WORDS, DEFINITIONS AND CONCEPTS IN DISCOURSES OF MATHEMATICS, TEACHING AND LEARNING

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PRELIMINARY THOUGHTS

The book *Mathematical Vocabulary* (DfES, 2000) represents the current official discourse of school mathematics in England, embodying the values, world view and practices that teachers are expected to adopt in their classrooms. The importance of language for children's learning is stated as the most important motivation for the publication and is presented as a simple and unquestionable fact. Thus:

> mathematical language is crucial to children's development of thinking. If children don't have the vocabulary to talk about division, or perimeters, or numerical difference, they cannot make progress in understanding these areas of mathematical knowledge. (p.1)

However, the only specific aspect of language identified is "vocabulary" – in fact, mathematical language appears to be identified with its vocabulary. The title of the book, its format (mainly consisting of lists of words) and the repeated emphasis on *vocabulary*, *terminology* and *words* (see the extract in Appendix 1 of the introductory article) construct an image of mathematical language as a collection of discrete terms. Although there are suggestions of language activities such as discussing, hypothesising, reading or writing instructions that hint at the complex functions of language in mathematics, these are presented only as "opportunities to develop [children's] mathematical vocabulary" (p.3) rather than as development of a more broadly conceptualised mathematical language.

In contrast to other kinds of language (described as *informal*), mathematical words are described as *technical* and *correct*. Teachers are exhorted to "explain their meanings" and to "sort out ambiguities or misconceptions" (p.2). The use of a mathematical dictionary is described as necessary in every classroom to be used by both children and teacher "to look up the meanings of words" (p.36). The relationship between word and meaning is thus constructed as one-to-one and as expressible in terms of other already known words. The meaning of the mathematical term appears to be identified with its dictionary definition and understanding of mathematical concepts is implicitly equated to understanding the words with which they are expressed.

The mathematical term under consideration in the classroom transcript (Appendix 2 of the introductory article) is *two-dimensional shape*. As advised by the NNS, I turned to a mathematical dictionary (Selkirk, 1990) and found three definitions for *dimension*, the one most relevant to this context being:

> the number of measures needed to give the place of any point in a given space, the number of coordinates needed to define a point in it. (p.170)

It seems unlikely that such a formal definition is accessible to Y5 children or very useful to their teacher. Moreover, even this definition is not entirely unambiguous, as the nature of the "given space" is left open. For example, the question of whether a circle is one-dimensional or two-dimensional (see turns 24-34) is not immediately resolvable.[i] This is not a weakness in the definition but a characteristic of the mathematical concept itself.

Most importantly, I question whether any definition can capture the richness of the mathematical thinking about dimensions that the children and teacher were engaged in during the lesson. Rather than producing an unambiguous meaning for this term, the talk of the children and the teacher constructs a multi-faceted notion of dimension. This includes:

- the idea of 2D as "flat" and 3D as "solid" (turns 7, 9);
- listing dimensions (breadth, height, etc.) invoking an implicit two-ness or three-ness (turns 6, 32);
- a notion of 3D involving something extra when compared to 2D (turn 9);
- the idea that "thickness" is characteristic of 3D (turn 41);
- diagrammatic representations of 2D (a square) and 3D (a 2D isometric drawing of a cube) (turn 9);
- imagining what might be meant by one- and even zero-dimensional objects (turns 14-19).

All of these aspects of the meanings of *two-dimensional* and *three-dimensional* seemed relevant, valid, and at some points, especially during the discussion of one- and zero-dimensions, mathematically sophisticated, though often incomplete or ambiguous (as in the listing of circumference, diameter and radius in identifying the dimensionality of a circle at turn 32). Yet at no point during the lesson did it seem possible or even appropriate to explain or to remove all ambiguities from the ways in which the words were being used or to establish a single 'correct' way of speaking and thinking about dimensionality.

