

Integrable delay-differential equations

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I, Bjorn Karl Berntson, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the work.

Abstract

Delay-differential equations are differential-difference equations in which the derivatives and shifts are taken with respect to the same variable. This thesis is concerned with these equations from the perspective of the theory of integrable systems, and more specifically, Painlevé equations. Both the classical Painlevé equations and their discrete analogues can be obtained as deautonomizations of equations solved by two-parameter families of elliptic functions. In analogy with this paradigm, we consider autonomous delay-differential equations solved by elliptic functions, delay-differential extensions of the Painlevé equations, and the interrelations between these classes of equations. We develop a method to identify delay-differential equations that admit families of elliptic solutions with at least two degrees of parametric freedom and apply it to two natural 16-parameter families of delay-differential equations. Some of the resulting equations are related to known models including the differential-difference sine-Gordon equation and the Volterra lattice; the corresponding new solutions to these and other equations are constructed in a number of examples. Other equations we have identified appear to be new.

Bäcklund transformations for the classical Painlevé equations provide a source of delay-differential Painlevé equations. These transformations were previously used to derive discrete Painlevé equations. We use similar methods to identify delay-differential equations with continuum limits to the first classical Painlevé equation. The equations we identify are solved by elliptic functions in particular limits corresponding to the autonomous limit of the classical first Painlevé equation.

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Chapter 1

Introduction

This thesis is concerned with nonlinear delay-differential equations from the perspective of integrable systems. Delay-differential equations have been widely studied from the perspective of dynamics and stability, but little is known about their integrability properties. On the other hand, the theory of integrable systems encompasses a broad range of concepts and methods, some of which apply naturally to the study of delay-differential equations. We emphasize that there is no universal definition of an integrable system. Rather, the term applies to systems in many different contexts that are in some sense exactly solvable. Integrability in particular settings is discussed in the subsequent chapter. In the context of delay-differential equations, it seems promising to pursue analogies with Painlevé equations.

The classical Painlevé equations are paradigmatic examples of integrable systems. The transcendents obeying these six nonlinear, nonautonomous, second-order ordinary differential equations can be viewed as generalizations of elliptic functions; appropriate autonomous limits of the Painlevé equations have elliptic functions as their general solutions. Similarly, discrete analogues of the classical Painlevé equations have been studied. Many such equations arise as deautonomizations of the symmetric Quispel-Roberts-Thompson (QRT) map, an 12-parameter difference equation whose general solution (up

to arbitrary periodic functions) is given in terms of the Jacobi sine function

$$u_n = \frac{\alpha \operatorname{sn}(\Omega n + z_0) + \beta}{\gamma \operatorname{sn}(\Omega n + z_0) + \delta} \quad (1.1)$$

for appropriate parameters: z_0 is free while the remaining parameters share a single degree of freedom. Integration of the symmetric QRT map is facilitated by a rational first integral $I = I(u_n, u_{n+1})$ whose numerator and denominator are quadratic in each of its arguments. Parameterization of the curve $I(u_n, u_{n+1}) = \text{const.}$ in terms of elliptic functions leads to (1.1).

In searching for delay-differential Painlevé equations, it would be convenient to have a class of autonomous equations solved by elliptic functions, i.e. an analogue of the symmetric QRT map. The main difficulty here is that there is no clear analogue of a first integral in the delay-differential setting. The symmetric QRT map arises from the condition $(E - 1)I(u_n, u_{n+1}) = 0$, where E is the shift map that evolves n to $n + 1$. If we promote n to a continuous variable, we could introduce an operator L , a linear combination of d/dn and $E - 1$, and consider equations $LI(u_n, u_{n+1}) = 0$. This integrates to $I(u_n, u_{n+1}) = \psi$, where $L\psi = 0$. Choosing ψ to be a constant, we see that the class of equations

$$F(u_n, u_{n+1}, u'_n, u'_{n+1}) + G(u_{n-1}, u_n, u_{n+1}) = 0 \quad (1.2)$$

(where $' = d/dn$) is solvable in terms of elliptic functions. Unfortunately, this asymmetric class does not contain any of the known delay-differential equations that admit elliptic solutions. For instance, the addition law for the Jacobi sine function (A.58a) is closely related to a Painlevé type equation identified in [68]:

$$au' = bu + u(\bar{u} - \underline{u}); \quad (1.3)$$

this equation is solved by the Jacobi sine function in a special case. However, neither of these equations are contained in (1.2), even after this class is generalized by a Möbius transformation of the dependent variable. Classes of equations corresponding to more general operators $L = f(E)d/dn + g(E)$ also

fail to contain these important examples.

In this thesis, we take a direct approach: we search for equations that admit elliptic solutions in the form (1.1) with at least two degrees of parametric freedom, in analogy with the autonomous differential equations that underlie the classical Painlevé equations and the symmetric QRT map. We will show that this ansatz contains all order-two elliptic functions: elliptic functions that take each value in the extended complex plane twice, counting multiplicity, within each period parallelogram. Our search is performed within two 16-parameter classes of delay-differential equations. The first of these consists of bi-Riccati equations that are simultaneously Riccati equations for the dependent variable and its upshift. This class has previously been studied in [32], where a number of equations with continuum limits to classical Painlevé equations were identified. The second class of equations involves a dependent variable and its downshift and upshift. While this class has not been studied before, it contains the examples described above and also includes traveling wave reductions of some known integrable differential-difference equations. Within these two classes, we classify all equations admitting order-two elliptic solutions and develop a method to identify a number of equations admitting multiparameter families of such solutions. Some of the equations we identify are related to known differential-difference equations. We exploit this connection to give new elliptic solutions to the Wadati lattice and Toda lattice equations. Other equations we identify appear to be new. Lastly, we show that Painlevé-type delay-differential equations can be obtained from particular Bäcklund transformations for the classical Painlevé equations. We identify two such equations and discuss their relationship to equations in our classification; in particular, we find their elliptic solutions in appropriate limits.

Chapter 2

Integrability in continuous and discrete systems

This chapter is concerned with several classes of nonlinear equations: ordinary differential, ordinary difference, partial differential, partial difference, and differential-difference. At the end of the chapter we discuss some basic properties of delay-differential equations, the main topic of this thesis. The term ‘integrability’ has been applied to each equation class we discuss, but given the fundamental differences between the classes, it is unsurprising that there is no universal definition of an integrable system. Even in a particular setting, it is often difficult to define integrability, and we will not attempt to do so. Instead, we will focus on some key structures typically associated with integrable systems. For our purposes, the singularity structure associated to an equation will be an important tool to isolate systems of interest. Those equations that admit elliptic function solutions are associated with particularly simple singularity structures.

The primary purpose of this chapter is to motivate the results reported in this thesis, which depend not only on features associated with integrable systems, but also on examples of well-known equations themselves. The Painlevé equations, Korteweg-de Vries equation, sine-Gordon equation, Volterra lattice, and Toda lattice will all be discussed. Transformations from these equations to themselves and other equations play an important role in establishing and

contextualizing results in subsequent chapters.

This introduction is organized as follows: single-variable equations are discussed first, followed by multivariable equations. Delay-differential equations and their place in the context of integrable systems are considered last.

2.1 Painlevé equations

In this section, we will discuss two classes of Painlevé equations: continuous and discrete. We first focus on the singularity analysis used to isolate the classical Painlevé equations before discussing some of the properties of these equations. We then move to the discrete case where we have opportunity to discuss the singularity confinement criterion and how this, together with the symmetric QRT map, leads to various discrete Painlevé equations.

2.1.1 Singularities in the complex plane

Here, we consider the singularities that can develop in solutions to ordinary differential equations (ODEs) in the complex plane. We restrict ourselves to ODEs of the form

$$\frac{d^n u}{dz^n} = f\left(z; u, \frac{du}{dz}, \dots, \frac{d^{n-1}u}{dz^{n-1}}\right), \quad (2.1)$$

where f is locally analytic—analytic with respect to each of its arguments in some common domain (a connected, open subset of the complex plane). In this case the ODE is locally represented as a first-order system

$$\frac{du_i}{dz} = f_i(z; u_1, \dots, u_n), \quad i = 1, \dots, n, \quad (2.2)$$

to which Cauchy's existence and uniqueness theorem applies.

Theorem 2.1.1 (Cauchy). *Suppose f_1, \dots, f_n are analytic functions on a domain $\Omega \subset \mathbb{C}^{n+1}$ and $(z_0, u_{1,0}, \dots, u_{n,0}) \in \Omega$. Then the system (2.2), together with the initial conditions*

$$u_i(z_0) = u_{i,0}, \quad i = 1, \dots, n, \quad (2.3)$$

admits a unique analytic solution in some neighborhood of z_0 .

In the standard proof of the theorem [44], the neighborhood of solution analyticity is constructed as a disk $D \subset \mathbb{C}$ centered at z_0 . Unsurprisingly, this disk is contained within $\Omega_1 \subset \mathbb{C}$, the domain of analyticity of f with respect to z .¹ Solutions are not guaranteed to be analytic outside of D , in fact singularities of two qualitatively distinct origins may arise. Anywhere outside of D , solutions may develop singularities whose locations depend on the initial conditions; such singularities are *movable*. Movable singularities arise due to singularities of f that involve the values (including the point at infinity) of u or its derivatives (i.e. are singularities whose location is not determined by the value of z alone). Outside of $\Omega_1 \supset D$ solutions possess *fixed* singularities wherever f is singular due only to the value of z . The concepts here are well-illustrated by an example due to Filipuk and Halburd [26]. In the first-order ODE

$$u' = \frac{u - u^3}{2(z + 1)} := f(z; u), \quad (2.4)$$

the function f is singular at $z = -1$, so we expect that any solution is also singular at this point. f is also singular when $u = \infty$, but this provides no information about where in the z -plane the corresponding solution singularity occurs. To determine the values of z for which $u = \infty$, we need to know the (particular) solution, which is in turn determined by the initial data. If we adjoin the initial condition $u(0) = u_0 \notin \{0, \pm 1\}$ to (2.4), a unique, analytic solution is given by

$$u(z) = u_0 \sqrt{\frac{1+z}{1+u_0^2 z}} \quad (2.5)$$

within the origin-centered open disk of radius $\min\{1, |u_0|^{-2}\}$. Outside of this disk we find, as expected, a singularity at $z = -1$. This singularity is an algebraic branch point. There is a second singularity (another algebraic branch

¹It is more convenient to discuss the singularity structure of f in (2.1) rather than the singularity structure of the f_i appearing in (2.2). The two are equivalent under the first assumption of Cauchy's theorem. Suppose we have a system (2.2); by assumption, the system can be put into the form (2.1), where f is analytic on Ω .

point) at $z = -u_0^{-2}$. This existence of this movable singularity was predicted by the behavior of f when $u = \infty$. It is interesting to consider the case when the initial data $u(z_0) = u_0 \in \{0, \pm 1\}$ is imposed on (2.4). In this case the solutions are analytic (in fact constant) on $\mathbb{C} \setminus \{-1\}$ and the singularity at $z = -1$ is removable. These solutions have no movable singularities, a fact that can actually be predicted from the ODE itself: if the domain of f is restricted to finite values of u , there is a single singular point at $z = -1$. Put another way, bounded solutions of (2.4) have only a fixed singularity at $z = -1$.

It is worth remarking that linear equations have only fixed singularities. For such an equation, (2.1) reduces to

$$\frac{d^n u}{dz^n} = a_0(z)u + a_1(z)\frac{du}{dz} + \cdots + a_{n-1}(z)\frac{d^{n-1}u}{dz^{n-1}}, \quad (2.6)$$

where the a_i are functions of z . The RHS of (2.6) is singular when either one of the coefficient functions is singular or when u is singular. By restricting to analytic coefficient functions and finite initial data, the following result [38, 44] may be established by direct construction of the solution.

Theorem 2.1.2. *Suppose a_0, \dots, a_{n-1} are analytic functions on a simply connected domain $\Omega \subset \mathbb{C}$. Then the system (2.6), together with the initial conditions*

$$u^{(i)}(z_0) = u_i \in \mathbb{C}, \quad i = 0, \dots, n-1, \quad (2.7)$$

admits a unique analytic solution in Ω .

We now return to nonlinear equations, again taking (2.4) as an example. We observe again that the solution (2.5) possesses movable algebraic branch points. It is natural to ask what other kinds of singularities can develop in solutions to equations in the same class. The question was first posed and investigated by Picard [65], thus initiating the study and classification of nonlinear ODEs on the basis of their singularity structures.

2.1.2 Painlevé-Gambier-Fuchs classification

The school Painlevé, Gambier, and L Fuchs studied nonlinear second-order ODEs from the perspective proposed by Picard. We first discuss the corresponding analysis of first-order equations.

2.1.2.1 First-order equations

We discuss the class of first-order rational equations

$$u' = R(z; u), \quad (2.8)$$

where R is rational in u with coefficients analytic in z . It turns out that for this class, the possible kinds of movable singularities are very restricted. For a more general class² of equations, Painlevé established the following.

Theorem 2.1.3 (Painlevé [63]). *Suppose the function $P(z; u, u')$ is polynomial in u and u' with coefficients analytic in z on some domain Ω . On Ω , any movable singularity of a solution to the equation*

$$P(z; u, u') = 0 \quad (2.9)$$

is either a pole or an algebraic branch point.

It is then natural to ask which equations admit only solutions that are singled-valued about their movable singularities, i.e. the only movable singularities are poles. It turns out there is only one such equation within the class (2.8).

Theorem 2.1.4 (Picard [65]). *Suppose that $R(z; u)$ is rational in u with coefficients analytic in z in some domain Ω . If the solutions of (2.8) have no movable algebraic branch points on Ω , then*

$$R(z; u) = a_0(z) + a_1(z)u + a_2(z)u^2, \quad (2.10)$$

²First-order rational equations are a subclass of the polynomial class (2.9), obtained when P is affine in its third argument.

where the coefficient functions a_0 , a_1 , and a_2 are analytic on Ω .

The equation identified by Picard is called a Riccati equation. A general Riccati equation has the form

$$u' = a_0(z) + a_1(z)u + a_2(z)u^2, \quad (2.11)$$

where a_0 , a_1 , and a_2 are locally analytic and a_2 is not identically zero (i.e. the equation is nonlinear). Riccati equations are linearizable: they can be solved in terms of linear, homogenous second-order ODEs. The change of variables

$$u = -\frac{v'}{a_2v} \quad (2.12)$$

in (2.11) leads to

$$v'' = \left(a_1 + \frac{a_2'}{a_2}\right)v' - a_0a_2v. \quad (2.13)$$

In these variables, it is easy to see why solutions of (2.11) have no algebraic branch points. As the equation for v is linear, v has no movable singularities. Therefore the only singularities of u arise from the transformation (2.12); u has a pole at any point where v has a zero. The zeroes of v depend in general on the initial conditions imposed on (2.13), so they give rise to movable poles of u .

Riccati equations also possess a nonlinear superposition principle. In particular, if we have four particular solutions u_1 , u_2 , u_3 , and u_4 to (2.11), their cross-ratio is constant:

$$\frac{(u_1 - u_2)(u_3 - u_4)}{(u_1 - u_3)(u_2 - u_4)} = \text{const.} \quad (2.14)$$

This relation can be verified by differentiating with respect to z and using the Riccati equation to eliminate derivatives. The superposition formula can be inverted for u_4 to obtain the general solution to (2.11) when three particular solutions are known. In the Riccati equation, we have seen how analysis of singularity structure leads to equations with special properties. We will see

the phenomenon again in second-order equations, to which we now turn.

2.1.2.2 Second-order equations

Here, we restrict ourselves to second-order rational equations

$$u'' = R(z; u, u'), \quad (2.15)$$

where R is rational in u and u' and locally analytic in z . The singularity structure for these equations is more complicated than in the first-order case (2.8). Solutions of second-order rational equations may develop movable singularities that are logarithmic branch points, transcendental branch points, or essential singularities, in addition to the possible singularities of first-order rational equations. A number of examples are given in [1].

As before, it is natural to look for equations in the class (2.8) with simple singularity structure. Equations with the same singularity structure as a Riccati equation are said to possess the Painlevé property.

Definition 2.1.1 (Painlevé property). *An ordinary differential equation has the Painlevé property if each movable singularity of each solution is a pole.*

It should be noted that some authors use a weaker definition for the Painlevé property, requiring only that each solution is single-valued about each movable singularity. This definition allows for movable essential points in addition to poles. The difference between these two definitions will not be significant in this thesis.

The Riccati equation is the only first-order rational ODE of the form (2.8) with the Painlevé property.³ The situation for rational second-order equations is more intricate. Here, there are fifty equations with the Painlevé property modulo the Möbius transformations

$$v(\zeta) = \frac{\alpha(z)u + \beta(z)}{\gamma(z)u + \delta(z)}, \quad \zeta = \zeta(z), \quad (2.16)$$

³The Weierstrass equation $(u')^2 = 4u^3 - g_2u - g_3$ possesses the Painlevé property, but is not contained in the class (2.8).

where $\alpha, \beta, \gamma, \delta$, and ζ are analytic and $\alpha\delta - \beta\gamma \neq 0$. The work of Painlevé [64], Gambier [30], and R. Fuchs [29] shows that six of these equations, called Painlevé equations, define new transcendents; the remaining 44 equations can be solved in terms of these new transcendents and known transcendents: elliptic functions and the solutions of linear ODEs and elliptic functions. The six Painlevé equations are:

$$P_I: \quad u'' = 6u^2 + z \quad (2.17a)$$

$$P_{II}: \quad u'' = 2u^3 + zu + a \quad (2.17b)$$

$$P_{III}: \quad u'' = \frac{1}{u}(u')^2 - \frac{1}{z}u' + \frac{au^2 + b}{z} + cu^3 + \frac{d}{u} \quad (2.17c)$$

$$P_{IV}: \quad u'' = \frac{1}{2u}(u')^2 + \frac{3}{2}u^3 + 4zu^2 + 2(z^2 - a)u + \frac{b}{u} \quad (2.17d)$$

$$P_V: \quad u'' = \left(\frac{1}{2u} + \frac{1}{u-1}\right)(u')^2 - \frac{1}{z}u' + \frac{(u-1)^2}{z^2} \left(au + \frac{b}{u}\right) + \frac{cu}{z} + \frac{du(u+1)}{u-1} \quad (2.17e)$$

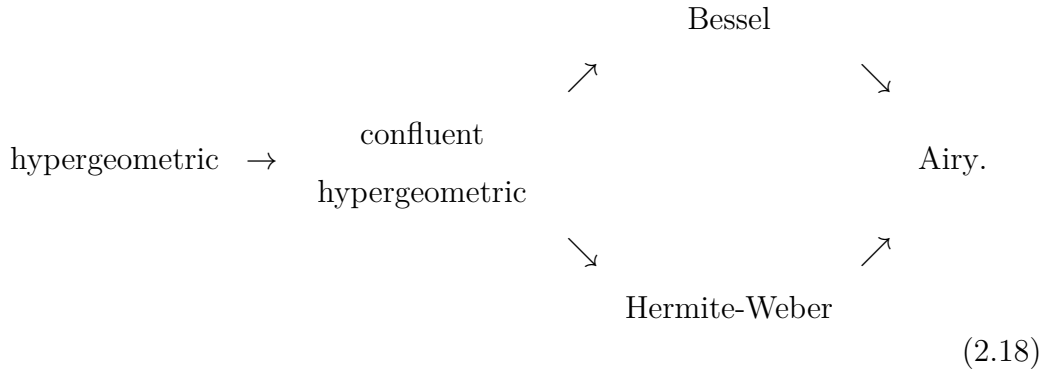
$$P_{VI}: \quad u'' = \frac{1}{2} \left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-z}\right)(u')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{u-z}\right)u' + \frac{u(u-1)(u-z)}{z^2(z-1)^2} \left[a + \frac{bz}{u^2} + \frac{c(z-1)}{(u-1)^2} + \frac{dz(z-1)}{(u-z)^2} \right]. \quad (2.17f)$$

Here, a, b, c , and d are arbitrary complex parameters. The Painlevé equations enjoy a number of remarkable properties; a review of these is given in [14]. Here, we will discuss their classical solutions, relationship to elliptic functions, and Bäcklund transformations.

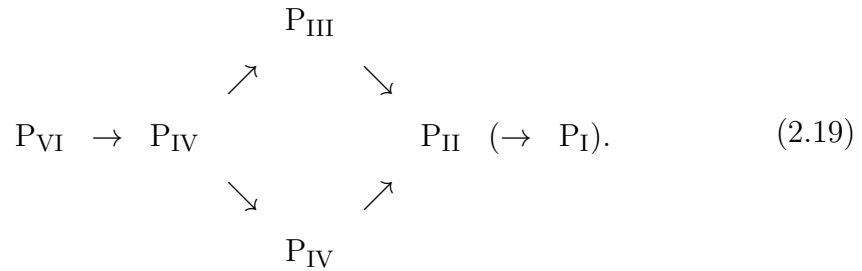
2.1.2.3 Classical solutions and degenerations

For particular parameters values, P_{II} - P_V (2.17b-2.17f) admit solutions in terms of classical special functions. Conversely, P_I (2.17a) admits no solutions in terms of known transcendents. The proofs of this result [60, 78] are technical, making use of differential Galois theory, and will not be repeated here.

We recall that the hypergeometric equation degenerates into a number of simpler linear equations, according to the following diagram:



Very similarly, the Painlevé equations $P_I - P_V$ are obtained from P_{VI} through successive degeneration:



Beyond this formal relationship, each Painlevé equation (besides P_I , which has no counterpart in the first diagram) possesses special solutions in terms of the corresponding linear special function appearing in the first diagram. The precise formulae for the degenerations in the above diagrams and special solutions of the Painlevé equations are given in [20].

2.1.2.4 Autonomous limits of Painlevé equations

The Painlevé equations are closely related to differential equations that describe elliptic functions. If any explicit dependence on z in (2.17) is replaced by a constant, the resulting equation can be integrated in terms of elliptic functions. Let us illustrate this phenomenon for P_I (2.17a). If we make the replacement $z \rightarrow p = \text{const.}$, we obtain the *autonomous limit*

$$u'' = 6u^2 + p. \tag{2.20}$$

Integrating this once and comparing with (A.8), we find the general solution

$$\wp(z + z_0; -2p, g_3). \quad (2.21)$$

In a less straightforward way, elliptic solutions may be constructed for the autonomous limits of the remaining Painlevé equations; further details are given in [44]. The asymptotics of the first two Painlevé equations have been shown to be similarly related to elliptic functions [49].

2.1.2.5 Bäcklund transformations

Each of the Painlevé equations besides the first involves at least one arbitrary complex parameter. An auto-Bäcklund transformation maps solutions of one of the Painlevé equations to a solution of the same equation with different parameters. More generally, a Bäcklund transformation may relate two distinct Painlevé transcendents, but we will not find use for such transformations in this thesis.

We will now give two examples that will find application later in the thesis. When $c = 0$ and $a = d = -1$, P_{III} (2.17c) reduces to

$$u'' = \frac{1}{u} (u')^2 - \frac{1}{z} u' + \frac{b - u^2}{z} - \frac{1}{u}. \quad (2.22)$$

Suppose now that $u(z; b)$ solves (2.22). Then we have the following solutions

$$u(z; b+2) = \frac{x[1 + u'(x; b)]}{u(z; b)^2} - \frac{b+1}{u(z; b)} \quad (2.23a)$$

$$u(z; b-2) = \frac{x[1 - u'(z; b)]}{u(z; b)^2} - \frac{b-1}{u(z; b)} \quad (2.23b)$$

for (2.22) when b is replaced by $b+2$ and $b-2$, respectively. Both (2.23a) and (2.23b) are auto-Bäcklund transformations for this special case of P_{III}. Let us consider another special case of P_{III} (2.17c) with $c = -d = 1$:

$$u'' = \frac{1}{u} (u')^2 - \frac{1}{z} u' + \frac{au^2 + b}{z} + u^3 - \frac{1}{u}. \quad (2.24)$$

If we denote solutions to this equation by $u(z; a, b)$, we have the following auto-Bäcklund transformations:

$$u(z; -a, -b) = -u(z; a, b) \quad (2.25a)$$

$$u(z; -b, -a) = \frac{1}{u(z; a, b)} \quad (2.25b)$$

$$u(z; -b-2, -a-2) = u(z; a, b) + \frac{(2+a+b)u(z; a, b)^2}{z[u'(z; a, b) + u(z; a, b)^2 + 1] - (1+b)u(z; a, b)}. \quad (2.25c)$$

2.1.3 Painlevé test

If an ODE has the Painlevé property, each solution must be locally described by a Laurent series containing a number of integration constants corresponding to the order of the equation. The Painlevé test is a sequence of steps to determine if an equation satisfies these necessary for the equation to have the Painlevé property. The test essentially consists of substitution of the series

$$u = \sum_{n=0}^{\infty} u_n (z - z_0)^{n+p} \quad (2.26)$$

into the equation; about any singular point z_0 we must verify that p is integral and that there is sufficient freedom in the expansion coefficients so that arbitrary initial data may be accounted for.

The philosophy behind the Painlevé test is that local singularity analysis can check for strong necessary conditions for a given equation to possess the Painlevé property. In particular, algebraic branching is easily detected through this analysis. Extensions of the method to detect logarithmic branching have been considered [66].

As an example, we perform the standard Painlevé test on P_I (2.17a). Possible values of p are determined by the leading order ansatz $u \sim u_0(z - z_0)^p$. We find that

$$u'' \sim p(p-1)u_0(z - z_0)^{p-2}, \quad u^2 \sim u_0^2(z - z_0)^{2p}. \quad (2.27)$$

Dominant balance occurs only when $p = -2$; by substitution into (2.17a) we find that $u_0 = 1$. We now consider the full Laurent expansion about the singular point z_0 :

$$u = \sum_{n=0}^{\infty} u_n (z - z_0)^{n-2} \quad (2.28a)$$

$$u'' = \frac{6u_0}{(z - z_0)^4} + \frac{2u_1}{(z - z_0)^3} + \sum_{n=0}^{\infty} u_n (z - z_0)^{n-4} \quad (2.28b)$$

$$u^2 = \sum_{n=0}^{\infty} \sum_{m=0}^n u_n u_{n-m} (z - z_0)^{n-4}. \quad (2.28c)$$

Given the initial value $u_0 = 1$ from the leading order analysis, the differential equation (2.17a) is used to generate a sequence of recurrence relations. We look for values of n where the expansion coefficient u_n is arbitrary. The initial iterates are calculated to be

$$u_0 = 1, \quad u_1 = u_2 = u_3 = 0, \quad u_4 = -\frac{z_0}{10}, \quad u_5 = -\frac{1}{6}. \quad (2.29)$$

At $n = 6$ we obtain the relation

$$u_3^2 + 2u_2u_4 + 2u_1u_5 + 2u_6(u_0 - 1) = 0, \quad (2.30)$$

so that u_6 is an arbitrary constant. The expansion also contains the arbitrary location of the pole, z_0 . No further arbitrary constants arise from the recurrence relations: for $n \geq 4$ we have the relations

$$(n+1)(n-6)u_n = \sum_{m=1}^{n-1} u_m u_{n-m} \quad (2.31)$$

and the polynomial multiplying u_n does not vanish for $n > 6$. We conclude that P_I (2.17a) passes the Painlevé test as a single-valued expression with two degrees of freedom:

$$u = \frac{1}{(z - z_0)^2} - \frac{z_0}{10}(z - z_0)^2 - \frac{1}{6}(z - z_0)^3 + u_6(z - z_0)^4 + \dots, \quad (2.32)$$

can be generated about z_0 .

