Measurement and measurement-related competences of five to eight-year-old children in a British primary school

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Abstract

Measuring with units requires underlying conceptual competences as well as practical and numerical abilities. The social context, notably in relation to language, undoubtedly plays its part in their acquisition. Existing research had investigated conceptual, practical, numerical and social aspects of the acquisition of measurement, but rarely together, or by the same group of children. The present research united both in a conceptually coherent structure, enabling a picture to be presented that was both broad and detailed. There was a specific focus on conceptual difficulties children may face that are identified in the psychological and educational literature.

Eighty-three five- to eight-year-old children were interviewed about their knowledge of measurement and participated in a comprehensive set of tasks designed to test their understanding of its language and concepts, their accuracy in making visual estimates, and their ability to measure length.

Results showed the children to have a lively appreciation of the importance of measurement, good understanding of its everyday language and concepts, and good ability to estimate length. Yet they were poor measurers. This unevenness in their accomplishments indicated underlying conceptual insecurity that was manifested in ineffective deployment of measurement instruments, but went beyond it. There was some evidence that ability in language and estimation were associated for the younger children, while estimation and measurement ability were associated for the older. An agenda for further investigation of the disjunctions identified by this research was outlined.

Declaration:

The work presented in this thesis is my own work

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Whoever limiting his worldly ambition finds satisfaction in the speculative life, has in the approval of an enlightened and competent judge a powerful incentive to labours.
Emmanuel Kant, 1781, Dedication of the Critique of Pure Reason.

Nobody thinks clearly, no matter what they pretend. Thinking's a dizzy business, a matter of catching as many of those foggy glimpses as you can and fitting them together the best you can.
Dashiell Hammet, 1929, The Dain Curse.

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Chapter 1

General introduction and review of the literature

1.1 Introduction

Measurement is inseparable from activities that between them support the bulk of human achievement. Agriculture, industry and trade, as well as pure and applied science, from molecular biology to astronomy, and from medicine to engineering all incorporate measurement. Across cultures, education systems recognize the central importance of measurement by prescribing it to be taught at an early stage of school mathematics curricula. However, learning to measure is not straightforward. The present research investigates the experience of measurement, the understanding of its language and concepts, and the estimation and measuring ability of primary-school-aged children.

Among measurement domains, length has a prototypical role. On scaled measuring instruments, for example, measurement is embodied as length irrespective of domain. Brown, Blondel, Simon & Black (1995: 145) argue that "length has a pre-eminent position among measures" because it matches the linearity of the number system itself. This prototypical character of length measurement recommended it as the central focus in the present research.

While length (including height and width) was the central focus, other measurement domains, such as weight and capacity, received a minor focus for the following reasons. Firstly, fundamental underpinning concepts are common to all measurement domains, and it was useful to test the generality of children's understanding of these concepts. Secondly, everyday experience of measurement spans various domains, and understanding of the breadth of children's knowledge was sought. Lastly, it was important to know whether children distinguished attributes specific to length from those appropriate to other measurement domains.

Numerical and non-numerical measurement

In the mathematics literature, measurement has not always been considered exclusively or even primarily numerical. Nagel (1930) cites pre-Cartesian geometry and arts such as cookery as domains in which "disciplined judgments" (Nagel, 1930: 313) may be made about quantities without the use of number. He conceptualises measurement very broadly, ranging from classification (because it requires evaluative comparisons between objects) to the scaling of extensive properties of matter. Here, "The raison d'être of numbers in measurement is the elimination of ambiguity in classification, and the achievement of uniformity in practice" (Nagel, 1930: 314). In contrast, Bergmann & Spence (1944) construe measurement more narrowly in terms of the "basic observations and measurements of the scientist". In their view, "physical measurement consists of the assignment of numbers to the objects or events of a physical dimension in accordance with certain rules" (1944: 1-2). Here, measurement is seen as a specifically numerical way of recording observed facts. This recalls the treatment of measurement in educational surveys: "an understanding of ... the use of numbers and measures to describe and compare mathematical and real-world objects" (IES National Centre for Statistics: National Assessment of Educational Progress, 2003) and in curricula such as the National Numeracy Strategy (Department for Education and Employment, 1999).

In the developmental literature, both aspects of measurement find their place. In Piaget's seminal account, for example, numerical and non-numerical measurement share underlying principles that are acquired before the ability to calculate numbers of standard units. (Piaget, Inhelder & Szeminska, 1960). Non-numerical as well as numerical measurement is therefore investigated in the present research.

Among aspects of length considered in the present research, units of length are the conceptual centre. The standard unit is the conventional expression of the conjunction of the numerical with the logical in measurement. "With a stable idea that a given length can be generated through unit iteration, an object being measured and the measuring device need not even look alike, but need only share a common internal unit" (Youniss, 1975: 235). While such a statement conveys the logical simplicity of the idea of the standard unit, the present research explores what psychological difficulties there might be in establishing this "stable idea".

Socio-cultural influences on the development of measurement concepts (Resnick & Singer, 1993) are also a strong theme in the research, in which children's experiences of measurement at home and at school, as well as characteristics of the everyday language of measurement, are examined.

Survey evidence of children's difficulties with measurement

National and international surveys of mathematics achievement, using samples of up to 10,000 or more, suggest that measurement skills, knowledge and understanding of up to one-third of seven to 16-year-old students are limited (Hart, 2004; Thompson & Preston, 2004; Brown *et al.*, 1995; Kouba, Brown, Carpenter, Lindquist, Silver &

Swafford, 1988; Hiebert, 1981). In particular, a surprisingly large proportion seem to lack fundamental measurement concepts that developmental research suggests are established, in some contexts, well in advance of secondary school age, such as conservation of length, transitivity, and measurement as iteration of a unit.

For example, asked which was the longer of two parallel lines whose ends were offset in the Piagetian paradigm and presented on squared paper, the responses of over ten percent of a large sample of students did not reflect the number of squares covered by each line, and suggested that they did not conserve length in this task (Hart, 2004). Similarly (in a smaller-scale study) a quarter of 11 to 15-year-olds failed to mark the start point of a clockwork toy when measuring the distance it travelled (Department of Education and Science, 1981). In a version of Piaget's 'towers' task one-third even of 10-to-13-year-olds failed to show understanding of transitivity (Brown *et al.*, 1995); and a minority in this small study did not iterate a 30-cm ruler when asked to measure a length in excess of 30 cm. Failure to apply the principle of the inverse relation between size and number of units occurred in some tasks (Brown *et al.*, 1995; Kouba, 1988). In another survey (Hiebert, 1981), difficulties in understanding that units can be subdivided were suggested by tasks where students apparently ignored the fact that one item, such as a 'stick' person, stood for several, or that one division stood for two degrees on a thermometer scale, and where part squares were counted as wholes, or ignored, in calculations of area. Difficulty in interpreting numbers in relation to quantity was exemplified by children who counted lines or spaces on a ruler rather than using the numbers (Brown *et al.*, 1995). There are many examples of poor estimation and inaccuracy of measurement (Hart, 2004; Hiebert, 1981).

While limited knowledge of standard units and their relationships, and limited ability to use measurement instruments (found in most surveys) seem readily remediable in school, failure to conserve length, to use an intermediate measure (showing understanding of transitivity) or to know that the smaller the unit used, the more of them are needed to measure a given quantity must, if interpretations of the results are valid, be a real cause for concern.

Such surveys are not without their critics (Silver & Kenney, 1993; Kamii & Lewis, 1991), and it is likely that some of the tests employed were not the best instruments for assessing mathematical understanding. Nevertheless, survey results in conjunction with the developmental literature can usefully indicate underlying conceptual difficulties in measurement, and can suggest useful smaller-scale in-depth investigations, of which the present research is an example. For instance, in addressing Research Question 6: *How well do children measure?* (see page 77) the present work investigated children's ability to conserve length when measuring, and it sought also to assess their understanding of transitivity as applied to measurement by observing their deployment of measuring instruments.

Measurement in the Primary Mathematics Framework

Mathematics in the UK National Curriculum for the first years of primary education has, for each year-group, expected learning outcomes for measurement. These are indicated in the *Primary Mathematics Framework* (PMF) (Department of Education, 2010; Department for Children, Schools and Families, 2006), are framed as the outcomes of problem-solving 'enquiries', and are typically integrated with other mathematics. As with the earlier National Numeracy Strategy (Department of Education and Skills, 1999), the objectives to which measurement activities contribute sometimes fail to acknowledge specific understanding required for the measurement component. Taking as examples the three school years from which children are drawn for the present research: one PMF objective for Year 1 is to "count reliably at least 20 objects, recognising that when rearranged the number of objects stays the same; estimate a number of objects that can be checked by counting". One corresponding learning outcome is: *I can find out how long a room is by counting the paces I take to cross it.* Here the specific understanding and potential for error involved in iterating a unit are unacknowledged; measurement is presented as a counting activity, perhaps preceded by estimation (Department for Children, Schools and Families, 2006: Year 1 Block D. Assessment focus: Ma3, Measures). The present research pursued the potential difficulty, overlooked here, of mapping number on to length, asking, in Research Question 5, *How well do children understand that a number may express length?*

In Year 2 an objective that specifically concerns measurement instruments runs: "Read the numbered divisions on a scale and interpret the divisions between them (e.g. on a scale from 0 to 25 with intervals of 1 shown but only the divisions 0, 5, 10, 15 and 20 numbered); use a ruler to draw and measure lines to the nearest centimetre". The corresponding "children's learning outcomes" are: *I can use a ruler or metre stick to measure how long something is. I can read numbers on a scale and work out the numbers between them.* (Year 2 Block C. Assessment focus: Ma 3, Measures). Conceptual understanding involved in reading and interpreting a scale in measurement is, if not taken for granted, at least unacknowledged here. To investigate this understanding, the present research asked, in Research Question 9b), *Do children understand that larger units may 'contain' smaller ones?* and in Research Question 10, *How well do children cope with fractional units?*

Choice of appropriate units and relationships between units are the main themes for Year 3: the latter was a major stumbling-block in at least one survey (Hart, 2004). The documents associated with the forthcoming new National Curriculum do not suggest there will be major changes (Department of Education, 2011). In asking: *Do children understand that there is an inverse relation between size and number of units?* (Research Question 9a) the present research identified perhaps the most fundamental relationship between units.

The concerns arising from the surveys suggest that the kinds of curricular objectives outlined above will not be achieved by children without conceptual understanding. The project *Supporting Students' Development of Measuring Conceptions: Analysing Students' Learning in Social Context*, undertaken by the National Council of Teachers of Mathematics (Gravemeijer, Bowers & Stephan, 2003) provides a research paradigm in this respect. The project had a solid base in developmental and semiotic theory. Like the PMF, it took classroom 'enquiry' as the pedagogic paradigm, and it systematically designed and tested a process by which children began to construct a principled understanding of units, using versions of the ruler as both instrument and concept.

While psychological studies specifically of measurement are few, concepts underlying measurement are powerful, apply well beyond the measurement domain, and have been thoroughly investigated in the developmental literature. The range of literature of relevance to measurement is therefore very broad. Selected for review are a number of substantial studies representing themes that are central to measurement and that suggested the investigations that follow. These themes are summarised at the end of the literature review, and are followed by twelve research questions to whose formulation they contributed. The research questions are set out on page 76. Throughout Chapter 1, the research questions are indicated (as they are above) wherever surveys, mathematical curricula, or research literature demonstrate the need to address them.

1.2 Review of the literature

This research investigates measurement-related skills, knowledge and understanding of children in the three years of English primary school that follow the reception year. The literature reviewed begins with the pioneering work of Piaget, and then focuses on the developmental and mathematics education literature that followed it.

First, the Piagetian framework in relation to measurement is described. Conceptual components of Piaget's framework, the methodology used to investigate their development, and the overall developmental structure to which they belong are evaluated. This framework provided the source of two lines of enquiry distinguishable in later work and pursued in the present research.

The first concerns non-numerical aspects of measurement, including underlying logical components such as transitivity. Non-numerical aspects of Piaget's work on measurement that retained value for researchers are discussed, and some theoretical developments from these. Developments that oppose the Piagetian account are also considered.

The second line of enquiry concerns numerical measurement, that is, measurement involving units. Studies of the development of numerical measurement are next considered, including research on children's ratio reasoning, since ratio reasoning is essential to understanding units.

Finally, the rationale for the present research is set out.

1.2.1 The Piagetian framework: fundamental conceptual components of measurement

Still the most substantial and important treatment of the development of measurement concepts, Piaget's account (Piaget, 1970; Piaget *et al.*, 1960; Piaget & Inhelder, 1954; 1956) provides an underpinning spatial and logical framework set in a context of broad intellectual development. His view of the essential components of measurement has continued to be used in the education literature, in the assessment of measurement competence (Hart, 2004; Brown *et al.*, 1995), to guide the teaching of measurement (Kamii & Clark, 1997; Steffe, 1971) and as underpinning for measurement (as well as other topics) in mathematics curricula (Department for Education and Employment, 1999; Williams & Shuard, 1982). The Piagetian framework is therefore reviewed in some detail.

In *The Child's Conception of Geometry* Piaget defines measurement as follows:

> To measure is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole: measurement is therefore a synthesis of subdivision and change of position. (Piaget *et al.*, 1960: 3)

While iteration of a unit is common to all numerical measurement, the "change of position" involved here defines measurement of length. The key logical prerequisite is a co-ordinated system of reference points in space and, for Piaget, mental representation of such a system is the key developmental prerequisite, so that "subdivision and change of position" can be successfully deployed.

This Euclidean conception of space presupposes a number of spatial and logico-mathematical concepts that are theoretically separable, although they sometimes overlap when operationalised in Piaget's studies. The spatial concepts are: object concept; a group considered here under the label 'uniformity of space'; and conservation of length. The logico-mathematical concepts (more specific to measurement) are: continuity and subdivision; ordination; transitivity; and subdivision and change of position. Together with the associated studies, these concepts are now outlined, broadly in order (according to Piaget) of logical complexity and of development. They are then evaluated in the light of later research. Finally Piaget's overall theoretical and methodological contribution to the study of measurement is summarised. The development of transitive inference is considered through Piaget's 'towers' study; fuller treatment of transitivity, which underpins the role of units in measurement, is reserved for later in the chapter.

1.2.1.1 Development of Euclidean spatial concepts

A conception of space that is stable because unified by co-ordinates affording fixed locations - a Euclidean conception of space - is identified by Piaget as the overarching precondition for measurement, which requires that objects are *moved* (whether set side by side for direct comparison, or compared indirectly using a third object as a measuring instrument). Objects cannot be measured if their size may change when they move; a Euclidean framework guarantees constancy of size. Piaget identified components of measurement logically implied by a Euclidean conception of space and argued that until seven or eight years of age, children cannot measure because their mental representations of space are not Euclidean but conform to the categories of topology.

Object concept

Euclidean space is independent of the perceptual frame of the individual. Objects seem to disappear when out of our line of vision, and to understand that they do not, we must conceive of space as independent of our own body (Piaget, 1954). This conception must be sustained despite the fact that we always perceive objects as relative to our own body, because always viewed from its standpoint.

Piaget noted (1954) that until about 9 months old, if a desired object was removed from an infant's visual field, there was no attempt to bring it into view. Older infants who searched successfully in the place where an object had been hidden still made the 'A not B error': if they first saw an object hidden in place A and subsequently in place B, they would persistently search at A. Piaget argued from such phenomena that until about eighteen months old, infants do not conceive of objects as moveable, independent entities in space. Even when this conception is achieved, objects are not entirely detached from their location, but continue to be associated with the position in which they were first placed, as a single entity.

> ...without being truly conceived as having several copies, the object may manifest itself to the child as assuming a limited number of distinct forms of a nature intermediate between unity and plurality. (Piaget, 1954: 63).

Without a notion of space as governed by fixed co-ordinates, it is argued, the infant cannot draw on such ideas as an object being *behind* a screen or *underneath* a pillow; hence when an object is occluded by another, it ceases to exist.

Subsequent research focused on infants tracking moving objects (e.g. Bower & Paterson, 1973; Bower, Broughton & Moore, 1971). Bower (1974) concluded that at issue was not object *permanence*, but object *identity*. Infants up to about five months old appeared to conceive an object occupying the same place, or continuing a path in the same direction, as the same object, despite several changes in appearance. Some transformations in appearance made in these experiments were transformations of size, substantiating Piaget's contention that for young children, objects may indeed change size when they move. However he did not consider perception of object boundaries, which also play a role (Spelke, 1990; de Schoenen and Bower, 1978; Neilson, 1977).

Investigation of the object concept continues apace: (Hespos & Baillargeon, 2001; Houdé, 2000; Aguiar & Baillargeon, 1999; Behr Moore & Meltzoff, 1999; Meltzoff & Moore, 1998; Wynn & Chiang, 1998; Munakata, McClelland, Johnson & Siegler, 1997). Piaget's overarching claim is actually substantiated by much of this later work: that is, the conception of space governed by fixed co-ordinates, required for measurement, is absent in infancy. His claim that this conception is not established till middle childhood, examined next, is not, however, similarly substantiated.

Conceptual development of uniformity of space

We need, if we are to measure, a mental representation of space as uniform, as a "common medium" (Piaget *et al.*, 1960: 70). We experience space, however, as differentiated – either as occupied by objects or merely as 'distance'.

Exploring children's conception of space, Piaget *et al.* (1960) placed two model trees or people about 50 cm apart on a table and asked whether they were close together or far apart. Until about seven years old, children changed their initial judgement of the distance between the two objects A and B when a screen or block was interposed between them, generally saying they were closer together. Distance between them was thought to be greater when the 'people' were of different heights, or on different levels. Distance was generally conserved, ignoring changes, at about seven years old.

Piaget had two interrelated explanations of these phenomena (Piaget *et al.*, 1960). First, space occupied by an object and 'unoccupied' space (distance) were different in kind for young children; hence following the interposition of a third object, the width of that object must be subtracted from the distance between the first two. The second concerned inability to represent linear order mentally as " a series of nesting intervals" (Piaget *et al.* 1960: 86)

> He reasons about AB if there is nothing between them, but if S is interposed, he thinks only on the new intervals AS and SB, while completely losing sight of the overall interval AB. (Piaget *et al.*, 1960:86)

By about seven years old, children are able to compare the length of an object with that of another, such as the edge of a table on which it lies (Piaget *et al.*, 1960). Now objects are conceived as contained by space and independent of it: whether 'filled' or 'empty', space is conceived as a homogeneous 'container' (Piaget & Inhelder, 1956).

If space is not apprehended as homogeneous by young children, who instead make a fundamental distinction between 'filled' and 'empty' space, then measurement of length as such will be impossible for them. A plausible alternative framework, however, that could account for findings such as Piaget's is provided by Gibson's (1979) theory of affordances. Gibson argued that perceptual information is processed functionally in terms of situational needs, and not in terms of separable, classifiable properties.

> If a terrestrial surface is nearly horizontal...nearly flat...sufficiently extended...and rigid...then the surface *affords support* for quadrupeds and bipeds....Note that the four properties listed - horizontal, flat, extended and rigid - would be *physical* properties of a surface if they were measured with the scales and standard units used in physics. As an affordance of support for a species of animal, however, they have to be measured *relative to the animal.* (Gibson, 1979: 127. Italics in the original.)

Such judgements of affordance can account, Gibson argued, for the posing and solving of measurement problems such as 'Will this stick fit across here?' without the use of number. Objects and the spaces available for occupation by them are perceived as a functional whole when a spatial judgement is demanded as a practical decision. Perceived changes in the affordances of a situation could underlie the type of response to interposed objects noted by Piaget and Inhelder. Gibson and Piaget agree, however, that early spatial judgements are perceptually bound, and made within an 'egocentric' framework

Topological spatial concepts

Until a fully Euclidean conception of space is established at about eight years old, Piaget argued, children's mental representations of objects in space conform to the categories of topology, a type of geometry that makes no use of size, distance and fixed position, but instead features relations of proximity, separation, enclosure and continuity, applied only to individual objects or shapes and their immediate relationships to each other. The claim is not that *perceptions* are topological in character (children easily navigate their spatial world as if Euclidean) but that their *mental representations* are. No a priori arguments are offered for this proposed difference.

To test the 'topological' thesis, Piaget and Inhelder (1956: 17-42) asked children to feel items behind a screen and then match them to real or pictured items they could see. They reasoned that to identify visually an item known only by touch, a match with some mental representation of the item must be made. (Thus, contradicting the rationale for the study, haptic *perception* of 'topological' shapes was taken as evidence of topological *mental representation*). Progressive differentiation of shapes with age was found, 'everyday' shapes being matched first, then 'topological' shapes, and lastly regular shapes. It is not clear, however, that specifically topological characteristics accounted for these differences, and there was no adult control group. Tasks closely resembling those of Piaget (e.g. Laurendeau & Pinard, 1970), claimed evidence of a similar progression, but investigations using different tasks have not found evidence that topological concepts consistently develop earlier (Kato, 1984; Martin, 1976b).

Horizontal and vertical axes; perspective-taking; constructing a straight line

A number of well-known studies were also offered as evidence of the late development of Euclidean spatial concepts. Investigating children's understanding of horizontal and vertical axes, Piaget and Inhelder (1956) asked children to describe or draw the level of the water in bottles tilted at various angles. Before about eight years old, children indicated various 'impossible' angles of the surface of the liquid to the container, in contrast to the plane surface they must often have observed. They drew objects or people on a hillside at right angles to the slope of the hill. Piaget and Inhelder argued that these children did not yet mentally represent a spatial frame of reference structured by horizontal and vertical axes, and their drawings reflected this. However, children's spontaneous drawings may not be good evidence for their mental representations. They may not intend their drawings to be fully representational (Freeman, 1980) and some young autistic children show early, skilful use of perspective in drawing (Wales, 1990).

In perspective-taking experiments (Piaget & Inhelder, 1956), children were shown a model village with a doll placed somewhere in it, then a duplicate village, rotated at some angle to the original. They were asked to place the doll correctly in the rotated model. Until seven or eight years old, children placed the doll more or less in accordance with their own perspective on the original model, showing, it was argued, inability to detach themselves from a spatial framework centred on the position of their own bodies and line of vision. Children's success on perspective-taking tasks, however, has been shown to depend on the type of task, with three to four-year-olds being successful with some (Spencer, Blades & Morsley, 1989).

Ability to construct a straight line (essential to Euclidean space with its fixed co-ordinates) is the final example considered. Consistently with his previous reasoning, Piaget distinguished between early ability to *recognize* a line as straight, and mental representation of a straight line. The task was to link "distant points by the interpolation of a series of points along a straight path" (Piaget & Inhelder, 1956: 155). Children were required to insert telegraph poles (matchsticks) along a straight highway with one pole already placed at each end. They also had to straighten curved or zigzag lines of 'poles'. Children under four years old were unable to construct the straight line requested, or to straighten a line. Between about four and seven years, they succeeded by making the line parallel to the straight edge of a table (showing physical capability of the task), but where this was not available, matchsticks were interpolated as close together as possible, the lines undulating. Finally, from about 7 years old, the children put themselves in a straight path with the two sticks, 'sighted' along the path and correctly interpolated the rest.

Laurendeau and Pinard (1970) replicated this task with children 2 to 10 years old. They found that even pre-schoolers could construct a straight row of objects parallel with a visible straight edge, but (like Piaget) not otherwise.

Piaget referred several times to children appearing not to know what the word 'straight' meant, but appeared to interpret this as symptomatic of their logical stage, rather than an independent factor in their performance. In many studies, in fact, Piaget left possible linguistic explanations of conceptual difficulties unexplored. The present research addressed this gap, investigating understanding of the language of length in some detail, asking, in Research Question 2, *Does the everyday language of length present any difficulties to children?* There was a special focus on language associated with the dimension of height, particularly in relation to the human figure. This was suggested by Piaget's study of distance disrupted by interposed blocks, described above.

Overall, the phenomena reviewed in the section above are not convincing as evidence of children's mental representation of space.

Development of geographical understanding

The literature on children's understanding and representation of their physical environment does not support Piaget's view of rather late development of the ability to use a Euclidean spatial framework, nor does it suggest a single frame of reference for all mature spatial reasoning. Rather, individuals' mental representations of spatial relationships seem to depend on their experience of spaces and on situational demands (Pick & Lockman, 1981). Hart (1981) found that pre-schoolers were able to make simple maps, with Euclidean characteristics, of the immediate environment of their homes and beyond. The character of these maps seemed to vary with the amount of experience children had had in moving around their locality. Darvizeh & Spencer, (1984) found that in route-learning studies using landmarks, three-year-olds were able to represent positions of objects in relation to each other along a route they had travelled, using symbols for buildings, trees and vehicles and connecting them with strips symbolising roads. Recent research shows four-year-olds spontaneously using angle and distance information from simple maps to locate real objects in three-dimensional space (Shusterman, Lee & Spelke 2008). Overall, evidence does not suggest late development of the conception of three-dimensional space, and this was not pursued in the present research. More specific pre-requisites for measurement are now considered.

1.2.1.2 Development of logical and mathematical concepts

Conservation of length

In the context of measurement of length, conservation of length within a Euclidean spatial framework is clearly essential, since "we cannot measure the length of one object without moving another, the length of which we know will not be altered by its movement." (Piaget *et al.*, 1960: 69). If we conserve length we also understand distance as reversible: the distance AB is equal to the distance BA, even though we may experience a given distance as 'longer' in one direction, for example because it is uphill (Piaget, 1970).

The principle tested in all Piaget's conservation studies is that quantities that are in fact equal retain their equality when appearances seem to challenge it. The studies were designed to investigate the extent to which children are able to ignore irrelevant aspects of the appearance of materials when judging quantity, or to take simultaneously into account more than one relevant perceptual aspect so as to arrive at a correct judgement.

Piaget *et al.*, (1960) investigated conservation of length when lengths of two items are not equal, but, misleadingly, end points coincide; and conversely, when length is equal, but end points do not coincide. In the first case, the children were shown a straight stick, and a curving plasticine 'snake' of greater length, but whose ends were made to coincide with those of the stick. In the second, ends of two parallel sticks of equal length at first coincided; then children saw one stick moved so that ends were offset. In both cases children younger than about seven judged lengths to be equal when end-points coincided. Even though they agreed that the straightened snake was longer than the stick, when the snake resumed its curves and its end points again coincided with those of the stick, younger children judged the lengths to be equal: the curves were not taken into account. When the endpoints of two parallel sticks were offset, the children judged a stick whose end protruded to be longer: the compensating retraction of the other endpoint was overlooked.

A further study implicated subdivision as a factor in conservation. Two equally long strips of paper were aligned with their endpoints coinciding, the children agreeing that they were equal in length. One strip was then cut into segments arranged in a line containing curves or angles. Children younger than about seven and a half said that the 'distance an ant would have to travel' had now changed. As before, younger children mentioned positions of the extremities of the lines as justification for non-conserving judgments. If there were curves or angles in the line, younger children pronounced the line longer than the straight version when the question was about 'distance to go', but not when the question was simply about length. Some counted the cut segments of the line, pronouncing some lines equal in length if they consisted of the same number of segments, although the size of the segments differed between the two lines. If one segment in a given line was markedly longer than all other segments, then *that* line was said to be longer overall than any line whose segments were closer in length. Older children made correct judgments of equality of length irrespective of the transformations and explained that what was essential had not changed.

Explanation of errors in these tasks due to deformation of lengths (zigzagging a strip) should be the familiar one of failure to compensate across positions of all endpoints, but *deformation* is confounded here with *unequal segmentation*, and in the attempt to explain errors due to both features, the account became rather obscure. Piaget stated that children failed to "[integrate] notions of intervals (parts of objects) with those relating to order and change of position" (Piaget *et al.*, 1960: 110). Two notions of length were used here: a) a series of nested parts, each containing all preceding parts and being contained in the next part in the succession (synonymous with 'subdivision': 116); and b) "a set of intervals between ordered points" (111). These two aspects must be integrated, he argued, for length to be conserved. Thus younger children are influenced by the ordinal aspects of the series of segments, and ignore inequalities of subdivision, or they are influenced by salient aspects of subdivision (such as an extra-long segment) because they do not fully understand subdivision as a synthesis of "order and change of position" (111). Although the explanation is rather obscure, difficulties regarding subdivision, as well as conservation, are clearly revealed by these tasks.

Piaget's tasks provided the basis of all later investigations of conservation; and because conservation of length is a logical pre-requisite of measurement, conservation tasks have been carried out alongside measurement tasks (Hart 2004; Petitto 1990; Boulton-Lewis 1987; Carpenter & Lewis 1976). In general, however, *specific* ways in which conservation ability may be bound up with measurement ability have not been examined. In the present study, conservation tasks were embedded in investigation of children's understanding of the language of length. Thus Research Questions 3a) and 3b) ask: *In the context of everyday length comparisons, do children conserve length?* and *If not, does this make any difference to their measurement ability?*

Piaget's focus on ordination is further considered below.

Continuity and subdivision

Piaget identified the paradox of continuity and subdivision - that a length or surface conserves its length or area, when subdivided – as a key conceptual element in the construction of Euclidean space. Here, understanding of continuity is defined as the ability to conceive a line or surface as composed of extensionless points (Piaget &

Inhelder, 1956). While reconciling continuity with subdivision mathematically, this idea is unhelpful as an account of how children may reconcile them.

A more persuasive account is given in the context specifically of measurement (Piaget *et al.*, 1960: 62). "Qualitative measuring" is distinguished from "a true metrical system" because in the former one entire object is compared with another, while in true measuring one subdivision of an object is abstracted as a unit and applied to the length of the object (creating further subdivisions). In the construction of number, Piaget argued, both cardination and ordination deal with discrete units, so the necessary synthesis is more easily achieved. In measurement like is not being synthesised with like in this way, and greater psychological difficulty can be expected.

> ... in the realm of measurement ...there is no automatic synthesis [of subdivision and change of position]. This is because length is continuous and not composed of discrete units. It can indeed be subdivided as we have seen... but the portions arrived at are immobile because they are connected so that it is difficult to compare them. The realisation that successive segments of a straight line are congruent therefore demands a greater degree of abstraction than the establishment of congruence between two separate moveable objects.
>
> (Piaget *et al.*, 1960: 60-61)

> In addressing Research Question 6: *How well do children measure?* the present research tested the claim that children would find "qualitative measuring" (where the lengths of two whole entities are compared) (Piaget *et al.*, 1960: 62) more difficult than using "a true metrical system" embodied in scaled measuring instruments.

Ordination

Piaget noted that ordination is logically presupposed in any comparison of quantities and hence in any form of measurement.

> If A, B and C are known to be successive points on a straight line, then, without measuring, we may deduce that the distance AB is shorter than the distance AC. This deduction follows from a knowledge of the order of points ABC and an understanding of the distances AB and AC which, being intervals, are symmetrical relations, but which correspond to the order of points, an asymmetrical relation. (Piaget *et al.*, 1960: 70)

Even with purely qualitative judgements of the form x is bigger than/longer than/deeper than... y, it must be understood a) that each item in a series differs in amount from earlier and later items, b) that judgements made in ordering them are relative: any item in a series (except the first and last) is simultaneously (for example) longer than the previous item and shorter than the succeeding one; and c) reversing a move in either direction in the series results in a return to the original position. Considerable intellectual demands on the child are, therefore, involved (although Piaget found children ordering items qualitatively, to correspond to those in an ordered display, by 5-6 years old.) (Piaget & Inhelder, 1956). Spatial and number concepts each contribute to concepts of measurement, and ordination was considered in Piaget's study both of space (Piaget & Inhelder, 1956; Piaget *et al.*, 1960) and of number (Piaget 1965).

In *The Child's Conception of Number* (Piaget 1965) the development of ordination and of cardination were dealt with together. Here, the focus was not on the spatial, but on the numeric properties of a series. Piaget considered that ordination and cardination (though logically interdependent) developed independently as they jointly apply to series. In one study, children were first shown (Piaget 1965: 97) a number of dolls and sticks that varied in length and asked to find the stick that belonged to each doll. To achieve this, the dolls had to be put in order of length and the sticks matched to them for length (or vice versa); or dolls and sticks ordered separately, the two lines afterwards being put in one-to-one correspondence. Only at about seven years old did children achieve complete success, carefully looking for the 'next largest/smallest' before laying an item and checking the whole length of each series for correctness with each addition.

In one version of the task, there was a consistent error of particular interest. The order of one of the rows was reversed, a doll was indicated and the child was asked which stick belonged to it. Children would identify a stick just before the one that correctly corresponded with the indicated doll. Piaget argued that they could not co-ordinate their knowledge that each item must have a unique *ordinal* position with their knowledge that the set of all items that preceded it in the series plus the focus item had a *cardinal* value. Hence they would count the set of items that preceded the focus item, and then number its ordinal position as a conceptually distinct operation, resulting in a difference of one. By about 7 years old, children counted the item whose position they were finding together with those before (or after) it in the series. For Piaget, this uniting of the cardinal and the ordinal represented a major step in the development of number operations including the ability to measure.

The present research, too, Research Question 4 asked *How well do children make ordinal comparisons of length?* but confined itself to eliciting qualitative judgements of *shorter, longer* and *the same.* This was because these particular kinds of ordinal judgements logically underpin numerical measurement, the central focus of the research.

Transitive inference: the 'towers' study

In measurement, transitivity is the fundamental logical principle underlying the use of the standard unit (Piaget, 1970). It is regarded by logicians as an axiom of measurement (Nagel 1930). If A and C are two objects whose length, say, is to be compared, transitive reasoning produces a deductive inference such as: If A > B and B > C, then A > C, where comparing B with C is logically equivalent to comparing A with C directly. B is the common or middle term through which the comparison is made. The required understanding is that B can represent the length of A for comparison with the length of C, removing the need for direct comparison between A and C. The role of B is what makes the inference transitive.

The 'towers' task (Piaget *et al.*, 1960) provides Piaget's most complete description of the psychological construction of measurement, shown as the construction of transitivity. Here the focus of interest is not direct comparison of the lengths of objects, but the middle term employed to compare lengths. The most familiar example of such a middle term is the ruler with its inscribed units, but in this study Piaget avoided conventional associations with measurement, using an unmarked stick and excluding, in the initial instructions, the word 'measure' itself, so as to investigate spontaneous measuring behaviour. The transitivity investigated was concrete and not formal (Smedslund, 1963): children's spontaneous behaviour and utterances were observed for evidence that implicit transitive inferences were being made.

The child was shown a tower of blocks of assorted sizes standing on a table, and was asked to build a second tower of the same height on a table two metres away and 90 cm lower than the first table. The assorted blocks he was given differed from those used in the first tower, so as to prevent equality being achieved through one-to-one matching of sizes. The child was given a supply of sticks and strips of paper which could serve as measuring devices, but was not told their purpose.

Piaget argued that subdivision and change of position (defining features of length measurement) were present at some level in every attempt to compare the heights of the two towers. For example, in visual comparison, actually looking from one tower to the other constitutes a virtual bringing together of the two towers (change of position), while the observation that one tower appears to be taller than the other is followed by visually referencing the shorter against the taller, decomposing the taller into the part estimated to be equal in height to the shorter and the remaining portion, so as to estimate by how much it is taller (subdivision).

Until the age of about 4 ½, children relied exclusively on visual comparison of the towers and showed no awareness of its fallibility, or of the possibility of using any other method of comparing their heights. Judgements were typically justified by saying simply 'I can see that it is so'. Visual comparison continued to be used thereafter, even as a check on later-developing strategies, but with some awareness of its limitations. The observation that a strategy is not discarded when a more sophisticated one is acquired is common in developmental studies (e.g. Siegler, 1996), and seemed also to apply to later strategies reported in the 'towers' study. In measurement, however, Piaget argued that strategies are not simply retained side by side, but that there is a retroactive effect. The child's gradual development of a co-ordinate system causes the perceptions and physical movements which gave rise to it to become better co-ordinated and more accurate; children become more aware of accuracy, and of the limitations and proneness to error of visual comparison. Thus older children are satisfied with visual comparison until they are asked to verify their judgment, when they notice that the bases of the towers are not in the same plane. This line of argument arises from Piaget's view that mature spatial concepts develop from a complex interplay between perception and movement; however the finding is also consistent with the view that children already possess the necessary understanding, but do not spontaneously recognize when they need to apply it (Bryant & Kopytinska 1976).

Piaget also gives a higher-level explanation in terms of childhood egocentrism, in relation to spatial concepts:

> They are supremely confident because they are supremely unaware of the empty space which separates the two objects. *It is the subject who, by the mere act of seeing, furnishes the only link between the objects* so that one is directly assimilated to the other. He therefore dispenses with the use of an objective measuring rule and with the structuration of space. (Piaget *et al.*,1960: 40. Italics added.)

In the study, if the use of the stick was suggested to the children, they might point it at their structure, or lay it across the top of a tower, but they did not attempt to use it as a measure. These children judged the comparative height of the two towers by reference to their tops only, and did not take into account the differing positions of their bases, exactly as the children in the studies of conservation of horizontal length took into account only the protrusion of the end of one stick in a pair, ignoring the compensating retraction of its other extremity. Children next attempted 'manual transfer' when they became aware that the bases of the towers were in different planes, and that visual comparison was untrustworthy. Now the children tried to move one tower across to stand by the other, to place the towers physically in the same plane, interpreted as a further stage in understanding space as a unity with fixed co-ordinates.

A child's use of his hands, leg or trunk to mark the height of one tower and then 'carry' it across to the other for comparison was seen as a further development. This 'body transfer', in Piaget's view well on the way to constituting a middle term by which the heights are compared, does not achieve this status because its function is still to *preserve* the height of the first tower, not yet understood as embodied in an independent measure. The cause of transition to true measuring is the child's dissatisfaction with the accuracy of his use of body transfer. Piaget argues that the effortful gestures involved require constant monitoring and adjustment so that, for example, the hands remain in the same plane while the height is 'carried' to the second tower. This increases awareness of horizontality and verticality and continues the work of the mental construction of Euclidean space. (Piaget *et al.*, 1960: 55).

What happens next, according to Piaget, is that the gesture of body transfer is mentally represented as "interiorised imitation" (Piaget *et al.*, 1960: 50). This representation is then projected on to a symbolic object, which finally becomes the independent intermediate term or measuring instrument. (Before this, children may accept a strip of paper or a stick as a measure, but may merely place it against the first tower and ask what to do next, or place it only against the second tower and say, 'Yes, that's right'. There is no concept of its function as a middle term). When the term is accepted as intermediate, there are at first limitations on its perceived usefulness: only if it proves to be exactly equal in height to that of the first tower do children accept it as a measure. Children's judgements are still "semi-perceptual" (Piaget *et al.*, 1960: 54). This phenomenon also reflects the absence of fully-developed ordination: the children are unable to understand length as a succession of intervals between points in an ordered series; their attention remains confined to the end-points of the whole length. Later, children are able to use the stick or strip of paper as a measure if it is longer than the first tower, marking off the height of the tower on the measure, but not if the measure is shorter than the tower. Later still they do accept the use of a shorter measure, but try to make up the full length required in makeshift fashion with another object. In both cases transitive inferences are being made in the form of "transitive congruence" (Piaget *et al.*, 1960: 60) It does not occur to the children to use a short length repeatedly as an iterated unit, or to mark equal divisions on their measure for the purpose of adding extra such units. Hence their understanding, Piaget says, is still qualitative.

In the present research, children were invited to say what they understood by 'measuring' and their descriptions of specific measuring experiences were elicited. By these means it was possible to look for evidence of the staged understanding pictured here by Piaget, and this was one objective of Research Question 1a): *What do children learn about measurement in their everyday social context,* and 1b) *How might this affect their conceptual understanding of measurement?*

Why should the final transition to quantitative measuring, as illustrated in the 'towers' study, and involving the iteration of a unit, prove so difficult, since the prerequisites of cardination and ordination are established? Piaget's view is that they are established for discrete units, not continuous quantities, and that continuous length is experienced by children as a different kind of thing from the units they are used to counting and ordering and hence not susceptible to the same treatment (Piaget *et al.*, 1960: 61). The present research investigated this claim by asking in Research Question 8: *Is there evidence that children conceptualise in different ways, units that are physically separate and units that form a scale?*

Finally, children do grasp and confidently iterate a unit such as one of the small bricks they have used to build their tower, to compare its height with that of the first tower, as a fully-developed middle term. Piaget is silent about the mechanism for this transition.

As an account of the way transitivity might typically develop, this is highly plausible. It incorporates at its explanatory best the Piagetian paradigm of physical activity working to expand childhood egocentrism. Change of position, and later subdivision, as key features of measurement, are strikingly present. However, as with other processes reviewed above, no psychological mechanism is offered for the decisive conceptual changes with which the account culminates, and the notion of mental representation involved, that of the schema, is left vague.

Bryant & Kopytinska (1976) tested the hypothesis that in the 'towers' study children failed to measure not because they did not understand that the height of the two towers could be compared through a middle term, but because they were satisfied with visual comparison. They predicted that children would measure (and hence show an understanding of transitivity) if the possibility of visual comparison were removed. Sixty five-year-olds and sixty six-year-olds who had all failed to measure in the 'towers' task were asked to determine the comparative depth of holes bored in wooden blocks. A stick of suitable length that would fit into the holes was provided. Most children used the stick as a measure without prompting, and most were correct in their judgments. However in another condition using transparent blocks, so that the depth of the holes could be seen, most children *still* measured. These blocks were, though, placed on different tables to discourage visual comparison, so it could still be argued that it is dissatisfaction with the quality of specific visual comparisons that leads to an increase in measurement. The perceived affordances of the situation (and the intrinsic interest of inserting sticks into holes) provides an equally plausible explanation. What this study does illustrate is that whether or not children will use an intervening measure varies with context and type of task. In the present work, contextual information was sought through Research Question 1a): *What do children learn about measurement in their everyday social context?* The effect of varying the type of task was examined by the combination of differing measuring devices and lengths of line used in answering Research Question 6: *How well do children measure?*

Subdivision and change of position

Piaget had formulated his definition of measuring in terms of subdivision and change of position (Piaget *et al.*, 1960), and in the 'towers' study, each developmental step embodied these ideas. In the studies of length conservation, various ways of subdividing length are shown to mislead children as to actual length. The idea of subdivision is thus central to Piaget's treatment of measurement, both as fundamental concept and as stumbling-block. It is indeed clear that inability to deal with subdivision must compromise a child's ability to compare lengths or to deploy units, but the nature of the psychological difficulties is not explored by Piaget. The present research explored important aspects of the subdivision of units by means of Research Questions 9b): *Do children understand that larger units may 'contain' smaller ones?* and 10: *How well do they cope with fractional units?* Research on subdivision of units is further considered later in this review.

1.2.1.3 Order of acquisition of components of measurement

An invariant order of acquisition of concepts underpinning measurement was proposed by Piaget governed by what he viewed as their increasing logical complexity. This order was repeatedly challenged (e.g. by Petitto, 1990; Boulton- Lewis, 1987; Braine, 1959) on the grounds that performance of a task at any level may reflect the state of other cognitive capacities that develop with age but are non-logical, such as working memory.

Boulton-Lewis (1987) tested length-measuring knowledge in 3-7 year-old children using three possible developmental sequences. The first was a purely logico-mathematical sequence based on Nagel's (1930) axioms of quantity. The second, an age-related sequence drawn from the literature on the development of logical reasoning that largely followed Piaget. The third classified the concepts in the Nagel sequence according to the information-processing and central processing requirements (M-space) of the types of task involved (Halford 1982; Halford & Wilson, 1980).

Children from 4 to 8 years old were tested individually on M-space and short-term memory measures and on tasks producing measures of all the variables in the identified sequences. Using a scalogram analysis, the results were put into a developmental sequence of length and number knowledge for the 80 participants, afterwards statistically tested to confirm sequence or co-occurrence of variables.

Four sets of associated variables emerged by age. The first set included ability to copy a horizontal line constructed with counters; recognition of equality or inequality of lengths of lines and ability to use the associated language; seriation by length (all from the literature sequence); subitising, and naming numbers 1 to 5. The second set consisted of recognition of length invariance, conservation of length, recognition of transitivity (from the literature sequence), simple number operations to 10 and knowing numbers greater than 10, and construction of a diagonal line. The third set to emerge included, surprisingly, recognition of one-to-one correspondence. This basic schema for counting thus emerged much later than was proposed in the logico-mathematical sequence. Using a ruler to compare the length of two pieces of ribbon also emerged at this point. Transitive reasoning occurred last, with no co-occurring variables.

The developmental sequence that emerged from this study bore little relationship to the Piagetian sequence, or to that derived from Nagel. It most resembled that predicted by the information-processing demands of the tasks. The study posed therefore a significant challenge to Piaget's account of the construction of measurement, which presupposed a straightforward relationship between logical simplicity or complexity and psychological ease or difficulty.

1.2.1.4 The Piagetian framework: summary

Because the components of measurement are *logically* mutually entailed, full co-ordination of these as a mental schema for measurement was seen by Piaget as both a necessary and sufficient condition for fully-developed understanding; there was no account of additional psychological factors. Development was conceived as a process of internalising physical actions: almost all the Piagetian studies reviewed above described understanding as arrived at in this way. When a schema was complete, it was said to be 'operational'. This term retained the notion of activity on the mental level; knowledge was not merely passive, but available for use in many contexts.

Piagetian methodology typically involved close observation and questioning of eight to ten individual children engaged in the tasks set. These were designed to help the experimenter to infer a child's thinking in respect of some component, or co-ordination of components, of measurement. Piaget would first give a theoretical interpretation of the meaning of children's typical actions (and sometimes justifications) in a given task. Illustrative details of specific actions of individual children within a given age-range were then reported, together with verbatim excerpts from the researcher's questioning about their actions, and of the children's responses and sometimes justifications for them. Finally the theoretical account was reiterated, and the detailed findings located precisely within it. This methodology resulted in uniquely detailed and rounded descriptions of aspects of the development of measurement, but their coherence depended on acceptance of the entire Piagetian logical framework. Outside this framework, later research used much larger samples, had more regard for issues of validity and reliability, but investigated narrower aspects of measurement. The framework indeed did not survive later empirical tests. Neither the general thesis that the construction of Euclidean space is a late accomplishment, nor the development of many of its component concepts, turned out to be as Piaget described. Nevertheless, the work continues, obliquely, to suggest fruitful lines of enquiry, as the following two examples show.

First, as underpinning for understanding continuous length, the concept of a line as a series of extensionless points (Piaget *et al.*, 1960) was surely unnecessary; nevertheless, this formulation drew attention to the paradox of continuity in the face of subdivision, and this represents a persistent difficulty for children learning to measure (Lamon 1996; Brown *et al.*, 1995). Second, errors children made when establishing a series (Piaget 1965) were ascribed to their incomplete integration of ordinal with cardinal number. Over-elaborate for its immediate purpose, this explanation nevertheless suggested a broad area of potential difficulty: understanding how a numbered sequence of units marks amount on a measuring device.

The opposing poles of Piaget's theory were perception and mental representation, but their roles in explanation lacked consistency. In conservation studies, perceptual features of displays were held responsible for obstructing the conception of invariance. In developing conceptions of space, children were said to perceive space threedimensionally, but represent it mentally as topological. In the development of transitivity, children abandoned exclusive reliance on perceptual methods of length comparison, to conceive length as invested in the middle term in transitive inference. Piaget offered no mechanism for transition to mental representations, or account of how these differed from perceptions. Moreover conscious awareness of mental representations varied in importance in his account. The development of transitive inference was described as spontaneous and unselfconscious. In other research, where children were asked to justify their judgements, not only conscious awareness but the ability to articulate it were considered a necessary part of understanding.

It has long been recognised that notions of perception and mental representation cannot in fact be separated and that unprocessed sensation is a myth (Russell, 1987). Conversely, some types of inference, long conceived as a characteristically mental activity, have been shown to be part and parcel of perception (Resnick, 1992; Spelke, 1990; Gibson, 1979; Bryant 1974). The notion of inference has been extended to the perceptual-mental activities of infants (Feigenson, Dehaene & Spelke, 2004). Meanwhile, research in the Piagetian developmental tradition has taken up the challenge of giving a theoretical account of mental representation including the role of conscious awareness. The work of Karmiloff-Smith (1992) is particularly relevant to the present research for the prominence it gives to procedural knowledge.

1.2.1.5 Later development of the notion of mental representation: Representational Redescription

Karmiloff-Smith's theory of representational redescription (RR) addressed the same phenomena as Piaget: the increasing breadth and general applicability of children's logical resources, of which the development of measurement is a paradigm, but focused on the mechanism for the changes in mental representation involved. It sought to "account for the way in which children's representations become progressively more manipulable and flexible [and] for the emergence of conscious access to knowledge" (1992: 17). In the model, procedural competence, conscious awareness of knowledge, and its accessibility to verbal report are all accounted for in the developmental process.

A four-phase cycle is proposed that recurs *within* domains during adulthood as well as childhood. In the first phase, data from the external environment are stored additively, with little processing (Implicit level), and drawn on to execute simple behavioural procedures, guided by immediate environmental feedback. By trial and error, for example, four-year-olds successfully balanced, on a fulcrum, blocks containing weights in different locations. (Karmiloff-Smith, 1984). In the second phase some implicit, procedural knowledge is mentally 're-described' independently of further environmental input. Some details are lost and the rest become permeable by similar features of other such stored knowledge to form the beginnings of cross-domain concepts at Explicit-1 level (E-1). In the third phase, E-2 level, 're-described' knowledge is available to conscious awareness but not to verbal formulation as a principle. Evidence for an E-2 level comes from the 'U-shaped curve', characteristic of knowledge acquisition in many domains, where the bottom of the curve reflects failures in a task previously correctly performed. Here, environmental feedback temporarily loses its influence during mental simplification and generalisation of previously-stored data. The resulting 'theory' (for example 'a block always balances at its geometrical centre') is then consciously applied and dominates behaviour, so that no adjustment is made to procedures that produce errors.

At phase four (E-3 level) *types of encoding* become capable of inter-relating, providing a basis that is sufficiently rich to allow linguistic expression. Now specific procedures may be verbally justified by reference to a principle. Increased conceptual flexibility permits new data from the external environment to be used to amend a theory.

Two departures from the Piagetian account are of particular note. Firstly, mental representations characteristic of any phase may co-exist and be cognitively available within their limits. Thus cognitive development is shorn of the rigidity of Piagetian structuralism. Secondly, at level E-2, children may be aware of a principle that they cannot yet explain; the status given by Piaget to correct justification is thereby removed as a hallmark of understanding. The RR theory has been extended, notably by Pine & Messer (2003; 1993) who confirmed Karmiloff-Smith's levels and identified additional, transitional levels, noting a complex role for language. As confirmation that complex representations may be present at different times in different domains, they also found five-year-olds to be capable of taking account of both weight and distance in the balance beam task. The role of language in conceptual advance on that task was also established, within the RR framework, by Philips & Tolmie (2007), who explored parents' verbal contributions to children's success. For Karmiloff-Smith, mastery of the relevant procedures is a pre-condition for initiation of level E-1 in any domain. This retains the Piagetian idea that physical activity is at the root of cognitive development

1.2.1.6 Piagetian individual constructivism and the socio-cultural perspective

Children construct their conceptual understanding of measurement within a specific socio-cultural environment that must influence the process, perhaps profoundly. Authors considered later in this review have attached considerable weight to socio-cultural influences (Miller, 1989; Resnick, 1992; Nunes, Light & Mason, 1993). For example, Resnick (1992: 107-108, italics in the original) characterises mathematical development as "both *situation-specific* and *situation-inclusive*": what is learned is attuned to the social demands of a specific situation, and to *all* its demands, not merely the mathematical aspects. For the educator who values the learning done by children in everyday life, the problem then becomes one of identifying reliable theoretical continuities between that learning and more formal mathematics instruction.

Aspects of the children's socio-cultural environment in relation to measurement, including the classroom environment, were investigated in the present research by means of Research Questions 1a): *What do children learn about measurement in their everyday social context, and* 1b) *How might this affect their conceptual understanding of measurement?* The expectation was that "a constitutive role in learning for...emergent processes which cannot be reduced to generalised structures" (Lave & Wenger 1991: 16) might well be revealed. However, the more thoroughgoing theoretical stance of these authors (Lave & Wenger 1991:17 Foreword) was resisted:

> In a classical structural analysis...understanding is seen to arise out of the mental operations of a subject on objective structures. Lave and Wenger reject this view of understanding insofar as they locate learning not in the acquisition of structure, but in the increased access of learners to participating roles in expert performances.

This perspective does not *exclude* unassisted individual learning, where required by the social (for example, the classroom) context, but apprenticeship is the favoured model. The present writer's perspective remains, however, in the individual constructivist tradition, and retains structuralist features. This is because where mathematical understanding is concerned, "participating roles in expert performances" do not seem equal to the task of overcoming the conceptual difficulties often involved; and also because the understanding of fundamental principles is to be contrasted with the mastery of procedures, however acquired, and however expertly performed. The perspective of Lave & Wenger does not seem to give adequate weight to these considerations.

1.2.2 Non-numerical components of measurement

Measuring using standard units is numerical, producing absolute judgements of length. Underlying numerical measurement, however, are perceptual judgements of amount, and fundamental, non-numerical kinds of reasoning about amounts. Theory and research relevant to these aspects are now reviewed. They include infants' responses to ratio characteristics of displayed amounts, relative judgements of size in visual estimation, and protoquantitative reasoning (Resnick, 1992). Because measuring instruments embody the principle of transitive inference, its extensive literature is also explored in this section. Finally Miller's (1989; 1984) hypothesis that measurement procedures may influence conceptions of amount is considered.

1.2.2.1 Ratio discrimination in infancy

A wealth of research has used habituation and looking time paradigms to explore infants' abilities to discriminate quantities (Feigenson, Dehaene & Spelke, 2004), including ratio characteristics of numerosities. With strict controls on non-numerical features of displays, six-months-olds discriminate between large collections of dots in the ratio 1:2. Thus they discriminate 8 versus 16, and 16 versus 32 dots, but not 8 versus 12 nor 16 versus 24. Ten-months-olds do discriminate between the latter collections, which are in the ratio of 2:3. Similar discriminability applies to sounds, and this similarity suggests representation of these ratios as such. An 'accumulator' analogue-magnitude model (Meck & Church, 1983) is proposed to account for such representation (Feigenson, Carey & Spelke, 2002). When an infant sees an array of items, each item allows a burst of energy into an accumulator. A 'gate' comes down after each item. After the last item in the set, the accumulated magnitude is stored in memory. This magnitude *represents* the number of items in the set, but is a continuous magnitude; that it represents them is shown by its ratio character.

Discrimination of *small* numerosities (1-3) is not made according to ratios. The findings suggest that different developmental factors may be involved in approximate representations of relative amounts (as ratios), from those involved in precise representations of small differences in numerosity. Findings also suggest (non-human primates appear to make similar representations) innate specification of some aspects of quantity (Feigenson *et al.*, 2004). The infancy studies provide developmental underpinning for findings regarding early relative judgements (Bryant, 1974), and some basis (if there are different developmental routes) for difficulties encountered in the integration of number with amount, considered later, in the section dealing with units.