An important aspect of the classroom dialogue in the extract is the implicit nature of the definition. An explicit definition of *two-dimensional shape* is never given, instead:

- some properties are named (flatness, width and length) – and properties a two-dimensional shape should not have (breadth, thickness);
- some examples (square) and non-examples (cube, line) are given;
- contrasts are constructed between two-dimensional shapes and shapes with other numbers of dimensions (e.g. turns 9, 41).

This perceived tension between the official discourse of the current mathematics curriculum and that of mathematical practice in a primary classroom prompted me to look more widely at the ways in which word-concept relationships are constructed in different mathematical practices. In what follows, I first review the role of definitions in mathematics itself, as discussed by mathematicians and mathematics educators. This is followed by an exploratory analysis of a small number of examples of definitions taken from published mathematics research papers and from school textbooks. The similarities and differences between these raise questions about relationships between school mathematics and the mathematics done by professional mathematicians (in universities and in industry) and about how the ways in which definitions are presented in school may affect students' access to higher mathematics.

THE ACADEMIC VIEW OF DEFINITION IN MATHEMATICS

The notion of *definition* has a privileged place in many mathematical practices, highlighted by the claim by mathematicians and mathematics educators that mathematical definitions are different from 'ordinary' definitions, as well as by its frequent association with terms such as *unambiguous*, *minimal* or *necessary and sufficient* that are highly valued in high status mathematical discourses.

Borasi, a mathematics educator who has undertaken research both at school level and with university mathematicians, lists the following "commonly accepted requirements for mathematical definitions":

> Precision in terminology. All the terms employed in the definition should have been previously defined, unless they are one of the few *undefined terms* assumed as a starting point in the axiomatic system one is working with. [ii]
>
> Isolation of the concept. All instances of the concept must meet all the requirements stated in its definition, while a non-instance will not satisfy at least one of them.
>
> Essentiality. Only terms and properties that are strictly necessary to distinguish the concept in question from others should be explicitly mentioned in the definition.
>
> Non-contradiction. All the properties stated in the definition should be able to coexist.
>
> Non-circularity. The definition should not use the term it is trying to define.
>
> (Borasi, 1992, pp.17-18)

In commenting on these requirements, she makes use of two criteria for justifying them. A definition of a given mathematical concept should:

> 1. Allow us to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency (by simply checking whether a potential candidate satisfies *all* the properties stated in the definition).

and should:

> 2. "Capture" and synthesise the mathematical essence of the concept (*all* the properties belonging to the concept should be logically derivable from those included in its definition).

The requirements listed do not seem peculiar to mathematics (apart perhaps from the formal acknowledgement of the role of undefined terms), though they may be applied rather more rigorously than in other domains. Borasi's criteria, however, hint at a role for definitions within mathematical practice that goes beyond both the record of usage of standard dictionaries and the technical taxonomising of common-sense phenomena identified by Wignell (1998) in the practices of natural and social sciences. Definitions in mathematics form a basis for logical derivation not only of those properties already known (perhaps in a common-sense way) to belong to the concept but also of new properties.

The notion that mathematics may be generated from definitions by logical deduction is strongly embedded in traditional methods of teaching mathematics at the university. Often characterised as *definition, theorem, proof*[iii], much exposition of mathematics to undergraduates has taken the form of the presentation of logical sequences of deduction from definitions, though this approach to teaching has been widely criticised by mathematics educators and by students themselves for its failure to help students to develop the concepts involved or, indeed, to learn how to derive proofs themselves (see, for example, Anderson et al., 2000; Burn, 2002). The definition, theorem, proof format is also strongly represented in published mathematics research reports. As Burn points out, however, this may not always represent the way that research mathematicians actually go about doing mathematics.

> The research mathematician may come to his results starting from special cases, which will appear as corollaries in the final version, from which he gets his ideas, which is worked with until he has a proof. Then the theorem is what has been proved. At this point he formulates his definitions so as to make the theorem and proof as neat as possible. (Burn, 2002, p.30)

During the early stages, the concepts the mathematician works with may thus not be formally defined but more or less intuitive, derived from special cases – concept images rather than concept definitions, to use Tall & Vinner's (1981) distinction.[iv] The construction of the formal definition and the consequent creation of a technical term is a deliberate creative act, aiming not simply to describe or "capture" a pre-existing concept but to shape that concept in a way that lends itself to particular purposes. Of course, this definition may subsequently be used to generate deductive sequences leading to the discovery of further theorems.