Let us consider the polynomial in n that appears on the LHS of (2.31). This is called a resonance polynomial, and its roots are the locations of arbitrary coefficients in the expansion (2.28a). One root is $n = 6$, the location of the arbitrary coefficient u_6 . The other root, $n = -1$, is called the universal resonance and corresponds to z_0 , the arbitrary location of the pole. The universal resonance is found in any expansion with a leading order pole. To see this, we perturb z_0 in (2.26) by a small value ϵ satisfying $|\epsilon| < |z - z_0|$ and perform a binomial expansion. At leading order we now have $u \sim p\epsilon u_0(z - z_0)^{p-1}$; the perturbation introduces a pole of order $1 - p$ at a position which corresponds to $n = -1$ in the expansion.

2.1.4 Quispel-Roberts-Thompson map

We now begin our discussion of discrete systems. Much like how differential equations for elliptic functions underlie the integrability of the Painlevé equations, a very general autonomous difference equation underlies the integrability of discrete Painlevé equations. We will first discuss this equation before showing how it, together with the notion of *singularity confinement*, is related to discrete analogues of the Painlevé equations.

Reductions of integrable differential-difference equations to pure difference equations were studied in [69, 70], where it was noted that all reductions were contained in a particular 18-parameter class of mappings. We construct this class as follows. Starting with two arbitrary matrices $A_0, A_1 \in \mathbb{C}^{3 \times 3}$, we define two vectors of functions:

$$f(v_n) = {}^t(f_1(v_n), f_2(v_n), f_3(v_n)) = (A_0 V_n) \wedge (A_1 V_n), \quad {}^t V_n = (v_n^2, v_n, 1) \quad (2.33a)$$

$$g(u_n) = {}^t(g_1(u_n), g_2(u_n), g_3(u_n)) = ({}^t A_0 U_n) \wedge ({}^t A_1 U_n), \quad {}^t U_n = (u_n^2, u_n, 1). \quad (2.33b)$$

Then, the system of equations

$$u_{n+1} = \frac{f_1(v_n) - u_n f_2(v_n)}{f_2(u_n) - u_n f_3(v_n)} \quad (2.34a)$$

$$v_{n+1} = \frac{g_1(u_{n+1}) - v_n g_2(u_{n+1})}{g_2(u_{n+1}) - v_n f_3(u_{n+1})} \quad (2.34b)$$

is called the Quispel-Roberts-Thompson (QRT) map [70]. The system (2.34) is equivalent to

$${}^tU_n(f(v_n) \wedge U_{n+1}) = {}^tV_n(g(u_n) \wedge V_{n+1}) = 0. \quad (2.35)$$

It follows from (2.33) that

$$\left({}^tU_n A_0 V_n\right) \left({}^tU_{n+1} A_1 V_n\right) - \left({}^tU_{n+1} A_0 V_n\right) \left({}^tU_n A_1 V_n\right) = 0 \quad (2.36a)$$

$$\left({}^tU_{n+1} A_0 V_n\right) \left({}^tU_{n+1} A_1 V_{n+1}\right) - \left({}^tU_{n+1} A_0 V_{n+1}\right) \left({}^tU_{n+1} A_1 V_n\right) = 0 \quad (2.36b)$$

This implies that

$$I(u_n, v_n) = \frac{{}^tU_n A_0 V_n}{{}^tU_n A_1 V_n} \quad (2.37)$$

is a conserved quantity under the discrete evolutions $u_n \rightarrow u_{n+1}$ or $v_n \rightarrow v_{n+1}$. Given a value for the integral I , the biquadratic family of curves defined by (2.37) can be parameterized in terms of Jacobi elliptic functions [71, 43]

When ${}^tA_0 = A_0$ and ${}^tA_1 = A_1$, we have the symmetric QRT map

$$u_{n+1} = \frac{f_1(u_n) - u_{n-1} f_2(u_n)}{f_2(u_n) - u_{n-1} f_3(u_n)}, \quad (2.38)$$

with f as in (2.33a). In this case, the invariant curves take the form

$$\begin{aligned} & a_1 u_n^2 u_{n+1}^2 + a_2 u_n u_{n+1} (u_n + u_{n+1}) + a_3 (u_n^2 + u_{n+1}^2), \\ & + a_4 u_n u_{n+1} + a_5 (u_n + u_{n+1}) + a_6 = 0 \end{aligned} \quad (2.39)$$

where the coefficients a_i depend on the entries of the matrices A_0 and A_1 and

the value of the integral (2.37). A Möbius transformation⁴

$$u_n = \frac{\alpha w_n + \beta}{\gamma w_n + \delta}, \quad u_{n+1} = \frac{\alpha w_{n+1} + \beta}{\gamma w_{n+1} + \delta} \quad (2.40)$$

can bring (2.39) into the canonical form [8]

$$w_n^2 w_{n+1}^2 + \tilde{a}_3 (w_n^2 + w_{n+1}^2) + \tilde{a}_4 w_n w_{n+1} + 1. \quad (2.41)$$

It remains only to parameterize (2.41) in terms of the Jacobi sine function; by setting

$$w_n = \tilde{\alpha} \operatorname{sn}(\Omega n + z_0 | m) \quad (2.42)$$

and using (A.58a), we find that (2.42) satisfies (2.41) when $\tilde{\alpha}^4 = m$ and the parameters m and Ω satisfy

$$\tilde{a}_3^2 = -\frac{m}{\operatorname{sn}^4(\Omega | m)}, \quad \tilde{a}_4 = -2\tilde{a}_3 \operatorname{cn}(\Omega | m) \operatorname{dn}(\Omega | m). \quad (2.43)$$

We thus obtain two-parameter family of solutions to (2.38):

$$u_n = \frac{\alpha m^{\frac{1}{4}} \operatorname{sn}(\Omega n + z_0 | m) + \beta}{\gamma m^{\frac{1}{4}} \operatorname{sn}(\Omega n + z_0 | m) + \delta}. \quad (2.44)$$

Here z_0 is an arbitrary complex constant; the remaining parameters depend on the entries of A_0 and A_1 and the value of the integral (2.37), which constitutes the second degree of freedom.

2.1.4.1 McMillan map

An interesting special case of the symmetric QRT map (2.38) is obtained when

$$A_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2a & 0 \\ -1 & 0 & 0 \end{bmatrix}. \quad (2.45)$$

⁴The expressions for the Möbius parameters in terms of the a_i are complicated. Full details are given in [71, 43].

The resulting map

$$u_{n-1} + u_{n+1} = \frac{2au_n}{1 - u_n^2} \quad (2.46)$$

is called the McMillan map. As a special case of the QRT map, this equation can be integrated in terms of elliptic functions. If we set $w_n = \text{sn}(\Omega n + z_0|m)$, it follows simply from (A.58a) that

$$w_{n-1} + w_{n+1} = \frac{2w_n \text{cn}(\Omega|m) \text{dn}(\Omega|m)}{1 - mw_n^2 \text{sn}^2(\Omega|m)}. \quad (2.47)$$

Then, it is easily seen that

$$u_n = \pm \sqrt{m} \text{sn}(\Omega|m) \text{sn}(\Omega n + z_0|m) \quad (2.48)$$

solves (2.46) when the parameters satisfy

$$\text{cn}(\Omega|m) \text{dn}(\Omega|m) = a. \quad (2.49)$$

A five-parameter generalization of the McMillan map is obtained A_1 is an arbitrary symmetric matrix and A_0 is the same as in (2.45). The result,

$$u_{n-1} + u_{n+1} = \frac{a_2 u_n^2 + a_4 u_n + a_5}{a_1 u_n^2 + a_2 u_n + a_3} \quad (2.50)$$

is called the McMillan family of maps. The solution of this map is more complicated than that of (2.46). However, as (2.50) is a special case of (2.38), its solution is given by (2.44).

2.1.5 Singularity confinement and discrete Painlevé equations

Let us begin by analyzing the singularity structure of the McMillan map (2.46). Taking the initial conditions

$$u_{n-1} = u, \quad u_n = \pm 1 + z, \quad (2.51)$$

where z is a small perturbation, the next iterate will develop a pole:

$$u_{n+1} = -\mp \frac{a}{z} - u \mp \frac{3a}{2} + O(z). \quad (2.52)$$

However, iterating further we find that

$$u_{n+2} = \mp 1 + z + O(z^2), \quad u_{n+3} = -u + O(z); \quad (2.53)$$

the behavior of u_{n+1} is such that a pole is avoided at u_{n+3} . Because the singularity at u_{n+1} does not propagate further and the initial condition influences the value of u_{n+3} , the singularity developed at u_{n+1} is said to be confined [33]. While that singularity confinement is not a well-defined property, it has proven to be a useful tool to identify integrability candidates, as we now illustrate. A nonautonomous generalization of the McMillan map is

$$u_{n-1} + u_{n+1} = \frac{a_n + b_n u_n}{1 - u_n^2}. \quad (2.54)$$

We will demand that the singularity structure of this equation follows that of its autonomous counterpart and this will lead to conditions on a_n and b_n . Assuming the same initial conditions as before (2.51) we compute

$$\begin{aligned} u_{n+1} &= \frac{\mp a_{n+1} - b_{n+1}}{2z} + O(1), \quad u_{n+2} = \mp 1 + O(z) \\ u_{n+3} &= \frac{(\pm a_{n+1} + b_{n+1})[a_{n+1} - a_{n+3} \pm (b_{n+1} - 2b_{n+2} + b_{n+3})]}{2z[a_{n+1} \pm (b_{n+1} - 2b_{n+2})]} + O(1) \end{aligned} \quad (2.55)$$

and so u_{n+3} will be regular if a_n is affine in $(-1)^n$ and b_n is affine in n . If we take a_n to be constant, the resulting map is

$$u_{n-1} + u_{n+1} = \frac{(an + b)u_n + c}{1 - u_n^2}. \quad (2.56)$$

This map has a formal continuum limit to P_{II} (2.17b): under the transformation

$$u_n = \epsilon w(z), \quad z = \epsilon n, \quad a = \epsilon, \quad b = 2, \quad c = \epsilon^3 \alpha, \quad (2.57)$$

we obtain precisely (2.17b) (with a replaced by α) in the limit $\epsilon \rightarrow 0$. Another interesting transformation of (2.56) is

$$u_n = 1 + \epsilon v_n, \quad a = 4 + 2\epsilon\tilde{a} - 2\epsilon^2\tilde{b}, \quad b = -4 - 2\epsilon\tilde{a}, \quad c = -2\epsilon^2\tilde{b}; \quad (2.58)$$

in the limit $\epsilon \rightarrow 0$, we obtain the equation

$$v_{n-1} + v_n + v_{n+1} = \frac{an + b}{v_n} + c. \quad (2.59)$$

This equation is a discrete version of P_I (2.17a): under the transformation

$$v_n = \frac{1}{6} - \epsilon^2 w(z), \quad z = \epsilon n, \quad \tilde{a} = -\frac{1}{18}\epsilon^5, \quad \tilde{b} = -\frac{1}{12}, \quad \tilde{c} = 1, \quad (2.60)$$

one obtains (2.17a) in the limit $\epsilon \rightarrow 0$.

Auto-Bäcklund transformations for the classical Painlevé equations are another source of discrete Painlevé equations [28]. Taking the difference of the auto-Bäcklund transformations in (2.23), we have

$$u(z; b-2) + u(z; b+2) = \frac{2z}{u(z; b)^2} - \frac{2b}{u(z; b)}. \quad (2.61)$$

If we make the transformation

$$v_n = u(z; b), \quad n = \frac{b}{2}, \quad (2.62)$$

and view z as a parameter, we obtain the discrete equation

$$v_{n-1} + v_{n+1} = \frac{z}{v_n^2} - \frac{4n}{v_n}. \quad (2.63)$$

This is another discrete P_I equation; it has a continuum limit to (2.17a) and a coalescence limit from (2.56). We also note that the autonomous form of (2.63) is in the McMillan family (2.50).

Ordinary difference equations are more fundamental than ODEs: multi-

ple discrete equations with the same continuum limit can be constructed. In order to qualify as a discrete Painlevé equation, it is required that the equation possess special properties in analogy with its continuous counterpart. A discussion of these properties is found in [31].

A partial list of discrete Painlevé equations [71], in analogy with the classical list (2.17), is

$$\text{dP}_I : u(u_{n+1} + u_n + u_{n-1}) = an + b + cu_n \quad (2.64a)$$

$$\text{dP}_{II} : u_{n+1} + u_{n-1} = \frac{(an + b)u_n + c}{1 - u_n^2} \quad (2.64b)$$

$$\text{qP}_{III} : u_{n+1}u_{n-1} = cd \frac{(u_n - aq^n)(u_n - bq^n)}{(u_n - c)(u_n - d)} \quad (2.64c)$$

$$\text{dP}_{IV} : (u_n + u_{n+1})(u_{n-1} + u_n) = \frac{(u_n^2 - a^2)(u_n^2 - b^2)}{(u_n - an - b)^2 - c^2} \quad (2.64d)$$

$$\text{qP}_V : (u_{n+1}u_n - 1)(u_nu_{n-1} - 1) = \quad (2.64e)$$

$$\begin{aligned} & cdq^{2n} \frac{(u_n - a)(u_n - 1/a)(u_n - b)(u_n - 1/b)}{(u_n - cq^n)(u_n - dq^n)} \\ \text{qP}_{VI} : & \frac{(u_nu_{n+1} - q_0^2q^{2n+1})(u_nu_{n-1} - q_0^2q^{2n-1})}{(u_{n+1}u_n - 1)(u_nu_{n-1} - 1)} = \quad (2.64f) \\ & \frac{(u_n - aq_0q^n)(u_n - q_0q^n/a)(u_n - bq_0q^n)(u_n - q_0q^n/b)}{(u_n - c)(u_n - 1/c)(u_n - d)(u_n - 1/d)}. \end{aligned}$$

Three of these equations are, as indicated by the notation, q -difference equations where the explicit dependence on the independent variable enters as an exponent. Differences [45] between these and difference equations where the independent variable enters as an affine function, e.g. (2.64a), are not important in this thesis. For our purposes, there are two salient features of the equations (2.64a-2.64f): each equation possesses a continuum limit to its corresponding continuous Painlevé equation and the autonomous limit of each equation is contained in the symmetric QRT map (2.38).

We remark that the classification of discrete Painlevé equations is more complicated than for the continuous Painlevé equations: there are multiple discrete Painlevé equations for each continuous Painlevé equation; the list

(2.64) is not canonical. A classification of discrete Painlevé equations was achieved by Sakai in [75]. The geometric approach taken there is beyond the scope of this thesis.

2.2 Partial differential equations

In this section, we will consider the integrability of partial differential equations (PDEs). We do not seek to give an exhaustive account of the theory, but rather to introduce a number of key equations that are related to results presented in the subsequent chapters. In particular, we will be interested in the Korteweg-de Vries (KdV) and sine-Gordon equations.

2.2.1 Korteweg-de Vries type equations

The KdV equation

$$u_t = 6uu_x + u_{xxx} \quad (2.65)$$

is the prototypical integrable PDE. It possesses a number of known exact solutions, a Bäcklund transformation, a Lax representation, and an infinity of conservation laws. We will focus on the first two properties.

We begin by taking a traveling wave reduction

$$w(z) = -\frac{1}{2}u(x, t) - \frac{p}{12}, \quad z = x - pt + x_0 \quad (2.66)$$

of (2.65) to obtain

$$w''' = 12ww', \quad (2.67)$$

where $' = d/dz$. After integrating twice, we obtain

$$(w')^2 = 4w^3 - g_2w - g_3 \quad (2.68)$$

where g_2 and g_3 are constants of integration. This is precisely the differential equation for the Weierstrass \wp function and hence we obtain

$$u(x, t) = -2\wp(x - pt + x_0; g_2, g_3) - \frac{p}{6} \quad (2.69)$$

as a traveling wave solution to the KdV equation. This solution can be expressed in terms of Jacobi elliptic functions. The result is the *cnoidal* wave

$$u(x, t) = -\frac{p}{6} + 2e_3 + 2(e_2 - e_3) \operatorname{cn}^2 \left(\sqrt{e_1 - e_2}(x - pt) + x_0 \mid m \right) \quad (2.70)$$

with m as in (A.51) and where the e_i satisfy (A.33-A.34). In the limit $m \rightarrow 0$ we recover the *soliton* solution

$$u(x, t) = 2k^2 \operatorname{sech}^2(kx + 4k^3t + x_0), \quad (2.71)$$

if we define $k = \sqrt{e_1 - e_2}$ and choose $p = 12e_3$.

The soliton solution we just obtained can actually be found from the auto-Bäcklund transformation for the (potential) KdV equation. The meaning of Bäcklund transformation in this context is similar to that in the case of Painlevé equations; it is taken to mean a relationship between solutions of any pair of PDEs and the specialized term auto-Bäcklund is used when we have a relationship between solutions of the same PDE (possibly with different parameters).

We now follow [45]. There is a Bäcklund transformation from the equation

$$v_t = 6\lambda v_x - 6v^2 v_x + v_{xxx} \quad (2.72)$$

to (2.65), called the Miura transformation [55]. Explicitly this transformation is given by

$$u = \lambda - v_x - v^2. \quad (2.73)$$

From the invariance of (2.72) under negation of v , we obtain another KdV solution

$$\tilde{u} = \lambda + v_x - v^2. \quad (2.74)$$

The Miura pair (2.73-2.74) is equivalent to

$$u + \tilde{u} = 2(\lambda - v^2) \quad (2.75a)$$

$$\tilde{u} - u = 2v_x. \quad (2.75b)$$

If we introduce a variable w satisfying $w_x = u$, we have

$$w_x + \tilde{w}_x = 2(\lambda - v^2) \quad (2.76a)$$

$$\tilde{w}_x - w_x = 2v_x, \quad (2.76b)$$

where w satisfies the potential KdV (pKdV) equation

$$w_t = 3w_x^2 + w_{xxx}. \quad (2.77)$$

In the new variables, v can be eliminated from (2.76) leading to the Bäcklund transformation for (2.77)

$$w_x + \tilde{w}_x = 2\lambda - \frac{1}{2}(w - \tilde{w})^2 \quad (2.78a)$$

$$w_t + \tilde{w}_t = -(\tilde{w} - w)(\tilde{w}_{xx} - w_{xx}) + 2(w_x^2 + 2w_x\tilde{w}_x + \tilde{w}_x^2). \quad (2.78b)$$

If we also consider the Bäcklund transformation with a new parameter μ (leading to a new solution \hat{w}):

$$w_x + \hat{w}_x = 2\mu - \frac{1}{2}(w - \hat{w})^2 \quad (2.79a)$$

$$w_t + \hat{w}_t = -(\hat{w} - w)(\hat{w}_{xx} - w_{xx}) + 2(w_x^2 + 2w_x\hat{w}_x + \hat{w}_x^2), \quad (2.79b)$$

we can construct a *superposition principle* from the compatibility of the two transformations (2.78-2.79)

$$(\hat{\tilde{w}} - w)(\hat{w} - \tilde{w}) = 4(\mu - \lambda). \quad (2.80)$$

We will use such equations to derive three-point differential difference equations in the next section.

To recover the soliton solution (2.71), we note that $w(x, t) = 0$ is a solution

of (2.77). We can use (2.78) to generate a new solution \tilde{w} . When $w = 0$, (2.78) reduces to

$$\tilde{w}_x = 2\lambda - \frac{1}{2}\tilde{w}^2 \quad (2.81a)$$

$$\tilde{w}_t = -\tilde{w}\tilde{w}_{xx} + 2\tilde{w}_x^2 \quad (2.81b)$$

with solution

$$w(x, t) = 2k \tanh(kx + k^3t + x_0), \quad (2.82)$$

where $k^2 = \lambda$. Differentiating this with respect to x , we obtain precisely (2.71), in accordance with the substitution used to obtain (2.76).

Bäcklund pairs can be constructed for other KdV type equations. One of these is the modified KdV (mKdV) equation, obtained by taking $\lambda = 0$ in (2.72):

$$u_t = -6u^2u_x + u_{xxx}. \quad (2.83)$$

The corresponding potential modified KdV equation (pmKdV) is

$$u_t = -3\frac{u_x u_{xx}}{u} + u_{xxx}, \quad (2.84)$$

which admits the transformation

$$u_x \tilde{u} + u \tilde{u}_x = \lambda (\tilde{u}^2 - u^2) \quad (2.85a)$$

$$u_t \tilde{u} + u \tilde{u}_t = 2\lambda (\tilde{u} \tilde{u}_{xx} - u u_{xx} - 2\tilde{u}_x^2 + 2u_x^2). \quad (2.85b)$$

The associated superposition principle is

$$\lambda (u \hat{u} - \tilde{u} \hat{\tilde{u}}) = \mu (u \tilde{u} - \hat{u} \hat{\tilde{u}}). \quad (2.86)$$

The Schwarzian KdV (SKdV) equation

$$u_t = -\frac{3}{2} \frac{u_{xx}^2}{u_x} + u_{xxx} \quad (2.87)$$

was identified using the Painlevé test for PDEs [82]. The equation is named for its invariance under Möbius transformations. It admits the Bäcklund pair

$$u_x \tilde{u}_x = \lambda (\tilde{u}^2 - u^2) \quad (2.88a)$$

$$u_x \tilde{u}_t + u_t \tilde{u}_x = u_{xx} \tilde{u}_{xx} + 2(\tilde{u} - u)(\tilde{u}_{xx} - u_{xx}) - 4\lambda(\tilde{u}_x - u_x)^2 \quad (2.88b)$$

and the superposition principle

$$\frac{(\hat{\tilde{u}} - \hat{u})(\tilde{u} - u)}{(\hat{\tilde{u}} - \tilde{u})(\hat{u} - u)} = \frac{\lambda^2}{\mu^2}. \quad (2.89)$$

Special solutions in analogy with (2.70-2.71) can also be found for the mKdV (2.83) and SKdV (2.87) equations.

2.2.2 Sine-Gordon equation

The sine-Gordon equation,

$$u_{xt} = \sin u, \quad (2.90)$$

is another well-known integrable PDE, but is not directly related to the KdV equation. This equation was obtained in the study of pseudospherical surfaces in \mathbb{R}^3 . Like the KdV type equations we considered, it admits soliton solutions and is actually the first equation for which a Bäcklund transformation was derived. There is an extensive literature on this equation; a geometric introduction is combined with more recent results in [73]. Special solutions are discussed in [12]. For our purposes, it suffices to write down the Bäcklund transformation:

$$\tilde{u}_x - u_x = 2\lambda \sin \frac{1}{2}(u + \tilde{u}) \quad (2.91a)$$

$$u_t + \tilde{u}_t = \frac{2}{\lambda} \sin \frac{1}{2}(\tilde{u} - u), \quad (2.91b)$$

which is closely related to some equations we obtain in subsequent chapters. The corresponding superposition principle is

$$\lambda \sin \left[\frac{1}{4} (\hat{u} + \hat{u} - \tilde{u} - u) \right] = \mu \sin \left[\frac{1}{4} (\hat{u} + \tilde{u} - \hat{u} - u) \right]. \quad (2.92)$$

2.3 Differential-difference equations

This section is devoted to differential-difference equations: equations of one discrete and one continuous variable. One important way these equations arise is from Bäcklund transformations for integrable PDEs. By reinterpreting the discrete variable in these as a separate continuous variable, reductions to delay-differential equations can be performed. We will make extensive use of such reductions later in this thesis.

2.3.1 Equations from Bäcklund transformations

Each of the components of the Bäcklund pairs discussed in the last chapter can be considered as a differential-difference equation. For definiteness, let us consider the spatial part of the pKdV Bäcklund pair (2.78a):

$$u'_n + u'_{n+1} = 2\lambda - \frac{1}{2} (u_n - u_{n+1})^2, \quad (2.93)$$

where $' = d/dx$ and we have relabeled the dependent variables. The idea here is that the process of generating solutions using the Bäcklund transformation can be iterated, giving us a countable infinity of solutions labeled by an integer n . For notational consistency in what follows, we will work with (2.93) in the form

$$w'_n + w'_{n+1} = \lambda + (w_{n+1} - w_n)^2. \quad (2.94)$$

For KdV type equations, we focus on the spatial components of the Bäcklund pairs, as they involve derivatives with respect to only one of the independent variables. Written as differential-difference equations, we have

$$w'_n w_{n+1} + w_n w'_{n+1} = \lambda (w_{n+1}^2 - w_n^2) \quad (2.95)$$

and

$$w'_n w'_{n+1} = \lambda (w_{n+1}^2 - w_n^2) \quad (2.96)$$

for the pmKdV (2.84) and SKdV (2.87) equations, respectively. For the sine-Gordon equation, each component of the Bäcklund transformation has explicit dependence on a single independent variable:

$$(u_{n+1} - u_n)_x = 2\lambda \sin \frac{1}{2}(u_{n+1} + u_n) \quad (2.97a)$$

$$(u_{n+1} + u_n)_t = \frac{2}{\lambda} \sin \frac{1}{2}(u_{n+1} - u_n). \quad (2.97b)$$

It is natural to consider the reductions $w_n(x) = u_n(x, t)$ and $w_n(t) = u_n(x, t)$ of (2.97a) and (2.97b), respectively, to obtain:

$$w'_{n+1} - w'_n = 2\lambda \sin \frac{1}{2}(w_{n+1} + w_n), \quad w_n = w_n(x), \quad ' = \frac{d}{dx} \quad (2.98a)$$

$$w'_{n+1} + w'_n = \frac{2}{\lambda} \sin \frac{1}{2}(w_{n+1} - w_n), \quad w_n = w_n(t), \quad ' = \frac{d}{dt}. \quad (2.98b)$$

These two differential-difference equations play an important role in the subsequent chapter.

2.3.1.1 Skew-continuum limits

For each of the KdV type equations we have analyzed, a ‘skew continuum limit’ of the superposition principle can be taken to obtain a three point differential-difference equation. To do this, a change of variables $(l, m) \rightarrow (l, n = l + m)$ is made and a continuum limit (in l) is taken as before. Full details are given in [45]. The resulting equations,

$$pw'_n - w'_n(w_{n+1} - w_{n-1}) = w_{n-1} - w_{n+1} \quad (2.99)$$

$$pw'_n = w_n \frac{w_{n+1} - w_{n-1}}{w_{n-1} + w_{n+1}} \quad (2.100)$$

$$pw'_n = \frac{(w_{n+1} - w_n)(w_n - w_{n-1})}{w_{n+1} - w_{n-1}}, \quad (2.101)$$

where $p \in \mathbb{C}$ is a free parameter (obtained by redefinition of Bäcklund parameters), are differential-difference analogues of (2.65), (2.83), and (2.87), respectively.