1.2.2.2 Relative versus absolute judgements of size

Piaget considered that relational reasoning, including relative judgements, develops from the co-ordination of separate cognitive competences previously acquired. Bryant (1974) proposed instead that the ability to make relative judgements develops *before* the ability to make absolute judgements, and that this sequence is the key to the development of concepts of size and number. Thus it is the size of objects in relation to their frame of reference that is 'given' in perception (cf. Gibson, 1979) and judgements of absolute size that are more difficult and develop later.

Bryant hypothesised that in early experiments to compare sizes of two stimuli (Kohler, 1918; Spence, 1937) participants were actually making *two* size comparisons: first of the size of each stimulus relative to its immediate background; and second of the relative sizes of stimuli to each other through their relationship to this common background.

Lawrenson & Bryant (1972) tested this hypothesis by comparing children's 'relative' and 'absolute' responses. Independent groups of four to six-year-olds undertook 'relative' and 'absolute' tasks. In the 'relative' task, squares of white card in a succession of pairs of different within-pair sizes were presented. Children in the 'relative' group were trained to respond sometimes to the larger, and sometimes to the smaller square in each pair. In the tests that followed, the children had to select either the larger or smaller square from each pair. The relative sizes of the two squares within each pair remained the same for each pair (one square being twice the area of the other) but the larger card in each pair was always the same absolute size as the smaller card in the succeeding pair.

The 'absolute' group was trained to select always the square that was the same absolute size as one of the squares in a preceding pair, but within a pair the correct choice was sometimes the larger and sometimes the smaller. Lawrenson and Bryant reported more errors overall on the 'absolute' than on the 'relative' task, irrespective of age, and took the results of the study as strong evidence for primary use of a relative code by young children.

Bryant argued on such evidence that young children recognize relationships such as 'larger than' and 'smaller than' before they recognize absolute size. This important claim, subsequently widely accepted (e.g. Resnick 1992; Resnick & Singer 1993; Nunes & Bryant 1996) and substantiated in relation to infants (Feigenson *et al.*, 2004) was rigorously tested in the present research in relation to measuring, where the tasks required separate relative and absolute judgements to be made. Research Question 6:

How well do children measure? examined both types of judgment. In estimation tasks (as well as language tasks), purely relative judgements were required and were examined by Research Question 4: *How well do children make ordinal comparisons of length?* Associations between success in relative and absolute judgements were then statistically tested. (Research question 11: *Are there associations among understanding the everyday language of length, ability to make ordinal comparisons of length, and the ability to measure?*)

Bryant discussed implicit perceptual comparisons, not inferences following reflection, whereas the Piagetian notion of co-ordination involved a degree of reasoning: in conservation tasks, for example, children's justification of their judgements informed the researchers' assessment of their understanding. Nevertheless Bryant's work gave to perception a fundamental role in developing understanding of amount, in contrast to the Piagetian view of perceptual salience as the source of errors.

1.2.2.3 Transitive inference

Standard units and measuring instruments on which they appear are physical embodiments of transitive inference, the notion that a specific length (for example 1cm) may be abstracted from any given instantiation of it and used to compare the lengths of two or more objects without the need to bring them together for direct comparison. Transitivity, an axiom of measurement (Nagel, 1930), is the logical foundation of all ordinal continua such as number, size and weight (Pears & Bryant, 1990) and hence must be grasped if the notion of a scale is to be understood (Markovits, Dumas & Malfait, 1995). Ability to make transitive inferences has been thoroughly investigated.

In typical transitivity studies involving length, a series of sticks A, B, C of increasing length is employed such that A>B>C. The sticks are typically colour-coded to make them individually memorable. Children are shown neighbouring sticks from the series in pairs, and are asked to judge which stick is longer or shorter. This is the 'training' phase and ensures that correct comparative judgements are being made. In the test phase, children are shown some stick from the series. This is removed and another is shown that was not its neighbour in the series; thus the child has never directly compared these two sticks. For example A, then C may be shown. The child is asked which is longer. To respond correctly, the child must make an inference based on the length of the missing middle stick B. Correct responses over a number of trials are taken as indicating that transitive inferences are being made.

Early studies

An early critic of Piaget's studies of transitivity, Braine (1959) identified many possible influences on task outcome that the studies did not control for, such as differing demands on perceptual discrimination by stimulus materials, level of vocabulary development, and memory demands of an extended series of comparisons. His own studies sought to eliminate some of these factors. For example, to control for memory of absolute length of sticks (which would make inference redundant) differences in length were made small and visual distractors were used. Children were trained to mastery with pairs of sticks in the first phase. For reliability, a minimum of forty trials was used in the test phase.

While Piaget believed that transitive inference and the ability to measure emerged together at about 7 or 8 years, Braine concluded that transitive inference appeared about two years earlier, and that ability to measure was achieved later. He also identified children's ability to manipulate measuring instruments and interest in doing so as important factors.

Smedslund (1963) identified several possible non-transitive explanations of Braine's results in the transitivity tasks and claimed to have found support for Piaget as to the age of acquisition of transitivity, but doubts were raised about whether his own studies showed genuine transitive inference (Braine, 1964). Later investigators (Murray & Youniss, 1968; Brainerd, 1973) used simpler protocols while similarly seeking to control for confounding factors.

The middle term

In the studies of Murray & Youniss (1968; Youniss & Murray, 1970) interest centred on the 'middle term' and its independence of the absolute length of the first term in children's judgments. In apparently inferring that A is longer than C, perhaps children merely ignored stick B and remembered that A was the longest in the series. Murray and Youniss dealt with this by adopting several versions of the relationship with the middle term: A>B>C, A>B=C and A=B>C. Only younger children were affected by version. Bryant & Trabasso (1971) noted that for a truly transitive inference to be made, the 'middle term' had to be understood as essentially relative: B is both longer than C and shorter than A. However, after making the initial comparisons A, B and B, C (prior to the inferential comparison) children could remember within each pair (the researchers argued) which term was shorter and which was longer; a response in accordance with this remembered judgement was thus indistinguishable from a correct inferential response. They proposed a longer sequence, A>B>C>D>E as an unambiguous test. Here, an inference was available that did not involve any anchoring to the absolute length of the end terms A and E: that is the inference B>C and C>D, so B>D.

Memory demands

A longer sequence, however, puts greater demands on children's memory for the terms in the series and their relations, so Bryant & Trabasso (1971) incorporated a thorough training period for memory of the four direct comparisons. Their results showed that in these circumstances 4, 5, and 6-year-olds performed well on all the inferences involved, including the crucial B, D comparison. Youniss & Furth (1973) pointed out, however, that memory had been confounded with other factors such as learning and attention. Riley & Trabasso (1974) confirmed with children of a similar age the importance of memory for the premises in transitivity, but Russell (1981) found that this was not a sufficient condition: in one study, one-third of participants justified incorrect inferences by reference to the correct premises, showing that these were both stored and retrieved from memory. Riley & Trabasso (1974) also found that inferential responses were at a higher level when questioning forced children to make a reversible judgment: for example that not only was A>B, but at the same time B<A. The importance of memory for the premises as a general factor in performance on reasoning tasks (Johnson-Laird, 1983) was investigated for transitive inference by Trabasso, Riley and Wilson (1975), who pointed out that transitive inference is a form of syllogistic reasoning. For them as for Johnson-Laird in his studies of adult reasoning, the key factor affecting success is the manner of representation of the premises of the inference in memory. The key question was: *by what process* does comparison of individual pairs of items lead to transitive inference? Riley & Trabasso (1974), using six sticks, suggested that items at each end of the series are again key.

Representation in memory

One possibility is that comparisons (premises) AB, BC and so on are represented separately in memory as ordered pairs, and so must be retrieved separately and then compared in distinct operations. The other possibility is that they are represented as a linear order and retrieved as such. The 'inferential' comparison is then made by reading it off from this series, and is not, therefore, actually inferential. Trabasso *et al.*, (1975) proposed the latter. They proposed that information about the longest and shortest pairs of sticks (AB and EF) is encoded first, and then pairs are encoded working inwards (BC, ED and finally CD). As evidence for this they found a serial position effect, whereby the greatest number of errors occurred in the pairs furthest from the ends of the series. Consistent with spatial encoding of the pairs as a series, they found better recall for pair 2,5 (two inferential steps) than for 2,4 or 3,5 (one inferential step). This held for younger and older children and for adults, and so, contrary to Piaget, did not appear to be developmental. Breslow (1981) suggested that although inference need not be involved if responses are 'read off' from a serial representation, inference may be essential in the *construction* of this representation in the training phase of studies. To verify correct placing of a stick in a series, (as in Piaget's studies) one would have to infer that all sticks to one side of the focus stick were shorter than it, and all those to the other were longer. This could not be done by repeatedly comparing adjacent sticks: transitive inference is involved.

The questions whether young children's responses to transitivity tasks indicate that they learn the absolute length of individual sticks rather than comparative length, whether each comparison they learn is stored independently or as part of a series, and whether representation of such a series itself entails that transitive inferences have been made, were not resolved (Thayer & Collyer, 1978; Miller, 1976; Harris & Bassett, 1975; De Boysson-Bardies & O'Reagan, 1973). Moreover the typically very large number of trials may have produced a training effect that casts doubt on the developmental interpretations made of the results. Pears and Bryant (1990) ingeniously bypassed the need to train memory for the premises in a transitivity task that investigated judgments about spatial position rather than length. The premises consisted of stacks of two blocks, A above B, B above C, C above D, D above E and E above F, in which the elements were represented by colours, for example A=yellow, B=blue. The child's task was to construct a stack of six blocks (sometimes fewer) in which the spatial relationships between the colours in the two-block stacks was preserved. All the block stacks were on view at once, removing the need for memory training, because absolute spatial positions were not preserved in the final tower and so direct comparisons would not help. The number of 4-5-year-old children apparently making correct inferences was well above chance.

Transitive inference as analogical reasoning

Halford (1992) construed transitive inference as a form of analogical reasoning, that is, mapping from a known source structure to an unknown target structure (Gentner, 1983). Here, the source structure is the left-to-right or top-to-bottom series familiar in everyday life; Halford cited evidence that constructing such a series often precedes transitive inferences. His example gave *top above middle above bottom* as a source structure and *John is fairer than Peter is fairer than Tom* as a target structure. Once the mapping has been made, *John is fairer than Tom* can simply be read off. Children have difficulty with transitive inferences, he suggested, because of the processing load involved in mapping. However, while mapping is a plausible model for a short, three-term series, more may be involved than increased processing load for a longer series where (Breslow, 1981) transitive inference may be involved in the initial construction of the series (however familiar). More fundamentally, if children are 'reading off' the inference (Trabasso *et al.*, 1975) they could in principle do this as easily from the target structure as from the source structure.

Using the mapping paradigm Goswami (1995) tested whether three- and four-year-olds could make transitive inferences involving two relations of relative size. From an array of three objects of their own, children identified an object that was 'the same' (according to type of object, spatial position of objects within the array, or absolute or relative size) as an object indicated in the experimenter's array of three. Transitive inference was required for success where the spatial position of objects was selected, because the children had to order the objects mentally before judging.

There was a very high success rate for both age groups. The lowest scores occurred where the objects in the child's array differed both in absolute size and in spatial position from those of the experimenter, but even here success was at around seventy per cent. This was additional evidence, therefore, that pre-school children can apparently make transitive inferences in some situations.

Sternberg used the case of transitive inference to consider how logical competence may be constructed psychologically. Making use of the mental models paradigm (Sternberg, 1980a), he attempted to explain use of the 'middle term' as use of a 'pivot element' consisting of the mental superimposition of term A on to term B for comparison with C.

Appraisal of the literature on transitive inference

A paradox emerges from the literature reviewed. While Piaget is silent as to how transitivity, as a decisive advance in logical development, is mentally represented, many who pursued this question tended to conclude that true inference was absent. Yet an individual who can explain how units on a measuring instrument represent the same quantity wherever that quantity is located has surely grasped a principle of considerable importance.

Research on transitive inference represents one attempt to give an account of deductive reasoning in terms of psychological process. Breslow (1981) argued that accounts reviewed above are reductionist and fail to do justice to the central position given by Piaget to the notion of logical competence. Youniss and Furth (1973: 315) made a similar point when they described correct 'transitive inferences' made after extensive memory training as purely perceptual and "sub-logical".

Two separate issues underlie this debate. The first concerns competence and performance. Cognitive factors such as attention and memory influence children's ability to make transitive inferences; so do features of an investigation, such as whether tasks 'make sense' to a child and whether the researcher's language is understood in the way intended (Donaldson, 1978). Some of these factors were addressed by the researchers reviewed above, with the aim of investigating purely logical competence. On a wider front, such efforts have resulted in claims of implicit logical ability quite early in childhood (e.g. Donaldson, 1978; Gelman & Gallistel, 1978).

The second issue is more fundamental: the problem of how more powerful logical structures can possibly arise from any combination of logically weaker components, and it applies to the whole Piagetian framework underpinning measurement. It remains intensely debated in domains such as linguistics, where the problem of the 'poverty of the stimulus' (Chomsky, 1980) remains unresolved, but perhaps its intractability is correctly viewed simply as the product of a category-mistake (Ryle, 1949); in this case, an epistemological question is illegitimately posed as a question about psychological processes.

The literature on transitive inference did not investigate children's understanding of transitivity in a measurement context, and indeed to do this specifically would be difficult. However in the present research, Research Question 6: *How well do children measure?* provided opportunities to observe children's handling of different measuring devices and consider how well they appeared to grasp the nature of the standard unit as embodying the length of the measured object.

1.2.2.4 Protoquantities

Resnick and Singer (1993) proposed a developmental sequence that encompasses children's competence in making qualitative judgements of amount, and their later application of number to quantities in measurement. These competencies find their place within four stages in the development of mathematical reasoning, which the authors summarise as follows:

> ...early abilities to reason non-numerically about the relations among amounts of physical material provide the child with a set of relational schemas that eventually apply to numerically quantified material and later to numbers as mathematical objects (Resnick & Singer 1993:109)... Eventually...mental entities [will] allow [students] to treat operators (e.g. addition...) not just as actions to be carried out on numbers, but as mathematical objects with properties of their own (110).

'Measurement' is the second of the stages, intermediate between non-numerical reasoning about quantities and purely formal reasoning with numbers (Resnick 1992), but this term does not distinguish between countable objects and continuous quantities; thus *units* of measurement are not specifically dealt with. It is rather the first, protoquantitative, stage that is of importance for the present research, including the discussion of direct and inverse 'proto-ratio reasoning'.

Resnick's stages of mathematical reasoning in relation to measurement

Protoquantities are the forerunners of units of any kind. They are amounts of material considered in qualitative terms. Protoquantitative reasoning is considered to develop spontaneously from everyday experience of quantities and the language used about them: to say that there are *more* boys than girls in the class, or that your house is *smaller* than mine are protoquantitative judgements. Such reasoning is employed, too, by expert adults such as engineers (Resnick & Singer, 1993). Both relative and absolute notions are included: for example, *bigger* as well as *big.* However, the mathematical operations associated with protoquantities - *increase, decrease, combine, separate, compare, order* - are all relative in character; the simplest form is described as direct perceptual comparison of different sizes of objects or sets. Protoquantitative reasoning can be inferential, (if some marbles have been removed from a bag, there are fewer marbles in the bag) and hypothetical (if some marbles were removed, there would be fewer) (Resnick & Singer, 1993; Resnick, 1992). Like Bryant and unlike Piaget, therefore, Resnick considers that logical inference develops early. The *combining* and *separating* operations in protoquantitative reasoning also permit *part/whole* reasoning such as knowing that if subsets are subdivided, there is no change in the whole set.

In the second stage, children apply counting principles (Gelman & Gallistel, 1978; Steffe, von Glaserfeld, Richards & Cobb, 1983), developed earlier and independently, to enumerate objects. Here, numbers *describe* objects (*seven apples*) or quantities of material (*ten centimetres of ribbon*) and are inseparable from them.

In the third stage, numbers are considered in themselves, rather than as attributes of physical objects or material. Their 'properties' take the form of relations with other numbers, and are dissociated from quantities of material. In the ultimate stage, operations on numbers (such as addition) themselves become objects of thought.

Protoquantitative ratio reasoning

Two types of non-numerical reasoning are described as possible forerunners of numerical ratio reasoning (involved in units): *fittingness* and *covariation.* The first is considered an *external ratio* after Vergnaud (1988) because different "measure spaces" (Resnick & Singer, 1993:112) are involved, such as beds and bears in the three bears story. Only the comparisons 'greater', 'less' and 'the same' are available to protoquantitative external ratio reasoning, and are judgements of affordance (beds are too small, too big or just right for a particular size of bear). Resnick & Singer also cite evidence of "implicit ratio reasoning" (112), where an object may be judged, say, big (for an egg) or small (for a cereal box). Here, ostensibly absolute judgements are actually relative.

An internal ratio is the proportion of a part to the whole in the same "measure space" and is involved in the understanding of units in measurement. Internal ratios are not easy for children to understand (Nunes, 2008; Piaget *et al.*, 1960), although Resnick and Singer did not comment on this. The inverse relation between size and number of units, an example of proportional reasoning that is central to measurement, causes particular difficulty. In the present research, children were invited to comment on a variety of everyday measuring devices whose scales displayed different units side by side, so that their understanding of the inverse relation could be assessed (Research question 9a): *Do children understand the inverse relation between size and number of units?*)

The protoquantitative *covariation* schema is applied to the correspondence of entities across two series, also familiar from everyday experience, such as 'the larger the child, the larger their clothes'. Inverse co-variation is also possible but somewhat more difficult (cf. Piaget, 1965).

Protoquantitative judgements are not necessarily relational (they include subitisation of small numerosities) and when relational, are not necessarily directly perceptual. Five-year-olds can correctly judge at better-than-chance the more numerous of two displays of dots and the longer of two strips of paper, where the two displays are not seen simultaneously (Cowan, 1982).

Implications for measurement of the notion of protoquantities

The work of Resnick (1992) as well as Bryant (1974) suggests, against Piaget, that there are many contexts in which 'co-ordinations' precede, rather than follow, absolute judgements, and that a variety of types of inference of relevance to measurement may be made by children in advance of the involvement of number in such judgements. One important co-ordination, however, is a later rather than earlier accomplishment, and that is the integration of early number competence with protoquantitative reasoning in measurement. There is evidence to suggest that this may not be an easy step (Nunes, Light & Mason, 1993; Petitto, 1990). While Resnick & Singer did not discuss potential difficulties with this integration, the present research explicitly investigated it. Research question 5: *How well do children understand that a number may express length?* was pursued in measurement tasks by partnering questions about the absolute lengths of lines (expressed as a number of units), with questions about comparative length of lines (relative, protoquantitative judgements). This required children to convert their protoquantitative judgements to numerical judgements.

1.2.2.5 How measurement procedures as 'tools of thought' may affect judgements of quantity: Miller's (1989) study

In his work on measurement with 3 to 10-year-olds, Miller (1989) considered the relationship between the use of certain measurement procedures and ability to conserve quantity. He hypothesised that such procedures may be 'tools for thought', in the sense that they may function as "schemes, organizing the domain over which they operate" (p589). Miller hypothesised that children's reasoning about amounts and their permissible transformations may be structured in terms of the measurement procedures they use, and in terms of the attributes those procedures measure. His definition of measurement as systematic use of some rule for comparing quantities (Stevens, 1975; 1946) permitted him to include counting among measurement procedures.

Miller's principal focus (1984, 1989) was Piaget's claim that a child must assemble all qualitative logical prerequisites of measurement before successful quantitative measurement can develop. Miller's counter-claim was that the learning of measurement procedures can be the basis for later understanding of measurement principles.

He used invariance of a given amount of material as the focus of a series of conservation studies in which the result of a measuring procedure appears to children either to contradict, or not, invariance of the given amount. His predictions were that where the experimenter transforms a fixed amount of material such that, when children use an appropriate measuring procedure, they obtain a result that disagrees with their pre-transformation judgement, children will tend to judge that the amount has changed. Where the transformation does not produce a new result following the application of the measuring procedure, children will tend to judge, correctly, that the amount has not changed.

Substantiation of these differential predictions would provide evidence that it is the measurement procedure that determines the way amount of material is represented in thought, and hence the logic that children apply to the task. Procedures determine principles, and not vice versa.

Number: children count a set of items. Two kinds of transformation are then made: a) An item is cut in half. The number of items has thus increased, but amount of material has not. The appropriate procedure (counting) suggests an increase in amount, so a child is predicted to say (incorrectly) that amount has increased. b) Some items are squashed, so that area has increased, but amount of material has not. The counting procedure suggests no change, so the child's (correct) verdict would be no increase in amount.

Length: when the relevant procedure is the measurement of length, if changing length also changes amount (e.g. adding extra material to make a strip of plasticine longer) then children will make the correct judgement that amount has increased. If changing length does not change amount (e.g. drawing out an existing strip of plasticine) then children are predicted to say incorrectly that amount has increased.

Area: When the relevant procedure is the measurement of area, if both area and amount are changed (adding material and also spreading out) then children will say amount has changed. If changing area does not change amount (spreading or squashing without adding material) then children will judge incorrectly that amount has increased.

The studies involved two toy turtles who had to be given an equal amount to eat. Their food was clay pieces representing sweets (to involve counting) spaghetti (to involve judging length) or fudge slabs (to involve judging area). Children were asked to make judgements in each of the three domains of measurement before and after transformation of the material in the different ways exemplified above.

Miller's predictions that there would be a higher proportion of non-conserving responses where information from a relevant measurement procedure conflicted with actual amount were broadly borne out in all age-groups.

This work provided evidence that young children have considerable ability to reason about measurement, but that they frequently do not know how to interpret the outcome of measurements they make. The 'tool' metaphor is elaborated in a socio-cultural framework: if measurement is a 'tool of thought', it will have developed in response to particular types of problems and thus exhibit 'functional fixedness' (Maier, 1931), that is, the applicability of its principles in other settings may not be appreciated. Thus, while some procedures may be available to children at an early age, such as judging how far a stick will go into a hole, the extent to which children can make use of the results of such a procedure will vary with the context. Miller's ingenious experiments provided theoretical support for the idea that instruments and procedures involved in measurement play a role in the development of underlying concepts. Equally interesting is the suggestion that the role they play may mislead children. The present research investigated whether differing measuring devices affected success in measuring length. (Research Question 7: *Do differences among measuring 'tools' affect ability to measure?*)

Miller's finding that counting may be a false cue to amount is particularly important in understanding units of length. In the present research, Research Questions 5: *How well do children understand that a number may express length?* and 9a): *Do children understand the inverse relation between size and number of units?* followed up Miller's finding.

1.2.2.6 Classroom development of 'tools of thought': an example

The idea of a 'tool of thought', where a practice and its characteristic tool enter into the conceptual development of their users was worked out in detail in a project to develop five-to-six-year-old pupils' understanding of measurement run by the National Council for Teachers of Mathematics (NCTM) (Gravemeijer *et al.*, 2003). The ruler or numberline constituted the overarching tool; during a series of sessions, it underwent various transformations or 'inscriptions', including footprint-units and other variously-sized and numbered units, according to the outcomes of pupil discussions. The *conceptual* insight of most interest arose from the notion of "measuring as an act of structuring space" (Gravemeijer *et al.*, 2003: 58) and that "the results of iterating [a unit] signify an accumulation of distances". Here the Piagetian order is reversed: measurement does not await a developed conception of space; rather it produces it. The subdivisions on the ruler, as the embodiment of the iterated unit, now reflect *accumulated changes of position*; physically constructed in the classroom, they may no longer represent a paradox. Difficulties commonly associated with them are, however, considered next.

1.2.3 Numerical components of measurement: units

In this section, studies of children's understanding of standard and non-standard units are reviewed. Piaget noted the difficulty children may face in the idea that a given length can be both one and many: that it can be subdivided yet retain its continuity (Piaget & Inhelder, 1956). Others have suggested that re-organisation of a whole conceptual field is required here. Subdivisions of a length in relation to the whole constitute ratios, expressed as a fractional number, and difficulties with units in measurement have been seen as one example of a general difficulty in moving from whole numbers to fractions (Brown *et al.*, 1995).

> While the concept of ratio can remain implicit when using non-standard units or reading off labeled units on scales, in order to operate with many subdivisions on scales, ...an explicit grasp of ratio...[is] required. (Brown *et al.*, 1995: 158-9)

The inverse relation between size and number of units in measurement both of extensive and intensive quantities is a case of special importance (Howe, Nunes, Bryant, Bell, & Desli, 2010). An idea that makes a strong showing in the literature on rational number is that opportunities for children to access what would be seen by adults as a single principle differ sharply according to context; teaching of principles needs to be carefully tailored in the light of this research, of which a selection is now reviewed.

1.2.3.1 Partitioning and unitising

Lamon's (1996; 1994) concept of 'unitising' suggests one way in which the transition from whole to rational numbers may be facilitated for children. Partitioning and 'unitising' (the use or creation of units) are inverse operations underlying rational number. Both are part of everyday experience: partitioning is about sharing amounts; unitising is about unifying them.

'Unitising' as a general strategy for the cognitive handling of quantities provides evidence of competent proportional reasoning in everyday life. Typically, reasoners choose their own units, which may be suggested by the social setting. Lamon's example is of canned drinks, where a single can, a six-pack or a whole case may be the unit selected.

'Unitising' empowers the reasoner to consider *simultaneously* a single aggregate and the items that make it up - for example a six-pack and the six cans of drink it contains. It provides a cognitive solution to the difficulty identified by Piaget of preserving the idea of continuity, while understanding that this can be represented as a series of discrete units. In Lamon's study, 9 to 12-year-olds partitioned food into fair shares. Some tasks presented countable items in composite units, like packs of drinks cartons, or chocolate bars marked in sections, while others presented whole units, such as pizzas, where children would be asked, for example, to share four pizzas between three people.

Lamon identified three unitising strategies: the *preserved pieces* strategy, the *mark all* strategy and the *distribution* strategy. In the *preserved pieces* strategy, three pizzas were left intact (three pizza-units) for distribution to the three recipients, while the last pizza was divided into three equal pieces (three one-third-pizza units) for distribution. In the *mark all* strategy, each pizza was marked into one-third-pizza units, but only one would be cut for distribution, the others being distributed as marked whole pizzas. In the *distribution* strategy, each pizza was both marked and cut in pieces for distribution. Economy of division varied: for sharing among three people, a pizza might be divided into six pieces; or it might be marked into six and then cut only into three. Lamon found an increase in economy according to age-group. However in the case of cans, or of a pack of chewing gum, all the items were drawn inside the pack. Lamon suggests that these were not yet *conceptually* single composite items seen as one unit.

Firm interpretation of such findings can be difficult, as shown by Goswami's (1995) study of 3 and 4-year-olds. Here children had three sizes of cup and the experimenter had corresponding pictured proportions of a whole object: of a pizza, shown in the pizza tin, of proportions of a drink, shown in the glass, and of different numbers of chocolates, shown in a box with spaces for eight. The children had to match the appropriate pictured food or drink proportion, to the cup of the same relative size in their own array. Almost all the 4-year-olds and over half the 3-year-olds told correctly which pictured proportion corresponded to which size cup, and so were apparently able to order the different pictured proportions of food or drink mentally. Goswami suggests, however, that children may have ignored, or not noticed, the fact that each proportion was part of a visible whole and considered the pictured proportions as 'honorary wholes'.

Lamon's studies suggest that children find it easy to decompose quantities into units, but only gradually learn to compose smaller units into larger ones. There is also a gradual progression from an emphasis on number and counting (*mark all* and *distributive* strategies) to conceptualisation of a continuous quantity that is also made up of units (*preserved pieces* strategy). The difference between the *mark all* and *distributive* strategies is particularly interesting. It suggests that children may need the reassurance of visual subdivision, but that they also understand, in a situation where an item really needs to be split and distributed, that visual subdivisions can be grouped into larger units.

The notion of 'unitising' is illuminating, and constitutes a novel approach to the paradox of 'continuity and subdivision' identified by Piaget & Inhelder (1956). It does not dissolve the paradox, however. In the present work, several research questions pursued this paradox and its instantiations. These are, Research Question 8: *Is there evidence that children conceptualise in different ways, units that are physically separate and units that form a scale?* Research Questions 9a): *Do children understand the inverse relation between size and number of units?* and 9b): *Do children understand that larger units may 'contain' smaller ones?* and Research Question 10: *How well do children cope with fractional units?*

Petitto's investigation of children's understanding of the way in which numbers represent length on a ruler, and of the equal interval principle, are considered next.

1.2.3.2 Petitto's numberline estimation study

Petitto (1990) investigated the relationship, in six to eight-year-olds, between counting, use of a ruler, and spatial concepts. She noted that while young children count proficiently on a numberline, they do not appear to understand the proportional properties that guarantee equality of its units, or that units represent lengths. Two studies explored development in understanding of a numberline during the first three years in school, and involved counting and arithmetic, length conservation, and understanding of equal intervals.

In the first study, children were scored as conservers or nonconservers of length on a standard Piagetian task, then proceeded to a numberline estimation task. Six lines were marked only at their endpoints, three being marked 0 and 10, and three 0 and 100. An arrow indicated an (unmarked) point at a different position on each line. Told that the numberline was like a ruler with marks omitted, the children had to supply the number to which the arrow would point if all the lines were marked in.

Petitto hypothesised a *sequence to proportion shift* in understanding a ruler: that numbers would be seen first as a sequence of points, and later as representing equal distances (construed as understanding proportionality). Her second and third hypotheses offered opposing mechanisms for any such shift. The *component skills hypothesis* proposed that acquisition of relevant counting and arithmetic skills contributes directly to use of the types of strategy that facilitate proportional understanding. If appearance of such strategies closely followed the teaching of counting and arithmetic skills, the *component skills hypothesis* would be supported. A *conceptual differentiation* hypothesis proposed that only after length conservation had been achieved would children use relevant numerical strategies learned in school. To conserve length is to differentiate length from position, Petitto argued, so conservers would be more likely to see numbers on a numberline as indicating a series of lengths than a sequence of positions.

Petitto used children's accuracy in estimating target values on the numberline as a measure of their use of a specific type of strategy. If children were using purely sequential strategies, their estimates should deviate progressively more from the target as its distance from an endpoint increased, and deviation should be greatest near the midpoint of the line. If the sole endpoint used by a child is 0, then deviations from the target should be greatest near the upper end of the line. The component skills hypothesis would be supported, Petitto argued, if younger children, being more familiar with smaller than larger numbers, showed higher levels of deviation from a target score in the higher ranges, because this would be evidence that their counting and arithmetical skills with small numbers affected their accuracy. She argued that the conceptual change hypothesis would be supported by a sudden rather than gradual change in accuracy that did not differ between large and small values, and was positively correlated with conservation of length.

In the first study, experimental groups reflected major relevant topic changes in the school curriculum as well as age differences relevant to cognitive developmental changes. Among the skills taught during the three years of the study were counting by 1s, 10s and 5s, estimation prior to counting and measuring, and construction and use of rulers by iterating a unit. All three were considered relevant to strategies the children might use during the tasks in the study.

Overall, there was wider variation in accuracy when the target was near the mid-point (compared with variation near end-points) for the two younger than for the two older groups; this was claimed as evidence of a sequence-to-proportion shift. In the 0-100 numberlines, this shift was also supported by the fact that levels of deviation for the midrange and highest targets were lowest for the oldest group. However, over- or under-estimation in the scores with the 0-10 numberline, did not support any of the study's hypotheses. There was no evidence of an effect of conservation on scores, and hence none for the conceptual differentiation hypothesis.

Abrupt change in the deviation scores had been proposed as evidence for the conceptual differentiation hypothesis. All significant differences in scores were, however, between groups with at least 12 months schooling separating them. Although strategy use did improve with age, and although children did progress from sequential to proportional reasoning, Petitto concluded that the findings were best accounted for by the component skills hypothesis, and that there is a 'gradualist' picture in which change follows school instruction, particularly in the application of arithmetic and counting skills.

In these ambitious experiments, Petitto rigorously pursued the relationship between counting and the development of proportional reasoning in relation to length, required for successful measurement with a ruler. However, the variables she included made her specific hypotheses rather difficult to test. The present research asked more simply: *How well do children understand that a number may express length?* (Research question 5) and tested this in measurement tasks by comparing qualitative judgements of the length of lines with judgements of the same lines expressed as numbers of units.

1.2.3.3 Equality of standard units: testing understanding of equal intervals

To function as a measuring instrument, a ruler must be divided into equal intervals. Petitto argued that this may be recognised as a convention, or may instead reflect principled understanding. She hypothesised, as before, that aspects of school instruction may indirectly affect principled understanding of units. For example, early school experience of measuring by iteration of a unit may encourage a view of measuring as a series of 'steps', and may fail to give children a sense of measuring a whole length as a single entity, while later work on fractions, with the notion of wholes divided into equal parts, may support them in understanding the role of subdivision in measurement. As before, she hypothesised that length conservation was a pre-requisite for any instructional effects on understanding of units.

The second study provided two tests of conventional aspects of equal intervals: the ability to recognize an equal interval ruler among rulers of unequal intervals, and recognition of the former as appropriate for a measuring task. It also provided a test of principled understanding by presenting a task where it would be convenient to abandon the equal interval ruler, to see whether there was resistance to doing this. The children were presented with two sets of five rulers (half with a range of 0-10 units; half with a range of 0-20) that were of equal intervals; of increasing or decreasing intervals; of alternating intervals (where each was half or double the width of its predecessor); and of irregular intervals. Each interval was numbered. Children were asked to identify from a simple description of the interval configuration, which ruler the researcher had in mind, and then to choose which ruler would be best for measuring. All either chose the equal interval ruler or said that any would be equally good.

There was no significant effect of age-group or of conservation status on choices of the equal interval ruler as best. A child was credited with understanding the equal interval principle if the equal interval ruler was preferred *consistently*. All children who satisfied this criterion were conservers. The author concluded that awareness that conventional rulers have equal intervals, and conservation of length are both necessary conditions for understanding the equal interval principle. However, since over half the children in her sample with these two accomplishments did not use the equal interval ruler consistently, they were not sufficient conditions. These studies suggest there may be a complex interaction between various school activities in mathematics and conceptual change in learning to measure, but they also illustrate the difficulty of tracking any influence such activities may have.

Nunes and Bryant (1996) tested whether 5 and 6-year-olds knew that standard units must be equal. A pictured ruler was marked similarly to those the children had in their classroom, but without numbers. The children were asked to write in the numbers. The experimenters observed whether children a) left equal spaces between their numbers; b) consistently left blank spaces between numbers (showing that they understood subdivision); and c) either started with 0, or left a blank space before 1 (showing that they understood that the spaces on the ruler were the units).

Of ninety-two children without task-related motor problems, 40% did not write their numbers in one-to-one correspondence with units. Half the children labelled only the cm marks, thereby acknowledging subdivision. The others labelled both cm and half cm marks (incidentally demonstrating that the children were not simply relying on their memory of rulers in the classroom, on which half-cms were not numbered.) Eighty-nine per cent wrote their 1 opposite the first line on the ruler and did not allow for 0, indicating that they did not, after all, understand how the ruler represented units. Of a sample of children interviewed about why they had placed their numbers where they did, only one had allowed for 0. He had done this by writing numbers *in the centimetre gaps* as he explained. The other children did not appear to understand the question, making responses such as *because that's where the numbers go* or simply checking that their number sequence was correct. The latter response seemed to indicate that children understood the task as just another way of implementing a counting procedure.

In a second task, children were asked whether rulers drawn in various ways were correctly drawn. Two rulers did not have equal intervals. Roughly the same number of children recognised these rulers as incorrect as had written their numbers at equal intervals on the pictured ruler in the first task. Four equal interval rulers were shown, two with imperial and two with metric markings. One 'metric' and one 'imperial' ruler started from 1, and the others from 0. About three-quarters of the children identified the metric rulers as correctly drawn, whereas less than half thought the rulers showing inches were correct, indicating a judgement on the basis of classroom familiarity with centimetre rulers rather than on the principled basis of equality of intervals. Children's understanding of the relationship between the size and number of units in general, of the way units are represented on rulers, and of the equality of units within a single scale were all investigated here.

The difficulties revealed, and those revealed in Petitto's second study, concerned understanding the intervals marked on a ruler. These were examined in a different, and in a sense, more direct manner in the present research. Here, children's measuring behaviour was observed for indications of strategies they may have been adopting (Research question: 6. *How well do children measure?*) and they were questioned about scales they were shown on everyday measuring devices. (Research Question 9a): *Do children understand the inverse relation between size and number of units?* and Research Question 10*: How well do they cope with fractional units?*)

1.2.3.4 Conversion of units

Nunes and Bryant (1996) went on to investigate children's understanding of relationships between units of different sizes. They noted particular difficulty in converting one unit value to another. This required understanding that smaller units may be 'contained' within larger ones and was the principle Lamon (1996) considered might be established by 'unitising'. They described a study (Davydov, 1982) in which children, shown two sizes of glass, were told that two small glasses held the same amount as one large one. Asked how many large glassfuls of liquid there would be in a collection of large and small glasses, the children responded with the total number of glasses. Nunes and Bryant noted that no numerical operations were required here: if a child understood units and was presented with two pieces of ribbon each 8 units long, but one measured in larger units than the other, they should judge that the ribbon measured in larger units is the longer ribbon even if unable to quantify the relationship between the units. This would be a relative and protoquantitative judgement. (The more difficult absolute judgement would involve finding a numerical value for the difference in length of the two ribbons, and this would necessitate expressing one of the units used in terms of the other). The hypothesis that children would know which was the longer of such a pair of ribbons without being able to quantify the difference was tested with five to seven-year-olds. Pairs of ribbons of similar colour but different lengths were assigned one ribbon to the experimenter, and the other to the child. Children were first familiarised with centimetres on a ruler. In the first block of trials children were given information about each pair of ribbons (the experimenter's and their own) entirely in centimetres. The information was sometimes of an absolute character (for example *I measured my ribbon and it was 6 cm long. Yours was 4 cm long*) and sometimes a mixture of absolute and relative information (*My ribbon is 6 cm long and yours is longer than 7 cm.*), as children might later need both.

In the second block, the experimenter's ribbons were measured in cm and the child's in inches. The children were first familiarised with the difference between cm and inches and with the fact that 5 cm was equal to about 2 inches. All information given was now absolute, but relative judgements would be made if children attempted to compare different units. In both blocks the children were asked: *Are the ribbons the same length, or is one longer than the other?* Feedback was provided by measuring the ribbons against each other after the test.

In the test where only cm were used, there was no difference in performance between the absolute and the absolute + relative conditions. Mean correct responses increased with age and quickly reached a ceiling. There were more mistakes in both age-groups where both cm and inches were used, but even the 5-year-olds performed better than chance, indicating, Nunes and Bryant argued, some sensitivity to size of units. A comparison was made between trials where the number of units was the same but the size was different, and trials where both number and size were different. In the first case children had only to reason that the larger units indicated the longer ribbon, but in the second units had to be somehow converted for a judgement to be made. Performance improved with age for both conditions, but for 5-year-olds was above chance only when the same units were used, and in the condition with different units, even 7-year-olds did not get maximum correct responses.

Nunes and Bryant concluded that children can manage simple relational reasoning (the larger the unit, the greater the length) when the number of units is the same, but find difficulty in converting units, even when this had been demonstrated, and visual reminders given. They concluded that despite children's familiarity with rulers, the way in which these represented units was not understood; that when, for example, children read a ruler and said *five centimetres* they were merely applying a procedure whose underlying logic they did not understand.

The present research pursued the question of conversion of units in an everyday context. Children saw common measuring devices on which, side by side, scales showed different units, and were asked to comment on the relationship between them. It also tested generality of understanding by including devices that measured weight and capacity as well as length. (Research Question 5: *How well do children understand that a number may express length?* and Research Question 9b): *Do children understand that larger units may 'contain' smaller ones?*)

Nunes, Light & Mason (1993), considered next, investigated the influence of familiar versus unfamiliar measuring instruments and techniques on understanding of units.

1.2.3.5 Conventional versus non-conventional measuring instruments

Nunes *et al.* (1993) argued that children's measuring performance is influenced both by their understanding of the principles of measurement and (like Miller, 1989) by the type of representation of quantity embodied in measuring tools. They predicted age-related developmental differences, and also differences between children according to different measuring instruments used.

Paired six to eight-year-olds in three groups were asked to compare the lengths of two lines. One group was given a conventional ruler marked with centimetres and fractions of centimetres; the second group, a broken ruler that began at the 4 cm mark, and the third, a length of string. The objective was to determine whether use of the conventional 'tool' would result in better performance than that achieved with the string. The measurement principle of iteration of units is 'ready-made' in a ruler, while the string had to be used in makeshift fashion as an iterated unit and was intended to assess understanding of the principle. The purpose of the broken ruler was to test understanding that the numbers on the ruler indeed signified iterated units: if this was so, then the number four would not be taken as representing four units.

Results showed no significant age effects. Children using the ruler did significantly better than those using the string; there were no significant differences between either of these conditions and the broken ruler condition. All the children in the ruler condition gave numerical responses, 84% of them correct. Sixty-three percent of the responses in the broken ruler condition were correct, despite the greater rigour required in achieving success. In the string condition, only 28% of the responses exemplified rigorous procedures, either of iteration (the string being moved along when shorter than the line) or folding (when longer). The results were claimed as clear evidence that use of a conventional measuring tool aids measurement.

Eight and ten-year-olds participated in a similar study comparing two irregular areas of wall. Here the conventional measuring 'tool' was not a physical instrument, but the length x width algorithm. Conventional rulers and glued strips of ten 1cm bricks were provided to help. The make-shift element consisted of a collection of single bricks, abundant for one group; for the other, less than needed to cover the area to be measured.

There were no overall differences in the proportion of correct responses in relation to age or condition, but older children made better use of 'intellectual' strategies, that is, versions of the width x height algorithm using the bricks. Children frequently shifted to use of bricks from initial unsuccessful use of the ruler; many used bricks straight away in the second trial.

Overall, results showed that the tool that directly measured amount of surface (the bricks) was favoured over the more conventional but more complicated tool (the algorithm) and its aids. Some children who used bricks did incorporate aspects of the multiplicative algorithm, such as counting rows rather than single bricks. Some with insufficient bricks, however, still 'counted', using a mixture of real and imaginary bricks. The study shows that the influence of cultural familiarity is complex and not always facilitative, and that its explanatory power may be limited. Nunes & Bryant suggested, more simply, that having a practical solution to hand may discourage intellectual performance.

The present research required children to use both conventional and non-conventional devices for estimation and measurement, to see whether success varied according to these features. (Research Question 7: *Do differences among measuring 'tools' affect ability to measure?*).

1.2.3.6 Cultural tools

Underlying the work of Petitto and explicit in that of Nunes and colleagues is the notion that conceptual knowledge is embodied in 'cultural tools' such as the ruler, and that familiarity with such tools and their conventional uses can support acquisition of that conceptual knowledge: mastering the procedure helps construct the principle. Their work demonstrates, however, the limitations of this affordance. Success depends not just on familiarity with the conventional tool, but on the practical skill required for its deployment relative to less familiar but easier methods (Nunes *et al.*, 1993). Miller (1989) argued that the very specificity of the context that facilitates success with such tools may hinder generalisation of any principles learned.

To understand the relationships that exist between units in measurement, understanding of ratio is required. The topic of ratio reasoning is next considered.

1.2.3.7 Ratio

The ratio relationships that exist between units in measurement, among other topics involving the shift from whole to rational numbers, can create considerable difficulties for school students in the middle years (Behr, Harel, Post & Lesh, 1992). The inverse relation between size and number of units is a special case of the ratio reasoning involved. Reasoning about intensive quantities is another (Howe, Nunes, Bryant, Bell, & Desli, 2010). Recent research has investigated contexts in which children of five years upwards can employ ratio reasoning in advance of teaching about fractions, and conditions for their success (Nunes & Bryant, 2009). One finding is that social 'sharing' situations provide an early, reliable model for certain kinds of fractions: although dividing a smaller by a larger number in general presents difficulties, young children readily understand that one or more cakes can be fairly shared between a larger number of children (Nunes, 2008). Another finding is that an external ratio, which holds between different 'measure spaces' (Resnick & Singer, 1993) seems easier to represent mentally than an internal ratio. Dividing one cake fairly between several children is understood because children can use *correspondences* between the parts of two wholes (the cake and the group of children). *Partitioning* e.g. subdividing a unit of length (same 'measure space') is more difficult, seemingly because only one whole is involved (Nunes, 2008). Understanding the iteration of a unit as "accumulation of distances" (Gravemeijer *et al.*, 2003: 58) is thus only one side of the coin; the paradox of units as a series of nested intervals (Piaget *et al.*, 1960) remains difficult to grasp.

In the present research, Research Question 9a): *Do children understand that larger units may 'contain' smaller ones?* and Research Question 10: *How well do children cope with fractional units?* pursued the issue of partitioning.

The inverse relation in the concept of division

Correa, Nunes & Bryant (1998) explored the understanding of five to seven-year-olds of the principle that the more parts into which a whole is divided, the greater the number of parts. The tasks did not require the manipulation of numbers.

The first experiment involved *partitive* division. The two divisors were sets of blue and pink toy rabbits. Dividends were sets of 12 or 24 plastic 'sweets'. A same-divisor condition presented equal sets of blue and pink rabbits, while in a different-divisor condition the sets were unequal. Children knew that within each set of rabbits, sweets were distributed equally; however the total allocated to each rabbit was hidden. The experimenter then pointed to a pink rabbit and asked whether it had more, less, or as many sweets as one of the blue rabbits, and asked children to justify their answers. In the different-divisor condition, two types of error were possible: that a rabbit received the same number of sweets irrespective of group; or that a rabbit in the larger group received more sweets. One-third of 5-year-olds, half the 6-year-olds and 90% of 7-year-olds justified responses by appeal to the inverse relation ('more rabbits, so fewer sweets each'). The minority 'more is more' justification ('more rabbits, so more sweets each') did not decrease over the age-span, indicating for these children a persistent bias towards addition.

The second experiment involved *quotative* division. Here the number of rabbits in each group was unknown, but was determined by the number of equal allocations of sweets that could be made from a fixed collection: if each rabbit is to have four sweets, and there are 12 sweets in all, only three rabbits can have sweets. Again there were 'same divisor' and 'different divisor' conditions; the 'different divisor' condition proved harder, and there was improvement with age. In the 'different divisor' trials, less than one-fifth of responses from 5-year-olds were justified in terms of the inverse relation (between the number of sweets each rabbit could receive and the number receiving sweets), while now nearly one-third were justified using the incorrect 'more is more' principle (the larger the number of sweets each rabbit receives, the more rabbits can have sweets).

Limits of children's ratio reasoning

These studies showed, as the authors commented, a remarkable ability in a sizeable proportion of a sample of young children to understand the inverse relation between divisor and quotient independently of computation. The fact that others were probably applying (incorrectly), also independently of computation, the 'more is more' principle of addition, is also of interest. Correa and colleagues (1998) pointed out that the procedure of 'sharing', familiar to children from a number of contexts, probably facilitated understanding of partitive division. In the present research, Research Question 9a): *Do children understand the inverse relation between size and number of units?* investigated the inverse relation specifically in relation to scaled measuring devices.

Kornilaki & Nunes (2005), however, pursuing the theme of the proposed facilitative role for procedures, investigated partitive and quotative division of discrete versus continuous quantities (fishes vs. portions of fish-cake for cats). There were differences in correctness of reasoning according to age and type of division required, but *not* according to discrete vs. continuous quantities. Since the 'sharing' procedure is known to be more difficult for continuous quantities (e.g. Lamon, 1996), this was evidence that principled understanding may be independent of procedures. Further evidence came from the work of Howe, Nunes and Bryant (2010). In this study of reasoning about intensive quantities, children's success increased with age, and was predicted by the intellectual demands of a problem. In the study, the children were increasingly able to take into account variables that were not especially salient in everyday experience and language usage.

In an early study, Carpenter & Lewis (1976) argued, too, that measurement principles may come to be understood not through measurement activities, but as a by-product of general conceptual development. If so, they reasoned, there should be children who understood that there was an inverse relation between unit size and number of units (this being a compensatory relationship characteristic of all conservation), but failed to understand that quantities could not be directly compared if measured in different units. In tasks with six and seven-year-olds, a majority indeed said that more small boxes than large would be needed to equal the lengths of lines. A similar finding with liquid quantities enabled Carpenter and Lewis to claim a general developmental influence. However, when discussing lines previously agreed to be of equal length, but each measured with boxes of a different size, the same sample was confused about their length, only 10% successfully compared lengths, and few mentioned both size *and* number of units in their justifications. Hart's (2004) findings were very similar for between one-third and two-thirds (decreasing with age) of 11 to 16-year-olds asked, in a large national survey, about two paths measured respectively with paperclips and walking sticks.

Overall, these studies give evidence of the ability of young children to apply mathematical principles independently of their embodiment in number operations, and hence of their ability to understand formal properties of specific tasks. They are evidence, in fact, of the power and sophistication of protoquantitative reasoning. However, despite some commonalities (Squire & Bryant, 2003), this ability seems confined to specific types of task. Furthermore, while some argue that procedural knowledge plays an important developmental role (Resnick & Singer, 1993; Karmiloff-Smith, 1992) others argue for an independent conceptual route to understanding. Thus, while children clearly have some insights to help them understand rational numbers (Nunes, Bryant, Hurry & Pretzlik, 2006; Nunes, Bryant, Pretzlik, Bell, Evans & Wade, 2004) few broad generalisations can as yet be made. The present research contributed to this knowledge specifically in relation to measurement. It did so by assessing children's success in measuring tasks where it was necessary to take account of fractional units. (Research Question 10: *How well do children cope with fractional units?)*

1.2.4 Procedural and conceptual knowledge

In the research reviewed above, the part played by routines and procedures in the acquisition of conceptual knowledge is a persistent theme. Miller (1989) makes perhaps the strongest claim for the priority of procedures: that they may *determine* mental representation of quantity. Using child-generated procedures, Gravemeijer and colleagues (2003) applied this idea in the classroom with some success. Procedures enter Karmiloff-Smith's developmental model at Level 1, as the basis of all subsequent conceptual change. The theoretical relationship between procedural and conceptual knowledge in these authors is clearly-defined, and enables them to predict what difficulties children are likely to encounter in developing conceptual understanding. In other writers the relationship is less clearly-defined. Petitto (1990) concluded, on only indirect evidence, that classroom practice in counting and arithmetical procedures was the likely major contributor to improved conceptual understanding of proportion in numberlines. Resnick & Singer (1993) considered that protoquantitative reasoning, embedded in everyday language and routines for comparing quantities, was the basis for increasingly abstract mathematical understanding, but did not discuss the nature and possible difficulty of developments from the routines to the abstractions. Evidence that everyday routines of sharing contribute to the understanding of units (Lamon, 1996) and of the inverse relation in the concept of division (Correa *et al.*, 1998) was naturally available only for the highly specific contexts in which this was tested. There is also evidence of conceptual development taking place independently of procedures (Kornilaki & Nunes, 2005; Carpenter & Lewis, 1976).

With regard to conceptual competences, Rittle-Johnson & Siegler (1998) considered two opposed accounts of those which appear early. The 'privileged domains' hypothesis (Geary, 1995) proposed an evolutionary source for basic arithmetical concepts that precede relevant social experience. The contrasting 'frequency of exposure' hypothesis (Briars & Siegler, 1984) argued a procedural source in the early opportunities infants have for observing and imitating procedures such as counting, and suggested that they extract mathematical principles from such experience (although causal mechanisms are not considered). More advanced mathematical competences are culturally specific and must be explicitly, often laboriously, learned from instruction.

Some competencies relevant to measurement may usefully be considered in the light of these hypotheses. For example, infants' early discrimination of ratio characteristics of magnitudes (Feigenson *et al.*, 2004) may facilitate ordinal comparison of length in early childhood. Observation of, or participation in, measurement procedures in the home may support the 'frequency of exposure' hypothesis, depending on the frequency of such exposure and what is extracted from it. The framing of the present research provided for consideration of these possible influences (Research Question 1a): *What do children learn about measurement in their everyday social context,* and 1b): *How might this affect their conceptual understanding of measurement?*).

Allied to procedures, and sometimes merging with them, is the proposed role of 'tools' in the construction of knowledge and understanding of measurement (Gravemeijer *et al.*, 2003; Nunes *et al.*, 1993; Miller 1989), because to learn the correct use of a tool is to learn a procedure. The role of 'tools' in conceptual understanding is, however, a narrower question than that of the role of procedures, and the distinction between conceptual and procedural knowledge did not seem sufficiently clear-cut, despite its prevalence in the literature, to structure the present research. In the context of mathematical development, Rittle-Johnson & Siegler (1998:77) define conceptual knowledge as "understanding the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (whether or not that knowledge is explicit). They define procedural knowledge as "action sequences for solving problems". In the present research, measuring itself is clearly an action sequence, but it cannot be undertaken without some conceptual understanding of how units work in measurement. In one underpinning competence investigated (understanding the everyday language of length) conceptual and socio-cognitive understanding are intertwined. In another, (ordinal comparison of length) perceptual skills are salient, and these are not well-described as an action sequence. The current research therefore made use of the conceptual/procedural contrast only when this seemed enlightening.

1.2.5 The present research

1.2.5.1 Key themes that provide the rationale

The developmental and educational literature reviewed was written from a variety of perspectives, and the associated research undertaken for a variety of purposes. Persistent themes can nevertheless be identified across this range of material that each suggest some aspect of measurement or its conceptual underpinnings that are potential sources of difficulty for children. These are described below and frame the present research.

Overall, evidence from the studies reviewed demonstrated the overriding influence on research outcomes of the type of task employed. Increasing rigour and sensitivity of design provided insights that were valuable, but increasingly various. Moreover different insights were contributed by different samples of children.

The present research attempted to investigate children's understanding of measurement in, so to say, a more unified way. It did this, firstly, by restricting itself to testing of children's abilities in *basic competencies* underpinning measurement and in *measuring itself*, using simple and robust tasks. Secondly, a single sample of children participated, enabling some insight into how these competencies might be associated. (Research

Question 11: *Are there associations among understanding the everyday language of length, ability to make ordinal comparisons of length, and the ability to measure?*) Thirdly, the design, in which the same tasks were used with three school year-groups, enabled year-group and age comparisons. (Research Question 12: *Do age and length of time in school make a difference to any of these abilities?*) To help understand how these competencies were acquired, the research also investigated the children's own accounts of the home and school contexts in which they experienced measurement.

As Piaget saw, the competencies are interrelated. However, the first three aspects below may be considered relatively independent themes, whereas those in later sections are harder to separate.

Conservation of length

Logically, measurement presupposes the principle of conservation: the lengths of objects cannot be compared, or measured with an instrument, if length may change when position changes. In the literature, the criterion of understanding this principle has generally been success in traditional Piagetian tasks, where children saw material transformed before being asked to renew a previous judgement of length (see also Petitto, 1990; Miller, 1989; Boulton-Lewis, 1987; Piaget, 1970; Piaget *et al.*, 1960; Braine, 1959). The term 'conservation' has also been applied more broadly in the literature, where the context required judgement of invariance of length, whether or not a transformation had been witnessed (e.g. Hart, 2004; Department of Education and Science, 1981). Surveys appear to show that children even of secondary school age lack some aspects of the ability to conserve length in this broader sense. For Petitto (1990:58) success in a traditional Piagetian task constituted grounds for expecting "the ability to conceptualise the distinction between position and length" (and hence how units of length are represented) on a numberline. So the term 'conservation' has been liberally interpreted, credited with a key role in the development of measurement, but rarely (since Piaget) given a central focus. In the present research, children's ability to conserve attributes of length across various transformations was thoroughly tested, and was then compared with their measurement ability.