A further characteristic of mathematical definitions is the possibility of multiple 'equivalent' definitions. I have used scare quotes for the term *equivalent* because, while two definitions may identify the same object, it is questionable whether they necessarily correspond to the same concept and they certainly lead to different forms of mathematical activity. Borasi gives the example of alternative definitions of a circle: the *metric* definition (focusing on the idea that all points on a circle are equidistant from a given centre), generally used at early stages of school mathematics, and the *analytic* definition (expressed in the form of an equation such as $(x-a)^2+(x-b)^2=r^2$), encountered by students at Advanced level. Either definition can be used to solve a problem such as "Find the circle passing through three given points" but the choice of definition makes a significant difference to the process of solution (Borasi, 1992, p.19).

Characteristics of the use of definitions in mathematics thus include:

- There exists a possibility of conflict with intuitive images of the concept being defined, especially with images formed by generalising from examples.
- Definitions form a generative basis for logical deduction, not only of known properties of the concept but of new properties.
- Definitions may be created deliberately in particular forms in order to facilitate the construction of theorems and proofs.
- A single object may be defined in several logically but not conceptually equivalent ways and such alternative definitions facilitate the generation of different types of theorems, proofs and solution methods.

These characteristics contribute to a relationship between definition and concept that appears dynamic and open to manipulation and decision making by mathematicians. This contrasts sharply with the static word-concept relationship constructed by the NNS advice.

ANALYSIS OF DEFINITIONS IN MATHEMATICAL TEXTS

In this section, I shall examine, compare and contrast the roles that definitions play in different mathematical practices applying a critical discourse analytic approach (Fairclough, 1992) to a small number of written texts. This analysis allows us to identify epistemological differences between discourses, variations in the ways in which the activity of the human mathematician is represented in relation to definitions, and tensions between the various discursive resources that teachers and students may draw on as new mathematical language is introduced. The texts come from three sources: an article published in an academic research journal and two school textbooks aimed at slightly different populations of students. By focusing on written texts, I am looking at only one aspect of the practices in which the texts arose. I would argue, however, that because of the high status of written language and the extent of writing activities in those practices, the analyses will have high relevance. Research papers are often taken to represent the official discourse of mathematics because of their important role in regulating the academic mathematics community, although there is also variation among them (Burton & Morgan, 2000) and of course there are other forms of academic mathematics practice that involve very different kinds of texts. My intention is to compare and contrast the place of definitions and the way in which relationships between word, definition and concept are constructed in this official discourse of mathematics with their place in school mathematics practices and to consider the extent to which the texts that inform school students' experience of mathematics serve to apprentice them to academic mathematics practices.

Rather than examining further texts related to primary mathematics, I have chosen textbooks designed for students in Key Stage 4 (aged 15-16). These represent the endpoint of mathematics education for many students and a transition to more advanced and specialist study for others. They may thus be seen to represent an eventual target towards which the Year 5 pupils in the classroom transcript and other primary pupils whose mathematical experience is shaped by the NNS are aiming. This provides a basis for considering the ways in which the approach to mathematical language recommended by the NNS provides an adequate and meaningful preparation for participation in more advanced mathematical practices.

The analysis uses tools drawn from systemic functional grammar (Halliday, 1985) selected to illuminate the ways in which the nature of mathematics and mathematical activity may be constructed through the texts presented to students. These are outlined in Table 1, identifying the questions used to interrogate a text and the grammatical tools that operationalise the resulting description. The first two questions in the table are related to Halliday's (1973) ideational function of language, concerned with the nature of our experience of the world, the next two to the interpersonal function, concerned with the identities of the participants and relationships between them, and the final question to the textual function, concerned with the way the text itself becomes a "living message". The description thus constructed allows us to address critical questions about how the text may contribute to possible readers' positioning in relation to mathematics and mathematical activity, asking in particular: *What is the nature of mathematics/ mathematical objects/ mathematical activity?* (using the first two questions in Table 1) and *Where do power and authority lie?* (using the second two questions) as well as specifically considering the role of definitions in the text and, by extension, in the practice. In the cases that follow, I do not present full grammatical descriptions but use the questions and tools outlined in Table 1 to highlight selected aspects that contribute significantly to addressing these critical questions and allow us to see most clearly the differences between the various texts. A fuller discussion of applications of this approach in mathematics education research may be found in (Morgan, 1996, 1998).