2.3.2 Volterra lattice

There is a Miura transformation [45] from (2.99) to the equation

$$v'_n = pv_n^2(v_{n+1} - v_{n-1}) \quad (2.102)$$

given by

$$v_n = -\frac{1}{p + u_{n-1} - u_{n+1}}. \quad (2.103)$$

(2.102) is equivalent to a special case ($q = 0$) of the modified Volterra lattice

$$v'_n = (v_n^2 - q^2)(v_{n+1} - v_{n-1}). \quad (2.104)$$

This equation [13] has a continuum limit to the mKdV equation (2.83) and a Miura transformation to the Volterra lattice:

$$w'_n = w_n(w_{n+1} - w_{n-1}) \quad (2.105)$$

given by

$$w_n = (v_n + q)(v_{n+1} - q). \quad (2.106)$$

The Volterra lattice has a continuum limit to the KdV equation (2.65). The Volterra lattice and the modified Volterra lattice have many properties associated with integrability. Historically, these equations first attracted interest due to their relationship with the Toda lattice, which we now discuss.

2.3.3 Toda lattice

The Toda lattice is a further well-known integrable differential-difference equation that is not known to arise from a Bäcklund transformation of an integrable PDE. It is however, closely related to the Volterra lattice we have just dis-

cussed. The equation

$$w_n'' = \exp(w_{n-1} - w_n) - \exp(w_n - w_{n+1}) \quad (2.107)$$

was introduced in [77] and a solution in terms of the Jacobi Z -function (A.73) was given. It is often convenient to use the *Flaschka variables* [27]

$$u_n = \frac{1}{2} \exp \frac{1}{2} (w_{n-1} - w_n) \quad (2.108a)$$

$$v_n = -\frac{1}{2} w_{n-1}', \quad (2.108b)$$

so that the Toda lattice is realized as the first-order system

$$u_n' = u_n (v_{n+1} - v_n) \quad (2.109a)$$

$$v_n' = 2(u_n^2 - u_{n-1}^2). \quad (2.109b)$$

In this form, there is a simple Miura transformation from the Volterra lattice; if w_n solves (2.105), then

$$u_n = \frac{1}{2} \sqrt{w_{2n} w_{2n+1}} \quad (2.110a)$$

$$v_n = \frac{1}{2} (w_{2n} + w_{2n-1}) \quad (2.110b)$$

solves (2.109).

2.4 Delay-differential equations

In this section, we begin to discuss delay-differential equations from the perspective of integrability by reviewing known results. In particular, we discuss an analogue of singularity confinement in this setting and discuss known, possibly integrable delay-differential equations.

2.4.1 Singularity confinement

As in the case of purely discrete equations, a notion of singularity confinement has been developed [72] and used to identify integrability candidates [32]. Here,

the notion of singularity confinement is a hybrid of the classical Painlevé test for ODEs and the singularity confinement criterion for discrete systems. We begin with a pair of definitions.

Definition 2.4.1 (Singularity sequence). *A sequence of Frobenius series (u_0, \dots, u_m) , with $m \geq r$ and*

$$u_k = \sum_{n=0}^{\infty} u_{k,n} (z - z_0)^{n+p_k}, \quad p_k \in \mathbb{Q}, \quad (2.111)$$

is called a singularity sequence.

Definition 2.4.2 (Admissible sequence). *A singularity sequence is an admissible sequence for an r -point delay-differential equation if $u(z + kh) = u_k, \dots, u(z + (m - r + 1)h) = u_{m-r+1}$ formally satisfy the k -upshifted delay-differential equation when u and its upshifts are viewed as independent variables, for each $k \in \{0, \dots, m - r + 1\}$.*

There is not a clear definition of what it means for a delay-differential equation to confine singularities. Roughly speaking, a delay-differential confines singularities if every admissible singularity sequence beginning with a Taylor series consists of only Laurent series (i.e. p_k is always integral) and is a subsequence of an admissible singularity sequence that terminates with a Taylor series. However, there may be an infinite number of singularity sequences to check. Moreover, one could, in principle, have an admissible repeating sequence such as

$$\left(\dots, \frac{a}{z} + O(1), b + O(z - z_0), \frac{a}{z} + O(1), b + O(z - z_0), \dots \right) \quad (2.112)$$

in which the singularity is not really confined. Nevertheless, the test has been successful in identifying equations; in [32] it is claimed that the simplest singularity sequences provide “essential constraints” needed to isolate integrability candidates. In the subsequent chapter,² we will show that admissibility of particular singularity sequences provides necessary conditions for an equation

to admit certain elliptic solutions.

For now we will demonstrate singularity confinement for a particular equation and particular singularity sequence. In doing so, it is convenient to use the following shorthand notation for terms in the singularity sequence (rg stands for regular):

$$\infty^p = \frac{u_0}{(z - z_0)^p} + O\left((z - z_0)^{-p+1}\right), \quad u_0 \neq 0, \quad p \in \mathbb{N} \quad (2.113a)$$

$$0^p = u_0(z - z_0)^p + O\left((z - z_0)^{p+1}\right), \quad u_0 \neq 0, \quad p \in \mathbb{N} \quad (2.113b)$$

$$\text{rg} = u_0 + O(z - z_0). \quad (2.113c)$$

Here each term is understood to be a Laurent series with arbitrary expansion coefficients. If there are multiple series with the same leading order, we will indicate that they are distinct with subscripts. When we have two simple poles or simple zeroes with leading coefficients that differ only by a sign, it will be convenient to use the notation

$$\infty_{\pm}^1 = \frac{\pm u_0}{z - z_0} + O(1), \quad (2.114a)$$

$$0_{\pm}^1 = \pm u_0(z - z_0) + O\left((z - z_0)^2\right). \quad (2.114b)$$

We now turn to the equation

$$u' + \bar{u}' = \bar{u}^2 - u^2. \quad (2.115)$$

This is a special case of an equation identified in [32]. We substitute an arbitrary Taylor series

$$u = \sum_{n=0}^{\infty} u_n z^n \quad (2.116)$$

into (2.115) to obtain an ODE; we perform the usual Painlevé test on the result. Taking

$$\bar{u} = \sum_{n=0}^{\infty} \bar{u}_n z^{n+p_1} \quad (2.117)$$

we find that either the principal part of the series vanishes or $p_1 = -1$, in which case $\bar{u}_{-1} = -1$ and $\bar{u}_0 = 0$. Assuming the latter, for $n \neq -1$ we have the recurrence relations

$$(n+2)\bar{u}_n = \sum_{j=0}^{n-1} (\bar{u}_j \bar{u}_{n-j-1} - u_j u_{n-j-1}) - n u_n, \quad (2.118)$$

The resonance polynomial on the LHS of (2.118) indicates that there is a resonance (the universal resonance) at $n = -2$, but there are no resonance conditions to check. Now supposing the expansion coefficients \bar{u}_n are known, we upshift (2.115) and take

$$\bar{\bar{u}} = \sum_{n=0}^{\infty} \bar{\bar{u}}_n z^{n+p_2}, \quad (2.119)$$

and find we must have $p_2 = -1$ and $\bar{\bar{u}}_{-1} = 1$ and $\bar{\bar{u}}_0 = \bar{u}_0 = 0$. For $n \neq -1$ the recurrence relations are

$$(n-2)\bar{\bar{u}}_n = \sum_{j=0}^{n-1} (\bar{\bar{u}}_j \bar{\bar{u}}_{n-j-1} - \bar{u}_j \bar{u}_{n-j-1}) - (n-2)\bar{\bar{u}}_n. \quad (2.120)$$

We see there is a resonance condition at $n = 2$:

$$\bar{\bar{u}}_0 \bar{\bar{u}}_1 - \bar{u}_0 \bar{u}_1 = 0 \quad (2.121)$$

which is satisfied because $\bar{\bar{u}}_0 = \bar{u}_0 = 0$. We upshift the equation once more and take

$$\bar{\bar{\bar{u}}} = \sum_{n=0}^{\infty} \bar{\bar{\bar{u}}}_n z^{n+p_3} \quad (2.122)$$

and find that either $p_3 = -1$ and $\bar{\bar{\bar{u}}}_{-1} = -1$ or $p_3 = 0$. In the latter case, we have the recurrence relations, valid for each n :

$$n\bar{\bar{\bar{u}}}_n = \sum_{j=0}^{n-1} (\bar{\bar{\bar{u}}}_j \bar{\bar{\bar{u}}}_{n-j-1} - \bar{\bar{u}}_j \bar{\bar{u}}_{n-j-1}) - (n+2)\bar{\bar{\bar{u}}}_n. \quad (2.123)$$

There is a resonance at $n = 0$. The condition is simply $\bar{\bar{\bar{u}}}_0 = 0$, which is satisfied.

We conclude that (2.115) admits the singularity sequence

$$\left(\text{rg}_1, \infty_{\pm}^1, \infty_{\mp}^1, \text{rg}_2\right) \quad (2.124)$$

where the first term is an arbitrary Taylor series (2.116) and the other terms are given, respectively, by (2.117, 2.119, 2.122) and the associated recurrence relations (2.118, 2.120, 2.123). We note that we only considered the case $p_3 = 0$ above. In the case $p_3 = -1$, we enter into a longer singularity sequence; we will not consider this sequence as it is precisely (2.124) that is closely related to the admittance of elliptic solutions. We will see later that in fact (2.115) admits a multiparameter family of such solutions.

2.4.2 Known equations

We will briefly discuss the delay-differential equations that have been obtained in the context of integrable systems and the role of such equations in this thesis. The first such equation,

$$au' = bu + u(\bar{u} - u). \quad (2.125)$$

was identified by Quispel, Capel, and Sahadevan [68] as a Lie symmetry reduction of the Volterra lattice (2.105). Here it was claimed that the equation (2.125) possesses a continuum limit to the first Painlevé equation (2.17a), though the given continuum limit was incorrect. We will discuss this equation in the fourth chapter of this thesis and give the correct continuum limit.

A number of equations were identified by Grammaticos, Ramani, and Moreira in [32]. They worked within the bi-Riccati class, which consists of equations that are separately Riccati equations (2.11) in both u and u' , and identified eight equations by means of a kind of singularity confinement:

$$u' + \bar{u}' = (u - \bar{u})^2 + b_1(u + \bar{u}) + b_2 + b_3 e^{2b_1 z} \quad (2.126)$$

$$u' + \bar{u}' = (u - \bar{u})^2 + b_1(u + \bar{u}) + b_2 + b_3 z \quad (2.127)$$

$$u' \bar{u} + u \bar{u}' = e^{\omega z} (b_1 u^2 + b_2 \bar{u}^2) \quad (2.128)$$

$$u'\bar{u} - u\bar{u}' = e^{\omega z} (b_1 + b_2 u^2 \bar{u}^2) \quad (2.129)$$

$$u' + \bar{u}' = \bar{u}^2 - u^2 + b_1(u + \bar{u}) + b_2 \quad (2.130)$$

$$u'\bar{u} - u\bar{u}' = -u^2 \bar{u}^2 + b(u + \bar{u}) \quad (2.131)$$

$$u'\bar{u} - u\bar{u}' = b_1 u + b_2 \bar{u} + b_3 u^2 \bar{u} + b_4 u \bar{u}^2 \quad (2.132)$$

$$u'\bar{u} + u\bar{u}' = b_1 u + b_2 \bar{u} + b_3 u^2 \bar{u} + b_4 u \bar{u}^2 + b_5. \quad (2.133)$$

Some of these equations possess continuum limits to classical Painlevé equations. Bi-Riccati equations are the subject of the subsequent chapter, in which we discuss elliptic solutions to some of the equations identified [32] and discuss the relationship between the notion of singularity confinement in [32] and elliptic functions.

Another P_I type equation has been identified by Joshi [46] as a direct reduction of the Toda lattice:

$$au + u' = u(\bar{v} - v), \quad av + v' + b = 2(u - \underline{u}^2). \quad (2.134)$$

A similar, but less general, equation was obtained by Levi and Winternitz as Lie symmetry reductions of the Toda lattice [51]. We will give solutions to a special case of (2.134) using results in the fourth chapter and the Miura transformation between the Volterra and Toda lattices (2.110).

Most recently, Halburd and Korhonen [35] used Nevanlinna theory to study delay-differential equations. The equations they identified are nonautonomous extensions of either (2.125) or

$$au' = bu + u^2(\bar{u} - \underline{u}), \quad (2.135)$$

which is a symmetry reduction of the semidiscrete modified Korteweg-de Vries equation. We obtain this equation in the final chapter from a Bäcklund transformation for the third Painlevé equation.

There have been several efforts to contextualize claimed integrable delay-

differential equations with respect to integrable systems at large. The bilinear forms of some of the known Painlevé type delay-differential equations were given in [11]. Some of these same equations were generalized to semidiscrete equations in [7]. An interesting formal relationship between delay-differential equations and integro-differential equations was established in [72].

Chapter 3

Bi-Riccati equations

The bi-Riccati family of delay-differential equations consists of equations that are separately Riccati equations for both the variable $u = u(z)$ and its upshift $\bar{u} = u(z + h)$. More explicitly, a generic equation in the bi-Riccati class takes the form

$${}^tUX\bar{U} = 0, \quad U = {}^t(1, u, u^2, u'), \quad X : \mathbb{C} \rightarrow \mathbb{C}^{4 \times 4}. \quad (3.1)$$

These equations were introduced and studied by Grammaticos, Ramani, and Moreira [32] from the perspective of the combined singularity confinement-Painlevé test [72] for differential-difference equations described in the previous chapter. Using this method, seven equations were identified; some of these equations possess continuum limits to Painlevé equations (2.17).

In this chapter, we will first discuss the relationship between the singularity structure of bi-Riccati equations and the existence of elliptic function solutions for these equations. The results in the first section provide a link between the results of [32] and the program we are pursuing in this thesis. As discussed in the introduction to this thesis, the construction of delay-differential QRT analogues is hampered by the absence of obvious conserved quantities in this new setting. Instead we directly seek autonomous delay-differential equations that admit elliptic solutions with two degrees of parametric freedom, as one finds in the general solutions of the symmetric QRT (2.38) map and autonomized Painlevé equations. The remainder of this chapter is devoted to performing

this search within the class (3.1). We restrict ourselves to elliptic solutions that may be expressed as a Möbius transformation of the Jacobi sine function, as one finds as the general solution of the symmetric QRT map (2.44). It is a relatively simple matter to identify all bi-Riccati equations, up to Möbius equivalence, that admit the Jacobi sine function as a solution with at least one degree of parametric freedom. Having done this, we develop a method to identify equations that admit such solutions with at least two degrees of parametric freedom. Classifying all such equations poses significant computational problems. Nonetheless, we are able to isolate five families of delay-differential equations that admit the desired elliptic solutions. Four of the equations we identify are related to known models, in particular semidiscrete sine-Gordon and Korteweg-de Vries equations. This relationship is discussed in detail and new solutions to the known equations are given.

3.1 Singularity structure of bi-Riccati equations

In seeking elliptic solutions for delay-differential equations, we begin with a general representation of an elliptic function [6]:

$$f(z) = A_0^0 + \sum_{n=1}^N A_n^1 \zeta(z - a_n; g_2, g_3) + \sum_{n=1}^N \sum_{m=2}^{M_n} A_n^m \wp^{(m-2)}(z - a_n; g_2, g_3). \quad (3.2)$$

A set of N pairs $\{(a_n, M_n) : 1 \leq n \leq N\}$, where a_n is the location of a pole and M_n is the multiplicity of that pole, together with (3.2), determines an elliptic function up to specification of the expansion coefficients A_n^m (the expansion coefficients A_n^1 are constrained by Cauchy's residue theorem). We immediately restrict ourselves to order-two elliptic functions. These functions take each value in the extended complex plane twice, counting multiplicity, in each period parallelogram. This simplest class of elliptic functions contains the most commonly used elliptic functions: the Weierstrass \wp -function and the twelve Jacobi elliptic functions. These functions are the most frequent examples of

elliptic solutions to integrable systems. If the order of the elliptic function is two, we must have $M_1 + \dots + M_N = 2$ in (3.2). By Cauchy's residue theorem, we must also have $A_1^1 + \dots + A_N^1 = 0$, so there are only two possibilities:

Case I: $N = 1, M_1 = 2$

$$f_1(z) = \alpha \wp(z - a; g_2, g_3) + \beta \quad (3.3)$$

Case II: $N = 2, M_1 = M_2 = 1$

$$f_2(z) = \alpha [\zeta(z - a_1; g_2, g_3) - \zeta(z - a_2; g_2, g_3)] + \beta, \quad (3.4)$$

Let us focus on the first case. Suppose f_1 solves an autonomous delay-differential equation. Using translational freedom in the independent variable z we can take $a = 0$ in (3.3) without loss of generality. The resulting function,

$$\alpha \wp(z; g_2, g_3) + \beta \quad (3.5)$$

possesses a double pole at $z = 0 \bmod \Lambda$, where $\Lambda = \Lambda(g_2, g_3)$ is the lattice associated to the elliptic function. The locations of the zeroes of the Weierstrass \wp -function are, in general, given by a very complicated formula [24]. On the other hand, the zeroes of the function

$$\wp(z; g_2, g_3) - \wp(h; g_2, g_3), \quad (3.6)$$

where $h \in \mathbb{C} \setminus \Lambda$ is constant, are very simple to characterize. In particular, the zeroes occur when $z = \pm h \bmod \Lambda$; the zeroes are simple unless $\wp'(h; g_2, g_3) = 0$ (equivalently $2h \in \Lambda$). Thus, in what follows, it will be more convenient to work with (3.6) versus \wp itself. By redefining the parameters appearing in (3.5),

$$\beta \rightarrow \beta - \alpha \wp(h; g_2, g_3), \quad (3.7)$$

we arrive at

$$\widetilde{f}_1(z) = \alpha [\wp(z; g_2, g_3) - \wp(h; g_2, g_3)] + \beta. \quad (3.8)$$

We will use this form of (3.5) to establish a relationship with particular singularity sequences. We will need the following definition.

Definition 3.1.1. *A function $g : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ contains a singularity sequence (u_0, \dots, u_n) with prescribed step size $h \in \mathbb{C} \setminus \{0\}$ if there exists $z_0 \in \mathbb{C}$ and $\epsilon > 0$ such that $g(z + mh - z_0) = u_m$ for each $m \in \{0, \dots, n\}$ in a punctured disk of radius ϵ about z_0 .*

We will now show that a generic order-two elliptic function, possibly after an affine transformation, must contain one of several particular singularity sequences that arise often in the application of singularity confinement. It follows that the admittance of these particular singularity sequences provides necessary conditions for an equation to admit an order-two elliptic solution. We will make use of the notation (2.113-2.114) introduced in the previous chapter. Ellipses are used to indicate that singularity sequence repeats.

Theorem 3.1.1. *Consider the functions*

$$f_1(z) = \alpha \wp(z - a; g_2, g_3) + \beta, \quad (3.9)$$

and

$$\widetilde{f}_1(z) = \alpha [\wp(z - a; g_2, g_3) - \wp(h; g_2, g_3)], \quad (3.10)$$

where $\alpha \neq 0$ and $\beta \neq -\alpha \wp(h; g_2, g_3)$. If $h \in \Lambda$, f_1 and \widetilde{f}_1 contain the singularity sequence

$$(\dots, \infty^2, \infty^2, \dots). \quad (3.11)$$

If $h \notin \Lambda$ and $2h \in \Lambda$, f_1 contains the singularity sequence

$$(\dots, \text{rg}, \infty^2, \text{rg}, \infty^2, \dots). \quad (3.12)$$

and \widetilde{f}_1 contains the singularity sequence

$$\left(\dots, 0^2, \infty^2, 0^2, \infty^2, \dots\right). \quad (3.13)$$

If $h \notin \Lambda$ and $2h \notin \Lambda$, f_1 contains the singularity sequence

$$\left(\text{rg}, \infty^2, \text{rg}\right). \quad (3.14)$$

and \widetilde{f}_1 contains the singularity sequence

$$\left(0_{\pm}^1, \infty^2, 0_{\mp}^1\right). \quad (3.15)$$

Proof. We may take $a = 0$ in (3.9-3.10) without loss of generality. In a punctured disk centered at $z = 0$ we have the Laurent series

$$f(z) = \frac{\alpha}{z^2} + O(1), \quad \widetilde{f}_1(z) = \frac{\alpha}{z^2} + O(1). \quad (3.16)$$

Near $z = \pm h$ there are three cases to consider.

Case 1: $h \in \Lambda$. If h is a period we have

$$f_1(z + nh) = \frac{\alpha}{z^2} + O(1), \quad n \in \mathbb{Z} \quad (3.17)$$

in a punctured disk about $z = 0$. Hence f_1 contains a singularity sequence of the form (3.11). An identical argument applies to \widetilde{f}_1 .

Case 2: $h \notin \Lambda$ and $2h \in \Lambda$. By assumption that h is a half-period, we have that

$$f_1(z + 2nh) = \frac{1}{z^2} + O(1), \quad n \in \mathbb{Z} \quad (3.18)$$

in a punctured disk about $z = 0$. By the same assumption, we have

$$f_1(z) = \alpha \wp(h; g_2, g_3) + \beta + O\left((z \mp h)^2\right) \quad (3.19)$$

near $z = \pm h$. It follows that

$$f_1(z + (2n - 1)h) = \alpha\wp(h; g_2, g_3) + \beta + O(z^2), \quad n \in \mathbb{Z} \quad (3.20)$$

near $z = 0$. The sequence consisting of Laurent series expansion of $f_1(z + nh)$ about $z = 0$, for $n \in \mathbb{Z}$, is contained in the function f_1 and the resulting sequence is of the form (3.13).

We now turn to the function \widetilde{f}_1 . By assumption that h is a half-period, we have that

$$\widetilde{f}_1(z + 2nh) = \frac{1}{z^2} + O(1), \quad n \in \mathbb{Z} \quad (3.21)$$

in a punctured disk about $z = 0$. By the same assumption, we have

$$\widetilde{f}_1(z) = \left[3\wp(h; g_2, g_3)^2 - \frac{g_2}{4} \right] (z \mp h)^2 + O((z \mp h)^3) \quad (3.22)$$

near $z = \pm h$. It follows that

$$\widetilde{f}_1(z + (2n - 1)h) = \left[3\wp(h; g_2, g_3)^2 - \frac{g_2}{4} \right] z^2 + O(z^3), \quad n \in \mathbb{Z} \quad (3.23)$$

near $z = 0$. The sequence consisting of Laurent series expansion of $\widetilde{f}_1(z + nh)$ about $z = 0$, for $n \in \mathbb{Z}$, is contained in the function $\widetilde{f}_1(z)$ and the resulting sequence is of the form (3.12).

Case 3: $h \notin \Lambda$ and $2h \notin \Lambda$. Near $z = \pm h$ we have

$$f_1(z) = \alpha\wp(h; g_2, g_3) + \beta + O(z \mp h), \quad (3.24)$$

which implies that

$$f_1(z \pm h) = \alpha\wp(h; g_2, g_3) + \beta + O(z) \quad (3.25)$$

near $z = 0$. In (3.26-3.27), $\wp'(h; g_2, g_3) \neq 0$ by the assumption that h is not a half-period. Then, the Laurent series expansions of $f_1(z - h)$, $f_1(z)$ and

$f_1(z+h)$ about $z=0$ are of the form (3.14).

We again return to the function \widetilde{f}_1 . Near $z = \pm h$ we have

$$\widetilde{f}_1(z) = \pm \wp'(h; g_2, g_3)(z \mp h) + O((z \mp h)^2), \quad (3.26)$$

which implies that

$$\widetilde{f}_1(z \pm h) = \pm \wp'(h; g_2, g_3)z + O(z^2) \quad (3.27)$$

near $z=0$. In (3.26-3.27), $\wp'(h; g_2, g_3) \neq 0$ by the assumption that h is not a half-period. Then, the Laurent series expansions of $\widetilde{f}_1(z-h)$, $\widetilde{f}_1(z)$ and $\widetilde{f}_1(z+h)$ about $z=0$ of the form (3.15). \square

Let us consider this result in the context of a particular delay-differential equation:

$$(\bar{u}')^2 - (u')^2 = 4(\bar{u}^3 - u^3) - g_2(\bar{u} - u). \quad (3.28)$$

This equation is solved by

$$u(z) = \wp(z; g_2, g_3), \quad (3.29)$$

where g_2 is fixed by the equation and g_3 is free. It follows from (3.1.1) that (3.28) must admit the singularity sequence (3.11), since g_3 can be chosen so that $h \in \Lambda$. Alternatively, g_3 can be chosen so that $h \notin \Lambda$, in which case (3.28), after an affine transformation, must admit either (3.13) or (3.15), depending on whether or not $2h \in \Lambda$.

Now we consider the second kind of order-two elliptic solution (3.4). In the context of autonomous delay-differential equations, we can take $a_1 = -a_2$ without loss of generality. Then we have

$$f_2(z) = \alpha [\zeta(z+a; g_2, g_3) - \zeta(z-a; g_2, g_3)] + \beta. \quad (3.30)$$

There are two special cases to consider. In the limit $a \rightarrow 0 \pmod{\Lambda}$, we obtain

(3.3) (with $a = 0$) after redefining α and β . When $2a \in \Lambda$, but $a \notin \Lambda$ we have $\wp'(a; g_2, g_3) = 0$. Then, by means of the addition law for the Weierstrass ζ -function (A.27), (3.30) can be rewritten:

$$f_2(z) = -\frac{\alpha\wp'(a; g_2, g_3)}{\wp(z, g_2, g_3) - \wp(a; g_2, g_3)} + \beta + 2\alpha\zeta(a; g_2, g_3) \quad (3.31)$$

and we see that f_2 reduces to a constant. So we assume that $a \notin \Lambda$ and $2a \notin \Lambda$. In this nondegenerate case (3.31) still holds. Under the further assumption $h \notin \Lambda$, this can be written as (the invariants appearing in the Weierstrass functions are suppressed)

$$f_2(z) = \frac{-\alpha\wp'(a) + [\beta + 2\alpha\zeta(a)][f(z) + \wp(h) - \wp(a)]}{f(z) + \wp(h) - \wp(a)}, \quad (3.32)$$

where $f(z)$ represents (3.6). Therefore, we could extend the previous theorem (3.1.1), by allowing full Möbius transformations instead of only affine transformations, to include (3.30). However, the singularity sequences associated with (3.30) are more easily characterized without reference to the Weierstrass \wp -function. In doing this, we will find use for the function

$$\zeta(z + a; g_2, g_3) - \zeta(z - a; g_2, g_3) - 2\zeta(a; g_2, g_3). \quad (3.33)$$

This function has double zeroes at $z = 0 \pmod{\Lambda}$.