Absolute versus relative judgements

Ordinal comparisons of length are the logical foundation for measurement with units. Except for judgements of 'same length', ordinal comparisons are relative judgements and measurement with units absolute. The mathematical operations that Resnick (1992) proposed as early-developing categories of protoquantitative reasoning are all relative in character, and indeed there is evidence of discrimination of relative magnitudes in the form of ratios in very early infancy (Feigenson *et al.*, 2004). Bryant (1974) showed that in some contexts young children did find relative judgements of size easier to make than absolute judgements; for example they made more correct judgements of 'larger' or 'smaller' in respect of two items than correct judgements of 'same size'. If the ability to make relative judgements is acquired early, it is important to identify any relationship between this and the ability to make absolute judgements, because children as young as 5 years are expected to be taught to estimate in absolute terms (*Primary Mathematics Framework,* Department for Children, Schools and Families, 2006: Year 1 Block D. Assessment focus: Ma3, Measures) and because older children perform poorly in some tasks that require absolute judgements involving units. Examples are numerical estimation tasks using Duplo blocks (Brown *et al.*, 1995) or mm (Department of Education & Science, 1981) as units. What is needed is comparison of children's ability to make ordinal comparisons of the length of objects (relative judgements) with their ability to measure *the same objects* (absolute judgements). This was done in the present research, where visual estimations and measurements of the length of the same lines were made. Children also made relative judgements in tests of their understanding of the everyday language and concepts of length.

The paradox of continuity and subdivision

Piaget and Inhelder (1956) suggested that the idea that a given length can be both continuous and at the same time subdivided will cause conceptual difficulties for children if they experience continuous length as different in kind from discrete units. Gravemeijer *et al.*, (2003) showed that encouraging children to assemble their own units into a continuous length could overcome this difficulty. Petitto (1990) and Nunes & Bryant (1996) provided evidence that children understood subdivisions on rulers in some contexts. Lamon (1996) considered this 'one-or-many?' paradox as broader, and that, developmentally, children may solve it by 'unitising', enabling them to contemplate simultaneously a single unit and the sub-units that make it up.

Piaget also suggested, more concretely, (Piaget *et al.*, 1960) that the nature of measurement itself, i.e. physical iteration of a unit, is more evident when a single unit can be physically moved along an object than when a continuous scale embodying the same unit is used to measure the object; hence the use of a scaled instrument may prove more difficult than the use of discrete units such as blocks. In contrast, Petitto (1990) suggested that if children are taught at first to measure length by iteration of a unit, they may cease to consider that they are measuring a single entity.

Here, an area of difficulty is identified without much convergence as to its nature. To pursue the enquiry, the present research asked children, over a large number of trials, to measure a continuous length with a ruler and tape measure (instruments embodying continuous scales) and to measure the same length with wooden cubes (discrete units). If different devices produced differences in performance, their character would be described and possible causes considered.

Ratio and proportion

Nunes and Bryant (1996) identified the notion that larger units may contain smaller ones as difficult for children. The surveys reviewed and the *Primary Mathematics Framework* testify to educational concern that children should be able to convert from one standard unit of measure to another. In the literature there is consensus that these ideas require an understanding of ratio and proportion.

Despite evidence of receptivity to ratio information in infancy (Feigenson *et al.,* 2004) and of both early (Resnick & Singer, 1993) and later (Correa *et al.*, 1998) ratio reasoning in limited contexts, the ability to handle ratio and proportion in school mathematics is generally considered a major hurdle of middle childhood (Behr *et al.,* 1992) of which surveys give evidence (Hiebert, 1981). Reluctance to convert units, and other difficulties involving the idea that larger units contain smaller ones, as well as the tendency to count numbered points on a ruler rather than spaces (Petitto, 1990), probably reflect the failures in proportional reasoning that also underlie difficulties with fractions (Brown *et al.,* 1995) and intensive quantities (Howe, Nunes, Bryant, Bell, & Desli, 2010).

The inverse relation between size and number of units is a special case of proportional reasoning that underlies all measuring with units and is of fundamental importance, but there are persistent reports of difficulties (Howe, Nunes, Bryant, Bell, & Desli, 2010); Nunes and Bryant, 1996; Carpenter and Lewis, 1976). In the National Assessment of Educational Progress Fourth Assessment (1988), no less than 50% of the 12-year-olds surveyed had difficulties in one task. In the present research, understanding of this inverse relation was explored in a realistic context as children commented on the scales shown on various measuring instruments.

Mapping number on to quantity

Quantitative measurement involves the application of number to amount. As abilities to be "assess[ed] for learning" in measurement, the Primary Mathematics Framework has a mixture of counting skills and qualitative comparisons, with no suggestion that the two processes may not be entirely congruent for children. *I can use counting to solve problems involving measures* as a "children's learning outcome" encapsulates this apparent lack of awareness. (Department for Children, Schools and Families, 2006: Year 1, Block D, Ma3, Measures). While some authors (Resnick and Singer, 1993; Petitto, 1990) expected development from qualitative comparisons of amount to the use of units of measure through application of counting skills, others have disputed this account. Lamon (1996) found a gradual development from over-dependence on number and counting to conceptualisation of a continuous quantity (that is, development in the opposite direction) while Miller (1989) showed that when number is not a valid cue to amount, children's early facility with number and counting will simply lead them astray. Nunes and Bryant (1996) found that children's labelling of a pictured unnumbered ruler did not necessarily show understanding of units, but often resembled a counting procedure. Carpenter and Lewis (1976) and Hart (2004) found children judging greater quantity on the basis of greater number, rather than greater size of units.

The present research followed up these somewhat disparate findings with an investigation of children's abilities to integrate qualitative and quantitative judgements of amount. Number was introduced (see Chapter 5) into a study of visual estimation by asking children to make comparisons of ordinal length with a given number of visible units. Later (and conversely), after measuring in units (see Chapter 6) they were asked to compare their numerical measurement with a given length in ordinal terms (i.e. shorter, longer, or same).

A different type of problem in numerical measurement is to do with its approximate nature, and with fractional units. In the present research, children were presented with a number of lines to measure that included fractional units, so as to see how they dealt with this situation.

Tools, procedures and principles

A diverse literature variously argues the degree and type of influence of procedural competence on conceptual understanding (Kornilaki & Nunes, 2005; Rittle-Johnson & Siegler, 1998; Karmiloff-Smith, 1992; Resnick, 1992). Allied to procedures, and sometimes merging with them, is the proposed role of 'tools' in the construction of knowledge and understanding of measurement (Gravemeijer *et al.*, 2003; Nunes *et al.*, 1993; Miller, 1989). The cultural specificity of tools is also an important theme. In the present study, children were asked to comment on and to use a range of everyday measuring devices as a means of exploring their measurement skill, knowledge and understanding. The complicated way in which these tools represented units was left intact, to see how children dealt with these culturally specific features.

Influence of social context

While the literature reviewed frequently asserts the socially situated nature of children's measurement ability, it rarely offers any social contextualisation for the research reported. Without some knowledge of the social context, however, it is very difficult to tell what factors might influence development of measurement. The present research sought that understanding by using information from the children themselves about their measurement experiences at home and in school as context for the research and to inform interpretation of the results.

1.2.5.2 Overview of the research

The themes discussed above were incorporated into four investigations with primary school participants. First, their *experience and general knowledge of measurement* were explored through interviews. Second, their understanding of the *language and concepts of comparative length* was investigated, then their ability to make *visual comparisons of ordinal length,* and finally their *measurement* ability. Associations among performance in language, visual comparison, and measurement tasks were then explored.

In the interviews, the children's language use was unrestricted. It was, however, highly constrained in the rest of the research by the design of the tasks, and specifically the form of the experimenter's questions and the verbal responses they invited. The advantages of a more expansive design were considered, (particularly Piaget's exploratory style of questioning and his invitation to justify responses) but it was decided that these were outweighed by the increased difficulty of interpreting data obtained in this way. The similarity across tasks of the language used, and invited, also facilitated comparison of the outcomes.

1.2.5.3 Research questions

The research questions were introduced in the literature review, where they addressed issues (brought together in the key themes above) that required further investigation. They are as follows:

In relation to the sample studied:

1. a) What do children learn about measurement in their everyday social context, and b) how might this affect their conceptual understanding of measurement?
2. Does the everyday language of length present any difficulties to children?
3. a) In the context of everyday length comparisons, do children conserve length? b) If not, does this make any difference to their measurement ability?
4. How well do children make ordinal comparisons of length?
5. How well do children understand that a number may express length?
6. How well do they measure?
7. Do differences among measuring 'tools' affect ability to measure?
8. Is there evidence that children conceptualise in different ways, units that are physically separate and units that form a scale?
9. a) Do children understand the inverse relation between size and number of units? b) Do they understand that larger units may 'contain' smaller ones?
10. How well do they cope with fractional units?
11. Are there associations among understanding the everyday language of length, ability to make ordinal comparisons of length, and the ability to measure?
12. Do age and length of time in school make a difference to any of these abilities?

Chapter 2

General method

For convenience, general aspects of the method are set out here. Aspects relating to specific phases of the research are described in the relevant chapters.

2.1 Sampling rationale

Five to eight-year-olds in three successive primary school years were selected for the research. This age-span had a Piagetian basis as the period during which concepts underlying measurement could be expected to mature (Piaget *et al.*, 1960). Concurrently children received instruction about measurement in accordance with the National Curriculum, so that on both counts age-related improvements in measurement might be expected. The research was based on successive waves of data collection from the same sample of children. This had the advantage of enabling examination of the relationship between different aspects of their performance.

2.2 Design

The research consisted of two interviews and three sets of tasks. All children were seen individually and all experienced each interview and task.

2.3 Participants

The participants were eighty-three children attending a non-denominational London state primary school. Families were mainly of British White and African-Caribbean heritage and of mainly low socio-economic status as indicated by the eligibility for free school meals, which was 65% of the total on roll. The participants consisted of three complete school classes in successive year-groups as follows.

Table 2.01 Age and gender of the participants

	Year 1	Year 2	Year 3[1]
Mean (SD) age (years) at first interview	5.66 (0.31)	7.22 (0.25)	8.37 (0.33)
Range	5.18 to 6.19	6.80 to 7.59	7.96 to 9.09
Boys	14	8	15
Girls	15	17	14
N	29	25	29

Note 1. Please refer to *Pattern and organization of sessions* below

As Table 2.01 shows, there were more than twice as many girls as boys among Year 2 children. This imbalance in Year 2 had the potential to create a confound between age and gender where there turned out to be gender differences, and made it especially important to check these. Gender differences were tested throughout the research, and results reported in the appropriate Results sections.

There were fewer participants in some aspects of the research than others, because children had left the school during the period of the research or were otherwise unavailable. No newcomers were added to the sample. The number of participants in each part of the research is set out in Table 2.02

Table 2.02 Phases of the research with number of children participating in each phase, according to year-group and gender

Phases (1 - 5)	Year 1			Year 2			Year 3		
	Boys	Girls	N	Boys	Girls	N	Boys	Girls	N
1. 1st interview	12	12	24	8	17	25	14	12	26
2. Language tasks	12	12	24	6	15	21	14	13	27
3. Estimation tasks	12	13	25	7	16	23	15	13	28
4. Measurement tasks	12	13	25	7	16	23	15	13	28
4. 2nd interview	12	14	26	6	15	21	14	12	26

2.4 Materials

Interviews

The first interview, which sought information about children's general knowledge and experience of measurement, employed an audio-recorder. The second interview, also audio-recorded, was structured around six scaled measuring devices in everyday use which children were shown and allowed to handle as they commented on various aspects of the devices. In the text, transcribed verbatim utterances are usually italicised.

Tasks

Tasks that investigated children's understanding of the language and concepts of measurement employed small plastic or metal toys and various items depicted on A4 cards. Tasks that examined their visual estimation and measurement abilities used A4 cards showing lines of various lengths. Various conventional and informal measuring devices were also used.

Full details of all materials are located in the sections that report the interviews and tasks. Appendix 1 shows examples of the materials.

2.5 Procedure

Setting

Children were seen individually in the school library, a quiet room relatively free of interruptions. For all sessions, the child and researcher sat facing each other about 90 cm apart at a circular table. Table and chairs were child-height.

Pattern and organization of sessions

The order of the two interviews and the three series of tasks was the same for each child. The first interview introduced children to the topic of measurement. Next came the series of tasks that investigated their understanding of everyday language and concepts of measurement. These could be accomplished fairly rapidly, and involved equipment that the children could be expected to find entertaining. The estimation and measurement tasks were generally more demanding. They required more time, care, and accurate physical manipulation of equipment by the children, and were scheduled next. It was considered that by this time, the topic of measurement would be engrossing the children and would support their persistence in these tasks. The second interview came last. It probed children's understanding of measurement more closely than the first, and specifically of measuring instruments and the units they showed, continuing but broadening the focus on instruments and their scales begun in the measurement tasks. In concluding the research, the second interview had the added benefit of re-affirming to the children the importance of their own knowledge and opinion.

The research took place during the Summer term of one school academic year, and the Autumn and Spring terms of the next. Since it was not possible to arrange for the Summer holiday to occur at the same point in the research schedule for all three classes, careful thought was given as to how this break could best be managed. Since the break was likely to have less effect on the older children, the following schedule was implemented. Year 3 children participated in the first interview and carried out the tasks on the language of measurement in the Summer term of one academic year. They undertook the estimation and measurement tasks and participated in the second interview in the Autumn term of the succeeding year, when, it should be noted, they entered Year 4. They are referred to as 'Year 3 children' throughout for the sake of simplicity. Year 2 children accomplished the entire sequence of work in the Summer term. Year 1 children completed the entire sequence in the Autumn and Spring terms.

Table 2.03 Phases of the research, number of responses required, number of sessions, and allocated time per session. All year-groups were similar.

Phases (1 - 5)	1. First interview	2. Language tasks	3. Estimation tasks	4. Measurement tasks	5. Second interview
Responses required	7 questions	75 responses	105 estimates	27 measurements	8 questions
Allocated sessions and time (minutes)	1 session (35 min)	1 session (15 min)	1 session (30 min)	1 to 2 sessions (40 min)	1 session (35 min)

General characteristics of the tasks and interviews

Table 2.03 presents an overview of the type and amount of work, and of the time taken, by the programme of research. It shows the five phases of the research, the number of sessions associated with each phase, and the time allocated to each session. 'Phase' refers to distinct aspects of the research, each phase consisting either of a series of tasks, or an interview. The results for each phase are reported in separate chapters of the thesis, except for the two interviews, which are reported in a single chapter. (This is because although the subject-matter of each interview was distinct, both interviews were about 'general knowledge and experience of measurement' and both contributed to the overall picture.) The sequence of phases (1-5) was invariant for each child.

'Session' has its usual meaning of a block of time spent on an activity with a single child, and 'time allocated' refers to typical session lengths as planned by the researcher. These turned out to be reasonably accurate in piloting. However, no aspect of the research was actually time limited: each child was allowed to spend as long as they needed to complete the tasks or interviews. There was thus no systematic variation in time allocated to children in different year-groups.

One session could be organized to be consecutive with another from the next phase of the research. Thus the first interview was generally followed up on the same day with the language tasks; and the estimation tasks with the measurement tasks. There were occasions where a session was interrupted at a suitable point by the researcher when she judged that a child was becoming tired. When this happened, the session was resumed as soon as feasible.

In the language tasks, the children reported in qualitative terms the comparative length, width or height of various items presented in displays. Altogether children made 75 comparisons, but as these were purely visual comparisons, this phase of the work was fairly rapidly accomplished, as Table 2.03 indicates. The 105 visual estimates were also accomplished more quickly than their number might suggest, again because they were perceptual judgements and required little manipulation of materials by the children. The measurement tasks (27 measurements) required the most effort and were the most time-consuming.

The researcher collected children individually from their classrooms; sessions were planned to fit in with class and individual work schedules.

2.6 Conduct of the researcher

The researcher had enjoyed twenty-two years' primary teaching experience, and this enabled good rapport with children and school staff as well as knowledge of classroom and school procedures, in which she participated when appropriate, so that her presence became accepted as routine. Her experience also informed her decisions concerning organization and management of the research, including minor adjustments in response to judgements of children's mood and level of engagement.

2.7 Ethical considerations

The support of the school head and relevant staff were obtained, and the consent of parents and carers of participants was sought by letter. No parent or carer withheld consent. The children were asked at the first session whether they were happy to participate. After initial shyness on the part of some, all agreed, and most thereafter came to sessions with enthusiasm. When during a long series of tasks there were signs of fatigue, a session was terminated and the child was told that the series would be finished another time, and was taken back to class. Occasional boredom and lapses of concentration, on the other hand, were countered with encouragement and persuasion to continue. The researcher relied on her knowledge of individual children to make suitable judgements.

2.8 Piloting

Interviews and tasks were piloted with a sample of children covering the same age-range in another school of similar intake, and minor adjustments made.

2.9 Analysis of data

All statistical tests were performed on a count of the number of correct responses made by children in their performance of tasks. Since the number of observations differed among tasks, descriptive statistics are generally reported as percentages correct, so as to facilitate comparisons.

Counts were compared using general linear models with repeated measures. Mauchly's test of sphericity was applied. If the test statistic was non-significant, sphericity assumed degrees of freedom were used. If the test statistic was significant, and Greenhouse-Geisser epsilon value $\leq .75$, then Greenhouse-Geisser degrees of freedom were used. Where Greenhouse-Geisser epsilon value $> .75$, Huyn-Feldt degrees of freedom were used. All t tests were 2-tailed unless otherwise stated. All post hoc tests were made using the Bonferroni adjustment.

Chapter 3

General knowledge and experience of measurement

3.1 Introduction

Essential features of measurement (for example dimensions and units) that children call spontaneously to mind can provide a broad indication of the current state of their conceptual elaboration of the domain. What they take to be the purposes of measurement and the kinds of measuring activities that they readily recall can help us understand the contexts and circumstances in which they acquire knowledge of it. Information about these contexts and circumstances may in turn suggest the kinds of influences that shape children's developing understanding of measurement and its underlying principles. The first phase of the research therefore explored the general knowledge of measurement of the children in the sample. This was done in two semi-structured interviews.

The first interview, in which the researcher engaged each child at their first meeting, focused on children's basic conceptualisation of measurement and the contexts in which this had developed. It elicited what children called to mind when they heard the term *measure* and cognate terms, and what experiences of measurement they recalled having at home and at school. The second interview focused, more narrowly, on children's knowledge about specific measuring devices, the dimensions of measurement with which the children associated them, and the units they measured. It also explored children's understanding of the general principle of the inverse relation between size and number of units in the context of specific measuring devices. Children were interviewed individually, each of the two interviews lasting about thirty-five minutes.

3.2 The first interview

The first interview explored the experiences the children reported having, or activities they reported witnessing, that were considered by them to be measuring activities. It sought information about the kinds of things children had measured or seen measured, about what objects they considered to be measuring devices and how they thought these measured. Dimensions of measurement that children spontaneously mentioned, such as length, weight and capacity/volume were noted, and also terms they mentioned that denoted aspects of these dimensions, such as *tall*, *wide, heavy* and *how much inside.*

Names of units of measurement employed by the children, and the dimensions they applied them to, were also noted.

The first interview also checked children's ability to make direct comparisons of length, in preparation for a later part of the research, where they were asked to make an extended series of such comparisons. In the interview, children were asked how they would compare two pencils of different lengths, and to demonstrate this. They were also asked how they would check the comparative heights of themselves and the experimenter.

The opportunity was also taken in this interview to bring to children's attention the distinction between the terms *taller* and *higher* in preparation for a later part of the research where they would compare vertical length of items. These were to be presented at different spatial levels to test conceptual understanding of intrinsic height of items (tallness) vs. height above ground. It was considered that while actually distinguishing between the concepts 'taller' and 'higher', some children might use one term - for example *taller* - for both. This is further discussed in Chapter 4. To offer children the opportunity to consider the distinction between the two terms, the difference in meaning was explained and illustrated in the first interview after they had been asked to compare their own height with that of the interviewer.

Finally, possible gender differences were considered in what was communicated by children about their knowledge and experience of measurement. The first interview was selected for this purpose because here, children were invited to talk about measurement in the broadest terms. Gender differences were considered, overall and for each year-group, by examining differences between boys and girls in their responses to the question *Can you tell me what 'measure' means?* and the questions that followed it about measurement at home and at school, that is, questions a) b) and c) in the schedule below. What was compared for boys and girls was a count for each child of the number of types of response classified in the manner shown in Tables 3.02; 3.04; and 3.07.

3.2.1 Method

Participants

The sample for the first interview consisted of seventy-five children, after exclusion of six children, on the advice of their class teacher, at an early stage of English acquisition (three each in Years 1 and 3); and two children (in Year 1) for whom there was a substantial amount of missing data due to a fault in the recording of the interview. There were twenty-four in Year 1; twenty-five in Year 2; and twenty-six in Year 3.

Materials and procedure

At the first interview, the investigator introduced herself as a researcher interested in measurement and in children's knowledge of it, and then invited the child to talk about their experiences of measurement. The interview schedule, set out below, started with a general question whose purpose was simply to introduce the term *measure* and see what ideas children spontaneously associated with it. Children were next invited to recall their experiences of measurement at home. The interview then moved on to ask about their experiences at school. Finally, children were invited to make a direct comparison of length and then asked about a specific height comparison.

The participants were allowed to be as expansive (or as reserved) as they wished, and were not interrupted, as information about measurement itself was sometimes embedded in a wider narrative that could give important insights into the social context in which measurement activities took place. If there was no immediate response to a question, five seconds were allowed to elapse before moving to the next (Black, Harrison, Lee, Marshall & Wiliam, 2002). Children were not prompted to elaborate their response except where this had been tautological, 'giving back' the words of the question (e.g. *measurement is when you measure something*). A neutral prompt such as *Could you say something more about that?* was then used. Prompts were not offered in other circumstances in the first interview, because it became clear at an early stage that children who were reluctant to respond became more so if the question was pursued at once. In these circumstances it was felt that persistence risked eliciting a response to the researcher's surmised expectations rather than to the question asked. However, questions were returned to as appropriate later in the interview. All interviews were tape recorded and transcribed.

Schedule

a) *I'm interested in what children of your age know about measuring. We do it in school and we sometimes need to do it at home. Can you tell me what 'measure' means? What do we mean by 'measure'?* If the child seemed unable to answer this question the following explanation was given before proceeding to the remaining questions. *What I mean by measuring is when you find out how big or small, or heavy something is, or how much of it there is.*

b) *Have you ever seen someone measuring something at home? Tell me about it.* Here children reported activities in which they had participated as well as those they had observed.

c) *Have you done any measuring at school? Tell me what you did.* Children interpreted 'you' in its collective sense, and described measuring they had seen classmates do as well as measuring they had done themselves.

d) Two pencils, lying side-by-side on the table, and differing in length by approximately 1 cm, are indicated. *Can you tell me just by looking, which of these two pencils is longer? ...Can you think of a way we could check which is longer? Can you show me?*

e) *Who is taller, you or me? How do you know?* If the child affirmed that the interviewer must be taller because she was an adult, the probe *Are adults always taller than children?* was used.

f) *How could we check?* If the child affirmed that no check was necessary, as the answer was obvious, the following was added *If you had a friend and you wanted to check who is taller, how would you do that?*

g) *If you stood on the table, would that make you taller than me?* If the child replied that they would be taller than the experimenter if they stood on the table, she responded: *Actually you wouldn't be taller. You would be **higher** if you stood on the table, but you wouldn't really be **taller**, because you didn't suddenly grow, did you?* As explained above, this part of the interview was intended as a 'seeding' experience in preparation for a subsequent part of the research; no check was made immediately afterwards on understanding of the distinction between *taller* and *higher.*

Classification of responses

Transcripts were carefully read, and patterns were sought within the children's responses. Separable, broad features of measurement discerned in them provided the basis for classification. In allocating children's utterances to categories, the overall meaning of an utterance was considered, and specific terms children used were subordinated to this. For example, as shown in Table 3.01, two specific responses to the question *Can you tell me what 'measure' means?* were *What size something is* and *Making sure a building is the right size for people.* While both mention *size,* in the first response the term was used as an attribute and described a generic dimension or aspect (c.f. 'What height something is' or 'how high something is'). In the second, *size* was used to express a relation, and the response was best classified with those that spoke of measurement as determining 'fit'.

A second reader was employed to validate categories, and to check reliability of allocation to them. There was complete agreement on categories. The reliability check was on a 25% sample for each question. The average inter-rater agreement on allocation of utterances to categories across all questions, and across both interviews, was 97%. Where raters did not agree on allocation of an utterance to a category, the utterance was discussed until agreement was reached. The same inter-rater procedure was adopted with all other questions.

Table 3.01 Categories of response to the question: *Can you tell me what 'measure' means?*

Response mentions or describes	Example response
A specific dimension (length, weight or capacity/volume) and/or one or more aspect of a dimension (e.g. tall, long; heavy; how much inside)	*Measurement tells you how tall or wide something is*
Non-specific dimensions (e.g. size) or aspects (e.g. big) only	*What size something is*
Measurement as determining 'fit'	*Making sure a building is the right size for people*
Measuring device - specific or generic (e.g. *thing with numbers*)	*You use a measure thing, the numbers tell you where it goes up to*
Specific unit	*It's centimetres, millimetres, metres*
Entity that may be measured	*We can measure tables, chairs, people*
Activity involving measuring	*If you want to make a drink for the baby*
Specific measuring activities	*In maths, if you measure something 'cos you don't know how long it is.*
Provision of a straight edge	*It's when you underline your work*
A tautology, not elaborated	*It's when you measure something*
No response, *Don't know*, or irrelevant response	*We write the letters down* (irrelevant response)

3.2.2 Results

Can you tell me what 'measure' means?

The purpose of this first question was to ascertain what, if any, ideas immediately came to mind for children when they heard the word *measure*, and what was looked for was any mention of measurement dimensions or aspects of dimensions; of measuring devices; of units of measurement; of entities that can be measured; or of activities involving measurement. Responses were classified according to these broad categories, which were not exclusive. For example, the response *When you're measuring a* [toy] *car you use a ruler. You need to write millimetres, centimetres or kilograms* was classified as having mentioned a measuring activity, a measuring device, and units.

Children's responses to the first question were typically brief. Ideas mentioned were occasionally mutually incongruent (for example, units mentioned were sometimes inappropriate to a dimension also mentioned). Classification of responses to this first question ignored such incongruence. So in the above example, the incongruity of *kilograms* was ignored.

Eight children (32%) in Year 3, three (12%) in Year 2 and six (26%) in Year 1 did not give an answer to this initial question, and were offered the explanation *What I mean by measuring is when you find out how big or small, or heavy something is, or how much of it there is.* In Year 1, the explanation generally followed silence on the part of a child (the usual wait of five seconds being allowed). In Years 2 and 3, the explanation was generally given when a child asked *What do you mean?* Table 3.02 shows the number of children mentioning the aspects of measurement listed.

A child who mentioned more than one instance included in a category was counted only once. In Table 3.02, the category 'a tautology, not elaborated' is amalgamated with 'no response, *don't know*, or irrelevant response'. Table 3.02 shows little apparent awareness about measurement on the part of Year 1 children, nearly half of whom failed to respond, or gave uninformative answers to this question. At most, a quarter of children in this year-group responded in accordance with any category in Table 3.02.

There was a shift in Year 2 to a higher overall level of awareness, but this was restricted to a narrow range of categories. While approximately the same pattern of responses characterised Year 3, overall they had rather less to say than children in Year 2. Less than half the children in these two year-groups mentioned a specific dimension or aspect; rather fewer mentioned something that could be measured. Nearly half those in

Year 2 mentioned a measuring device, but in Year 3, nearly as many children mentioned using a ruler for drawing straight lines as mentioned a measuring device.

Across the whole sample, almost the only devices mentioned were rulers, metre sticks and tape measures, and almost the only dimension mentioned was length and its associated aspects of height, length and width.

Table 3.02 *Can you tell me what 'measure' means?* In each year-group, the number of children responding according to the categories in Table 3.01

Mentions at least one of the following features of measurement	Year 1	Year 2	Year 3
Specific dimension (or aspect of dimension)			
Length (height, length, width)	5	10	12
Weight	0	2	1
Capacity	0	2	0
Non-specific dimension or aspect only	5	2	1
No dimension mentioned	14	12	13
Measurement as determining 'fit'	0	1	2
Measuring device			
Ruler	0	8	4
Metre stick	0	3	2
Tape measure	1	3	1
Measuring jug	0	3	0
Height scale used in clinics	1	0	0
No device mentioned	22	14	21
Units			
Specific unit mentioned	0	7	4
No units mentioned	24	18	22
Entity that may be measured			
Specific entity mentioned	6	10	9
No entities mentioned	18	15	17
Activity involving measuring	1	4	5
Specific measuring activity	3	1	1
No activities mentioned	20	20	20
Uninformative or irrelevant responses			
Mentions drawing a straight edge	0	0	4
Tautology, no response, *Don't know*, or irrelevant response	10	3	6
Total uninformative/irrelevant responses	10	3	10
N	24	25	26

As a broad criterion of children's elaboration of the notion of measurement, five features from Table 3.02 that were considered fundamental to measurement were selected, and for each child a count was made of how many of these five features they mentioned in response to the question *Can you tell me what 'measure' means?* Four of the features chosen were: specific dimension or aspect; measuring device; unit of measurement; and entity that may be measured. Mention *either* of an activity involving measurement *or* of a specific measuring activity counted as the fifth feature. (No child mentioned both.) Occurrence of any of the five features was counted only once for a child, even if the child mentioned more than one instance of that feature.

Table 3.03 sets out the number of children in each year-group who mentioned the number of features given.

Table 3.03 Number of children in each year-group mentioning a given number of features (0-5)

	Year		
	1	2	3
Features			
0	13	4	5
1	7	7	13
2	2	6	4
3	2	6	2
4	0	2	1
5	0	0	1
Mean[1]	0.2	0.4	0.5
(SD)	(0.4)	(0.5)	(0.5)
N	24	25	26

Note 1. = mean number of features mentioned per child.

Over half the children in Year 1 mentioned none of the five features of measurement selected, as Table 3.03 shows. The majority of children in Years 2 and 3 mentioned at least one feature, although in Year 3, only a few children mentioned more than one. Children in Year 2 were more evenly divided, with over 70% of children mentioning one, two, or three features in roughly equal numbers. Differences between the year-groups as to number of features mentioned were, however, non-significant.

Measurement at home

Have you ever seen someone measuring something at home? Tell me about it.

Those children who recalled measuring activities out of school usually needed little prompting to give a full account of their experience. Some Year 1 children whose accounts were hard to follow received neutral probes such as *I didn't quite understand that* to encourage them to elucidate their responses.

Twelve children (50%) in Year 1, eighteen (72%) in Year 2 and twenty-four (92%) in Year 3 said they had seen measuring at home, and gave relevant responses. The accounts of these fifty-four children (68%) were typically full and circumstantial. Table 3.04 summarises the content of responses under the general headings of 'measurement activity', 'measuring devices', 'dimensions' and 'units', with a number of sub-categories. Table 3.05 sets out for each child the number of activities, items measured, and measuring devices mentioned by each child. Table 3.06 furnishes examples of children's descriptions. Tables 3.04 and 3.06 give a quite striking impression of the wealth of situations in out-of-school environments in which the children encountered measurement.

Table 3.04 indicates a wide range of measuring activities, with as many as thirteen out of twenty-four Year 3 children recalling measuring up at home for floor or wall coverings. Congruent with responses to Question 1, it was measurement of length that featured in most activities; most devices that were mentioned measured length (the tape measure being by far the most familiar to children) and most aspects mentioned were those of length. Few children referred to units, but of those who did, most referred to correct units for the dimension. Where children mentioned a measuring device, some also gave a recognisable account of how it was deployed. Twelve children in Year 3, ten in Year 2, and six in Year 1 did so.

Table 3.05 shows that, of those children in Years 2 and 3 who recalled measuring at home, and who described at least one type of measuring activity that took place there, most described only one. Only six Year 1 children mentioned measurement activities, but of these, four mentioned more than one type of activity. Year 2 children mentioned slightly more types of activity, on average, than those in Year 3. However, these differences were not significant.

Most children in Years 2 and 3 recalled one or two types of item being measured at home, with progressively fewer children mentioning more. In Year 1, most children recalled only one type of item. There was no significant difference between year-groups. Of those mentioning measuring devices, only two children mentioned more than two. There was, again, no significant difference between year-groups.

Table 3.04 *Have you ever seen someone measuring at home?* The number in each year-group mentioning the listed type of activity, device, dimension, and unit.

	Year		
	1	**2**	**3**
Type of activity			
Fitting floor or wall covering or curtains	1	3	13
Fitting/buying clothes	2	3	4
Measuring space for furniture, people or animals[1]	4	7	5
Food or drink preparation	4	4	4
Determining height (person, plant, object)	6	2	2
Instructional activity	0	3	1
Other activity	3	7	8
No activity stated	4	1	1
Appropriate measurement device and use			
Tape	10	7	19
Ruler or metre stick	3	3	3
Scales	3	4	4
Measuring jug	4	3	2
Visual comparison	0	0	1
Direct comparison	0	1	1
Other device or use of device			
Spirit level	0	0	3
Ruler (as straight edge)	2	1	0
No device mentioned	3	6	2
Mention of aspects appropriate for the dimensions of			
Length	5	7	9
Weight	2	3	2
No specific aspect mentioned	9	13	10
Units (correct for dimension)			
m, cm, mm, foot, inch, step	2	4	7
kg, ton, stone, oz	1	4	3
l, ml	0	0	1
Units (incorrect for dimension)			
m, cm, mm,	0	1	1
kg, g	0	0	1
l, ml	0	0	1
No units mentioned	10	13	14
N	12	18	24

Note 1. or conversely, measuring furniture, people or animals for the available space.

Table 3.05 Measurement at home: the number of children in each year-group mentioning activities, items, and devices, according to the number of types mentioned

	Year		
	1	**2**	**3**
Activities			
0	0	0	3
1	2	9	10
2	1	2	0
3	2	1	3
4	1	1	0
Mean (SD)[2]	2.3 (1.2)	1.5 (1)	1.2 (1.0)
N	6	13	16
Items[1]			
0	0	0	0
1	7	8	8
2	1	6	8
3	1	3	3
4	1	1	1
5	1	0	1
6	0	0	1
Mean (SD)[2]	1.9 (1.5)	1.8 (0.9)	2.2 (1.4)
N	11	18	22
Devices			
0	0	0	2
1	5	8	10
2	4	3	6
3	1	0	1
Mean (SD)[2]	1.6 (0.7)	1.3 (0.5)	1.3 (0.8)
N	10	11	19

Note 1. e.g. person, article of furniture, cooking ingredient. Several items of the same type mentioned by one child are counted as one item. 2. Mean activities, items or devices mentioned per child.

The quotations in Table 3.06 give a vivid picture of the children's personal interest in the measurement experiences in the home setting that they reported, and their specificity. For Year 1 children there was a sense that the personal nature of the experience was closely bound up with their relationship with the adult doing the measuring – being allowed on the roller coaster, for example, or holding the screws for the model ship. In Years 2 and 3 the focus was more on the technical details of the measurement procedure. Here there is a sense of the importance of units, though often little sense of the accuracy that is the reason for their use.

Table 3.06 Examples of measuring activity at home as described by the children

Year 1	My mum measured me to see if I could go on the roller coaster. You've got to be about 5 foot and I was more than 5 foot. I was 6 foot, and so I could go on there. We've got a jug with numbers to see how much water to put in the sand pit...because you can make better sand castles. My Dad uses it [tape]. And my mum can sew a skirt or trousers. And my dad can do what my mum can do. They share it [tape]. My granddad can use it [tape] for making something. He made a...model ship. It's in my bedroom. I gave him the screws. The doctor pulled it down on our head. It told...how tall I am. She put my feet on it. And it told how tall my feet were. My dad measured my bunk bed ...to find out the size. The size is 21, 3. [Did he see that on the tape?] He thought it was 88, 9.
Year 2	My sister's weight. You step on these toilet scales. I helped my mum with cooking. We measured flour and sugar...with scales. Mum measured the photo so she could get the right size frame, with that thing that you pull out. My dad was measuring a window...cos someone smashed one... [with] this little thing that's round and goes up. He looked at the number and it was about ten hundred kg. My mum measured peas in the garden with a tape. She wanted to know how long they were. My nan done it all the way to the other end of the room to see if there was enough space for the dog she was going to buy. [Was it a big dog?] [Indicates about 1 m with hands]. About 30 m long. It was heavier than me. That tape thing...my mum held it up to her head and I counted from the bottom and I knew how small she was...she was 186 m. Nearly as tall as you. Mum's always measuring. She puts this liquid [oil] in a box, and then there's a wood across and then another box, and she puts a block in it. She sees what's heavier. You get this jug if you want to make milk and you put it in and you put the water in and see if it's past the 1, and then you pour it into the baby's bottle. If you have a baby...you measure the cot and measure the room to see if it will fit.
Year 3	In the shop…I have to stand like a scarecrow. My mum puts the dress at the back of me to measure how long it is. He was trying to put a wooden floor, but if you put all of them down, and you've got a long piece and a little gap, you have to measure it, and measure the hole... and then cut it off. My brother was trying to make a map of the underground and he knew how big the underground was, but the paper was only small, so he had to measure how long the paper would be. He had to add more paper.

> We didn't measure the flour and sugar with the jug, we measured it with spoonfuls. Cos each spoon would be 5 ml. so we put 5 of those.
>
> I want my wall to be painted different colours, pink, red, purple, green. So my dad has to measure about 5 or 6 cm of it, and then he just puts a little bit of paint.
>
> When my dad makes pancakes he always...measures them in weight..,and when he sees which one's heavier, he always thinks it's worse I don't know why...he throws it away.

Measurement at school

Have you done any measuring at school? Tell me what you did.

Children who recalled measuring activities in school typically began by listing classroom objects and fittings that they had measured, moving on to give details about the deployment of the device used. There was little circumstantial detail. Eight children (32%) in Year 1, twenty-two (88%) in Year 2 and twenty-four (92%) in Year 3 recalled measuring experiences at school, and gave relevant responses to this question. While rather fewer children in Year 1, and rather more in Year 2 recalled measuring at school than at home, the number of children who recalled measuring in each setting (fifty-four) was the same overall. However the range of activities recalled and of types of item measured was much narrower at school than at home, as Tables 3.07 and 3.08 indicate.

It was useful to think about school activities that children reported as having two types of purpose, which were a) measurement for its own sake and b) measurement as part of a broader classroom project. The former included measuring the length of arms or legs, of items of classroom furniture, or of a line previously drawn. As part of a project, children recalled measuring water to mix with substances such as sand or clay, apparently to study porosity, measuring equal amounts of water to give to plants, and measuring distances travelled by toy cars down an inclined plane. Two children also mentioned measuring for the specific purpose of practising for Key Stage 2 Standard Attainment Tests. These latter responses were categorised as 'measurement for its own sake'. Responses were also classified according to mention of items measured, measuring devices used, terms used that were appropriate to given aspects of dimensions mentioned (such as 'tall or 'long' for length; 'light or 'heavy' for weight) or were inappropriate; and units that were appropriate or inappropriate to the dimensions they were applied to. Table 3.07 sets out this information.

Table 3.07 *Have you done any measuring at school?* The number in each year-group mentioning the listed purposes, items measured, device, dimensions, and units

	Year		
	1	**2**	**3**
Purpose of measurement			
For its own sake	8	22	24
As part of a broader classroom project	0	3	7
No purpose mentioned	0	0	0
Item measured			
Playground or classroom space (including floor or walls)	2	4	5
School furniture/fittings/equipment	5	16	18
Items for classroom project	0	3	7
Person or body part	2	6	8
Fluid	0	4	10
Drawn lines or shapes	0	5	2
No items mentioned	0	0	1
Measuring device			
Iterated body part	5	0	4
Discrete units (cubes; 'short rulers')	0	3	1
Ruler or m stick	4	17	21
Measuring jug	0	4	15
Visual comparison	0	1	0
Direct comparison	0	1	0
Other use			
Ruler (as straight edge)	0	2	2
No device mentioned	1	1	0
Terms appropriate to			
Length	1	5	5
Weight	0	0	3
Inappropriate	0	0	4
Non-specific	1	3	1
No terms mentioned	6	13	12
Units (correct for dimension)			
m, cm, mm, foot, inch, step	4	10	16
kg, ton, stone, oz	0	0	0
l, ml	0	0	0
Units (incorrect for dimension)			
m, cm, mm	0	1	4
kg, g	0	1	2
l, ml	0	0	0
No units mentioned	2	7	5
N	24	25	26

Table 3.07 indicates that measurement at school was typically recalled as undertaken for its own sake and rarely as part of a wider classroom project. Most items children remembered measuring at school consisted of classroom furniture or equipment, classmates' height or the length of their arms, legs, hands or feet. In each year-group, at least twice as many children mentioned units as had done so when recalling the home context. As then, few mentioned incorrect units for the dimension. Most activities, as in the home context, involved measurement of length, but the most commonly mentioned devices used at school were the ruler and metre stick and (as recalled by Year 3 children) measuring jugs. In mentioning devices used at school, eight children in Year 3, thirteen in Year 2, and two in Year 1 gave a fair account of how they were deployed.

Table 3.08 shows the number of children who mentioned different types of item measured and different measuring devices, according to the number of types mentioned. (Inappropriate devices and units, and terms appropriate to various dimensions, listed in Table 3.07, are excluded from this summary because few children mentioned any. Many children mentioned appropriate units, but all were units of length, so this category was likewise excluded.)

Table 3.08 shows that few children recalled measurement done at school for a purpose beyond the measuring exercise itself. Year-groups did not differ significantly in this. While rather fewer children recalled items measured in the home setting than at school, the range of items measured was narrower at school (compare Tables 3.07 and 3.08). There was again no significant difference between year-groups. However, many more children in Years 2 and 3 mentioned one or more types of measuring device they used at school than mentioned any devices used in the home setting. Year 3 children recalled significantly more types of device used at school than did Year 2 children ($t(42) = -3.64$; $p = .0007$); probably reflecting, by that stage, accumulated practice. The use at school of iterated body parts (such as hand-breadths) and separated units was recalled by a few children; neither figured among devices used at home. Three children in Year 3 and six in Year 2 also mentioned iterating rulers or metre sticks; of these, one child in Year 2 and three in Year 3 mentioned adding the number of units produced.

Table 3.08 Measurement at school. Number of children in each year-group recalling the listed number of purposes[1], types of item, and types of device.

	Year		
Purpose	**1**	**2**	**3**
1[a]	8	19	17
2[b]	0	3	7
Mean (SD)[1]	1 (0)	1.1 (0.4)	1.3 (0.5)
Items			
1	7	14	3
2	1	7	13
3	0	0	8
4	0	1	0
Mean (SD)[1]	1.1 (0.4)	1.5 (0.7)	2.2 (0.7)
Devices			
0	1	1	0
1	5	16	9
2	1	5	15
3	1	0	0
Mean (SD)[2]	1.3 (0.9)	1.2 (0.5)	1.6 (0.5)
N	8	22	24

Note 1. = maximum two: measurement for its own sake; measurement as part of a classroom project. [a] = one purpose mentioned; [b] = both purposes mentioned. 2.= Mean number of purposes, items or devices mentioned per child.

Table 3.09 gives examples of children's descriptions of measuring activities in school. It illustrates that recall was of a much narrower range of measurement activities performed in school than at home, but it also illustrates (in Years 2 and 3) greater awareness of the techniques of using measuring devices, which are sometimes described with considerable precision. Where numbers or units were mentioned, these represented plausible quantities for the entities measured.

Table 3.09 Examples of measuring activity at school as described by the children

Year 1	We measured [pictured]dinosaurs and trees with cubes
	...the classroom with our feet
	...the table with our hands and who had the longest hair
	...the floor with paper cut-outs of our feet
	...people, when I was in the nursery
Year 2	...pencils or lines
	...how long our finger could go on a line
	contd.
	...windows, chairs, doors, books. When it kept going up we had to put our fingers on the numbers.
	...the wall. We had a big ruler and we kept moving it up.
	...the door. It was quite long. We used our normal rulers. We had to bring it up one by one.
	...our teacher. We had to use a ruler that was longer than we normally use.
	...jugs, we got some water, fill it to the top, look, write on the board how much there was inside.
	...a table, a door...if it was longer than a little ruler we had to do it again...and write...how long it was, how many times we had used the ruler.
	...capacity...[teacher] wrote how much we put in the jug and the most was 11. How much capacity will fit in cups and jugs.
	In a practice SATS test there was measuring lines and we had to make one that was a bit longer.
	...a skipping rope...with a long ruler. You can only measure skipping ropes, people and big boxes.
	There was all sorts of jugs and water. [Teacher] had a little tin cup. She poured it into a jug, and then ... into another jug, and we knew that the bigger jug took more water.
	...the board, how long. We used a couple of rulers...I was the only one who knew how to add the numbers.
	...the table. We used a ruler. It was 30. Then we put it down. 30 again. So 60. Then we put the ruler down again. 90.
	...a car...we used a 100 cm ruler. You push [the car], let it go, and see the number, how far it went. My one went to 47 cm.
Year 3	We all got a piece of paper and drew loads of shapes and measured the size...some people used rulers, some...tape measures...some used a computer
	[teacher] asked us to measure this line and it was 6 cm. But [teacher] measured it and said it was 4cm. So I measured it and it was 4.

...the table. You put your finger where the 30 is, pick it up, put the ruler there, count how much it is, if you've got another 17, it'll be 47.

...my teacher. I got this measuring stick and she stood up and I put the stick above her...and I didn't know what number she was cos she was taller than the stick.

...We got different kinds of soil...and we put the same amount of water in every jug to see how much it soaked in...see it's like what you get on rulers. It's like that on the jug.

contd.

...we put little things like these [toys] on the gram thing. It would go higher if it was lighter and lower if it was heavier

Comparison of responses according to gender

As already stated, gender differences were examined by considering differences in responses between boys and girls in the first interview (overall and for each year-group) to the question *Can you tell me what 'measure' means?* and the questions that followed it about measurement at home and at school. What was compared for boys and girls was a count for each child of the total number of types of response they made, as labelled in Tables 3.02; 3.04; and 3.07.

There was no significant effect of gender. Between 2 and 19 types of response per child were elicited by the three questions. The mean number of types of response for boys ($M = 7.6$, $SD = 4.88$), and the mean number for girls ($M = 8.6$, $SD = 5.66$) were not significantly different ($t_{73} = -0.84$, $p = .39$). The three year-groups were also tested separately and again did not differ significantly. For Year 1, $M_{boys} = 4.4$, $SD = 3.87$; $M_{girls} = 3.4$, $SD = 4.58$; $p = .51$. For Year 2, $M_{boys} = 8.1$, $SD = 4.49$; $M_{girls} = 10.2$, $SD = 4.97$; $p = .31$. For Year 3, $M_{boys} = 10.2$, $SD = 4.54$; $M_{girls} = 11.3$, $SD = 4.33$; $p = .57$.

Comparing length and height

The first question was: *How could we check which pencil is longer ? Can you show me?* Children a) demonstrated direct comparison of the pencils, (either carefully aligning one set of pencil ends, or omitting to do so) or b) stated that they would simply look, or would use a ruler. To the question *How do you know I am taller than you?* the responses could be classified as tautological (*because you are)* or as appealing to visual comparison (*I can see you are*) or to a status criterion (age, occupation). Responses to

How could we check ? mentioned either direct comparisons of respective height, the use of a measuring device, or both. As well as eliciting yes/no responses, the question *Would you be taller than me if you stood on the table?* elicited qualified 'yes' responses, in which some recognition was given to the fact that the judgement was not straightforward. Table 3.10 sets out the numbers of children responding in these various categories.

To the question *Can you tell me just by looking, which of these two pencils is longer?* Each child answered in the affirmative, and indicated one of the pencils. When asked how they would check this, a large majority of children in each year-group stated that they would place the pencils together to determine which was the longest and did so with endpoints correctly aligned, as Table 3.10 shows. All correctly identified the longer pencil. A few stood the pencils on end, an alternative way of ensuring end-points coincided. A few in each age-group failed to align endpoints of the pencils correctly; two were satisfied with a visual comparison that did not involve bringing the two pencils together. Four children said they would measure the pencils with a ruler, but did not give a clear account or demonstration of how they would determine which pencil was longer, having measured them.

In response to the next question, all the children correctly affirmed that the experimenter was taller than themselves. To the question *How do you know?* A majority in Years 1 and 2 responded that they knew because the experimenter was an adult, a teacher, a lady, went to college and the like. Somewhat fewer used this status criterion in Year 3, and only in Year 3 did many children qualify their response when prompted by the researcher (*Are adults always taller than children?*) by giving examples of exceptions to this generalisation. (Five children in Year 2 and one child in Year 1 also did so). One child affirmed that you could tell by the colour of an adult's hair whether they would be shorter than a child, and explained that elderly people became shorter again, and that when one was elderly, one's hair was grey.

When asked how they would check their height compared with that of the experimenter (or a friend), about half in each year-group (more in Year 2) said both parties should stand up and a direct comparison be made, while more than one-third (less in Year 2) either said they would measure each with a ruler or tape, or mentioned this method as an alternative to direct comparison. Mention of use of the ruler or tape was therefore more frequent for measuring people's height than it had been for comparing the length of pencils.

Table 3.10 Comparing lengths of pencils and heights of participant and experimenter. Number of children responding as indicated.

	Year		
	1	2	3
How could we check which pencil is longer? Can you show me?			
'You can see just by looking'	0	2	0
Children compared pencils directly, ends aligned	20	17	22
Children compared pencils directly, ends not aligned	4	3	3
'Measure them with a ruler'	1	2	1
Child demonstrated iteration of a unit	0	1	0
No response/ don't know	0	0	0
How do you know I am taller than you?			
'Because you are'	6	2	3
'I can see you are'	0	4	2
'You are older/adult/teacher' (unqualified)	13	14	8
'You are older/adult/teacher' (qualified in response to *Are adults always older than children?*)	1	5	10
Other	2	0	1
No response/ don't know	2	0	2
How could we check ?			
Direct comparison ('Stand up together and look – or place hand on head - to check')	12	16	14
'Use tape/ruler'	7	3	7
Both direct comparison and measure	2	0	2
No need/don't know	3	6	3
N	24	25	26
Would you be taller than me if you stood on the table?			
'Yes'	14	6	12
'No'	8	1	10
Qualified Yes	0	4	2
Don't know/confused response	1	1	3
N	24	13[1]	26

Note 1. Due to time-tabling constraints that affected Year 2, a large number of children were not asked this last question.

Of the children who were asked whether standing on a table would make them taller than the experimenter, about half said that it would. The rest (except in Year 1, where the proportion was one third) said that standing on a table would not make them taller, or said that it would make them taller, but qualified this statement by adding *but I would be cheating*, or *only because your feet were still on the floor* or other comments indicating that this was not a straightforward case of being taller. As already stated, all children were debriefed by being told *Well you would be higher, but you wouldn't really* *be taller, because you wouldn't suddenly have grown, would you?* or *Quite right: you would be higher, but you wouldn't really be taller.*

3.2.3 Discussion

In the context of this first interview, while few children elaborated the *notion* of measurement beyond a very mundane level, many gave vivid and detailed accounts of measurement activities they had experienced at home, and older children recalled with some precision the more constrained process of measuring at school. The interview suggested striking differences between home and school environments as far as measurement was concerned. There was evidence of some confusion when children were asked specifically about direct comparisons of height.

When asked what the word *measure* meant, more than half the children in Year 1 did not respond, said they did not know, gave an irrelevant response, or said that it was 'when you measure something', and did not elaborate. Among the rest of Year 1, rather less than half mentioned dimensions or aspects, and fifty per cent of these used non-specific terms such as 'big, and 'size'. About half mentioned items that could be measured; none mentioned units of any kind. (Table 3.02). Only four children mentioned more than one feature of measurement according to the classification used (Table 3.03). Children failing to respond positively were told that measurement was *when you find out how big or small, or heavy something is, or how much of it there is* before the remaining questions were put. Following this information, over half the children in Year 1 said that they had seen measuring at home, and just under one third recalled measuring in the classroom.

Given the much higher proportion of positive responses overall in Years 2 and 3, it is fair to say that Year 1 children, as a group, probably had little background knowledge of measurement at that stage in their lives. However, as Table 3.04 indicates, the *range* of measuring activities, measuring devices, and dimensions and aspects of measurement encountered in the home mentioned by those Year 1 children who did respond was much the same as it was for Years 2 and 3. This was less true of the school setting (see Table 3.06). Here, for Year 1, there was a narrower range in all categories, except units. (Units of length were the only ones recalled in the school setting in any year-group.) This probably reflected the lesser experience of measurement activities that Year 1 children had had in school.

There were few differences overall between Years 2 and 3; those tested were non-significant at $p < .05$. Most could give some account of the notion of measurement when asked the meaning of the term, although six children in Year 3 and three in Year 2 failed to do so. The most frequently mentioned features were dimension and aspects – overwhelmingly in the dimension of length – and items that could be measured. Less than half the children in Year 2, and less than one fifth in Year 3 mentioned measuring devices, and of the latter as many mentioned using the ruler to draw a straight line as to measure. (Several Year 3 children mentioned being asked to draw a straight line, and then measure it, as familiar classroom work, so perhaps the two activities were jointly conceived of as measurement). Units were mentioned by few.

Rather fewer Year 2 children (72%) recalled measurement activities at home than at school (88%); for Year 3 children the proportion was 92% in relation to both settings. Year 3 children had been more involved than Year 2 in measuring up for carpets, tiling, wallpaper or curtains in the home. For all year-groups the retractable tape was by far the most familiar measuring device in the home setting. Units of length and weight were mentioned by a few children in Years 2 and 3. Few children mentioned more than one or two features of measurement in any of the categories identified (see Tables 3.05 and 3.08).

Which features of measurement children mentioned were of course strongly influenced by the specific activities they had experienced and were reporting. For example, as Table 3.04 shows, activities associated with redecorating the home, acquiring new furniture, the fitting of new clothes or determining height predominated in the home setting, and this probably accounts for the fact that measurement was thought of predominantly in terms of length. Features of measurement (as categorised here) that children readily called to mind were in fact modest in number. Units, even units of length, were rarely mentioned. However, the children's reported experience of measurement taken as a whole was a rich one, and presented two important contrasts between home and school. Firstly, in the home, reported measurement activities generally took place in the context of important family events such as moving house, buying new furniture or equipment and painting and decorating, or were associated with events that were special, or were pleasurable to the individual concerned. At school, measurement was typically recalled as carried out for its own sake, on a narrow range of classroom objects or of classroom or playground dimensions. Here, recording the result of measuring was frequently seen as the most important part of the exercise. In one response - 'we measure with a piece of paper' - the recording of measurement stood in for the whole process: investigation revealed that this statement described writing down the measurement, not using the piece of paper as a measuring device.