Table 1: Analytic Tools.

Descriptive questions:	Grammatical tools:
Who or what are the actors and where does agency lie?	What objects and humans are present? How are active or passive voice used?
What are the processes?	Relational, material, mental/behavioural?
What are the roles of the author and reader and what is the relationship between them?	How are personal pronouns used? In what kinds of processes are author and reader actors?
Describe the modality.	Modal verbs, adverbs, adjectives
How is the status of 'definition' established textually?	Given/New structures[v]; how cohesion is achieved.

Definitions in a research paper

In two extracts from the same mathematics research paper[vi], published in a standard academic journal, we can see a break from the orthodox expectation of a one-to-one relationship between concept and definition and the construction of definition as a (possibly contestable) product of human endeavour. The first extract comes from the introductory section of the paper.

Extract 1

> In the first section of the paper we give a somewhat non-standard definition of the Hecke algebra as a subquotient of the group algebra, which is easily seen to be equivalent to the usual definitions. This viewpoint makes the actions of the Hecke algebra on cohomology more or less transparent (see Lemma 1.1), as well as being adapted for our intended applications (e.g., Lemma 5.1).[vii]

Extract 2 is taken from a later section of the same paper in which findings and the reasoning leading to them is presented using the definition, theorem, proof format discussed above.

Extract 2

> We recall the definition of a *G*-functor (Green [6]). …
>
> **Definition**. A *G*-functor *F* = (*F*, *R*, *I*, *C*) over *k* consists of a *k*-module *F*(*H*) corresponding to each subgroup H of G and the following operations: […]
>
> Satisfying the following axioms […]
>
> **Definition.** A *G*-functor is said to be *cohomological* if it satisfies
>
> (C) $I_H^K R_H^K(x) = [K : H]x \text{ whenever } H \leq K, x \in F(K)$

An analysis of the two extracts, structured by the questions identified in Table 1 above, is presented in Table 2 and is discussed below.

Table 2: Analysis of extracts of a mathematics research paper

	Extract 1	Extract 2
Actors & Agency	*we give a … definition* – human agency is explicit at first, though the definition then *is easily seen* – the passive voice obscuring agency. *This viewpoint* is presented as an actor in its own right, completing a shift from active human agency to metaphorical agency of the definition itself.	Human agency is present in the first sentence, recalling the definition, but is obscured by use of the passive voice *is said to be* in the second definition. The citation of the mathematician *Green* may be considered to ascribe agency to him as originator of the definition.
Processes	Mental process *see* would normally	Mental process *we recall*.

	require a sentient agent but here is in the passive voice. Material process *make transparent* is performed by the abstract *viewpoint*.	Behavioural process *is said*, here in the passive voice. Relational *A G-functor … consists of* [a collection of its parts]
Author & Reader	The (single) author uses *we* in a way that cannot include his reader as it refers to his act of writing the definition. This is a widely, though not universally, observed convention in mathematics research papers (see Burton & Morgan, 2000). The statement that the definition is adapted for *our intended applications* establishes the author's ownership of the material presented in the paper.	*We recall* in this case may be read as an inclusive use of *we* orienting the reader to knowledge that is available to them as members of the academic community.
Modality	The suggestion that the definition *is easily seen to be equivalent* may serve to assert authority over the reader. If an individual reader cannot see, it must arise from their own inadequacy. Modifications *somewhat non-standard* and *more or less*, on the other hand, reduce the strength of the claims in this section.	The modality throughout is absolute. At this point in the paper, definitions are not open to question.
Textual status of *definition*	As might be expected in an introductory paragraph, the thematic structure of the first sentence orients the reader to the organisation of the paper. Subsequently, the *viewpoint* or definition is itself positioned thematically.	In the first definition, the word *G-functor* is given and the description of its properties is the new information, providing the properties that distinguish this concept from others. In the second, this order is reversed. A given object *is said to be cohomological* – a pre-existing concept is given a new name,

