 4.3 define the massless Dirac operator uniquely modulo the gauge transformation (5.2), (5.3). See also Section 4.4 for a discussion of spin structure.

5.2 Two-term perturbation formulae for small eigenvalues

The standard metric $(g_0)_{\alpha\beta}(y)$ on $\mathbb{S}^3$ is obtained by restricting the Euclidean metric from $\mathbb{R}^4$ to $\mathbb{S}^3$. For the standard metric the spectrum of the (massless) Dirac operator on $\mathbb{S}^3$ has been computed by different authors using different methods [47, 48, 6, 7] and reads as follows: the eigenvalues are

$$\pm\left(k+\frac{1}{2}\right), \qquad k=1,2,\dots,$$

with multiplicity $k(k+1)$.

We now perturb the metric, i.e. consider a metric $g_{\alpha\beta}=g_{\alpha\beta}(y;\epsilon)$ whose components are smooth functions of local coordinates y^α, $\alpha=1,2,3$, and small real parameter ϵ and which turns into the standard metric for $\epsilon=0$:

$$g_{\alpha\beta}(y;0)=(g_0)_{\alpha\beta}(y).$$

Let $\lambda_+(\epsilon)$ and $\lambda_-(\epsilon)$ be the lowest, in terms of absolute value, positive and negative eigenvalues of the Dirac operator $W(\epsilon)$. Our aim is to derive the asymptotic expansions

$$\lambda_\pm(\epsilon)=\pm\frac{3}{2}+\lambda_\pm^{(1)}\epsilon+\lambda_\pm^{(2)}\epsilon^2+O(\epsilon^3) \qquad \text{as} \qquad \epsilon\to 0. \tag{5.4}$$

Note that $\lambda_\pm(\epsilon)$ are double eigenvalues which cannot split because eigenvalues of the Dirac operator have even multiplicity. Note also that the arguments presented in [17] apply to any double eigenvalue of the Dirac operator on any closed orientable Riemannian 3-manifold, so we know a priori that $\lambda_\pm(\epsilon)$ admit the asymptotic expansions (5.4). The issue at hand is the evaluation of the asymptotic coefficients $\lambda_\pm^{(1)}$ and $\lambda_\pm^{(2)}$.

Let

$$V(\epsilon):=\int_{\mathbb{S}^3}\rho(\epsilon)\,\mathrm{d}y\,, \tag{5.5}$$

$\mathrm{d}y=\mathrm{d}y^1\mathrm{d}y^2\mathrm{d}y^3$, be the Riemannian volume of our manifold. Here

$$\rho(\epsilon)=\rho(y;\epsilon):=\sqrt{\det g_{\mu\nu}(y;\epsilon)}$$

is the Riemannian density for the perturbed metric.

Then

$$V(\epsilon)=V^{(0)}+V^{(1)}\epsilon+O(\epsilon^2) \qquad \text{as} \qquad \epsilon\to 0, \tag{5.6}$$

where

$$V^{(0)} = \int_{\mathbb{S}^3} \rho_0 \, \mathrm{d}y \; = \; 2\pi^2 \tag{5.7}$$

is the volume of the unperturbed 3-sphere,

$$\rho_0 = \rho_0(y) := \sqrt{\det(g_0)_{\mu\nu}(y)}$$

is the standard Riemannian density on the 3-sphere,

$$V^{(1)} = \frac{1}{2} \int_{\mathbb{S}^3} h_{\alpha\beta} \, (g_0)^{\alpha\beta} \, \rho_0 \, \mathrm{d}y \tag{5.8}$$

and

$$h_{\alpha\beta} := \left. \frac{\partial g_{\alpha\beta}}{\partial \epsilon} \right|_{\epsilon=0} . \tag{5.9}$$

Theorem 5.2.1. *We have*

$$\lambda_{\pm}^{(1)} = \mp \frac{1}{4\pi^2} V^{(1)}. \tag{5.10}$$

We see that the dependence of the two lowest eigenvalues, $\lambda_{\pm}(\epsilon)$, on the small parameter ϵ is, in the first approximation, very simple: it is determined by the change of volume only. As expected, an increase of the volume of the resonator (volume of our Riemannian manifold) leads to a decrease of the two lowest natural frequencies (absolute values of the two lowest eigenvalues). Furthermore, formulae (5.6), (5.7) and (5.10) imply

$$\frac{\lambda_{\pm}^{(1)}}{\lambda_{\pm}^{(0)}} = -\frac{1}{3} \frac{V^{(1)}}{V^{(0)}} \, , \tag{5.11}$$

where by $\lambda_{\pm}^{(0)} = \pm \frac{3}{2}$ we denoted the unperturbed values of the two lowest eigenvalues. Now put $\ell(\epsilon) := (V(\epsilon))^{1/3} = \ell^{(0)} \left(1 + \frac{1}{3} \frac{V^{(1)}}{V^{(0)}} \epsilon + O(\epsilon^2) \right)$, where $\ell^{(0)} = \ell(0) = (2\pi^2)^{1/3}$. The quantity $\ell(\epsilon)$ can be interpreted as the characteristic length of our Riemannian manifold. It is easy to see that formula (5.11) is equivalent to the statement

$$\lambda_{\pm}(\epsilon) = \frac{\lambda_{\pm}^{(0)}}{\ell(\epsilon)} + O(\epsilon^2) \qquad \text{as} \qquad \epsilon \to 0,$$

which shows that in the first approximation the two lowest eigenvalues are inversely proportional to the characteristic length of our Riemannian manifold.

An important topic in the spectral theory of first order elliptic systems is the issue of spectral asymmetry [1–4, 17], i.e. asymmetry of the spectrum about zero. From a physics perspective spectral asymmetry describes the difference between a particle (in our case massless neutrino) and an antiparticle (in our case massless antineutrino).

Formulae (5.4) and (5.10) imply

$$\lambda_+(\epsilon) + \lambda_-(\epsilon) = \left(\lambda_+^{(2)} + \lambda_-^{(2)}\right)\epsilon^2 + O(\epsilon^3) \qquad \text{as} \qquad \epsilon \to 0,$$

which means that there is no spectral asymmetry in the first approximation in ϵ but there may be spectral asymmetry in terms quadratic in ϵ.

We will now evaluate the asymptotic coefficients $\lambda_\pm^{(2)}$. We will do this under the simplifying assumption that the Riemannian density does not depend on ϵ:

$$\sqrt{\det g_{\mu\nu}(y;\epsilon)} = \sqrt{\det(g_0)_{\mu\nu}(y)}\,, \tag{5.12}$$

so that $V^{(1)} = 0$. In mechanics such a deformation is called *shear*. Then Theorem 5.2.1 implies $\lambda_\pm^{(1)} = 0$, so formula (5.4) now reads

$$\lambda_\pm(\epsilon) = \pm\frac{3}{2} + \lambda_\pm^{(2)}\epsilon^2 + O(\epsilon^3) \qquad \text{as} \qquad \epsilon \to 0. \tag{5.13}$$

In order to evaluate the asymptotic coefficients $\lambda_\pm^{(2)}$ we need to introduce triples of special vector fields $(K_\pm)_j$, $j = 1, 2, 3$. For their definitions and properties see Appendix B.2. Here we mention only that these are triples of orthonormal Killing fields with respect to the standard (unperturbed) metric.

Put

$$(h_\pm)_{jk} := h_{\alpha\beta}\,(K_\pm)_j{}^\alpha\,(K_\pm)_k{}^\beta, \tag{5.14}$$

where $h_{\alpha\beta}$ is the real symmetric tensor from (5.9). Note that the elements of the 3×3 real symmetric matrix-function $(h_\pm)_{jk}(y)$ are scalars, i.e. they do not change under changes of local coordinates y. Further on we sometimes raise and lower frame indices (see Section 4.3) and we do this using the Euclidean metric. This means, in particular, that raising a frame index in $(h_\pm)_{jk}$ does not change anything.

Put also

$$(L_\pm)_j := (K_\pm)_j{}^\alpha\,\frac{\partial}{\partial y^\alpha}\,, \qquad j = 1, 2, 3. \tag{5.15}$$

The operators (5.15) are first order linear differential operators acting on scalar fields over $\mathbb{S}^3$. The fact that our $(K_\pm)_j$ are Killing vector fields implies that the operators (5.15) are formally anti-self-adjoint with respect to the standard inner product on scalar fields over $\mathbb{S}^3$. It is also easy to see that our operators $(L_\pm)_j$, $j = 1, 2, 3$, satisfy the commutator identities

$$[(L_\pm)_j, (L_\pm)_k] = \mp 2\varepsilon_{jkl}(L_\pm)_l\,, \tag{5.16}$$

where ε_{jkl} is the totally antisymmetric quantity, $\varepsilon_{123} := +1$.

Let Δ be the Laplacian on scalar fields over $\mathbb{S}^3$ with standard (unperturbed) metric. Our Δ is a nonpositive operator, so our definition agrees with the one from basic calculus. By $(-\Delta)^{-1}$ we shall denote the pseudoinverse of the non-negative differential operator $-\Delta$, see Appendix B.3 for explicit definition. Obviously, $(-\Delta)^{-1}$ is a classical pseudodifferential operator of order minus two. This follows from the facts that the parametrix of Δ is a classical pseudodifferential operator of order minus two and the projection onto the kernel of the Laplacian is a smoothing operator, which follows from the elliptic regularity theorem.

Theorem 5.2.2. *Under the assumption* (5.12) *we have*

$$\lambda_{\pm}^{(2)} = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} P_{\pm}\, \rho_0 \, \mathrm{d}y \,, \tag{5.17}$$

where

$$\begin{aligned} P_{\pm} = {} & \pm \frac{1}{4} (h_{\pm})_{jk} (h_{\pm})_{jk} \\ & - \frac{1}{16} \varepsilon_{qks} (h_{\pm})_{jq} \left[(L_{\pm})_s (h_{\pm})_{jk} \right] \\ & \pm \frac{1}{8} (h_{\pm})_{ks} \left[(-\Delta)^{-1} (L_{\pm})_s (L_{\pm})_j (h_{\pm})_{jk} \right] \\ & - \frac{1}{16} \varepsilon_{qks} (h_{\pm})_{rq} \left[(-\Delta)^{-1} (L_{\pm})_r (L_{\pm})_s (L_{\pm})_j (h_{\pm})_{jk} \right] . \end{aligned} \tag{5.18}$$

Theorem 5.2.2 warrants the following remarks.