Secondly, most activities in the home were described in terms of affordances – of the fit of an item (or person or animal) to available space. The use of the tape was prominently involved in achieving this fit, but was not the focus of the activity. At school, by contrast, the ruler or metre stick and the units marked on it were the focus of the activity; *what* was measured (except on the seemingly rather rare occasions when measurement formed part of a wider project) was of limited consequence. The use of iterated or discrete units (Table 3.07) was occasionally recalled as happening at school, but never in the home context. Moreover the category 'ruler or m stick' in that table sometimes included cases where the child mentioned iterating a ruler or metre stick as a unit and counting the iterations or, in some cases, adding together the units shown on the device that were produced in successive iterations (examples can be found in Table 3.09). Furthermore the limitations of iterating informal units such as body parts - the problems of different-sized feet, the difficulty of ensuring there were no gaps – were sometimes hinted at, if not fully explained. It seemed, then, that while the school setting achieved a focus on units that was not apparent at home, children had a firmly-rooted sense of important purposes for which measurement was undertaken in 'real life' that was not reflected in school. What they said about measurement at home also reflected an appreciation of the difficulties this enterprise presented at times – the difficulties of achieving a good fit where this really mattered. Some of the comments in Table 3.06, particularly of Year 3, illustrate this.

Table 3.06 gives examples of real-life problems that require measurement for their solution. Many of these problems are at once everyday and novel in character. Cognitive effort is required to achieve the desired outcome, which supplies the problem-solving impetus. If measurement in the classroom is typically devoid of any purpose beyond itself, then attempts to teach it are unlikely to sustain children's active mental engagement. As already suggested they may, for example, fail to persist with the precision in carrying out measurement procedures that is essential for accuracy.

The two checks of understanding of direct comparison of length (pencils, and height of child/experimenter) showed the great majority of children to be familiar with a reasonably accurate method of direct comparison. However, asked to estimate whether they themselves or the experimenter were taller, and asked to justify their response, most children did so by appeal to the adult's age or grown-up status rather than by reference to actual height. Although fewer children gave this justification in Year 3 than in other age-groups, and although nearly half the children in that year-group showed greater understanding by qualifying their response, the overall finding did suggest that where the human figure was concerned, some confusion was to be expected about how height is described.

Finally, the check on understanding of the term *taller* (when the child was asked whether standing on the table would make them taller), showed that a majority of children in each year-group failed to make the distinction between intrinsic height and height above ground when the term *taller* was used – again, in this particular example, in the context of the human figure. The children were briefly trained in the correct uses of *taller* and *higher*, and this conceptual and linguistic distinction was further examined in a later part of the research.

3.3 The second interview

The purpose of the second interview was to obtain a picture of the children's knowledge about scaled measuring devices in common use and the ways in which these measured. Particular interest centred on the nature of children's understanding of the numbered units shown on the devices. Children were shown a ruler, a tape measure, a measuring jug, and a non-digital clock, wall thermometer and kitchen weighing machine. On each device (except for the clock) two separated scales were shown whose units were labelled, numbered and subdivided, and the scales themselves differed in general appearance. The information displayed on the devices was therefore complex. A decision was taken not to simplify this information, as the degree of complication was considered typical of everyday measuring instruments, and it was of interest to see how well children coped with it. [1].

The interview also afforded an opportunity to explore children's understanding of the inverse relation between unit size and number of units (Nunes & Bryant 1996; Carpenter & Lewis 1976). It was to this end that the devices selected (except for the clock) each displayed two separate numbered scales, usually side by side, whose major units differed in size. On the ruler, for example, 30 marked and numbered cm were shown on one edge, while 300 mm (marked and numbered in tens) were shown on the other edge. On the weighing machine, 8 oz were marked on a scale on the left of the rectangular display panel, and 240 g were marked on a separate scale on the right. In this way, a) units of different sizes, and b) different numbers of units (clearly numbered) were fully visible and could easily be compared. (On the tape measure, cm and inch scales were on reverse faces of the device). The inverse relation was explored for the ruler, tape measure, measuring jug and weighing machine, but was omitted for the clock and thermometer, the former because minute and hour scales do not appear separately on clocks, and the latter because it was apparent in piloting that the children were generally unfamiliar with units of temperature.

Finally, because all types of quantity measured by visible scales are represented as length, it was thought possible that these visible scales might confuse children as to what was actually measured. Devices were therefore selected that measured weight, volume, time and temperature, as well as length; it was of interest to see whether children would associate length with devices other than the ruler and tape measure.

Questioning focused on whether children could name a device, considered it to be a measuring device, and could name the units shown on it and the dimensions to which they applied. Children were also asked to describe how four of the devices were used. This was not done for the ruler and tape measure, as children would be asked to demonstrate their use in a later phase of the research. As each device was introduced, children were encouraged to handle it and to comment on its markings and uses as they wished.

[1]A fluid oz scale on the measuring jug was masked as it was felt the units named would be confused with oz units of weight on the weighing machine. A litre scale and a pint scale remained visible on the jug.
[2] Cm and inch scales were not displayed side by side on the tape measure, but were on reverse faces.

3.3.1 Method

Participants

The sample comprised seventy-three children, twenty-six in Year 1, twenty-one in Year 2, and 26 in Year 3.

Materials and procedure

a) General questions about the devices and the units they showed

In the second interview, the first four questions sought the same kind of information about each measuring device as had been asked in the first interview about measurement in general. After children had handled and commented on each device, they were asked to name it. They were then asked whether they thought it could be used for measuring, and if so, how it was used, and what could be measured with it. The next question, *What do the numbers tell us?* was designed to elicit knowledge of the dimension measured by the instrument; and the next, *Do you know what these letters stand for?* (the experimenter indicating abbreviated unit labels that appeared on the instruments) invited children to name the units measured.

b) Questions investigating understanding of the inverse relation The children were carefully prepared for the two important questions designed to probe their understanding of the inverse relation between size and number of units. These questions simply asked which of two visible units of different sizes on each device was the greater (or smaller) and how the child knew this. These routine questions happened to have produced, during piloting, unexpected responses relevant to children's understanding of the inverse relation, and were retained without change. They were asked in relation to the ruler, tape, weighing machine and measuring jug. In the case of the ruler (used here by way of example) the first question was *So which is longer, 1cm or 1 mm?* and the second *How do you know?*

Thus, as noted above, the children had been allowed to become thoroughly familiar with the ruler throughout the first part of the interview. They were then directed to its cm edge, and asked *Can you show me how long 1 cm is?* If it was not clear from a child's gesture that they intended to indicate the space between two marks that were 1 cm apart on the ruler, the experimenter moved a pencil-tip across the space indicated by the child, from one cm mark to the next, and said *Yes, the space from here to here is 1 cm, isn't it?* If the child indicated a line or number rather than a space, the cm space was carefully indicated, with the words *Well actually, the cm is the space, not the line/number.*

The child was then invited to identify a second cm - *Can you show me another cm?* – and the follow-up explanation given again if necessary. The process was repeated once more, so that the children indicated or were shown three separate cm spaces in all.

The child was then directed to the other edge of the ruler, and asked to indicate the length of 1 mm. (Here their indication could be only approximate, and references to the closeness of the marks or the tiny space involved were expected.) The procedure that followed was the same as it had been for the cm.

With the ruler still in full view, the experimenter then asked a) *So which is longer, 1cm or 1 mm?* and b) *How do you know?* Whether the greater or smaller unit was mentioned first was systematically varied from child to child, and (within-child) between devices. The children had just had their attention directed to the physical size of the units named, and so, other things being equal, it was natural to expect the correct response to question a), and obvious justifications of this response (*How do you know?*) were that one could *see* which unit was longer, or that one knew which was longer (or shorter) because of the greater (or lesser) space it took up. If, however, the numbering or other information on the device were to mislead a child or divert them from this key information, errors might be expected.

Questioning about the relative size of units was invariable in form across children and devices. In other respects, the second interview was semi-structured: children sometimes gave information spontaneously so that certain questions were unnecessary; at other times a question initiated an extended conversation that was pursued by the experimenter because it afforded additional insight into a child's understanding. An excerpt from such a conversation is given at Appendix 2. The children were shown the measuring devices one at a time in random order. The interview was tape recorded and transcribed.

The full interview schedule is set out below for the ruler. Under *Tape*, and headings for all the other devices, only variants on the schedule relevant to each device are shown. Some information relevant to follow-up questions for all devices, suitably indicated, is reported with the schedule for the ruler.

Ruler

A 30 cm transparent plastic ruler was used, of the type routinely used in the school where the research took place. One edge was marked in cm, half cm and mm, with cm marks numbered. Only millimetres were marked on the other edge and were numbered in tens. The scale began 0.5 cm from each end of the ruler, whose full dimensions were 31cm x 3.6 cm.

Interview schedule and procedure for ruler

The experimenter showed the child the ruler.

1. *What's this?*

2. *Can we measure with it?*

3. *What could we measure with it?*

4. *What do the numbers tell us?* If a child replied that the numbers told us about cm or mm, they were re-directed as follows: *Do they tell us how heavy* [item] *is? How long it is? How much room it takes up?* The order of last three questions was systematically varied between children. Nineteen children in Year 1, fifteen in Year 2, and twenty in Year 3 were offered these prompts for one or more devices. Their responses were inspected for evidence of primacy or recency effects favouring alternatives offered in the first or third positions. There was a high proportion of correct responses irrespective of position, and an even distribution of incorrect responses between the three positions. This indicated that any recency and primacy effects were unlikely to have influenced the data.

5. *Do you know what these letters stand for?* [cm, mm] The name of a unit was supplied if the child did not do so.

6. *Can you show me how long 1 cm is?* The experimenter confirmed/corrected and indicated three 1 cm spaces to ensure that the size of the unit was attended to.

7. *Can you show me how long 1 mm is?* The experimenter confirmed/corrected and indicated three 1 mm spaces.

8. *So which is longer, 1cm or 1mm?* The order of these options was reversed between children. The ruler remained in full view.

Tape measure

A white fibreglass dressmaker's tape measure showing inches and cm. One side of the tape was marked with cm and mm, with cm numbered. The reverse side of the tape was marked in inches and eighths of inches, with inches numbered with particularly large numerals. Both scales began at the extreme left-hand end of the tape. The full dimensions of the tape were 152.4 cm x 1 cm. The last marked units were 150 cm and 59 inches. All numbers were correctly oriented for reading when the tape was vertical with respect to the reader.

Interview schedule and procedure for tape measure

The interview schedule and procedure were as for the ruler, except that inches and cm were mentioned instead of cm and mm.

Kitchen weighing machine

A white plastic machine 10 cm high, 8.5 cm deep and 6 cm wide, with a pan 11 x 11 x 5 cm deep, set on a sprung stand on top of the machine. Two separated vertical scales, with a clear space of 7 mm between them, appeared side by side on the front of the machine, an oz scale to the left, and a g scale to the right. An indicator moved downwards horizontally across both scales when material was added to the pan. Each scale was divided and numbered on its right, the g scale being divided into 5g divisions and numbered every 20g from 0 to 240g; the oz scale divided into ¼ oz divisions, and every ounce numbered. Numbers therefore ranged from 0 to 8. The marks labelled '0' at the top of each scale were level, but the gram scale extended below that of the ounces.

Interview schedule and procedure for weighing machine

The first three questions were as for the ruler. After question 5, where children were asked to name units printed in abbreviated form on the two scales, the experimenter asked the child to show how the machine was used. Small plastic toys and other objects suited to the size of the scale pan were available. After they had placed material in the pan, children were asked to read the amount indicated on each scale. The horizontal indicator on the weighing machine crossed both scales, and the intention here was to offer children the opportunity of seeing that the same amount could be expressed in units of different sizes (yielding different numbers of units) to prepare the ground for question 8, where they had to attend to the difference in size of units.

Measuring jug

A translucent, plastic jug of one-litre capacity, showing two scales, one on either side of a single central spine. One scale showed 1 litre and the other 2 pints. Half a litre and 1 litre were marked on the litre scale, which was also divided into 50-ml divisions and numbered every 100 ml. The pint scale was divided into sixteenths of a pint up to the 1 pint mark, and into quarter-pints from the 1 pint to the 2 pint mark. The whole scale was numbered in quarter pints (¼, ½,.....1¾, 2). A further fluid oz scale was masked as it was felt that the units named would be confused with oz units of weight on the weighing machine.

Interview schedule and procedure for measuring jug

Questions 1 to 4 were as for the ruler. Question 5 took the form *Can you read these words?* and followed up with: *Do you know what they mean?* (The words 'litre' and 'pint' were printed on the jug, and read to the child if necessary.) The experimenter then asked how the jug was used for measuring.

The experimenter's demonstration of the size of a unit (questions 6 and 7) differed for the jug from that for the ruler, tape measure and weighing machine. The jug's full capacity was just 1 litre, so children could be shown only one 1-litre unit instead of several. However, two pints were marked on the jug, and these were separately indicated.

Clock

A 'travelling' clock, 5.5 cm x 5.5 cm x 2.8 cm in depth; diameter of the face 4.7 cm. As well as hour and minute hands, there were a second hand and alarm indicator. The circular dial was divided into hours, subdivided into minutes. Hours were boldly numbered in digits $5mm^2$. Batteries were installed and the second hand moved visibly.

Interview schedule and procedure for clock

As for the other devices, children were first asked whether clocks measured anything, and if so, what. It was anticipated, however, that many would say that clocks did not measure anything, but instead told the time. The idea of time as measurable, and thence as involving units, was therefore introduced to all children by question 3: *How does a clock tell us how much time has passed?* This question was intended to lead naturally to consideration of the functions of the hour, minute and second hands (and of the alarm indicator). The question *What do the numbers tell us?* (with re-directing questions as necessary, set out among the questions for 'ruler') then followed on as before, and then *How does a clock work?*

No unit names were printed on the clock, and there was only one scale. The subsequent questions as set out in relation to the ruler were therefore inapplicable.

Wall thermometer

The glass bulb and tube containing a red fluid were mounted in a plastic case. A Fahrenheit scale was printed on one side of the glass tube and a Celsius scale on the other. The Fahrenheit scale showed 2^{o} divisions, extended from -40 ^{o} at the bottom of the scale to 130 ^{o} at the top, and was numbered from 120 ^{o} to 40 ^{o} every 20 ^{o}. The Celsius scale showed 1^{o} divisions, extended from -40 ^{o} at the bottom to 55 ^{o} at the top, and was numbered from -40 ^{o} to 50 ^{o} every 10 ^{o}. Zero degrees were therefore marked part of the way up each scale. Degrees below zero were not marked with a minus sign on either scale.

Interview schedule and procedure for thermometer

It was expected that children would have seen and would understand the function of a thermometer and how it worked in general terms, but would be unfamiliar with F and C scales and their relationship. The same first four questions were therefore asked for the thermometer as for the ruler, and the children were also asked how a thermometer worked. They were not asked about relationships between the units, as piloting had shown this to be too difficult for them.

3.3.2 Results

Table 3.11 classifies children's responses to the first three questions. Table 3.12 classifies an important subset of responses to Question 3. Table 3.13 classifies the responses to the question *How does it work?* asked in relation to four of the devices. Table 3.14 classifies the responses to Question 4, and Table 3.15 the responses to Question 5.

Can this device measure? What could we measure with it?

Table 3.11 *Can this device measure? What could we measure with it?* Number of children in each year-group who named the device shown, affirmed that it was a measuring device, and identified entities that could be measured with it

		Year	
	1	**2**	**3**
Ruler			
Device named	26	21	26
Can it measure? - Yes	25	20	24
What can it measure?			
Identifies as measurable ≥ 1 countable item	21	18	23
No information relevant to measurement given	0	0	0
Tape measure			
Device named	15	14	16
Can it measure? - Yes	26	21	26
What can it measure?			
Identifies as measurable ≥ 1 countable item	22	21	19
No information relevant to measurement given	0	0	0
Measuring jug			
Device named ('measuring jug' [2])	2	4	3
Can it measure? - Yes	16	18	23
What can it measure?			
Identifies as measurable ≥ 1 liquid or uncountable solid	14	11	16
Identifies as measurable ≥ 1 inappropriate entity[3]	3	5	6
No information relevant to measurement given	0	0	0
Kitchen weighing machine			
Device named ('weighing machine' or 'scales'[4])	16	21	26
Can it measure? - Yes	15	16	26

What can it measure?			
Identifies as measurable ≥ 1 weighable solid or collection	0	16[1]	20
Identifies as measurable ≥ 1 inappropriate entity [5]	0	2[1]	0
No information relevant to measurement given	0	0	0
Clock			
Device named	26	21	26
Can it measure? What can it measure?			
Yes: it measures time	0	1	3
Yes: it measures length	1	0	0
No: it tells the time (at a given moment)	15	18	21
No information relevant to measurement given	10	3	2
Thermometer			
Device named	0	4	4
Can it measure? What can it measure?			
Yes: it measures heat/cold/temperature	0	6	4
Yes: it measures length	4	0	0
No: it tells how hot or cold it is	1	16	12
No: it tells that you are ill/has to do with blood	4	2	9
Other associations	5	0	0
No information relevant to measurement given	7	2	3
N	26	21	26

Notes 1. Totals to > 16 because two children who said the machine could not measure went on to give examples of how it did; 2. 'measuring jar', 'measuring pot' and 'measuring cup' were also accepted; 3. for example, large countable items such as pieces of fruit; 4.'weighter' and 'thing for weighing' were also accepted; 5. for example, a person.

Table 3.11 shows that all children named the ruler, clock, and (in Years 2 and 3) the weighing machine. Just over half in each year-group named the tape measure. Very few named the measuring jug or thermometer. Eight children in each year-group called the former simply a 'jug', 'pot' 'jar' or 'cup', labels which gave no indication of a measurement function. Seven children in Year 3, four in Year 2 and two children in Year 1 mentioned the word 'temperature' when asked what the thermometer was.

Children who did not know or could not recall the name of a device, however, were often able to describe what they took to be its function. This information included a) the type of entity children considered could be measured by the device, and b) a description of how the device was operated. These two categories of information are presented respectively in Table 3.11 under the heading *Can it measure?* and in Table 3.13 under the heading *How does it work?*

In general, the appearance of the scales on the weighing machine, the measuring jug, the clock and the thermometer did not suggest to children that these devices measured length. This was the case only for the clock and thermometer, and only among the youngest children. Two children in Year 1 suggested that one could *measure this* with the clock, and held one of its straight edges against a nearby object (book and tape recorder). This could be seen as a sensible answer if the side of the clock was being used as a non-standard unit; however, there was no attempt in either case to iterate the unit along the object 'measured'.

Two other children in Year 1 affirmed that the thermometer was for measuring the length of the foot (presumably they were reminded of the device used for that purpose when children are measured for new shoes) and one described it as a 'blood ruler'. References by a number of children to blood, or to the thermometer as indicating illness – for example *It's a blood thing. It tells you you're ill* and *You put your blood through it to see if it's all right* or *The line goes up to show how sick you are* - seemed to be prompted by the association of the taking of one's temperature with illness and medical settings, and by the colour of the fluid in the thermometer shown. Neither the mention of measuring feet nor the association with illness suggests that the scales on the thermometer were actually thought to measure length; rather it was the general appearance of the device that seemed to prompt these associations. One Year 1 child, for example, obviously reminded of a metronome, said that she had 'one of those' on her piano, and it ticked.

Some associations seemed more complex. *It goes red for fire*, probably prompted by the colour of the fluid, seemed to unite the ideas of heat, of red for danger, and of the thermometer as an indicator or scale. Two other respondents, having correctly said that the red fluid showed you how hot it was, added that if you wanted to know how cold it was, you needed a thermometer with blue fluid, probably associating these colours with the coding used on water taps. One of these two also turned the thermometer so that the bulb was at the top and said that it was like a try-your-strength machine in a fun fair. Sometimes two distinct functions would be mentioned: for example, one of the children who associated a thermometer with measuring one's feet also said that you put it in your mouth when you were sick. Three children had apparently seen a thermometer explode in a television cartoon and, having described its function broadly correctly, went on to state that when the line of fluid reached the top of the device, it exploded. Several others mentioned explosions in relation to thermometers in more general terms.

By contrast, the five children in Years 2 and 3 who did not associate the thermometer with the measurement of heat or indication of illness, said simply that they did not know what it was for.

It should be noted that the response 'anything' to the question *What can* [this device] *measure?* was not credited as acceptable, and children who responded in this way are not enumerated in Table 3.11.

One type of response to the question *What can* [this device] *measure?* in relation to the 30 cm ruler and occasionally to the tape, was of particular interest. This was the usually firmly uttered assertion that *only small things* - that is, only things no longer than the device - could be measured with it. This response was frequent only in Year 3, where it was made by half the children. Two interpretations of such a response suggest themselves. Piaget remarks in the towers study (1960: 60) that when children have accepted a stick or strip of paper as a measuring device, they at first do so only if the device exceeds the height of the tower to be measured. Later they do accept a measure that is shorter than the tower, but try to make up the full length in a makeshift way with another object. Piaget argues that although the children are no longer trying to bring the two towers together in a direct comparison and do accept an intermediate measure, their understanding of units is still not established, because it does not occur to them either to iterate a short strip as a unit, or to mark a longer strip in equal units and use that as a measure.

One interpretation of children's comments in the present interview might thus be that by a number of children the ruler or tape was not fully understood as the embodiment of units that could be indefinitely iterated (the length of the device being merely conventional) but rather that measurement was still seen as a direct comparison between two objects - the ruler (or tape) and the item measured. A second possible interpretation is that the children were taking too literally a common injunction (Department for Education and Employment 1999; Department for Children, Schools and Families 2006) to select the best measuring device for the job in hand: the ruler for small objects, and the tape for walls, doors and furniture (which, as they reported in the first interview, were invariably measured with a tape at home). If the first interpretation is correct, and general developmental factors were at work, one would expect this response to be commonest among the youngest children. The fact that it was commonest in Year 3 makes the second explanation the more plausible, since these children had experienced more measurement instruction in school.

To better understand each such response, the researcher followed it up with the question *Isn't there **any** way you could measure something longer if you just had this ruler/ tape?* The three main types of response to this further question are set out in Table 3.12.

Table 3.12 Number of children in each year-group affirming the ruler or tape could not measure items longer than itself, and responses at follow-up

	Year		
	1	**2**	**3**
Ruler or tape cannot measure items longer than itself	4	4	13
Q: Isn't there *any* way it could do so?			
No	2	1	1
Yes – supplement the device with another item	0	0	5
Yes – iterate the device and add the units	1	1	6
Other/unclear	1	2	1
N	26	21	26

It will be seen that in Year 3 responses were evenly divided between describing a strategy of unit iteration and describing the makeshift strategy noted by Piaget.

How does it work?

Table 3.13 *How does it work?* Number of children in each year-group mentioning the given features of the weighing machine, measuring jug, clock and thermometer.

		Year	
Device and features of its use	**1**	**2**	**3**
Weighing machine			
pan moves downwards	16	15	23
indicator moves to a number	12	16	21
read the number	12	17	15
Number of above features mentioned			
3	7	5	9
2	5	2	4
1	1	3	1
0	6	0	1
Measuring jug			
pour fluid into jug	17	18	18
ensure fluid is level with requisite line	12	16	14
read the number	13	18	14
Number of above features mentioned			
3	8	13	10
2	8	5	8
1	2	3	0
0	8	0	8
Clock (*How do we know how much time has passed?)*			
Hand named (or function described)			
hour hand	1	6	12
minute hand	1	5	12
second hand	1	5	13

relative movement of hands	1	2	13
Number of above features mentioned			
4	0	2	10
3	1	3	2
2	0	0	1
1	1	1	2
0	24	15	11
Thermometer			
either fluid rises when it is hot *or* fluid falls when it is cold	3	3	2
both occur	0	0	6
no response	23	18	18
N	26	21	26

The sub-categories used in Table 3.13 describe basic actions necessary in using the device (weighing machine and jug) and basic knowledge of how the device measures (clock and thermometer). These sub-categories were chosen in advance of the children's responses. Children were not asked *How does it work?* about the ruler or tape measure, since they would be asked to demonstrate this in the measurement tasks. Table 3.13 shows that over half the children in each year-group could mention some features of how one weighed material on a kitchen weighing machine, or measured liquid in a measuring jug, although in Year 1 almost one-third failed to mention any such features, and somewhat surprisingly, a similar number of Year 3 children were at a loss as far as the measuring jug was concerned.

In the case of the weighing machine, all were given the opportunity to weigh small items and were then asked to report the weight as shown on the two scales. All were able to report the numbers where the indicator stopped; no child spontaneously added the labels 'grams' and 'ounces'; the researcher supplied these.

In the case of the clock, it was thought that *How does it work?* might divert children's attention to the mechanism of the device, and so here, *How do we know how much time has passed?* did duty for the former question: the focus was on whether children would describe the function of the hands on the clock face. In Year 3, about half the children could name and/or describe the function of all three hands on the clock, and their relative movements, (although four others in Year 3 were of the view that the function of the fast-moving second hand was to haul round the other two hands). Nine children in Year 1, three in Year 2 and one in Year 3 spoke almost exclusively about the alarm function on the clock, usually for waking them in the morning, but also for reminding them of other important daily events. One suggested that the purpose of the hands on the clock was to 'get you to the time when the alarm went off', and although this was explicitly stated only once, this order of importance among the clock's hands could be inferred from what other children said. More generally, it became apparent that while some children were interested in how clocks told the time, were focused on telling what they knew of this and the part played by the various hands and markings on the clock, and were aware of the relevance of these to the notion of time passing, others did not think of the clock as recording the *passing* of time at all. (One child in Year 3 succinctly remarked, *It doesn't tell you how much time has passed. It tells you what time it is right now.*) Rather, they thought of the clock primarily as marking important daily events, such as getting up, going to work or school, or having lunch, so that the 'times' at which these happened were thought of as, in a sense, absolute. Only two children in Year 3 and one in Year 2 spoke in this way, but fifteen in Year 1 did so.

In the case of the thermometer, knowing that a rise or fall of the column of fluid indicated a rise or fall in temperature was considered the key information.

Overall, very few children could supply much detail about how one used either clocks or thermometers.

What do the numbers tell us?

Table 3.14 *What do the numbers tell us?*[1] Number of children in each year-group responding according to the categories listed

	Year		
	1	2	3
Ruler			
Mention of			
appropriate dimension (e.g. 'how long')	12	12	18
non-specific dimension ('how much there is')	2	0	0
inappropriate or mixed dimensions	3	1	1
don't know/no response	9	8	7
Tape measure			
Mention of			
appropriate dimension (e.g. 'how long')	11	12	17
non-specific dimension ('how much there is')	0	0	1
inappropriate or mixed dimensions	4	1	2
don't know/no response	11	8	6
Measuring jug			
Mention of			
appropriate dimension (e.g. 'how much inside')	5	7	7
non-specific ('how much there is')	4	4	7

inappropriate dimension: 'how long'	6	2	5
other inappropriate or mixed dimensions	4	2	4
don't know/no response	7	6	3
Kitchen weighing machine			
Mention of			
appropriate dimension (e.g. 'how heavy')	15	11	17
non-specific ('how much there is')	5	2	2
inappropriate dimension: 'how long'	0	1	1
other inappropriate or mixed dimensions	5	2	1
don't know/no response	1	5	5
Clock[2]			
Mention of			
appropriate dimension (e.g. 'how much time has passed')	0	0	0
hours only	0	0	2
hours and minutes	0	0	8
don't know/no response	26	20	11
Thermometer			
Mention that			
either fluid rises when hot *or* fluid falls when cold	0	3	2
both occur	0	0	6
no response	23	18	18
N	26	21	26

Note 1. No child gave a positive response to this question in regard to the thermometer, so it is omitted.
2. No child responded in accordance with the subheadings used for the other devices, so except for the first, these are omitted.

Table 3.14 shows that in the case of the ruler, tape measure and weighing machine, nearly half the children in each year-group, and sometimes more than half, named an appropriate dimension in connection with the device that measured it, and relatively few mentioned inappropriate dimensions, or a mixture of appropriate and inappropriate dimensions.

In the case of the jug, it was necessary to refer to 'the amount inside', 'the amount of room it takes up', or 'its capacity' to be credited with an appropriate dimension, and this was difficult for most children. The non-specific *the numbers tell us how much there is* was not accepted. Many children – often about one third in a year-group – did not respond to this question.

What do the letters stand for?

Table 3.15 shows that centimetres, millimetres and grams were the only units readily named by children from their abbreviated forms. Few children in Year 1 were able to name any units.

Table 3.15 *What do the letters stand for?* Number of children in each year-group responding to unit names abbreviated on the measuring devices.

	Year		
	1	**2**	**3**
Ruler			
cm	2	18	24
mm	3	18	22
units incorrectly named	2	2	2
don't know/no response	19	0	0
Tape measure			
inches	0	7	11
cm	3	21	25
units incorrectly named	2	5	5
don't know/no response	22	2	1
Measuring jug			
litres/ml	1	12	5
pints[1]	0	2	6
units incorrectly named	0	0	0
don't know/no response	25	7	15
Weighing machine			
oz	0	0	1
g	0	16	16
N	26	21	26

Note 1. This was an unfamiliar word to many children, often pronounced to rhyme with 'mints'.

Questions 6, 7 and 8.

Ruler*: Can you show me how long 1 cm is? Can you show me how long 1 mm is? So which is longer, 1cm or 1mm?*

The preparation for this important series of questions was thorough. It is fully explained in the Method section for the second interview and summarised here: After being allowed to handle and to comment on any features of the measuring devices they wished, each child was asked to indicate the length of 1 cm on the ruler, and then asked to identify two more individual cm-lengths. If a child indicated a number or line rather than a space, the experimenter corrected the child by indicating a cm space and stating that the space, and not the line or number, was the actual centimetre. The process was repeated twice more. The same procedure was adopted in respect of the mm scale, the point of a sharp pencil being use to indicate the length of individual mm. The experimenter then asked *So which is longer, 1cm or 1mm?* When the child responded, the experimenter asked, *How do you know?* With changes appropriate to the device and the units involved, the same procedure was adopted for the tape measure, the weighing machine and the measuring jug.

For each device, Table 3.16 shows the children's responses and their justifications.

Table 3.16 Children's judgements about the greater unit on each measuring device and their justifications, according to year-group

Question	Response	Year 1	Year 2	Year 3
Ruler				
Which is longer?	1 cm	5	13	13
How do you know?	It takes up more space	2	6	7
	The numerals are bigger	0	1	0
	There are no little lines	0	1	1
Which is longer?	1 mm	16	6	11
How do you know?	The numbers are larger	9	1	6
	The numerals are bigger	1	0	0
	There are more mm	1	0	0
	Don't know/No response	4	1	2
	Justification inconsistent with response [1]	2	3	0
Tape measure				
Which is longer?	1 inch	14	8	13
How do you know?	It takes up more space	3	2	12
	The numerals are bigger	3	0	2
	There are more inches	0	1	0
	Justification inconsistent with response	3	0	0
Which is longer?	1 cm	11	10	12
How do you know?	The numbers are larger	2	2	5
	It takes up more space	2	0	0
	The numerals are bigger	0	1	0
	There are more cm	3	5	0
	Don't know/No response	0	1	1
	Justification inconsistent with response	1	3	5

Weighing machine				
Which is heavier?	1 g	19	14	16
How do you know?	The numbers are larger	9	10	11
	There are more grams	1	1	2
Which is heavier?	1 oz	5	4	8
How do you know?	It takes up more space	1	0	1
	The numbers are larger	1	0	0
	Don't know/No response	2	3	9
	Justification inconsistent with response	1	1	0
Measuring Jug				
Which has more inside?	1 litre	16	15	20
How do you know?	The numbers are larger, (or there are more of them)	7	2	4
	It takes up more space	1	0	1
	Seen pints & litres before	0	1	3
Which has more inside?	1 pint	3	2	5
How do you know?	The numbers are larger, or there are more of them	1	1	0
	Don't know/No response	5	2	1
	Justification inconsistent with response	1	2	2
N		26	21	26

Note 1. = (for example) *1 cm is shorter than 1 mm because there are more mm; 1 g is heavier than 1 oz because there are not so many oz.*

Table 3.16 shows that for every device, a substantial number of children affirmed that the larger of the two units that they had just examined was in fact the smaller, and frequently justified their response by appealing to the greater number of the smaller units.

The ruler was of 30 cm and, numbered in tens on its other edge, 300 mm. Table 3.16 shows that half the children in Year 3, rather more than half in Year 2 and about one fifth in Year 1 affirmed that 1 cm on the ruler was longer than 1 mm, and about half of those in each year-group justified this correct response by saying that 1 cm occupied more space than 1 mm. However, a majority of Year 1 children and a large minority in Year 3 affirmed that, to the contrary, 1mm was longer than 1 cm. About half of these respondents in Years 1 and 3 justified this incorrect response by reference to the larger number of mm than of cm shown on the ruler. A smaller proportion of Year 2 children, but still more than a quarter, affirmed that mm were larger and justified this opinion by reference to their larger number.

The tape measure showed cm on one face and inches on the other. In this case more than half the Year 1 children correctly affirmed that 1 inch was longer than 1 cm, but only three of these were able to supply a correct justification for their opinion. About half the Year 3 children judged relative length of the units correctly and also justified their opinion correctly. Rather less than one third of Year 2 children judged correctly, and few could justify their response. Nearly half the children in each year-group judged that 1 cm was longer than 1 inch. Reasons were similar to those advanced to support incorrect opinions in relation to the ruler, or were absent; no type of reason predominated.

In relation to the grams and ounces shown on the weighing machine, there was an even higher proportion of erroneous judgments. This was despite the experience children had been given of placing items in the pan of the machine and reading off the weight shown in both g and oz. (It will be recalled that the indicator crossed both scales so that there was a clear visual indication that the same weight could be expressed in units of different sizes.) Most incorrect judgments were justified by reference to the larger number of grams shown on the scale. The gram scale was longer than the scale showing oz, and this may have encouraged the predominance of this justification. Few children said that the ounce was the larger unit, and almost none who did offered a justification for this view.

Well over half the children in each year-group correctly said that a litre was more than a pint. In this they may have been responding to the absolute size of the litre as represented on the jug; it was by far the largest unit on any scale they had been shown. Greater familiarity with the litre may also have played its part. (The pint was a seemingly unfamiliar unit that was sometimes pronounced to rhyme with 'mint'). This possibility is revisited in the general discussion. However, of the justifications (which were few) the largest number were inconsistent with correct judgements that had been made: children said that a litre was more than a pint because *more* litres than pints were shown on the jug. In fact only one litre mark appeared on the jug, since it was of one-litre capacity. Millilitres, marked in 100s, appeared prominently below the one-litre mark, and it is likely that children were judging that there were more *ml* than pints.

3.3.3 Discussion

The second interview centred on a number of measuring devices. About these, overall, the children were knowledgeable. However, there was clear evidence of confusion about the relationship between the size and number of units shown on the instruments.

Although there were many occasions on which children said that they did not know the answer to specific questions about the devices they were shown, or chose not to respond, there were few examples of individuals being entirely mistaken about the function of a device, the type of entity that it could measure, or the dimension it measured. Overall, children had good knowledge of the uses of the ruler, tape measure, weighing machine and measuring jug, the ways in which they measured, and the dimensions they measured, as the commentary on Tables 3.11; 3.13 and 3.14 indicates. They knew less about these matters in relation to the clock, and very little in relation to the thermometer. There was little evidence that visible scales led children to assimilate non-length dimensions to the dimension of length.

With regard to the clock, only four children considered that clocks measure time, and although many of the older children were able to name and talk about the movements of the hands (Table 3.13), and some were eager to demonstrate what they knew about telling the time, few (Table 3.14) associated the numbers with hours, minutes and seconds as units of time. Rather, the numbers associated with the clock were considered as markers for the sequence of important daily events. Thus understanding of time as an interval scale, stating "time 'at which' " was well on the way, while time as a ratio scale (Haylock & Cockburn, 1989: 72) was little understood. The thermometer was the least familiar device, and the reach of the ideas it prompted - the association with blood; the metronome - suggested the extent to which the overall appearance of a measurement device, together with the social contexts with which it is associated, may be recruited to explain its function, and the same data suggest that from the appearance of a device and the context in which it is encountered it may not be at all obvious to children that the device has anything to do with measurement.

Of greatest interest were indications of fundamental misconceptions with regard to units. There was the suggestion by some children that the ability of a device to measure the length of an object might be limited in an absolute sense by its own length, indicating that the idea, basic to measurement, of iterating a unit or collection of units might not be fully understood.

More striking, however, was the large number of children across the age-range who seemingly ignored the physical size of units they had just been examining, to assert that a unit's size accorded with the number of them that was displayed on the device that embodied them, and not with the extent of the space they occupied. This was asserted across devices, whether length was actually measured, or was used to symbolise the dimension measured. This finding suggested that the inverse relation between size and number of units was far from understood by many of these children.

3.4 General discussion

As intended, the two interviews offered, firstly, useful evidence of the children's knowledge of the essential features of measurement and of what they took to be its purposes.

Secondly, the interviews gave evidence of the contexts in which this knowledge was acquired. The first interview showed that between the two main contexts of acquisition – home and school - there was a sharp contrast in the type of measuring experience that children had had, and the type of importance they attached to it. At home, measurement seemed to be conceptualised predominantly in terms of affordances, and this was done in the service of important events of everyday life. At school the preoccupation was with the handling of rulers and metre sticks and measurement was done for the purpose of reporting the numbers of units involved. The interview thus showed the contrasting sets of conditions under which children constructed their conceptual understanding of measurement.

Thirdly, a more general contrast, a contrast that applied well beyond the measurement domain, could be discerned from the interviews. It concerned two types of understanding that children are expected to acquire, often from the very same experience. These two types of understanding are discussed below under the labels 'conceptual' and 'social'. Development of measurement concepts may be influenced by the degree of children's success in untangling 'conceptual' and 'social' information embedded in their experience.

Finally and more specifically, understanding of a fundamental principle of measurement, the inverse relation between the size and number of units was shown to be insecure for many of the children.

Reflection on the character of the interviews

It is accepted that the picture of children's general knowledge and understanding given in this chapter is likely to be incomplete, and this is discussed before further consideration of the findings.

The first interview sought to establish what children spontaneously called to mind when they were asked about measurement, and although they were given ample time to respond, and encouraged where appropriate with neutral prompts, they were not pressed to be more explicit than they chose. A more direct style of questioning was considered but not adopted for this interview, since there were clear signs that the reticence of some children would have been increased by this course of action. In these conditions it was particularly difficult to know how far responses of individuals might represent the full extent of their knowledge about measurement, and a non-response, a *don't know*, or an apparently irrelevant response could not be taken as an indication of absence of knowledge. However, it did seem fair to conclude that what a child said was an indication of what was most salient for them about the notion of measurement, or, when there was no positive response, of the fact that it was not salient at all. It also seemed reasonable to assume that differences among year-groups in the degree of elaboration of responses could be taken to indicate either a comparatively greater or lesser degree of knowledge about measurement, or of availability of language to talk about it. The same principles applied to the second interview, although here, fewer children were reticent and dialogue with the researcher was on the whole more extensive. This was probably because the measuring instruments they were handling provided a specific focus for their talk.

In both interviews some children were far from reticent and did not restrict their responses to the specific questions of the researcher, imparting instead a wide range of information about measurement in the broader context of their lives.

In sum, whether children said much or little, the picture of their knowledge and experience of measurement given in this chapter is likely to underestimate it. Nevertheless this picture succeeded in providing evidence of the range of knowledge that these children had about measurement, of the contexts in which it was acquired, and also of the *kind* of job children face in developing good understanding of measurement principles that are embedded in home and school experience.

General knowledge about features of measurement

In the first interview, few children mentioned more than one or two features of measurement (dimensions, measuring devices, measurable entities, or measuring activities). Most talk was of length, and given this predominance, some confusion might have been expected as to what was actually measured with respect to other domains whose scales represent units as length. In the second interview this was investigated by asking children to comment on scaled instruments that measured length, weight, volume, time and temperature, all of which represented their units as length. Where children recognised instruments as measuring devices, there turned out to be little confusion about what they measured, suggesting a store of tacit knowledge accumulated from everyday experience and contrasting with the explicit knowledge characteristic of the classroom.

Units of any kind were rarely mentioned in the first interview. Spontaneous mention of units was not recorded in the second interview, but when asked *What do the letters* [on the devices shown] *stand for?* children could usually name only centimetres, millimetres and grams from their shortened form. Children in Years 2 and 3 often recalled the lengths of specific objects measured in the classroom and expressed these as numbers of (generally unnamed) units. Sometimes they even recalled the outcome of iterating the ruler or metre stick and adding the numbers of units produced. Alongside this evidence of competence, however, about half the children in Year 3 gave some evidence of a belief that the length measured by a ruler or tape could not exceed the length of the instrument. Pressed on the question, about half of these children stated that the instrument could be iterated and the units added; however the rest either suggested supplementing the ruler or tape with some other object not marked with units, or had no suggestion to make. There was some indication here that understanding of measurement as iterating a standard unit, and of the ruler as embodying this iteration, was not secure. Furthermore, there was clear evidence of confusion about the relationship between size and number of units when these were indicated on the instruments shown in the second interview. This is further discussed below.

Measurement at home and at school

The first interview demonstrated that children's experience of measurement outside school was a rich one, and was embedded in a wide range of interesting activities. The predominant model of measurement of length was that of fitting an item to the space available for it; the conception was that of affordance. Two possible consequences of extensive experience of measurement activities that take the form of calculating or estimating affordances suggest themselves. On the one hand such experiences may have a generally beneficial impact on certain kinds of estimation ability: children may become adept at visual comparisons of length. Children's visual estimation capabilities were tested in a later part of this research. On the other hand, the same experience of such direct comparisons of length and their success may encourage a 'satisficing' stance (Simon 1979), and motivation to be precise in the matter of deploying measuring devices and interpreting amount expressed in units may be adversely affected. (It was suggested above that motivation to be precise may also be adversely affected by lack of interest in typical classroom measurement activities.)

The language used to describe measurement in the home was direct and vivid; where units were named, they seemed to be deployed to embellish the narrative of which they were a part, and were frequently inappropriate to the domain to which they were applied. Numbers of units given were often wildly in excess of a sensible estimate for the entity named; the large numbers seemed to be used to mark the importance of the activity in which they figured.

Descriptions of measurement at school suggested, by contrast, typically rather mundane activities, centering on the instrument involved (usually the ruler or metre stick), and on the noting and recording of the number of units that was the product of the activity. These activities were usually more carefully described than those in the home setting, and there were often precise accounts of the handling of the measuring instrument and of the outcomes of specific instances of measuring in terms of numbers of units. These precise accounts suggested that the classroom experience was more likely than that of the home to foster understanding of units, although the narrow range of classroom activities and the fact that identical rulers and metre sticks were used throughout the school left room for doubt as to how far that understanding would be generalised.

The conceptual and the social

In the course of development, specific experiences become represented with some degree and form of generality. Both children and adults are expected to abstract general principles from a wealth of specific daily experiences, and in the case of children, the development of conceptual understanding - of transitivity and conservation for example – that underlies many types of reasoning has been extensively studied. There has been much less focus on the development of children's understanding of specific social situations and their implications (such as what to do at a birthday party, or how to conduct oneself in a particular type of classroom activity) or on how the conceptual and the social might interact in specific contexts. The interviews in the present research suggest a lack of continuity between the 'conceptual' and the 'social' in relation to measurement concepts that may adversely affect their development.

For example, the way in which, in an out-of-school context, large numbers and the names of comparatively large units may be used to embellish a measurement story of personal importance to the teller has already been mentioned. In Table 3.06, the dog *about thirty metres long* and the new window that measured *about ten hundred kilograms* are cases in point. We might say that numbers and unit names are here put to the 'social' use of marking the importance of the event. Exaggeration is licensed in such social contexts; how do children come to recognise this as exaggeration, and adjust to the classroom experience, where it is unacceptable inaccuracy? In contrast, Table 3.09 sees Year 2 and Year 3 children, disciplined by school experience of the measuring activity itself, refer to plausible numbers of units (albeit usually unnamed), and how these might be arrived at by addition. The contrast these data suggest between home and school situations as to the role units play for children, together with the prominence of measuring in the home in terms of affordances, indicate that teachers might have an uphill struggle in persuading children of the importance of accuracy in measuring. More fundamentally, precision may be difficult to establish as essential to the concept of a unit in measurement.

Overall, this contrast suggests that features of measurement knowledge acquired in the home and school contexts are not likely to support each other, and that the type of measurement done in school may not be seen as relevant beyond that setting. There is a broad parallel here with science education, where children's own frameworks for explaining physical phenomena, acquired outside the classroom, may obstruct understanding of the science account taught in the classroom (Driver, Guesne & Tiberghien, 1985).

Next, consider the responses when children were asked whether they or the experimenter was the taller, and how they knew. *How do you know?* elicited children's social knowledge that those who were older and whom they perceived as of higher status would be taller. This is effective use of social knowledge and those who made use of it were likely to be right about the relative heights in most circumstances. The next question *How could you check that?* was superficially similar, in that both were requests to justify the judgment. However the first question activated social knowledge, while the second elicited practical techniques for comparing physical height. This illustrated quite well how superficially similar language may signal for children quite different areas of knowledge.

Finally, consider the kinds of observations made by children about the clock and the thermometer, the first very familiar to children; the second quite unfamiliar. These two devices differed from the others used in the interview in two ways. Both measure entities that are invisible and intangible, and both measure 'passively': unlike the other devices, neither requires physical manipulation in order to measure. In fact few children considered them to be measuring devices.

Possibly because of this, the social functions of both the clock and the thermometer figured prominently in what children had to say about them. A 'time' on the clock was seen as labelling an important event in the daily routine (rather than as a measure), and the alarm indicator was accordingly prominent in discussion because it was a reminder of such events. On the other hand associations prompted by the thermometer, about which the children knew little, were more diffuse. Nevertheless, these associations were not haphazard. The red colour of the fluid in the thermometer was associated with heat, with blood, and with danger. The word 'temperature' was associated with being ill. Blood, having a temperature, and hospitals were associated together, while the idea that the device would explode when the column of liquid reached the top of the scale, apparently initiated by a television cartoon, also contributed to what seemed to be a vague association of thermometers with danger. Responses to the thermometer illustrate how readily available everyday knowledge, social in origin, may be recruited (especially by younger children) to invest what is little-known with meaning.

Understanding of the inverse relation between size and number of units

Responses on the relative size of units also illustrated how readily use is made of easily available knowledge, in this case the knowledge that larger numbers usually mean 'more'.

Many children across the age-range repeatedly, after closely examining the relative size of units, asserted that the smaller of the two units they had just examined was in fact the larger, on the grounds that larger numbers were associated with it on the measuring device. On the face of it, the way in which these children seemingly ignored the evidence of their eyes was extraordinary; it seemed that the impression made on them by large numbers completely overwhelmed the meaning of those numbers in relation to the units of amount that they signified.

Miller (1989) contended that the procedure for measuring an amount (here, counting units of length) affects children's judgements of amount. Thus if counting does not change the amount of material – as happens when smaller units are used (or existing units subdivided) –children will not conserve quantity and their judgements will be wrong, because counting misleads in this situation. If this is so, then the proposed influence must be very strong, considering the emphasis in the interview on comparing the actual spaces occupied by the larger and smaller units.

The possibility that the more familiar unit was nominated as being 'larger' - in importance, as it were - (the justification in terms of the number of units being ex post facto) was considered but rejected. This suggestion was perhaps plausible where grams were said to weigh more than ounces, since grams were by far the better-known unit of weight to the children, and Table 3.16 shows that they were identified as being the heavier by three times as many children as identified ounces as the heavier. Yet in connection with the ruler, the same table shows more children saying that 1 mm was longer than 1 cm than said the reverse, when throughout the research, the unit with which the children were clearly most familiar was the cm. Furthermore, slightly *more* children, looking at the tape measure, correctly said that an inch was longer than a cm than said the reverse, yet the inch was by far the lesser-known unit, and some had never heard of it. The litre was clearly more familiar to the children than the pint, and many correctly said that a litre was more than a pint, with, however, incorrect justifications in terms of there being a larger number of litres. (Since the jug was of one-litre capacity, only one litre mark appeared on it). Here, as already argued, the most likely explanation was that children were actually attending to the many ml marks rather than to the single litre mark; the large number of 100 ml marks was much the most salient feature of this scale on the jug. This would support the general thesis that the impression made on the children by large numbers, in itself obscured the meaning of those numbers in relation to the units they signified. The argument from familiarity of the unit becomes least plausible, however, in the light of the substantial number of children who correctly identified the greater (or lesser) unit, irrespective of their relative familiarity, and justified this by indicating that they could *see* which was greater.

This finding suggested that many of the children did not understand the inverse relation between size and number of units. It is possible that this task is made harder by the complexity of the display of units found on common measuring devices, and suggests firstly that the principle involved in the inverse relation should be explicitly taught, and secondly that this might best be done using simplified displays of units.

The interviews established broad differences in the contexts in which children learned about measurement, reflected in general characteristics of the language they used when recalling those settings. The research next focused more closely on childrens' understanding of specific everyday language and concepts of ordinal length.

Chapter 4

The language and concepts of ordinal length

4.1 Introduction

The previous chapter gave evidence that contrasting social contexts influence children's developing understanding of measurement and it was the children's use of everyday language, particularly in the first interview, that provided this evidence. In the Discussion section of that chapter, examples were given of the ways in which the children's social uses of the measurement language of several domains may contrast unhelpfully with the conceptual focus of the same language in school. Rhetorical devices such as exaggeration (applied to units) provided examples.

Does the everyday language of the domain of length present any specific difficulties to children? The research continued by examining children's understanding of everyday language and concepts used specifically in length comparisons, a choice made in consideration of the tasks to follow, which required visual estimation and measurement specifically of length. The expectation was that while the everyday language of length provides essential conceptual underpinnings for measurement, the different social contexts in which particular terms are used, as well as other subtleties of use, would make it difficult for children to extract their conceptual content. Resnick summed this up (1992:107-8) by saying that children learned "both less than and more than mathematics on each such [social] occasion".

4.2 Ordinal length comparisons

How well do children make ordinal comparisons of length? The everyday language and concepts investigated and reported in this chapter are those at work in ordinal length comparisons such as *The green line is longer than the black one.* The understanding of what it is for one object to be shorter, longer or the same length as another has to be secure before units of length can enable us to be more precise about such comparisons, because simply knowing that one number is greater than another can be entirely misleading when considering how numbers of units represent length (Correa *et al.*, 1998; Nunes & Bryant 1996; Carpenter & Lewis, 1976). Thus the work reported in the current chapter asked children to respond to questions like *Which is longer? Which is shorter? Which is the same length?* The research attempted to distinguish children's linguistic from their conceptual competence in relation to such questions by keeping questions constant, while varying the materials about which they were asked.

Thus the first objective was to investigate children's understanding of the everyday *language* of length. The second was to explore children's *conceptual* understanding of ordinal length comparisons. In the context of everyday length comparisons, do children conserve length? Here a lead was taken from Piaget's (1960) formulation of the understanding required: that changing an object's position in space does not change its length. The position in space of the materials used was systematically varied: the question was whether the children's judgements of length would vary accordingly, or whether they would remain constant across different spatial arrangements. This was therefore a test of conservation in the broader sense described in 1.2.5.1, and because of its fundamental importance to measurement, given greater prominence in the present research than in the work reviewed there.

Finally, to assess *generality* of findings, a wide range of materials was used. Below, a balance has been sought between giving full and detailed description of all the materials, and an overview sufficient to inform understanding of what was done. Examples of materials are included in appendices.

Except in the final interview, participants were more constrained from this phase of the research onward, both by the tasks involved, and by the narrower focus and increased directness of the questioning. The constraints, as already stated, were designed to facilitate comparison of the outcomes of the tasks, and to limit potential difficulties of interpretation of the results.

4.3 The language of length

The National Numeracy Strategy (1999) required 5-year-olds in UK schools to be able, by the end of their first school year, to use terms like *longer*, *shorter* and *taller* to compare the lengths of "two, then three or more" objects (Department for Education and Employment, 1999:22). In its successor document, the Primary Mathematics Framework (Department for Children, Schools and Families, 2006:Year 1 Block D, Ma3, Measures) assessment for Year 1 emphasises problem-solving *How did you find out which of these two objects was the...shorter?* and documents associated with the new National Curriculum available to date (Qualifications and Curriculum Development Agency, 2010) do not suggest major changes.

These formulations do not appear to anticipate any difficulties in the everyday use of the language of length and may underestimate them, providing an additional impetus to the present investigation, which examined children's understanding of three essential attributes of length: height, length and width.

The language of ordinal comparison

The research tested children's ability to make five types of ordinal comparison, labelled here for brevity 'least', 'less', 'same', 'more' and 'most'. For example, children were shown displays of toy figures of different heights and asked to identify who was, from head to feet, *the shortest* ('least'); who was *shorter* ('less'), *the same height* ('same'), or *taller* ('more') than another figure; and who was *the tallest* ('most'). The same five types of comparison were tested in relation to length, to width, and to height above ground.

Bias in favour of terms that express greater quantity

In these types of comparison, one possible general influence is the bias in language usage noted by Haylock and Cockburn (1989) in favour of terms that express greater, rather than lesser, quantity; an example, perhaps, of the much greater frequency of positively-toned than negatively-toned words in a number of languages (Zajonc 1968). Thus, children tend to hear and speak of who has more, rather than who has fewer, sweets; we are more likely to compare the heights of two people by saying that one is taller, rather than that one is shorter. It was possible, therefore, that children would find it easier to make comparisons of greater than of lesser length, due to increased exposure to the relevant language. A number of more specific ways in which the everyday language of height, length and width may be difficult for children to interpret are discussed next.

The language of height, length and width in everyday life

First, children may find it difficult to determine what is meant by height, by length and by width. *Height* refers to length from top to bottom where these extremities differ in appearance (roof to ground level of a building; head to feet of a person). However, even when a person is lying down, their *height* is still the distance from head to feet. Where top and bottom do not differ in appearance (a block; a tube), *height* refers to the longest dimension when that dimension is perpendicular to ('standing on') the surface that supports the object. When the block or tube is lying at full length on the supporting surface, however, the longest dimension is instead referred to as its *length*.

Understanding of the relevant convention with respect to human figures was tested in the present research by asking children to compare the heights of figures some of whom were standing, and some lying down.

Width refers to length from side to side. So the length of a pen is from top to bottom, and its width is from side to side. The length of a car is from front to back, its width is from side to side, and its height is from top to bottom. Children were asked to compare the lengths of toy cars whose width and height differed as well as their length, offering potential distractors and increasing the demands of the task. They were also asked to judge the width of pens and pencils. This was expected to be easier because here, length is proportionally much greater than width, making the two attributes relatively easy to distinguish.