Theorem 3.1.2. *Consider the functions*

$$f_2(z) = \alpha[\zeta(z - a_1; g_2, g_3) - \zeta(z - a_2; g_2, g_3)] + \beta \quad (3.34)$$

and

$$\widetilde{f}_2(z) = \alpha[\zeta(z + a; g_2, g_3) - \zeta(z - a; g_2, g_3) - 2\zeta(a; g_2, g_3)], \quad (3.35)$$

where $a \notin \Lambda$, $2a \notin \Lambda$, $\alpha \neq 0$, and $\beta \neq -2\alpha\zeta(a; g_2, g_3)$. If $h \in \Lambda$, f_2 and \widetilde{f}_2 contain

the singularity sequences

$$\left(\dots, \in \text{cn}(\Omega h|m) \text{dn}(\Omega h|m) \text{fty}_+^1, \infty_+^1, \dots \right) \text{ and } \left(\dots, \infty_-^1, \infty_-^1, \dots \right). \quad (3.36)$$

If $h \notin \Lambda$ and $h = 2a \pmod{\Lambda}$, f_2 and \widetilde{f}_2 contain the sequence

$$\left(\text{rg}_1, \infty_{\pm}^1, \infty_{\mp}^1, \text{rg}_2 \right). \quad (3.37)$$

If $h \notin \Lambda$ and $h \neq a \pmod{\Lambda}$ and $h \neq 2a \pmod{\Lambda}$, f_2 and \widetilde{f}_2 contain the sequence

$$\left(\text{rg}_1, \infty^1, \text{rg}_2 \right). \quad (3.38)$$

If $h \notin \Lambda$ and $h = a \pmod{\Lambda}$, f_2 contains the sequence

$$\left(\infty_{\pm}^1, \text{rg}, \infty_{\mp}^1 \right) \quad (3.39)$$

and \widetilde{f}_2 contains the sequence

$$\left(\infty_{\pm}^1, 0^2, \infty_{\mp}^1 \right). \quad (3.40)$$

Proof. We proceed by cases. We can take $a_1 = -a_2 = -a$ in (3.34-3.35) without loss of generality.

Case 1: $h \in \Lambda$. We have the Laurent expansions

$$f_2(z) = \mp \frac{\alpha}{z \pm a} + O(1) \quad (3.41)$$

in punctured disks about $z = -a$ and $z = a$, respectively. The assumption that h is a period implies that

$$f_2(z \mp a + nh) = \mp \frac{\alpha}{z \pm a} + O(1), \quad n \in \mathbb{Z} \quad (3.42)$$

in a punctured disk about $z = 0$. The sequences consisting of the Laurent series for $f_2(z - a + nh)$ and $f_2(z + a + nh)$ about $z = 0$ are of the form of those in

(3.36), i.e. the coefficients of the leading order terms differ only by a sign. An identical argument applies to $\widetilde{f}_2(z)$.

Case 2: $h \notin \Lambda$ and $h = 2a \bmod \Lambda$ In a punctured disk about $z = a$ we have

$$f_2(z) = -\frac{\alpha}{z-a} + O(1); \quad (3.43)$$

it follows that

$$f_2(z+a) = -\frac{\alpha}{z} + O(1). \quad (3.44)$$

in a punctured disk about $z = 0$. By assumption, $z = a+h$ is neither a zero nor a pole (and so is a regular point). By assumption and the parity of f we have

$$f_2(z+a-h) = f_2(z-a) = f_2(z+a) = \frac{\alpha}{z} + O(1). \quad (3.45)$$

in a punctured disk about $z = 0$. Again by parity, it follows that $z = -a-h = a-2h$ is a regular point. The sequence consisting of the Laurent series expansions of $f_2(z+a-2h)$, $f_2(z+a-h)$, $f_2(z+a)$, and $f_2(z+a+h)$ about $z = 0$ is of the form (3.37). An identical argument applies to g .

Case 3: $h \notin \Lambda$ and $h \neq a \bmod \Lambda$ and $h \neq 2a \bmod \Lambda$. In this case we also have (3.44) in a punctured disk about $z = 0$. By assumption, $z = a \pm h$ are regular points. Then, the sequence consisting of the Laurent series expansions of $f_2(z+a-h)$, $f_2(z+a)$, and $f_2(z+a+h)$ about $z = 0$ is of the form (3.38). An identical argument applies to \widetilde{f}_2 .

Case 4: $h \notin \Lambda$ and $h = a \bmod \Lambda$. Near $z = 0$ we have

$$f_2(z) = -\alpha\wp'(a; g_2, g_3)z^2 + O(z^4) \quad (3.46)$$

in a punctured disk about $z = h$ we have

$$f_2(z) = -\frac{\alpha}{z-h} + O(1). \quad (3.47)$$

It follows that

$$f_2(z+h) = -\frac{\alpha}{z} + O(1) \quad (3.48)$$

in a punctured disk about $z = 0$. By the parity of f_2 , it follows that

$$f_2(z-h) = \frac{\alpha}{z} + O(1) \quad (3.49)$$

in a punctured disk about $z = 0$. The sequence consisting of the Laurent series expansions of $f_2(z-h)$, $f_2(z)$, and $f_2(z+h)$ about $z = 0$ is of the form (3.40).

We now turn to the function \widetilde{f}_2 . Near $z = 0$ we have

$$\widetilde{f}_2(z) = \beta + 2\alpha\zeta(a; g_2, g_3) + O(z^2) \quad (3.50)$$

in a punctured disk about $z = h$ we have

$$\widetilde{f}_2(z) = -\frac{\alpha}{z-h} + O(1). \quad (3.51)$$

It follows that

$$\widetilde{f}_2(z+h) = -\frac{\alpha}{z} + O(1) \quad (3.52)$$

in a punctured disk about $z = 0$. By the parity of \widetilde{f}_2 , it follows that

$$\widetilde{f}_2(z-h) = \frac{\alpha}{z} + O(1) \quad (3.53)$$

in a punctured disk about $z = 0$. The sequence consisting of the Laurent series expansions of $\widetilde{f}_2(z-h)$, $\widetilde{f}_2(z)$, and $\widetilde{f}_2(z+h)$ about $z = 0$ is of the form (3.39).

□

It is worth discussing the sequences (3.37) and (3.38) in greater detail. By the parity of f in (3.33), it is easily seen that if

$$\text{rg}_1 = \sum_{n=0}^{\infty} u_n z^n \quad (3.54)$$

in (3.37), we must have

$$\text{rg}_2 = \sum_{n=0}^{\infty} (-1)^n u_n z^n \quad (3.55)$$

in the same sequence. For the same reason if

$$\text{rg}_1 = \sum_{n=0}^{\infty} u_n z^n, \quad \text{rg}_2 = \sum_{n=0}^{\infty} v_n z^n \quad (3.56)$$

in (3.38) with positive residue, the equation must also admit a sequence with negative residue and

$$\text{rg}'_1 = \sum_{n=0}^{\infty} (-1)^n v_n z^n, \quad \text{rg}'_2 = \sum_{n=0}^{\infty} (-1)^n u_n z^n \quad (3.57)$$

with the same expansion coefficients as in (3.56).

We can now return briefly to an example (2.115) discussed in the previous chapter. The equation

$$u' + \bar{u}' = \bar{u}^2 - u^2 \quad (3.58)$$

was shown to admit the singularity sequence (3.37), which, by theorem (3.1.2), is a necessary condition for an equation to admit an order-two elliptic solution of the form (3.34) (with $h = 2a \pmod{\Lambda}$). Later in this chapter, we construct a solution of this form for the equation (3.58).

We conclude this section by reiterating its intent: to establish a relationship between the singularity sequences often used to isolate integrable equations and the elliptic solutions that are the focus of this thesis. This explains the fact that a number of known examples are recovered in our classification performed in the subsequent section. These are then discussed as examples later in this chapter.

3.2 Order-two elliptic solutions

The main result of this section is a list of five equations, solvable by order-two elliptic functions with at least two degrees of parametric freedom. In order to achieve this result, we first identify all equations, in the form of a

vector subspace of 4×4 matrices X in (3.1), that admit order-two elliptic solutions with at least a single degree of parametric freedom. Having done this, a search may be performed within the vector subspace for equations that admit multiparameter elliptic solutions.

We begin by classifying all equations in the bi-Riccati class (3.1) that are solved by order-two elliptic functions. We will show that we may use the Möbius-Jacobi representation

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0 | m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0 | m) + \delta}, \quad (3.59a)$$

$$\alpha\delta - \beta\gamma = 1 \quad (3.59b)$$

for any such function. The parameter $z_0 \in \mathbb{C}$ is free by the translational symmetry inherent in autonomous equations. For this reason, we omit it in intermediate calculations.

Any order-two elliptic function can be written in the form (3.59). In the previous section, we established that any order-two elliptic function can be written as a Möbius transformation of the Weierstrass \wp -function. Therefore, if we can show that there exist parameters $\alpha, \beta, \gamma, \delta, z_0, \Omega$ and m such that

$$\wp(z; g_2, g_3) = \frac{\alpha \operatorname{sn}(\Omega z + z_0 | m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0 | m) + \delta} \quad (3.60a)$$

$$\alpha\delta - \beta\gamma \neq 0, \quad (3.60b)$$

we can take Möbius transformation of both sides of (3.60a) to represent an arbitrary order-two elliptic function as a Möbius transformation of a Jacobi sine function. As our ultimate interest is in solutions to autonomous delay-differential equations, we regard two elliptic functions as equivalent when they differ by a constant translation of the independent variable. Thus, it suffices to show that the differential equations

$$(u')^2 = \Omega^2 (1 - u^2) (1 - mu^2) \quad (3.61a)$$

$$(v')^2 = 4v^3 - g_2v - g_3, \quad (3.61b)$$

solved by $\operatorname{sn}(\Omega z + z_0|m)$ and $\wp(z + z_0; g_2, g_3)$, respectively, are Möbius-equivalent. We set $u = (\alpha v + \beta)/(\gamma v + \delta)$ in (3.61a) and equate the result with (3.61b) to give us a system of equations in powers of v . After imposing $\alpha\delta - \beta\gamma = 1$, we obtain the solution¹

$$\begin{aligned} \alpha^2 &= \frac{\Omega(1 + 14m + m^2)}{6g_2(1 - m)}, & \beta &= \frac{5 - m}{6\alpha(1 - m)}, & \gamma &= \alpha \\ \Omega^2 &= \frac{2g_2^2(1 + m)(1 - 34m + m^2)}{3g_3(1 + 14m + m^2)^2}, & 1 + 14m + m^2 &= \frac{12g_2}{\Omega^4}. \end{aligned} \quad (3.62)$$

Thus we have established that all order-two elliptic functions can be expressed in Möbius-Jacobi form (3.59) and we will next find all such solutions of bi-Riccati equations.

We now observe that the bi-Riccati class is form-invariant under Möbius transformations of the dependent variable. Under the change of variables

$$u \rightarrow \frac{\alpha u + \beta}{\gamma u + \delta}, \quad \alpha\delta - \beta\gamma = 1, \quad (3.63)$$

(3.1) becomes

$${}^tU^tMXM\bar{U} = 0, \quad U = (1, u, u^2, u'), \quad X : \mathbb{C} \rightarrow \mathbb{C}^{4 \times 4} \quad (3.64)$$

where

$$M = \begin{bmatrix} \delta^2 & 2\gamma\delta & \gamma^2 & 0 \\ \beta\delta & \alpha\delta + \beta\gamma & \alpha\gamma & 0 \\ \beta^2 & 2\alpha\beta & \alpha^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.65)$$

Specializing to the case of Jacobi sine solutions, if (3.59) solves (3.1), then

¹The solution is written somewhat implicitly to simplify the presentation. Note that by squaring the fourth equation and substituting it into the fifth, we obtain an eighth-order polynomial equation for m in terms of given quantities (g_2 and g_3). Supposing this can be solved for m , we can sequentially solve for Ω , α , β , and γ .

$\operatorname{sn}(\Omega z|m)$ solves (3.64). But (3.64) is equivalent to (3.1) via a redefinition of X :

$$X \rightarrow {}^t M^{-1} X M^{-1}. \quad (3.66)$$

Therefore we restrict ourselves to the problem of finding all $X \in \mathbb{C}^{4 \times 4}$ such that

$${}^t U X \bar{U} = 0 \quad (3.67a)$$

$$U = \left(1, \operatorname{sn}(\Omega z|m), \operatorname{sn}^2(\Omega z|m), \operatorname{sn}(\Omega z|m)'\right). \quad (3.67b)$$

In doing this, we will require standard identities for Jacobian elliptic functions, in particular the differential relations (A.54), algebraic relations (A.55), differential equations (A.56-A.57), and the addition law (A.58a). To simplify the presentation we also introduce the notation²

$$u = \operatorname{sn}(\Omega z|m), \quad s = \operatorname{sn}(\Omega h|m), \quad c = \operatorname{cn}(\Omega h|m) \quad d = \operatorname{dn}(\Omega h|m). \quad (3.68)$$

We now return to (3.67). By means of the addition law (A.58a), we see that $U^t X \bar{U}$ in (3.67a) is rational in u and its first and second derivatives and linear in each entry of X . By clearing out denominators and using differential equations (A.56a) and (A.57a) to replace u'' and $(u')^2$, respectively, we arrive at a polynomial equation

$$P(u; X) + u' Q(u; X) = 0, \quad (3.69)$$

where P and Q are seventh- and fifth-order in u , respectively, with coefficients rational in Ω , m , s , c , d and linear in the entries of X . According to (A.57a), u and u' are linearly independent; by treating them as formal variables we see that

$$L = P(u; X) + u' Q(u; X) \quad (3.70)$$

²By our notation, quantities such as cd indicate $\operatorname{cn}(\Omega h|m)\operatorname{dn}(\Omega h|m)$ and not the Jacobi cd -function $\operatorname{cn}(\Omega h|m)/\operatorname{dn}(\Omega h|m)$.

is a linear map from $\mathbb{C}^{4 \times 4}$ to a twelve-dimensional subspace of $\mathbb{C}[u] \oplus u'\mathbb{C}[u]$. Choosing the standard bases for both vector spaces, the kernel of this map can be computed by Gauss-Jordan elimination on the matrix representation of L . We will choose a particular basis for this kernel where each basis vector has at most one nonzero entry in the fourth row or column of the matrix. Through (3.1) this corresponds to the appearance of at most one derivative in the bi-Riccati delay-differential equation. Viewed in this way, seven of our basis elements, X^1, \dots, X^7 ,³ contain a single derivative while one basis element, X^0 , corresponds to a purely discrete equation:

$$\ker L = \text{span} \{X^0, X^1, X^2, X^3, X^4, X^5, X^6, X^7\}, \quad (3.71)$$

where

$$X^0 = \frac{\Omega}{s} \begin{bmatrix} s^2 & 0 & -1 & 0 \\ 0 & 2cd & 0 & 0 \\ -1 & 0 & ms^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.72a)$$

$$X^1 = \frac{1}{s} \begin{bmatrix} 0 & -\Omega & 0 & 0 \\ cd\Omega & 0 & 0 & 0 \\ 0 & ms^2\Omega & 0 & 0 \\ s & 0 & 0 & 0 \end{bmatrix} \quad (3.72b)$$

$$X^2 = \frac{1}{s} \begin{bmatrix} -\Omega & 0 & 0 & 0 \\ 0 & -cd\Omega & 0 & 0 \\ \Omega & 0 & 0 & 0 \\ 0 & s & 0 & 0 \end{bmatrix} \quad (3.72c)$$

³Here, and in what follows, superscripts on X , x , λ , and ϕ are labels. This is done so that subscripts may indicate partial derivatives or evaluation at a particular value of the deformation parameter ϵ , as explained below.

$$X^3 = \frac{1}{s} \begin{bmatrix} 0 & -\Omega & 0 & 0 \\ 0 & 0 & -cd\Omega & 0 \\ 0 & \Omega & 0 & 0 \\ 0 & 0 & s & 0 \end{bmatrix} \quad (3.72d)$$

$$X^4 = \frac{1}{s^2} \begin{bmatrix} 0 & 0 & -cd\Omega^2 & 0 \\ 0 & [2 - (1+m)s^2]\Omega^2 & 0 & 0 \\ -cd\Omega^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^2 \end{bmatrix} \quad (3.72e)$$

$$X^5 = \frac{1}{s} \begin{bmatrix} 0 & -cd\Omega & 0 & s \\ \Omega & 0 & -ms^2\Omega & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.72f)$$

$$X^6 = \frac{1}{s} \begin{bmatrix} \Omega s^2 & 0 & -\Omega & 0 \\ 0 & cd\Omega & 0 & s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.72g)$$

$$X^7 = \frac{1}{s} \begin{bmatrix} 0 & 0 & 0 & 0 \\ s^2 & 0 & -\Omega & 0 \\ 0 & cd\Omega & 0 & s \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.72h)$$

A bi-Riccati equation (3.1) with constant X is solved by $u = \operatorname{sn}(\Omega z + z_0|m)$ if and only if $X \in \ker L$, for which we have given an explicit basis. The eight basis elements correspond to a seven parameter family of bi-Riccati equations: a generic $X \in \ker L$ may be represented by a set of eight expansion coefficients $\{\lambda^n\}$ together with the basis (3.72). The corresponding bi-Riccati equation (3.1) is homogenous in the expansion coefficients, and so we may divide through by any nonzero λ^n leaving seven degrees of freedom.

We will now discuss the existence of multiparameter Möbius-Jacobi families of solutions to autonomous bi-Riccati equations. In particular, we search

for equations that admit solutions

$$u(z; \varepsilon) = \frac{\alpha(\varepsilon) \operatorname{sn}(\Omega(\varepsilon)z + z_0 | m(\varepsilon)) + \beta(\varepsilon)}{\gamma(\varepsilon) \operatorname{sn}(\Omega(\varepsilon)z + z_0 | m(\varepsilon)) + \delta(\varepsilon)}, \quad (3.73a)$$

$$\alpha(\varepsilon)\delta(\varepsilon) - \beta(\varepsilon)\gamma(\varepsilon) \neq 0, \quad (3.73b)$$

where at least one of the parameters α , β , γ , δ , Ω , and m has nontrivial dependence on the auxiliary variable ε . This parametric freedom is in addition to the translational symmetry represented by z_0 (which, again, will be omitted in intermediate calculations). Suppose (3.73) solves an autonomous bi-Riccati equation

$${}^tU(\varepsilon)X_0U(\varepsilon) = 0, \quad X_0 \in \mathbb{C}^{4 \times 4}, \quad (3.74)$$

where we have emphasized the ε -dependence of U through the parameters in (3.73). It follows that $u(z) = \operatorname{sn}(\Omega(\varepsilon)z | m(\varepsilon))$ solves

$${}^tU(\varepsilon){}^tM(\varepsilon)X_0M(\varepsilon)U(\varepsilon) = 0, \quad (3.75)$$

where

$$M(\varepsilon) = \begin{bmatrix} \delta(\varepsilon)^2 & 2\gamma(\varepsilon)\delta(\varepsilon) & \gamma(\varepsilon)^2 & 0 \\ \beta(\varepsilon)\delta(\varepsilon) & \alpha(\varepsilon)\delta(\varepsilon) + \beta(\varepsilon)\gamma(\varepsilon) & \alpha(\varepsilon)\gamma(\varepsilon) & 0 \\ \beta(\varepsilon)^2 & 2\alpha(\varepsilon)\beta(\varepsilon) & \alpha(\varepsilon)^2 & 0 \\ 0 & 0 & 0 & \alpha(\varepsilon)\delta(\varepsilon) - \beta(\varepsilon)\gamma(\varepsilon) \end{bmatrix}. \quad (3.76)$$

We seek to identify matrices X_0 and parameterizations α , β , γ , δ , Ω , and m such that (3.75) is satisfied in some neighborhood of $\varepsilon = 0$.

As before, we perform our classification up to a constant Möbius transformation: by the redefinition

$$X_0 \rightarrow {}^tM(0)^{-1}X_0M(0)^{-1} \quad (3.77)$$

we may take

$$\alpha(0) = \delta(0) = 1, \quad \beta(0) = \gamma(0) = 0, \quad (3.78)$$

i.e. the Möbius transformation (3.76) reduces to the identity at $\varepsilon = 0$. We will also parameterize the Möbius transformation by $\delta(\varepsilon) = 1$:

$$M(\varepsilon) = \begin{bmatrix} 1 & 2\gamma(\varepsilon) & \gamma(\varepsilon)^2 & 0 \\ \beta(\varepsilon) & \alpha(\varepsilon) + \beta(\varepsilon)\gamma(\varepsilon) & \alpha(\varepsilon)\gamma(\varepsilon) & 0 \\ \beta(\varepsilon)^2 & 2\alpha(\varepsilon)\beta(\varepsilon) & \alpha(\varepsilon)^2 & 0 \\ 0 & 0 & 0 & \alpha(\varepsilon) - \beta(\varepsilon)\gamma(\varepsilon) \end{bmatrix}. \quad (3.79)$$

Because (3.78) implies that $M(\varepsilon)$ is locally invertible near $\varepsilon = 0$, this parameterization is locally equivalent to (3.59b). Returning now to (3.75) with $M(\varepsilon)$ as in (3.79), we define

$$X(\varepsilon) = {}^t M(\varepsilon) X_0 M(\varepsilon) \quad (3.80)$$

so that (3.75) becomes

$${}^t U(\varepsilon) X(\varepsilon) \bar{U}(\varepsilon) = 0. \quad (3.81)$$

We view this as an equation for $X(\varepsilon)$. As a consequence of (3.72), the solution space of (3.81), for $X(\varepsilon)$, consists of elements

$$\sum_{n=0}^7 \lambda^n X^n = \begin{bmatrix} \Omega s (\lambda^0 - \lambda^2 + \lambda^6) & -\frac{\Omega}{s} (\lambda^1 + s^2 \lambda^3 + cd \lambda^5) & -\frac{\Omega}{s} (\lambda^0 + \frac{\Omega cd}{s} \lambda^4 + \lambda^6) & \lambda^5 \\ \frac{\Omega}{s} (cd \lambda^1 + \lambda^5 + s^2 \lambda^7) & \frac{\Omega cd}{s} (2\lambda^0 - \lambda^2 + \lambda^6) + \frac{\Omega^2 (c^2 + d^2)}{s^2} \lambda^4 & -\frac{\Omega}{s} (cd \lambda^3 + ms^2 \lambda^5 + \lambda^7) & \lambda^6 \\ -\frac{\Omega}{s} (\lambda^0 - \lambda^2 + \frac{cd \Omega}{s} \lambda^4) & \frac{\Omega}{s} (ms^2 \lambda^1 + \lambda^3 + cd \lambda^7) & \Omega ms \lambda^0 & \lambda^7 \\ \lambda^1 & \lambda^2 & \lambda^3 & \lambda^4 \end{bmatrix}. \quad (3.82)$$

The parameters Ω and m (and consequently s , c , and d) above have suppressed dependence on ε . The solution space is parameterized by expansion coefficients λ^i . We use the notation x^{ij} for the entries of the matrix $X(\varepsilon)$ and x_0^{ij} for the

entries of X_0 . If $X(\varepsilon)$ satisfies (3.81), we have

$$\lambda^1 = x^{41}, \lambda^2 = x^{42}, \lambda^3 = x^{43}, \lambda^4 = x^{44}, \lambda^5 = x^{14}, \lambda^6 = x^{24}, \lambda^7 = x^{34} \quad (3.83)$$

and

$$\lambda^0 = \frac{x^{33}}{\Omega ms}. \quad (3.84)$$

The remaining entries of $X(\varepsilon)$ (those not appearing in (3.83-3.84)) can be parameterized in terms of the x^{ij} appearing in (3.83-3.84) if $X(\varepsilon) \in \ker L$.

This parameterization is given by

$$\phi^n(X(\varepsilon), \Omega(\varepsilon), m(\varepsilon)) = 0, \quad n = 1, \dots, 8, \quad (3.85)$$

where

$$\phi^1 = x^{11} - \frac{1}{m}x^{33} + \Omega s(x^{42} - x^{24}) \quad (3.86a)$$

$$\phi^2 = x^{12} + \frac{\Omega}{s}(x^{41} + s^2x^{43} + cdx^{14}) \quad (3.86b)$$

$$\phi^3 = x^{13} + \frac{1}{ms^2}x^{33} + \frac{\Omega^2 cd}{s^2}x^{44} + \frac{\Omega}{s}x^{24} \quad (3.86c)$$

$$\phi^4 = x^{21} - \frac{\Omega}{s}(cdx^{41} + x^{14} + s^2x^{34}) \quad (3.86d)$$

$$\phi^5 = x^{22} - \frac{2cd}{ms^2}x^{33} + \frac{\Omega cd}{s}(x^{42} - x^{24}) - \frac{\Omega^2(c^2 + d^2)}{s^2}x^{44} \quad (3.86e)$$

$$\phi^6 = x^{23} + \frac{\Omega}{s}(cdx^{43} + ms^2x^{14} + x^{43}) \quad (3.86f)$$

$$\phi^7 = x^{31} + \frac{1}{ms^2}x^{33} - \frac{\Omega}{s}x^{42} + \frac{\Omega^2 cd}{s^2}x^{44} \quad (3.86g)$$

$$\phi^8 = x^{32} - \frac{\Omega}{s}(ms^2x^{41} + x^{43} + cdx^{34}). \quad (3.86h)$$

Recalling that the x^{ij} depend on the Möbius parameters via (3.80), we view ϕ (with components (3.86)) as a map from \mathbb{C}^5 (the space of tuples $(\alpha, \beta, \gamma, \Omega, m)$) to \mathbb{C}^8 , parameterized by the matrix X_0 . It is clear that $X(\varepsilon)$ satisfies (3.81)

and only if (3.85) holds. In particular, at $\varepsilon = 0$ we must have

$$\phi^n(X_0, \Omega_0, m_0) = 0, \quad n = 1, \dots, 8, \quad (3.87)$$

where $\Omega_0 = \Omega(0)$ and $m_0 = m(0)$ (going forward, we will extend this notation to $s_0 = \text{sn}(\Omega_0 h | m_0)$, etc.). This gives us a parameterization of the matrix X_0 . We can view (3.85), whenever (3.87) holds, as a symmetry condition. Substituting (3.87) into (3.85) leads to a complicated system which is not immediately solvable, although one trivial solution is known:

$$\alpha = 1, \quad \beta = 0, \quad \gamma = 0, \quad \Omega = \Omega_0, \quad m = m_0 \quad (3.88)$$

Thus we focus on a linearized symmetry condition; by a Taylor expansion of (3.85) about the trivial solution, we obtain:

$$\begin{aligned} \phi(X(\varepsilon), \Omega(\varepsilon), m(\varepsilon)) &= \phi(X_0, \Omega_0, m_0) + \varepsilon \left(\frac{\partial \phi}{\partial \Omega} \frac{d\Omega}{d\varepsilon} + \frac{\partial \phi}{\partial m} \frac{dm}{d\varepsilon} \right) \Big|_{\varepsilon=0} \\ &+ \varepsilon \sum_{1 \leq i, j \leq 4} \left[\frac{\partial \phi}{\partial x^{ij}} \left(\frac{\partial x^{ij}}{\partial \alpha} \frac{d\alpha}{d\varepsilon} + \frac{\partial x^{ij}}{\partial \beta} \frac{d\beta}{d\varepsilon} + \frac{\partial x^{ij}}{\partial \gamma} \frac{d\gamma}{d\varepsilon} \right) \right] \Big|_{\varepsilon=0} + O(\varepsilon^2). \end{aligned} \quad (3.89)$$

If (3.85) is satisfied in some neighborhood of $\varepsilon = 0$, it is necessary that the order- ε^1 term above vanish (the order- ε^0 term vanishes by assumption). To discuss this further it is convenient to introduce the the Jacobian

$$J = \frac{\partial(\phi^1, \dots, \phi^8)}{\partial(\alpha, \beta, \gamma, \Omega, m)} \Big|_{\varepsilon=0} \quad (3.90)$$

and write (3.89) as

$$\phi(X(\varepsilon), \Omega(\varepsilon), m(\varepsilon)) = \phi(X_0, \Omega_0, m_0) + \varepsilon \frac{\partial(\phi^1, \dots, \phi^8)}{\partial(\alpha, \beta, \gamma, \Omega, m)} \Big|_{\varepsilon=0} \begin{bmatrix} \frac{d\alpha}{d\varepsilon} \Big|_{\varepsilon=0} \\ \frac{d\beta}{d\varepsilon} \Big|_{\varepsilon=0} \\ \frac{d\gamma}{d\varepsilon} \Big|_{\varepsilon=0} \\ \frac{d\Omega}{d\varepsilon} \Big|_{\varepsilon=0} \\ \frac{dm}{d\varepsilon} \Big|_{\varepsilon=0} \end{bmatrix} + O(\varepsilon^2). \quad (3.91)$$

In this form, we see that the order- ε^1 term can vanish either trivially (when the column vector consisting of ε -derivatives is identically zero) or nontrivially, when the Jacobian is rank-deficient (has a rank less than its maximal rank of 5). As a consequence of (3.86), each entry of the Jacobian is linear in the entries of X_0 , but (3.87) can be used to eliminate eight of these. As a result, the Jacobian depends on eight parameters x_0^{ij} ; our aim is to determine the conditions on these parameters so the Jacobian is rank-deficient.