Remark 5.2.3.

(a) We chose the factor $\frac{1}{2\pi^2}$ in the RHS of (5.17) based on the observation that the volume of the unperturbed 3-sphere is $2\pi^2$, see formula (5.7). This will simplify the comparison with the appropriate formulae previously derived for the 3-torus, see item (f) below, and it will also simply calculations that will be carried out in Subsection 5.6.3.

(b) The terms in the RHS of (5.18) are written in such an order that each subsequent term has an extra appearance of a first order differential operator $L_{\pm}$.

(c) The operators $(-\Delta)^{-1}(L_{\pm})_s(L_{\pm})_j$ and $(-\Delta)^{-1}(L_{\pm})_r(L_{\pm})_s(L_{\pm})_j$ appearing in the last two terms in the RHS of (5.18) are pseudodifferential operators of order 0 and 1 respectively.

(d) The fact that the operators $(L_{\pm})_j$, $j = 1, 2, 3$, are formally anti-self-adjoint with respect to the standard inner product on $\mathbb{S}^3$ implies that for any scalar field $f : \mathbb{S}^3 \to \mathbb{C}$ we have

$$\int_{\mathbb{S}^3} [(L_{\pm})_j f] \, \rho_0 \, \mathrm{d}y = 0 \, . \tag{5.19}$$

Formula (5.19) implies that in the last two terms in the RHS of (5.18) the operator $(-\Delta)^{-1}$ acts on functions from $(\operatorname{Ker} \Delta)^{\perp}$.

(e) The operators $(L_{\pm})_j$ commute with the scalar Laplacian, hence, they also commute with $(-\Delta)^{-1}$. Therefore, the last two terms in the RHS of (5.18) can be written in a number of equivalent ways.

(f) The second and fourth terms in the RHS of (5.18) have a structure similar to that of formula (2.5) from [17]. In fact, if one adjusts notation to agree with that of [17], then it turns out that the second and fourth terms in the RHS of (5.18) are, in effect, an equivalent way of writing formula (2.5) from [17]. See Subsection 5.2.1 for more details.

(g) The first and third terms in the RHS of (5.18) do not have an analogue for the case of the 3-torus [17]. Their appearance is due to the curvature of the 3-sphere.

(h) The first term in the RHS of (5.18) can be rewritten as

$$\pm \, \frac{1}{4} h_{\mu\nu} \, h_{\sigma\tau} \, (g_0)^{\mu\sigma} \, (g_0)^{\nu\tau} \, , \tag{5.20}$$

which means that this term does not feel the Killing vector fields $(K_{\pm})_j$, $j = 1, 2, 3$, and, hence, does not contribute to spectral asymmetry. Put

$$\widetilde{h}_{\mu\nu} := h_{\mu\nu} - \frac{1}{3} \, \delta_{\mu\nu} \, h_{\sigma\tau} \, (g_0)^{\sigma\tau} \, ,$$

which is the part of the deformation tensor $h_{\mu\nu}$ describing shear (deformation preserving Riemannian density). Formula (5.12) implies $h_{\sigma\tau} \, (g_0)^{\sigma\tau} = 0$, so in our case $\widetilde{h}_{\mu\nu} = h_{\mu\nu}$ and the expression (5.20) takes the form

$$\pm \frac{1}{4} \widetilde{h}_{\mu\nu} \, \widetilde{h}_{\sigma\tau} \, (g_0)^{\mu\sigma} \, (g_0)^{\nu\tau} \, .$$

Such an expression describes the elastic potential energy generated by shear, see formula (4.3) in [37].

(i) For a generic perturbation of the metric we expect

$$\lambda_+^{(2)} + \lambda_-^{(2)} \neq 0, \tag{5.21}$$

which means that we expect spectral asymmetry in terms quadratic in ϵ. An example illustrating the inequality (5.21) will be provided in Subsection 5.6.3: see formulae (5.88) and (5.89).

(j) Let us expand the metric tensor in powers of the small parameter ϵ up to quadratic terms:

$$g_{\alpha\beta}(y;\epsilon) = (g_0)_{\alpha\beta}(y) + h_{\alpha\beta}(y)\,\epsilon + k_{\alpha\beta}(y)\,\epsilon^2 + O(\epsilon^3) \qquad \text{as} \qquad \epsilon \to 0.$$

Here the tensor $h_{\alpha\beta}$ is defined by (5.9) whereas $k_{\alpha\beta} := \frac{1}{2}\,\frac{\partial^2 g_{\alpha\beta}}{\partial\epsilon^2}\Big|_{\epsilon=0}$. One would expect the coefficients $\lambda_\pm^{(2)}$ in the asymptotic expansions (5.13) of the lowest eigenvalues to depend on the tensor $k_{\alpha\beta}$, but Theorem 5.2.2 tells us that it is not the case. Here a rough explanation is that the only way the tensor $k_{\alpha\beta}$ can appear in the formulae for $\lambda_\pm^{(2)}$ is linearly, however, condition (5.12) ensures that the linear terms in the map

$$\text{perturbation of metric} \quad \to \quad \text{perturbation of lowest eigenvalues}$$

vanish.

5.2.1 Comparison with the 3-torus

If we leave only the second and fourth terms in the RHS of (5.18), substitute this expression into (5.17), drop the subscripts $\pm$ and use (5.7), we get

$$\lambda^{(2)} = -\frac{1}{16V^{(0)}}\,\varepsilon_{qks}\int_{\mathbb{S}^3}\Big(h_{jq}\,[L_s h_{jk}] + h_{rq}\,\Big[(-\Delta)^{-1}L_r L_s L_j h_{jk}\Big]\Big)\,\rho_0\,\mathrm{d}y\,. \tag{5.22}$$

Formula (5.22) coincides with the result from [17, Theorem 2.1] if we put $V^{(0)} = (2\pi)^3$ (volume of the unperturbed torus), $\rho_0 = 1$ and $L_j = \delta_j{}^\alpha\partial_\alpha$, with ∂_α denoting partial differentiation in the αth cyclic coordinate on the 3-torus.

5.3 Preparatory material

In this section we present auxiliary results which will be used later in the proofs of Theorems 5.2.1 and 5.2.2. Both theorems offer a choice of signs, so for the sake of brevity we present all our preparatory material in a form adapted to the case of upper signs.

5.3.1 The unperturbed Dirac operator

Suppose that $\epsilon = 0$, i.e. suppose that we are working with the standard (unperturbed) metric. It is convenient to write the (massless) Dirac operator using the triple of vector fields $(K_+)_j$, $j = 1, 2, 3$, defined in Appendix B.2 as our frame, see Section 4.3 for the definition of a frame. Straightforward calculations show that in this case the Dirac operator reads

$$W^{(0)} = -is^j (L_+)_j + \frac{3}{2} I \,, \tag{5.23}$$

where s^j are the standard Pauli matrices (4.16), $(L_+)_j$ are the scalar first order linear differential operators (5.15) and I is the 2×2 identity matrix. The superscript in $W^{(0)}$ indicates that the metric is unperturbed.

Let $v^{(0)}$ be a normalised eigenfunction corresponding to the eigenvalue $+\frac{3}{2}$ of the unperturbed Dirac operator (5.23). For example, one can take

$$v^{(0)} = \frac{1}{\sqrt{2}\,\pi} \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{5.24}$$

Here one can replace $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ by any other constant complex 2-column of unit norm. The freedom in the choice of $v^{(0)}$ is due to the fact that $+\frac{3}{2}$ is a double eigenvalue of the unperturbed Dirac operator. The choice of a particular $v^{(0)}$ does not affect subsequent calculations, what matters is that $v^{(0)}$ is a constant spinor.

Choosing the optimal frame (gauge) is crucial for our subsequent arguments because we will heavily use the fact that the eigenspinor $v^{(0)}$ of the unperturbed Dirac operator is a constant spinor. See also Remark B.2.1(c).

Observe that the triple of vector fields $(K_+)_j{}^\alpha$ uniquely defines a triple of covector fields $(K_+)^j{}_\alpha$: the relation between the two is specified by the condition

$$(K_+)_j{}^\alpha (K_+)^k{}_\alpha = \delta_j{}^k, \qquad j, k = 1, 2, 3. \tag{5.25}$$

Of course, the covector $(K_+)^j{}_\alpha$ is obtained by lowering the tensor index in the vector $(K_+)_j{}^\alpha$ by means of the standard metric on $\mathbb{S}^3$. Here the position of the frame index j, as a subscript or superscript, is irrelevant.

In the next subsection we will make use of both the vector fields $(K_+)_j{}^\alpha$ and the covector fields $(K_+)^j{}_\alpha$.

5.3.2 The perturbed Dirac operator

Let $e_j{}^\alpha(y;\epsilon)$ be a frame corresponding to the perturbed metric $g_{\alpha\beta}(y;\epsilon)$, see Section 4.3 for the definition of a frame. This frame can be written as

$$e_j{}^\alpha(y;\epsilon) = (c_+)_j{}^k(y;\epsilon)\,(K_+)_k{}^\alpha(y)\,, \tag{5.26}$$

where $(c_+)_j{}^k$ are some real scalar fields. Without loss of generality we choose to work with frames satisfying the symmetry condition

$$(c_+)_j{}^k = (c_+)_k{}^j, \tag{5.27}$$

which can always be achieved by an application of a gauge transformation — multiplication by a 3×3 orthogonal matrix-function. Then formulae (5.9), (5.14), (5.26) and (5.27) imply

$$(c_+)_j{}^k(y;\epsilon) = \delta_j{}^k - \frac{\epsilon}{2}(h_+)_{jk}(y) + O(\epsilon^2)\,. \tag{5.28}$$

The frame (5.26) uniquely defines the corresponding coframe $e^j{}_\alpha$ analogously to (5.25):

$$e_j{}^\alpha e^k{}_\alpha = \delta_j{}^k, \qquad j,k=1,2,3. \tag{5.29}$$

Of course, the covector $e^j{}_\alpha$ is obtained by lowering the tensor index in the vector $e_j{}^\alpha$ by means of the perturbed metric $g_{\alpha\beta}(y;\epsilon)$ on $\mathbb{S}^3$. Formulae (5.26), (5.25) and (5.29) imply

$$e^j{}_\alpha(y;\epsilon) = (d_+)^j{}_k(y;\epsilon)\,(K_+)^k{}_\alpha(y)\,, \tag{5.30}$$

where

$$(c_+)_j{}^k(d_+)^l{}_k = \delta_j{}^l\,, \qquad j,l=1,2,3. \tag{5.31}$$

By (5.31) and (5.27), the matrix of scalar coefficients $(d_+)^j{}_k$ is also symmetric,

$$(d_+)^j{}_k = (d_+)^k{}_j\,. \tag{5.32}$$

Formulae (5.28), (5.31), (5.27) and (5.32) give

$$(d_+)^j{}_k(y;\epsilon) = \delta^j{}_k + \frac{\epsilon}{2}(h_+)_{jk}(y) + O(\epsilon^2)\,. \tag{5.33}$$