Understanding the difference between height from head to feet and height above ground

Second, children may experience special difficulties in relation to height. When speaking of the height of something or someone, one may be referring to length from top to bottom or to height above the ground. In the latter sense, a short person might, for example, be higher than a taller one, or a cottage higher than a lighthouse. Piaget and colleagues noted (1960) that in comparing the heights of two towers, children appeared to take into account only the tops of the towers, ignoring the differing levels of their bases. One view of this behaviour might be that children were interpreting the instructions as requiring judgment of height above the ground, rather than comparison of the two towers from top to bottom. The English translation (Piaget *et al.*, 1960) supports this view, as the instructions to the children seem either ambiguous (first instruction) or vague (the phrase that follows):

> ...the experimenter uses phrases such as: "You make a tower the same height as mine", or simply "the same as mine" &c. (Piaget *et al.*, 1960: 30)

However, the original is not ambiguous in this way:

> «Tu vas faire une tour de la même grandeur que la mienne» ou « la même que la mienne &c » (Piaget, Inhelder & Szeminska 1948: 44)

where the grammar, at least of the first sentence, clearly indicates intrinsic height or size. Moreover, while the Concise Oxford English Dictionary (Pearsall, 2001) lists *height* as having two physical meanings: *1. the measurement of someone or something from head to foot or from base to top. 2. elevation above ground*, Le Petit Larousse Dictionnaire (Karoubi, Maire & Ouvrard, 2008) lists the sole physical meaning of *grandeur* as *dimension de quelque chose, taille, étendue.* Elevation is not a possible implication here. (Le Petit Larousse lists *hauteur* as having the same physical meaning as the first listed above for *height* by the Concise Oxford Dictionary: *dimension de quelque chose de sa base à son sommet* but lists a different expression - *à hauteur de* - as indicating height above ground.)

It seems, then, that the difficulty encountered by the children in Piaget's study may not have been entirely due to the language used (although it is true that the vagueness of *la même que la mienne &c* does not suggest particular care in this respect on the part of the experimenters) but may have involved, as Piaget argued, conceptual factors. This suggests that English-speaking children might encounter conceptual as well as linguistic difficulties. They may not distinguish height from head to feet from height above ground, and may accordingly make global judgements. This might be particularly likely before experience has enabled them to elaborate the different ways in which height is referred to in English. In the present research, during the first interview, for example, children in all three year-groups appeared to judge that standing on a table would make them taller.

To investigate the extent to which global judgments of this type might be made, children were asked in the research to compare the heights (tallness) of toy figures some of whom were standing on a block. The raised figures were actually the shorter among those that children were asked to compare, so that children were tested with a potential conflict between height of a figure from top to bottom and its height above the ground.

It is also possible that, once having distinguished height from top to bottom from height above ground, children then switch to answering *all* questions about height as if they referred to height from top to bottom. To test for this, further tasks were devised that required children to compare heights of different items above ground. Shorter items were raised to a higher level above ground than taller items, thus inducing a potential conflict between overall distance above ground and height from top to bottom. If children understood both types of question - those about height from top to bottom, and those about height above ground – they would make both kinds of comparison correctly.

Finally, the language of length may be used metaphorically, where no physical reference is intended, as in the expressions 'the height of bad manners' and 'a tall story'.

So children may not understand linguistic conventions governing talk about length that are taken for granted by speakers, or the extent to which interpretation of terms describing length depends on context.

4.4 Conceptual ability

The ability to conserve length, ignoring irrelevant features of spatial presentation when judging comparative length, is a conceptual ability. Piaget and colleagues (1960) noted that until the age of about seven years, children sometimes judged distance between toy people to be greater when the people were of different heights, or their level was raised. In comparing the heights of two towers, as already mentioned, children ignored the differing levels of the bases of the towers. They judged wrongly when end-points of parallel lines whose length was to be compared were displaced. If two lines of equal length were each segmented into unequal pieces, the line in which an especially long segment was included was judged to be the longer line.

In the present research, children's ability to ignore such irrelevant features of spatial presentation was tested with two- and three-dimensional displays in which the relative level of items differed, end-points were displaced, or the orientation of items differed.

4.5 Separation of language and conceptual ability

The work reported in this chapter sought to separate linguistic from conceptual knowledge in children's understanding. It did this by asking them to make height, length and width comparisons under two conditions. In the first (simple) condition, the perceptual comparisons involved were straightforward. Within each display, items to be compared were presented on the same level, and were aligned with a common end point and parallel with each other. In the second (complex) condition, items that were similar to those used in simple displays were rearranged so that the alignment of their end points changed, their orientation varied, or the relative level of items differed.

In both simple and complex conditions, the experimenter's questions to children were exactly the same. Since this was so, it was reasoned that if responses in the simple condition were mainly correct, but children made appreciably more errors in the complex condition, then their understanding of the language involved was not likely to be responsible. In that case, their difficulties must be due to aspects of the displays themselves. Because the same materials were used in simple as in complex displays, and the relative length of items in each complex display was similar to that in the corresponding simple display, it was the different spatial arrangement of items presented that was likely to be the cause of difficulty.

Finally, evidence (Bryant, 1974) that relative judgments of size are easier than absolute judgements would suggest that there would be more correct relative judgements of greater or less length, than absolute judgements of equal length

4.6 Factors investigated

The three main factors investigated were (1) type of comparison (five types of comparison, reflected in the five types of question asked); (2) attribute (three attributes: height, length and width, reflected in the spatial relationships among the items displayed, as well as in the questions asked about them); (3) complexity of display (two levels: 'simple' and 'complex' as described above).

As an additional factor to test generality of understanding, displays were presented in two formats: 3D format (real items) and 2D format (line drawings of similar items).

4.7 Method

4.7.1 Sample

Seventy-two of the children who were involved in all other aspects of the research participated, twenty-four children in Year 1; twenty-one in Year 2 and twenty-seven in Year 3.

4.7.2 Summary of materials and procedure

Table 4.01 presents an overview of the materials used in relation to each attribute of length investigated. Table 4.02 sets out the wording of the questions used in relation to the same attributes. Table 4.03 shows the number of comparisons children made according to attribute and type of display.

Table 4.01 Summary of materials used to make comparisons of height from head to feet, of length, of width, and of height above ground

Attribute			
Height from head to feet	**Length**	**Width**	**Height above ground**
2D			
Pictured people of	Lines of various	Pictured writing	People of various

various heights; some raised on a block	lengths printed on card	implements of various widths	heights pictured on steps	Continued... .....
3D				
Toy people of various heights; some raised on a block	Toy cars of various lengths	Writing implements of various widths	Toy people of various heights standing on steps	A model of three hillocks of various heights, surmounted by flag-poles of various heights

Table 4.02 Questions inviting comparisons of height from head to feet, of length, of width, and of height above ground, according to type of comparison

	Wording of the questions according to attribute			
Type of comparison	**Height (from head to feet)**	**Length**	**Width**	**Height (above ground)**
Least	*Who is the shortest?*	*Which one is the shortest?*	*Which one is the narrowest /thinnest*?*	*Whose head/ which flag is the lowest?*
Less	*Who is shorter than him/ her?*	*Which is shorter than that one?*	*Which is narrower /thinner* than that one?*	*Whose head/which flag is lower than that one?*
Same	*Who is as tall as him/her?*	*Which is as long as that one?*	*Which is as wide as that one?*	*Whose head/which flag is at the same height as that one?*
More	*Who is taller than him/her?*	*Which is longer than that one?*	*Which is wider than that one?*	*Whose head/which flag is higher than that one?*
Most	*Who is the tallest?*	*Which is the longest?*	*Which is the widest?*	*Whose head/which flag is the highest?*

**Thinner* and *thinnest* were offered along with *narrower* and *narrowest* as the latter terms were unfamiliar to some children.

An example of how these questions were deployed is given under the heading 'Details of procedure' below.

4.7.3 Details of materials

Height from head to feet

Children were asked to compare the heights of people in line drawings (2D format) and of plastic toy people (3D format). In the simple condition (in both formats) combinations of figures of different heights standing at the same level were shown, with an appropriate figure indicated where the question required it. In the complex condition (again in both formats) either the shortest figure was presented standing on a block, or the tallest figure was presented lying down. Figures in line drawings ranged from 3.1 cm to 6.8 cm, and plastic toys from 5.1 cm to 7 cm in height.

Length

Children were questioned about lines of differing lengths printed on A4 card (2D format) and about toy cars of assorted lengths (3D format). In the simple condition, lines were parallel and with left end-points aligned. In the complex condition, lines either appeared with endpoints displaced, or were presented at right angles to each other. Toy cars that in the simple condition were parallel 'parked', with their rear bumpers against a 'wall', were re-arranged in the complex condition in configurations similar to the complex line displays. Lines ranged from 6.5 to 9.5 cm; cars from 6.7 to 10.2 cm in length .

Width

Children were questioned about line drawings of pencils and crayons (2D format). Real writing implements such as pens, a pencil and board writers were also shown (3D format). In the simple condition, the items were presented parallel and in close proximity to each other. In the complex condition, items were presented further apart. Their endpoints were displaced in some arrays, while in others they were orientated to each other at angles between 45° and 135 °. Items in line drawings ranged from 0.6 to 5 cm in width; real items from 0.6 cm to 1.7 cm.

Height above ground

Ability to ignore the length of different components that separately contribute to height above the ground, and to judge *total* height above the ground was tested using three displays. In a pictured (2D) and a toy (3D) format, a set of four people of differing heights was presented standing on a flight of steps, with the shorter people standing on the higher steps. Children were asked: *Whose head is higher?* and cognate questions. A further (3D) test presented a model of four hillocks, each surmounted by a flag. Each hillock and each flagpole was of a different height, with shorter flagpoles standing on taller hillocks. Here, children were asked: *Which flag is higher?* and cognate questions. These three displays had no simple condition, since the type of height they were used to investigate was by nature complex. Heights of toy people together with the steps on which they were raised ranged from 11 cm to 16.1 cm, and of their pictured equivalents from 6.7 cm to 9.1 cm; heights of flags ranged from 7.5 to 10 cm.

Presentation of experimental displays

For 2D displays, one set of ten cards was used for comparisons of height from head to feet, another set of ten for length, and a further set of ten for width. Half the cards in each set showed a simple arrangement of items and half showed a complex arrangement. There were three items on each card.

For 3D displays, three items were drawn from a pool of six, among which at least two were of equal height (or length, or width) so as to provide appropriate material for questions such as *Who is the same height as…?*

For 'height above ground' one set of four people on steps (3D), one card showing the pictured equivalent (2D), and one model (3D) sufficed for all questions.

Participants were seated opposite the experimenter, but no orientation was prescribed to the viewer, so children were free to view the displays from any angle. For 2D displays this could make little practical difference, since the viewpoint was fixed by the drawings. 3D displays, however, were liable to be viewed from a wider range of angles.

A record sheet for each child listed the displays to be shown and was used to record their responses.

Table 4.03 Number of comparisons made by each child in relation to attribute and type of display (75 comparisons per child)

	Number of comparisons			
	Displays			
Attribute	Simple		Complex	
	2D	3D	2D	3D
Height from head to feet	5	5	5	5
Length	5	5	5	5
Width	5	5	5	5
Height above ground			5	5, 5

4.7.4 Details of procedure

Comparisons of height from head to feet are used as an example.

Children were asked to compare the heights of toy people (3D displays), and the experimenter selected and arranged three toys from the six available so as to permit the required comparison. She then pointed to the appropriate toy in a display and asked, for example, *Who is shorter than him (her)?* noting the response. Displays were then rearranged, and toys in successive displays were appropriately indicated. In 2D displays, the experimenter indicated an appropriate pictured figure. Questions were varied according to type of comparison and according to attribute as shown in Table 4.02.

Most children saw all the arrays in one session. (Those who did not, always saw simple and complex counterparts in a single session.) For 3D displays, the experimenter made, on the record sheet, a 'blind' selection of a specific question to be asked next and then selected items for the appropriate display. For 2D presentations, each set of five cards (corresponding to the five questions), was shuffled before blind selection of a card. Children made the five comparisons for simple displays, followed by the five corresponding comparisons for complex displays.

The experimenter said to each child: *I am going to show you some toys, some pens and pencils and some pictures, and ask you some questions about them*, and then proceeded with the questions. Minimum changes were made to the wording of questions to ask, as appropriate, about height from head to feet, about length, about width, and about height above ground (see Table 4.02). Children's response to each question was noted on their record sheet.

4.7.5 Scoring

Responses were scored correct or incorrect.

4.8 Results

The results from the three displays associated with height above ground had no 'simple' version, and so could not be included in the analysis of those from the main series of displays. They were analysed separately.

Main series of displays: height from head to feet; length; width

The main series of displays tested comparisons of height from head to feet; length and width. In view of the girl/boy imbalance in Year 2 noted in Chapter 2, a preliminary analysis examined gender effects across the whole sample. There was no significant effect of gender or significant interaction of gender with any other factor, including year-group. Gender was therefore excluded from the analysis. Mean correct comparisons according to year-group, type of comparison, attribute, format and complexity are shown in Table 4.04. Related analysis of variance is shown in Table 4.05.

Table 4.04 reports a high proportion of correct comparisons, showing that children had good understanding of the language and concepts of length comparison.

An initial mixed analysis of variance showed significant main effects for year-group and for each of the four between-subjects factors shown in Table 4.04 ($p < .01$).

There were also a number of significant interactions. Higher-order interactions involving either year-group or format, however, were mainly non-significant, and all had very small F ratios; so to facilitate analysis of interactions, year-group and format were removed from the principal anova and analysed separately. Thus, two analyses of variance were performed as follows:

a) a three-way within-subjects analysis. Factors were i) type of comparison, with five levels: least, less, same, more, and most; ii) attribute, with three levels: height from head to feet, length, and width; iii) complexity, with two levels: simple and complex. This analysis is reported in Table 4.05.

b) a two-way mixed analysis with year-group as between-subjects variable and format as within-subjects variable. This is also reported in Table 4.05.

Table 4.04 Mean (SD) percentage of correct comparisons in the main series of displays according to year-group, type of comparison, attribute, complexity and format.

	Year 1	Year 2	Year 3	All
Type of comparison				
Least (*shortest/narrowest*)	69 (18)	83 (8)	86 (11)	79 (15)
Less (*shorter/narrower*)	88 (13)	92 (9)	97 (7)	92 (10)
Same (*the same height/length/width)*	73 (21)	82 (21)	81 (19)	79 (20)
More (*taller/longer/wider*)	84 (16)	88 (14)	90 (12)	88 (14)
Most (*tallest/longest/widest*)	71 (17)	83 (8)	86 (11)	80 (14)
Attribute				
Height	73 (12)	76 (13)	84 (11)	78 (13)
Length	82 (16)	88 (11)	91 (10)	87 (13)
Width	77 (21)	93 (10)	90 (16)	87 (18)
Complexity				
Simple	84 (12)	93 (7)	95 (6)	91 (10)
Complex	79 (17)	89 (12)	88 (12)	85 (14)
Format				
2-D	79 (13)	87 (8)	89 (10)	85 (11)
3D	75 (14)	84 (10)	87 (7)	82 (12)
All	77 (18)	86 (14)	88 (14)	84 (16)
N	24	21	27	72

Note. Each child made 60 comparisons.

Table 4.05 Analysis of variance for correct comparisons according to a) type of comparison, attribute and complexity and b) year-group and format

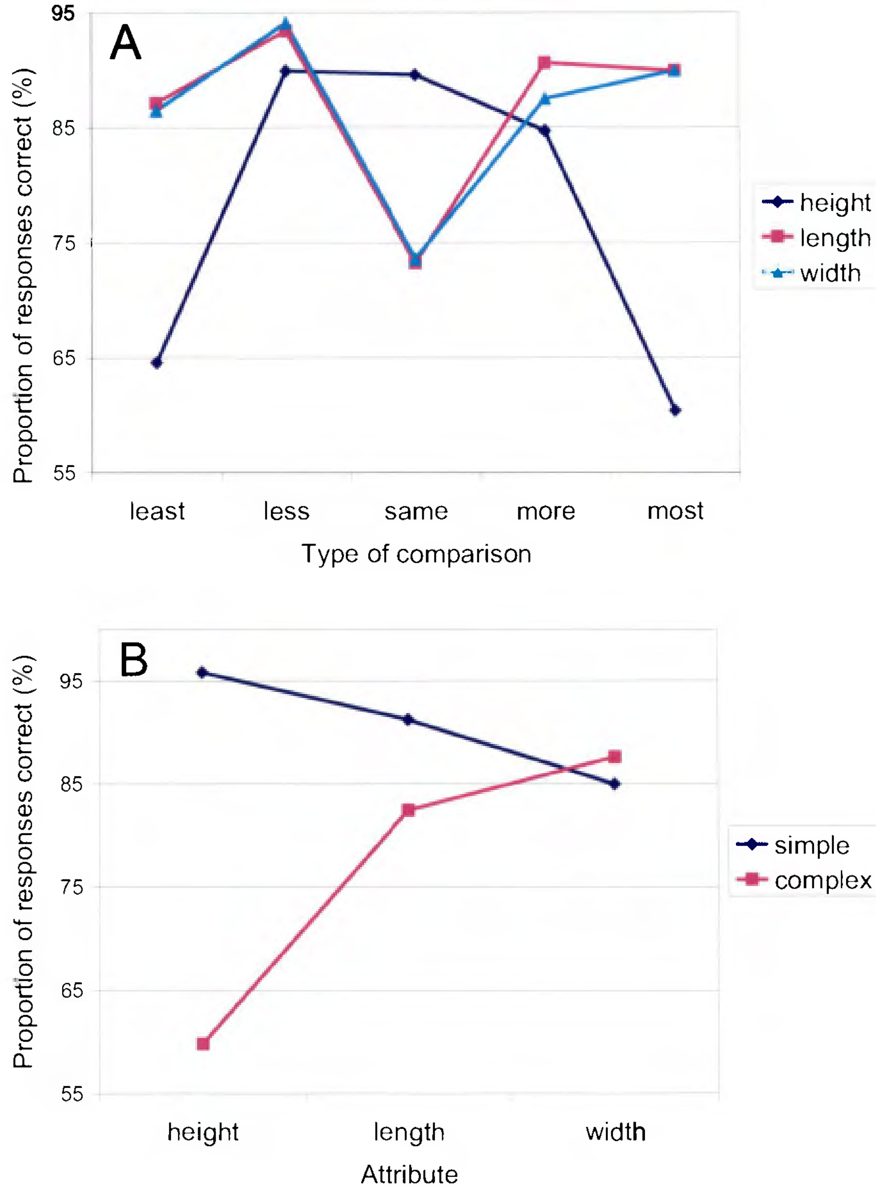
Source	Error *df*	*df*	Error Mean Square	*F*	*p*
a) Type of comparison (T)	284 [a]	2.6[a]	2.45	18.49	<.01
Attribute (A)	122.1 [b]	1.7[b]	2.14	12.03	<.01
Complexity (C)	71	1	10.63	82.92	<.01
T x A	12.3	5.3[a]	2.31	25.80	<.01
T x C	4.3	4	1.08	24.14	<.01
A x C	14.2	2	7.09	106.12	<.01
T x A x C	7.9	6.9[b]	1.17	24.31	<.01
b) Year-group (Y)	69	2	0.18	9.37	<.01
Format (F)	69	1	0.03	9.07	<.01
Y x F	69	2	0.003	0.42	.66

Note.[a] = Greenhouse-Geisser df used; [b] =Huynh-Feldt df used.

As Tables 4.04 and 4.05 show, children's judgments were significantly and substantially more accurate with simple than with complex displays. There were main effects for year-group, type of comparison and attribute. Post hoc tests showed that overall, children in Year 2 and Year 3 made significantly more correct comparisons than those in Year 1; that the length and the width of items were compared significantly better than the height; and that there were significantly more successful comparisons identifying less (*shorter* or *narrower*) and more (*taller, longer* or *wider*) than with the other three types of comparison (least, most and same). Overall, children performed significantly better with displays in 2D format than with 3D displays. Means plots were used to investigate the significant three-way interaction between type of comparison, attribute and complexity and are shown in Figure 4.01.

Inspection of the two plots in Figure 4.01 indicates two likely sources for the three-way interaction observed. First, height from head to feet behaved entirely differently from length or width according to type of comparison. As Figure 4.01(A) shows, there were far fewer correct responses to the questions *Who is the shortest?* and *Who is the tallest?* than to the corresponding questions for length and width. For height, there were also more correct responses to *Who is as tall as him?* than for similar comparisons involving equal length or width. Second, 'simple' differs markedly from 'complex' according to attribute. Figure 4.01 (B) shows performance in simple and complex conditions to be similar for width, to diverge somewhat for length, and to be widely different for height, with performance at near maximum in the simple condition, while at a much lower level in the complex condition.

Figure 4.01 Mean correct responses for (A) attributes of length according to type of comparison; (B) level of complexity according to attribute



Second series of displays: height above ground

The second series (three displays) tested comparisons of height above ground. All three displays presented conflicting cues about height (and hence all were complex displays). It will be recalled that, relative to others in the same display, toy people or flags were placed such that their height above ground contrasted with their height from head to feet (for flagpoles, from top to bottom). For example, the *shortest* in a display of four toy people stood on a step that raised its head *above* those of the others. Thus to answer the question *Whose head is the highest?* correctly, it was necessary to ignore height from head to feet.

Table 4.06 Mean (SD) percentage of correct judgements in the three tests of height above ground according to year-group and type of comparison

Displays of height above ground	Year 1	Year 2	Year 3	All
Toy people on steps (2-D)	63 (34)	69 (38)	79 (29)	71 (34)
Toy people on steps (3-D)	69 (31)	67 (42)	83 (31)	74 (35)
Flags on hillocks (3-D)	80 (22)	88 (20)	96 (13)	88 (20)
Type of comparison				
Lowest	67 (34)	73 (39)	84 (28)	75 (34)
Lower	69 (33)	73 (33)	88 (25)	77 (31)
At the same height	60 (26)	67 (32)	80 (23)	72 (28)
Higher	81 (24)	79 (31)	89 (21)	83 (25)
Highest	76 (29)	79 (27)	90 (22)	82 (26)
Total	71 (21)	74 (29)	86 (21)	77 (24)
N	24	21	27	72

Note. Each child made 15 comparisons in the three tests of height above ground.

A preliminary analysis of variance of the tasks comparing height above ground, again undertaken to check any effect of imbalance between the sexes in Year 2, showed no significant effect of gender, or significant interaction of gender with any other factor including year-group; so the analysis was repeated without gender, as a three-way mixed analysis with year-group as between-subjects factor and display (the three different displays constituting the three levels) and type of comparison (with five levels as shown in Table 4.06) as the two within-subjects factors. This analysis is reported in Table 4.07.

Table 4.07 Analysis of variance of the three tests of comparison of height above ground

Source	Error *df*	Error mean square	*df*	*F*	*p*
Between subjects					
Year-group (Y)	69	0.22	2	3.11	.05
Within subjects					
Display (D)	126.4[b]	0.27	1.83[b]	13.46	<.01
D x Y			3.66[b]	0.36	.23
Type of comparison (T)	245.7[b]	0.11	3.56[b]	7.29	<.01
T x Y			7.12[b]	0.48	.85
D x T	389.8[a]	0.12	3.65[a]	7.92	<.01
D x T x Y			11.30[a]	2.02	.03

Note.[a] = Greenhouse-Geisser df used; [b] =Huynh-Feldt df used.

As Table 4.07 shows, there were significant main effects of display and of type of comparison. Post hoc tests showed significantly better performance with the model of flags set on hillocks than with either the real or the pictured displays of people on steps ($p < .05$). There were significantly more correct responses to the question *Which is higher?* (a comparison of more) than to *Which is lowest?* (least) and also significantly more to *Which is higher?* and *Which is highest?* (most) than to *Which is at the same height?* (same).There were two significant interactions, also shown in Table 4.07. The three-way interaction involving display, type of comparison and year-group had a very small F ratio; moreover there had been no main effect of year-group. It was therefore considered that any interaction effect of year-group was likely to be marginal. Accordingly the two-way interaction between display and type of comparison was selected for investigation. This was done with the help of the means plot shown in Figure 4.02.

Figure 4.02 Mean correct responses for type of comparison according to display.

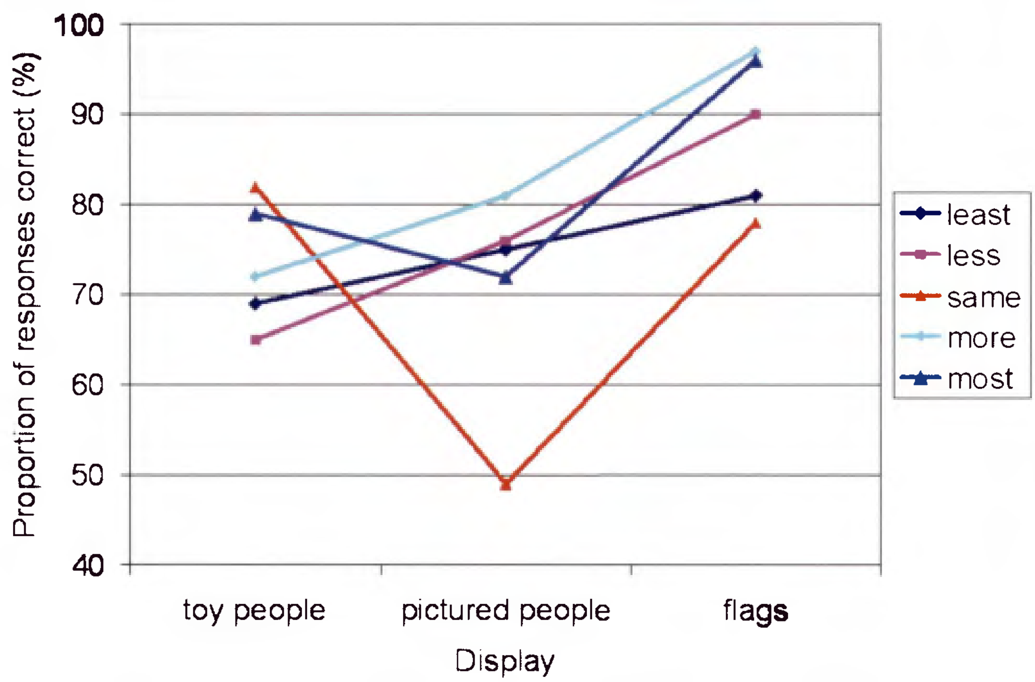


Figure 4.02 shows the likeliest source of the interaction to be a very depressed performance (compared with other comparisons) when children judged 'same height' in the display that pictured people on steps.

The three types of height display compared

Table 4.08 brings together data for all three types of height display children saw in the main and second series, i.e. height from head to feet (simple), height from head to feet (complex) and height above ground. It shows that children performed best (in fact at near maximum) in simple comparisons of height from head to feet. They performed next best comparing height above ground and they performed worst of all in complex comparisons of height from head to feet. All differences were significant at $p < .05$. Among complex height displays and among displays of height above ground there were no significant differences between year-groups.

Table 4.08 Mean (SD) percentage of correct judgements in all types of height display, according to year-group.

All types of height display	Year 1	Year 2	Year 3	All
From head to feet (simple)	92 (10)	97 (10)	99 (0)	96 (10)
From head to feet (complex)	53 (21)	56 (27)	69 (22)	60 (24)
Height above ground (all displays complex)	71 (21)	74 (29)	86 (21)	77(24)
N	24	21	27	72

The differences reported in Table 4.08 were further investigated.

Complex height comparisons involving human figures

During the first interview, most children had correctly aligned the ends of pencils before comparing their length, showing understanding that the same starting point must be used when comparing the length of two objects. When it came to comparing their own height with that of the experimenter, however, while most children said that the experimenter was taller, they then affirmed that they themselves would be the taller if they stood on a table, suggesting that the understanding shown did not generalise to the human figure, at least in this context.

The opportunity was now taken to look at the human figure in a fresh context, and see whether complex displays involving human figures presented greater difficulties than other complex displays of height. Two of the three displays of height above ground used human figures. The third (the display with flags) was separated from them and used as the contrast. Correct responses for a) the display with flags were compared with b) amalgamated correct responses for the other complex height displays that used human figures (Table 4.08). Mean correct responses for the display with flags was 88%, significantly greater than for displays with human figures at 68% (t = 6.96; df = 71; $p <$ .01). It should be borne in mind that children made in a) only five comparisons while in b) they made twenty comparisons.

Further investigation of performance with a) complex height displays and b) displays of height above ground

a) In relation to complex displays of height from head to feet, children performed substantially and significantly worse when responding to the two questions *Who is the shortest?* (34% correct) and *Who is the tallest?* (25% correct) (that is, in comparisons of least and most) than to the other three questions: *Who is shorter?* (82%), *Who is the same height?* (83%) and *Who is taller?* (76%) that is, in comparisons of less, same and more ($p < .01$).

It was when answering the questions *Who is the shortest?* and *Who is the tallest?* that children had to judge height from head to feet of figures raised on a block. In these displays, the real or pictured block was a prominent feature, making the figure placed on it the most salient figure in that display. As already stated, it was the shortest figure in the display that was placed on the block, and its head was now the highest of the three in the display by a considerable margin. By these arrangements the children's attention was deliberately drawn to a cue that conflicted with the height of the figure from head to feet. Thus when asked *Who is the shortest?* children who made a correct response might first attend to the figure in the display whose *head* was the lowest, before considering and correctly judging the shortest from head to feet, that is, the figure on the block. Conversely, when asked *Who is the tallest?*, they might attend to, but then disregard the joint height of the figure and the block. They would finally consider only the height of that figure from head to feet.

In contrast, the questions *Who is shorter? Who is the same height?* or *Who is taller?* were asked where a figure was made to lie down and children had to compare its height with that of two standing figures. Here too a conflicting cue was provided, in that one figure in a display was presented in an atypical posture for judging height, and where height might be confused with length. Nevertheless, here, there was a high success rate. Children were, on the whole, not misled about height from head to feet when presented with a figure that was lying down. Only where they were required to distinguish between the height of a figure from head to feet, and its level when raised on a block was there any substantial difficulty.

b) The three displays of height above ground presented the reverse requirement. That requirement was to *ignore* the distinction between the height of the figure (or flag) and the step (or hillock) on which it was raised, attending only to total height. In these three displays, as already reported, children did significantly better than in complex height displays.

To examine children's understanding of these aspects of height more closely, responses to specific height displays were selected for closer scrutiny. The display with flags was omitted, since here there was a high level of success; only the results from displays that employed human figures were used, as follows: a) the four displays of figures raised on blocks in the complex height condition (height from head to feet), where the height of the block had to be ignored; b) the four displays of figures standing on steps (height above ground), where the height of a figure had to be considered together with that of the step.

It was suggested in the introduction to this chapter that younger children in particular might be likely to judge height in a global manner, and not discriminate, for example, between the height of a figure from head to feet, and its height combined with that of the block on which it stood. If global judgements of this kind were being made by a child, one would expect errors when judging complex displays of height from head to feet, combined with correct responses when judging height above ground, where aggregating the height of a figure with that of the object on which it was raised was a condition of success. This combination is labelled 'Category 1'.

It was further suggested that when a child had learned to judge a figure's height from head to feet separately from that of the object on which it is raised, this knowledge might then be applied across the board. Thus, when asked to judge height above ground, such a child would ignore the height of the step on which a figure stood, and attend only to its height from head to feet. If this type of over-discrimination was occurring, one would expect correct judgements in complex displays of height from head to feet, combined with errors in judging height above ground. This combination is labelled 'Category 2'.

Children who had learned to make judgements that were appropriate to each context would judge correctly in both situations (Category 3).

The numbers of children giving correct and incorrect responses, and the pattern of correct and incorrect responses they gave, for height from head to feet (figures on blocks) and for displays of height above ground (figures on steps), were set out in a cross-tabulation matrix. For each child, the number of judgements made in conformity with one of the three categories described above was then counted. Each child saw a total of eight displays that were relevant to this question, and hence made a total of eight judgements, in relation to these two aspects of height, so that seven or more judgements in conformity with one category was above chance level ($p < .05$) for that category. Children who met this criterion were allocated to the appropriate category (Table 4.09).

As Table 4.09 shows, a majority of those meeting the criterion were in Category 1, making global judgements on every, or almost every occasion, whether appropriate or not. Only three over-discriminated (Category 2). Eleven made judgements appropriate to each context, of whom eight were children in Year 3.

Table 4.09 The number of children in each year-group whose judgements of height conformed to categories as stated.

Year	Category 1: errors in judging height from head to feet, combined with correct responses in judging height above ground	Category 2: correct judgements of height from head to feet, combined with errors in judging height above ground	Category 3: judgements appropriate to each context	Children who met 'greater than chance' criterion	N
1	10	1	1	12	24
2	11	1	2	14	21
3	11	1	8	20	27

4.9 Discussion

Main findings

The work reported in this chapter showed fundamental prerequisites of measurement to be well established among the children investigated. They were good at making comparisons of length in a wide range of contexts, successfully ignoring a considerable number of distracting cues, and they were good comprehenders of the everyday language used to make those comparisons. This finding supports Resnick's description of the scope and robust nature of children's protoquantitative reasoning, since all the required judgements were protoquantitative (Resnick & Singer, 1993). More broadly, it contributes a further context in which children's accomplishments in making relative judgements are demonstrated.

The children saw each experimental display in a simple condition, where length comparisons were expected to be easy, and in a complex condition, which presented them with various conflicting cues about length to negotiate. The language used by the experimenter to question the children was held constant across simple and complex displays. It was considered that if children did well with the simple displays, but appreciably worse with the complex displays, the language used could be ruled out as a cause of difficulty. Instead, difficulties associated with spatial aspects of the displays would be the likely cause. In contrast, a large number of errors in both simple and complex conditions would suggest that difficulties associated with the language of comparative length were the main cause. High levels of success in both conditions would indicate that neither language nor spatial arrangement was a problem for these children.

Children did attain a high level of success in both simple and complex conditions overall, giving evidence that neither the language used, nor spatial arrangement presented any great problem to them. The subtleties of the everyday language of length did not, on the whole, elude them, and there was very little failure to conserve length. In their everyday understanding of length there was, therefore, a sound basis for school work in estimation and measurement. Even the youngest children did well, with only an 11% overall mean difference (which was, however, significant) between Year 1 and Year 3 in the main series of tasks (Table 4.04). Mean correct responses were higher in each successive year, although Year 2 and Year 3 did not differ significantly.

Performance was slightly but significantly better overall with displays in 2D than in 3D format in the main series of displays; that is, children found it easier to compare height, length or width of lines or items in line drawings than to compare these attributes of real objects. This slight difference was probably due to the greater opportunity for misjudgement permitted by children's freedom to view 3D displays from any angle, while the viewing angle is fixed by the nature of line drawings. In the second series of displays (the three displays of height above ground) on the other hand, there was a much higher level of success with the complicated model of flags on hillocks, and a slightly higher level of success with 3D displays of people, than with the 2D display in that series (Table 4.06). The unexpected success with the model is considered below, while the lesser success with the 2D than the 3D display of people was probably an artefact of the line drawing used.

There was no evidence that comparisons of more were easier for children than comparisons of less due to bias in language usage, as suggested by Haylock and Cockburn (1989).

Length and width were compared significantly better than height from head to feet. Differences in performance according to type of comparison (least, less, same, more and most) and according to attribute (height, length and width), were largely due to poorer judgements of height in certain contexts, which were further investigated.

Poorer performance in relation to aspects of height

Children comparing the height of figures in simple displays were successful over ninety per cent of the time irrespective of year-group. Performance on complex versions of the displays was much worse. It will be recalled that these complex formats were designed to test children's ability to distinguish whether a figure was taller or shorter than another, or merely situated higher or lower than another, and that this investigation was suggested by the work of Piaget *et al.*, (1960), where children comparing the heights of towers appeared to ignore the fact that the bases of the towers were not at the same level. Such phenomena formed part of Piaget's case that children elaborate only slowly the conceptual properties of three-dimensional space. He expressed this by saying that at first children do not understand that the size of an object does not vary when its position changes. An alternative explanation, discussed in the introduction to this chapter, concerns the language of height, and suggests that children fail to discriminate the various meanings that can attach to questions about height in different contexts.

Where children were asked to compare the heights of figures from head to feet, level was manipulated in complex versions of displays by standing one figure on a block, or making it lie down. As already explained, to compare height correctly, a child had to ignore these features of presentation, and attend only to the height of the figure from head to feet. In the case of the raised figures, it was considered that younger children might tend to make global judgements of the total height of the figure and the block, due to lack of familiarity with the difference between the height of a figure in the sense of its tallness, and its height in the sense of its distance above ground. Displays featuring a figure that was lying down tested understanding of the same two senses of height.

The three displays of height above ground, it will be remembered, were designed to test the converse possibility: that children may indeed learn when intrinsic height of items is being referred to, irrespective of their level above ground, but then apply this knowledge indiscriminately, so that when asked about height above ground (*Whose head is higher?... Which flag is the lowest?*) they judge, for example, height from head to feet, or from the bottom of the flagpole to the top of the flag, and fail to take into account the height of the base on which the item is raised.

If children in this research understood both types of question - those about height from top to bottom, and those about height above ground – they would make both kinds of comparison correctly.

In fact by far the greatest difficulty for children was in dissociating the height of a figure from a block on which it stood. The figures that were lying down in complex height displays caused no special problems. The three tests of height above ground were significantly better performed overall than complex tests of height from head to feet. This was surprising, because there were more conflicting cues in the latter comparisons. Here, four figures were seen in the same display on steps of different heights, whereas in complex height displays there were only three figures to compare, of which only one was raised relative to the others. The model of flags on hillocks had even more potential to confuse. Here again there were four items to be viewed simultaneously, and again intrinsic height 'contradicted' the height to which the items were raised. But additionally, there was no obvious point from which to view this model, and children had to determine their own perspective on it.

It seemed that there was particular difficulty with displays involving raised human figures. Children's responses specifically to raised figures (figures on blocks in the main series of displays, and figures on steps in the three tests of height above ground) were therefore examined by a crosstabulation in which the number of children making each possible combination of correct responses to these two types of display was recorded (Table 4.09). Over one-third of the children in each year-group had specific difficulty in considering separate components of height when this was appropriate (in many cases having no success at all) while making successful comparisons when a judgement of aggregate height was called for.

So figures raised on blocks were the sole substantial problem. Taken together, the findings for all the height displays suggest a bias towards the global in children's judgements. Such a bias would have served them well in the complicated displays of figures on steps and flags on hillocks, where the various ways in which height above ground was partitioned had to be ignored, and would account for their greater success with these displays. The fact that the height of figures that had been made to lie down was well-judged also supports the view that it was the *partitioning* of height that proved the stumbling block: no partitioning was necessary with the recumbent figures, who presented the same type of conflicting cue to height as lines or cars placed at right angles did to length. In neither height nor length displays did this type of conflict present much difficulty.

Whether the failure to partition overall height was conceptual in character, or whether it was due to difficulties with the language involved is difficult to determine, but two considerations suggest the latter. Firstly, children who had erroneously identified a figure on a block as the tallest would occasionally add *because he's on the block.* They would nevertheless reiterate when asked, that the block really did make him taller. Others would occasionally signal uncertainty by saying *no-one is the tallest*, or *he's cheating*. These occurrences suggest an imperfect understanding of the different ways in which the language of height is used in certain circumstances, rather than a genuine failure to partition height between the block and the figure. Secondly, performance on height above ground was significantly better for the display using flags on hillocks than for the two displays using people on steps. A bias towards global height judgements should have produced much the same result in both types of display, or even (since the display with flags was more complicated) a better performance with the figures on steps. It seemed that height in relation to human figures may have involved ambiguities for the children not evoked by other objects whose height was judged, and that these ambiguities were to do with language usage. Children may know, for example, that people sometimes wear high-heeled shoes 'to make them taller'.

It may seem that questioning children about specific height judgements would have been the most direct way of clarifying these issues, and this was a tempting option. But to ask for reasons for all judgements would not have made sense (*Why do you think that car is the longest?* inviting the response *Because it is!*), while to ask only about displays that included raised figures risked signalling to children that these specific judgements had been unsatisfactory in some way, and might have caused them to change their judgements to satisfy the experimenter. The shortcomings of the approach adopted – that of making inferences about understanding from a pattern of responses - are indeed acknowledged, but it was considered that on the whole, the less intrusive method would provide more reliable data.

In the tests reported in this chapter, children in Years 2 and 3 performed significantly better overall than children in Year 1, but no significant differences were found between year-groups either for complex height displays or for the three displays of height above ground. That is, in the only aspects of the tests that proved difficult, there was no significant increase in accuracy in successive year-groups. This could have been due to characteristics of the specific sample studied, but for these children at least, neither increased time in school nor more general developmental factors made the contribution to greater understanding that might have been expected.

These findings suggest that, as far as height, length and width are concerned, it is height that has the greatest potential to confuse, and that it is height that should, of the three attributes, receive most attention in primary mathematics teaching.

The preceding chapter established important general characteristics of children's conceptions of measurement, through examination of the children's everyday language as they recalled their measurement experiences. Overall, the language resources at their disposal proved to be rich, although often deployed with freedom rather than accuracy. The current chapter continued the focus on everyday language, looking closely and specifically at the understanding of language and concepts associated with ordinal length comparisons. The children coped very well with most of the potential pitfalls here, so that one basis for successful numerical measurement – understanding of ordinal comparison - was shown to be in place. The research now moved to look at children's ability to *estimate* length. Here, some of the language used in the current chapter was again employed, (*shorter; longer; same*) but height and width were not considered; the focus was on the attribute of length. The estimation procedure had some numerical features, and measuring devices were introduced.

Chapter 5

Visual Estimation of Ordinal Length

5.1 Introduction

The ability to make comparisons of ordinal length must be secure if children are to go on to understand how lengths of items can be compared using standard units. The previous chapter investigated children's understanding of everyday language and concepts of ordinal length, because it was possible that such everyday language and the social contexts in which it was used might obstruct access to the conceptual content. Chapter 4 reported little evidence of difficulty. Overall, the children coped remarkably well in making ordinal comparisons in which they estimated the relative height, length and width of various figures and objects.

The question *How well do children make ordinal comparisons of length?* was asked again in relation to the work reported in the current chapter, but now the focus was on their *visual estimation ability* as opposed to their understanding of the language and concepts involved. The judgements were again relative. Here, children made more judgements, and only of length; and they were made in a narrower range of contexts. This further investigation of ordinal ability, as logical underpinning for measurement, was also important because of the prominence of estimation in classroom activities.

Estimating and measuring

Estimating and measuring are often found as a pair among classroom activities in British primary schools, with estimation either preceding or following measurement. One rationale given to children for pairing estimation with measurement is that the greater accuracy of measurement serves as a corrective to the inexactness of estimation. Another, where estimation is encouraged as a check on measurement, is that approximate agreement between the outcomes of the two procedures can indicate that measuring was appropriately executed.

Whatever the rationale for estimation, it may be assumed by educators that estimation is a more intuitive and thus a less demanding procedure than measurement. This may not be the case, especially when children are expected to retrieve mental representations of familiar objects, distances or units (Department of Education and Employment, 1999) to compare them with the length of an entity that they can see.

The type of estimation reported in the current chapter did not give children the task of retrieving such mental representations. This was because information about the children's ability to make the ordinal comparisons that underlie measurement could be provided just as well by a less demanding type of estimation, in which two entities to be compared are both physically present. This type of estimation was used here.

There was good reason to expect that children would be successful at estimation of this kind. Developmental literature indicated that young children may be adept at making approximate judgements where no direct numerical comparisons are required. Resnick's (1992) notion of protoquantities and Bryant's (1974) earlier argument that relative judgments of size (involving qualitative comparison) are successfully made earlier than absolute judgements, both suggest that we might expect children to be good qualitative estimators of length; and research with infants suggests that ability to make certain types of qualitative ratio judgement emerges very early indeed. (Feigenson *et al.,* 2004; Feigenson *et al.,* 2002). In Chapter 3 it was reported that children's experiences of measuring at home seem frequently to take the form of judging affordances - the degree of 'fit' of one object to another (Gibson, 1979); Chapter 4 reported their high level of success in making qualitative comparisons of height, length and width. Together, these sources suggested that qualitative estimation may develop early, and that one basis for measurement with units might thus be secure.

Some of the varied materials used in the estimation and measurement tasks in the present research were common to both sets of tasks. Employed in the *estimation* tasks, the variety among these materials provided a range of conditions under which children estimated, to test the robustness of their estimation ability.

In the *measurement* tasks, the varying of the materials had a more specific purpose. The various features instantiated potential difficulties embedded in measurement concepts. These potential difficulties concerned a) the ways in which number maps on to quantity, b) notions of separation and continuity and c) subdivision of units and were suggested by the developmental literature (Resnick, 1992; Miller, 1989; Piaget *et al.*, 1960). These features are described below. For convenience, their rationale in the measurement tasks is also described.

Outline and rationale

Twenty-one lines of varying lengths were displayed at a fixed distance from the child, who was asked to estimate whether each line was shorter, longer or the same length as five different types of item, referred to hereafter as *the comparators.* Thus the materials consisted on the one hand of a collection of lines, and on the other of a group of five comparators. The collection of lines was large and varied so as to provide a thorough test of estimation ability.

The five comparators were i) the whole length of one unmarked strip of card; ii) a number of 1 cm wooden cubes; iii) a number of units on a second strip of card that was marked and numbered in divisions of 1 cm (but not labelled 'cm'); and finally a number of cm indicated iv) on a ruler and v) on a tape measure. It was these five comparators that featured the range of conditions under which children estimated, and provided the grounds for potential conceptual difficulties when they measured. How the comparators incorporated these features is now explained.

The features of the comparators

In her characterisation of mathematical development, Resnick (1992) had identified a transition between *qualitative* judgements of quantity (protoquantities), and numerical quantification of amount. The present estimation study was designed with this transition in mind, to begin investigation (continued in the measurement study) of the question *How well do children understand that a number may express an amount?* This was done in the following way. While the comparisons that children were asked to make continued to be qualitative (being ordinal comparisons of shorter/longer/same), they were now asked to make most of these estimates in relation to a given number of cubes, a given number of unnamed units, or a given number of centimetres on a ruler or tape. Thus number was introduced for the first time in this research. Here, although comparison of *two numerical quantities* was not required, it was of interest to see whether the introduction of units *in itself* presented any difficulty. Children's comparisons with the unmarked strip of card (no units) provided the required contrast.

Miller (1989) and Piaget and colleagues (1960) had demonstrated a difficulty involving children's judgement of qualitative amount. They had shown that children did not necessarily conserve amount when a quantity of material was divided into several pieces. Additionally, there had been speculation that children might have conceptual difficulties with the notion that a continuous length could be split into separate pieces and yet continue to be constitutive of the same continuous length (Piaget *et al.*, 1960) or, conversely, (and more broadly) that a *single* aggregate consists simultaneously of the items that make it up (Lamon, 1996). The requirement in the present research to use comparators that *separated* units of length (the cubes) on the one hand and that *subdivided* continuous units of length (on the ruler and tape measure) on the other presented children with concrete examples of this paradox, and it was of interest to see whether there would be any differences in success levels when using these comparators.

So three of the comparators (the cubes, the ruler and the tape measure) were intended to suggest these areas of potential difficulty, while two (the unmarked strip of card, and the strip marked at 1 cm intervals with no subdivisions) were free of them. The unmarked strip of card made no use of units at all. (Children were simply asked to say whether a given line was longer, shorter, or about the same length as the entire strip). The strip of card marked and numbered at 1 cm intervals embodied continuous but un-named units that were not subdivided. The collection of 1cm wooden cubes constituted individual units that had to be conceptualised as a continuous length before a valid estimate could be made. Finally the ruler and tape measure, both marked with centimetres subdivided into millimetres, showed continuous, subdivided, named units. It was of interest to see whether, when the length of a line was estimated against comparators that did not differ among themselves in length, but did differ in the respects described, children would perform worse with some than with others.

Comparing estimation and measurement

When children had *estimated* the length of a set of lines in relation to the five comparators listed above, they were asked to use three of these devices (the cubes, the ruler and the tape measure) to *measure* nine of the lines. These three devices preserved the key differences regarding separation, continuity and subdivision discussed above; they were reduced to three (along with the reduction in the number of lines) because the measurement task was laborious and there was a concern to maintain children's interest and concentration.

The use of comparators as measuring devices enabled direct comparison of the children's estimation and measurement skills. It also provided the possibility of eliminating certain variables from examination of their measuring ability. Thus if the differing features of cubes, ruler and tape measure had no effect on children's success when used in estimating, then any differences according to these devices when they were used for measurement could be ascribed to their characteristics specifically as measuring devices.

The type of questioning used in the estimation study (whether a line was longer, shorter, or the same length) was retained for part of the measurement task and had two advantages. Firstly it facilitated comparison of the outcome of the tasks, since an important independent variable was retained. Secondly, it was intended to recall the type of language likely to be familiar from the home setting (where measurement was to do with affordances), and measurement in this setting had been shown (in the first interview) to engage children's interest.

5.2 Method

5.2.1 Sample

Seventy-six of the children who were involved in all other aspects of the research participated, twenty-five children in Year 1; twenty-three in Year 2 and twenty-eight in Year 3.

5.2.2 Materials

Lines

The twenty-one lines whose lengths were estimated constituted three sets of seven: one set of lines that were overall rather 'short' (coloured red), one set of lines of 'medium' length (coloured purple), and one set of 'long' lines (coloured black). The lines in each set varied in length around a 'reference' line; all other lines in that set were either shorter or longer than this reference line (Table 5.02).

The reference line in each set was equal in length to the comparators for that set. So the reference line in the 'short' set was 5 cm long, and children were asked to estimate the length of the lines in that set by comparison with a) an unmarked strip of card 5 cm long, b) 5 x 1cm cubes; c) 5 units on the marked strip; and d) and e) 5 cm on the ruler and tape measure. (Thus the correct estimate of ordinal length when the reference line was compared with one of these comparators would be *the same*. Correct estimates for the other lines in the set would be either *shorter* or *longer*). The reference line and comparators for the 'medium' and 'long' sets of lines were respectively 12 cm and 19 cm long.

In each set, the six lines either longer or shorter than the reference line varied proportionately to it and were shorter than the reference line by approximately 6% (line 3), 30% (line 2) and 60% (line 1); and longer than it by approximately 4% (line 5), 40% (line 6) and 60% (line 7). The actual length of the lines is set out in Table 5.02. They were of 0.5 cm width and each was presented on a separate sheet of A4 white card. As a factor in the design, these sets of lines are referred to as *set* hereafter. Line length is referred to as *length*.

Comparators

1. Three strips of blue card 2 cm x 30 cm long, on each of which the central portion (with which children compared the length of lines) was black. On each strip the black portion was of equal length to the reference line in one of the three sets. Thus the lengths of the black portions were 5cm (for comparison with the set of red lines), 12 cm (for the purple set) and 19 cm (for the black set).

2. A collection of 1 cm^3 wooden cubes

3. A strip of white card 4cm wide x 30 cm long, marked and numbered at 1 cm intervals (but not marked 'cm')

4. The conventional 30 cm transparent plastic cm and mm ruler commonly used in the school where the research took place, used in the second interview, and described briefly in Chapter 3. The full dimensions of the ruler were 3.6 x 31 cm. Divisions of 1 cm, ½ cm and 1 mm were marked on one edge (marks extended respectively 6 mm, 4 mm and 3 mm from the edge). Only cm were numbered. On the other edge, the pattern of markings was identical. Ten-millimetre divisions extended 6 mm from the edge of the ruler and were numbered 10 to 300; 5 mm (un-numbered) divisions extended 4 mm from the edge, and 1 mm (un-numbered) divisions 3 mm from the edge. The 0 cm mark occurred 7 mm from the end of the ruler at which the centimetre numbering began; the 300 mm mark appeared at the same end on the opposite edge. The 0 mm and 30 cm marks occurred 5 cm from the end of the ruler where the millimetre numbering began. The marks and numbering together extended to a depth of 1cm across the face of the ruler on each edge. The letters 'cm' were printed below the first centimetre space, and 'mm' below the first millimetre space on their respective sides of the ruler.

5. A fibreglass centimetre-and-inch dressmaker's tape measure, also used in the second interview, and described briefly in Chapter 3. Cm were displayed on one face, and inches on the other. Only the centimetre face of the tape was used for estimation and measurement. The full dimensions of the tape were 152.4 cm x 1 cm. The last marked cm units were 150 cm. One-centimetre divisions extended the full width of the tape, except for a central space printed with the relevant number of cm. Half-centimetre divisions (extending 5mm from one edge) and one-millimetre divisions (extending 3mm from the same edge) were marked but not numbered. The numbers were correctly positioned for reading when the tape was perpendicular to the reader. The zero position was at the extremity of the tape. A semi-circular stud sealed the tape fabric at both ends of the tape. The words and letters DEAN, FIBREGLASS TAPE, MADE IN

ENGLAND and CM were printed near the start of the tape. A summary of key features of the comparators is set out in Table 5.01.

Table 5.01 Comparators in relation to which the lengths of lines were estimated

Comparator	Characteristics
A plain, unmarked strip	No units – undivided length
A number of 1cm wooden cubes	Separated and un-numbered units
A cardboard strip marked and numbered at 1 cm intervals, but not labelled 'cm'	Continuous units that were numbered, but not named. The units were not subdivided.
A ruler marked with cm and mm	Continuous, subdivided, numbered and named units
A tape measure marked with cm and mm	Continuous, subdivided, numbered and named units

5.2.3 Procedure

Sessions were approximately 30 minutes long.

The experimenter verified the children's understanding of the terms *definitely longer, definitely shorter,* and *about the same length* by presenting books of obviously different lengths, or of similar lengths, in appropriate pairs before starting the tasks. No child gave evidence of difficulty in understanding these terms when questioned about them in this context. The experimenter then said *Today we are going to estimate, that is, guess, the length of some lines. We are going to use different things to estimate their length. Here is the first thing* [e.g. plain unmarked strip]. *I am going to hold up some lines, one at a time, and I want you to tell me whether the line is definitely longer, definitely shorter,* or *about the same length as this strip.*

The comparator was placed on the table in front of the child, where the child was free to handle it. The experimenter then successively held up each of the seven lines in one set, at the child's eye level and at a distance of approximately 80 cm from the child, asking each time *Is this line definitely longer, definitely shorter, or about the same?* [as the comparator given].

For example, for the red lines (reference line 5 cm long), a child was offered the plain unmarked strip, 5 cm long, for comparison, one at a time, with each of the lines in the set. The process of comparison was then repeated with the same set of lines but a fresh comparator, the child being asked, successively, to select 5 cubes, find 5 units on the marked strip, 5 cm on the ruler and 5 cm on the tape measure, and to compare the length of each of the lines with that of the comparator. Each time the child was asked to say whether a line was *definitely longer, definitely shorter,* or *about the same length* as the given comparator (unmarked strip; a given number of cubes), or as a number of units identified on the comparator (the 30 cm marked strip, the ruler or the tape measure). Thus the experimenter said: *Is the line definitely longer, definitely shorter,* or *about the same length as 5 cubes? ...as 5 units on your strip? ...as 5 cm on the ruler? ...as 5 cm on the tape measure?*

All seven lines in a set (see Table 5.02) were presented for estimation of their length in relation to one comparator, before moving to the next comparator.

The three sets of lines: red, purple and black, were presented one set at a time. The order of presentation of the sets, and of the comparators, was systematically varied between children, and the lines within a set were presented in random order for each child.

In the few cases where children, for the purposes of the comparison, were unable to count out a prescribed number of cubes accurately, or identify and name one- or two-digit numerals on other comparators, the experimenter performed this task on the child's behalf. No special effort in relation to number was therefore required by the task. Most children made 105 estimates.