		although the order here is not consistent as the properties that allow the new name to be used are listed afterwards. Bold headings mark definitions as important. These and the label (C) will be referred back to in the proofs that follow later in the paper.

The claims of the paper depend on the idea, made explicit in Extract 1, that the same object (the Hecke algebra) can be defined in alternative ways. The modification *somewhat non-standard* implies that definitions are not unique but at the same time that there exist privileged definitions that are generally acknowledged/valued by the community. A standard definition is likely to be known (or at least readily accessible) to the expected reader of this paper. The modality of *somewhat* defers apologetically to the community values but this is tempered by the strong authority claim that the new definition can *easily* be seen to be equivalent. In this paragraph, therefore, the author is establishing his identity within the community, acknowledging the priority of established knowledge while claiming novelty, validity and utility for his own work.

The metaphor of alternative definition as a "viewpoint" is consistent with the discussion of multiple equivalent definitions above. A definition is not identical with the object but is a way of looking at an independently existing object. The choice of a particular definition is presented both in relation to general community values (transparency) and as a personal or contextual matter, related to "our intended applications". It is thus possible to make judgements about the value of definitions based not only on the global, structural criteria identified by Borasi but also on more local criteria related to the problem currently under consideration, in a way similar to that described by Burn (2002).

In Extract 2, the possibility of alternatives is not considered, though the citation ascribes ownership of the first definition, suggesting that the definition may not be commonly known but was an original product of the cited mathematician. The structure of the definition itself as an absolute statement of the constituent parts of the G-functor and of its necessary properties gives no hint of its origin. Did this object pre-exist its definition in some common-sense way or was it entirely a product of Green's imagination? Thus it is not clear whether this definition is an invention or a discovery; it is simply a statement of properties. The second definition, on the other hand, is presented as a human construction – or at least a human decision about how to name the kind of object described. Whatever the origin of the definitions, their structural importance in the mathematical arguments constructed in the paper is marked by the use of bold headings and labels.

The presentation of definitions in this research paper thus include features similar to those identified in the previous section:

- there can be different definitions of the same object;
- choices between definitions may be made on the basis of utility;
- definitions play an important role in the formation of mathematical argument.
- they are the product of human activity, though it is not always clear whether this is the construction of new objects or naming of pre-existing objects;

In addition, it appears that various definitions of the same object have different standings within the mathematical community and may need more or less justification by an author.

Definitions in school mathematics texts

The ways in which definitions appear in school mathematics texts vary significantly with the type of mathematics involved and with the age of the intended student-readers. At lower levels, in spite of the NNS advice, most new terms seem to be introduced by naming and by exemplification rather than by explanation or definition. Given the limited space available in this paper, it is not possible to review the different approaches in detail, so I have chosen to focus on two examples from Key Stage 4 (Years 10 and 11), the stage at which students are expected increasingly to engage in formal mathematical reasoning, including the use of definitions (DfEE, 1999). The two examples chosen as a starting point are taken from textbooks in the same series, written by the same authors, intended for students in Key Stage 4 preparing for GCSE examinations at Intermediate and at Higher level (public examinations taken at age 16+ at the end of compulsory education, set at different levels for students with different expected levels of attainment). Both examples present definitions of trigonometric concepts, though at different levels.[viii]

GCSE Intermediate Textbook

> In Investigation 15:1, you found that the ratio $\frac{\text{shortest side}}{\text{longest side}}$ i.e. $\frac{\text{opposite}}{\text{hypotenuse}}$ is the same for each of these triangles.
>
> This ratio is given a special name. It is called the sine of 40° or sine 40°.
>
> The ratio $\frac{\text{adjacent}}{\text{hypotenuse}}$ is called cosine 40°. The ratio $\frac{\text{opposite}}{\text{adjacent}}$ is called tangent 40°.
>
> The abbreviations sin, cos, tan are used for sine, cosine, tangent.
>
> The ratios sin A, cos A, tan A are called trigonometrical ratios, or trig. ratios.