Let $W(\epsilon)$ be the perturbed Dirac operator and let $W_{1/2}(\epsilon)$ be the corresponding perturbed Dirac operator on half-densities. According to (4.26), the two are related as

$$W(\epsilon) = (\rho(\epsilon))^{-1/2}\, W_{1/2}(\epsilon)\, (\rho(\epsilon))^{1/2}. \tag{5.34}$$

Lemma 5.3.1. *The perturbed Dirac operator on half-densities $W_{1/2}(\epsilon)$ acts on 2-columns of complex-valued half-densities $v_{1/2}$ as*

$$\begin{aligned} v_{1/2} \mapsto -\frac{i}{2} s^j\, (\rho_0)^{1/2} \left[(c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k\right] (\rho_0)^{-1/2}\, v_{1/2} \\ + (W_{1/2}(\epsilon))_{\mathrm{sub}}\, v_{1/2}\,, \end{aligned} \tag{5.35}$$

where $(W_{1/2}(\epsilon))_{\mathrm{sub}}$ is its subprincipal symbol.

Let us emphasise that the Riemannian density appearing in (5.35) is the unperturbed density ρ_0 and not the perturbed density $\rho(\epsilon)$ as in (5.34).

Proof of Lemma 5.3.1. Formulae (4.19) and (5.34) tell us that the principal symbol of the operator $W_{1/2}(\epsilon)$ is $\sigma^\alpha(y;\epsilon)p_\alpha$. Using formulae (2.45), (5.26) and (5.15) we can rewrite this principal symbol as

$$-\,i s^j\, (c_+)_j{}^k [(L_+)_k]_{\mathrm{prin}}\,. \tag{5.36}$$

But (5.36) is also the principal symbol of the operator (5.35), so the proof reduces to proving that the operator

$$v_{1/2} \mapsto -\frac{i}{2} s^j\, (\rho_0)^{1/2} \left[(c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k\right] (\rho_0)^{-1/2}\, v_{1/2}$$

has a zero subprincipal symbol. By [45, Proposition 2.1.13] it is sufficient to prove that the operators $(\rho_0)^{1/2}(L_+)_k(\rho_0)^{-1/2}$, $k = 1, 2, 3$, have zero subprincipal symbols. But the latter is a consequence of (5.15) and the fact that our $(K_+)_k{}^\alpha$, being Killing vector fields with respect to the unperturbed metric, are divergence-free.

An alternative approach is as follows. We want to show that

$$(L_{+,\frac{1}{2}})_k := (\rho_0)^{\frac{1}{2}} (L_+)_k (\rho_0)^{-\frac{1}{2}}$$

has zero subprincipal symbol. We first write down its full symbol:

$$(L_{+,\frac{1}{2}})_k(y,\eta) = i(K_+)_k{}^\alpha\eta_\alpha + \rho_0^{\frac{1}{2}}(K_+)_k{}^\alpha\partial_{y^\alpha}(\rho_0^{-\frac{1}{2}})\,.$$

Then the subprincipal symbol is

$$\begin{aligned}[(L_{+,\frac{1}{2}})_k]_{sub} &= \rho_0^{\frac{1}{2}}(K_+)_k{}^\alpha\partial_{y^\alpha}(\rho_0^{-\frac{1}{2}}) - \frac{1}{2i}\partial_{y^\alpha}\partial_{\eta_\alpha}(i(K_+)_k{}^\beta\eta_\beta)\\ &= \rho_0^{\frac{1}{2}}(K_+)_k{}^\alpha\partial_{y^\alpha}(\rho_0^{-\frac{1}{2}}) - \frac{1}{2}\partial_{y^\beta}(K_+)_k{}^\beta\\ &= (K_+)_k{}^\alpha\Big(\rho_0^{\frac{1}{2}}\partial_{y^\alpha}(\rho_0^{-\frac{1}{2}}) - \frac{1}{2}(K_+)^l{}_\alpha\partial_{y^\beta}(K_+)_l{}^\beta\Big)\,. \quad (5.37)\end{aligned}$$

Now, let K_+ denote the matrix with entries $(K_+)_j{}^\beta$. Note that

$$\begin{aligned}\rho_0^{\frac{1}{2}}\partial_{y^\alpha}(\rho_0^{-\frac{1}{2}}) &= \frac{1}{2}\partial_{y^\alpha}(\ln\rho_0^{-1}) = \frac{1}{2}\partial_{y^\alpha}\Big[\ln\det(K_+)\Big] = \frac{1}{2}\partial_{y^\alpha}\Big[\operatorname{tr}\ln(K_+)\Big]\\ &= \frac{1}{2}\operatorname{tr}\Big[\partial_{y^\alpha}\ln(K_+)\Big] = \frac{1}{2}\operatorname{tr}\Big[(K_+^{-1})\partial_{y^\alpha}(K_+)\Big] = \frac{1}{2}(K_+)^j{}_\beta\partial_{y^\alpha}(K_+)_j{}^\beta\,. \quad (5.38)\end{aligned}$$

Hence, equation (5.37) becomes

$$\begin{aligned}[(L_{+,\frac{1}{2}})_k]_{sub} &= (K_+)_k{}^\alpha\Big(\frac{1}{2}(K_+)^j{}_\beta\partial_{y^\alpha}(K_+)_j{}^\beta - \frac{1}{2}(K_+)^l{}_\alpha\partial_{y^\beta}(K_+)_l{}^\beta\Big)\\ &= (K_+)_k{}^\alpha\Big(\frac{1}{2}(K_+)_j{}^\beta\partial_{y^\beta}(K_+)^j{}_\alpha - \frac{1}{2}(K_+)_j{}^\beta\partial_{y^\alpha}(K_+)^j{}_\beta\Big)\\ = \frac{1}{2}(K_+)_k{}^\alpha(K_+)_j{}^\beta&\Big(\partial_{y^\beta}(K_+)^j{}_\alpha - \partial_{y^\alpha}(K_+)^j{}_\beta\Big) = \frac{1}{2}d(K_+)^j\Big((K_+)_k,(K_+)_j\Big)\,. \quad (5.39)\end{aligned}$$

Now, we view $(K_+)_j$ as vector fields and observe that we have the following identity

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])\,,$$

for any vector fields X, Y and any 1-form ω. Therefore, equation (5.39) becomes

$$\begin{aligned}[(L_{+,\frac{1}{2}})_k]_{sub} &= \frac{1}{2}d(K_+)^j\Big((K_+)_k,(K_+)_j\Big)\\ = \frac{1}{2}\Big[(K_+)_k\Big((K_+)^j\Big((K_+)_j\Big)\Big) &- (K_+)_j\Big((K_+)^j\Big((K_+)_k\Big)\Big) - (K_+)^j\Big([(K_+)_k,(K_+)_j]\Big)\Big]\\ &= -\frac{1}{2}(K_+)^j\Big([(K_+)_k,(K_+)_j]\Big). \quad (5.40)\end{aligned}$$

Since $[(K_+)_k, (K_+)_j] = -2\varepsilon_{kji}(K_+)_i$, expression (5.40) reduces to

$$[(L_{+,\frac{1}{2}})_k]_{sub} = \varepsilon_{kji}(K_+)^j\Big((K_+)_i\Big) = \varepsilon_{kji}\delta^{ij} = 0\,,$$

which ends the proof. □

According to [15, formulae (6.1) and (8.1)] the explicit formula for the subprincipal symbol of the Dirac operator on half-densities reads

$$(W_{1/2}(\epsilon))_{\text{sub}} = If(\epsilon), \tag{5.41}$$

where I is the 2×2 identity matrix and $f(\epsilon) = f(y;\epsilon)$ is the scalar function

$$\begin{aligned} f(\epsilon) := \frac{\delta_{kl}}{4\rho(\epsilon)}\Big[&e^k{}_1\,\frac{\partial e^l{}_3}{\partial y^2} + e^k{}_2\,\frac{\partial e^l{}_1}{\partial y^3} + e^k{}_3\,\frac{\partial e^l{}_2}{\partial y^1}\\ &- e^k{}_1\,\frac{\partial e^l{}_2}{\partial y^3} - e^k{}_2\,\frac{\partial e^l{}_3}{\partial y^1} - e^k{}_3\,\frac{\partial e^l{}_1}{\partial y^2}\Big]\,, \end{aligned} \tag{5.42}$$

with $e^k{}_j = e^k{}_j(y;\epsilon)$.

Combining formulae (5.34), (5.35), (5.41) and (5.42) we conclude that the perturbed Dirac operator $W(\epsilon)$ acts on 2-columns of complex-valued scalar fields v as

$$v \mapsto -\frac{i}{2}s^j\sqrt{\frac{\rho_0}{\rho(\epsilon)}}\left[(c_+)_j{}^k(L_+)_k + (L_+)_k(c_+)_j{}^k\right]\sqrt{\frac{\rho(\epsilon)}{\rho_0}}\,v \;+\; f(\epsilon)\,v\,. \tag{5.43}$$

Of course, when $\epsilon = 0$ formulae (5.43) and (5.42) turn into formula (5.23) with $W^{(0)} = W(0)$.

5.3.3 Half-densities versus scalar fields

Given a pair of 2-columns of complex-valued half-densities, $v_{1/2}$ and $w_{1/2}$, we define their inner product as

$$\langle v_{1/2}\,, w_{1/2}\rangle := \int_{\mathbb{S}^3}(w_{1/2})^*\,v_{1/2}\,\mathrm{d}y\,. \tag{5.44}$$

The advantage of (5.44) over (5.1) is that the inner product (5.44) does not depend on the metric. Consequently, if we work with half-densities, perturbations of the metric will not change our Hilbert space. And, unsurprisingly, the perturbation process described in [17, Section 4] was written in terms of half-densities.