5.2.4 Criteria used for assessing children's judgements

As well as correct responses that used phrases modelled by the experimenter's questions (such as *It's definitely longer*), other responses that expressed acceptable estimates were also scored correct. Criteria were carefully considered so as to accommodate the range of expressions a child might use to express an acceptable judgement. Thus a child's estimate of the length of line 1 or line 2 was considered correct if they said that these lines were *definitely shorter, shorter,* or *a bit shorter* than the comparator. Their estimate of line 3, which was much closer in length to (though still shorter than) the reference line, was considered correct if they said this line was *about the same* or *the same* as the comparator and also if they said it was *definitely shorter*, *shorter* or *a bit shorter*. Their estimate of the length of the reference line, line 4, was considered correct only if they said it was *about the same* or *the same* as the comparator. Line 5 was scored correct if a child said either that this line was *definitely longer*, *longer* or *a bit longer* than the comparator, or if they said it was *about the same* or *the same*. Estimates of lines 6 and 7 were considered correct if a child said it was *definitely longer, longer,* or *a bit longer* than the comparator. Two per cent of responses used other formulations. These were easily assimilated to one of the foregoing categories. These criteria are summarised in the right-hand column of Table 5.02.

Table 5.02. Lines whose lengths were estimated

	Length (cm)			
	Short (red)	**Medium (purple)**	**Long (black)**	**Acceptable judgments against comparator**
Line 1	2	5	7.6	definitely shorter shorter a bit shorter
Line 2	3	7	11.4	
Line 3	4.75	11.5	18.5	definitely shorter shorter a bit shorter same about the same
Line 4	5 (reference)	12 (reference)	19 (reference)	same about the same
Line 5	5.3	12.7	20	definitely longer longer a bit longer same about the same
Line 6	6.5	15.6	24.5	definitely longer longer a bit longer
Line 7	8	19.2	29	

Regular checks were made to ensure that the child was judging the length of the line, and not that of the comparator; for example that a judgment of *it's definitely shorter* referred to the shortness of the line relative to the comparator, and not to the shortness of the comparator relative to the line. The check was made by saying, for example, *Are you saying the **line** is shorter, or **the cubes**?*

5.3 Results

Missing data

Three children, for whom there was a large amount of missing data due to extended absence from school, were removed from the analysis. The few remaining missing observations occurred because of premature termination of a session due to school demands and amounted to 38 missing observations out of 7980.

In a repeated measures design, analysis would result in the loss of an unacceptably large amount of information in view of the small proportion of scores that the missing values represented. Missing data were therefore substituted as follows. It was noted that for 22 of the missing values, 70% or more of values present for that item in that year-group were the same. For a further 6 missing values, between 50% and 69% of values were the same, and for the remaining 10 missing values, between 49% and 38% were the same. In view of this, missing values were assigned the modal response for that particular variable for that year-group.

Analysis of accuracy

Table 5.03 sets out means according to year-group, comparator, set and length. As before, on account of the girl/boy imbalance in Year 2, a preliminary analysis of variance included gender as a factor. There was no significant effect of gender, or significant interaction of gender with any other factor, including year-group. Gender was therefore excluded from further analysis.

Table 5.03 Mean (SD) percentage of correct estimates according to year-group, comparator, line set, and line length

	All	Year 1	Year 2	Year 3
Comparator				
Plain strip	81 (9)	77(9)	82 (8)	85 (7)
Cubes	81 (11)	76 (11)	81 (13)	85 (9)
Marked strip	83 (10)	79 (10)	84 (13)	85 (7)
Ruler	81 (12)	78 (11)	80 (15)	86 (10)
Tape measure	81 (11)	77(12)	80 (11)	86 (10)
Set				
Red	79 (10)	74 (9)	80 (11)	83 (8)
Purple	82 (10)	79 (10)	81 (10)	86 (8)
Black	80 (9)	77 (10)	80 (9)	84 (9)
Length				
Line 1	98 (4)	98 (3)	98 (5)	99 (2)
Line 2	90 (10)	89 (12)	90 (10)	91 (9)
Line 3	79 (17)	76 (21)	80 (14)	80 (15)
Line 4	49 (24)	41 (21)	51 (29)	55 (20)
Line 5	83 (19)	74 (21)	84 (21)	90 (12)
Line 6	79 (19)	74 (16)	76 (23)	85 (17)
Line 7	93 (10)	88 (12)	91 (11)	98 (5)
N	76	25	23	28

Table 5.03 shows, for all comparators and all sets, high mean correct estimates by children in all year-groups. There was no year-group in which children obtained less than 74% of responses correct according to comparator or set, and standard deviations indicate a rather narrow range of scores, except for line 4.

To determine whether accuracy of estimates varied according to these factors, a four-way mixed analysis of variance was conducted with year-group as between-subjects factor and comparator, set and length as within-subjects factors. The only factors for which there were significant main effects were year-group F (2, 73) = 8.30, p = .001) and length F (2.99, 218.17) = 86.13, p < .001).

Post hoc tests showed significantly better estimation by Year 3 children than by those in Year 1 (p < .01).

Post hoc tests also showed that there were significantly more correct estimates of line 1 (the shortest) than of all other lines; and of line 7 (the longest) than of all other lines except line 2. There were no significant differences between lines 3, 5 and 6, or between line 5 and line 2. Line 4 (equal in length to the comparator) was judged significantly worse than all other lines.

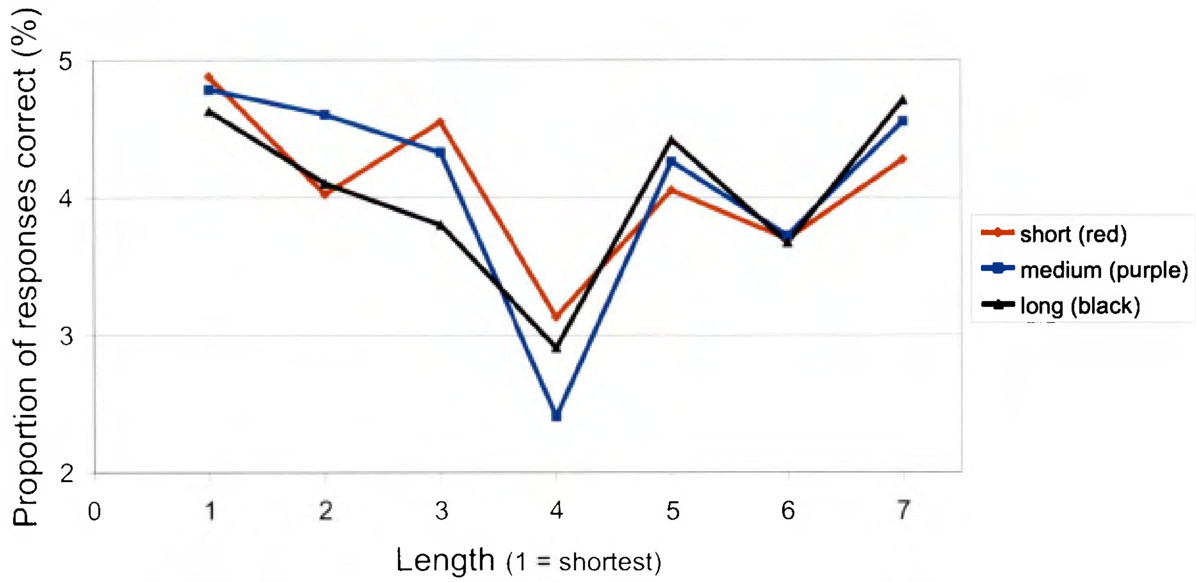
Although set was not significant overall, it was involved in an interaction with length F (7.50, 547.7) = 10.28, p < .01) as well as in a three-way interaction with length and year-group F (15.01, 547.7) = 1.83, p = .03). As there was a very small F ratio for the latter interaction, and no interaction of year-group with any other factor, it was considered likely that year-group made only a marginal contribution to these interactions and so the involvement of year-group was not pursued.

Although type of comparator was not significant overall, there was a significant interaction between comparator and length F (13.60, 992.5) = 3.04, p < .01).

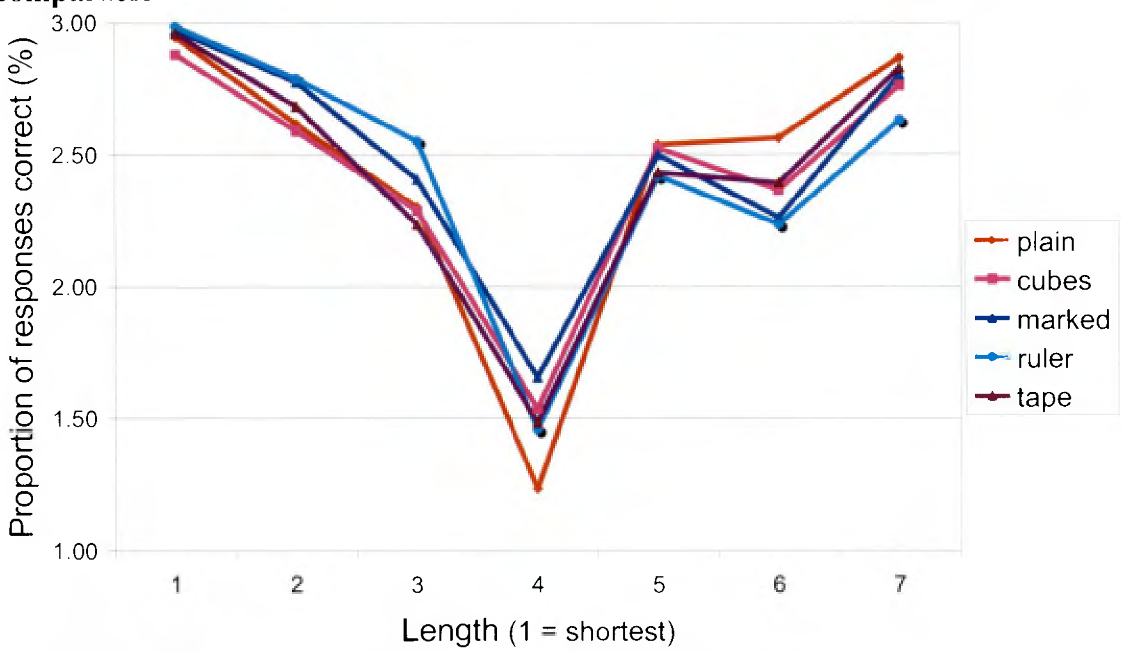
Means plots were obtained to investigate the set-length and comparator-length interactions (Figure 5.01).

Figure 5.01

(a) Mean proportion (%) of estimates* correct, according to line length and set.



(b) Mean proportion (%) of estimates* correct, according to line length and comparator



Note: *out of five estimates

Figure 5.01(a) shows that while scores for the purple and black sets were progressively lower for each of lines 1, 2 and 3, scores for the red set fell for line 2 and then rose sharply for line 3. Further, scores for the red set ranked highest for lines 1, 3 and 4, but ranked lowest for line 5. This is the most likely explanation for the interaction between length and set. In the scores for each comparator, shown in 5.01(b), there was a tendency, from line 1 to line 7, for scores that were initially higher to become progressively lower in the rank order, best illustrated by the line indicating the score for ruler. These two interactions made only a minor contribution to the overall picture regarding children's estimates and were therefore not further considered.

Dealing with the latitude allowed in criteria for correctness

Lines 3, 4 and 5 in each set were close to each other in length, and it was expected that of the alternatives offered the most likely estimate for all of them would be *about the same* (i.e. about the same length as the comparator). However, line 3 was actually slightly shorter, and line 5 slightly longer than their comparator (Table 5.02), so that estimates of *shorter* and *longer* respectively were also counted as correct responses. Because it was considered important to assess children's ability to judge equality of length, responses accepted for line 4 were restricted to *same* and *about the same*. While reasonable in essence, this strategy resulted in broader success criteria for lines 3 and 5 than for line 4, and could have contributed to the lower level of success in judging line 4.

The analysis of variance was therefore re-run a) with lines 3 and 5 excluded, and b) separately for lines 3 and 5. As in the full analysis, reported above, there was no main effect of comparator, and none of set, when lines 3 and 5 were excluded. In the separate analysis of lines 3 and 5 there was a main effect for set (F (2, 139.9) = 6.36 $p < .01$); judgements were significantly better for the red (short) set. In both a) and b), as before, Year 3 children did significantly better than those in Year 1 (for a), F (2, 73) = 7.34, $p < .01$); for b) F (2, 73) = 5.50, $p < .01$).

The comparative lack of success with line 4 was further investigated by looking at the nature of the errors made by a fifty per cent sample of each year-group. Among the errors of children in Year 1 the proportions of judgements of *shorter* and *longer* were equal. Among children in Years 2 and 3, however, there were more than twice as many judgements of *longer* than of *shorter*.

5.4 Discussion

This type of estimation task had two key features. The first was that comparator and line compared were both physically present, and there was no need to make use of mental representations of length. This contrasted, for example, with Petitto's (1990) numberline studies, where children had to retrieve representations of sequence or proportions before they could make a judgement. The second was that children were not asked to quantify their comparisons, and were simply asked to say whether they thought lines were definitely longer, definitely shorter, or about the same length as their comparators. Clearly in these circumstances, these children were good estimators, even at five years old. Success increased with age, but all year-groups attained a high proportion of correct responses. There was no evidence of any differences in performance according to gender.

In the introduction to this chapter it was recalled that in the home context (on the evidence of the first interview) children typically encountered measurement of length as the estimation or calculation of affordances: of the fit of an object to the space available. Children were legitimate peripheral participators (Lave & Wenger, 1991) in these activities or interested observers of them (in the double sense that the activities were interesting, and that the children themselves often had a stake in the outcome) and it was argued that these factors might contribute to an early aptitude for visual estimation of relative length. The main finding of the current chapter is congruent with this proposal, as well as with indications, also mentioned in the introduction, in the developmental literature (Feigenson *et al.*, 2004; 2002, Resnick, 1992; Bryant, 1974).

There was slight evidence of variation in success with different comparators, but no overall difference according to comparator. In general terms this suggested that ability to judge relative length, in this case in the face of a number of distractors presented by physical differences between comparators, was very robust at the ages studied. More particularly, the features of comparators that had been selected to suggest difficulties that might be encountered when children contemplated *units* (separation, continuity and subdivision) proved no stumbling block, that is, the mere appearance of these features, in a non-measurement context, did not in itself confuse children. Nor, in itself, did the introduction of number: performance with the cubes, ruler and tape measure, where specific numbers of cubes or of cm were used in comparisons, was not significantly worse than it was with the unmarked strip. These findings partially address the theme at the end of Chapter 1 that is labelled 'mapping number on to quantity', where it was noted that some authors saw a natural transition from qualitative comparisons of length to comparisons involving units (Lamon, 1996; Resnick and Singer, 1993; Petitto, 1990) facilitated by the development of counting, while Miller (1989) expected the counting procedure, by its very dominance, to lead children astray when applied to continuous quantities. Although measurement itself was not involved, the results reported in this chapter, as far as they went, were consistent with the former view.

There was overall no significant difference in performance according to set, that is, according to the absolute length of the lines judged. This accords with the early findings of Bryant (1974) regarding the comparative ease for both children and adults of making relative as opposed to absolute judgements of size, and is a particularly striking example of it. The length of the three 'reference' lines for the red, purple and black sets differed from set to set by seven cm, and within sets, the longest and shortest lines differed from the reference line by as little as 3 cm (red set) and as much as 10 cm or more (black set). While children saw, therefore, a very wide range of lines as regards absolute length, good judgement of their relative lengths remained constant.

Performance did differ significantly according to the ratio of the length of lines to the length of their comparator. There were significantly more correct estimates of line 1 (the shortest in each set) than of all other lines; and of line 7 (the longest in each set) than of all other lines except line 2. There were no significant differences among line 3, line 5 (in every set the two closest in length to the comparator) and line 6, and none between line 5 and line 2. Line 4 (equal in length to the comparator) was judged significantly worse than all other lines.

The overall picture is that there was a gradual improvement as the difference in length between line and comparator increased. The lines that were furthest in length from the comparator (lines 1 and 7 in every set) were the easiest to estimate correctly (with the exception that there was no significant difference between line 2 and line 7) undoubtedly due to the much greater discriminability of lines 1 and 7 from the comparator (Ross, 1997). The degree of success for lines that were second most distant from the comparator (lines 2 and & 6) 'overlapped' to some extent with that for lines that were closest to it (lines 3 and 5), but overall, the same picture held. The overlap is consistent with the 'accumulator' model of the mental representation of numbers as continuous magnitudes (see 1.3.1), in which the representation is dependent on discrimination of 'signal' from 'noise'. Siegler & Opfer (2003) point out that as magnitudes approach each other in size, there will be overlap, and discrimination will be harder. This model seems to fit the continuous magnitudes presented by the lines used in the present research rather well, and if discriminative capacity, as evidence seems to show, develops in early infancy, it would help to account for the high degree of success in estimation shown by the children in the present study.

The line (line 4) that was exactly the same length as the comparator proved hardest for children to judge correctly, with less than 50% success overall. This was not surprising, since in this case there was no difference to discriminate. In fact it represented the same kind of absolute judgement required of children by Lawrenson & Bryant (1972) and found by them to be poorly performed. The type of error made when judging line 4 in the present research was investigated, and while children in Year 1 were found to judge this line to be shorter than its comparator just as often as they judged it to be longer, children in Years 2 and 3 made twice as many erroneous judgements of *longer* than of *shorter*. Perhaps, in the absence of a discriminable difference, a general positive bias in favour of larger quantities was at work (Haylock and Cockburn, 1989) although there was no sign of this among the findings reported in the previous chapter, and no particular reason why this should be evident only among older children.

Given the high degree of overall success irrespective of comparator or of absolute length of lines, together with the differences found according to length of line, we can summarise children's performance on this estimation task as both robust and nuanced.

Differences among comparators made no overall difference to children's success in estimating. We may therefore be confident that any differences according to comparator when these were used as measuring devices could be ascribed to their characteristics specifically as measuring devices. It is to the measurement tasks that we now turn.

Chapter 6

Measurement

6.1 Introduction

The analysis of children's visual estimation performance using all seven lines in each set (twenty-one lines in all) is considered to have given a thorough, substantial picture of children's estimation ability from 5 to 8 years old in the population and in the tasks studied. It was found that that the children were good estimators, and that their success did not differ significantly according to the type of comparator against which length of lines was estimated, or, overall, according to the absolute length of those lines (that is, according to whether the lines belonged to the 'short', 'medium' or 'long' set). It did differ according to length of line relative to the comparator within sets. Since differences among comparators had made no overall difference to children's success in estimating, any differences according to comparators when these were used as measuring devices could be ascribed to their characteristics specifically as measuring devices. *Do* differences among measuring tools affect ability to measure? Is there any evidence from children's handling of different devices that they consider units that are physically separate and units that form a continuous scale to be incommensurable? Do children falter when confronted with subdivided units? The materials children used in the measuring tasks were designed to investigate these questions.

Careful thought was given to the type of measurement task employed. The research already reported showed that children had much more to say about their experience of measurement outside school than within school, and their recall of the former was much more vivid and personal. The measurement tasks were designed to make sense to children in the light of this out-of-school experience, the character of which had been broadly apparent in piloting. Thus in measuring up for floor or wall covering, a person may measure with a tape (in conventional units) the length, height or width to be covered, and then measure in the same way the available covering material. If this work is done collaboratively, one person may measure the space to be covered and announce the number of units that represent the height, length or width, while another measures the available material and announces the degree of fit (too short, too long, or just right).

The three-part task given to the children was designed to suggest some of these features. Thus the experimenter asked the children a) to measure a line with the device supplied. Next they were asked b) whether that line was longer, shorter, or the same length as a given number of units. Here the children had to compare their measurement with the number of units mentioned. This task, therefore, investigated how well children understood that a number (of units) may express an amount. The number used was that of the 'reference' length used in the estimation tasks – five, twelve or nineteen units – associated respectively with the red, purple and black set of lines. (The procedure is more fully described below). Finally c), the children were asked, as an absolute judgement, how long the line actually was.

So children were asked to make both a numerical measurement and a comparison. Conceptually, when a child had measured a line, and the experimenter asked, *Is that shorter, longer, or the same length as* (say) *12 cm?* the question tested understanding that ordinal judgement (*shorter, longer* or *same*) was logically implied by a statement of length measured in units. The comparison thus examined one aspect of the application of number to quantity embedded in measurement practices.

Measurements were deliberately limited in number because it was clear that a more extensive series would exhaust some children, as measuring demands more time and persistence than estimation. Of the twenty-one lines whose length was estimated, nine were measured and of the five comparators used in estimation, three were selected as measuring devices. Thus nine lines were measured with each of three devices, making twenty-seven measurements in all (Table 6.01).

The lines selected for measurement were lines 3, 4 and 5 (those that were closest to each other in length) in each of the three sets. The numerical lengths of five of these nine lines included fractional units (Table 6.01) and required children to deal with the subdivision of units in an immediate and practical way. Of the five comparators used in estimation, the cubes, the ruler and the tape measure were selected as measuring devices. Between them, these three devices embodied all the areas of potential difficulty that were of special interest in this research. The cubes constituted separated, unnumbered units, while the units shown on the ruler and tape measure were numbered, continuous and sub-divided. These specific devices were chosen, too, because each was known (on the evidence of what children said at interview and of information from their teachers) to be familiar to the children as an everyday item either at home or at school, and (on the same evidence) most children had handled and manipulated cubes and rulers and some had used tape measures. However, familiarity does not necessarily lead to expertise. Each device needed to be handled and interpreted differently both because they were physically different, and because they represented units in different ways. It was anticipated, therefore, that when children came to use comparators as measuring devices, there would indeed be significant differences according to the device used.

It was expected that competent handling of the devices would become harder as line length increased, so that there would be an effect of set, and 'long' lines (black set) would be harder to measure than 'short' lines (red set).

It was also expected that there would be a general improvement with increasing age. Children in Years 2 and 3 were expected to be more successful overall than those in Year 1. The latter were comparatively inexperienced in measuring with the ruler and tape measure, and although their teacher indicated that they had used a number of different non-conventional units for measurement including cubes, these young children were expected to have difficulty in manipulating 1 cm cubes, which were rather small. Better success was expected in Years 2 and 3 due to these older children's greater experience in measuring in school, and their likely greater experience in measuring out of school.

Table 6.01 The lines within each set used for the measurement tasks.

	Length (cm)		
	Short (red set)	**Medium (purple set)**	**Long (black set)**
Line 3	4.75	11.5	18.5
Line 4	5 (reference)	12 (reference)	19 (reference)
Line 5	5.3	12.7	20

6.2 Method

6.2.1 Sample

Seventy-six of the children who were involved in all other aspects of the research participated, twenty-five children in Year 1; twenty-three in Year 2 and twenty-eight in Year 3.

6.2.2 Materials

The collection of wooden 1cm cubes, the 30cm ruler and the tape measure that were used in the estimation task were used as measuring devices.

On the ruler, the edge to which the children were directed was marked in cm, half cm and mm, with cm marks numbered. (Millimetres were numbered in tens on the other edge.) The scale began 0.5 cm from the end of the ruler. Numbers on the ruler were correctly oriented for reading when the ruler was horizontal with respect to the reader.

The face of the tape measure to which the children were directed was marked, as the ruler was, with cm and mm, with cm numbered. Here, the numbers were correctly oriented for reading when the tape was vertical with respect to the reader. The scale began at the end of the tape. (The reverse face of the tape was marked in inches and eighths of inches). Further details of the ruler and tape were given in Chapter 5.

The 'reference' line in each set (line 4), together with the line next shortest (line 3), and the line next longest to it (line 5) (see Table 6.01) were measured. Each line was presented to the child horizontally, as before, on a sheet of A4 card.

Written records were kept of children's measurements and of the ways in which they deployed measuring devices.

6.2.3 Procedure

The same seating arrangements in the school library were observed as for all other tasks. Measurement followed on from the estimation tasks and were, as previously stated, about 40 minutes in length.

The children were asked to measure all three lines in all three sets with all three devices. Thus each child made nine measurements with each device. The children received the sets in a random order. Within each set, lines 3, 4 and 5 were randomly presented, randomisation being achieved for both set and line by shuffling the cards used. The order of the measuring devices was also randomised, this time by blind selection of the word 'cubes', 'ruler', or 'tape' printed on a sheet of paper.

After the child had completed the estimation task, the experimenter handed to the child one of the cards with a line on it and said *Can you measure this line for me now, please?* and offered the ruler, the tape measure or a collection of cubes. After the child had measured a line with the device, the experimenter asked ***Is** the line longer, shorter, or the same length?* [as the appropriate 'reference' length in units].

For example, the experimenter gave the child a collection of cubes that was greater than twelve and asked ***Is** the line longer, shorter, or the same length as twelve cubes?* (Twelve cubes were the 'reference' length for the purple set; five and nineteen cubes, respectively, for the red and black sets.) If the child miscounted, the experimenter supplied the right number of cubes, but did nothing further to help: the children arranged the cubes for measurement as they saw fit. If the ruler or tape was being used the experimenter asked ***Is** the line longer than 12 cm, shorter than 12 cm, or 12 cm long?* If the child had difficulty in identifying 12 cm on the ruler or tape, the experimenter located the mark for them, but again did nothing further. The number of cm mentioned was changed to five or nineteen as appropriate for the red and black sets.

The child's response to the question was recorded in writing. The experimenter then asked *How long **is*** it [the line]*?* The response was similarly recorded. Ways in which measuring devices were deployed were also noted in writing.

Children were asked to report comparisons and numerical measurements. Their report was recorded for each measurement.

6.2.4 Analysis of results

In all analyses, correct responses were totalled across the twenty-seven measurements, and subdivisions of the total were used for finer-grained analysis.

Reporting of the analysis follows the order in which the measurement procedure was carried out by the children, that is, 1. physically measuring a line; 2. saying whether the line was shorter, longer, or the same length as the number of units mentioned by the researcher; 3. stating the absolute length of the line. Accordingly the three parts of the analysis concern:

1. The correctness of positioning of the measuring devices

2. The correctness of comparisons

3. The correctness of numerical measurement

Positioning of the measuring devices

Children were scored on their alignment of the measuring device with the appropriate end of the line to be measured. Correct alignment required that the zero position on the scale (ruler, tape), or the placing of the outer edge of the first cube, coincided with the end of the line. Also recorded were additional aspects of the deployment of devices that were considered relevant to the errors in measurement that children made.

Comparisons

A child's response that a line was shorter, longer or the same length as the number of units mentioned by the experimenter was termed their **comparison** because they compared their own numerical measurement with a number of units given orally by the experimenter (a relative judgement).

Numerical measurement

After they had measured a line and had made their comparison, children were asked how long the line actually was. Their response to this question was termed their **numerical measurement** (an absolute judgement). A numerical measurement was scored correct if it was accurate to within 2.5 mm of the actual length of the line. This was considered a reasonable degree of tolerance given the age of the children, and had the advantage of being able to accommodate phrases incorporating the term *quarter* (2.5 mm being equal to ¼ cm), since *quarter* (as well as *half*) are commonly used by many children to express fractional units. Success with this task required a number of abilities including the reading of the scale and the taking into account of fractional units.

An example

A child measuring the 4.75 cm line from the 1 cm mark on the ruler or tape, and reporting the line length as 5.75 cm (or *five and three-quarters cm*), would be scored incorrect for numerical measurement and incorrect for positioning the device with respect to the starting point for measurement. They would, however, be scored correct on their comparison if they said that the line they had measured as 5.75 cm was longer than the five units mentioned by the experimenter. Thus correctness of the comparison was based on the measurement they made, rather than on the actual length of the line.

Factor structure

As explained above, of the seven lines in each set used in the estimation tasks, only the three 'central' lines in each set were used for the measurement tasks. While the twenty-one lines used in the estimation tasks had overlapped in length between sets (for example black lines 1 and 2 were shorter than purple lines 3 to 7, (see Table 5.02) there were no overlaps among the nine lines retained, and the sets to which they belonged now became 'islands', the differences between sets being much more marked. Moreover the three lines retained in each set were very close in length (within a set, no difference was greater than 1 cm). Thus, while the remaining lines within each set differed enough to provide different measurement tasks, the major differences in line length became those between the three sets. In these circumstances, it was decided to discard 'line' as a factor in the analysis of variance, so that two within-subjects factors remained: device and set.

Additional data collected: acknowledgement of fractional units

One of the hypotheses of this research was that children would have difficulties with the notion that units of length can be subdivided. Each standard line and just one other (that of 20 cm) were a whole number of units (cm) in length (Table 6.01). All other lines embodied fractional units and were used to investigate the children's manner of reporting, or failing to report, incomplete units. Where children acknowledged fractional units, a record was kept of the ways in which they did so. (see Appendix 3).

6.3 Results

6.3.1 Missing data

The same three children who had been removed from the Estimation analysis due to a large amount of missing data were removed from the Measurement analysis. Missing Measurement data after this amounted a total 54 out of 2052 observations, again due to premature termination of a session due to school demands. To avoid further loss of data in a repeated measures design, these missing values were replaced. A random allocation procedure was used in a manner that preserved the proportion of correct and incorrect judgements for the variable, within the year-group for which they were missing. In the majority of cases, this was a single substitution within a variable within one year-group. In total there was only one occasion where as many as four substitutions were made. Thus the substitutions are unlikely to have introduced significant differences into the results.

6.3.2 Understanding of the term 'measure'

Four children asked what was meant, or appeared not to know what to do, when asked to measure a line. These were all in Year 1. The children were given appropriate guidance.

6.3.3 Preliminary analyses

As before, on account of the girl/boy imbalance in Year 2, gender was initially included as a variable in all three analyses of variance (positioning the device; the comparisons;

and numerical measurement) but, as before, produced no main effect or interaction with any other factor including year-group, and so was excluded from all further analyses.

6.3.4 Positioning the device

How children handled the devices in making their measurements went a long way towards explaining their degree of success in measuring, as well as providing information about their understanding of the way in which units were embodied and represented in the different devices. The most important decision children had to make concerned the positioning of the measuring device in relation to the line to be measured, and the most important aspect of that was whether or not (cubes), the first cube was aligned with the end of the line to be measured, and (ruler and tape) whether or not the zero mark was so aligned. There were other important aspects of positioning. In the case of measurement with cubes, it was necessary to remove gaps in the line of cubes, to ensure that the line of cubes was straight, and to make sure that if the cubes shifted as they were laid, compensating adjustments were made to maintain correct alignment. In the case of both cubes and tape, it was necessary that children had a clear view of the line to be measured, and did not obscure their own view, for example by covering the line with the measuring device so that the full length of the line was not visible to them when the measurement was made. (The ruler was transparent, so in this case it did not matter if the line was covered).

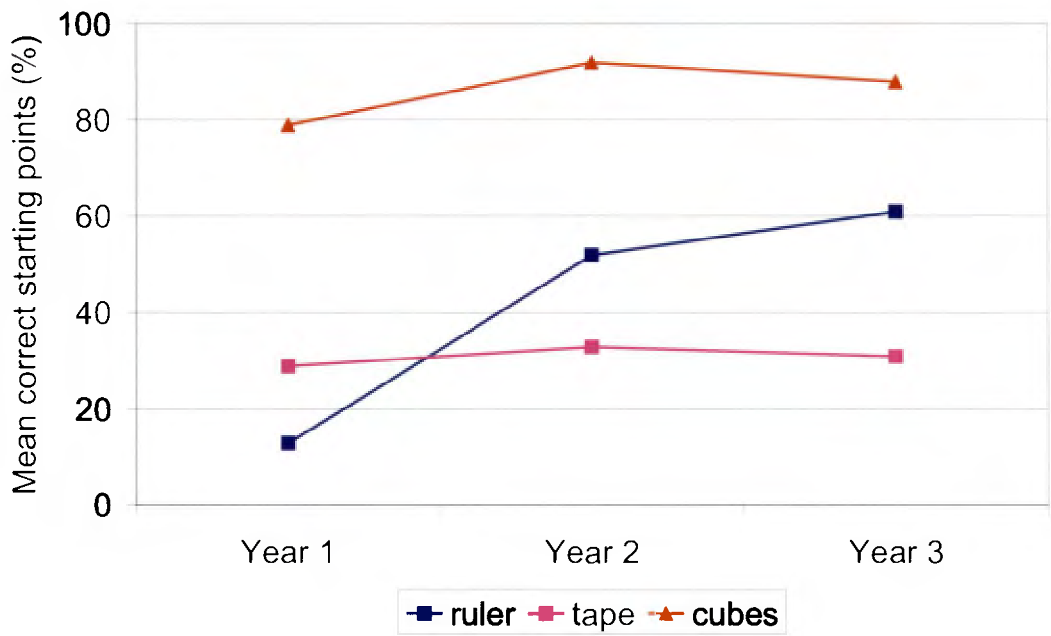
What the children did in respect of the **starting point** of their measurement is reported first. Whether gaps were closed and correct maintenance of a straight line of cubes was attempted is reported next. Finally, whether the full length of the line measured was visible to the child when they declared their measurement is reported.

Table 6.02 and Figure 6.01 show the mean proportion of measurements (out of 27) where children used the correct starting point on the measuring device. Overall the children made a poor selection of starting point, except when measuring with cubes.

Table 6.02 Mean (SD) proportion (%) of measurements where children used the correct starting point on the measuring device.

	Year 1	Year 2	Year 3	All years
Measuring device				
Ruler	13 (21)	47 (37)	61 (36)	40 (38)
Tape measure	29 (32)	31 (32)	31 (32)	30 (31)
Cubes	79 (25)	90 (10)	88 (16)	86 (19)
All correct starting points	40 (17)	56 (19)	60 (19)	52 (20)
N	25	23	28	76

Figure 6.01. Mean proportion (%) of measurements where children used the correct starting point on the measuring device.



Note. Error bars to show SD are not provided because their overlapping heights would have confused the figure.

In all year-groups the correct starting point was much more likely to be selected for cubes (in approximately 80% to 90% of measurements) than when using the tape measure (approximately 30% of measurements) or ruler (13% to 61% of measurements). Table 6.02 and Figure 6.01 indicate little difference between year-groups for cubes and tape, but more use of correct starting points for the ruler in each successive year-group. Large standard deviations, especially for ruler and tape measure, indicate that individual children differed widely in their degree of correct use of starting points.

A two-way analysis of variance had year-group and measuring device as factors. Significant main effects were found for year-group (F (2, 73) = 9.28, p < .001) and device (F (1.9, 138.92) = 96.13, p < .001) and there was a significant interaction between them (F 4, 138.92) = 6.38, p < .001).

Post hoc tests showed that overall, children in Year 2 and Year 3 selected correct starting points significantly more often than children in Year 1 (p < .01) with no significant difference between children in Years 2 and 3. The tests also confirmed that, overall, there were significantly more correct starting points using the cubes than the ruler or tape measure (p <. 01), with no significant difference between ruler and tape.

The significant interaction between device and year-group was accounted for (Figure 6.01) by the very poor selection of correct starting point on the ruler by Year 1 and the comparatively much better performance by Year 3. Differences between year-groups for the other two devices were not so marked.

Separate analyses of variance for each year-group (Year 1 (F (2, 48) = 49.78, p <.001); Year 2 (F (2, 44) = 29.37, p <.001); Year 3 (F (2, 54) = 31.87, p <.001)) confirmed that when measuring with the cubes Year 1 and Year 2 children used significantly more correct starting points than with the ruler and tape (p <.01), with no significant difference between ruler and tape. Year 3 children were most likely to select the correct starting point when using the cubes, and least likely to do so when using the tape. For Year 3 there were significant differences among all three devices (p < .01). Separate analyses of variance for each device (ruler (F (2, 73) = 6.25, p <.001); tape (F (2, 73) = .09, *NS*); cubes (F (2, 73) = 3.33, p = .04) and post hoc tests showed that when using the ruler, Year 2 and Year 3 children selected the correct starting point significantly more often than those in Year 1 (p < .01). When using the cubes, Year 2 selected the correct starting point significantly more often than Year 1 (p = .04) with no other significant differences. There were no significant differences among year-groups for starting points with the tape.

6.3.4.1 Consistency of use of correct starting points

Even when children used the correct starting point, very few did so consistently except when using the cubes. Half the children in Years 2 and 3 always aligned the first cube correctly, and almost all did so in more than fifty per cent of the measurements. Even in Year 1, more than one third of the children always aligned the first cube correctly.

Performance with the ruler was much worse. More than half the children in Year 1 never aligned the ruler correctly, and only five did so in more than one tenth of their measurements. In Year 2, there was a much more even spread among children of degrees of consistency in correct alignment of the ruler. Three children never used the correct starting point, four always did so, and the remainder of the children were more or less evenly distributed between these two extremes. In Year 3 there were five children who never used the correct starting point with the ruler, but overall children used a higher proportion of correct starting points than in Year 2. Year on year, children used the ruler increasingly correctly, but even in Year 3, only five children always did so.

In selecting the correct starting point for the tape measure, there was little difference between the year-groups. Only three children in the whole sample always used the correct starting point. Half the children in Year 3, almost half in Year 2 and over one-third in Year 1 never used it. The remainder were, as with the ruler, fairly evenly spread between the two extremes.

6.3.4.2 Predominant starting points

The children made nine measurements with the ruler and nine with the tape measure. The main points from which children started their measure were (in addition to 0 cm, the correct starting point) -0.5cm, 0.5 cm and 1cm, but many used a variety of non-typical starting points. In the case of children in Years 2 and 3, these were often poor alignments where children were two or three millimetres off one of the typical starting points. For children in Year 1 the use of a non-typical starting point usually reflected a completely haphazard approach, with no detectable strategy, and many non-typical starting points employed. In Years 2 and 3, for a given measurement, children typically took up several starting points before settling on one, and in those year-groups both initial and later starting points typically conformed to one of the predominant categories. Thus a Year 2 or 3 child might start from 1cm, move to 0 cm, and move back to 1 cm before finally making their measurement. In all cases the last starting point taken up was the one recorded.

Ruler

On the ruler, those who measured from -0.5 cm were in fact measuring from the very end of the ruler: the scale began half a centimetre from the end of the device. Those measuring from 0.5 cm began from the first of the prominent half-cm marks to be seen along its edge. A variety of non-typical starting points were also used.

Consistency of starting point for all nine measurements with the ruler was very rare. More than half the children in Year 1 measured from the end of the ruler (-0.5 cm) some of the time. In roughly equal proportions, about 40% sometimes measured from the correct starting point (0 cm), from the 0.5 cm mark and from the 1 cm mark. Ninety-two per cent of children in Year 1 used other starting points some or all of the time.

More than one third of the children in Year 2 sometimes measured from the end of the ruler, and a similar proportion also sometimes measured from the first half-cm mark. Over half sometimes measured from the 1 cm mark. A majority (78%) sometimes measured from the correct starting point; 48% did so most of the time. Sixty-five per cent of children in Year 2 occasionally measured from other starting points.

In Year 3, only 14% of children ever measured from the end of the ruler (-0.5 cm), and 21% from the first half-cm mark. Sixty-four per cent sometimes measured from the correct starting point, and half did so most of the time. Thirty-nine per cent occasionally measured from other starting points.

Four children (one in each of Years 1 and 2 and two in Year 3) identified on the ruler the number mentioned by the experimenter (5, 12 or 19) and used that as their starting point.

Overall the number of children who sometimes used the correct starting point on the ruler increased in successive year-groups.

Tape measure

Consistency of starting points using the tape measure was just as rare as it was with the ruler. Around half the children in Years 1 and 3 – more (70%) in Year 2 - sometimes used the 0.5 cm point on the tape to begin their measurement, but most of these did so only occasionally. Measuring from -0.5 cm was infrequent, since the scale started from the end of the tape; the three children who used this starting point placed the end of the tape 0.5cm from the left-hand end of the line being measured.

When using the tape, fewer children than was the case with the ruler ever used the correct starting point. Only one child in each year-group consistently did so, and there was little difference among year-groups as to the number of children who sometimes did so. In this, too, the tape differed from the ruler, where an increase from year to year was discernible. With the tape, more than half the children in each year-group sometimes measured from the 1 cm mark, and surprisingly, over one third of children in Year 3 frequently or consistently did so.

Fifteen children, almost evenly divided across year-groups, sometimes measured from the number (5, 12 or 19) mentioned by the experimenter. This was three times as many as had done so when using the ruler.

Most children in Year 1, more than half in Year 2, and just over one-third in Year 3 measured from a variety of non-typical starting points with the tape measure. In the case of many children in Year 1, this was visibly the result of unsuccessful attempts to bring this long, loosely hanging, unwieldy device into sensible contact with the line to be measured.

6.3.4.3 Measuring with cubes: other adjustments

For each measurement with cubes, Table 6.03 shows the number of children who, as well as aligning the first cube correctly, removed any gaps in the line of cubes, ensured that the line of cubes was straight, and sought to make compensating adjustments to maintain correct alignment if the cubes shifted as they were laid. It can be seen that although more than half the children in each year-group observed all these precautions for the majority of their nine measurements with cubes, no more than five in any year-group did so with complete consistency. In this, the older children were no better than the younger overall.

Table 6.03 Number of children in each year-group making all necessary adjustments when measuring with cubes

	Measurements				
	In all nine	In 5 to 8	In 1 to 4	In none	N
Year 1	5	13	7	0	25
Year 2	3	17	3	0	23
Year 3	5	18	5	0	28

A two-way analysis of variance with year-group and cubes as factors showed (F (2, 146) = 10.03, $p < .001$) that when children measured lines in the red (shortest) set with the cubes, they attempted all necessary adjustments significantly more often than when they measured purple or black lines (i.e. those in the longer sets) ($p < .001$). There was no main effect of year-group, but there was a significant interaction involving year-group. Separate analyses for the three school years showed that children in Year 1 were significantly more careful when they measured the red lines (they attempted to make all necessary adjustments in 76% of measurements) than when they measured the purple ones (all necessary adjustments attempted in 53% of measurements): (F (1.71, 141.1) = 4.87, p = .02). The same was true of Year 2. All necessary adjustments were attempted in 89% of measurements of the red lines and 65% of the purple: (F (2, 44) = 4.15, p = .02). Surprisingly, neither red nor purple differed significantly from black – the longest lines – where children persisted in their efforts to make all the necessary adjustments in 71% of measurements (Year 1) and in 69% (Year 2). In Year 3, red lines were again the most carefully measured, but here the significant differences (F (2, 54) = 8.75, p = .001) were between red (84% with all adjustments made) and black (51%) (p = .002); and between purple (77%) and black (p = .01). Thus the longest lines were least carefully measured in Year 3. However, large standard deviations in all year-groups for black lines indicated that different children persisted to different degrees.

6.3.4.4 Obscuring the line

A number of children partially or completely covered the line they were measuring with the measuring device. In the case of the ruler, which was transparent, the line was still visible, so that its full length could be seen. The cubes or tape were both wider than the line they measured and children frequently completely obscured a line with them, so that they could no longer see it. Sometimes this was avoided. Table 6.04 shows, for the children in each year-group, the incidence of obscuring the line with the measuring device.

In the case of cubes, these were often stacked *on* the line, rather than parallel to and in contact with it. This did not indeed make accurate measurement impossible (it was virtually impossible with the tape in these circumstances) but it made it difficult. The child could not see whether the first cube laid had shifted from its alignment with the left-hand end of the line, nor could they see by how much a cube overlapped at the other end of the line without, in both cases, lifting and replacing cubes. This made adjustment of the line of cubes more complicated and less successful than it need have been, and judgements about equality or non-equality of length, more complicated and less reliable. Table 6.04 shows that the majority of children deployed the cubes in such a way as to create these difficulties, and that a similar number of children did so in each year-group.

Table 6.04 Number of children who obscured the line with the measuring device: incidence in each year-group

	Children obscuring the line to be measured **Cubes** Never	 Sometimes	 Always	N
Year 1	6	9	10	25
Year 2	2	10	11	23
Year 3	5	13	10	28
	Never	**Tape** Sometimes	Always	**N**
Year 1	4	7	14	25
Year 2	20	2	1	23
Year 3	25	2	1	28

As for the tape, despite having carefully identified their chosen starting point on this device, and aligned it often carefully with the left-hand end of the line to be measured, children sometimes covered the line with the tape and had to peer under it to determine the position of the other end of the line in relation to the scale. Sometimes they made a guess. This behaviour was rare in Years 2 and 3, but common in Year 1, as Table 6.04 shows.

6.3.5 The comparisons

These were the child's report, after measuring a line, as to whether the line was longer, shorter, or the same length as the 'reference' number of units mentioned by the experimenter. This report was called the child's 'comparison' because they compared their own numerical measurement with a number of units given orally by the experimenter. Correctness of comparisons was thus judged on the basis of the measurement made, rather than on the actual length of the line measured.

6.3.5.1 Analysis of accuracy of comparisons

The means of correct comparisons according to year-group, measuring device and set were calculated and are shown in Table 6.05. Overall performance was poor, success being in the region of 50% although children in Years 2 and 3 did rather better than this with the ruler.

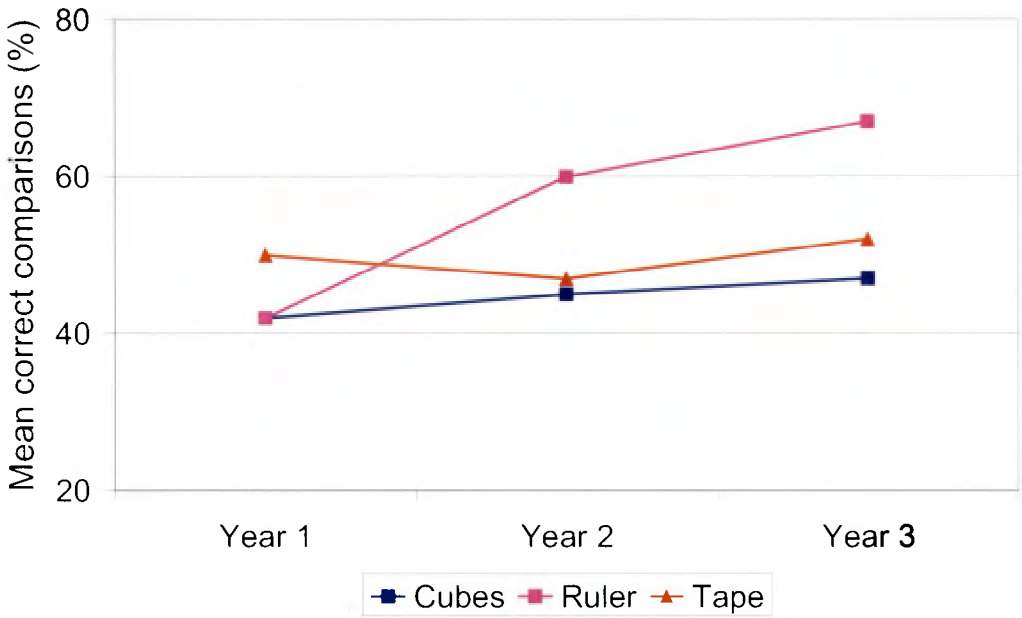
Table 6.05 Mean (SD) proportion (%) of correct comparisons made by children according to year-group, device and set.

	Year 1	Year 2	Year 3	All years
Measuring device				
Ruler	42 (23)	60 (28)	67 (30)	56 (29)
Tape measure	50 (25)	47 (24)	52 (28)	50 (26)
Cubes	42 (24)	45 (21)	47 (23)	45 (23)
Set				
red	44 (20)	47 (26)	56 (28)	49 (26)
purple	45 (28)	50 (24)	53 (27)	50 (27)
black	45 (24)	55 (26)	57 (29)	52 (27)
All comparisons	44 (24)	51 (25)	55 (28)	50 (26)
N	25	23	28	76

Note: Each child made 27 comparisons.

A three-way analysis of variance with year-group, device and set as factors found significant main effects of year-group ($F (2,73) = 6.70$; $p < .001$) and device ($F (2,146) = 8.06$; $p < .001$) and a significant interaction between device and year-group ($F (4,146) = 3.83$, $p = .005$). Post hoc tests showed children in Year 1 to be significantly less successful in their comparisons than those in Years 2 and 3 ($p = .03$). They also showed the comparisons to be significantly worse with the cubes than with the ruler ($p = .001$). The interaction, as Figure 6.02 indicates, was explained by the fact that while Years 2 and 3 performed similarly, that is, much better with the ruler than with the cubes or

Figure 6.02 Correct comparisons: interaction between device and year-group



tape, Year 1 children performed no better with the ruler than with the cubes, and rather less well than with the tape. This picture was confirmed by analyses of variance performed separately for the three year-groups, which showed no significant difference among the devices for children in Year 1, while children in both Year 2 (F (2, 44 = 5.50; p = .007) and Year 3 (F (2, 54 = 7.64; p = .001) made significantly more correct comparisons when they used the ruler than when they used the tape or the cubes ($p <$ .05).

6.3.6 Numerical measurement

For each line, the numerical measurement each child actually reported was entered as data in decimal notation. If the child's utterance was not wholly numerical but could sensibly be interpreted in this way, such an interpretation was made. For example if, having measured a line with the ruler, a child said *five and two little lines,* '5.2' was entered. Where numerical interpretation was not straightforward - where, for example the child said *five and a bit* – no numerical measurement was entered, and their utterance was counted as an error (although a separate record of such non-numerical acknowledgements of fractional units was kept). Cubes are non-subdivided units, and the number of expressions available to children here to express fractional units numerically was in practice limited to *and a half* or *and a quarter* or *and three quarters* (recorded as .5, .25 and .75). If children declined to give a numerical measurement, either replying, when asked how long the line was, that they did not know, or simply reiterating that it was longer, shorter, or the same length as the number of units supplied by the experimenter, a zero was entered and the response counted as an error.

Numerical measurements were counted as correct if a child reported a measurement within 2.5 mm of the actual length of the line. Children were scored correct or incorrect irrespective of their mention of units.

6.3.6.1 Analysis of correctness

Each child made twenty-seven numerical measurements. The data giving their performance according to year-group, device and set are shown in Table 6.06.

On average, under one third of numerical measurements were correct by the 2.5 mm criterion, ranging from 21% in Year 1 to 36% in Year 3 (Table 6.06). A three-way analysis of variance with one between-subjects factor (year-group) and two within-subjects factors (measuring device and set) was performed. There were significant main effects for all three factors: year-group (F (2,73) = 7.88, p = .001); device (F

(1.87,136.3) = 13.15, $p < .001$) and set (F (2,146) = 11.10, $p < .001$). There was also a significant interaction between device and set (F (4,4) = 2.47, $p = .045$). Post hoc tests showed that Year 1 children made significantly fewer correct measurements than those

Table 6.06 Mean (SD) proportion (%) of correct numerical measurements made by children according to year-group, device and set.

	Year 1	Year 2	Year 3	All
Measuring device				
Ruler	24 (29)	43 (29)	52 (29)	40 (29)
Tape measure	23 (20)	28 (20)	29 (20)	27 (20)
Cubes	16 (16)	29 (15)	25 (15)	24 (16)
Set				
red	26 (22)	36 (22)	41 (21)	34 (22)
purple	17 (17)	27(17)	26 (16)	23 (17)
black	20 (19)	37 (19)	40 (19)	32 (18)
All numerical measurements	21 (14)	33 (14)	36 (14)	30 (18)
N	25	23	28	76

Note: Each child made 27 measurements.

in Year 2 ($p = .02$) and Year 3 ($p < .001$); and that significantly more correct measurements were made with the ruler than with the tape or the cubes ($p < .01$ in both cases), with no significant difference between tape and cubes. (This partly reflected the findings for the comparisons, where overall performance was worse with the cubes than the ruler.) Post hoc tests showed that the medium-length (purple) lines were significantly worse measured than either the short (red) lines ($p < .001$) or the long (black) ones ($p = .001$). Since the significant interaction between device and set had a very small F ratio, it was considered that any interaction effect was likely to be marginal, and was not further investigated.

6.3.6.2 Failure to report a numerical measurement

Among the numerical measurements recorded as incorrect were an overall average of 23% of responses where children did not give a numerical measurement when requested, either repeating their previous response that the line was shorter, longer, or the same length as the number of units mentioned by the experimenter, or saying that they did not know how long it was. This proportion was smaller in each successive year-group and was 35% in Year 1; 20% in Year 2 and 16% in Year 3. There were significant differences (F (2,73) = 10.87, $p < .001$) between Year 1 and Year 2, and between Year 1 and Year 3 ($p < .01$ in both cases).

6.3.6.3 Response to lines that embodied fractional units

A prediction of the research was that children would have difficulty in dealing with fractional units. Each child measured fifteen lines that were not a whole number of units long, and their reports of these measurements were expected to give evidence of such difficulty.

When children ignored fractional units, they either reported whole numbers, or no numerical measurement at all, simply reiterating that a line was shorter, longer, or the same length as the number of units mentioned by the experimenter.

When children did acknowledge fractional units, they did so in a number of ways that included mention of halves and quarters; mention of a number of units additional to cm, sometimes implicitly, such as *five and three* (5cm and 3mm); numerical allusions to subdivisions on the ruler and tape such as *...and two little lines* or *five points before twelve*; and expressions that could not be interpreted numerically, such as *nearly six* or *twelve and a bit*. Sometimes an expression mixed these categories, such as *four and a half and two millimetres* (see Appendix 3). Table 6.07 shows acknowledgment of fractional units however acknowledged. It can be seen that that no child acknowledged fractional units on more than twelve occasions out of a possible fifteen, while approximately half the children in Years 2 and 3 did so less than fifty per cent of the time. Only one child in Year 1 acknowledged fractional units on more than fifty per cent of possible occasions, and more than half never acknowledged them.

Table 6.07 Number of children in each year-group acknowledging fractional units.

Year	13 to 15	8 to 12	1 to 7	None	N
1	0	1	10	14	25
2	0	10	12	1	23
3	0	14	13	1	28
Total	0	25	35	16	76

Note: acknowledgements out of a possible 15

Evidence of adjustment of starting point

If children dislike dealing with fractional units, it is likely that they will take measures to avoid having to do so. In the present research, where children using the ruler and tape were often observed to make several changes of starting point before settling on one, and where their final decisions represented a wide variety of starting points, it seemed a reasonable hypothesis that a starting point might be selected with a view to avoiding fractional units. To test this, an examination was made of the measurement of certain lines. These were lines that involved fractional units, but would enable a whole number of units to be reported if, when measuring, an adjustment of 0.5 cm were made from the correct starting point. The result was compared with the outcome of measuring those lines that also involved fractional units, but where a whole number would *not* be the outcome if the half-centimetre adjustment were made. The half-centimetre adjustment was chosen because as a fraction it was familiar to the children, and thus likely to attract them, and because half-cm were easily visible on the ruler and tape.

Inspection of Table 6.01 shows that (a) if line 3 in the purple set and line 5 in the black set were measured from -0.5 cm or from 0.5 cm on the ruler or tape, a whole number could be reported (11 or 12 cm in the first case and 18 or 19 cm in the second). By contrast, (b), if line 3 or line 5 in the red set, or line 5 in the purple set were measured from -0.5 cm or from 0.5 cm, this would *not* enable a child to report a whole number. (The total number of measurements in (a) and (b) above is therefore ten).