In computing the Jacobian, we require derivatives of (3.80) with respect to the Möbius parameters (we use the notation $M_\alpha = \partial M / \partial \alpha$, etc.):

$$\frac{\partial X}{\partial \alpha} = {}^t M_\alpha {}^t M^{-1} X + X M^{-1} M_\alpha \quad (3.92a)$$

$$\frac{\partial X}{\partial \beta} = {}^t M_\beta {}^t M^{-1} X + X M^{-1} M_\beta \quad (3.92b)$$

$$\frac{\partial X}{\partial \gamma} = {}^t M_\gamma {}^t M^{-1} X + X M^{-1} M_\gamma. \quad (3.92c)$$

At $\varepsilon = 0$, M and X reduce to the identity and X_0 , respectively, and we obtain

$$\frac{\partial X}{\partial \alpha} \Big|_{\varepsilon=0} = {}^t M_\alpha|_{\varepsilon=0} X_0 + X_0 M_\alpha|_{\varepsilon=0} \quad (3.93a)$$

$$\frac{\partial X}{\partial \beta} \Big|_{\varepsilon=0} = {}^t M_\beta|_{\varepsilon=0} X_0 + X_0 M_\beta|_{\varepsilon=0} \quad (3.93b)$$

$$\frac{\partial X}{\partial \gamma} \Big|_{\varepsilon=0} = {}^t M_\gamma|_{\varepsilon=0} X_0 + X_0 M_\gamma|_{\varepsilon=0}, \quad (3.93c)$$

where

$$M_\alpha|_{\varepsilon=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.94a)$$

$$M_\beta|_{\varepsilon=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.94b)$$

$$M_\gamma|_{\varepsilon=0} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (3.94c)$$

We arrive at

$$\left. \frac{\partial X}{\partial \alpha} \right|_{\varepsilon=0} = \begin{bmatrix} 0 & x_0^{12} & 2x_0^{13} & x_0^{14} \\ x_0^{21} & 2x_0^{22} & 3x_0^{23} & 2x_0^{24} \\ 2x_0^{31} & 3x_0^{32} & 4x_0^{33} & 3x_0^{34} \\ x_0^{41} & 2x_0^{42} & 3x_0^{43} & 2x_0^{44} \end{bmatrix} \quad (3.95a)$$

$$\left. \frac{\partial X}{\partial \beta} \right|_{\varepsilon=0} = \begin{bmatrix} x_0^{12} + x_0^{21} & 2x_0^{13} + x_0^{22} & x_0^{23} & x_0^{24} \\ x_0^{22} + 2x_0^{31} & 2x_0^{23} + 2x_0^{32} & 2x_0^{33} & 2x_0^{34} \\ x_0^{32} & 2x_0^{33} & 0 & 0 \\ x_0^{42} & 2x_0^{43} & 0 & 0 \end{bmatrix} \quad (3.95b)$$

$$\left. \frac{\partial X}{\partial \gamma} \right|_{\varepsilon=0} = \begin{bmatrix} 0 & 2x_0^{11} & x_0^{12} & 0 \\ 2x_0^{11} & 2x_0^{12} + 2x_0^{21} & 2x_0^{13} + x_0^{22} & 2x_0^{14} \\ x_0^{21} & x_0^{22} + x_0^{31} & x_0^{23} + x_0^{32} & x_0^{24} \\ 0 & 2x_0^{41} & x_0^{42} & 0 \end{bmatrix}. \quad (3.95c)$$

We will also require derivatives of the Jacobi elliptic functions with respect

to the modulus [50]:

$$\frac{\partial}{\partial m} \operatorname{sn}(z|m) = \frac{\operatorname{cn}(z|m) \operatorname{dn}(z|m)}{2m(1-m)} [(1-m)z - \mathcal{E}(z|m) + m \operatorname{sn}(z|m) \operatorname{cd}(z|m)] \quad (3.96a)$$

$$\frac{\partial}{\partial m} \operatorname{cn}(z|m) = -\frac{\operatorname{sn}(z|m) \operatorname{dn}(z|m)}{2m(1-m)} [(1-m)z - \mathcal{E}(z|m) + m \operatorname{sn}(z|m) \operatorname{cd}(z|m)] \quad (3.96b)$$

$$\frac{\partial}{\partial m} \operatorname{dn}(z|m) = -\frac{\operatorname{sn}(z|m) \operatorname{cn}(z|m)}{2(1-m)} [(1-m)z - \mathcal{E}(z|m) + \operatorname{dn}(z|m) \operatorname{sc}(z|m)] \quad (3.96c)$$

where $\mathcal{E}(z|m)$ is the Jacobian \mathcal{E} -function

$$\mathcal{E}(z|m) := \int_0^z \operatorname{dn}^2(t|m) dt. \quad (3.97)$$

If we introduce the quantity

$$\Xi = \frac{\Omega h(1-m) - \mathcal{E}(\Omega h|m)}{2hm(1-m)} + \frac{\operatorname{sc}}{2hd(1-m)}, \quad (3.98)$$

the Ω and m derivatives of s , c , and d can be related in a very simple way:

$$\frac{\partial s}{\partial m} = \Xi \frac{\partial s}{\partial \Omega}, \quad \frac{\partial c}{\partial m} = \Xi \frac{\partial c}{\partial \Omega}, \quad \frac{\partial d}{\partial m} = \left(\Xi + \frac{s}{2hmcd} \right) \frac{\partial d}{\partial \Omega}. \quad (3.99)$$

We are now prepared to write down the components of the Jacobian (3.90).

To do this, we express each component of ϕ as

$$\phi^n = \sum_{i,j=0}^3 x^{ij} p_{ij}^n(\Omega, m) q_{ij}^n(s, c, d), \quad n = 1, \dots, 8. \quad (3.100)$$

Here the dependence on the parameters Ω and m enters explicitly only through the coefficients p_{ij}^n , whereas the q_{ij}^n carry only implicit dependence on these parameters through s , c , and d . The x^{ij} depend on the Möbius parameters

through (3.80). The derivative with respect to α is computed as

$$\frac{\partial \phi^n}{\partial \alpha} = \sum_{i,j=0}^3 \frac{\partial x^{ij}}{\partial \alpha} p_{ij}^n q_{ij}^n. \quad (3.101)$$

At $\epsilon = 0$, the derivatives $\partial x^{ij}/\partial \alpha$ can be found from (3.95) and the condition (3.87), together with the (3.86), ensures that the resulting expression can be written in terms of the eight x_0^{ij} appearing in (3.83-3.84). Derivatives with respect to β and γ are calculated analogously. The derivative with respect to Ω is

$$\frac{\partial \phi^n}{\partial \Omega} = \sum_{i,j=0}^3 x^{ij} \left[\frac{\partial p_{ij}^n}{\partial \Omega} q_{ij}^n + p_{ij}^n \left(\frac{\partial q_{ij}^n}{\partial s} \frac{\partial s}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial c} \frac{\partial c}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial d} \frac{\partial d}{\partial \Omega} \right) \right] \quad (3.102)$$

and the derivative with respect to m is

$$\frac{\partial \phi^n}{\partial m} = \sum_{i,j=0}^3 x^{ij} \left[\frac{\partial p_{ij}^n}{\partial m} q_{ij}^n + p_{ij}^n \left(\frac{\partial q_{ij}^n}{\partial s} \frac{\partial s}{\partial m} + \frac{\partial q_{ij}^n}{\partial c} \frac{\partial c}{\partial m} + \frac{\partial q_{ij}^n}{\partial d} \frac{\partial d}{\partial m} \right) \right]. \quad (3.103)$$

Using (3.99), the latter can be written in terms of Ω derivatives and the quantity Ξ :

$$\frac{\partial \phi^n}{\partial m} = \sum_{i,j=0}^3 x^{ij} \left[\frac{\partial p_{ij}^n}{\partial m} q_{ij}^n + \frac{s(\text{cd})^{-1}}{2hm} p_{ij}^n \frac{\partial q_{ij}^n}{\partial d} + \Xi p_{ij}^n \left(\frac{\partial q_{ij}^n}{\partial s} \frac{\partial s}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial c} \frac{\partial c}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial d} \frac{\partial d}{\partial \Omega} \right) \right]. \quad (3.104)$$

Consider now the linear combination

$$\frac{\partial \phi}{\partial m} + A^\Omega \frac{\partial \phi}{\partial \Omega} + A^\alpha \frac{\partial \phi}{\partial \alpha} + A^\beta \frac{\partial \phi}{\partial \beta} + A^\gamma \frac{\partial \phi}{\partial \gamma}; \quad (3.105)$$

if this is zero for some choice of the A -coefficients the Jacobian is rank-deficient. In this case, it is clear from the Gauss-Jordan elimination algorithm and the structure of ϕ_m that each A must be affine in Ξ , i.e. $A^\Omega = A_0^\Omega + A_1^\Omega \Xi$ and similarly for other parameters. Hence,

$$\frac{\partial \phi^n}{\partial m} + A^\Omega \frac{\partial \phi^n}{\partial \Omega} + A^\alpha \frac{\partial \phi^n}{\partial \alpha} + A^\beta \frac{\partial \phi^n}{\partial \beta} + A^\gamma \frac{\partial \phi^n}{\partial \gamma} =$$

$$\begin{aligned}
& \sum_{i,j=0}^3 \left\{ x^{ij} \left[\left(\frac{\partial p_{jk}^i}{\partial m} + A_0^\Omega \frac{\partial p_{ij}^n}{\partial \Omega} \right) q_{ij}^n + \frac{s(\text{cd})^{-1}}{2hm} p_{ij}^n \frac{\partial q_{ij}^n}{\partial d} \right] \right. \\
& \quad \left. + x^{ij} A_0^\Omega p_{ij}^n \left(\frac{\partial q_{ij}^n}{\partial s} \frac{\partial s}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial c} \frac{\partial c}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial d} \frac{\partial d}{\partial \Omega} \right) + \left(A_0^\alpha \frac{\partial x^{ij}}{\partial \alpha} + A_0^\beta \frac{\partial x^{ij}}{\partial \beta} + A_0^\gamma \frac{\partial x^{ij}}{\partial \gamma} \right) p_{ij}^n q_{ij}^n \right\} \\
& + \Xi \sum_{i,j=0}^3 \left\{ x^{ij} \left[A_1^\Omega \frac{\partial p_{ij}^n}{\partial \Omega} q_{ij}^n + (1 + A_1^\Omega) p_{ij}^n \left(\frac{\partial q_{ij}^n}{\partial s} \frac{\partial s}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial c} \frac{\partial c}{\partial \Omega} + \frac{\partial q_{ij}^n}{\partial d} \frac{\partial d}{\partial \Omega} \right) \right] \right. \\
& \quad \left. + \left(A_0^\alpha \frac{\partial x^{ij}}{\partial \alpha} + A_0^\beta \frac{\partial x^{ij}}{\partial \beta} + A_0^\gamma \frac{\partial x^{ij}}{\partial \gamma} \right) p_{ij}^n q_{ij}^n \right\}. \tag{3.106}
\end{aligned}$$

There are two qualitatively distinct ways for the above expression to vanish. One possibility is that both sums vanish simultaneously. Otherwise the equation (with the LHS set to zero) can be solved for Ξ and hence \mathcal{E} . In particular, Ξ must be expressible as

$$\mathcal{E}(\Omega h|m) = 2hm(1-m) \frac{S_0^i}{S_1^i} + \Omega h(1-m) + m^{-1} \frac{sc}{d}, \tag{3.107}$$

where S_0^i and S_1^i are the first and second sums appearing in (3.106), respectively. The Jacobi \mathcal{E} -function satisfies an identity involving elliptic integrals. In particular, the function is quasi-periodic with period $4K$ (where $K = K(m)$ is the complete elliptic integral of the first kind (A.67)) and remainder $4E$ (where $E = E(m)$ is the complete elliptic integral of the second kind (A.68)):

$$\mathcal{E}(z + 4K|m) = \mathcal{E}(z|m) + 4E(m). \tag{3.108}$$

Let $\hat{h} = h + 4\Omega^{-1}K(m)$ so that $\mathcal{E}(\Omega \hat{h}|m) = \mathcal{E}(\Omega h|m) + 4E(m)$. If (3.107) holds, S_0^i and S_1^i must satisfy

$$2m(1-m) \left(\hat{h} \frac{\hat{S}_0^i}{\hat{S}_1^i} - h \frac{S_0^i}{S_1^i} \right) + 4(1-m)K(m) = 4E(m), \tag{3.109}$$

where $\hat{\cdot}$ indicates evaluation at \hat{h} . In particular, the quantity $\hat{h} \hat{S}_0^i / \hat{S}_1^i - h S_0^i / S_1^i$ must be independent of Ω and h , for each i . While not impossible, this condition is not satisfied for any known examples (to be discussed towards the

conclusion of this chapter) and so we will pursue only the mechanism of rank drop where the sums in (3.106) vanish simultaneously.

The general condition for the rank of the Jacobian to drop is

$$A^m \frac{\partial \phi}{\partial m} + A^\Omega \frac{\partial \phi}{\partial \Omega} + A^\alpha \frac{\partial \phi}{\partial \alpha} + A^\beta \frac{\partial \phi}{\partial \beta} + A^\gamma \frac{\partial \phi}{\partial \gamma} = 0 \quad (3.110)$$

for some choice of A -coefficients. There are two cases to consider: either A^m is zero or A^m is nonzero (in which case we take it to be one and the linear combination reduces to (3.105)). In the first case, the rank of the reduced Jacobian

$$J^{\text{red}} = \frac{\partial(\phi^1, \dots, \phi^8)}{\partial(\alpha, \beta, \gamma, \Omega)} \quad (3.111)$$

must drop ($\text{rk} J^{\text{red}} \leq 3$). In the second case, it is convenient to work with the column-equivalent matrix

$$\tilde{J} = JC \quad (3.112)$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\Xi & 1 \end{bmatrix}. \quad (3.113)$$

This matrix has the block structure

$$\tilde{J} = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix}, \quad (3.114)$$

where B_1 and B_2 are each 4×2 matrices and B_3 and B_4 are 4×3 matrices. Each submatrix depends on only four of the eight parameters x_0^{14} , x_0^{41} , x_0^{34} , x_0^{43} , x_0^{24} , x_0^{42} , x_0^{33} , and x_0^{44} . B_1 and B_4 depend only on the parameters x_0^{14} , x_0^{41} , x_0^{34} , and x_0^{43} . B_2 and B_3 depend only on x_0^{24} , x_0^{42} , x_0^{33} , and x_0^{44} .

By our discussion above, we search for rank deficiency conditions where both sums in (3.106) vanish simultaneously. To do this, we introduce an 8×6

extended Jacobian matrix \tilde{J}^{ext} as follows. Note that the components of the fifth column of \tilde{J} can each be written as

$$\tilde{J}_{i,5} = \tilde{J}_{i,5}^0 + \Xi \tilde{J}_{i,5}^1, \quad (3.115)$$

where $J_{i,5}^0$ and $J_{i,5}^1$ are independent of Ξ . We define \tilde{J}^{ext} so the first four columns of this matrix coincide with those of \tilde{J} while the fifth and sixth columns are given by $\tilde{J}_{i,5}^{\text{ext}} = \tilde{J}_{i,5}^0$ and $\tilde{J}_{i,6}^{\text{ext}} = \tilde{J}_{i,5}^1$, respectively. The resulting matrix has the block structure

$$\tilde{J}^{\text{ext}} = \begin{bmatrix} B_1 & B_3^{\text{ext}} \\ B_2 & B_4^{\text{ext}} \end{bmatrix}, \quad (3.116)$$

with B_1 and B_2 as before. B_3^{ext} and B_4^{ext} are 4×4 matrices with the same x_0^{ij} dependences as B_3 and B_4 , respectively. As \tilde{J}^{ext} is the principal tool in the computations that follow, we will write its components explicitly. In order to present the matrix in a reasonable way and also for future analysis, we use the notation $(\cdot|\cdot)$ to indicate column-wise concatenation. In particular, to present B_3^{ext} we split it into a pair of 4×2 matrices according to $B_3^{\text{ext}} = ({}_1B_3^{\text{ext}}|{}_2B_3^{\text{ext}})$ and similarly for B_4^{ext} .

$$\begin{aligned}
B_1 &= \frac{\Omega_0}{s_0} \left[\begin{array}{l} (c_0 d_0 - 1) (x_0^{41} - x_0^{14}) + s_0^2 (x_0^{43} - x_0^{34}) \\ - m_0 s_0^2 x_0^{14} + x_0^{34} - c_0 d_0 x_0^{43} \\ 2(-x_0^{34} + m_0 s_0^2 x_0^{41} - x_0^{14} + x_0^{43}) \\ c_0 d_0 x_0^{34} + m_0 s_0^2 x_0^{41} - x_0^{43} \end{array} \right. \\
&\quad \left. \frac{1}{m_0} [m_0 s_0^2 (x_0^{41} - x_0^{14}) - (c_0 d_0 - 1) (x_0^{43} - x_0^{34})] \right. \\
&\quad \left. \frac{1}{m_0 s_0^2} [(c_0 d_0 - 1) (x_0^{34} - m_0 s_0^2 x_0^{14}) - (-1 + c_0 d_0 + m_0 s_0^4) x_0^{43}] \right. \\
&\quad \left. \frac{2}{m_0 s_0^2} [(-1 + c_0 d_0 + (1 + m_0) s_0^2) (x_0^{34} - x_0^{43}) - m_0 s_0^2 (c_0 d_0 - 1) (x_0^{14} - x_0^{41})] \right. \\
&\quad \left. \frac{1}{m s_0^2} [(c_0 d_0 - 1) (m_0 s_0^3 x_0^{41} - x_0^{43}) + (-1 + c_0 d_0 + m_0 s_0^4) x_0^{34}] \right] \\
&\quad (3.117) \\
B_2 &= \left[\begin{array}{l} \frac{2(c_0 d_0 - 1) x_0^{33} + \Omega_0 m_0 [s_0 (c_0 d_0 - 1) (2x_0^{24} - x_0^{42}) + (2 - 2c_0 d_0 + (m_0 - 1) s_0^2) x_0^{44}]}{m_0 s_0^2} \\ \frac{2(c_0 d_0 - 1) x_0^{33} + \Omega_0 m_0 [s_0 (c_0 d_0 - 1) (x_0^{24} - 2x_0^{42}) + (2 - 2c_0 d_0 + (m_0 - 1) s_0^2) x_0^{44}]}{m_0 s_0^2} \\ 2x_0^{33} + \Omega_0 m_0 s_0 x_0^{24} \\ 2x_0^{33} - \Omega_0 m_0 s_0 x_0^{42} \\ \frac{2(c_0 d_0 - 1) x_0^{33} + \Omega_0 m_0 [s_0 (c_0 d_0 - 1) x_0^{24} - (-2 + 2c_0 d_0 + (1 - m_0) s_0^2) x_0^{44}]}{m_0 s_0^2} \\ \frac{2(c_0 d_0 - 1) x_0^{33} + \Omega_0 m_0 [s_0 (c_0 d_0 - 1) x_0^{42} - (-2 + 2c_0 d_0 + (1 - m_0) s_0^2) x_0^{44}]}{m_0 s_0^2} \end{array} \right] \\
&\quad (3.118)
\end{aligned}$$

$${}_1B_3^{\text{ext}} = \left[\begin{aligned} & -\frac{4x_0^{33}}{m_0} + 2\Omega_0 s_0 \left(x_0^{42} - x_0^{24} \right) \\ & \frac{2x_0^{33}}{m_0 s_0^2} - \frac{m_0 s_0^3}{4c_0 d_0 x_0^{33}} \\ & -\frac{ms^2}{2x_0^{33}} - \frac{2x_0^{33}}{m_0 s_0^2} \end{aligned} \right] \frac{(s_0 + h\Omega_0 c_0 d_0) \left(x_0^{42} - x_0^{24} \right)}{m_0 s_0^2 x_0^{24} - 2hc_0 d_0 x_0^{33} + \Omega_0 m_0 s_0 c_0 d_0 \left(2x_0^{44} - hx_0^{24} \right) + h\Omega_0 m_0 \left(-2 + (1+m_0)s_0^2 \right) x_0^{44}} \\
 & \frac{2h \left((1+m_0)s_0^2 - 2 \right) x_0^{33} + m_0 \left(s_0^2 c_0 d_0 + hs_0 \left(1 - m_0 s_0^4 \right) \right) \left(x_0^{24} - x_0^{42} \right) + 2m\Omega_0 \left(-2s_0 + (1+m_0)s_0^3 + 2hs\Omega_0 m_0 c_0 d_0 \right) x_0^{44}}{m_0 s_0^3} \\
 & \frac{m_0 s_0^2 x_0^{42} - 2hc_0 d_0 x_0^{33} + \Omega_0 m_0 s_0 c_0 d_0 \left(2x_0^{44} + hx_0^{42} \right) + \Omega_0 m_0 h \left(-2 + (1+m_0)s_0^2 \right) x_0^{44}}{m_0 s_0^3} \end{aligned} \right] \quad (3.119a)$$

$${}_2B_3^{\text{ext}} = \left[\begin{aligned} & s_0 \left(x_0^{24} - x_0^{42} \right) \\ & -\frac{x_0^{24}}{s_0} + \frac{2\Omega_0 c_0 d_0 x_0^{44}}{s_0^2} \\ & \frac{c_0 d_0}{s} \left(x_0^{42} - x_0^{24} \right) + \frac{2\Omega_0 \left(-2 + (1+m_0)s_0^2 \right) x_0^{44}}{s_0} \\ & \frac{x_0^{42}}{s} + \frac{2\Omega_0 c_0 d_0 x_0^{44}}{s_0^2} \end{aligned} \right] \frac{\frac{x_0^{33}}{m_0} - \frac{x_0^{33}}{m_0 s_0^2} - \frac{\Omega_0^2 c_0 x_0^{44}}{2d_0}}{m_0 s_0^2 x_0^{24} + \frac{\Omega_0 s_0 c_0 d_0}{2} \left(x_0^{24} - x_0^{42} \right) - \frac{x_0^{33}}{m^2 s_0^2} - \frac{\Omega_0^2 c_0 x_0^{44}}{2d_0}} \quad (3.119b)$$

$${}_1B_4^{\text{ext}} = \left[\begin{aligned} & 2\Omega_0 s_0 x_0^{43} \\ & -2\Omega_0 s_0 x^{34} \\ & -2m_0 \Omega_0 s_0 x_0^{14} \\ & 2m_0 \Omega_0 s_0 x_0^{41} \end{aligned} \right] \frac{\left[\begin{aligned} & -\Omega_0 h \left(1 - m_0 s_0^4 \right) + s_0 c_0 d_0 \left| x_0^{14} + \left(s_0 - \Omega_0 h c_0 d_0 \right) x_0^{41} + \left(s_0^3 + \Omega_0 h s_0^2 c_0 d_0 \right) x_0^{43} \right. \\ & \left. \left(-s_0 + \Omega_0 h c_0 d_0 \right) x_0^{14} + \left(-s_0 c_0 d_0 + \Omega_0 h \left(1 - m_0 s_0^4 \right) \right) x_0^{41} - \left(s_0^3 + \Omega_0 h s_0^2 c_0 d_0 \right) x_0^{34} \right. \\ & \left. \frac{s_0^2}{s^2} \left(s_0 + \Omega_0 c_0 d_0 \right) x_0^{14} \left(s_0 - \Omega_0 h c_0 d_0 \right) x_0^{34} \left(s_0 c_0 d_0 - \Omega_0 h \left(1 - m_0 s_0^4 \right) \right) x_0^{43} \right. \\ & \left. - m_0 s_0^2 \left(s_0 + \Omega_0 h c_0 d_0 \right) x_0^{41} - \left(s_0 c_0 d_0 - \Omega_0 h \left(1 - m_0 s_0^4 \right) \right) x_0^{34} - \left(s_0 - \Omega_0 h c_0 d_0 \right) x_0^{43} \right. \\ & \left. \frac{s_0^2}{s_0} \right]}{s_0} \end{aligned} \right] \quad (3.120a)$$

$${}_2B_4^{\text{ext}} = \left[\begin{aligned} & -\frac{c_0 d_0}{s_0} x_0^{14} - \frac{1}{s_0} x_0^{41} - s_0 x_0^{43} \\ & \frac{1}{s_0} x_0^{14} + s x_0^{34} + \frac{c_0 d_0}{s_0} x_0^{41} \\ & -m_0 s_0 x_0^{14} - \frac{1}{s_0} x_0^{34} + \frac{c_0 d_0}{s_0} x_0^{43} \\ & \frac{c_0 d_0}{s_0} x_0^{34} + m_0 s_0 x_0^{41} + \frac{1}{s_0} x_0^{43} \end{aligned} \right] \frac{\Omega_0 s_0 c_0 x_0^{14} - \frac{\Omega_0 s_0 c_0 x_0^{14}}{d_0} - \frac{\Omega_0 s_0 c_0}{d_0} \Omega_0 s_0 x_0^{14} - \frac{\Omega_0 s_0 c_0 x_0^{43}}{2d_0} - \frac{\Omega_0 s_0 c_0 x_0^{34}}{2d_0} - \Omega_0 s_0 x_0^{41}}{s_0} \quad (3.120b)$$

We can exploit the block structure of \tilde{J}^{ext} in the Gauss-Jordan elimination process. When $x_0^{24} = x_0^{42} = x_0^{33} = x_0^{44} = 0$, we have $B_2 = 0_{4 \times 2}$, $B_3 = 0_{4 \times 3}$, and $B_3^{\text{ext}} = 0_{4 \times 4}$. When $x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$, we have $B_1 = 0_{4 \times 2}$, $B_4 = 0_{4 \times 3}$, and $B_4^{\text{ext}} = 0_{4 \times 4}$. Therefore, identifying rank-deficiency conditions for the individual B -matrices can lead to rank-deficiency conditions for the Jacobian. When $x_0^{24} = x_0^{42} = x_0^{33} = x_0^{44} = 0$ and one of B_2, B_3 is rank-deficient, the Jacobian is rank-deficient, and similarly for the case $x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$.