The explicit formula for the action of the operator $W_{1/2}(\epsilon)$ reads

$$v_{1/2} \mapsto -\frac{i}{2} s^j \, (\rho_0)^{1/2} \left[(c_+)_j{}^k (L_+)_k + (L_+)_k (c_+)_j{}^k \right] (\rho_0)^{-1/2} \, v_{1/2} + f(\epsilon) \, v_{1/2} \,, \tag{5.45}$$

where $f(\epsilon)$ is the scalar function (5.42).

Formulae (5.45) and (5.42) give us a convenient explicit representation of the perturbed Dirac operator on half-densities $W_{1/2}(\epsilon)$. We will use this representation in the next two sections when proving Theorems 5.2.1 and 5.2.2.

When $\epsilon = 0$ formulae (5.45) and (5.42) turn into

$$v_{1/2} \mapsto -i s^j (\rho_0)^{1/2} (L_+)_k (\rho_0)^{-1/2} v_{1/2} + \frac{3}{2} v_{1/2} \,,$$

which is the action of the unperturbed Dirac operator on half-densities $W^{(0)}_{1/2} = W_{1/2}(0)$. The normalised eigenfunction of the operator $W^{(0)}_{1/2}$ corresponding to the eigenvalue $+\frac{3}{2}$ reads

$$v^{(0)}_{1/2} = \rho_0^{1/2} v^{(0)}, \tag{5.46}$$

where $v^{(0)}$ is given by formula (5.24).

5.3.4 Asymptotic process

Let us expand our Dirac operator on half-densities in powers of ϵ,

$$W_{1/2}(\epsilon) = W^{(0)}_{1/2} + \epsilon W^{(1)}_{1/2} + \epsilon^2 W^{(2)}_{1/2} + \dots . \tag{5.47}$$

Then, according to [17, formulae (4.12) and (4.13)], formula (5.4) holds with

$$\lambda^{(1)}_+ = \left\langle \, W^{(1)}_{1/2} \, v^{(0)}_{1/2} \, , v^{(0)}_{1/2} \, \right\rangle, \tag{5.48}$$

$$\lambda^{(2)}_+ = \left\langle \, W^{(2)}_{1/2} \, v^{(0)}_{1/2} \, , v^{(0)}_{1/2} \, \right\rangle - \left\langle \, \left(W^{(1)}_{1/2} - \lambda^{(1)}_+ I \right) Q_{1/2} \left(W^{(1)}_{1/2} - \lambda^{(1)}_+ I \right) v^{(0)}_{1/2} \, , v^{(0)}_{1/2} \, \right\rangle, \tag{5.49}$$

where $Q_{1/2}$ is the pseudoinverse of the operator $W^{(0)}_{1/2} - \frac{3}{2} I$. See [17, Section 3] for definition of pseudoinverse.

Lemma 5.3.2. *We have*

$$\left\langle \, W_{1/2}(\epsilon) \, v^{(0)}_{1/2} \, , v^{(0)}_{1/2} \, \right\rangle = \left\langle \, f(\epsilon) \, v^{(0)}_{1/2} \, , v^{(0)}_{1/2} \, \right\rangle = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} f(\epsilon) \, \rho_0 \, \mathrm{d}y \,. \tag{5.50}$$

Proof. Substituting (5.45), (5.46) and (5.24) into the LHS of (5.50) and using Remark 5.2.3(d), we see that the terms with $(L_+)_k$ integrate to zero, which leaves us with the RHS of (5.50). □

Let us now expand the scalar function $f(\epsilon)$ in powers of our ϵ,

$$f(\epsilon) = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \dots . \tag{5.51}$$

Here, of course, $f^{(0)} = f(0) = \frac{3}{2}$.

Formulae (5.50), (5.47) and (5.51) imply

$$\left\langle W^{(n)}_{1/2}\, v^{(0)}_{1/2}\,, v^{(0)}_{1/2} \right\rangle = \left\langle f^{(n)}\, v^{(0)}_{1/2}\,, v^{(0)}_{1/2} \right\rangle = \frac{1}{2\pi^2}\int_{\mathbb{S}^3} f^{(n)}\,\rho_0\,\mathrm{d}y\,, \qquad n = 0, 1, \dots .$$

Then formulae (5.48) and (5.49) become

$$\lambda^{(1)}_+ = \frac{1}{2\pi^2}\int_{\mathbb{S}^3} f^{(1)}\,\rho_0\,\mathrm{d}y\,, \tag{5.52}$$

$$\begin{aligned}\lambda^{(2)}_+ &= \frac{1}{2\pi^2}\int_{\mathbb{S}^3} f^{(2)}\,\rho_0\,\mathrm{d}y \;-\left\langle \left(W^{(1)}_{1/2} - \lambda^{(1)}_+ I\right) Q_{1/2}\left(W^{(1)}_{1/2} - \lambda^{(1)}_+ I\right) v^{(0)}_{1/2}\,, v^{(0)}_{1/2} \right\rangle \\ &= \frac{1}{2\pi^2}\int_{\mathbb{S}^3} f^{(2)}\,\rho_0\,\mathrm{d}y \;-\left\langle Q_{1/2}\left(W^{(1)}_{1/2} - \lambda^{(1)}_+ I\right) v^{(0)}_{1/2}\,, \left(W^{(1)}_{1/2} - \lambda^{(1)}_+ I\right) v^{(0)}_{1/2} \right\rangle .\end{aligned} \tag{5.53}$$

5.4 Proof of Theorem 5.2.1

We prove Theorem 5.2.1 for the case of upper signs.

We have

$$\rho(\epsilon) = \rho_0\left(1 + \frac{\epsilon}{2} h_{\alpha\beta}\,(g_0)^{\alpha\beta} + O(\epsilon^2)\right). \tag{5.54}$$

Using formulae (5.30), (5.32) and (5.33), we get

$$\begin{aligned}&\delta_{kl}\left[e^k{}_1\,\frac{\partial e^l{}_3}{\partial y^2} + e^k{}_2\,\frac{\partial e^l{}_1}{\partial y^3} + e^k{}_3\,\frac{\partial e^l{}_2}{\partial y^1} - e^k{}_1\,\frac{\partial e^l{}_2}{\partial y^3} - e^k{}_2\,\frac{\partial e^l{}_3}{\partial y^1} - e^k{}_3\,\frac{\partial e^l{}_1}{\partial y^2}\right] \\ &\qquad = 6\rho_0\left(1 + \frac{\epsilon}{3}(h_+)_{jj} + O(\epsilon^2)\right) = 6\rho_0\left(1 + \frac{\epsilon}{3} h_{\alpha\beta}\,(g_0)^{\alpha\beta} + O(\epsilon^2)\right).\end{aligned} \tag{5.55}$$

Substitution of (5.54) and (5.55) into (5.42) gives us

$$f^{(1)} = -\frac{1}{4} h_{\alpha\beta}\,(g_0)^{\alpha\beta}. \tag{5.56}$$

Finally, substituting (5.56) into (5.52) and using (5.8), we arrive at (5.10). □

5.5 Proof of Theorem 5.2.2

We prove Theorem 5.2.2 for the case of upper signs.

Recall also that we are proving this theorem under the assumption (5.12). This implies, in particular, that

$$\lambda_+^{(1)} = 0. \tag{5.57}$$

With account of (5.57), in order to use formula (5.53) we require the expressions for the scalar function $f^{(2)}$ and for $W_{1/2}^{(1)}\, v_{1/2}^{(0)}$.

Substituting (5.30) and (5.33) into (5.42) and using (5.12) and (5.32), we get

$$f^{(1)} = 0, \tag{5.58}$$

$$f^{(2)} = \frac{1}{4}(h_+)_{jk}(h_+)_{jk} - \frac{1}{16}\varepsilon_{qks}(h_+)_{jq}\left[(L_+)_s(h_+)_{jk}\right]. \tag{5.59}$$

Examination of formulae (5.45), (5.28) and (5.58) gives us the explicit formula for the action of the operator $W_{1/2}^{(1)}$:

$$v_{1/2} \mapsto \frac{i}{4} s^j\, (\rho_0)^{1/2}\left[(h_+)_{jk}(L_+)_k + (L_+)_k(h_+)_{jk}\right](\rho_0)^{-1/2}\, v_{1/2}. \tag{5.60}$$

Acting with the operator (5.60) on the eigenfunction (5.46) of the unperturbed massless Dirac operator on half-densities, we obtain

$$W_{1/2}^{(1)}\, v_{1/2}^{(0)} = \frac{i}{4}(\rho_0)^{1/2}\, s^j\, v^{(0)}\left[(L_+)_k(h_+)_{jk}\right]. \tag{5.61}$$

Using formula (5.61), we get

$$\begin{aligned} &-\left\langle\, Q_{1/2}\, W_{1/2}^{(1)}\, v_{1/2}^{(0)}\,, W_{1/2}^{(1)}\, v_{1/2}^{(0)}\,\right\rangle \\ &= -\frac{1}{16}\int_{\mathbb{S}^3}\Big[(L_+)_r(h_+)_{qr}\Big]\left(\Big[\Big[v^{(0)}\Big]^* s^q\, (\rho_0)^{-1/2}\, Q_{1/2}\, (\rho_0)^{1/2}\, s^j\, v^{(0)}\Big]\Big[(L_+)_k(h_+)_{jk}\Big]\right)\rho_0\, \mathrm{d}y\,. \end{aligned} \tag{5.62}$$

But $(\rho_0)^{-1/2}\, Q_{1/2}\, (\rho_0)^{1/2} = Q$, the pseudoinverse of the operator $W^{(0)} - \frac{3}{2}I$. Hence, formula (5.62) simplifies and reads now

$$\begin{aligned} &-\left\langle\, Q_{1/2}\, W_{1/2}^{(1)}\, v_{1/2}^{(0)}\,, W_{1/2}^{(1)}\, v_{1/2}^{(0)}\,\right\rangle \\ &= -\frac{1}{16}\int_{\mathbb{S}^3}\Big[(L_+)_r(h_+)_{qr}\Big]\left(\Big[\Big[v^{(0)}\Big]^* s^q\, Q\, s^j\, v^{(0)}\Big]\Big[(L_+)_k(h_+)_{jk}\Big]\right)\rho_0\, \mathrm{d}y\,. \end{aligned} \tag{5.63}$$

Observe now that we have the identity

$$\left(W^{(0)} - \frac{1}{2}I\right)^2 = (-\Delta + 1)I, \tag{5.64}$$

where I is the 2×2 identity matrix and Δ is the Laplacian on scalar fields over $\mathbb{S}^3$ with standard metric. Formula (5.64) can be established by direct substitution of (5.23) and the use of the commutator formula (5.16). Formula (5.64) appears also as Lemma 2 in [6].