The number of starting points from -0.5 cm or from 0.5 cm for (a) and for (b) were calculated. In Year 3, on average, 22% of children adjusted their starting point by half a centimetre when this would allow them to report a whole number (a), compared with 13% when it would not (b). In Year 2 the averages were 26% and 13% respectively, and in Year 1, 22% and 23%. The difference (calculated on counts of those children giving the two responses for the ten measurements) was significant for Year 2 ($t = 2.48$, $df = 8$, $p = .04$). These data provide some indication that in Years 2 and 3, some choice of incorrect starting points may have been motivated by the desire to avoid fractional units. This is supported by the very similar proportions for (a) and (b) in Year 1, where children's very poor measurement performance would not suggest any strategy at all in their measuring behaviour.

Other evidence

A second source of evidence was offered by those instances where children, measuring with the ruler or the tape, declined to report a numerical measurement, and either said they did not know how long the line was, or simply reiterated that the line was longer, shorter, or the same length as the number of units mentioned by the experimenter. By all three year-groups, significantly more non-numerical responses were made for lines that incorporated fractional units than for lines that were a whole number of units in length.

In Year 3, an average of 14% of responses on lines incorporating fractional units gave no numerical measurement, compared with 7% on lines that were a whole number of units in length (t = 2.20, df = 11, $p < .01$); in Year 2 the corresponding averages were 21% and 13% ; (t = 2.13, df = 15, $p = .03$); and in Year 1 they were 34% and 22% (t = 2.14, df = 14, $p = .02$). These figures, including a particularly high proportion in Year 1, suggest that some children may have 'played safe' by declining to give a numerical measurement when confronted with fractional units that they did not know how to report. The fact that this tendency lessened in successive year-groups supports this interpretation, since it is likely that older children had more resources they could call upon to help them report fractional units.

6.4 Discussion

6.4.1 Main finding

Overall, the children's measuring ability was found to be poor, with little improvement from year-group to year-group except when measuring with the ruler.

6.4.2 Understanding of the term 'measurement'

It is worthy of comment that when handed a collection of cubes, a ruler or a tape measure, and asked to measure a line, only four children in the entire sample ever showed themselves at a loss as to what to do. With these exceptions (all in Year 1) every child aligned cubes, ruler or tape in the same plane as the line to be measured, and whichever starting point was used on the device, usually began their measurement at the left-hand end of the line. (The exception was measurement with cubes, where cubes might be stacked at various positions on the line before being moved to the left-hand end.) Although some children struggled with the measurement device, none asked for help. So children knew what measuring looked like, and were confident in undertaking it. Their measuring ability, however, was poor.

6.4.2 Declining to report a numerical measurement

Although very few children declared any difficulty when they were asked to measure, there was a substantial number of measurements where there was no numerical response. This was true of just over one-third of measurements in Year 1, one-fifth in Year 2, and 16% in Year 3. It has been argued that a proportion of these non-numerical responses may have been due to a desire to avoid dealing with fractional units. This desire, however, if present, may have been just one aspect of a general lack of confidence that developed only as a child became fully involved in the measuring operation, and their lack of hesitation when asked to measure may itself have been one index of their lack of understanding of what was entailed in measurement.

6.4.3 Success in measuring lines of differing lengths

It had been expected that the shortest lines (red set) would be better measured than the longest (black set) because correct handling of the measuring devices would be more demanding when measuring longer lines. This did not occur. Instead, on average, there was no difference in the degree of correct measurement of the longest and shortest sets of lines, and both were better measured by all year-groups than lines of medium length (purple set) (although the difference was smaller for Year 1). It is possible that red lines were better measured simply because they were shorter, while black lines, especially for Years 2 and 3, drew out the children's better efforts because they were visibly more challenging. This difference should be set in the context of poor measurement performance overall, however, where accuracy was nearly always below fifty per cent.

6.4.4 Measuring with cubes

In the great majority of measurements using cubes, with little difference between year-groups, children correctly aligned the first cube with the end of the line to be measured. However, there was poor consistency in ensuring that the length of the line to be measured was matched by the joint length of the cubes used to measure it. That is, the conservation of length essential to measurement was poorly observed. Though more than half the children in each year-group did frequently remove gaps in the line of cubes, ensure that the line was straight, and restore proper alignment if the cubes moved, no more than five children in each class invariably did so. In this, the older children were no better than the younger overall (Table 6.03). Children in Year 2 and Year 3 were, however, less careful in placing their cubes when there were eighteen or more cubes to be placed (lines in the black set), than when only five were needed, whereas for children in Year 1 there was no difference. This suggested that older children did understand what was needed, but became bored, or their concentration lapsed, during the lengthier task, whereas children in Year 1 either found it hard to manipulate even five cubes with the care required, or did not see the need for care.

However, even when they had only five cubes to attend to, children in Years 2 and 3 failed to take all necessary care with about a quarter of their measurements. It is not possible to say, however, whether measurement with cubes was simply not perceived by older children as a task worthy of their best efforts, or whether they really did not understand the principle of conservation of length as it applied in this measurement task.

The practice of covering the line to be measured with the measuring device, such that the length of the line could not be viewed, is puzzling. Cubes were often stacked *on* the line, rather than parallel to and in contact with it. Table 6.04 shows that the great majority of children always or sometimes obscured the line in this way, and that the older children did this just as often as the younger. This practice bore no discernible relation to how well the cubes were handled in other respects; for example children who had laid cubes with great care might then have to lift the outermost ones to determine the whereabouts of the ends of the lines. It was as if the idea that two lengths could be *equivalent* was not fully accepted, and that instead, the children were trying to *unify* the length to be measured with the instrument used to measure it. This suggests that the idea that measurement is about equivalence between a number of units and the length of an item being measured was not really understood. A general argument could of course be made that we expect inappropriate behaviour where an activity seems without purpose, and that the children could not see the purpose of measuring lines. However, children had, during their interviews, given plenty of reasons for measurement in everyday life at home that seemed good to them. They knew that, when they were asked to measure the lines, the experimenter wanted to know how well they measured, and their confidence in going about the task did not suggest that they found it meaningless or puzzling.

The correct starting point was selected much more often for the cubes than for the ruler or tape measure. However, in view of the inefficiency of other strategies for deployment of the cubes discussed above, it may be that to call the placing of the first cube exactly at the end of the line 'correct alignment for measurement' may be to over-interpret a simple act of matching.

Correct numerical measurements with cubes were on average less than 25% across year-groups and only at 25% for children in Year 3 (Table 6.06). Although subdivisions were not available on the cubes to help children meet the criterion of accuracy to within 2.5mm, older children used the terms *a quarter, half,* and *three-quarters* on many occasions during the research, and use of one of these terms would have satisfied the 2.5 mm criterion for any of the lines measured (Table 6.01), so that lack of subdivisions on the cubes was not an insuperable obstacle to accurate measurement. Overall, poor numerical measurement with cubes seems to have been due to failure to marshal consistently the various skills involved, but there is evidence, too, of lack of understanding of underlying principles.

Children were scored for the correctness of their comparisons, irrespective of the correctness of the numerical measurements on which they were based. To be successful, children had to make a mental comparison of two numerical quantities and convert the result into a comparative judgment. This was not, on the face of it, a difficult task, and yet on average only 45% of children overall made the judgement correctly, and only 47% of children in Year 3 did so (Table 6.05). Absolute judgements are known to be more difficult for children than relative judgements (Bryant, 1974), and here children were asked to compare two sets of units expressed as absolute numbers to produce a relative judgement (*shorter* or *longer*), so perhaps the task was rather more difficult than it appeared. However, it is also possible that some children had not thought of the numbers they were considering as units of length; that is, they knew that one of the numbers was larger than the other but were confused as to what this implied about length. If this is what happened, it contributes to the picture of confusion about basic aspects of measurement.

6.4.5 Measuring with the ruler

Children in Year 1 clearly had very little idea about how to measure with a ruler, with an average of only 13% correct starting points. On average, over half the measurements were made from the correct starting point in Year 2, and over two-thirds in Year 3, (although there was no significant difference between these two year-groups). Of the three devices, only the ruler showed much improvement in use of the correct starting point after Year 1, strongly suggesting that greater familiarity and frequency of use of the ruler in later school years accounted for improvements, especially as the ruler used in the study was of the type most commonly used in the school.

Several specific starting points other than the zero position on the scale were favoured when children measured with the ruler, indicated in Table 6.04. The most commonly chosen starting point other than the correct one for children in Years 2 and 3 was the 1 cm mark, indicating lack of understanding that numbers on a ruler label units of length on the device.

More than a quarter of the children sometimes measured from the 0.5 cm mark. Half-centimetres were marked with a longer line than the 1 mm lines, so these were prominent divisions on the ruler, and this was the first such line. It is difficult to see why this starting point should be attractive, although some evidence was found that avoiding fractional units may have been implicated on some occasions. Four children sometimes found on the ruler the number of units (5, 12 or 19) mentioned by the experimenter and aligned this mark at the start of the line to be measured; this happened more frequently when the tape was used. It suggests considerable confusion, with children simply making use of a number they knew to be relevant in some way.

As Table 6.02 shows, very few children consistently used a single starting point for all nine measurements and between 11 and 15 children in each year-group used a variety of starting points that were not among the most favoured. This variety must indicate that they had no settled principle for the use of the ruler. A hypothesis that children adjusted their starting point so as to avoid dealing with fractional units was tested and evidence was found that this might have happened in some cases.

Table 6.02 does however show that children in Years 2 and 3 made a larger number of measurements from the zero mark on the ruler scale than from other marks, and although it could be argued that only complete consistency would indicate understanding of the principle of measuring from the start of the scale, increasing use of the zero mark must at the very least indicate that a preference was being established, probably as a result of instruction and increasing practice in measuring with a ruler.

Given the frequency of measurement from incorrect starting points, considerable inaccuracy in reported measurements was to be expected, and as Table 6.06 shows, measurements that were correct by the 2.5 mm criterion ranged from 13% in Year 1 to only 51% in Year 3, with no significant difference in accuracy between Years 2 and 3. Also contributing to the low level of accuracy were failure to offer a numerical measurement when invited to do so, and reporting whole numbers when confronted with fractional units. Table 6.07 shows that no children consistently reported fractional units when appropriate, and nearly half the children in Year 2 and Year 3 did so on less than half the possible occasions. There is evidence that these three causes of error may have been interrelated. That is, on some occasions children may have adjusted their starting points to avoid dealing with fractional units, and may have inappropriately reported whole numbers for the same reason.

Nevertheless, only when measuring with the ruler were children in the older year-groups appreciably more accurate than children in Year 1 (Figure 6.03). This suggests that increased experience in measuring with the ruler as well as increased instruction results in improvement. However, the much smaller extent to which children in the older year-groups were more accurate with the cubes or the tape measure than children in Year 1 suggests that any knowledge of measurement gained from practice with the ruler was not generalised to other devices.

For children in Years 2 and 3, comparisons using the ruler were significantly better than those using the tape or cubes. However, there was no significant difference in correctness between tape and cubes, and none for children in Year 1 between any of the devices. Speculation in the literature (Petitto, 1990; Piaget *et al.*, 1960) that separate units may differ in their difficulty for children from scaled measuring devices thus received no support from this research.

It will be recalled that correctness of the comparisons was assessed independently of correctness of numerical measurements. For cubes and tape, correctness of the comparisons was on average 45% to 50%, and differed little between year-groups, or between the two devices. It has already been suggested that this uniformly poor performance of a relatively simple task may have been due to children's difficulties in associating the numbers they saw on the device and heard from the experimenter with the judgments of length they were asked to make. If older children were no better than younger in making this association where cubes and tape were concerned, there is no obvious reason why they should have been better with the ruler. It seems that greater familiarity with the ruler must have facilitated this outcome with the older children, but it is not clear by what mechanism this could have happened.

6.4.6 Measuring with the tape

No more than one third of measurements with the tape were made from the correct starting point on the scale, and in this there were no significant differences between year-groups. Considering that the scale started at the end of the tape, so that only matching this to the end of the line was required for success, this result was surprising, and suggested that, even though many had said at interview that they had seen tape measures used at home, few had used one much themselves. The way in which the end of the tape was sealed with a flat semi-circular stud which projected some way towards the 1 cm mark, may for some have proved a distraction. About half the children in each year-group sometimes measured from near the 0.5 cm position (Table 6.02). Approximately the same number of children sometimes measured from the 1 cm mark with the tape as with the ruler, but a greater number than with the ruler – fourteen – sometimes began their measurement from the 5, 12 or 19 cm mark. If this type of choice did indicate real confusion, then it is understandable that this would have been greater with the unfamiliar tape than with the ruler. Children sometimes covered the line to be measured with the tape, as they did with the cubes, so that the line they were measuring was not visible to them, but in the case of the tape, this behaviour was practically confined to Year 1.

Mean correct numerical measurements with the tape were very low and reached 25% only in Year 3, with no significant difference between Years 2 and 3. As already suggested, the desire to avoid fractional units (data for ruler and tape were combined to investigate this) may have contributed to this lack of success.

Not surprisingly, Year 1 children were at sea when handling the tape, finding its length and lack of rigidity very difficult to handle, and sometimes seemingly preoccupied by the long series of numerals printed on it. Children in Year 2 and especially Year 3, however, manipulated the tape better and might have been expected to be able to use of knowledge of the ruler and its scale in measuring with the tape. This did not occur.

Comparisons using the tape were about 50% correct, did not differ significantly from correct comparisons for cubes, or (like cubes) among year-groups. As with cubes, this surprisingly poor performance of an apparently simple task is hard to explain for the older children, except by some degree of dissociation between numbered units on the tape and the notion of length.

When, in making their comparisons, children were asked to convert an absolute judgement (their measurement in units) to a relative judgement, (that is, whether the line was shorter, longer or the same length as a number of units mentioned by the experimenter) no more than half did so successfully. Yet the reverse procedure (reported in Chapter 5) – responding in relative terms when a specific number of units was mentioned – showed a very high success rate. This seems to be a case, seen so often in the literature, of sensitivity to the exact form of the task. Apparent similarity of cognitive demand does not necessarily predict performance.

Practice in measuring with the ruler (as suggested above) and not sheer familiarity with it as a 'cultural tool' seems likely to have accounted for the improved performance over the others of the oldest year-group; the difficulty of coping with its numbering and subdivisions produced poor results even with Year 3.

Units of length are similarly represented on tape measures and rulers. It might be expected that children who have practiced measuring with a ruler will make use of this knowledge when asked to measure with a tape measure, with which they have had less practice. The fact that accuracy with the tape measure was so poor indicates that if there was any transfer of knowledge, its effect was slight.

Chapter 7

Associations between measurement and measurement-related abilities

7.1 Introduction

The overall structure of the research reported in Chapters 4 to 6 was informed by the *logical* relationship between measurement and measurement-related concepts. Thus the language and concepts of ordinal comparisons of length (Chapter 4) logically underpin estimation of ordinal length (Chapter 5), while ordinal comparisons underpin measurement involving units (Chapter 6). An important outstanding question, however, was whether the corresponding language, estimation and measurement abilities supported each other psychologically.

To help understand this, statistical tests of association between the outcomes of the three sets of tasks (reported respectively in Chapters 4, 5 and 6) were conducted. It will be recalled that these tasks tested a) children's understanding of terms used to describe ordinal length (such as *shorter, widest, as tall as*) together with the concepts such **language** expressed (Chapter 4); b) their success in **estimating** whether lines were shorter, longer, or the same length as various comparators, such as a number of cubes, or a number of cm on a ruler (Chapter 5); and c) their **measurement** ability (Chapter 6). Measurement ability was tested in two ways, reported in Chapter 6 as 'comparisons' and 'numerical measurement'. In their comparisons, after measuring a line in units, children were asked to say whether that line was shorter, longer, or of the same length as a number of units mentioned by the experimenter (that is, to *compare* their measurement with the number of units mentioned). In their numerical measurements, they were asked, as an absolute judgement, how long (in units) the line they had measured actually was. Comparisons were thus logically dependent on numerical measurement. Because of the logical priority of numerical measurement, and because the absolute judgments it called for are the typical outcome of measurement, only the **numerical measurement** scores were used in the tests of association.

Several subsets among the tasks were of particular interest in exploring associations. The **'complex' language** tasks in Chapter 4 constituted one such subset, because they were at the same time tests of conservation of length (see the description in Chapter 4). It had been noted that, since Piaget, the literature had not given the central place to the psychological relationship between conservation of length and measurement concepts that their logical relationship merited. Association between performance on complex language tasks and that on numerical measurement was therefore investigated.

Wherever feasible, in the remainder of this chapter, the phrase 'the language and concepts of ordinal length' will be shortened to 'language'; 'visual estimation of ordinal length' will be shortened to 'estimation'; and 'numerical measurement' will be shortened to 'measurement'.

7.2 Method

Sample

The complete data required for correlational analysis were available for 68 children: twenty-one in Year 1; twenty-one in Year 2; and twenty-six in Year 3.

7.2.1 Internal reliability testing

Valid statistical comparisons could be made only where performance on each series of tasks could be assumed to reflect a coherent set of competencies in those performing them. It will be recalled that, so as to assess generality of understanding, the variety of materials employed had been wide, and it was possible that this variety affected the extent to which the tasks within a series tapped similar competencies. Internal reliability was therefore computed separately for scores on the tasks reported in Chapter 4 (language); Chapter 5 (estimation); and Chapter 6 (measurement). For this purpose, each child's mean score across a) all language tasks, b) all estimation tasks and c) all numerical measurement tasks were calculated and used to calculate Cronbach's alpha. Acceptable alpha values (α (language) = .85; α (estimation) = .73; α (numerical measurement) = .76) were obtained for the three sets of tasks. It would have been interesting to explore correlations between performance in estimation and measurement tasks further for the individual measuring devices (ruler, cubes and tape measure). However, the outcomes of these three subsets of tasks did not achieve acceptable alpha values (perhaps because the means were of smaller numbers of scores) and so were not subjected to tests of association.

Next, each child's mean score across 'complex' language tasks (those constituting tests of conservation of length) were used to calculate α, and achieved a satisfactory value (α = .82).

Correlations were thus examined among scores for language, estimation and numerical measurement. Scores just for 'complex' language tasks were examined for correlation with those for numerical measurement. Association of scores for language, estimation and numerical measurement with children's ages was also explored.

7.2.2 Correlations

Language, estimation and measurement scores were first examined for correlation with the children's ages at the date on which each child undertook the relevant tasks. Pairwise correlations of scores for language/estimation, language/measurement and estimation/measurement were then conducted.

7.3 Results

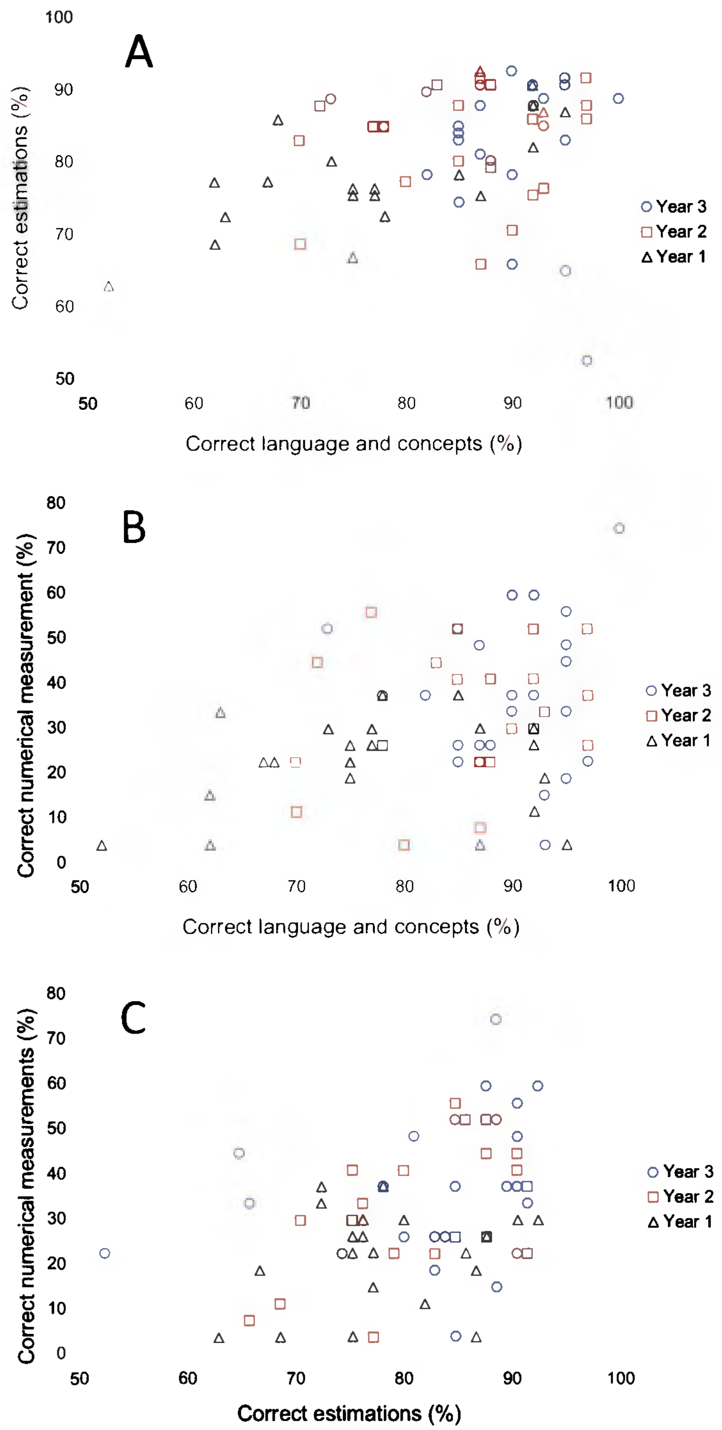
7.3.1 Age

With all year-groups included, there was a moderate, positive, significant correlation of age with language scores (r (66) = .46; $p < .01$). The same was true, to a lesser extent, of measurement (r (66) = .36; p = .01). Estimation scores showed a weak, non-significant association with age (r (66) = .22; p = .07). The same calculations performed within each year-group with language, measurement and estimation showed for all three variables very weak ($r < .2$), non-significant associations with age.

7.3.2 Language/ Estimation

The associations between children's language and estimation scores are plotted in 7.01A. It can be seen that the scores from the three year-groups overlapped substantially, and for children in all three years, there was a wide range of scores for both abilities. For the combined data set (all three year-groups) the correlation between language and estimation was r (66) = .33, $p < .01$. There was a somewhat weaker partial correlation when age was controlled for; this was weaker still, and just missed significance (p = .052), when both age and measurement were controlled for (Table 7.01). The correlations were repeated separately for the three year-groups. For Years 2 and 3 there were only very weak, non-significant zero-order associations that changed little when age and measurement were controlled for. In contrast, in Year 1 there was a strong and highly significant association almost unaffected by age or measurement (Table 7.01). This indicated a close association between understanding the everyday language and concepts of length and the ability to make visual estimations, limited to five to six-year-olds.

Figure 7.01 Scatterplots of children's scores for language against estimation (A); for language against measurement (B); and for estimation against measurement (C).



Note: Each data point represents a child's mean score for the variables labelled on the two axes.

7.3.3 Language/ Numerical measurement

The associations between children's language and measurement scores are plotted in Figure 7.01B. Substantial overlap between the scores of the three year-groups can again be seen. For the combined data set (all three year-groups) the correlation between

Table 7.01 Pearson's coefficients for zero order and partial correlations among a) language of ordinal length, b) visual estimation of ordinal length and c) numerical measurement.

	Variables controlled for	All	Y1	Y2	Y3
		Estimation			
Language					
All	None	.33**			
	Age	.28*			
	Age and measurement	.24			
Year 1	None		.72**		
	Age		.71*		
	Age and measurement		.70*		
Year 2	None			.11	
	Age			.09	
	Age and measurement			-.04	
Year 3	None				-.15
	Age				-.13
	Age and measurement				-.17
		Measurement			
Language					
All	None	.30*			
	Age	.15			
	Age and estimation	.06			
Year 1	None		.12		
	Age		.15		
	Age and estimation		-.06		
Year 2	None			.23	
	Age			.22	
	Age and estimation			.21	
Year 3	None				.05
	Age				.12
	Age and estimation				.16
Estimation					
All	None	.39**			
	Age	.35**			
	Age and language	.33**			
Year 1	None		.22		
	Age		.27		
	Age and language		.24		
Year 2	None			.54*	
	Age			.55*	
	Age and language			.54*	
Year 3	None				.26
	Age				.25
	Age and language				.27

* $p<.05$; ** $p\leq.01$

Note 1. In partial correlations, coefficients are shown with i) age controlled for and ii) age and the third variable controlled for.

Note 2. Coefficients are given for the whole sample $N = 68$) and separately for the three year-groups (Y1, $N = 21$; Y2, $N = 21$; Y3, $N = 26$).

language and measurement was r (66) = .30, p = .01. Re-calculation controlling first for age and then for age and estimation, however, revealed only very weak non-significant associations between language and measurement (Table 7.01). The partial correlations were repeated for the three year-groups separately and were, at best, weak. None were significant (Table 7.01).

7.3.4 Measurement/ Estimation

The associations between children's measurement and estimation scores are plotted in Figure 7.01C, where a positive relation between measurement and estimation scores is evident. There is again no clear differentiation among the three year-groups as their ranges of abilities overlap considerably. For the combined data set the correlation between measurement and estimation was r (66) = .39, $p < .01$. This correlation remained modest and highly significant when recalculated controlling for age, and then for age and language (Table 7.01). The partial correlations were repeated separately for the three year-groups. For Years 1 and 3 there was only a weak non-significant association between measurement and estimation when controlling for age and then for language and age. However, in Year 2 there was a moderate, positive, significant association (Table 7.01).

7.3.5 Complex language tasks/numerical measurement

There were only very weak, non-significant correlations between complex language tasks and numerical measurement both overall, and for any year-group, with or without controlling for age. (In the zero order correlations, $r < .2$ except for Year 2 where $r = .27$).

7.4 Discussion

Previous chapters showed that, overall, children had good understanding of everyday language and concepts of measurement; that their visual estimation ability was good; but that their measurement ability was poor. In relation to some aspects of their performance there was substantial variation within year-groups; analysis of variance showed, however, that there was typically an overall improvement between Years 1 and 3.

7.4.1 Correlations with age

Across the whole sample, correlations performed for the three types of ability with children's age showed a moderate association with language scores and a more modest association with measurement scores. There was however no significant association of age with estimation scores, despite the fact that analysis of variance had identified a significantly higher mean score for estimation in Year 3 than in Year 1. This effect was explained by the great variation in ability across the year-groups.

When correlations of the language, estimation and measurement scores with children's age were performed separately for the three year-groups, there were no significant associations. Thus, despite an overall association of language and measurement scores with age, *within each year-group* older children were no more successful than younger and a wide range of language and measurement proficiency was indicated. So for these two abilities, improvement was not so much age-related as year-group related, suggesting, as one would expect, an effect of teaching and of (perhaps cohort-specific) life experience.

7.4.2 Correlations among language, estimation and measurement scores.

There were modest, positive associations among language, estimation and measurement scores. However when age and the third factor were controlled for, only measurement and estimation scores were significantly correlated, and then only modestly. There was no significant association between language and measurement, either overall, or for any year-group, suggesting that language ability in itself had little bearing on measurement ability. There was, however, a strong positive correlation of language with estimation ability that was confined to Year 1, and a moderate positive correlation of estimation ability with *measurement* ability confined to Year 2 (Table 7.01). This suggested a ladder effect, whereby a degree of proficiency with the everyday language and concepts of measurement supported ability in estimation of ordinal length in Year 1, and proficiency in estimation supported measurement ability in Year 2, (although it is possible that the direction of support was from measurement to estimation in Year 2: this is discussed later). These suggestions are now further explored.

For Year 1, the strong association between understanding of language on the one hand, and estimation ability on the other constituted the lower rung of the ladder: these abilities were not associated for Years 2 and 3. Both sets of tasks involved ordinal judgements and used questions of a similar structure, but in the language tasks, children saw a wide range of everyday materials, and had to respond to a much broader range of 'length' vocabulary than was employed in the estimation tasks. In the latter, while language and materials were more restricted, the tasks themselves were more demanding, using many lines of different lengths, different types of comparator, and involving units. It seemed that for the youngest group of children, the highly-structured language tasks (reinforced by the extent to which they made ordinal comparisons routinely in everyday life) did tap abilities which supported the more demanding comparisons required by the estimation tasks, and that this support was no longer needed by children from the two older year-groups for whom language and estimation abilities were by then independent.

When it came to associations between the ability to make ordinal estimations of length and measurement ability, none were significant *except* for Year 2, where there was a moderate positive correlation. This effect constituted the upper rung of the ladder, whereby ordinal estimation supported numerical measurement.

At least two interpretations are possible as to the direction of the effect. The first is that logical order was the most influential factor: ordinal comparisons underpin measurement with units. According to this interpretation, in Year 2 the extent to which children made correct estimations supported their measurement ability, while by Year 3, this effect was complete, and the two abilities were independent. In Year 1, measurement ability was at too low a level to benefit from any such support.

The second possibility was that improvements in measurement drove improvements in estimation in Year 2, facilitated by overall improvements in arithmetical skills feeding into measurement practices. Incidental evidence of arithmetic skills being deployed in classroom measurement activities was given in Interview 1. The argument would be that by Year 3, measurement and arithmetic skills were developing independently, and that in Year 1, children were at too early a stage in both skills for either to support the other. That measurement could be supporting estimation looks implausible, however, in the light of very good scores for estimation, contrasting with very poor scores for measurement in all year-groups.

The 'ladder' remains a hypothesis, since the abilities featuring in it may in fact be independent of each other, or accounted for by other factors, and this is a third possibility. Nevertheless, the design of the research incorporated the idea that language was likely to underpin estimation, and estimation likely to underpin measurement, psychologically as well as logically, so that it is a persuasive hypothesis.

The correlational pattern overall provides a rich basis for reflection on how the curricular arrangement of topics leading up to measurement might best be organized, and on how judicious selection of language and materials can support the logical transitions children need to make.

Surprisingly, no association was found between ability in the 'complex' language tasks (which were also conservation tasks) and numerical measurement, despite the logical relationship between conservation of length and the ability to measure. However, there was a high level of success on the part of all year-groups in the language tasks, suggesting that most children had surpassed the conceptual requirements of conservation (as measured by these traditional tasks) that were needed for measurement, and this probably explained the lack of association. The finding leaves unexplained, however, examples of non-conservation of length in relation specifically to measurement identified in surveys (e.g. Hart 2004; Department of Education and Science 1981) and in the present research. This is considered further in the Discussion chapter.

Chapter 8

Discussion

8.1 Introduction

The present research investigated children's ability in measurement and its allied competences. Studies reviewed in Chapter 1 provided valuable but very various insights into this ability. In consideration of this, the present research restricted itself to an integrated set of tasks designed to assess basic measurement and measurement-related abilities and offered to the same group of children. The research investigated the children's understanding of the everyday language and concepts of length, their ability to estimate ordinal length, and their ability to measure in units. The same experimental tasks were suitable for three successive primary school year-groups and enabled comparison of their abilities. The fact that a single sample of children participated enabled exploration of how these abilities might be associated. To help understand how the abilities were acquired, the research also investigated the children's own accounts of the home and school contexts in which they experienced measurement. The results of each interview and each series of tasks were discussed in Chapters 3 to 7. The present chapter identifies and discusses the findings that were judged to be the most important. It does so in relation to the research questions and key theoretical themes summarised at the end of Chapter 1, and to the literature reviewed in that chapter. It also indicates how future investigation could be taken forward, and considers some classroom implications of the present findings.

The most important findings of this research were all, as it happened, somewhat unexpected, particularly when considered together. Firstly, children turned out to be poor measurers. Secondly, and in contrast, they showed very good ability in several competences that logically underpin measurement, and might have been expected to support it. Thirdly, performance in these underpinning competences was unexpectedly and extraordinarily robust. It varied little across many different tasks indicating, to this extent, conceptual security. Fourthly, while the children's understanding of measurement did not mature with age and two to three years of schooling to anything like the extent that might have been expected, there was nevertheless evidence of a developmental process from the underpinning competences to measurement itself. Finally, the influence of 'real life' experience on understanding of measurement was mixed. On balance, there were probably more ways in which it hindered than helped. Each of these findings is discussed below.

8.2 Main findings

8.2.1 Children as measurers

Children in this research measured surprisingly poorly. Those in Years 2 and 3 measured better with the ruler than with the other two devices, but even in these year-groups only about half these measurements were correct by the criterion set. While the immediate causes of the children's poor performance, as observed, arose from their handling of the measuring devices, these procedural errors suggested conceptual difficulties. Firstly, children were notably inconsistent in their choice of starting point on the ruler or tape to begin their measurement. This suggested no settled idea (whether correct or incorrect) about the *relationship* of units on these instruments to the length to be measured. Secondly, supporting evidence for the frailty of this understanding came from the way in which children often covered the line to be measured with a device, so that *matching* units on the device to the length to be measured could not be achieved. Having noticed that matching could not be achieved, children did not change their strategy. This suggested that the idea of *equivalence* between the units of length shown on the device and the length measured, if this idea existed, was overtaken by the inclination to create a *single* length by superimposition. Operational transitive inference seemed to be lacking here. Thirdly, there was the opinion of some of the older children in the second interview, that the ruler could not measure an item longer than itself. Half the children expressing this opinion could not suggest a way of dealing with the situation that involved using the ruler to add more units (cf. Brown *et al.*, 1995). The units were not, seemingly, 'detachable' from the ruler that embodied them.

The subdivisions on the ruler appeared, as predicted, to be a source of difficulty, when these were required to measure a line that was not a whole number of units long. This difficulty was reflected in children often apparently choosing not to give a numerical measurement at all, or to state a number of whole units, when fractional units were involved.

When the cubes were used as units, the failure of many children to be consistent in closing up the gaps between them similarly suggested lack of understanding that *equivalence* between the length of the line of cubes and the length to be measured was essential. It seems fair to describe this as failure to conserve length, not as in traditional conservation tasks, but in the 'applied' sense exemplified in surveys of performance in mathematics and science. Thus, in one survey, children failed to mark the start point of a clockwork toy when measuring the distance it travelled (Department of Education and Science, 1981). In another, children were asked to compare the length of parallel lines drawn on squared paper. Because the ends of the lines were offset, direct comparison of the two lines was impossible, but still it did not occur to some children to count the number of squares crossed by each line. The line lengths were compared visually, and errors resulted (Hart, 2004). As in the failure to place cubes correctly (in the present research), these are examples of non-conservation of length in practical contexts.

Measurements with the cubes and tape measure were performed equally poorly, and were worse performed than those using the ruler. This provided no support for the view that continuous scales (represented by the tape measure and ruler) would make the meaning of units more difficult to grasp than discrete units (the cubes) (Piaget *et al.*, 1960), or indeed for the reverse view (Petitto, 1990). It also suggested that children could not make use of conceptual knowledge they may have gained through practice with the ruler when it came to dealing with the units embodied in the other devices. The notion of functional fixedness (Casler & Kelemen 2005; Miller 1989) whereby the function of a tool, learned as a social norm (Rakoczy, Hamann, Warneken & Tomasello, 2010) may for that very reason obstruct full understanding of its properties and their 'portability' to other contexts, seems to apply here. The deployment of the calculator in the classroom (where the nature of the tool might invite greater flexibility) has encountered similar problems (Guin & Trouche, 1999). Even where the far richer resources of computer environments have been developed, as tools, specifically to instantiate mathematical principles (Noss & Hoyles, 1992), it has not always been clear to the developers whether children participating in the computer activities have understood those activities as mathematically structured.

The idea, therefore, that conventional measurement instruments, as 'cultural tools' with which children are familiar, may both assist them to measure correctly and also support their acquisition of underlying logical principles (Nunes *et al.*, 1993) receives little support from the present research. The first interview showed the measuring tape to be very familiar from out-of-school contexts, but this did not help children use it correctly when measuring. While performance with the familiar ruler was better among the older children with longer experience of using it than among the younger children, its subdivisions seemed to be avoided by many. In the second interview, the fact that the greater numbers on the ruler were displayed in conjunction with the smaller units seemed to disrupt any understanding that there is an inverse relation between size and number of units. In this case, the physical appearance of the measuring instrument, for all its familiarity, actually obstructed understanding of the concepts it embodied. These findings suggest that basic ideas (such as the fact that larger numbers do not necessarily indicate greater amount) should be explicitly taught and exemplified, and that the benefits of familiarity cannot be taken for granted. Teaching of measurement may, for example, profit from being routinely supported at first by simplified, if unfamiliar, devices such as rulers marked with units that are not subdivided or numbered. Children would then need to rely on *counting* the number of units that were equal to the length to be measured. The marking of numerals on the ruler could then be introduced and explained as a check on the counting of units, and in due course the conventional use of numerals on rulers could be presented in the light of a convenience for measurers.

Evidence of conceptual difficulties about the relationship between number and quantity came from children's 'comparisons', in which less than half the responses were correct, even in Year 3. When children had measured a line and declared what they took to be its length as a number (with or without mentioning units), they were asked whether the line was therefore shorter, longer, or the same length as a 'reference' number of units mentioned by the experimenter. It was this comparison of two numbers that was so poorly performed. On the reasonable assumption that most of the children were easily able to tell which was the greater of two numbers under twenty (20 cm being the maximum length of a line and 19 cm the largest 'reference' number), it must have been uncertainty about how these numbers mapped on to the physical length of a line (enabling them to say which of two lines was, for example, shorter) that led to errors.

Finally, and most strikingly, there was clear evidence from the second interview that at least one-third even of the older children did not understand that there is an inverse relation between the size and number of units required to measure a given length. This simple and fundamental principle of the proportional reasoning involved in measurement thus eluded many children in this sample. Among the themes summarised at the end of Chapter 1, that on ratio and proportion considered a range of research in which children were successful in some contexts and unsuccessful in others. The findings of the present research, in which children simply commented on the scales on common measuring devices, indicated unequivocally that many would find difficulty with everyday measurement tasks.

8.2.2 Competences underpinning measurement

Language

Children proved to be expert comprehenders of the everyday *language* of comparative length, an ability not examined in the previous investigations reviewed. Piaget (Piaget *et al.*, 1960) discussed the understanding of ordination as a logical pre-requisite for relative judgements of length; Bryant (1974) established that relative judgements are made early and are easier to make than absolute judgements; Resnick (1992) proposed categories of protoquantitative reasoning that develop early, underlie all reasoning about quantities, and are all relative in character; while Feigenson *et al.*, (2004) argued for an ability to discriminate ratios between quantities in early infancy. Furthermore, the present research showed that experience of measurement in the home favours judgements of affordance or 'fit', which are relative judgements. All this evidence pointed to the likelihood of early competence in judging relative length, but no evidence was offered about the role of language in these relative judgements. (This was not, of course, possible for the infancy studies).

The present research investigated such language in a dedicated series of tasks. It investigated children's ability to understand the everyday language of length (that is, of the attributes of height, length, and width) as they made relative judgements in the differing contexts of a substantial number of displays. Some of the subtleties of the way in which we use language that may have misled children as to the meaning of these attributes were described in the introduction to Chapter 4.

Despite these subtleties, children in all age-groups proved to be excellent comprehenders of such everyday language. They were highly successful in making comparisons of height, length, and width in a wide range of contexts, the sole difficulty concerning certain judgements of height where the human figure was involved. This was no mean achievement, and meant that one likely barrier to the understanding of measurement, that is, the everyday language involved, could be ruled out.

Conservation of length

The children were also good conservers of length in the traditional sense. The language tasks incorporated a 'complex' condition in which the spatial arrangement of a substantial number of displays tested conservation of length in its traditional Piagetian form. With the sole exception regarding height already mentioned, these tests posed remarkably few difficulties to children.

The competence with which conserving judgements were made should be given its full weight. Conservation tests have long figured (either centrally or peripherally) in studies of the development of measurement (Piaget *et al.*, 1960; Boulton-Lewis, 1987; Miller, 1989; Petitto, 1990), as well as (in the broader sense explained above) in mathematics surveys (Department of Education and Science, 1981; Hart, 2004) because conservation is indispensible to any ability to judge amount. Performance on such tests has frequently been found to be weak. By contrast, performance in the present research showed the children to have this building block firmly in place. This did not guarantee, however, that they would conserve length in the broader sense, as required by the measurement tasks, and they frequently failed to do so.

Visual estimation of length

Children were consistently very good estimators of ordinal length throughout a long and demanding series of comparisons. They were not confused when asked to make their ordinal comparisons in relation to a given number of units. They were successful regardless of the absolute length of the line whose length was judged. Most surprisingly, the differences between the 'comparators' against which the length of lines was judged, designed to be challenging, proved no obstacle. These five comparators comprised one plain unmarked strip, three devices marked and numbered in various ways, and a collection of wooden cubes. Children were equally successful with all of them. Their performance did differ according to the ratio of the length of a line to that of its specific comparator: the greater the ratio, the greater the likelihood of being correct. This was to be expected in view of the greater discriminability of that difference in length (Ross, 1997), and was reflected in poorer performance in correctly estimating that certain lines were the *same* length as their comparator (where there was no difference to discriminate).

Lack of influence of the materials used

In language and estimation tasks, children's performance was, surprisingly, consistently good in the face of a wide variety of experimental materials. In measurement tasks, performance was somewhat better with the ruler than with the other devices, but was pretty consistently poor. Conceptual difficulties arising *at the contextual level* were therefore few.

In Chapters 4 and 5, the rather lengthy description of materials used for each series of tasks reflected the care that had been taken to include features designed to test in many different contexts the conceptual understanding examined. Thus the displays used in the language tasks (for example, human figures, toy cars, writing implements in both 2-D and 3-D format) and the different spatial orientations in which they were arranged in the 'complex' displays, required children to apply nuanced language conventions regarding height, length, and width correctly for successful performance. In the estimation tasks an elaborate combination of line lengths with five comparators that exemplified (or not) discrete and continuous units, subdivisions and numbers were meant to assemble various factors that previous research had identified as presenting difficulties to children. In the event, this variety produced a surprisingly uniform result. In the language tasks, only very few, very specific displays to do with human height were found difficult; other variations were easily dealt with. In the estimation tasks, neither the type of comparator nor the absolute length of lines affected the outcome, which was uniformly successful, except for lines close in length, or of equal length, to their comparator. Performance on the measurement tasks was uniformly poor, with performance with the ruler by older children achieving only a modestly higher level.

It has already been remarked that in existing research on measurement, different types of tasks and materials produced different outcomes. Few studies encompassed the range of variation in materials used in the present research, and above all not *with the same participants*. The single sample used in the present research revealed how stable across contextual variation children's performance can be. It suggests that children may leave behind conceptual difficulties arising at the contextual level, while overarching conceptual difficulties, such as those associated with the very notion of units, in many cases remain. One way of looking at these results is to say that they amplify those of earlier studies (Resnick & Singer, 1993; Bryant, 1974) that concluded that relative judgments are made earlier in development and are easier to make than the difficult absolute judgements characteristic of measurement with units. The present research brings into focus the detailed texture of that ease and that difficulty.

8.2.3 Age

While overall there was improved performance with age, this was a great deal less, and less clear-cut, than might have been expected. Nevertheless, there was evidence of a 'ladder effect' between competencies from year-group to year-group.

The most typical finding in each series of tasks was a significant difference between the performance of children in Years 1 and 3, but no significant difference between the two younger or the two older year-groups. Among the results of some sub-sets of the tasks (for example in the tests of 'height above ground' and in measurement performance with tape and cubes) the degree of success was practically the same for all three year-groups. There was little difference, either, when it came to interpretative or procedural errors (which nonetheless had conceptual implications), such as identifying the smaller unit as larger (in the second interview); using an incorrect starting-point for measuring with the cubes and tape; and obscuring a line to be measured with the measuring device.

When the outcomes of the three main series of tasks (language, estimation and measurement) were correlated with chronological age, there were significant, moderate, positive correlations for language and measurement, but these disappeared when the calculations were performed for each year-group separately. What accounted for this was substantial variation in performance according to age that crossed year-group boundaries, so that only when the full range of ages was considered could any trend be picked out. Together with the significant differences in performance frequently found only between Years 1 and 3, this presented no clear picture of the influence of development and education on measurement ability.

When age and the third factor were controlled for in partial correlations, however, an important developmental effect came into focus. Ability in the language and estimation tasks correlated strongly and positively for Year 1 children, while estimation and measurement ability were moderately positively correlated for Year 2. This suggested a 'ladder' whereby facility with the language of comparative length supported children, at an early stage, in making accurate estimations of comparative length, while estimation ability in turn facilitated accuracy in measurement by children at a later stage.

In the case of Year 1, then, there was a strong association between understanding of language on the one hand, and estimation ability on the other. Both sets of tasks involved ordinal judgements and used questions of a similar, closely-prescribed, structure, but there the resemblance ended. In the language tasks, a wide range of everyday real and pictured items was used, and questioning was about height and width as well as length. In the estimation tasks, only the lengths of lines were compared, many more comparisons were made, and numbers and units were involved. Nevertheless, it seemed that for the youngest group of children, the language tasks (reinforced no doubt by the extent to which they made ordinal comparisons routinely in everyday life) did tap abilities which supported the more demanding comparisons required by the estimation tasks.

8.2.4 The social and the conceptual

Finally, the question of broadest scope and potential interest was: what do children learn about measurement in their everyday social context - in 'real life' - and how might this affect their conceptual understanding of measurement? The terms 'social' and 'conceptual' are of broad connotation and their respective applications in the present research were not straightforward. Nevertheless, the terms represent a contrast that is of considerable consequence for education. Below, the term 'social' is especially broadly construed, and includes classroom experience. The 'conceptual' in measurement is taken to include, for example, conservation of length, transitivity, and logical characteristics of units.

Resnick's succinct remark that at every opportunity for mathematics learning, children learn "both less than and more than mathematics" (Resnick, 1992: 107) captures the nature of the contrast between the social and the conceptual, as well as the difficulty of separating the two. Applied to measurement, the *less* and *more* neatly present the paradox that although everyday life throws up a wealth of social situations involving measurement, conceptual aspects are not likely to be especially salient. This is so even when measurement is at the heart of a social activity, such as (in the present research) when buying new clothes or furniture, cooking, and home decorating. The value to children of those social activities consisted in their importance in the children's lives and in the pleasure they gave. Less was likely to be learned about measurement itself because the focus was on the social function it served. More than measurement was learned because that activity was embedded in an event of greater social substance.

In the general discussion at the end of Chapter 3, it was argued that guided experience of measurement in school had probably led to the greater precision observed when children in Years 2 and 3 spoke of units than when those in Year 1 did so. Yet precision does not necessarily imply conceptual understanding. Even in school, where the social setting is overtly dedicated to learning, and the focus is clearly on measurement itself, children's perceptions of the social meaning of an activity may still influence what is learned. In the present research, recall of occasions in school when measurement took place was strikingly specific, and it was the specificity of the procedure that seemed to be the focus. Precisely how the ruler or metre stick was manipulated on a particular occasion, specific numbers of units and their addition, and, frequently, *writing down the outcome* of a measurement were what children in Years 2 and 3 talked about. The 'social' value children attached to these activities might be described as 'getting the job done'; reflections like *...so we knew the bigger jug held more water* (Table 3.09) were rare. If the goal of an activity is to complete a procedure, it is not difficult to see how conceptual understanding could be bypassed. For example, fifty per cent of the children in Year 3 stated that only 'small things' could be measured with a ruler (Chapter 3). While half of these, when pressed, spoke of measuring a larger object by iterating the ruler and adding the units, just as many could only suggest making up the full length of a longer object by placing some other item at the end of the ruler. These children had no ready access to the idea of measurement as the iteration of a unit; probably the customary way of deploying the standard school ruler had obscured that. Thus at first sight, neither home nor school terrain seemed necessarily to favour conceptual understanding: the conceptual content of children's experiences risked being overwhelmed by the social in both settings.

This discouraging evidence was balanced, however, by evidence of positive features, particularly of the home setting. First, as already noted, most measurement activities in the home served to determine the fit of an item to available space. These featured the affordances and relative judgements identified by Gibson (1979) and Bryant (1974) as forerunners of the absolute judgements characteristic of measurement in units. Referring to the same class of ability, Resnick & Singer (1993) spoke of the protoquantitative relationship of *fittingness* and considered this to be the basis of ratio reasoning, also essential to understanding units. The home setting thus provided children with the opportunity to acquire important competences underpinning measurement, and did so in circumstances where familiarity and the enthusiasm frequently associated with the activities were likely to make them a favourable basis for future learning. Secondly, measurement typical of the home setting occurred as part of problem-solving activities and strategies. This was particularly evident in the comments of children in Years 2 and 3 (Chapter 3), for example in the description of laying a wooden floor, or of mapping the London underground system. This approach was highly valued by the *Primary Mathematics Framework* (Department for Children, Schools and Families, 2006) which expected learning to occur as the outcome of problem-solving 'enquiries'. Enjoyment of problem-solving activities in the home may encourage a positive attitude to problem-solving in school, including the persistence required, and the present research gave evidence that the home setting could nurture this engagement.

In the interviews, what one might call the *demands of the narrative* gave evidence of an additional social influence. One example again concerned units, whose whole purpose is to make quantification precise. In telling the story of an important event that involved measurement, quantities expressed in units were often absurdly large, especially in Years 1 and 2. (Year 3 children were more precise). The younger children never exaggerated in the direction of *smaller* numbers of units than were plausible, so that this was not simply a case of increasing accuracy with age. Rather, large numbers seemed to be used as markers of the importance of an event. An alternative explanation of this finding is that this early favouring of larger numbers was an example of a general bias in language usage towards terms expressing greater, rather than lesser, quantity noted by Haylock and Cockburn (1989). There was no sign of this bias among the children's ordinal comparisons in Chapter 4. In the estimation tasks reported in Chapter 5, children in Years 2 and 3 did make twice as many erroneous judgements of *longer* than of *shorter*, but children in Year 1 showed no such bias. Overall, this explanation is not persuasive.

A second example of the influence of narrative was the construction of meaning around the lesser-known measuring instruments children were shown in the second interview, notably the clock and thermometer. The social function of the clock was so pervasive that its marking of important events (Haylock and Cockburn, 1989) in the 'narrative' of daily life took complete precedence over its recording of the passing of time and the visible units in which this was measured. The unfamiliar thermometer prompted in many children a network of associations reminiscent of Vygotsky's (1986) 'chain complexes' (forerunners by some distance of fully-developed concepts) in an effort to characterise its function. Thus its red ink was associated with blood; blood and the word 'temperature' were associated with illness and hospitals; while the colour red and the idea of rising heat, boosted by memories of a television cartoon, prompted the notion of an explosion.

It is possible that the cognitive importance of this type of narration by individuals has been underestimated. In the literature, where a tradition stemming from the work of Vygotsky has prioritised social over individual construction of meaning (e.g. Hoyles, Healy & Sutherland 1991; Noss & Hoyles 1992; Sfard 2001), the focus in classroom research has been on dialogic features of cognitive construction where partners in the process have typically been seen equally as *producers*. The potential represented by the *receptive* role of listener (likely to be an attentive adult) has been largely ignored. This role can be useful in two ways. Firstly, a narrative is directed at an audience and the listener provides that audience. The narrative situation requires an effort to engage, to convince, perhaps to justify a stance, all conducive to the exercise of reasoning skills, which have an important part to play in conceptual development. Secondly, such narrations are likely to give evidence of both social and conceptual components of the narrator's thinking, and did so in the current research. The narrations, (sometimes seen, as in the current research, in the context of an interview) offer an opportunity to the reflective listener to privilege the development of the conceptual components by skilful questions or prompts. The science education literature has provided many hundreds of examples of 'interviews about instances' and 'interviews about events', (Driver, Guesne & Tiberghien 1985; Driver, Squires, Rushworth & Wood-Robinson 1994) where children's explanations of science phenomena are elicited in individual interviews. However, the focus was on children's non-conventional science schemas, and the purpose was to identify commonalities among these. Little importance seems to have been attached to the social component of what children may have said, and the transcripts of some interviews show little attempt to understand what may have been of importance to children (e.g. Vosniadou & Brewer 1992). With the help of suitably trained adults both in and out of school, it seems feasible to plan one-to-one conversations where socially-embedded conceptualisations are accepted and valued but also refined so as to foster cognitive advance.

An influence that was 'social' in some sense but was hard to pin down concerned height and the human figure. In the first interview, when asked how they knew the researcher was taller than themselves, a majority in each year-group said that she must be because she was an adult or a 'teacher'; and when asked whether adults were *always* taller than children, only in Year 3 did an appreciable number (just over one-third) allow that there could be exceptions. It seemed that a 'status' criterion was involved in this judgement of height. Inconsistently, however, about half the children in each year-group said that they themselves would be taller if they stood on a table. A detailed analysis in Chapter 4 of children's comparative judgements of 'tallness' revealed that, among various complicated displays, only artificially raised human figures gave rise to erroneous judgements of greater height. It seemed that the notion of height in relation to human figures incorporated some metaphorical flavour that influenced the children's ordinal judgements.

Thorough attunement to the social context is indispensible to understanding the subtleties of language usage. Chapter 4 described in some detail potential conceptual difficulties that might be engendered by the everyday language of length, but these turned out not to be difficulties for the children concerned. Instead (with the exception in certain circumstances of human height, already mentioned) their ordinal judgements as responses to questions about height, length and width turned out to be excellent. Their mastery of the receptive language skills needed for correct responses was fully equal to the conceptual demands of the tasks, including the tests of conservation that constituted the 'complex' tasks in this series. What they had learned from their social context about the everyday language of length was entirely supportive of their conceptual development.

Thus there were two aspects to the sheer vitality of the social, as reflected in children's productive and receptive language in this research, reported respectively in Chapters 3 and 4. On the one hand measurement in the social context of the home could be highly motivating; it produced experiences of measurement that could form a sound basis for quantification with units in school; and it provided problem-solving opportunities as the basis for strategy development. Social aspects of school procedures encouraged an appetite for detail and accuracy. The known, everyday language of length, acquired in a variety of social contexts, was navigated with ease.

When children were confronted with the unknown, however, the picture was rather different. A drive to produce meaning took control, sometimes with positive effects for learning, and sometimes with negative. In one context, for example, it helped children to articulate their knowledge of fractional units (see Appendix 3), while in another it led them to decline the suggestion that thermometers measured anything, in favour of a view that the instrument indicated whether you were ill, or was to do with blood. But in neither case did the drive to create meaning seem a natural ally of the reflection necessary for conceptual understanding. The social requirement to communicate an effective narrative, of which the first interview gave numerous examples, was similarly likely to override reflection. In one way or another, it seemed, social resources stood ever-ready to fill the gaps in conceptual understanding, and it is the difficult task of pedagogy to help children to make a distinction between the two. It was suggested above that advantage could be taken of children's narrations to strengthen burgeoning conceptual elements within them. But the idea that teachers can build on children's informal experience in any straightforward way must be considered problematic.

8.2.5 Reflections on earlier research

In this section, perspectives in the literature reviewed in Chapter 1 are revisited.