GCSE Higher Textbook

> The ratios $\sin\theta$ and $\cos\theta$ may be defined in relation to the lengths of the sides of a right-angled triangle.
>
> $\sin\theta$ is defined as $\frac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
>
> $\cos\theta$ is defined as $\frac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
>
> Since $\theta < 90$, $\sin\theta$ and $\cos\theta$ defined in this way only have meaning for angles less than 90°.
>
> We will now look at an alternative definition for $\sin\theta$ and $\cos\theta$ which has meaning for angles of any size. (…) This gives the following alternative definition for the ratios $\cos\theta$ and $\sin\theta$.
>
> The ratios $\cos\theta$ and $\sin\theta$ may be defined as the coordinates of a point P where OP makes an angle of θ with the positive x-axis and is of length 1. Defined in this way, the ratios $\cos\theta$ and $\sin\theta$ have meaning for angles of any size.

An analysis is given in Table 3, laid out to facilitate comparison of the two texts.

Table 3: Analysis of GCSE Texts

	GCSE Intermediate	GCSE Higher
Actors & Agency	The ratio *is given a special name*, *is called* and *the abbreviations … are used* – passive voice obscures agency	*The ratios … may be defined* – passive voice Sin and cos *have meaning*
Processes	Material process *found* by student Behavioural processes *call*, *use* which would normally require a sentient agent but here are in the passive voice	Behavioural processes *define* and *look* Relational (intensive) *have* [*meaning*]
Author & Reader	*You found …* – student agent in practical activity	*We will look* – is this the authors or is it an inclusive *we*? In either case, there is some human agency here and it is possible to read this as an expression of solidarity.
Modality	Generally neutral i.e. absolute modality (these are given facts – no questions asked) The ratio is given *a special name* – stressing the importance of the new vocabulary	Modification of verbs to reduce level of certainty – *may be defined*. This opens up the possibility of alternative ways of doing things - and the possibility that the student might be able to make choices. Similar adverbial and adjectival modifications: *defined in this way*, *an alternative definition*
Textual status of *definition*	All sentences except the first have unmarked word order: the ratio (found by the student) is the given knowledge; the mathematical terminology is the new. Move from a specific example of a concrete object (the ratio of opposite to hypotenuse in a 40° triangle) to giving a	In the final sentence, word order is marked by positioning the adverbial phrase *defined in this way* in the 'given knowledge' position. The form of the definition is presented as changing the meaning of the object – thus definition precedes object/concept *Since* $\theta < 90$ in a thematic position

	name to this object and to extending this naming to general similar objects – thus the object/concept pre-exists the naming of it Cohesion achieved by repetition of *the ratio* and its cognates in the thematic position, presenting the text as a collection of facts about *the ratio* – description.	presents the text as a process of logical argument.

Some significant differences between the two texts are apparent from this analysis. Considering the nature of mathematics and mathematical activity in the context of definition, in both texts agency in the act of naming or defining is obscured by use of the passive voice but the types of activity in which human actors are agents are different. In the Intermediate text, the student herself is presented as having been involved in an earlier practical activity. In the Higher text, there is no practical activity but *we* are engaged in the intellectual activity of looking at an alternative definition. The forms of the two texts themselves also contribute to differences in the type of activity that is constructed as mathematical. The Intermediate text is essentially descriptive, starting with what is known about a specific concrete example and extending the description to naming a more general set of similar objects. The object/concept of the ratio between two sides of a triangle is established as the outcome of practical activity before it is named. This order is reversed in the Higher text: the choice of an alternative definition changes the nature of the object being defined. This text also uses structures that highlight the formation of a logical argument – an aspect of mathematical activity absent from the Intermediate extract.