Formula (5.64) implies

$$Q = (-\Delta)^{-1}\left(W^{(0)} + \frac{1}{2}I\right) = (-\Delta)^{-1}\left(-is^l(L_+)_l + 2I\right). \tag{5.65}$$

Formula (5.65), in turn, gives us the following representation for the scalar pseudodifferential operator $\left[v^{(0)}\right]^* s^q\, Q\, s^j\, v^{(0)}$:

$$\begin{aligned}&\left[v^{(0)}\right]^* s^q\, Q\, s^j\, v^{(0)}\\ &= 2\left(\left[v^{(0)}\right]^* s^q\, s^j\, v^{(0)}\right)(-\Delta)^{-1} - i\left(\left[v^{(0)}\right]^* s^q\, s^l\, s^j\, v^{(0)}\right)(-\Delta)^{-1}(L_+)_l\,.\end{aligned} \tag{5.66}$$

Substituting (5.66) into (5.63), we get

$$\begin{aligned}&-\left\langle Q_{1/2}\, W^{(1)}_{1/2}\, v^{(0)}_{1/2}, W^{(1)}_{1/2}\, v^{(0)}_{1/2}\right\rangle\\ &\quad= -\frac{1}{8}\left(\left[v^{(0)}\right]^* s^q\, s^j\, v^{(0)}\right)\int_{\mathbb{S}^3}\Big[(L_+)_r(h_+)_{qr}\Big]\Big[(-\Delta)^{-1}(L_+)_k(h_+)_{jk}\Big]\rho_0\,\mathrm{d}y\\ &\quad+ \frac{1}{16}\left(i\left[v^{(0)}\right]^* s^q\, s^l\, s^j\, v^{(0)}\right)\int_{\mathbb{S}^3}\Big[(L_+)_r(h_+)_{qr}\Big]\Big[(-\Delta)^{-1}(L_+)_l(L_+)_k(h_+)_{jk}\Big]\rho_0\,\mathrm{d}y\\ &\quad= \frac{1}{8}\,\mathrm{Re}\left(\left[v^{(0)}\right]^* s^q\, s^j\, v^{(0)}\right)\int_{\mathbb{S}^3}(h_+)_{qr}\Big[(-\Delta)^{-1}(L_+)_r(L_+)_k(h_+)_{jk}\Big]\rho_0\,\mathrm{d}y\\ &\quad- \frac{1}{16}\,\mathrm{Re}\left(i\left[v^{(0)}\right]^* s^q\, s^l\, s^j\, v^{(0)}\right)\int_{\mathbb{S}^3}(h_+)_{qr}\Big[(-\Delta)^{-1}(L_+)_r(L_+)_l(L_+)_k(h_+)_{jk}\Big]\rho_0\,\mathrm{d}y\,.\end{aligned} \tag{5.67}$$

But

$$\begin{aligned}\mathrm{Re}\left(\left[v^{(0)}\right]^* s^q\, s^j\, v^{(0)}\right) &= \frac{1}{2}\left(\left[v^{(0)}\right]^*\left(s^q s^j + s^j s^q\right)v^{(0)}\right)\\ &= \delta^{qj}\left(\left[v^{(0)}\right]^* I\, v^{(0)}\right) = \frac{1}{2\pi^2}\,\delta^{qj},\\ \mathrm{Re}\left(i\left[v^{(0)}\right]^* s^q\, s^l\, s^j\, v^{(0)}\right) &= \frac{i}{2}\left(\left[v^{(0)}\right]^*\left(s^q s^l s^j - s^j s^l s^q\right)v^{(0)}\right)\\ &= -\varepsilon^{qlj}\left(\left[v^{(0)}\right]^* I\, v^{(0)}\right) = -\frac{1}{2\pi^2}\,\varepsilon^{qlj},\end{aligned}$$

where we made use of (5.24). Hence, formula (5.67) simplifies and reads now

$$-\Big\langle Q_{1/2}\, W^{(1)}_{1/2}\, v^{(0)}_{1/2}\,, W^{(1)}_{1/2}\, v^{(0)}_{1/2}\Big\rangle = \frac{1}{16\pi^2}\int_{\mathbb{S}^3}(h_+)_{jr}\Big[(-\Delta)^{-1}(L_+)_r(L_+)_k(h_+)_{jk}\Big]\rho_0\,\mathrm{d}y + \frac{1}{32\pi^2}\,\varepsilon_{qlj}\int_{\mathbb{S}^3}(h_+)_{qr}\Big[(-\Delta)^{-1}(L_+)_r(L_+)_l(L_+)_k(h_+)_{jk}\Big]\rho_0\,\mathrm{d}y\,. \tag{5.68}$$

Substituting (5.59) and (5.68) into (5.53) we arrive at (5.17), (5.18) with upper signs. □

5.6 Generalized Berger spheres

A left-handed generalized Berger sphere is a 3-sphere equipped with metric

$$g_{\alpha\beta} = C_{jk}(K_+)^j{}_\alpha(K_+)^k{}_\beta\,, \tag{5.69}$$

where $C = (C_{jk})_{j,k=1}^3$ is a **constant** 3×3 positive real symmetric matrix and $(K_+)^j{}_\alpha$, $j = 1, 2, 3$, are our special covector fields defined in accordance with Section 5.2 and formula (5.25). One can, of course, define in a similar fashion right-handed generalized Berger spheres: these involve the covector fields $(K_-)^j{}_\alpha$, $j = 1, 2, 3$. However, in this section, as in [28], we restrict our analysis to left-handed ones.

One can always perform a rotation in $\mathbb{R}^4$ so that (5.69) turns to

$$g_{\alpha\beta} = \sum_{j=1}^3 a_j^2(K_+)^j{}_\alpha(K_+)^j{}_\beta\,, \tag{5.70}$$

where a_j, $j = 1, 2, 3$, are some positive constants. In formula (5.70) the $(K_+)^j{}_\alpha$, $j = 1, 2, 3$, are new covector fields defined in the new Cartesian coordinate system in accordance with formulae (B.4) and (5.25). Of course, a_j^2 are the eigenvalues of the matrix C. Further on we assume that our generalized Berger metric has the form (5.70).

To the authors' knowledge, metrics of the type (5.70) were first considered in Section 3 of [28]. The expression "generalized Berger sphere" first appears in [27]. The standard (as opposed to the generalized) Berger sphere corresponds to the case $a_2 = a_3 = 1$, and the standard sphere corresponds to the case $a_1 = a_2 = a_3 = 1$.

For future reference, let us give the formula for the Riemannian volume (5.5) of the generalized Berger sphere:

$$V = 2\pi^2 a_1 a_2 a_3 \,. \tag{5.71}$$

5.6.1 Dirac operator on generalized Berger spheres

The remarkable feature of generalized Berger spheres is that for these metrics the calculation of eigenvalues of the (massless) Dirac operator reduces to finding roots of polynomials.

The Dirac operator (5.43) corresponding to the generalized Berger metric reads

$$W = -i \sum_{j=1}^{3} \frac{1}{a_j} s^j (L_+)_j + \nu I \,, \tag{5.72}$$

where

$$\nu = \frac{a_1^2 + a_2^2 + a_3^2}{2 a_1 a_2 a_3} \,. \tag{5.73}$$

In writing (5.72) we followed the convention of choosing the symmetric gauge, see formulae (5.27) and (5.32). The constant (5.73) was written down by means of a careful application of formula (5.42).

Note that formula (5.72) appears also in Proposition 3.1 of [28].

Examination of formula (5.72) shows that $\lambda = \nu$ is an eigenvalue of the Dirac operator, with the corresponding eigenspinors being constant spinors.

In order to calculate other eigenvalues of the Dirac operator it is convenient to extend our spinor field from $\mathbb{S}^3$ to a neighbourhood of $\mathbb{S}^3$ in $\mathbb{R}^4$ and rewrite the operator in Cartesian coordinates. Substituting (B.4) into (5.72), we get

$$\mathbf{W} = -i \sum_{j=1}^{3} \frac{1}{a_j} s^j (\mathbf{L}_+)_j + \nu I \,, \tag{5.74}$$

where

$$\begin{aligned} (\mathbf{L}_+)_1 &= -\mathbf{x}^4 \partial_1 - \mathbf{x}^3 \partial_2 + \mathbf{x}^2 \partial_3 + \mathbf{x}^1 \partial_4 \,, \\ (\mathbf{L}_+)_2 &= \mathbf{x}^3 \partial_1 - \mathbf{x}^4 \partial_2 - \mathbf{x}^1 \partial_3 + \mathbf{x}^2 \partial_4 \,, \\ (\mathbf{L}_+)_3 &= -\mathbf{x}^2 \partial_1 + \mathbf{x}^1 \partial_2 - \mathbf{x}^4 \partial_3 + \mathbf{x}^3 \partial_4 \,. \end{aligned} \tag{5.75}$$

Here the way to work with the Cartesian representation of the Dirac operator is to act with (5.74), (5.75) on a spinor field defined in a neighbourhood of $\mathbb{S}^3$ and then restrict the result to (B.1). It is easy to see that under this procedure the resulting spinor field

on $\mathbb{S}^3$ does not depend on the way we extended our original spinor field from $\mathbb{S}^3$ to a neighbourhood of $\mathbb{S}^3$ in $\mathbb{R}^4$.

The operators (5.75) commute with the scalar Laplacian in $\mathbb{R}^4$. This implies that these operators map homogeneous harmonic polynomials of degree k to homogeneous harmonic polynomials of degree less than or equal to k. Hence, the eigenspinors of the Dirac operator can be written in terms of homogeneous harmonic polynomials. Of course, the restriction of homogeneous harmonic polynomials to the 3-sphere (B.1) gives spherical functions, but we find working with polynomials in Cartesian coordinates more convenient than working with spherical functions in spherical coordinates (B.2).