The paradoxical role of a logico-mathematical framework

The first reflection is far from new, but it retains its importance because a bias it identifies is perennially attractive. That is, the high status of logico-mathematical theory seems to exert a kind of gravitational pull on the study of the psychological development of mathematical understanding. Clearly the mathematics involved in measurement must provide the *concepts* whose development is studied, but frequently the mathematics also supplies the types of *explanation* offered for experimental results, where no independent argument or evidence is offered for their validity, and where the psychological component is missing. Thus, Piaget (1965) explains a certain type of error in seriation tasks by stating that children have not yet integrated their separate understanding of ordinal and cardinal number. This integration also logically underlies ability to understand units. Yet in the 'towers' task, at a stage where such integration is supposed to be complete, it still does not occur to children to iterate a makeshift unit in order to measure. At this point Piaget must introduce a further layer of explanation: he contends that the continuous length to be measured is not experienced as comparable with separate units (Piaget *et al.*, 1960). It is not clear how either the first or the later explanation could be justified empirically, but the logical allure of such explanations can divert the researcher from a critique of their relevance.

A second and particularly important example is furnished by the definition of measurement adopted by the same author and woven into every aspect of his account:

> To measure is to take out of a whole one element, taken as a unit, and to transpose this unit on the remainder of a whole: measurement is therefore a synthesis of subdivision and change of position. (Piaget *et al.*, 1960:3)

The issue here is not that of relevance: this definition, after all, does capture the logical essence of iterating a unit in measurement. However, it also suggests that the unit remains part of the object measured, rather than separate but equivalent to part of its length. This is not obviously consistent with transitive inference as a fundamental feature of measurement, where the length of the object measured and the length embodied in the measuring device are logically distinct, and so available for comparison. Now this apparent contradiction can easily be dealt with within an overall logical framework. But the researcher accepting these conceptual guides can be left at a loss when their features are played out in concrete terms. In the present research, apparent failure to treat the line measured and the measuring device as independent embodiments of length led children to cover up the line with the device, hampering the whole measurement operation, and raising doubts about their understanding of transitive inference. Some also insisted that long objects could not be measured with 'short' rulers; measurement by iteration of the ruler as a unit or group of units apparently did not occur to them. Which of the two types of difficulty should be seen as more fundamental? Should such findings indeed be seen as entirely separable psychologically, and each type of finding be investigated as an independent phenomenon? More recent research has generally taken the latter approach, but it has led to a field containing many isolated studies.

Researchers who avoid acceptance of the Piagetian logical framework can still accept the plentiful a priori guidance it offers as to 'where to look' for fruitful areas of empirical enquiry. In the first example given above, discussion of the theoretical integration of ordinal with cardinal number suggests that children may indeed struggle to grasp how a number sequence marks amount on a measuring device. So it proved in the present research.

The value of surveys

Concerns about the insecurity of concepts underlying measurement, identified in surveys of mathematical performance, were mostly borne out by the present research. While children conserved length with ease in traditional conservation tasks (reported in Chapter 4), they failed to do so in the context of the measurement tasks, just as they had in practical contexts used in surveys (e.g. Department of Education and Science, 1981). The measurement tasks and the interviews in the present research also provided examples of apparent failure to understand measurement as iteration of a unit; of transitivity (Brown *et al.*, 1995); of the inverse relation between size and number of units (Brown *et al.*, 1995; Kouba *et al.*, 1988); of sub-division of units, and that larger units may be made up of smaller ones (Hiebert, 1981); and of general inaccuracy in measurement (Hiebert, 1981). Despite doubts about exactly what can be learned from them, therefore (Kamii and Lewis, 1991; Silver and Kenney, 1993), such surveys (like the Piagetian heritage) have demonstrated their value in suggesting to researchers 'where to look'.

Cultural tools

The general Vygotskian precept that learning is by means of 'cultural tools' has inspired a number of studies in measurement research (e.g. Nunes *et al.*, 1993; Miller, 1989) but there is little consistent evidence that conceptual understanding is advanced by the *prominence* of the tool within the given culture. Children performed better with a more familiar device (the ruler in the present research and in Nunes *et al.* (1993), but worse with the familiar algorithm for calculating area in the latter study, and with the tape measure in the current research. Moreover the degree of any beneficial effect varies with different samples: in the current research, though measuring with the ruler compared favourably with the cubes and tape, it was still very poor compared with that of the Nunes *et al.* sample. Miller notes that socially-induced 'functional fixedness' may inhibit the extending of any benefit to new contexts. Potential benefits of practices like 'unitising' using materials common in a given social environment (Lamon, 1994) remain speculative. It seems that in teaching about units, each device should be assessed for its pedagogic merits, by conceptual rather than social criteria, in a specific teaching context. The present research, together with a wealth of Piagetian studies (e.g Piaget *et al.*, 1960:110), demonstrate that salient aspects of the representation of units may distort conceptual content, and that what may turn out to be salient is not easy to predict. This seems to apply equally to conventional and experimental representations of units.

Learning in informal contexts

The present research investigated children's experience of informal contexts in which they learned about measurement, and expected evidence of "a constitutive role in learning for improvisation...and emergent processes" (Lave & Wenger, 1991: 16). There was indeed evidence of learning in informal settings, but this did not appear to favour conceptual development in any straightforward way, and indeed sometimes appeared to obstruct it. This was discussed in some detail earlier in this chapter. This finding, however, should be set beside other evidence that sound mathematical models are indeed developed in non-formal settings where a person's real livelihood is involved (Nunes, 2010).

Relative judgements

Early development of relative judgements, which were quite sophisticated in some domains (Feigenson *et al.*, 2004; Resnick 1992; Gibson 1979; Bryant 1974; Carpenter & Lewis, 1976;) was amply confirmed in the present research by the language and estimation results.

Units

Difficulties regarding units, of which the most fundamental were to do with the general relationship between larger and smaller units (Lamon, 1996; Nunes and Bryant, 1996; Davydov,1982; Carpenter & Lewis, 1976; Piaget *et al.*, 1960) were all reflected, in one way or another, in the findings of the present research.

Principles and procedures

Finally, the debate about the relationship between conceptual and procedural knowledge, and whether there is evidence that procedures promote conceptual learning runs through much of the research reviewed. Resnick & Singer (1993); Karmiloff-Smith (1992) and Piaget *et al.* (1960) all accord procedural knowledge a fundamental role in conceptual development. Gravemeijer *et al.* (2003) and Lamon (1994) considered child-created procedures to be particularly efficacious in promoting conceptual understanding. In contrast, Kornilaki & Nunes (2005); Miller (1989) and Carpenter & Lewis (1976) provided evidence that conceptual understanding might develop independently of procedures. The present research did not engage with this debate as such; however the findings did strongly suggest that *flawed* measuring procedures were the result of conceptual confusion.

8.2.6 The developmental and the instructional: summary

What classroom implications might, in general terms, be suggested by the results of the present research?

As stated in Chapter 1, the research was conducted from an individual constructivist perspective, because socio-cultural accounts seemed unable to explain how fundamental conceptual difficulties might be overcome at the individual level, except by what amounted to procedural mastery. This seemed an inadequate mechanism. On the other hand, while the influence of social factors is offered at its periphery, traditional individual constructivism gives a rather solipsistic account of cognitive development with little to say about the mechanisms of learning. Learning occurs, however, and (in a broad sense) always in a social context.

It was argued earlier in this chapter that the dominance of the social over the cognitive in children's everyday lives and language has not, on the evidence of the present research, always been an ally of cognitive advance. So the task for education (long acknowledged) is to harness dimensions of social experience and activity in and out of school that are potentially beneficial for learning. The following summarises indications in the present research as to how this might be done. It also identifies conceptual difficulties that may be developmental in character that are relatively intractable.

Such intractability, discussed in the first section of this chapter, was demonstrated in the present research by apparent failure to understand some fundamental conceptual underpinnings of measurement. These were: the inverse relation between size and number of units; the idea that larger units may 'contain' smaller ones; operational conservation; operational transitive inference; and how numbers may express amount. It seemed that for some children neither general developmental factors nor planned classroom experiences had conferred secure understanding. Previous research, reviewed in Chapter 1, confirms the intractability of difficulties in establishing these conceptual underpinnings. Chapters 4 and 5 demonstrated that where children *were* conceptually secure, they were so across many novel instantiations of the concepts that they understood. A tentative conclusion might be that where there *are* intractable conceptual problems, there is little point in attempting to overcome them indirectly by offering children a range of materials, or by teaching a range of activities, designed to induce understanding. Such instantiations may be used for their intrinsic interest and to consolidate understanding, but the emphasis should be on frequent, simple explanation of the principle involved, such as the fact (already mentioned) that a larger number does not necessarily mean a greater amount of anything. Such 'up front' explanations seem rather neglected, or deemed unnecessary; for example in the curricular objectives for measurement in the Primary Mathematics Framework, 2006, discussed in the General Introduction to this research.

To turn next to the correlations of performance on language, estimation and measurement tasks with age, reported in Chapter 7: there were no significant correlations with children's age *within any year-group*. In itself this does not suggest an important role for developmental factors in the abilities tested by these tasks. On the other hand there was (except for estimation), a modest, positive and significant correlation with age *across the sample as a whole*. Together with the analyses in Chapters 4, 5 and 6, which typically show significantly better performance in Year 3 than in Year 1, this suggests an effect of teaching, and perhaps of the children's everyday experience in other settings, rather than of development. The correlational evidence is far from conclusive, however, and it could just be that development is slow, and that is why only a weak relationship is discernible across a sample with this profile.

But there is certainly no evidence of strong developmental shifts in understanding, and there may be, therefore, good scope for effective instruction to make a difference.

In considering effective instruction, it is worth recalling that in Chapter 3, the current research showed school measurement activities to encourage precision, but to be detached from the purposes of measurement in the wider world, and probably rather boring. By contrast, measurement activities at home were in the service of interesting goals. As far as possible, this should be the rule in the classroom too.

Reference was made above to the harnessing of the social in the service of the cognitive. It was suggested earlier in this chapter that the importance of the *listener* to

children's narratives had been underestimated. The narratives considered were those about events of social importance to the children, where cognitive content was overridden by the purely social interest of the material. (Excerpts from such narratives were set out in Chapter 3). It was argued that a predominantly dialogic view of language interactions had neglected the role of the reflective listener, and how cognitively useful elements of children's narratives might be noted and encouraged by such a listener. This is an idea (and a practice) worthy of developing within the classroom and outside it.

Meanwhile the dialogic model is rapidly developing, to the benefit of classroom learning. Researchers are currently constructing and testing classroom interactions, particularly among peers, that may advance conceptual understanding and are pursuing theoretical understanding of how and why such interactions work, when they do.

8.2.7 Directions for future research

This research featured intensive work with three year-groups in a single school involved in the same experimental tasks. By these means, a very full picture of the range of the abilities of this sample in measurement and cognate skills was achieved, and investigation of possible associations among these was made possible. Experience with this repeated measures design, together with the results, provides clear indicators as to how work in the field could be taken forward.

It is safe to say that future research should focus on measurement itself. The poor measurement ability shown in the present results did not appear to be underlain by confusion about qualitative comparisons of length, either in language and conceptual terms, or in estimating length visually: children were very successful in both language and estimation studies. In the former, the one anomalous difficulty regarding the height of human beings was clearly enough defined not to require further investigation. In relation to the two younger age-groups, the ways in which language and estimation ability may mediate measurement ability for individuals, suggested by the correlational analysis, also deserve further study.

As far as measurement is concerned, we find in the children's apparent confusion about some of its basic conceptual aspects a clear agenda for further investigation. These basic aspects are both non-numerical and numerical, and some relate to defining characteristics of separable units and of scaled devices *as such*. This being the case, if aspects of the current design were retained, the cubes and the ruler respectively could serve as exemplars, and the tape measure be dropped. The number of lines measured could be reduced to a small set between, say, 11cm and 13 cm in length (the 'medium-length' set in the current research) since there was no significant difference in children's success between measuring much longer and much shorter lines. These reductions in treatment levels would substantially shorten the period of time needed for data collection, reducing the scope for intrusion of extraneous variables, and for the possibility of any training effect. In further investigations, the first job would be to determine whether the present findings regarding measurement are replicated with an enlarged sample that includes older children and children from a broader range of social and cultural backgrounds. Going up the present age-range rather than down is indicated because the correlational analysis showed performance on the measurement tasks for the oldest group of children to be independent of their language and estimation ability, so that a focus on measurement among, say, seven to nine-year-olds would be examining the nature of relatively settled concepts. Finally and ideally, a longitudinal study, retaining the strengths of the repeated measures design, would enable a greater focus on developmental questions.

Non-numerical problems of measurement

The length of an object and the units that are used to equal its length when it is measured are conceptually equivalent: without being *identical* with its length, the units *conserve* the length of the object being measured, and do so when they are physically separated from it. As discussed at the start of this chapter, some of the behaviours and statements of the children suggested that many did not wholly distinguish equivalence from identity, but seemed to be *identifying* the units with the length to be measured. If such behaviours and statements are replicated, their meaning should be pursued. The listener role referred to above in connection with narratives would be important here; some version of the 'think aloud technique' (van Someren, Barnard & Sandberg, 1994) could be employed.

Numerical problems of measurement

a) The inverse relation between size and number of units is a logical principle in measurement that is not essentially numerical in character, but apparent failure to understand it in the present research was always demonstrated in relation to visible numerals. Further research here does not seem to be necessary. The children's responses (in the second interview) were generally unambiguous; so were their justifications of them. The further work needed is work in the classroom where necessary, probably by direct explanation of this form of the 'more is more' fallacy (cf. Piaget *et al.,* 1960; Correa *et al.*, 1998).

b) The transition from qualitative to quantitative expressions of amount has been identified as a key transition for children (Resnick & Singer, 1993; Piaget *et al.* 1960;) When children were asked in the present research to say whether their measurement in units meant that the line was shorter, longer or the same length as a number of units given by the experimenter, no more than half the responses were correct. Many declined to make this relative judgement at all. It was suggested above that some had not fully worked out that the numbers they were considering actually expressed length. But one must tread warily. Children were very successful in making similar relative judgements, also involving number, *prior* to their own measurements (see Chapter 5). There seems to be great sensitivity here to details of the task, particularly, perhaps, to the exact language used by the experimenter. It had been hoped that by using the same experimental materials (ruler, cubes and tape) for both the relative and the absolute judgements involved, irrelevant influences would be removed, but this precaution proved insufficient. This is an important area for continued research, as children are expected to be able to estimate length in absolute terms very early on their school career (*Primary Mathematics Framework,* Department for Children, Schools and Families, 2006: Year 1 Block D. Assessment focus: Ma3, Measures). However, careful thought would need to be given to avoiding results that could be an artefact of the experimental situation and the language used.

c) Children's utter confusion about how to align a device, even the familiar ruler, when measuring length, seems best addressed jointly by research and pedagogy, beginning with investigation of how well children understand the basis of measurement as iteration of a single unit and proceeding, where necessary, to practice this form of measurement. Devices with simplified, initially un-numbered scales could then be gradually introduced, as suggested earlier. The routine use of fully conventional devices such as the ruler, replete with subdivisions, would, where children seemed at all insecure, be introduced to them with considerable care. The difficulties presented to children in the current research by the need to take account of fractional units and the extent to which (it was suspected) they took evasive action underlines the necessity of careful nurturing of basic understanding, quite possibly into late primary education. The project of Gravemeijer *et al.*, (2003), reviewed in Chapter 1, is an excellent example of the construction in the classroom of understanding of units.

8.2.8 Concluding remarks

The current findings raise some intriguing questions of a general character. Why does sound conceptual understanding of amount at the ordinal level fail to be generally supportive of measurement with units? How is it that successful performance across a wide variety of materials is the rule for ordinal comparisons, while the physical appearance of some of the same materials seems to be a decisive factor in failures of measurement itself? What exactly is the influence of specific social experiences of measurement on conceptual understanding, either at home or in school? It is hoped that future research will cast light on these questions.

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Appendices

Appendix 1

Materials used in the research

Figure A.01 Ruler

Figure A.02 Tape measure

Figure A.03 Cubes

Figure A.04 Thermometer

Figure A.05 Weighing machine

Figure A.06 Measuring jug

Figure A.07 Clock

Figure A.08 Marked and numbered strip

Figure A.09 Unmarked strips

Figure A.10 Toy figures

Figure A.11 'Steps' for raising toy figures

Figure A.12 Block for raising toy figures

Figure A.13 Toy cars

Figure A.14 2D figures. A: simple; B: complex. *Who is the tallest?*

Figure A.15 2D crayons. A: simple; B: complex. *Which is the widest?*

Figure A.16 Lines. A: simple; *Which line is shorter than the red one?* B: complex: *Which line is shorter than the pink one?*

Figure A.17 Lines whose lengths were estimated (red set).

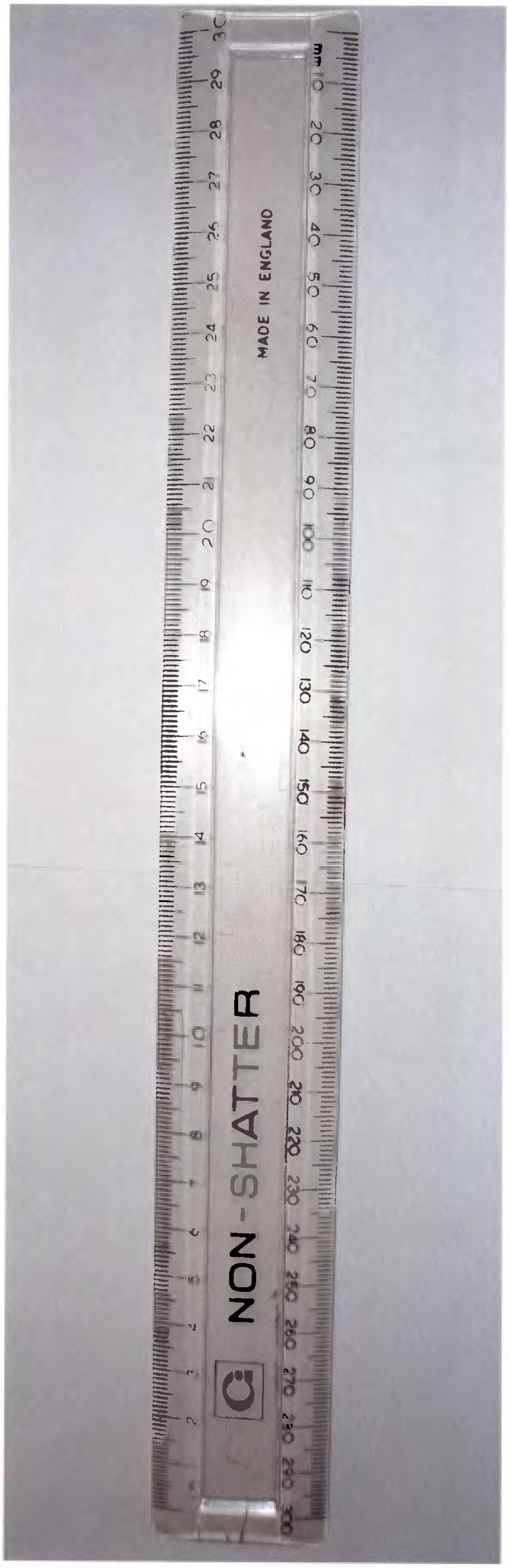

Figure A.01 Ruler

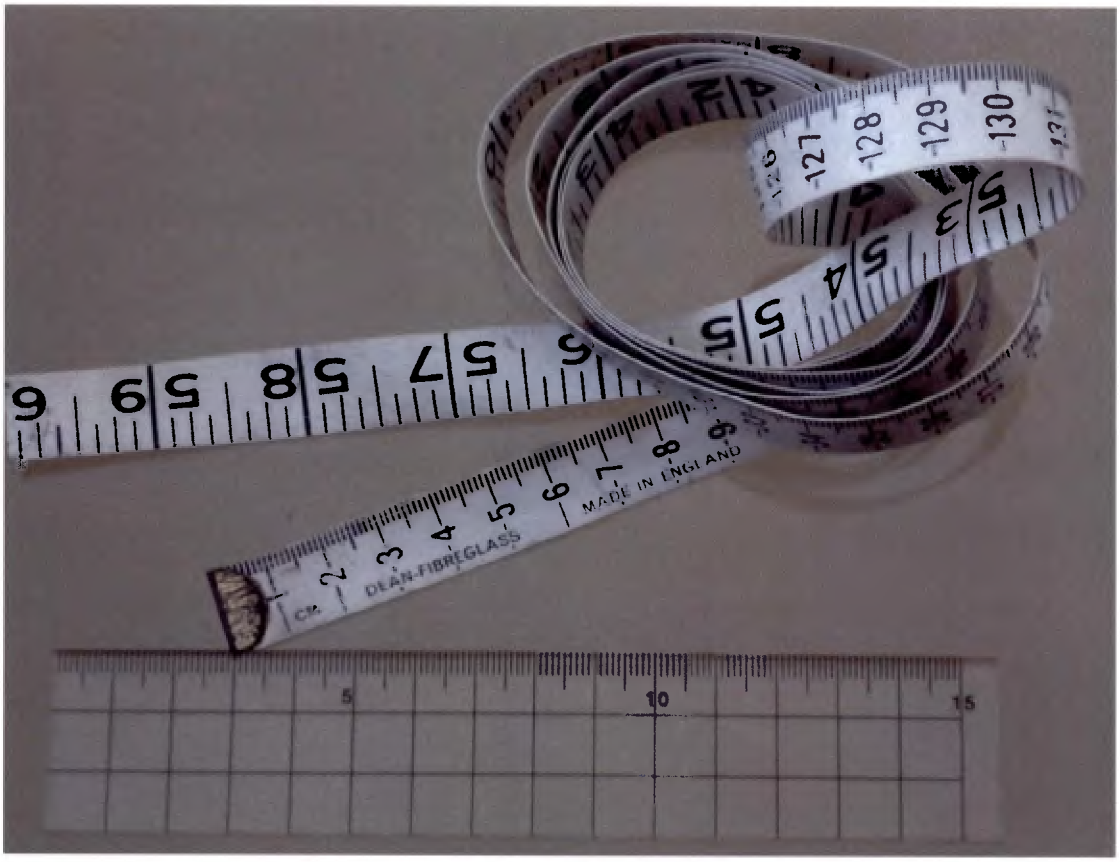

Figure A.02 Tape measure

Figure A.03 Cubes

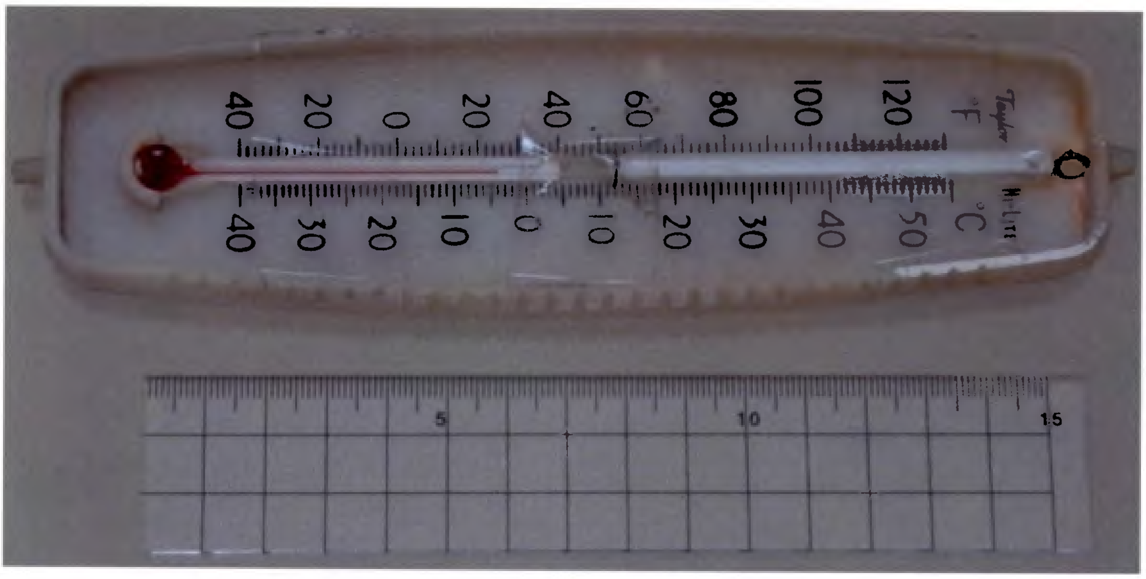

Figure A.04 Thermometer

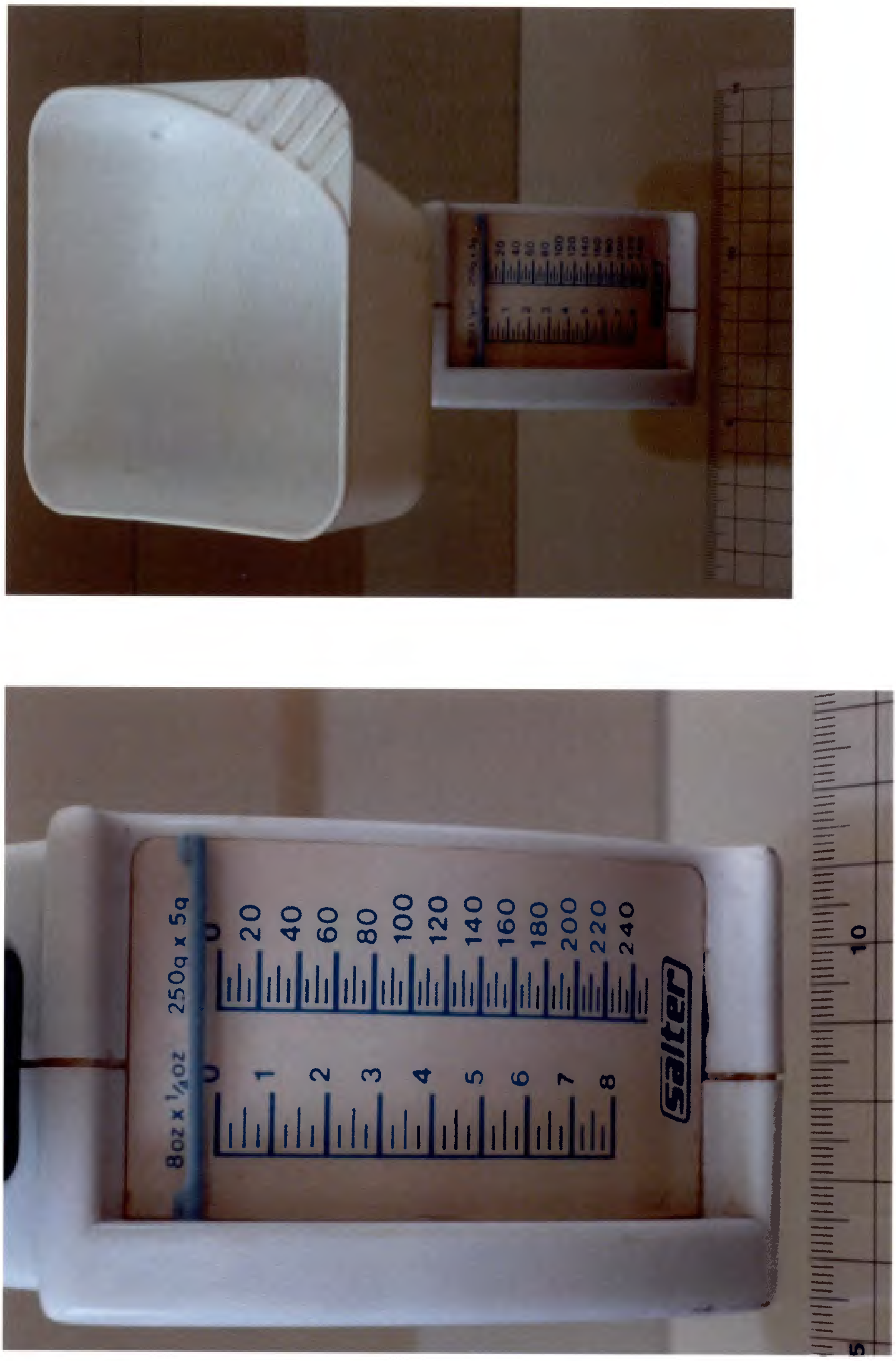

Figure A.05 Weighing machine

Figure A.06 Measuring jug

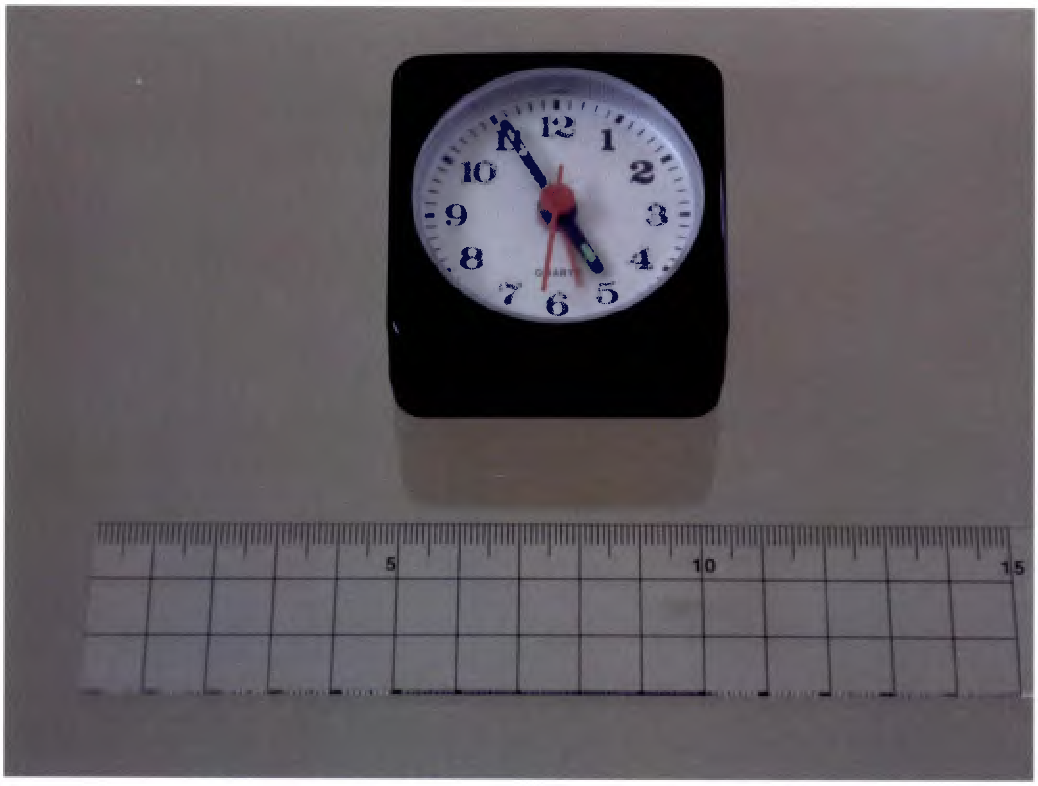

Figure A.07 Clock

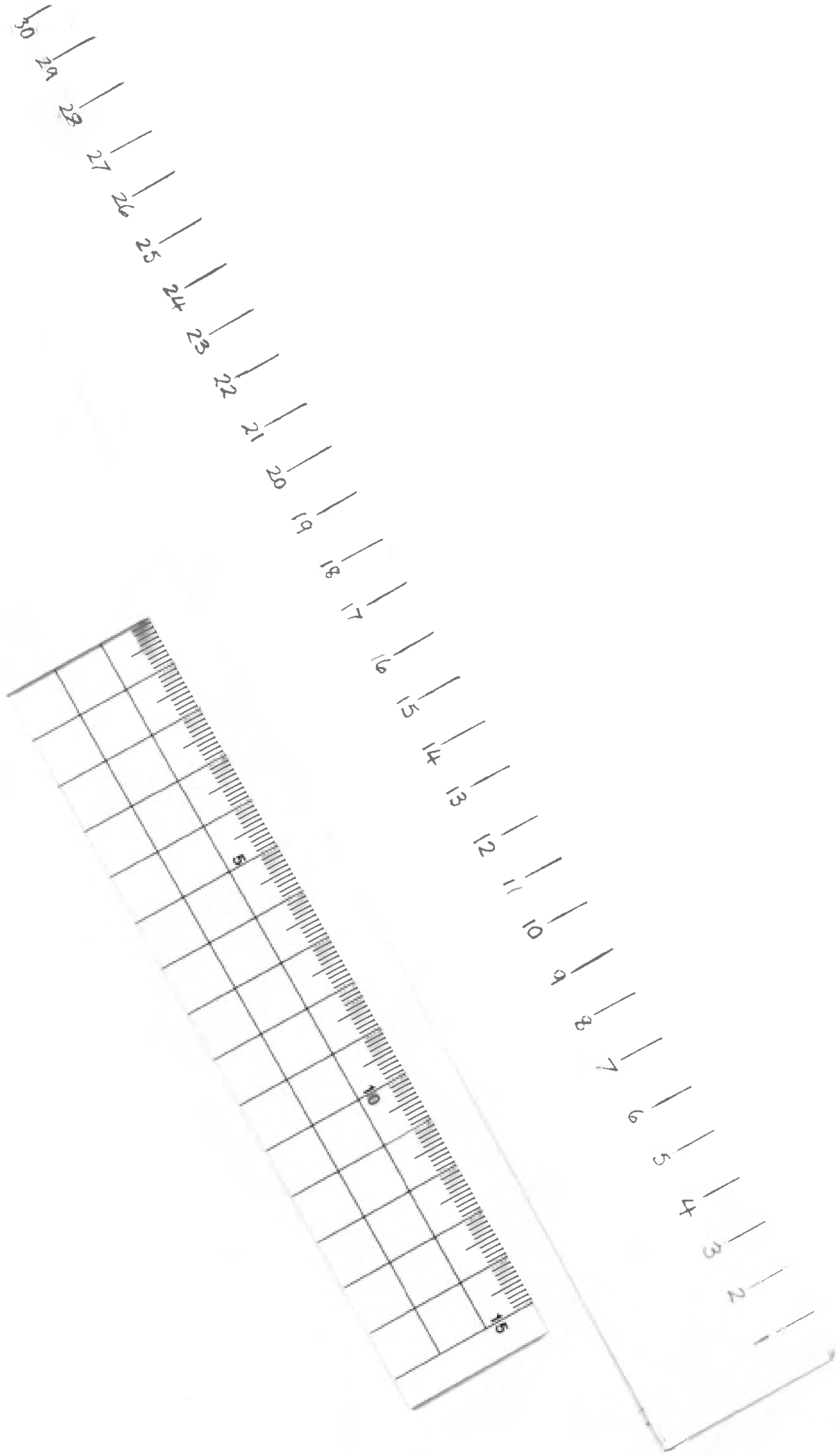

Figure A.08 Marked and numbered strip

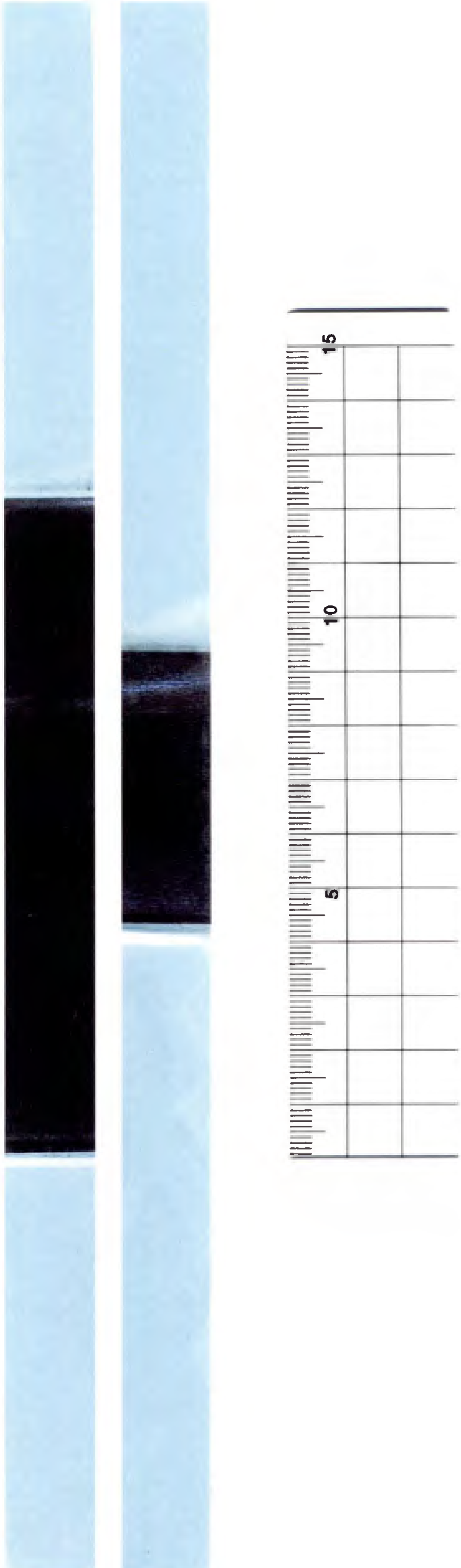

Figure A.09 Unmarked strips

Figure A.10 Toy figures

Figure A.11 'Steps' for raising toy figures

Figure A.12 Block for raising toy figures

Figure A.13 Toy cars

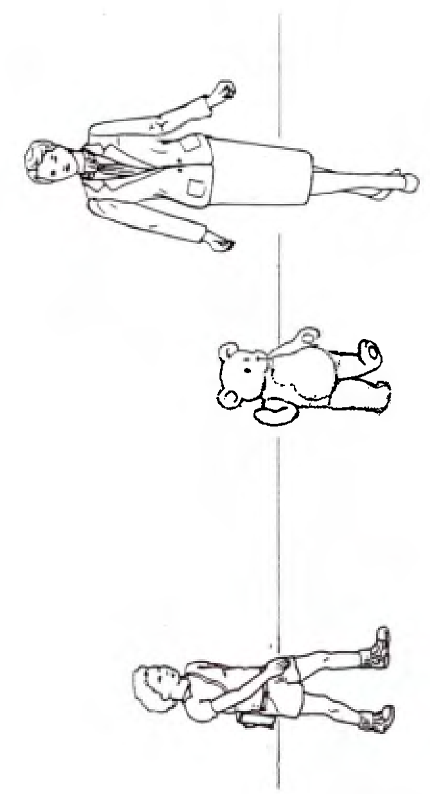

Figure A.14 (A) 2-D figures (simple) *Who is the tallest?*

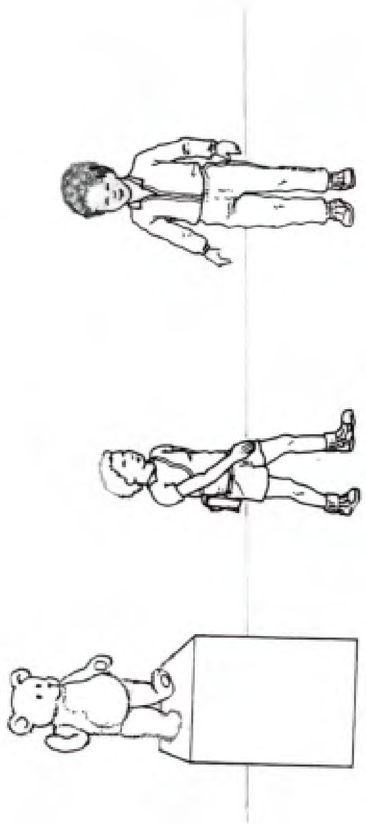

Figure A.14 (B) 2D figures. 2-D figures (complex) *Who is the tallest?*

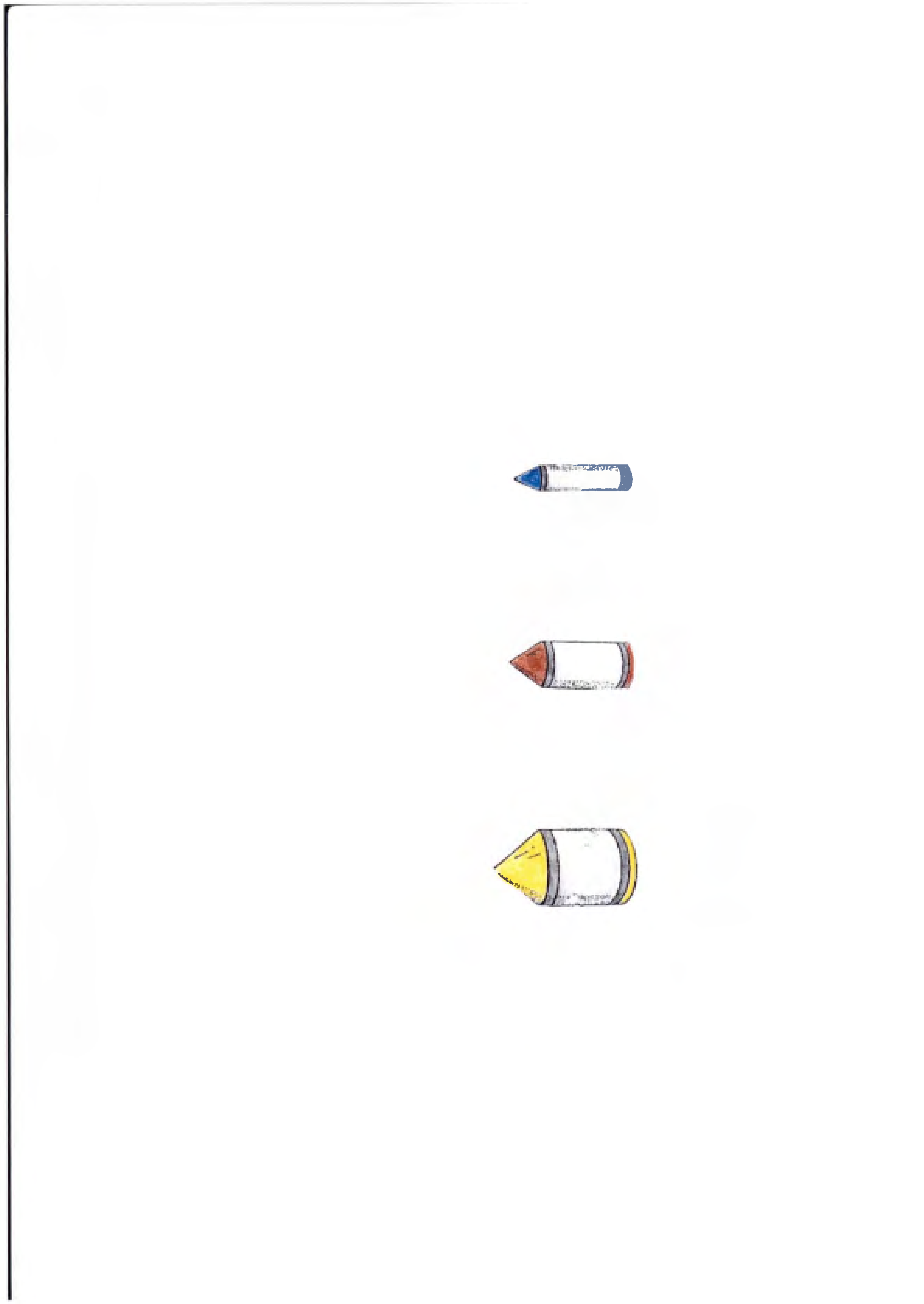

Figure A.15 (A) 2D crayons (simple) *Which is the widest?*

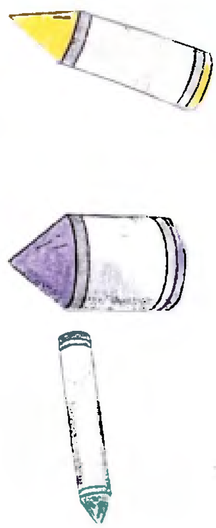

Figure A.15 (B) 2D crayons (complex). *Which is the widest?*

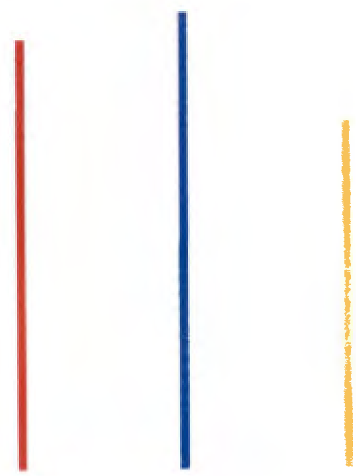

Figure A.16 (A): simple. *Which line is shorter than the red one?*

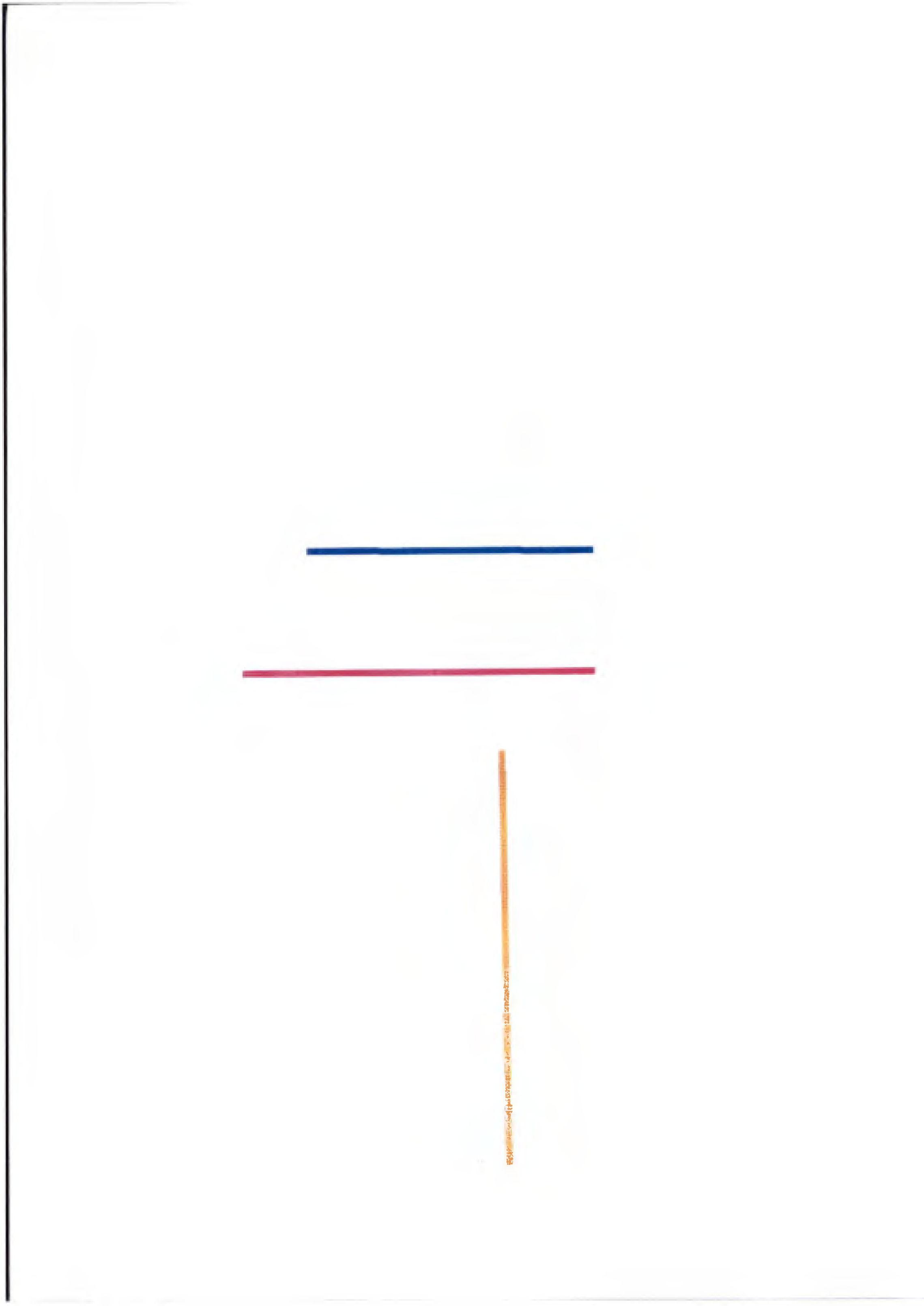

Figure A.16 (B): complex. *Which line is shorter than the pink one?*

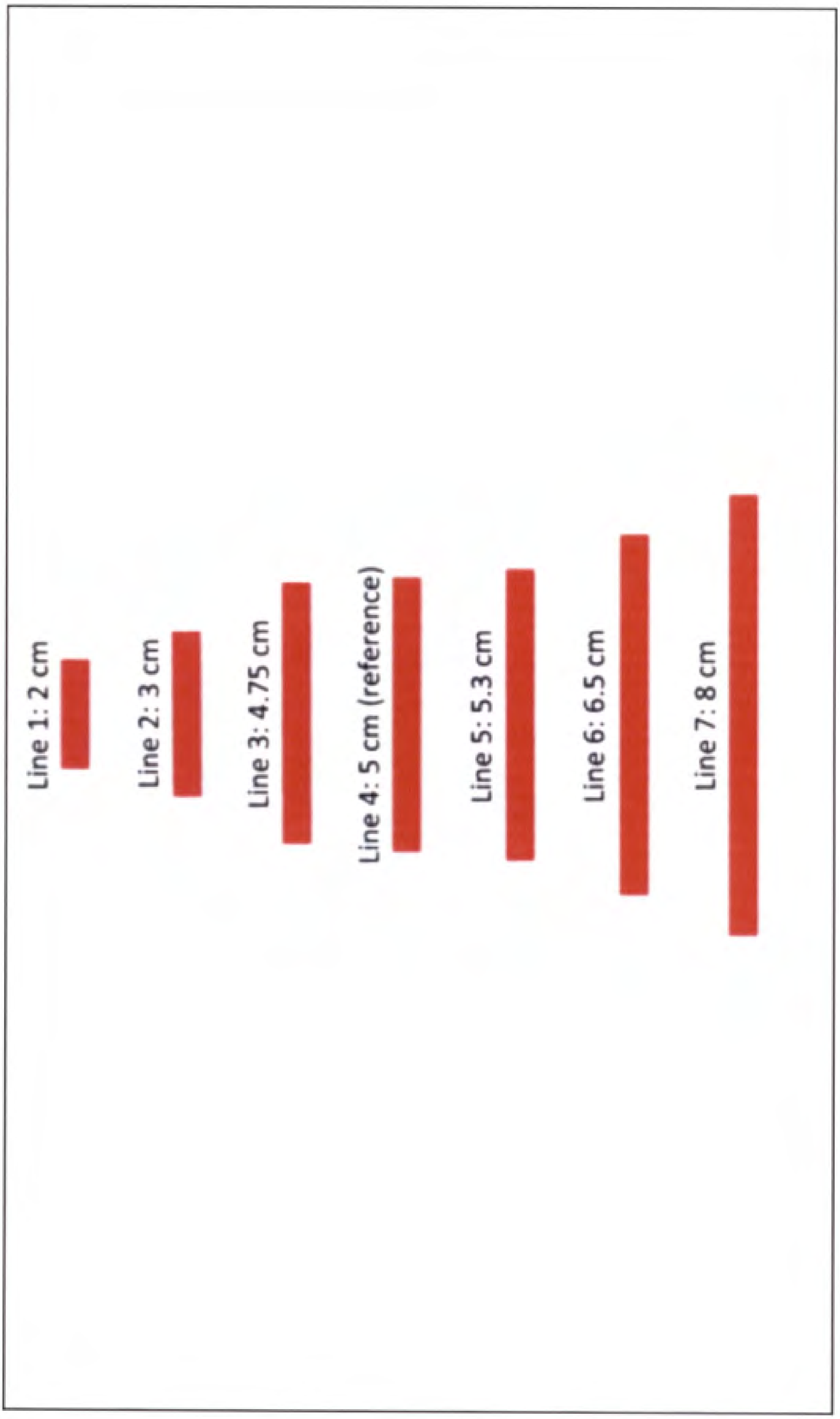

Figure A.17 Lines whose lengths were estimated (red set).

Appendix Two

Excerpt from the second interview with a Year 2 child, illustrating partial understanding of the relationship between the two scales on the weighing machine.

Experimenter: here you've got two different scales, two different ways of measuring, one there [indicates gram scale] and one there [indicates ounce scale]. Can you see the little g here? [indicates g scale].

Marie-Claire: Grams?

E: Exactly. This scale is measuring the weight in grams and this one, can you see that little 'oz'? It stands for ounces. [Explains oz in terms of parts of a lb (fruit or veg)]. It's a different way of measuring weight. So here we've got grams, here [indicates g scale] and here we've got ounces [indicates oz scale]. Do you want to weigh these? [toys]. [M-C places about 40g of toys in pan]. How much do they weigh?

M-C: Between the 40 and the 60.

E: And what about oz?

M-C: Between the 1 and the 2.

E: So which do you think weighs more, 1 gram or 1 oz?

M-C: Gram

E: Why?

M-C: It's cos they're – I'm looking at the numbers.

E: OK. Tell me about the numbers.

M-C: Cos 20's big, 40's big, 80's big, 100's big.

E: What about these? [indicates oz scale].

M-C: They're quite small for a number, but it's the same kind of thing to tell you some things.

E: When you say it's the same kind of thing even though the number's small, what do you mean?

M-C: It will still tell you the measurement even though it's different numbers. It won't tell you the same, but it's kind of the less of the same.

E: Can you explain that a bit?

M-C: You see when you see this bit here [indicates g scale] that's a bigger number than that bit [indicates oz scale]. Say if I put my hand there and it goes up to there – it goes up to the 4 [indicates 4 on the oz scale]. It goes up to 100 [on the g scale], and it goes up to the 4, and the 3, and 80 and 100, 120.... When it gets to 120 it will tell you what the grams are…and it'll tell you the same, not the same thing, but it will tell you [how]…heavy it is in oz.

E [*Removes scales*] If I weighed some flour in there, and it came to 2 oz, about how many grams do you think it would be?

M-C: 40?

E: And if I weighed some flour, and it came to 5 oz, about how many grams do you think it would be?

M-C: Can I use my hands? [Counts on fingers] 80?

Appendix 3

Some ways in which children reported fractional units when they measured lines.

	Child	Actual Measurement (cm)	Reported as
Year 1	1	12.5	A bit longer. To the middle line.
			A bit longer. Up to the big line near the 12.
	2	12.5	Halfway to 13
	3	21.3	21 and 3 steps to go
Year 2	1	12.7	A quarter to 13
	2	12.5	12 middle
		4.6	4 cm and 6 halves
		5.2	5 and 2 of them lines
	3	12.5	12 and 13
	4	13.5	14 and a little bit less
	5	13.6	4 more bits, it goes to 14
		12.6	4 more mm, it makes 13
		5.6	Half 6
	6	11.5	Half 11
	7	6.5	Half 6 cm
	8	12.5	120.5 mm [i.e would be .5 if in *cm*]
	9	5.2	About 1 more, a little chip from 5
	10	11.5	About 12 cm, 5 points before it
	11	18.75	Just a little bit to 19. 19 and a quarter.
Year 3	1	4.7	4 and a half and 2 mm
	2	5.3	5, 3. [i.e. 5cm and 3mm]
	3	11.5	Line in the middle from 12
	4	11.5	Half of 11
		11.5	Half of 12
	5	4.6	4 cm and 6 halves
	6	12.7	12 rounded up to 13
	7	18.5	5 points after 18, so 18 and a half