The second major difference arises from the modality of the two texts. While the Intermediate text lays down a set of absolute and unquestionable facts to be accepted by the student-reader, the Higher text allows uncertainty and alternatives, opening up the possibility that the student-reader herself might choose between the two definitions. The student entered for the Higher level examination is thus constructed as a potential initiate into the practices of creative and purposeful definition that academic mathematicians engage in.

DISCUSSION

The examples analysed here were not selected in a systematic way so it would be inappropriate to draw firm conclusions about differences between various types of texts. The analysis does, however, raise some questions and hypotheses about the ways that definitions are presented at different levels and the roles that they play in different mathematical practices. The extracts from the research paper confirm the characteristics of mathematical definition identified in the literature on advanced mathematical thinking, in particular their role in argument and the possibility of purposeful choice between alternative definitions of the same object. These characteristics may also be seen in the extract from the textbook for Higher level students. The definitions encountered by the Intermediate level students involve naming and formalising a pre-existing concept, playing a role much closer to that characteristic of definitions in natural and social sciences (Wignell, 1998). Similar differential access to mathematical practices is identified by Dowling (1998) in his analysis of a differentiated textbook scheme. In that case, the ‘lower’ students were constructed as engaged in ‘everyday’ practices and were denied access to esoteric mathematical practices.

The NNS characterisation of the relationship between mathematical vocabulary, meaning and understanding contains no hint of the logical, generative or creative aspects of mathematical definition. Of course, it is necessary to consider whether this reflects essential differences between different types of mathematical concept met at different stages of learning mathematics. Perhaps only the concepts encountered at more advanced levels lend themselves to these forms of mathematical activity? A counter-example to this suggestion, familiar to many teachers of mathematics at both primary and secondary levels, is the concept of *rectangle* and the debate that arises in classrooms about whether a square is a rectangle (resulting, perhaps, from trying to find the rectangle with the largest area for a given perimeter). Resolving this debate involves making a choice about the precise definition of rectangle to be used and then engaging in logical argument based on the chosen definition. It also seems to involve the sort of conflict between concept image and concept definition identified by researchers in advanced mathematical thinking (e.g., Tall & Vinner, 1981).

Returning to the primary classroom transcript, here too we can see younger students involved in forms of mathematical practice that go beyond those suggested by the NNS booklet. While no formal definition of dimension is given in the lesson, the discussion towards the end of the transcript shows participants using their implicit definitions to form arguments about whether particular shapes (and even whole classes of potentially constructible shapes) fulfil the necessary conditions to be classified as two-dimensional (turns 41-56). As we have seen, this form of reasoning is one of the important ways in which definitions are used in mathematics and plays a significant role in mathematical reasoning and proof. If formalised, it could match the conventional order of presentation of definition, theorem, proof found in academic papers. At the same time, the ambiguity and multiplicity of meanings at play in this classroom provide a setting for argumentation that seems likely to contribute more to the children's developing understanding of the concepts involved – and of mathematical activity itself – than any "clear explanation" could.

Another interesting – and mathematical – aspect of the transcript is the generative activity. Starting from the children's conceptions of two and three dimensions, derived at least in part from experience with concrete objects, new, increasingly abstract, objects are conceived with one dimension and no dimensions. Again, the adaptation and extension of definitions into new domains is an important way in which new mathematics is created. These children, like those using the Higher level GCSE text, are being inducted into creative mathematical practices. The one-to-one word-to-meaning relationship apparent in the NNS official discourse thus seems neither to reflect the way in which mathematical words and meanings are related in practice nor to provide any inkling of the powerful and productive role that definitions can play in mathematics.

The image of vocabulary and concept development presented by the NNS is thus both restrictive in the model it presents of language use and inadequate to describe what actually happens in classrooms. The differences identified between the treatment of definition of trigonometric ratios in Higher and Intermediate textbooks also suggest that opportunities to experience some characteristically mathematical aspects of the use of definitions may be being restricted for many students. There is a need to look more thoroughly and critically at the ways in which concepts and vocabulary are introduced to younger and less advanced students in classroom practice and in texts and curriculum guidance and to consider alternatives that may serve to introduce them to powerful forms of mathematical thinking.