By the rank-nullity theorem, we are free to work with the transposed matrix

$${}^t\tilde{J}^{\text{ext}} = \begin{bmatrix} {}^tB_1 & {}^tB_2 \\ {}^tB_3^{\text{ext}} & {}^tB_4^{\text{ext}} \end{bmatrix}. \quad (3.121)$$

In the next section, we will completely identify the conditions under which the blocks of the above matrix are rank-deficient. To motivate this process, we recall the relationship between the ranks of the blocks of ${}^t\tilde{J}^{\text{ext}}$ and the Jacobian (3.90) and the reduced Jacobian (3.111). When $({}^tB_1 | {}^tB_2)$ is rank-deficient, the reduced Jacobian (3.111) is rank deficient. The reduced Jacobian is also rank-deficient when the first two rows of $({}^tB_3^{\text{ext}} | {}^tB_4^{\text{ext}})$ are linearly dependent. If the first two rows of $({}^tB_3^{\text{ext}} | {}^tB_4^{\text{ext}})$ are linearly independent, the rank of $({}^tB_3^{\text{ext}} | {}^tB_4^{\text{ext}})$ must drop by two to ensure that the rank of the Jacobian drops.

3.2.1 Rank drop analysis

3.2.1.1 B_1

The rank of B_1 can drop in two qualitatively different ways. First, one of the two rows of tB_1 can vanish. The first row, consisting of derivatives with respect to β , vanishes when

$$x_0^{41} = x_0^{14}, \quad (3.122a)$$

$$x_0^{34} = x_0^{43} = \frac{m_0 x_0^{14}}{s_0^2 (1 - c_0 d_0)}. \quad (3.122b)$$

The second row, consisting of γ -derivatives, vanishes when

$$x_0^{41} = x_0^{14} \quad (3.123a)$$

$$x_0^{34} = x_0^{43} = \frac{1 - c_0 d_0}{s_0^2} x_0^{14}. \quad (3.123b)$$

Alternatively, there could be a linear relationship between two nonzero rows. In this case, we assume that the second equalities in (3.122b) and (3.123b) do not hold and find that we must have

$$x_0^{14} = x_0^{41}, \quad x_0^{34} = x_0^{43}. \quad (3.124)$$

We see that (3.122) and (3.123) are just special cases of this more general condition.

3.2.1.2 B_2

The conditions under which B_2 is rank-deficient are more complicated than those for B_1 . Again, the simplest mechanism for the rank to drop is when a single row of ${}^t B_2$ vanishes. The first row vanishes when

$$x_0^{42} = -x_0^{24}, \quad x_0^{33} = -\frac{\Omega_0 m_0 s_0}{2} x_0^{24}, \quad x_0^{44} = \frac{2(c_0 d_0 - 1)}{\Omega_0 (2(c_0 d_0 - 1) + (1 + m_0) s_0^2)} x_0^{24} \quad (3.125)$$

and the second row vanishes when

$$x_0^{42} = -x_0^{24}, \quad x_0^{33} = -\frac{3\Omega_0 m_0 s_0}{2} x_0^{24}, \quad x_0^{44} = \frac{2(c_0 d_0 - 1)}{\Omega_0 (2(c_0 d_0 + (1 + m_0) s_0^2))} x_0^{24}. \quad (3.126)$$

The rank also drops when the rows of ${}^t B_2$ are nonzero scalar multiples of each other. The general condition for this to happen is

$$x_0^{42} = -x_0^{24} \quad (3.127a)$$

$$m_0 s_0^4 (2x_0^{33} + \Omega_0 m_0 s_0 x_0^{24}) (2x_0^{33} + 3\Omega_0 m_0 s_0 x_0^{24}) =$$

$$\left[(2 - 2c_0 d_0) x_0^{33} + \Omega_0 m_0 \left((1 - c_0 d_0) x_0^{24} + \Omega_0 (2c_0 d_0 - 2 + (1 + m_0) s_0^2) x_0^{44} \right) \right] \times$$

$$\left[(2 - 2c_0d_0)x_0^{33} + \Omega_0m_0 \left(3(1 - c_0d_0)x_0^{24} + \Omega_0(2c_0d_0 - 2 + (1 + m_0)s_0^2)x_0^{44} \right) \right]. \quad (3.127b)$$

Similarly to the case of B_1 , (3.125-3.126) are contained as special cases.

3.2.1.3 B_3

We recall that $\text{rk}B_3 \leq 2$ is a necessary condition for the corresponding bi-Riccati equation to admit a multiparameter family of solutions. The rank of the reduced Jacobian drops when either the first or second row of tB_3 vanishes or when there is a nonzero linear relationship between the first two rows. Otherwise it is necessary to have $\text{rk}B_3^{\text{ext}} \leq 2$ to obtain a multiparameter family of solutions. In either case it is required that the determinant of B_3^{ext} vanishes. If the rank of B_3^{ext} is to drop by at least two, the determinant of B_3^{ext} and each of its 3×3 minors must vanish simultaneously. There is a remarkably simple formula for the former:

$$\begin{aligned} \det B_3^{\text{ext}} &= \frac{2h(1-m_0)}{m_0^4s_0^4} (x_0^{24} + x_0^{42}) \left[2x_0^{33} + \Omega_0m_0s_0 (x_0^{24} - x_0^{42}) \right] \\ &\quad \times \left[(x_0^{33})^2 + \Omega_0m_0s_0x_0^{33} (x_0^{42} - x_0^{24}) + \Omega_0^4m_0^3s_0^4 (x_0^{44})^2 \right]. \end{aligned} \quad (3.128)$$

Solutions to $\det B_3^{\text{ext}} = 0$ are sufficient conditions for $\text{rk}B_3^{\text{ext}} \leq 3$ and necessary conditions for $\text{rk}B_3^{\text{ext}} \leq 2$. For each such solution we use Gauss-Jordan elimination on ${}^tB_3^{\text{ext}}$ and observe whether or not some linear combination (with at least one nonzero coefficient) of the first two rows is zero. If not, we use the minors of B_3^{ext} to obtain conditions for $\text{rk}B_3^{\text{ext}} \leq 2$.

When the first factor, $x_0^{24} + x_0^{42}$, vanishes the first two rows of ${}^tB_3^{\text{ext}}$ are linearly independent. Therefore we seek conditions under which $\text{rk}B_3^{\text{ext}} \leq 2$. We assume that x_0^{33} and x_0^{44} are nonzero to obtain

$$x_0^{42} = -x_0^{24} \quad (3.129a)$$

$$\begin{aligned} x_0^{33} &= x_0^{24}\Omega_0m_0 \left[2\Omega_0c_0(s_0^2 - 1)(m_0s_0^2 - m_0 - 1)x_0^{24}x_0^{44} \right. \\ &\quad \left. + s_0d_0 \left((1 + m_0 - 2m_0s_0^2)(x_0^{24})^2 - \Omega_0^2(1 - m_0)^2(x_0^{44})^2 \right) \right] \end{aligned} \quad (3.129b)$$

$$\times \left[d_0 (2m_0 s_0^2 - m_0 - 1) (x_0^{24})^2 + \Omega_0^2 d_0 (1 - m_0)^2 (x_0^{44})^2 \right]^{-1}$$

and one of

$$x_0^{44} = \frac{s_0}{\Omega_0(c_0 + d_0)} x_0^{24} \quad (3.130a)$$

$$x_0^{44} = \frac{s_0}{\Omega_0(c_0 - d_0)} x_0^{24} \quad (3.130b)$$

$$x_0^{44} = -\frac{s_0}{\Omega_0(c_0 + d_0)} x_0^{24} \quad (3.130c)$$

$$x_0^{44} = -\frac{s_0}{\Omega_0(c_0 - d_0)} x_0^{24}. \quad (3.130d)$$

Here we actually have four distinct equations determined by the permutation of signs chosen in the formula for x_0^{44} . For each of these we can substitute this formula into that for x_0^{33} to obtain a formula for x_0^{33} in terms of x_0^{24} alone. Alternatively, when $x_0^{44} = 0$, we obtain

$$x_0^{42} = -x_0^{24}, \quad x_0^{33} = -\Omega_0 m_0 s_0 x_0^{24}, \quad x_0^{44} = 0. \quad (3.131)$$

and when $x_0^{33} = 0$, we have

$$x_0^{24} = x_0^{42} = x_0^{33} = 0, \quad x_0^{44} = 1. \quad (3.132)$$

There is no solution when $x_0^{33} = x_0^{44} = 0$.

Next we consider the vanishing of the second factor:

$$2x_0^{33} + \Omega_0 m_0 s_0 (x_0^{24} - x_0^{42}) = 0. \quad (3.133)$$

One solution to this equation is

$$x_0^{42} = x_0^{24}, \quad x_0^{33} = 0. \quad (3.134)$$

This condition (we emphasize that x_0^{44} is arbitrary) leads to the vanishing of the first row of ${}^t B_3$. The first two rows of ${}^t B_3$ are linearly independent for all

other solutions to (3.133). Now looking for conditions where $\text{rk} B_3^{\text{ext}} \leq 2$, we recover (3.130) and (3.131) but also generate a single new solution

$$x_0^{42} = x_0^{24}, \quad x_0^{33} = 0, \quad x_0^{44} = 0, \quad (3.135)$$

which is just a special case of (3.134).

When the third factor, $(x_0^{33})^2 + \Omega_0 m_0 s_0 x^{33} (x_0^{42} - x_0^{24}) + \Omega_0^4 m_0^3 s_0^4 (x_0^{44})^2$, vanishes we again obtain (3.134) as a condition for linear dependence of the first two rows. Seeking conditions where $\text{rk} B_3^{\text{ext}} \leq 2$, we find a discrete class of solutions where

$$x_0^{42} = -x_0^{24} \quad (3.136)$$

and x^{33} and x^{44} is given by any one of

$$x_0^{33} = \frac{2\Omega_0 m_0^2 s_0^5}{(1+c_0)(1+d_0) - m_0 s_0^4} x_0^{24}, \quad x_0^{44} = -\frac{1+c_0+d_0+c_0 d_0}{\Omega_0^2 m_0^2 s_0^4} x_0^{33} \quad (3.137a)$$

$$x_0^{33} = \frac{2\Omega_0 m_0^2 s_0^5}{(1-c_0)(1-d_0) - m_0 s_0^4} x_0^{24}, \quad x_0^{44} = -\frac{1-c_0-d_0+c_0 d_0}{\Omega_0^2 m_0^2 s_0^4} x_0^{33} \quad (3.137b)$$

$$x_0^{33} = \frac{2\Omega_0 m_0^2 s_0^5}{(1-c_0)^2(1+d_0)^2 - m_0 s_0^4} x_0^{24}, \quad x_0^{44} = \frac{1-c_0+d_0-c_0 d_0}{\Omega_0^2 m_0^2 s_0^4} x_0^{33} \quad (3.137c)$$

$$x_0^{33} = \frac{2\Omega_0 m_0^2 s_0^5}{(1+c_0)^2(1-d_0)^2 - m_0 s_0^4} x_0^{24}, \quad x_0^{44} = \frac{1+c_0-d_0-c_0 d_0}{\Omega_0^2 m_0^2 s_0^4} x_0^{33}. \quad (3.137d)$$

3.2.1.4 B_4

Equations associated to B_4 and admitting multiparameter families of Möbius-Jacobi solutions are obtained using the same method as before. Again we obtain a simple expression for the relevant determinant

$$\det B_4^{\text{ext}} = 4h\Omega_0^3(1-m_0)s_0 \left(x_0^{14} x_0^{43} - x_0^{34} x_0^{41} \right) \left(x_0^{14} x_0^{34} - x_0^{41} x_0^{43} \right). \quad (3.138)$$

Beginning with either factor involving the x_0^{ij} leads to the same results. The first two rows of ${}^t B_4$ are linearly dependent when

$$x_0^{41} = x_0^{14}, \quad x_0^{43} = x_0^{34}, \quad (3.139a)$$

$$\begin{aligned} & \left[s_0(1 + c_0 d_0) + h(m_0 s_0^4 - c_0 d_0 - 1) \right] \left[(x_0^{14})^2 + m_0 (x_0^{34})^2 \right] \\ & + 2m_0 s_0^2 (s_0 + h\Omega_0 c_0 d_0) x_0^{14} x_0^{34} = 0 \end{aligned} \quad (3.139b)$$

or when

$$x_0^{41} = -x_0^{14}, \quad x_0^{43} = -x_0^{34}, \quad (3.140a)$$

$$\begin{aligned} & \left[s_0(c_0 d_0 - 1) + h(m_0 s_0^4 + c_0 d_0 - 1) \right] \left[(x_0^{14})^2 + m_0 (x_0^{34})^2 \right] \\ & - 2m_0 s_0^2 (s_0 + h\Omega_0 c_0 d_0) x_0^{14} x_0^{34} = 0. \end{aligned} \quad (3.140b)$$

Now we search for conditions where $\text{rk} B_4^{\text{ext}} \leq 2$. The first such condition is

$$x_0^{41} = x_0^{14}, \quad x_0^{43} = x_0^{34}; \quad (3.141)$$

we remark that this condition is identical to (3.124) and as such also renders B_1 rank-deficient. The second condition is

$$x_0^{41} = -x_0^{14}, \quad x_0^{43} = -x_0^{34}. \quad (3.142)$$

We observe that (3.141) and (3.142) contain (3.139) and (3.140), respectively, as special cases.

3.2.1.5 B_1 and B_2

Like its constituent matrices, the matrix resulting from concatenating ${}^t B_1$ and ${}^t B_2$ can be rank-deficient in qualitatively distinct ways. The first way, in which a full row of $({}^t B_1 | {}^t B_2)$ vanishes, occurs when the same rows of ${}^t B_1$ and ${}^t B_2$ vanish simultaneously. The conditions under which this happens are easily found from our previous results. In particular, the first row of $({}^t B_1 | {}^t B_2)$ vanishes when

$$x_0^{41} = x_0^{14}, \quad x_0^{34} = x_0^{43} = \frac{m_0(1 + c_0 d_0)x^{14}}{1 + m_0 - m_0 s_0^2},$$

$$x_0^{42} = -x_0^{24}, \quad x_0^{33} = -\frac{\Omega_0 m_0 s_0}{2} x_0^{24}, \quad x_0^{44} = \frac{2(c_0 d_0 - 1)}{\Omega_0 (-2 + 2c_0 d_0 + (1 + m_0) s_0^2)} x_0^{24} \quad (3.143)$$

and the second row vanishes when

$$\begin{aligned} x_0^{41} &= x_0^{14}, \quad x_0^{34} = x_0^{43} = \frac{1 - c_0 d_0}{s_0^2} x_0^{14}, \\ x_0^{42} &= -x_0^{24}, \quad x_0^{33} = -\frac{3\Omega_0 m_0 s_0}{2} x_0^{24}, \quad x_0^{44} = \frac{2(c_0 d_0 - 1)}{\Omega_0 (-2 + 2c_0 d_0 + (1 + m_0) s_0^2)} x_0^{24}. \end{aligned} \quad (3.144)$$

The rank of $({}^t B_1 | {}^t B_2)$ also drops when there is a nonzero linear relationship between its rows. If this is the case, there is a matrix $R \in \text{GL}(2, \mathbb{C})$ so that one of the rows of $R({}^t B_1 | {}^t B_2)$ is identically zero. But

$$R({}^t B_1 | {}^t B_2) = (R {}^t B_1 | R {}^t B_2) \quad (3.145)$$

and the conditions (and hence the explicit R matrices) for $R {}^t B_1$ and $R {}^t B_2$ to have vanishing second rows are known. Let R_1 and R_2 be, respectively, the matrices that render the second rows of ${}^t B_1$ and ${}^t B_2$ zero and leave the first rows unchanged. These matrices are unique up to rescaling of the second row; we choose representatives where the $(2, 2)$ entry is unity. We have

$$R_1 = \begin{bmatrix} 1 & 0 \\ \frac{(1 - c_0 d_0) x_0^{14} - s_0^2 x_0^{34}}{m_0 s_0^2 x_0^{14} - (1 - c_0 d_0) x_0^{34}} & 1 \end{bmatrix} \quad (3.146a)$$

$$R_2 = \begin{bmatrix} 1 & 0 \\ \frac{2(1 - c_0 d_0) x_0^{33} + \Omega_0 m_0 [s_0(1 - c_0 d_0) x_0^{24} + \Omega_0 ((1 + m_0) s_0^2 + 2c_0 d_0 - 2) x_0^{44}]}{m_0 s_0^2 (2x_0^{33} + \Omega_0 m_0 x_0^{24})} & 1 \end{bmatrix}. \quad (3.146b)$$

By equating R_1 and R_2 , we arrive at the set of conditions:

$$\begin{aligned} x_0^{41} &= x_0^{14}, \quad x_0^{43} = x_0^{34}, \quad x_0^{42} = -x_0^{24}, \\ m_0 s_0^4 (2x_0^{33} + \Omega_0 m_0 s_0 x_0^{24}) (2x_0^{33} + 3\Omega_0 m_0 s_0 x_0^{24}) &= \end{aligned}$$

$$\begin{aligned} & \left[(2 - 2c_0d_0)x_0^{33} + \Omega_0m_0 \left((1 - c_0d_0)x_0^{24} + \Omega_0(2c_0d_0 - 2 + (1 + m_0)s_0^2)x_0^{44} \right) \right] \times \\ & \left[(2 - 2c_0d_0)x_0^{33} + \Omega_0m_0 \left(3(1 - c_0d_0)x_0^{24} + \Omega_0(2c_0d_0 - 2 + (1 + m_0)s_0^2)x_0^{44} \right) \right], \end{aligned} \quad (3.147)$$

together with

$$\begin{aligned} & \frac{(1 - c_0d_0)x_0^{14} - s_0^2x_0^{34}}{m_0s_0^2x_0^{14} - (1 - c_0d_0)x_0^{34}} = \\ & \frac{2(1 - c_0d_0)x_0^{33} + \Omega_0m_0 \left[s_0(1 - c_0d_0)x_0^{24} + \Omega_0 \left((1 + m_0)s_0^2 + 2c_0d_0 - 2 \right) x_0^{44} \right]}{m_0s_0^2 \left(2x_0^{33} + \Omega_0m_0x_0^{24} \right)}. \end{aligned} \quad (3.148)$$

3.2.1.6 B_3 and B_4

As was done for B_1 and B_2 , we first seek conditions under which there is a linear relationship between the first two rows of the matrix $({}^tB_3|{}^tB_4)$. We recall from previous analysis that there is a single instance where the first two rows of B_3 are linearly dependent (3.134)—in this case the first row of B_3 is zero. There are a pair of conditions (3.139,3.140) where the first two rows of B_4 are linearly dependent, but neither of these result in the vanishing of the first row of B_4 . We conclude there is no condition leading to the linear dependence of the first two rows of $({}^tB_3|{}^tB_4)$.

We seek conditions under which the rank of $({}^tB_3|{}^tB_4)$ drops by two. We recall that there are several ways ((3.129-3.130), (3.131), (3.132), (3.135), (3.136-3.137)) whereby the rank of B_3 drops by two, but only a pair of ways ((3.141) and (3.142)) in which the rank of B_4 drops by two. There are ten cases in all to consider, each of which can be analyzed by a method analogous to that for $({}^tB_3|{}^tB_4)$. Each condition for the rank of B_3 or B_4 to drop by two is associated with a sequence of elementary row operations R . For each pair of conditions (one for B_3 and one for B_4), we can equate these matrices. Unfortunately, there are no nontrivial solutions to any of these equations.

3.2.2 Equations and solutions

In the previous section, we identified some conditions under which the rank of the Jacobian drops, a necessary condition for the corresponding equation to admit a multiparameter family of solutions. Accordingly, not all of the rank-deficiency conditions must lead to equations with the desired solution structure. Each condition can be substituted into the system of equations (3.85, 3.87), where we seek a nontrivial solution with at least one extra degree of freedom (in addition to the freedom in z_0). In total, we identify five equations resulting from particular combinations of parameters x_0^{ij} appearing in (3.1) and (3.72).

Case I: $x_0^{14} = x_0^{41}$, $x_0^{34} = x_0^{43}$, $x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$

This condition (3.124) causes the rank of both B_1 and B_4 (3.141) to drop. The matrix corresponding to the condition is

$$x_0^{14} (X_0^1 + X_0^5) + x_0^{34} (X_0^3 + X_0^7) = \frac{\Omega_0}{s_0} \begin{bmatrix} 0 & -(1 + c_0 d_0) x_0^{14} - s_0^2 x_0^{34} & 0 & \frac{s_0}{\Omega_0} x_0^{14} \\ (1 + c_0 d_0) x_0^{14} + s_0^2 x_0^{34} & 0 & -m_0 s_0^2 x_0^{14} - \frac{\Omega_0}{s_0} (1 + c_0 d_0) x_0^{34} & 0 \\ 0 & m_0 s_0^2 x_0^{14} + (1 + c_0 d_0) x_0^{34} & 0 & \frac{s_0}{\Omega_0} x_0^{34} \\ \frac{s_0}{\Omega_0} x_0^{14} & 0 & \frac{s_0}{\Omega_0} x_0^{34} & 0 \end{bmatrix}, \quad (3.149)$$

leading to the equation

$$x_0^{14} (u' + \bar{u}') + x_0^{34} (u^2 \bar{u}' + u' \bar{u}^2) = \frac{\Omega_0}{s_0} [(1 + c_0 d_0) x_0^{14} + s_0^2 x_0^{34}] (\bar{u} - u) + \frac{\Omega_0}{s_0} [m_0 s_0^2 x_0^{14} + (1 + c_0 d_0) x_0^{34}] (u \bar{u}^2 - u^2 \bar{u}). \quad (3.150)$$

With the redefinition of parameters

$$a_1 = x_0^{14}, \quad a_2 = x_0^{34},$$

$$b_1 = \frac{\Omega_0}{s_0} [a_1(1 + c_0 d_0) + a_2 s_0^2], \quad b_2 = \frac{\Omega_0}{s_0} [a_1 m_0 s_0^2 + a_2(1 + c_0 d_0)], \quad (3.151)$$

we obtain the equation

$$a_1 (u' + \bar{u}') + a_2 (u^2 \bar{u}' + u' \bar{u}^2) = b_1 (\bar{u} - u) + b_2 (u \bar{u}^2 - u^2 \bar{u}). \quad (3.152)$$

Rather than demonstrate that the mapping (3.151) is surjective, we will show that (3.152), which is at least as general as (3.150), admits a multiparameter family of elliptic solutions (under a technical assumption). The solution to (3.152) is

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0 | m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0 | m) + \delta}, \quad (3.153)$$

subject to the conditions

$$s(\alpha \beta b_2 + \gamma \delta b_1) = 2\Omega(\alpha \beta a_2 + \gamma \delta a_1) \quad (3.154a)$$

$$\frac{s}{\Omega} (\beta^2 b_2 + \delta^2 b_1) = s^2 (\alpha^2 a_2 + \gamma^2 a_1) + \operatorname{cd} (\beta^2 a_2 + \delta^2 a_1) + \beta^2 a_2 + \delta^2 a_1 \quad (3.154b)$$

$$\frac{s}{\Omega} (\alpha^2 b_2 + \gamma^2 b_1) = m s^2 (\beta^2 a_2 + \delta^2 a_1) + \operatorname{cd} (\alpha^2 a_2 + \gamma^2 a_1) + \alpha^2 a_2 + \gamma^2 a_1. \quad (3.154c)$$

These conditions are too complicated for us to work with: it is difficult to determine under what conditions they have a solution. However, if we seek solutions with $\beta = \gamma = 0$ and $\delta = 1$, only two of the conditions (3.154) survive:

$$b_1 = \frac{\Omega}{s} [a_1(1 + \operatorname{cd}) + a_2 \alpha^2 s^2] \quad (3.155a)$$

$$b_2 \alpha^2 = \frac{\Omega}{s} [a_1 m s^2 + a_2 \alpha^2 (1 + \operatorname{cd})]. \quad (3.155b)$$

We assume now that a_1 , a_2 , b_1 , and b_2 are given. We suppose first that $a_2 = 0$. Then the first condition (3.155a) is independent of α . Assuming $h \in \mathbb{C} \setminus \{0\}$ is

fixed, we choose $m \in \mathbb{C} \setminus \{0, 1\}$ so that

$$f(\Omega) = a_1 \Omega \operatorname{ns}(\Omega h | m) [1 + \operatorname{cn}(\Omega h | m) \operatorname{dn}(\Omega h | m)] \quad (3.156)$$

is a nonconstant meromorphic function of Ω when $a_1 \neq 0$ (when $a_1 = 0$ we must also have $b_1 = b_2 = 0$ corresponding to a trivial equation). Such a function has at most two omitted values ($f(\Omega) = w$ has a solution for all but two $w \in \mathbb{C} \cup \infty$). Then we may solve $f(\Omega) = b_1$ for Ω provided that b_1 is not an omitted value of f . The result may then be substituted into (3.155b) and α can be determined.

Now we assume $a_2 \neq 0$, so that we may solve for α^2 in (3.155a). Doing so and substituting the result into (3.155b) leads to (3.163b) leads to

$$b_1 b_2 s^2 - (a_1 b_2 + a_2 b_1) \Omega (cd - 1) + 2a_1 a_2 \Omega^2 (1 + m) s^2 = 0. \quad (3.157)$$

We fix $m \in \mathbb{C} \setminus \{0, 1\}$. The LHS of (3.157) is a nonconstant meromorphic function of Ω . Provided that 0 is not an omitted value of this function, (3.157) may be solved for Ω . The result may then be substituted into (3.155b) and α can be determined. We conclude that, subject to technical assumptions on the range of certain meromorphic functions, (3.152) generically admits a two-parameter family of order-two elliptic solutions.