Let us seek an eigenspinor which is linear in Cartesian coordinates $\mathbf{x}^{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = 1, 2, 3, 4$. Such an eigenspinor is determined by eight complex constants and finding the corresponding eigenvalues reduces to finding the eigenvalues of a particular 8×8 Hermitian matrix. Explicit calculations (which we omit for the sake of brevity) show that the characteristic polynomial of this 8×8 Hermitian matrix is the square of a polynomial of degree four whose roots are

$$\nu - \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3}\,, \tag{5.76}$$

$$\nu - \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}\,, \tag{5.77}$$

$$\nu + \frac{1}{a_1} - \frac{1}{a_2} + \frac{1}{a_3}\,, \tag{5.78}$$

$$\nu + \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3}\,. \tag{5.79}$$

One can repeat the above procedure for homogeneous harmonic polynomials of degree $n = 2, 3, \ldots$, thus reducing the problem of finding eigenvalues of the Dirac operator on a generalized Berger sphere to finding roots of polynomials. See Appendix B.4 for further details.

5.6.2 Testing Theorem 5.2.1 on generalized Berger spheres

From now on we will assume that the positive constants a_j are close to 1. This assumption will allow us to identify the lowest, in terms of modulus, positive and negative eigenvalues of the Dirac operator.

When $a_1 = a_2 = a_3 = 1$ the expression (5.73) takes the value $+\frac{3}{2}$ and the expression (5.76) takes the value $-\frac{3}{2}$. Hence,

$$\lambda_+ = \nu \tag{5.80}$$

is the lowest positive eigenvalue of the Dirac operator and

$$\lambda_- = \nu - \frac{1}{a_1} - \frac{1}{a_2} - \frac{1}{a_3} \tag{5.81}$$

is the lowest, in terms of modulus, negative eigenvalue of the Dirac operator. Recall that ν is given by formula (5.73). As to the expressions (5.77)–(5.79), their values are close to $+\frac{5}{2}$.

In this subsection and the next one we assume that the constants a_j appearing in formula (5.70) are smooth functions of the small parameter ϵ and that $a_j(0) = 1$.

Expanding (5.71), (5.80) and (5.81) in powers of ϵ, we get

$$V(\epsilon) = 2\pi^2 \left(1 + (a_1' + a_2' + a_3')\epsilon + O(\epsilon^2)\right), \tag{5.82}$$

$$\lambda_\pm(\epsilon) = \pm\frac{3}{2} \mp \frac{1}{2}(a_1' + a_2' + a_3')\epsilon + O(\epsilon^2), \tag{5.83}$$

where

$$a_j' := \left.\frac{da_j}{d\epsilon}\right|_{\epsilon=0}.$$

Formulae (5.4), (5.6), (5.7), (5.82) and (5.83) imply (5.10). Thus, we are in agreement with Theorem 5.2.1.

5.6.3 Testing Theorem 5.2.2 on generalized Berger spheres

In this subsection we make the additional assumption

$$a_1(\epsilon)\, a_2(\epsilon)\, a_3(\epsilon) = 1, \tag{5.84}$$

which ensures the preservation of Riemannian volume (5.71) under perturbations. But generalized Berger spheres are homogeneous Riemannian spaces, so preservation of Riemannian volume is equivalent to preservation of Riemannian density. Hence, (5.84) implies (5.12), which is required for testing Theorem 5.2.2.

For future reference note that formula (5.84) implies

$$a_1' + a_2' + a_3' = 0, \tag{5.85}$$

$$a_1'' + a_2'' + a_3'' + 2(a_1'a_2' + a_2'a_3' + a_3'a_1') = 0, \tag{5.86}$$

where

$$a_j'' := \left.\frac{d^2 a_j}{d\epsilon^2}\right|_{\epsilon=0} . \tag{5.87}$$

Expanding (5.80) and (5.81) in powers of ϵ and using formulae (5.85) and (5.86), we get

$$\lambda_+(\epsilon) = \frac{3}{2} + \left((a_1')^2 + (a_2')^2 + (a_3')^2\right)\epsilon^2 + O(\epsilon^3)\,, \tag{5.88}$$

$$\lambda_-(\epsilon) = -\frac{3}{2} + \frac{1}{2}\left((a_1')^2 + (a_2')^2 + (a_3')^2\right)\epsilon^2 + O(\epsilon^3)\,. \tag{5.89}$$

Note that the second derivatives (5.87) do not appear in formulae (5.88) and (5.89), which is in agreement with Remark 5.2.3(j).

We first test whether formula (5.88) agrees with Theorem 5.2.2. Calculating the scalars (5.14) with upper sign, we get

$$(h_+)_{jk} = 2\sum_{l=1}^{3} a_l'\,\delta_{lj}\,\delta_{lk}\,. \tag{5.90}$$

Formulae (5.18) and (5.90) imply

$$P_+ = (a_1')^2 + (a_2')^2 + (a_3')^2. \tag{5.91}$$

Substituting (5.91) into (5.17) and using (5.7), we get $\lambda_+^{(2)} = (a_1')^2 + (a_2')^2 + (a_3')^2$, which is in agreement with (5.88).

In the remainder of this subsection we test whether formula (5.89) agrees with Theorem 5.2.2. This is trickier because the scalar fields $(h_-)_{jk}$ are not constant.

Consider the matrix-function

$$\mathbf{O}_{jk} = \begin{pmatrix} (\mathbf{x}^1)^2 - (\mathbf{x}^2)^2 - (\mathbf{x}^3)^2 + (\mathbf{x}^4)^2 & 2(\mathbf{x}^1\mathbf{x}^2 - \mathbf{x}^3\mathbf{x}^4) & 2(\mathbf{x}^1\mathbf{x}^3 + \mathbf{x}^2\mathbf{x}^4) \\ 2(\mathbf{x}^1\mathbf{x}^2 + \mathbf{x}^3\mathbf{x}^4) & -(\mathbf{x}^1)^2 + (\mathbf{x}^2)^2 - (\mathbf{x}^3)^2 + (\mathbf{x}^4)^2 & 2(\mathbf{x}^2\mathbf{x}^3 - \mathbf{x}^1\mathbf{x}^4) \\ 2(\mathbf{x}^1\mathbf{x}^3 - \mathbf{x}^2\mathbf{x}^4) & 2(\mathbf{x}^1\mathbf{x}^4 + \mathbf{x}^2\mathbf{x}^3) & -(\mathbf{x}^1)^2 - (\mathbf{x}^2)^2 + (\mathbf{x}^3)^2 + (\mathbf{x}^4)^2 \end{pmatrix} \tag{5.92}$$

whose elements are homogeneous harmonic quadratic polynomials. Let O be the restriction of the above matrix-function to the 3-sphere (B.1). Note that the matrix-function O is orthogonal. Let us denote the elements the matrix-function O by O_{jk}, with the first subscript enumerating rows and the second enumerating columns. The two sets of scalar fields, $(h_+)_{jk}$ and $(h_-)_{jk}$, are related as

$$(h_-)_{il} = O_{ij}(h_+)_{jk}O_{lk}\,. \tag{5.93}$$

Substitution of (5.90) into (5.93) gives us explicit formulae for the scalar fields $(h_-)_{jk}$.

We now need to substitute (5.93) into the formula for P_-, see (5.18).

Observe that the (spherical) functions O_{jk} satisfy the identity

$$(L_-)_i\, O_{jk} = 2\varepsilon_{ijl}\, O_{lk}\,. \tag{5.94}$$

Justification. Recall that in the 4-dimensional setting we have

$$(\mathbf{K}_-)_j{}^{\boldsymbol{\alpha}} = \mathbf{O}_j{}^k (\mathbf{K}_+)_k{}^{\boldsymbol{\alpha}}\,, \tag{5.95}$$

where the $(\mathbf{K}_\pm)_j{}^{\boldsymbol{\alpha}}$ are defined by formula (B.4). Let O_{jk} and $(K_\pm)_j$ be the restrictions of $\mathbf{O}_{jk}$ and $(\mathbf{K}_\pm)_j$ to the 3-sphere and $g_0 = g(0)$ be the unperturbed metric on the 3-sphere. Now, we have

$$g_0\Big((K_-)_j, (K_-)_k\Big) = g_0\Big((K_+)_j, (K_+)_k\Big) = \delta_{jk}$$

and

$$O_j{}^k = O_{jk} = g_0\Big((K_-)_j, (K_+)_k\Big)\,. \tag{5.96}$$

Here we view the $(K_\pm)_j$ as vector fields on the 3-sphere, which is the same as working with the operators $(L_\pm)_j$ on the 3-sphere. To simplify notation, we now use g for g_0 and ∇ for ∇^0. Thus,

$$\begin{aligned}(L_-)_p(O_{mn}) &= (K_-)_p\, g\Big((K_-)_m, (K_+)_n\Big)\\ &= g\Big(\nabla_{(K_-)_p}(K_-)_m, (K_+)_n\Big) + g\Big((K_-)_m, \nabla_{(K_-)_p}(K_+)_n\Big)\,.\end{aligned} \tag{5.97}$$

Then we use the formula

$$\nabla_X Y - \nabla_Y X = [X, Y]\,, \tag{5.98}$$

for our torsion-free Levi-Civita connection and the formula

$$[(K_-)_j, (K_+)_k] = 0 \quad \text{for} \quad j, k = 1, 2, 3\,.$$

Thus, equation (5.97) becomes

$$(L_-)_p(O_{mn}) = g\Big(\nabla_{(K_-)_p}(K_-)_m, (K_+)_n\Big) - g\Big((K_-)_m, \nabla_{(K_+)_n}(K_-)_p\Big) \tag{5.99}$$

Recall also that

$$[(K_-)_j, (K_-)_k] = 2\varepsilon_{jkl}(K_-)_l\,.$$

Combining the above formula with equation (5.98), we get

$$\nabla_{(K_-)_p}(K_-)_m = \nabla_{(K_-)_m}(K_-)_p + 2\varepsilon_{pmq}(K_-)_q\,. \tag{5.100}$$

Substitution of (5.100) into equation (5.99) gives

$$\begin{aligned}(L_-)_p(O_{mn}) &= g\Big(\nabla_{(K_-)_m}(K_-)_p + 2\varepsilon_{pmq}(K_-)_q, (K_+)_n\Big) - g\Big((K_-)_m, \nabla_{(K_+)_n}(K_-)_p\Big)\\ &= 2\varepsilon_{pmq}O_{qn} + \Big[g\Big(\nabla_{(K_-)_m}(K_-)_p, (K_+)_n\Big) - g\Big((K_-)_m, \nabla_{(K_+)_n}(K_-)_p\Big)\Big]\,.\end{aligned} \tag{5.101}$$