Equally the emphasis put on vocabulary by the NNS presents a restricted image of the nature of mathematical language itself. The analysis of mathematical texts that I have offered in this article, by considering the grammatical construction of mathematical meanings and of relationships between the authors and readers mathematical texts, also demonstrates some of the ways in which mathematical language consists of more than just specialist vocabulary. Learning to engage in mathematical discourse thus involves learning more than definitions of mathematical words. Taking definitions as just one example of a type of mathematical text, their formation and their incorporation into mathematical arguments are fundamental mathematical activities that take place in language. Induction into mathematical practices must involve students in developing ways of speaking and writing that enable them to engage in these activities. The importance of mathematical language has been recognised by the NNS and current teaching practice at both primary and secondary levels in English schools now involves considerable effort to incorporate 'key words' into lesson plans and into the classroom environment. While I am not convinced that official endorsement is enough to change classroom practice for the better, recognition of the broader nature of mathematical language and what may be done with it might create opportunities for teachers and students to develop greater awareness of the ways in which they can use language to do mathematics.

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NOTES

[i] If the circle is seen as a part of a more substantial space such as a plane then two measures will be needed to define a point on it. However, if the circle itself constitutes the entire space, only one measure is needed, i.e. the distance from a fixed point on the circle in a positive direction around the circumference.

[ii] In this context, *undefined term* does not mean a non-scientific or 'everyday' term. It refers to the basic objects of an abstract mathematical system.

[iii] In this form of presentation, one or more definitions are given, the theorem to be proved is stated, then an argument is made showing how the definitions (together with other theorems that have already been established in this way) logically imply the theorem.

[iv] A concept image is a more or less intuitive concept, generally derived from experience of a number of examples and from analogies with informal or everyday concepts, visual images and language. The corresponding concept definition does not simply describe a concept image but may even conflict with it. A typical example of such conflict arises with the concept of the *limit* of a sequence. For many students, their concept image includes the notion that a limit is never reached and that subsequent terms of a given sequence will get closer and closer to its limit. The concept definition, however, is formulated to include constant sequences, all of whose terms are equal to the limit of the sequence.

[v] Halliday notes that, grammatically, definitions may be set up "facing both ways", using constructions: "*a* is defined as *x*" or "*x* is called *a*" (1993, p.73). In the first of these constructions, the word itself is 'given' while the description of the concept is presented as 'new'; this order is consistent with a creative function of definition. The act of defining brings the mathematical concept into being and provides it formally with the properties that distinguish it from other concepts. The second construction, by contrast, suggests that the concept pre-exists its definition; by being given a technical name, a 'common sense' or more intuitive technical concept is "translated" into specialised knowledge (Martin, 1993, p.209).

[vi] The details of the sample of mathematics research papers from which this paper is taken are given in (Burton & Morgan, 2000).

[vii] I will not attempt to clarify the meaning of this passage for the non-mathematician reader. It may be helpful, however, to recognise that the term *algebra* as used here does not refer to the

kind of manipulation of letters experienced within school mathematics. It refers to an abstract structure consisting of elements that may be combined in specified ways. An algebra is thus an object rather than a field of activity.

[viii] The textbook, of course, is not the only source of definition for students. In most classrooms, the text is likely to be mediated by the teacher and this will affect the ways in which students interact with the text themselves. As students construct their understandings of the nature of mathematics and mathematical activity and of their own identities in relation to mathematics they will draw to different extents on the textbook, the teacher's speech and actions and on their previous experiences. However, where teachers are insecure in their own subject knowledge they are likely to rely heavily on the forms of definition and argumentation that are provided for them in published resources. Haggarty and Pepin (2002) note that in England, while students themselves make relatively little use of textbooks, their teachers use them extensively in planning lessons. Textbooks thus have a strong influence, whether direct or indirect, on students' experience of mathematics.