Case II: $x_0^{41} = -x_0^{14}$, $x_0^{43} = -x_0^{34}$, $x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$

This condition (3.142) leads to the matrix

$$x_0^{14} (X_0^1 - X_0^5) + x_0^{34} (X_0^3 - X_0^7) = \begin{bmatrix} 0 & (c_0 d_0 - 1)x_0^{14} - s_0^2 x_0^{34} & 0 & -\frac{s_0}{\Omega_0} x_0^{14} \\ \frac{\Omega_0}{s_0} (c_0 d_0 - 1)x_0^{14} - s_0^2 x_0^{34} & 0 & m_0 s_0^2 x_0^{14} - (c_0 d_0 - 1)x_0^{34} & 0 \\ 0 & m_0 s_0^2 x_0^{14} - (c_0 d_0 - 1)x_0^{34} & 0 & -\frac{s_0}{\Omega_0} x_0^{34} \\ \frac{s_0}{\Omega_0} x_0^{14} & 0 & \frac{s_0}{\Omega_0} x_0^{34} & 0 \end{bmatrix} \quad (3.158)$$

and the equation

$$\begin{aligned} x_0^{14} (\bar{u}' - u') + x_0^{34} (u' \bar{u}^2 - u^2 \bar{u}') &= \frac{\Omega_0}{s_0} [-(c_0 d_0 - 1)x_0^{14} + s_0^2 x_0^{34}] (u + \bar{u}) \\ &+ \frac{\Omega_0}{s_0} [-m_0 s_0^2 x_0^{14} + (c_0 d_0 - 1)x_0^{34}] (u^2 \bar{u} + u \bar{u}^2). \end{aligned} \quad (3.159)$$

This equation is very similar to the previous one (3.150). Again we redefine parameters

$$\begin{aligned} a_1 &= x_0^{14}, \quad a_2 = x_0^{34}, \\ b_1 &= \frac{\Omega_0}{s_0} [a_1(c_0 d_0 - 1) - a_2 s_0^2], \quad b_2 = \frac{\Omega_0}{s_0} [a_1 m_0 s_0^2 - a_2(c_0 d_0 - 1)] \end{aligned} \quad (3.160)$$

to obtain

$$a_1 (\bar{u}' - u') + a_2 (u' \bar{u}^2 - u^2 \bar{u}') + b_1 (u + \bar{u}) + b_2 (u^2 \bar{u} + u \bar{u}^2) = 0. \quad (3.161)$$

Rather than demonstrate (3.159) and (3.161) are equivalent, we will show that the latter, which is not less general than the former, admits an multiparameter family of order-two elliptic solutions. Despite its similarity to (3.152), (3.161) does not possess full Möbius-Jacobi solutions (3.153) but only solutions

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m), \quad (3.162)$$

subject to

$$b_1 = \frac{\Omega}{s} [a_1(cd - 1) - a_2 \alpha^2 s^2] \quad (3.163a)$$

$$b_2 \alpha^2 = \frac{\Omega}{s} [a_1 m s^2 - a_2 \alpha^2 (cd - 1)]. \quad (3.163b)$$

Assume first that $a_2 = 0$. Then the condition (3.163a) is independent of α .

Assuming $h \in \mathbb{C} \setminus \{0\}$ is given, we choose $m \in \mathbb{C} \setminus \{0, 1\}$. Then

$$f(\Omega) = a_1 \Omega \operatorname{ns}(\Omega h | m) [1 - \operatorname{cn}(\Omega h | m) \operatorname{dn}(\Omega h | m)] \quad (3.164)$$

is a nonconstant meromorphic function of Ω when $a_1 \neq 0$ (when $a_1 = 0$ we must also have $b_1 = b_2 = 0$ corresponding to a trivial equation). We may solve $f(\Omega) = b_1$ for Ω provided that b_1 is not one of the omitted values of f . The result may then be substituted into (3.155) and α can be determined.

Now we assume $a_2 \neq 0$, so that we may solve for α^2 in (3.163a). Doing so and substituting the result into (3.163b) leads to

$$b_1 b_2 s^2 - (a_1 b_2 + a_2 b_1) \Omega (cd - 1) - a_1 a_2 \Omega^2 (c + d)^2 = 0. \quad (3.165)$$

We fix $m \in \mathbb{C} \setminus \{0, 1\}$. The LHS of (3.165) is a nonconstant meromorphic function of Ω . Provided that 0 is not an omitted value of this function, (3.165) may be solved for Ω . The result may then be substituted into (3.155) and α can be determined. Again we conclude that, subject to technical assumptions on the range of certain meromorphic functions, (3.161) generically admits a two-parameter family of order-two elliptic solutions.

The remaining equations arise from the rank of B_3 dropping.

Case III: $x_0^{42} - x_0^{24}$, $x_0^{33} = -\Omega_0 m_0 s_0 x_0^{24}$, $x_0^{44} = x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$

This condition (3.131), is associated with the matrix

$$X_0^0 + X_0^2 - X_0^6 = \begin{bmatrix} -\Omega_0 s_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \Omega_0 m_0 s_0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (3.166)$$

and corresponds to the equation

$$u' \bar{u} - u \bar{u}' = \Omega_0 s_0 (1 - m_0 u^2 \bar{u}^2). \quad (3.167)$$

We make a redefinition of the parameters:

$$b_1 = \Omega_0 s_0, \quad b_2 = -\Omega_0 m_0 s_0, \quad (3.168)$$

after which (3.167) becomes

$$u'\bar{u} - u\bar{u}' = b_1 + b_2 u^2 \bar{u}^2. \quad (3.169)$$

To show that (3.169) with arbitrary b_1 and b_2 is equivalent to (3.167) we would need to demonstrate that the equations (3.168) can be solved for Ω_0 and m_0 , for any b_1 and b_2 . We will instead take the approach that (3.169) is at least as general as (3.167) and demonstrate directly that (3.169) admits a multiparameter family of order-two elliptic solutions. (3.169) is solved by

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m) \quad (3.170)$$

when the parameters satisfy

$$b_1 - 2\alpha^2 \Omega s - \frac{b_2 \alpha^4}{m} = 0 \quad (3.171a)$$

$$b_2 \alpha^2 + \Omega m s = 0. \quad (3.171b)$$

We first assume that $b_2 = 0$. Then we have $\Omega m s = 0$, which in all cases leads to trivial solutions. When b_2 is nonzero, we can solve (3.171b) for α^2 and substitute the result into (3.171a) to obtain:

$$\Omega^2 m s^2 = -b_1 b_2. \quad (3.172)$$

The LHS of (3.172),

$$f(\Omega) = \Omega^2 m \operatorname{sn}^2(\Omega h | m), \quad (3.173)$$

is a non-constant elliptic function when $h \in \mathbb{C} \setminus \{0\}$ and $m \in \mathbb{C}$ are fixed. Provided that $b_1 b_2$ is not an omitted value of f , (3.172) can be solved for Ω .

The result, together with the fixed value of h and chosen value of m , can be substituted into (3.171b) and a value for α may be obtained. We see that (3.171) possesses a two-parameter family of solutions (3.170-3.171), subject to our assumption on $b_1 b_2$ and (3.173).

Case IV: $x_0^{42} = x_0^{24}$, $x_0^{33} = x_0^{44} = x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$

Another equation is obtained from (3.135) with matrix

$$X_0^2 + X_0^6 = \begin{bmatrix} 0 & 0 & -\frac{\Omega_0}{s_0} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\Omega_0}{s_0} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (3.174)$$

The equation,

$$u'\bar{u} + u\bar{u}' = \frac{\Omega_0}{s_0} (\bar{u}^2 - u^2), \quad (3.175)$$

has a scaling symmetry leading to a second degree of freedom besides z_0 in the solution. We make the replacement $b = \Omega_0/s_0$ in (3.175) and obtain

$$u'\bar{u} + u\bar{u}' = b(\bar{u}^2 - u^2). \quad (3.176)$$

This equation is solved by

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m_0), \quad (3.177)$$

when the constraint

$$\frac{\Omega}{s} = b \quad (3.178)$$

is satisfied. The LHS of (3.178),

$$f(\Omega) = \Omega n s(\Omega h | m), \quad (3.179)$$

is a non-constant meromorphic function of Ω when $m \in \mathbb{C}$ and $h \in \mathbb{C} \setminus \{0\}$ are

fixed. Under these conditions, there are at most two values of $b \in \mathbb{C}$ such that $f(\Omega) = b$ has no solution in \mathbb{C} . Assuming b is not one of these values, (3.178) can be solved for Ω and (3.177-3.178) is a three-parameter family of solutions to (3.176).

Case V: $x_0^{42} = x_0^{24}$, $x_0^{33} = x_0^{14} = x_0^{41} = x_0^{34} = x_0^{43} = 0$

Lastly, we consider the condition (3.134) to obtain a generalization of the previous equation (3.175) (where x_0^{44} is not zero, but arbitrary). The associated matrix is

$$x_0^{24} (X_0^2 + X_0^6) + x_0^{44} X_0^4 = \begin{bmatrix} 0 & 0 & -\frac{\Omega_0}{s_0} x_0^{24} - \frac{\Omega_0^2 c_0 d_0}{s_0^2} x_0^{44} & 0 \\ 0 & \frac{\Omega_0^2 (2 - s_0^2 - m_0 s_0^2)}{s_0^2} x_0^{44} & 0 & x_0^{24} \\ \frac{\Omega_0}{s_0} x_0^{24} - \frac{\Omega_0^2 c_0 d_0}{s_0^2} x_0^{44} & 0 & 0 & 0 \\ 0 & x_0^{24} & 0 & x_0^{44} \end{bmatrix} \quad (3.180)$$

so the equation becomes

$$x_0^{24} (u' \bar{u} + u \bar{u}') + x_0^{44} u' \bar{u}' = \frac{\Omega_0}{s_0} x_0^{24} (\bar{u}^2 - u^2) + \frac{\Omega_0^2 c_0 d_0}{s_0^2} x_0^{44} (u^2 + \bar{u}^2) - \frac{\Omega_0^2 (2 - s_0^2 - m_0 s_0^2)}{s_0^2} x_0^{44} u \bar{u}. \quad (3.181)$$

As in the case of (3.175), there is a scaling symmetry. Let us redefine parameters according to

$$a_1 = x_0^{24}, \quad a_2 = x_0^{44}, \\ b_1 = a_1 \frac{\Omega_0}{s_0}, \quad b_2 = a_2 \frac{\Omega_0^2 c_0 d_0}{s_0^2}, \quad b_3 = -a_2 \frac{\Omega_0^2 (c_0^2 + d_0^2)}{s_0^2}, \quad (3.182)$$

so as to obtain

$$a_1 (u' \bar{u} + u \bar{u}') + a_2 u' \bar{u}' = b_1 (\bar{u}^2 - u^2) + b_2 (u^2 + \bar{u}^2) + b_3 u \bar{u}. \quad (3.183)$$

The parameters in appearing in (3.183) are not independent. An algebraic relationship between the parameters may be established as follows. Using the identities $c_0^2 = 1 - s_0^2$ and $d_0^2 = 1 - m_0 s_0^2$ we can derive algebraic relationships between b_1 and b_2 , b_1 and b_3 , and b_2 and b_3 :

$$\Omega_0^2 a_1^4 b_2^2 = (b_1^2 - \Omega_0^2 a_1^2) (b_1^2 - \Omega_0^2 m_0 a_1^2) \quad (3.184a)$$

$$a_1^2 = -a_2 [2b_1^2 - \Omega_0^2 (1 + m_0) a_1^2] \quad (3.184b)$$

$$4a_2 b_2 = \Omega_0^2 \left[(\Omega_0^2 (1 + m_0) a_2 - b_3)^2 - 4\Omega_0^2 a_2^2 \right] \left[(\Omega_0^2 (1 + m_0) a_2 - b_3)^2 - 4\Omega_0^2 m_0 a_2^2 \right]. \quad (3.184c)$$

Using (3.184a) and (3.184b), we can obtain expressions for Ω_0 and m_0 in terms of only a_1 , a_2 , b_1 , and b_2 (the former involves a square root). In principle we can substitute these expressions into (3.184c) and, after manipulation, obtain a algebraic relationship between the parameters appearing in (3.183) that is independent of Ω_0 and m_0 (and s_0 , c_0 , and d_0). There are two issues if we wish to parameterize the equation (3.181) in terms of the new parameters (3.182). The first is that the algebraic relation is too complicated (and for this reason we do not include it here) to be of practical use. The second is that the process of obtaining the algebraic relation introduces spurious solutions: (3.183), even when subject to the algebraic relation, is more general than (3.181). For these reasons, we will work with equation (3.181) directly. Its solution is

$$u(z) = \alpha \operatorname{sn}(\Omega_0 z + z_0 | m_0) \quad (3.185)$$

with parameters as in (3.181). This is a two-parameter family of solutions: α and z_0 are free.

3.3 Examples

In this section, we consider a number of known delay-differential and differential-difference equations. Some of these equations were recovered in our rank drop analysis in the previous section, possibly after a Möbius trans-

formation. Other known models do not admit order-two elliptic solutions. We show this, and construct other solutions for these equations when appropriate.

3.3.1 Grammaticos-Ramani-Moreira equations

A total of eight equations were identified in [32]; four of these are autonomous:

$$u' + \bar{u}' = \bar{u}^2 - u^2 + b_1(u + \bar{u}) + b_2 \quad (3.186)$$

$$u'\bar{u} - u\bar{u}' = -u^2\bar{u}^2 + b(u + \bar{u}) \quad (3.187)$$

$$u'\bar{u} - u\bar{u}' = b_1u + b_2\bar{u} + b_3u^2\bar{u} + b_4u\bar{u}^2 \quad (3.188)$$

$$u'\bar{u} + u\bar{u}' = b_1u + b_2\bar{u} + b_3u^2\bar{u} + b_4u\bar{u}^2 + b_5, \quad (3.189)$$

while the remaining four are non-autonomous:

$$u' + \bar{u}' = (u - \bar{u})^2 + b_1(u + \bar{u}) + b_2 + b_3e^{2b_1z} \quad (3.190)$$

$$u' + \bar{u}' = (u - \bar{u})^2 + b_1(u + \bar{u}) + b_2 + b_3z \quad (3.191)$$

$$u'\bar{u} + u\bar{u}' = e^{\omega z} (b_1u^2 + b_2\bar{u}^2) \quad (3.192)$$

$$u'\bar{u} - u\bar{u}' = e^{\omega z} (b_1 + b_2u^2\bar{u}^2). \quad (3.193)$$

The autonomous forms of these equations are obtained by setting z to a constant wherever it appears explicitly. Without loss of generality, we set $z = 0$ and redefine parameters to obtain

$$u' + \bar{u}' = (u - \bar{u})^2 + b_1(u + \bar{u}) + b_2 \quad (3.194)$$

$$u'\bar{u} + u\bar{u}' = b_1u^2 + b_2\bar{u}^2 \quad (3.195)$$

$$u'\bar{u} - u\bar{u}' = b_1 + b_2u^2\bar{u}^2 \quad (3.196)$$

Note that (3.194) is the autonomous limit of both (3.190) and (3.191). The final two equations are closely related to equations we have identified.

We begin with (3.195), for which the corresponding matrix is

$$X = \begin{bmatrix} 0 & 0 & -b_2 & 0 \\ 0 & 0 & 0 & 1 \\ -b_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (3.197)$$

When $b_1 = -b_2 = -b$, this is precisely (3.176) in our classification. The solution in this case is given by (3.177-3.178). In fact (3.195) only admits order-two elliptic solutions when it reduces to (3.176), as is seen by solving the system of equations

$${}^tMXM = \sum_{n=0}^7 \lambda^n X^n \quad (3.198)$$

for the λ^n , the Möbius parameters, and the elliptic parameters; (3.198) has a nontrivial solution if and only if $b_1 = -b_2$. In doing so, we obtain the solution (3.177-3.178) and also the solution

$$u(z) = \beta \operatorname{ns}(\Omega z + z_0 | m), \quad (3.199)$$

which is subject to (3.178). This solution was not previously found because it is associated with a Möbius transformation that is not identity-connected.

Now we turn to (3.196), which is identical to (3.169). The solution to this equation is (3.170-3.171). No additional solutions for this equation are found from (3.198) with X as in (3.166)

The remaining equations identified in [32] (in their autonomous forms) were not isolated in our classification. Nevertheless they may be Möbius-equivalent to our identified equations. It suffices to say that this is not the case for (3.187), while for (3.188-3.189) this happens only when the parameters p_i satisfy complicated conditions. We will discuss the remaining equations in greater detail.

The equation (3.186) is associated with the matrix

$$X = \begin{bmatrix} -b_2 & -b_1 & 1 & 1 \\ -b_1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (3.200)$$

Solving (3.198) with X , as above, we find that (3.186) admits a multiparameter family of order-two elliptic solutions if and only if $b_1 = b_2 = 0$, in which case (3.200) is equivalent via a constant Möbius transformation to $X^1 + X^7$, i.e. a special case of (3.161). The solution to (3.186) with $b_1 = b_2 = 0$ is given by

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0 | m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0 | m) + 1} \quad (3.201)$$

with

$$(\alpha\delta + \beta\gamma)s + 2\Omega\gamma\delta = 0 \quad (3.202a)$$

$$2\beta\delta s + \gamma^2\Omega s^2 + \delta^2\Omega(1 + cd) = 0 \quad (3.202b)$$

$$2\alpha\gamma s + \delta^2\Omega m s^2 + \gamma^2\Omega(1 + cd) = 0. \quad (3.202c)$$

In principle, a multiparameter (subject to the usual caveat regarding omitted values) solution to (3.186) could also be constructed from the solution (3.162-3.163) of (3.161). We will not pursue this as the solution can be represented in a simpler way in terms of the Weierstrass ζ -function. A similar result was obtained in [10]. Taking

$$x_1 = \Omega z + z_0, \quad x_2 = -\Omega z - \Omega h - z_0, \quad x_3 = \Omega h \quad (3.203)$$

in (A.29) and using properties of the ζ -function, we find that

$$\begin{aligned} & [\zeta(\Omega z + \Omega h + z_0; g_2, g_3) - \zeta(\Omega z + 2\Omega h + z_0; g_2, g_3) + \zeta(\Omega h; g_2, g_3)]^2 \\ & - [\zeta(\Omega z + z_0; g_2, g_3) - \zeta(\Omega z + \Omega h + z_0; g_2, g_3) + \zeta(\Omega h; g_2, g_3)]^2 \end{aligned}$$

$$-\zeta'(\Omega z + z_0; g_2, g_3) - \zeta'(\Omega z + 2\Omega h + z_0; g_2, g_3) = 0. \quad (3.204)$$

It follows that

$$u(z) = \Omega [\zeta(\Omega z + z_0; g_2, g_3) - \zeta(\Omega z + \Omega h + z_0; g_2, g_3) + \zeta(\Omega h; g_2, g_3)] \quad (3.205)$$

solves (3.186) with $p_1 = p_2 = 0$. Using the scaling properties of the ζ -function (A.31), we are able to write this solution in the standard form (3.4)

$$u(z) = \zeta(z + z_0; g'_2, g'_3) - \zeta(z + h + z_0; g'_2, g'_3) + \zeta(h; g'_2, g'_3), \quad (3.206)$$

where $g'_2 = \Omega^4 g_2$ and $g'_3 = \Omega^6 g_3$. In (3.206), the parameters z_0 , g'_2 , and g'_3 are free. By means of (A.27) and (3.62) we find that (3.202) and (3.206) are equivalent.

We lastly consider the equation (3.194). This equation admits no order-two elliptic solutions, but a special case does admit solutions in terms of the Weierstrass ζ -function. We consider the case $b_1 = 0$, so the equation becomes

$$u' + \bar{u}' = (u - \bar{u})^2 + b_2. \quad (3.207)$$

Note that this equation is invariant under a translation of the dependent variable. Again starting from (A.29) and (3.203), we find that

$$\begin{aligned} & [\zeta(\Omega z + z_0; g_2, g_3) - \zeta(\Omega z + \Omega h + z_0; g_2, g_3) + \zeta(\Omega h; g_2, g_3)]^2 = \\ & -\zeta'(\Omega z + z_0; g_2, g_3) - \zeta'(\Omega z + \Omega h + z_0; g_2, g_3) - \zeta'(\Omega h; g_2, g_3). \end{aligned} \quad (3.208)$$

It follows that

$$u(z) = -\Omega \zeta(\Omega z + z_0; g_2, g_3) + \beta \quad (3.209)$$

solves (3.207) provided that the constraints

$$\zeta(\Omega h; g_2, g_3) = 0, \quad \wp(\Omega h; g_2, g_3) = b_2 \quad (3.210)$$

are satisfied. Note that this solution is not elliptic.

3.3.2 Sine-Gordon type equations

Four of the equations we have identified are related to a semidiscrete sine-Gordon equation studied by Orfanidis [61]. This equation is essentially the spatial part of the Bäcklund transformation (2.91a). We redefine the parameters in (2.91a) to achieve the form given in [9]:

$$\theta'_{n+1} - \theta'_n = 4\lambda \sin \frac{\theta_n + \theta_{n+1}}{2}, \quad (3.211)$$

where $\phi_n = \phi_n(t)$ and $' = d/dt$. By introducing a new dependent variable

$$w_n = a \exp \frac{i\theta_n}{2}, \quad (3.212)$$

and redefining the parameter λ according to

$$a^2 \lambda = pb_1, \quad \frac{\lambda}{a^2} = -pb_2, \quad (3.213)$$

we arrive at

$$w'_n w_{n+1} - w_n w'_{n+1} = p(b_1 + b_2 w_n^2 w_{n+1}^2). \quad (3.214)$$

Now taking a traveling wave reduction

$$w_n(t) = u(z), \quad z = nh + pt + z_0, \quad (3.215)$$

we obtain precisely (3.169) or (3.196). Making use of the solution (3.170-3.171), we find that

$$w_n(t) = \alpha \operatorname{sn}(\Omega(nh + pt) + z_0 | m) \quad (3.216)$$

solves (3.214) when the parameters satisfy (3.171) and

$$\theta_n(t) = -2i \log \left[\frac{\alpha}{a} \operatorname{sn}(\Omega(nh + pt) + z_0 | m) \right] \quad (3.217)$$

solves (3.211) when the parameters satisfy (3.171) and (3.213).

A special case of (3.159) may also be obtained from (3.211). Under the change of variables

$$\theta_n = \pm 4 \arctan \left(\sqrt{\frac{b_2}{b_1}} w_n \right), \quad (3.218)$$

(3.211) becomes

$$\begin{aligned} b_1 (w'_{n+1} - w'_n) + b_2 (w'_n w_{n+1}^2 - w_n^2 w'_{n+1}) = \\ 2\lambda \left[b_1 (w_n + w_{n+1}) + b_2 (w_n w_{n+1}^2 + w_n^2 w_{n+1}) \right]. \end{aligned} \quad (3.219)$$

Using the same traveling wave reduction (3.215), we obtain

$$b_1 (\bar{u}' - u') + b_2 (u' \bar{u}^2 - u^2 \bar{u}') = \frac{2\lambda}{p} \left[b_1 (u + \bar{u}) + b_2 (u \bar{u}^2 + u^2 \bar{u}) \right]. \quad (3.220)$$

While this equation is not as general as (3.161), we may use previous results to obtain a solution:

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m), \quad (3.221)$$

subject to

$$\frac{2\lambda b_1}{p} = \frac{\Omega}{s} \left[b_1 (cd - 1) - b_2 \alpha^2 s^2 \right] \quad (3.222a)$$

$$\frac{2\lambda b_2 \alpha^2}{p} = \frac{\Omega}{s} \left[b_1 m s^2 - b_2 \alpha^2 (cd - 1) \right]. \quad (3.222b)$$

The argument that this is generically a two-parameter family of solutions is essentially identical to that in the context of (3.161). The corresponding solution to (3.219) is

$$w_n(t) = \alpha \operatorname{sn}(\Omega(nh + t) + z_0 | m), \quad (3.223)$$

also subject to (3.222). We finally obtain a solution to (3.211):

$$\theta_n(t) = \pm 4 \arctan \left[\alpha \sqrt{\frac{b_2}{b_1}} \operatorname{sn}(\Omega(nh + t) + z_0 | m) \right], \quad (3.224)$$

where the parameters satisfy (3.222).

Now we consider the temporal part of the Bäcklund transformation: (2.91b). Redefining parameters, we obtain

$$\theta'_{n+1} + \theta'_n = 4\lambda \sin \frac{\theta_{n+1} - \theta_n}{2}, \quad (3.225)$$

in analogy with (3.211). The change of variables

$$w_n = \exp \frac{i\theta_n}{2} \quad (3.226)$$

leads to

$$w'_n w_{n+1} + w_n w'_{n+1} = \lambda (w_{n+1}^2 - w_n^2), \quad (3.227)$$

which precisely (2.95). Taking the traveling wave reduction (3.215), we arrive at the delay-differential equation

$$u'\bar{u} + u\bar{u}' = \frac{\lambda}{p} (\bar{u}^2 - u^2), \quad (3.228)$$

which is equivalent to (3.175) and is a special case of (3.195). We thus obtain the solutions

$$w_n(t) = \alpha \operatorname{sn}(\Omega(nh + qt) + z_0 | m) \quad (3.229)$$

$$w_n(t) = \beta \operatorname{ns}(\Omega(nh + qt) + z_0 | m) \quad (3.230)$$

to (2.95) and

$$\theta_n(t) = -2i \log [\alpha \operatorname{sn}(\Omega(nh + qt) + z_0 | m)] \quad (3.231)$$

$$\theta_n(t) = -2i \log [\beta \operatorname{ns}(\Omega(nh + qt) + z_0 | m)] \quad (3.232)$$

to (3.225), where α and β are free and the remaining parameters satisfy

$$\frac{\Omega}{s} = \frac{\lambda}{p}. \quad (3.233)$$

Now applying the change of variables (3.218) to (3.225), we obtain

$$b_1(w'_n + w'_{n+1}) + b_2(w'_n w_{n+1}^2 + w_n^2 w'_{n+1}) = \quad (3.234)$$

$$2\lambda \left[b_1(w_{n+1} - w_n) + b_2(w_n w_{n+1}^2 - w_n^2 w_{n+1}) \right] \quad (3.235)$$

and a traveling wave reduction (3.215) leads to

$$b_1(u' + \bar{u}') + b_2(u'\bar{u}^2 + u^2\bar{u}') = \frac{2\lambda}{p} \left[b_1(\bar{u} - u) + b_2(u\bar{u}^2 - u^2\bar{u}) \right], \quad (3.236)$$

a special case of (3.152). The solution to (3.236) is

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta}, \quad (3.237)$$

subject to

$$\lambda s(\alpha\beta b_2 + \gamma\delta b_1) = p\Omega(\alpha\beta b_2 + \gamma\delta b_1) \quad (3.238a)$$

$$\frac{2\lambda s}{p\Omega} (\beta^2 b_2 + \delta^2 b_1) = s^2 (\alpha^2 b_2 + \gamma^2 b_1) + \operatorname{cd}(\beta^2 b_2 + \delta^2 b_1) + \beta^2 b_2 + \delta^2 b_1 \quad (3.238b)$$

$$\frac{2\lambda s}{p\Omega} (\alpha^2 b_2 + \gamma^2 b_1) = m s^2 (\beta^2 b_2 + \delta^2 b_1) + \operatorname{cd}(\alpha^2 b_2 + \gamma^2 b_1) + \alpha^2 b_2 + \gamma^2 b_1. \quad (3.238c)$$

The corresponding solutions to (3.235) and (3.225) are, respectively,

$$w_n(t) = \frac{\alpha \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta}, \quad (3.239)$$

and

$$\theta_n(t) = \pm 4 \arctan \left[\sqrt{\frac{b_2}{b_1}} \frac{\alpha \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta} \right], \quad (3.240)$$

both subject to (3.238).