Now, let $T_{mpn} = g\Big(\nabla_{(K_-)_m}(K_-)_p, (K_+)_n\Big) - g\Big((K_-)_m, \nabla_{(K_+)_n}(K_-)_p\Big)$. Then we have

$$\begin{aligned}O_{mr}\,T_{mpn} &= O_{mr}\,g\Big(\nabla_{(K_-)_m}(K_-)_p, (K_+)_n\Big) - O_{mr}\,g\Big((K_-)_m, \nabla_{(K_+)_n}(K_-)_p\Big)\\ &= g\Big(\nabla_{(K_+)_r}(K_-)_p, (K_+)_n\Big) - g\Big((K_+)_r, \nabla_{(K_+)_n}(K_-)_p\Big)\\ &= g\Big(\nabla_{(K_+)_r}(K_-)_p, (K_+)_n\Big) + g\Big((K_+)_r, \nabla_{(K_-)_p}(K_+)_n\Big)\\ &= g\Big(\nabla_{(K_+)_r}(K_-)_p, (K_+)_n\Big) - g\Big(\nabla_{(K_-)_p}(K_+)_r, (K_+)_n\Big)\\ &= g\Big(\nabla_{(K_+)_r}(K_-)_p - \nabla_{(K_-)_p}(K_+)_r, (K_+)_n\Big)\\ &= g\Big([(K_+)_r, (K_-)_p], (K_+)_n\Big)\\ &= 0\,,\end{aligned} \tag{5.102}$$

which implies $T_{mpn} = 0$. Thus,

$$(L_-)_p(O_{mn}) = 2\varepsilon_{pmq}O_{qn}\,. \tag{5.103}$$

□

Formulae (5.93) and (5.94) and the fact that the matrix of constants $(h_+)_{jk}$ is symmetric imply $(L_-)_s(L_-)_j(h_-)_{jk} = 0$, so the last two terms in the RHS of (5.18) vanish, giving us

$$P_- = -\frac{1}{4}(h_-)_{jk}(h_-)_{jk} - \frac{1}{16}\varepsilon_{qks}(h_-)_{jq}\left[(L_-)_s(h_-)_{jk}\right]. \tag{5.104}$$

We examine the two terms in the RHS of (5.104) separately. As the matrix O is orthogonal, we have, with account of (5.90),

$$-\frac{1}{4}(h_-)_{jk}(h_-)_{jk} = -\frac{1}{4}(h_+)_{jk}(h_+)_{jk} = -\Big[(a_1')^2 + (a_2')^2 + (a_3')^2\Big]. \tag{5.105}$$

The other term is evaluated by substituting (5.93), using the identity (5.94) and the fact that our perturbation of the metric is pointwise trace-free $(h_\pm)_{jj} = 0$, which gives us

$$-\frac{1}{16}\varepsilon_{qks}(h_-)_{jq}\left[(L_-)_s(h_-)_{jk}\right] = \frac{3}{8}(h_+)_{jk}(h_+)_{jk} = \frac{3}{2}\Big[(a_1')^2 + (a_2')^2 + (a_3')^2\Big]. \tag{5.106}$$

Substituting (5.105) and (5.106) into the RHS of (5.104), we arrive at

$$P_- = \frac{1}{2}\Big[(a_1')^2 + (a_2')^2 + (a_3')^2\Big]. \tag{5.107}$$

Substituting (5.107) into (5.17) and using (5.7), we get $\lambda_-^{(2)} = \frac{1}{2}[(a_1')^2 + (a_2')^2 + (a_3')^2]$, which is in agreement with (5.89).

Appendix A

Correction function f

In this appendix we present some technical calculations needed for the proof of Proposition 2.5.2.

A.1 Calculations involving the correction function f and axial torsion

Let us start with formula (2.69):

$$s^m s^n = \sum_{p=1}^{4} \Big[(\delta^m{}_4 \eta^{np} + \eta^{mp} \delta^n{}_4 - \delta^m{}_4 \delta^n{}_4 \eta^{p4}) \\ + (1 - \delta^m{}_4 - \delta^n{}_4 + \delta^m{}_4 \delta^n{}_4) \left(i(1 - \delta^p{}_4) \varepsilon^{mnp} + \delta^{mn} \eta^{p4} \right) \Big] s_p \,,$$

Then we have

$$\sigma^\alpha \sigma^\beta = s^m s^n e_m{}^\alpha e_n{}^\beta = (e_4{}^\alpha e_i{}^\beta + e_i{}^\alpha e_4{}^\beta) s^i + e_4{}^\alpha e_4{}^\beta s^4 + (i\varepsilon^{ijk} s_k + \delta^{ij} s^4) e_i{}^\alpha e_j{}^\beta \,. \quad \text{(A.1)}$$

Using the transformation (2.33) and equation (2.69), we get

$$\tilde{\sigma}^\alpha \sigma^\beta = s^m s^n \tilde{e}_m{}^\alpha e_n{}^\beta = (e_4{}^\alpha e_i{}^\beta - e_i{}^\alpha e_4{}^\beta) s^i + e_4{}^\alpha e_4{}^\beta s^4 - (i\varepsilon^{ijk} s_k + \delta^{ij} s^4) e_i{}^\alpha e_j{}^\beta \,. \quad \text{(A.2)}$$

Now, we have

$$
\begin{aligned}
&\sigma^\gamma\tilde{\sigma}^\alpha\sigma^\beta = e_m{}^\gamma s^m\tilde{\sigma}^\alpha\sigma^\beta = e_4{}^\gamma\tilde{\sigma}^\alpha\sigma^\beta + e_i{}^\gamma s^i\tilde{\sigma}^\alpha\sigma^\beta \\
=&e_4{}^\gamma\Big[(e_4{}^\alpha e_i{}^\beta - e_i{}^\alpha e_4{}^\beta)s^i + e_4{}^\alpha e_4{}^\beta s^4 - (i\varepsilon^{ijk}s_k + \delta^{ij}s^4)e_i{}^\alpha e_j{}^\beta\Big] \\
&+ e_i{}^\gamma s^i\Big[(e_4{}^\alpha e_j{}^\beta - e_j{}^\alpha e_4{}^\beta)s^j + e_4{}^\alpha e_4{}^\beta s^4 - (i\varepsilon^{jkl}s_l + \delta^{jk}s^4)e_j{}^\alpha e_k{}^\beta\Big] \\
=&e_4{}^\gamma\Big[(e_4{}^\alpha e_i{}^\beta - e_i{}^\alpha e_4{}^\beta)s^i + e_4{}^\alpha e_4{}^\beta s^4 - (i\varepsilon^{ijk}s_k + \delta^{ij}s^4)e_i{}^\alpha e_j{}^\beta\Big] \\
&+ e_i{}^\gamma\Big\{(e_4{}^\alpha e_j{}^\beta - e_j{}^\alpha e_4{}^\beta)(i\varepsilon^{ijk}s_k + \delta^{ij}s^4) + e_4{}^\alpha e_4{}^\beta s^i \\
&\qquad\qquad - \Big[i\varepsilon^{jkl}(i\varepsilon^{ill'}s_{l'} + \delta^{il}s^4) + \delta^{jk}s^i\Big]e_j{}^\alpha e_k{}^\beta\Big\} \qquad \text{(A.3)} \\
=&\Big(e_4{}^\gamma e_4{}^\alpha e_m{}^\beta s^m - e_4{}^\gamma e_4{}^\beta e_m{}^\alpha s^m - \delta^{ij}e_i{}^\alpha e_j{}^\beta e_m{}^\gamma s^m \\
&\qquad\qquad + \delta^{ij}e_i{}^\gamma e_j{}^\beta e_m{}^\alpha s^m - \delta^{ij}e_i{}^\gamma e_j{}^\alpha e_m{}^\beta s^m + e_4{}^\alpha e_4^\beta e_m^\gamma s^m\Big) \\
&+ i\varepsilon^{ijk}\Big(-e_4{}^\gamma e_i{}^\alpha e_j{}^\beta s_k + e_i{}^\gamma e_4{}^\alpha e_j{}^\beta s_k - e_i{}^\gamma e_j{}^\alpha e_4{}^\beta s_k + e_i{}^\gamma e_j{}^\alpha e_k{}^\beta s_4\Big), \qquad \text{(A.4)}
\end{aligned}
$$

where we used the following identities in transitioning from (A.3) to (A.4):

$$
\begin{cases}
s^i s^j = i\varepsilon^{ijk}s_k + \delta^{ij}s^4\,, \\
\varepsilon^{ijk}\varepsilon^{i'j'k} = \delta^{ii'}\delta^{jj'} - \delta^{ij'}\delta^{ji'}\,.
\end{cases} \qquad \text{(A.5)}
$$

Now, we define a new inner product in dimension four as follows:

$$
< e^\alpha, e^\beta >_\eta := e_m{}^\alpha e_n{}^\beta\eta^{mn} = e_m{}^\alpha e^{m\beta} =< e^\beta, e^\alpha >_\eta \ .
$$

Also, let us denote the part without ε^{ijk} in the expression (A.4) by $A^{\gamma\alpha\beta}$. Then

$$
A^{\gamma\alpha\beta} =< e^\gamma, e^\beta >_\eta e_m{}^\alpha s^m - < e^\alpha, e^\gamma >_\eta e_m{}^\beta s^m - < e^\alpha, e^\beta >_\eta e_m{}^\gamma s^m. \qquad \text{(A.6)}
$$

Observing that the quantity $A^{\gamma\alpha\beta}$ is symmetric in γ and β, we arrive at formula (2.71).

A.2 Examining equation (2.78)

In this section we shall prove the identity (2.78). For the benefit of the reader we repeat this identity below:

$$
g(\nabla^{\mathrm{T}}_{e_p}e_m, e_n)s^m\wedge s^n\wedge s^p = (e_m{}^\rho)_{x^\gamma}e^m{}_\alpha\, g_{\rho\beta}\,\sigma^\alpha\wedge\sigma^\beta\wedge\sigma^\gamma + 3\times\frac{1}{3!}T^{\mathrm{ax}}_{\alpha\beta\gamma}\sigma^\alpha\wedge\sigma^\beta\wedge\sigma^\gamma\,. \qquad \text{(A.7)}
$$

Let us start with

$$g(\nabla_{e_p} e_m, e_n) = g\Big([e_p{}^{\gamma}(e_m{}^{\alpha})_{x^\gamma} + e_p{}^{\lambda} e_m{}^{\rho}\Gamma^{\alpha}_{\lambda\rho}]\frac{\partial}{\partial x^\alpha}, e_n{}^{\beta}\frac{\partial}{\partial x^\beta}\Big)$$
$$= (e_m{}^{\alpha})_{x^\gamma} e_{n\alpha} e_p{}^{\gamma} + e_m{}^{\rho} e_n{}^{\beta} e_p{}^{\lambda} g_{\alpha\beta}\Gamma^{\alpha}_{\lambda\rho}\,. \quad \text{(A.8)}$$

Multiplying both sides of (A.8) by $s^m \wedge s^n \wedge s^p$ and summing over repeated indices, we get

$$\begin{aligned}
& g(\nabla_{e_p} e_m, e_n) s^m \wedge s^n \wedge s^p - (e_m{}^{\alpha})_{x^\gamma} e_{n\alpha} e_p{}^{\gamma} s^m \wedge s^n \wedge s^p \\
=& e_m{}^{\rho} e_n{}^{\beta} e_p{}^{\lambda} g_{\alpha\beta}\Gamma^{\alpha}_{\lambda\rho} s^m \wedge s^n \wedge s^p \\
=& \frac{1}{2} e_m{}^{\rho} e_n{}^{\beta} e_p{}^{\lambda} g_{\alpha\beta}(\Gamma^{\alpha}_{\lambda\rho} - \Gamma^{\alpha}_{\rho\lambda}) s^m \wedge s^n \wedge s^p \\
=& \frac{1}{2} e_m{}^{\rho} e_n{}^{\beta} e_p{}^{\lambda} T_{\beta\lambda\rho} s^m \wedge s^n \wedge s^p \\
=& \frac{1}{2} \times 3! \times \frac{1}{3!} T^{\text{ax}}_{\alpha\beta\gamma} \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma \\
=& 3 \times \frac{1}{3!} T^{\text{ax}}_{\alpha\beta\gamma} \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma\,. \qquad \text{(A.9)}
\end{aligned}$$

Appendix B

The 3-sphere

In this appendix we list some auxiliary facts related to the 3-sphere.

B.1 Orientation

The unit 3-sphere, $\mathbb{S}^3$, is the hypersurface in $\mathbb{R}^4$ defined by the equation

$$\|\mathbf{x}\| = 1, \tag{B.1}$$

where $\|\cdot\|$ is the standard Euclidean norm. Spherical coordinates

$$\begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \mathbf{x}^3 \\ \mathbf{x}^4 \end{pmatrix} = \begin{pmatrix} \cos y^1 \\ \sin y^1 \cos y^2 \\ \sin y^1 \sin y^2 \cos y^3 \\ \sin y^1 \sin y^2 \sin y^3 \end{pmatrix}, \quad y^1, y^2 \in (0, \pi), \quad y^3 \in [0, 2\pi), \tag{B.2}$$

are an example of local coordinates on $\mathbb{S}^3$. We define the orientation of spherical coordinates (B.2) to be positive.

Consider a triple of orthonormal (with respect to the given metric g) smooth real vector fields e_j, $j = 1, 2, 3$. Each vector $e_j(y)$ has coordinate components ${e_j}^\alpha(y)$, $\alpha = 1, 2, 3$. The triple of vector fields e_j, $j = 1, 2, 3$, is called an *orthonormal frame*. We assume that

$$\det {e_j}^\alpha > 0\,, \tag{B.3}$$

which means that the orientation of our frame agrees with the orientation of our local coordinates.

B.2 Special vector fields on the 3-sphere

Working in $\mathbb{R}^4$ and using Cartesian coordinates, consider the triple of vector fields $(\mathbf{K}_\pm)_j{}^{\boldsymbol{\alpha}}$, $j = 1, 2, 3$, $\boldsymbol{\alpha} = 1, 2, 3, 4$, defined as

$$\begin{aligned}(\mathbf{K}_\pm)_1 &= \begin{pmatrix} -\mathbf{x}^4 & \mp\mathbf{x}^3 & \pm\mathbf{x}^2 & \mathbf{x}^1 \end{pmatrix}, \\ (\mathbf{K}_\pm)_2 &= \begin{pmatrix} \pm\mathbf{x}^3 & -\mathbf{x}^4 & \mp\mathbf{x}^1 & \mathbf{x}^2 \end{pmatrix}, \\ (\mathbf{K}_\pm)_3 &= \begin{pmatrix} \mp\mathbf{x}^2 & \pm\mathbf{x}^1 & -\mathbf{x}^4 & \mathbf{x}^3 \end{pmatrix}.\end{aligned} \tag{B.4}$$

Observe that the vector fields (B.4) are tangent to the 3-sphere (B.1), so let us denote by $(K_\pm)_j{}^\alpha$ the restrictions of the vector fields (B.4) to the 3-sphere. Here the tensor index $\alpha = 1, 2, 3$ corresponds to local coordinates y^α on $\mathbb{S}^3$. Note that we have $\det\{(K_\pm)_j{}^\alpha\} > 0$, which is in agreement with (B.3).

Remark B.2.1. The vector fields $(K_\pm)_j$, $j = 1, 2, 3$, constructed above are special because with the standard metric on $\mathbb{S}^3$ they possess the following properties.

(a) The vector fields $(K_\pm)_j$ are orthonormal,

(b) The vector fields $(K_\pm)_j$ are Killing vector fields.

(c) If we write down the Dirac operator $W_\pm$ using $(K_\pm)_j$ as a frame, then the eigenspinors corresponding to the eigenvalue $\pm\frac{3}{2}$ are constant spinors. Of course, for a given operator W_+ or W_- one cannot have constant eigenspinors for eigenvalues $+\frac{3}{2}$ **and** $-\frac{3}{2}$ because this would contradict the fact that eigenspinors corresponding to different eigenvalues are orthogonal.

Note that the operators W_+ and W_- defined in Remark B.2.1(c) are related as $W_- = R^* W_+ R$, where $R : \mathbb{S}^3 \to \mathrm{SU}(2)$ is the restriction of the matrix-function

$$\pm \begin{pmatrix} \mathbf{x}^4 + i\mathbf{x}^3 & \mathbf{x}^2 + i\mathbf{x}^1 \\ -\mathbf{x}^2 + i\mathbf{x}^1 & \mathbf{x}^4 - i\mathbf{x}^3 \end{pmatrix}$$

to the 3-sphere (B.1).

B.3 The scalar Laplacian and its pseudoinverse

In this appendix we work on the 3-sphere equipped with standard metric $(g_0)_{\alpha\beta}(y)$.

Let f be a smooth scalar function on $\mathbb{S}^3$. Then there exists a unique sequence of homogeneous harmonic polynomials $p_n(\mathbf{x})$ of degree $n = 0, 1, 2, \ldots$ such that the series

$\sum_{n=0}^{+\infty} p_n(\mathbf{x})$ converges uniformly, together with all its partial derivatives, on the closed unit ball in $\mathbb{R}^4$, and coincides with f on $\mathbb{S}^3$.

It is known that the eigenvalues of the operator $-\Delta$ acting on $\mathbb{S}^3$ are $n(n+2)$, $n = 0, 1, 2, \ldots$, and their multiplicity is $(n+1)^2$, which is the dimension of the vector space of homogeneous harmonic polynomials of degree n. The explicit formula for the action of the operator $(-\Delta)^{-1}$, the pseudoinverse of $-\Delta$, on our function f is

$$(-\Delta)^{-1} f = \sum_{n=1}^{+\infty} \frac{p_n(\mathbf{x})}{n(n+2)} \Bigg|_{\|\mathbf{x}\|=1}.$$

B.4 Eigenvalues for generalized Berger spheres

Here we give further explicit expressions for the eigenvalues using the procedure from Section 5.6.1 where we apply the operator to harmonic polynomials of degree n. For convenience we seek eigenvalues μ of the operator $\widetilde{\mathbf{W}} = \mathbf{W} - \nu I$ obtained by dropping the constant term from (5.74).

Let $\kappa = (\kappa_1, \kappa_2, \kappa_3) \in \{\pm 1\}^3$, and let

$$N_+ := \{\kappa \in \{\pm 1\}^3 : \kappa_1\kappa_2\kappa_3 = +1\} = \{(1,1,1), (1,-1,-1), (-1,1,-1), (-1,-1,1)\}.$$

For each $n \leq 4$ we give below an explicit formula for the characteristic polynomial $\chi_n(\mu)$ whose roots give the eigenvalues of $\widetilde{\mathbf{W}}$. For $n \geq 5$ formulae become too cumbersome to list, and we do not have a general formula yet.

Degree $n = 0$.

$$\chi_0(\mu) = \mu^2.$$

Degree $n = 1$.

$$\chi_1(\mu) = \prod_{\kappa \in N_+} \left[\mu + \sum_{j=1}^{3} \frac{\kappa_j}{a_j}\right]^2.$$

See also formulae (5.76)– (5.79).

Degree $n = 2$.

$$\chi_2(\mu) = \left[\mu^3 - \left(4 \sum_{j=1}^{3} a_j^{-2}\right) \mu + \frac{16}{\prod_{j=1}^{3} a_j}\right]^6.$$

Degree $n = 3$.

$$\chi_3(\mu) = \prod_{\kappa \in N_+} \left[\mu^2 - \left(\sum_{j=1}^{3} \frac{\kappa_j}{a_j} \right) \mu - 3 \left(\sum_{j=1}^{3} a_j^{-2} - \sum_{\substack{j,k=1 \\ j \neq k}}^{3} \frac{\kappa_j \kappa_k}{a_j a_k} \right) \right]^4 .$$

Degree $n = 4$.

$$\chi_4(\mu) = \left[\mu^5 - \left(20 \sum_{j=1}^{3} a_j^{-2} \right) \mu^3 + \left(\frac{80}{\prod_{j=1}^{3} a_j} \right) \mu^2 \right.$$
$$\left. + 64 \left(\sum_{j=1}^{3} a_j^{-4} + 2 \sum_{\substack{j,k=1 \\ j \neq k}}^{3} a_j^{-2} a_k^{-2} \right) \mu - 768 \, \frac{\sum_{j=1}^{3} a_j^{-2}}{\prod_{j=1}^{3} a_j} \right]^{10} .$$

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