3.3.3 Korteweg-de Vries type equations

In this section we consider order-two elliptic solutions of semidiscrete KdV type equations (2.94-2.96). In fact, we have already discussed one of these, the lattice pmKdV equation (2.95), in the previous section. We begin with (2.94), for which a traveling wave reduction (3.215) leads to

$$u' + \bar{u}' = \frac{\lambda}{p} + \frac{1}{p}(\bar{u} - u)^2 \quad (3.241)$$

after a relabeling of parameter. This equation admits no order-two elliptic solutions. However, when $\lambda = 0$ it is equivalent to (3.207). In this case, we obtain the solution

$$u(z) = \Omega\zeta(\Omega z + z_0; g_2, g_3) + \beta \quad (3.242)$$

to (3.241) and

$$w_n(t) = \Omega\zeta[\Omega(nh + qt) + z_0; g_2, g_3] + \beta \quad (3.243)$$

to (2.94), where the parameters in each satisfy

$$\zeta(\Omega h; g_2, g_3) = 0, \quad \wp(\Omega h; g_2, g_3) = \frac{1}{p}. \quad (3.244)$$

We next turn to a Schwarzian KdV equation (2.96). The corresponding delay-differential equation under (3.215) is

$$u'\bar{u}' = \frac{\lambda}{p}(u - \bar{u})^2, \quad (3.245)$$

which has the matrix representation

$$X = \begin{bmatrix} 0 & 0 & -\frac{\lambda}{p} & 0 \\ 0 & 2\frac{\lambda}{p} & 0 & 0 \\ -\frac{\lambda}{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (3.246)$$

The solution is found to be

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta}, \quad (3.247)$$

where $\alpha\delta - \beta\gamma \neq 0$ and

$$\lambda s^2 - p\Omega^2 cd = 0 \quad (3.248a)$$

$$2\lambda s^2 - p\Omega^2 [2 - (1+m)s^2] = 0. \quad (3.248b)$$

Substituting the second condition into the first and using standard identities, we find that $c = d$. One solution to this transcendental equation is $m = 1$, in which case the solution becomes

$$u(z) = \frac{\alpha \operatorname{sech} \Omega z + \beta}{\gamma \operatorname{sech} \Omega z + \delta}, \quad (3.249)$$

subject to $\alpha\delta - \beta\gamma = 0$ (we have absorbed the translational freedom represented by z_0 into the Möbius parameters) and

$$\lambda \sinh^2 \Omega h = p\Omega^2. \quad (3.250)$$

By the Schwarzian nature of (3.245), this solution can be written as

$$u(z) = \frac{\alpha \exp \Omega z + \beta}{\gamma \exp \Omega z + \delta}; \quad (3.251)$$

the corresponding solution to (2.96) is

$$w_n(t) = \frac{\alpha \exp \Omega(nh + pt) + \beta}{\gamma \exp \Omega(nh + pt) + \delta}. \quad (3.252)$$

Both of these are subject to the nondegeneracy condition $\alpha\delta - \beta\gamma \neq 0$ and the constraint (3.250). This simple example illustrates how our method can be used to obtain elementary solutions to bi-Riccati equations.

Chapter 4

Three-point equations

In this chapter we study a class of three-point delay-differential equations:

$${}^tUXV = 0, \quad U = {}^t(1, u, u^2, u'), \quad V = {}^t\left(1, \frac{\bar{u} + u}{2}, \bar{u}u, \frac{\bar{u} - u}{2}\right), \quad X : \mathbb{C} \rightarrow \mathbb{C}^{4 \times 4}. \quad (4.1)$$

Unlike the bi-Riccati class (3.1), this particular class of equations has not been studied before, though it contains several well-known examples. These include the McMillan map, traveling wave reductions of the Wadati lattice, and a Painlevé-I type delay-differential equation obtained by Quispel, Capel, and Sahadevan [68]. One particular feature of (4.1) is that it contains a subclass of the QRT map (2.34), namely all symmetric QRT maps (2.38) in which the degree in u of each f_i is at most two.

The results on the relationship between singularity sequences and elliptic solutions in the previous chapter apply directly to equations in the class (4.1). However, there has been no systematic study of (4.1) from the perspective of singularity confinement, and there are very few examples to which we can apply the singularity structure results of the previous chapter. We are aware of only two such examples: singularity confinement in the sense of [72] has been applied to a Painlevé-I type delay-differential equation in [68] and the symmetric QRT map, whose degree-two subclass is contained in (4.1), has been studied from the usual singularity confinement perspective. It is actually more natural to relate our results to these examples from other directions. We

will show that the Painlevé-I type equation in [68] is an extension of a one of the equations we identify in this chapter. QRT maps in the class (4.1) are also discussed for their direct relation to equations we isolate in what follows. For these reasons, we will not discuss singularity structure any further in this chapter.

The structure of this chapter follows that of the previous chapter very closely. The main result is again a list of families of equations in the class (4.1) that admit multiparameter families of order-two elliptic solutions. The process of obtaining this list is analogous to that in the last chapter. We first identify the vector subspace of $\mathbb{C}^{4 \times 4}$ corresponding to equations (4.1) admitting order-two elliptic solutions with at least one degree of parametric freedom. As in the case of bi-Riccati equations, the dimension of this subspace is eight, corresponding to a seven-parameter family of three-point equations (4.1). Using the same methods introduced and applied to bi-Riccati equations in the previous chapter, we isolate five families of equations within (4.1) that admit order-two elliptic solutions with at least two degrees of parametric freedom. Our results are then applied to a number of examples.

4.1 Order-two elliptic solutions

As in the case of bi-Riccati equations, we will identify all autonomous equations in the class (4.1) that admit order-two elliptic solutions. The solutions are again sought in the form (3.59). Our chosen class of equations has the desirable property of being Möbius equivalent; in fact it transforms identically to the bi-Riccati class under a general fractional linear transformation. Therefore we perform our classification up to Möbius equivalence. Applying the same methods as before, we find that an equation

$${}^tUXV = 0, \quad U = {}^t(1, u, u^2, u'), \quad V = {}^t\left(1, \frac{\bar{u} + u}{2}, \bar{u}u, \frac{\bar{u} - u}{2}\right), \quad X \in \mathbb{C}^{4 \times 4} \quad (4.2)$$

is solved by $u = \text{sn}(\Omega z + z_0|m)$ if and only if

$$X \in \text{span} \{X^1, X^2, X^3, X^4, X^5, X^6, X^7, X^8\}, \quad (4.3)$$

where

$$X^1 = \begin{bmatrix} 0 & 0 & 0 & -\frac{\Omega}{s} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega ms \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (4.4a)$$

$$X^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\Omega cd}{s} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (4.4b)$$

$$X^3 = \begin{bmatrix} 0 & 0 & 0 & \Omega s \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Omega s^{-1} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (4.4c)$$

$$X^4 = \begin{bmatrix} \frac{\Omega s(c^2+d^2)}{ms^4-1} & 0 & \frac{\Omega c^2 d^2}{s(ms^4-1)} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\Omega}{s} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.4d)$$

$$X^5 = \begin{bmatrix} -\frac{s^2 cd}{ms^4-1} & 0 & \frac{cd}{ms^4-1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.4e)$$

$$X^6 = \begin{bmatrix} 0 & \frac{ms^4-1}{ms^2 cd} & 0 & 0 \\ \frac{1}{ms^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.4f)$$

$$X^7 = \begin{bmatrix} 0 & -\frac{1}{ms^2} & 0 & 0 \\ \frac{cd}{ms^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.4g)$$

$$X^8 = \begin{bmatrix} -\frac{1}{m} & 0 & -\frac{1}{ms^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{ms^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (4.4h)$$

It follows immediately that an equation in the class (4.2) is solved by (3.59) if and only if

$${}^tMXM \in \text{span}\{X^1, X^2, X^3, X^4, X^5, X^6, X^7, X^8\}. \quad (4.5)$$

A generic element of the span of (4.4) can be written as

$$\sum_{i=1}^8 \lambda^i X^i = \begin{bmatrix} \frac{\Omega s(c^2+d^2)\lambda^4 + cds^2\lambda^5}{ms^4-1} - \frac{1}{m}\lambda^8 & \frac{ms^4-1}{mcds^2}\lambda^6 - \frac{1}{ms^2}\lambda^7 & \frac{\Omega c^2 d^2}{s(ms^4-1)}\lambda^4 + \frac{cd}{ms^4-1}\lambda^5 - \frac{1}{ms^2}\lambda^8 & -\frac{\Omega}{s}\lambda^1 + \Omega s\lambda^3 \\ \frac{1}{ms^2}\lambda^6 + \frac{cd}{ms^2}\lambda^7 & \lambda^5 & \lambda^6 & -\frac{\Omega cd}{s}\lambda^2 \\ \frac{\Omega}{s}\lambda^4 + \frac{1}{ms^2}\lambda^8 & \lambda^7 & \lambda^8 & \Omega ms\lambda^1 - \frac{\Omega}{s}\lambda^3 \\ \lambda^1 & \lambda^2 & \lambda^3 & \lambda^4 \end{bmatrix}; \quad (4.6)$$

from this we construct the function ϕ , with components

$$\phi^1 = x^{14} + \frac{\Omega}{s}x^{41} - \Omega s x^{43} \quad (4.7a)$$

$$\phi^2 = x^{24} + \frac{\Omega cd}{s}x^{42} \quad (4.7b)$$

$$\phi^3 = x^{34} - \Omega ms x^{41} + \frac{\Omega}{s}x^{43} \quad (4.7c)$$

$$\phi^4 = x^{31} - \frac{\Omega}{ms}x^{44} - \frac{cd}{ms^2}x^{33} \quad (4.7d)$$

$$\phi^5 = x^{11} - \frac{\Omega s (c^2 + d^2)}{ms^4 - 1} x^{44} - \frac{cds^2}{ms^4 - 1} x^{22} + \frac{1}{m} x^{33} \quad (4.7e)$$

$$\phi^6 = x^{13} - \frac{\Omega c^2 d^2}{s(ms^4 - 1)} x^{44} - \frac{cd}{ms^4 - 1} x^{22} + \frac{1}{ms^2} x^{33} \quad (4.7f)$$

$$\phi^7 = x^{12} - \frac{ms^4 - 1}{mcds^2} x^{23} + \frac{1}{ms^2} x^{32} \quad (4.7g)$$

$$\phi^8 = x^{21} - \frac{1}{ms^2} x^{23} - \frac{cd}{ms^2} x^{32}. \quad (4.7h)$$

We again work with an extended Jacobian matrix, defined as before as

$$\tilde{J}^{\text{ext}} = {}^t [D ({}^t C^t J)], \quad (4.8)$$

where J as in (3.90) with ϕ as in (4.7), C as in (3.113), and

$$D = \left[\begin{array}{c|cc} I_4 & & 0_{4 \times 2} \\ \hline & 1 - \Xi \frac{\partial}{\partial \Xi} & 0 \\ 0_{2 \times 4} & & 0 \quad \frac{\partial}{\partial \Xi} \end{array} \right]. \quad (4.9)$$

The resulting matrix has the block structure

$$\tilde{J}^{\text{ext}} = \begin{bmatrix} A \\ B \end{bmatrix}, \quad B = \begin{bmatrix} B_1 & B_3 \\ B_2 & B_4 \end{bmatrix}. \quad (4.10)$$

Here A is a 3×6 matrix with dependence on x_0^{41} , x_0^{42} , and x_0^{43} , B_1 is a 3×2 matrix with dependence on x_0^{23} and x_0^{32} , B_2 is a 2×2 matrix with dependence on x_0^{22} , x_0^{33} , and x_0^{44} , B_3 is a 3×4 matrix with dependence on x_0^{23} and x_0^{32} , and B_4 is a 2×4 matrix with dependence on x_0^{22} , x_0^{33} , and x_0^{44} .

$$\begin{aligned}
{}^tB_3 = & \left[\begin{array}{l} -\frac{2x_0^{33}}{m_0s_0^2} \\ 2hc_0d_0x_0^{33} + m_0s_0(h\Omega_0c_0d_0 - s_0)x_0^{44} \\ \frac{m_0s_0^3}{x_0^{44}} \\ \frac{s_0}{m_0s_0^2} \\ \frac{x_0^{33}}{m_0^2s_0} \end{array} \right] + \left[\begin{array}{l} 4(\Omega_0m_0s_0^4 - 1)x_0^{33} + 2\Omega_0m_0s_0^2c_0d_0x_0^{22} - 2\Omega_0m_0s_0[2 - (1 + m_0s_0^2)] \\ \frac{m_0(m_0s_0^4 - 1)x_0^{44}}{m_0(m_0s_0^4 - 1)} \\ \frac{s_0[2 - (1 + m_0)s_0^2]x_0^{44}}{m_0s_0^4 - 1} \\ \frac{2x_0^{33}}{m_0s_0^2} \end{array} \right] \\
& \qquad \qquad \qquad (B_3)_{22} \qquad \qquad \qquad (B_3)_{32} \qquad \qquad \qquad (B_3)_{34} \\
& \qquad \qquad \qquad \frac{c_0^2d_0^2x_0^{44}}{s_0(m_0s_0^4 - 1)}
\end{aligned} \tag{4.14}$$

$$\begin{aligned}
(B_3)_{22} &= \frac{hs_0[-2 + 3(1 + m_0)s_0^2 - 6m_0s_0^4 + m_0(1 + m_0)s_0^6]x_0^{22}}{(m_0s_0^4 - 1)^2} \\
&+ \frac{[s_0(-2 + (1 + m_0)s_0^2)(m_0s_0^4 - 1) - hc_0d_0(-2 + 3(1 + m_0)s_0^2 - 6m_0s_0^4 + m_0(1 + m_0)s_0^6)]x_0^{44}}{(m_0s_0^4 - 1)^2} \\
(B_3)_{32} &= \frac{hs_0[-1 + m_0(-1 + 6s_0^2 - 3[1 + m_0]s_0^4 + 2m_0s_0^6)]x_0^{22}}{(m_0s_0^4 - 1)^2} - \frac{2hc_0d_0[1 + m_0s_0^4(-2 + m_0s_0^4)]x_0^{33}}{m_0s_0^3(m_0s_0^4 - 1)^2} \\
&+ \frac{[-m_0s_0^2c_0^2d_0^2(-1 + m_0s_0^4) + h\Omega_0m_0s_0c_0d_0(-1 - [1 + m_0]s_0^2 + 8m_0s_0^4 - 3m_0[1 + m_0]s_0^6 + m_0^2s_0^8)]}{m_0s_0^3(m_0s_0^4 - 1)^2}
\end{aligned}$$

$$(B_3)_{24} = \frac{s_0^4c_0(1 - 2s_0^2 + m_0s_0^4)x_0^{22}}{2d_0(m_0s_0^4 - 1)^2} - \frac{x_0^{33}}{m_0} - \frac{\Omega_0s_0^3c_0^4}{m_0(m_0s_0^4 - 1)^2}$$

$$(B_3)_{34} = -\frac{s_0^2c_0(1 - 2s_0^2 + m_0s_0^4)x_0^{22}}{2d_0(m_0s_0^4 - 1)^2} - \frac{x_0^{33}}{m_0^2s_0} + \frac{\Omega_0s_0c_0^4}{(m_0s_0^4 - 1)^2}$$

$${}^tB_4 = \left[\begin{array}{l} 2(1 - m_0s_0^4)x_0^{23} + 2c_0d_0x_0^{32} \\ h[-2 + 3(1 + m_0s_0^2 - 6m_0s_0^4 + m_0(1 + m_0)s_0^6]x_0^{23} - 2hc_0d_0x_0^{32} + [2 - (1 + m_0)s_0^2]x_0^{32} \\ \frac{m_0s_0^2c_0d_0}{m_0s_0^3c_0^2d_0^2} \\ 0 \\ -\frac{2 + m_0s_0^2(m_0s_0^4 - 3)}{2m_0^2s_0^2c_0d_0}x_0^{23} - \frac{x_0^{32}}{m_0^2s_0^2} \\ -\frac{2c_0d_0x_0^{23} + c_0^2(2 - m_0s_0^2)x_0^{32}}{2m_0s_0^2c_0d_0} \end{array} \right] \\
\qquad \qquad \qquad 0 \qquad \qquad \qquad 0 \qquad \qquad \qquad 0
\end{aligned} \tag{4.15}$$

4.1.1 Rank drop analysis

We now seek conditions under which the rank of the Jacobian drops. As before, we look for conditions where the first four columns of \tilde{J}^{ext} are rank-deficient or the rank of \tilde{J}^{ext} drops by two. We first observe that if $x_0^{23} = x_0^{32} = x_0^{22} = x_0^{33} = x_0^{44}$, then $B = 0_{5 \times 6}$ and the rank of the Jacobian is at most three. Therefore no analysis on the matrix A is required. On the other hand, if $x_0^{41} = x_0^{42} = x_0^{43} = 0$, we have $A = 0_{3 \times 6}$ which is not sufficient to cause the rank of the Jacobian to drop. Therefore a detailed analysis of the submatrices of the matrix B is required to determine conditions under which the rank drops.

4.1.1.1 B_1

The matrix B_1 has a maximal rank of 2. Either row of ${}^t B_1$ vanishes if and only if $x_0^{23} = x_0^{32} = 0$. We now analyze the 2×2 minors of B_1 . The first two minors have homogenous linear factors in x_0^{23} and $x_0^{32} = 0$ with only the trivial solution. We conclude that B_1 has full rank unless $x_0^{23} = x_0^{32} = 0$, in which case the rank is zero.

4.1.1.2 B_2

The first row of B_2 vanishes when¹

$$\begin{aligned}
 x_0^{22} = 1, \quad x_0^{33} &= \frac{m_0 [2 - c_0 d_0 - (1 + m_0) s_0^2] [2 - 2c_0 d_0 - (1 + m_0) s_0^2]}{2 [2 - (1 + m_0) s_0^2] [1 + m_0 (2s_0^2 - 1)]}, \\
 x_0^{44} &= \frac{m_0 s_0^2 [c_0 d_0 - 1 + 2(1 + c_0 d_0) m_0 s_0^4 - 2m_0 (1 + m_0) s_0^6 + m_0^2 s_0^8]}{2\Omega_0 m_0 s_0 [c_0 d_0 - 1 + (1 + m_0) s_0^2 - 3m_0 s_0^4 c_0 d_0 + m_0 (1 + m_0) (c_0 d_0 - 1) s_0^6 + m_0^2 s_0^8]} \\
 &\quad - \frac{2(c_0 d_0 - 1) (m_0 s_0^4 - 1)^2 x_0^{33}}{2\Omega_0 m_0 s_0 [c_0 d_0 - 1 + (1 + m_0) s_0^2 - 3m_0 s_0^4 c_0 d_0 + m_0 (1 + m_0) (c_0 d_0 - 1) s_0^6 + m_0^2 s_0^8]}
 \end{aligned} \tag{4.16}$$

and the second row vanishes when

$$x_0^{22} = 1, \quad x_0^{33} = \frac{m_0 c_0 d_0 [-2 + 2c_0 d_0 + (1 + m_0) s_0^2]}{2\Omega_0 [2 - (1 + m_0) s_0^2] [1 - m_0 (2s_0^2 - 1)]},$$

¹The condition is written implicitly to simplify the presentation.

$$x_0^{44} = \frac{2c_0^3 d_0^3 - [2 - (1 + m_0)s_0^2] (m_0 s_0^4)}{2s_0 [2 - (1 + m_0)s_0^2] [1 - m_0 (2s_0^2 - 1)]}. \quad (4.17)$$

More generally, the two rows are linearly dependent when $\det B_2 = 0$. The resulting quadratic equation does not simplify; it suffices to say that the vanishing of the quadratic does not lead to any equations with multiparameter order-two elliptic solutions.

4.1.1.3 B_3

The first row of ${}^t B_3$ vanishes when

$$x_0^{22} = -\frac{\Omega_0(c_0^2 + d_0^2)}{s_0 c_0 d_0}, \quad x_0^{33} = 0, \quad x_0^{44} = 1, \quad (4.18)$$

and the second row of ${}^t B_3$ only when $x_0^{22} = x_0^{33} = x_0^{44} = 0$. More generally the first two rows of ${}^t B_3$ are linearly dependent when the x_0^{22} , x_0^{33} and x_0^{44} satisfy a complicated quadratic equation. However, the only solution of interest to this equation is (4.18).

If the first two rows of ${}^t B_3$ are linearly independent, we search for conditions under which $\text{rk} B_3 \leq 2$. For this to happen, all 3×3 minors of ${}^t B_3$ must vanish. The 3×3 minor corresponding to the last three columns of B_3 factors into a product of linear homogenous terms, leading to the condition

$$\begin{aligned} & x_0^{44} \left[m_0^2 s_0^2 c_0^3 d_0 x_0^{22} + (m_0 s_0^4 - 1) (2m_0 s_0^2 - m_0 - 1) x_0^{33} + \Omega_0 m_0^2 s_0 c_0^2 (c_0^2 + d_0^2) x_0^{44} \right] \\ & \times \left[m_0 s_0^2 (1 + m_0 s_0^4 - 2s_0^2) x_0^{22} + 2c_0 d_0 (1 - m_0 s_0^4) x_0^{33} + 2\Omega_0 m_0 s_0 c_0^3 d_0 x_0^{44} \right] = 0. \end{aligned} \quad (4.19)$$

The vanishing of the first factor causes the other minors to vanish; the rank of the Jacobian drops when

$$x_0^{44} = 0 \quad (4.20)$$

and x_0^{22} and x_0^{33} are arbitrary. The vanishing of the second factor does not lead to the vanishing of the other minors. From the third factor we obtain the

condition

$$x_0^{22} = -\frac{2\Omega_0 c_0 d_0}{s_0}, \quad x_0^{33} = \frac{\Omega_0 m_0 s_0^3 d_0^2}{m_0 s_0^4 - 1}, \quad x_0^{44} = 1 \quad (4.21)$$

that causes all minors to vanish and the rank of the Jacobian to drop.

4.1.1.4 B_4

No analysis of B_4 is required because the rank of the Jacobian drops generically when all x_0^{ij} but those appearing in B_4 (and B_1) vanish. We see in the next section that for arbitrary x_0^{23} and x_0^{32} , this leads to an equation with a multiparameter family of order-two elliptic solutions.

4.1.2 Equations and solutions

From the results in the previous section, we obtain five equations that admit multiparameter families of Möbius-Jacobi solutions. As in the previous chapter, we consider five cases of the parameters appearing in the matrix X_0 .

Case I: $x_0^{23} = x_0^{32} = x_0^{22} = x_0^{33} = x_0^{44} = 0$

The first case is associated with the matrix A ; when all x_0^{ij} are zero besides those appearing in A , we obtain

$$x_0^{41} X_0^1 + x_0^{42} X_0^2 + x_0^{43} X_0^3 = \begin{bmatrix} 0 & 0 & 0 & -\frac{\Omega_0}{s_0} x_0^{41} + \Omega_0 s_0 x_0^{43} \\ 0 & 0 & 0 & -\frac{\Omega_0 c_0 d_0}{s_0} x_0^{42} \\ 0 & 0 & 0 & \Omega_0 m_0 s_0 x_0^{41} - \frac{\Omega_0}{s_0} x_0^{43} \\ x_0^{41} & x_0^{42} & x_0^{43} & 0 \end{bmatrix} \quad (4.22)$$

with associated equation (after relabeling $x_0^{41} \rightarrow a_1$, $x_0^{42} \rightarrow a_2$, $x_0^{43} \rightarrow a_3$)

$$\frac{u'}{\bar{u} - \underline{u}} = \frac{\frac{\Omega_0}{s_0} a_1 + \Omega_0 s_0 a_2 + u \frac{\Omega_0 c_0 d_0}{s_0} a_2 - u^2 \left(\Omega_0 m_0 s_0 a_1 - \frac{\Omega_0}{s_0} a_2 \right)}{2a_1 + a_2(\bar{u} - \underline{u}) + 2a_3 \underline{u} \bar{u}}. \quad (4.23)$$

This equation has the solution

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0 | m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0 | m) + \delta} \quad (4.24)$$

when the parameters satisfy

$$\begin{aligned} \Omega s \left(\gamma^2 a_1 + \alpha \gamma a_2 + \alpha^2 a_3 \right) - \frac{\Omega}{s} \left(\delta^2 a_1 + \beta \delta a_2 + \beta^2 a_3 \right) = \\ \frac{\Omega_0}{s_0} \left[\beta^2 \left(m_0 s_0^2 a_1 - a_2 \right) - \beta \delta c_0 d_0 a_2 - \delta^2 \left(a_1 - s_0^2 a_2 \right) \right] \end{aligned} \quad (4.25a)$$

$$\begin{aligned} \frac{\Omega c d}{s} \left(2 \gamma \delta a_1 + (\alpha \delta + \beta \gamma) a_2 + 2 \alpha \beta a_3 \right) = \\ \frac{\Omega_0}{s_0} \left[2 \left(\gamma \delta - \alpha \beta m_0 s_0^2 \right) a_1 + (\alpha \delta + \beta \gamma) c_0 d_0 a_2 + 2 \left(\alpha \beta - \gamma \delta s_0^2 \right) \right] \end{aligned} \quad (4.25b)$$

$$\begin{aligned} \frac{\Omega}{s} \left[\left(\delta^2 m s^2 - \gamma^2 \right) a_1 + \left(\beta \delta m s^2 - \alpha \gamma \right) a_2 + \left(\beta^2 m s^2 - \alpha^2 \right) a_3 \right] = \\ \Omega_0 s_0 \left(\alpha^2 m_0 x_0^{41} + \gamma^2 a_3 \right) + \frac{\Omega_0}{s_0} \left(\gamma^2 a_1 + \alpha \gamma c_0 d_0 a_2 + \alpha^2 a_3 \right). \end{aligned} \quad (4.25c)$$

It is difficult to parameterize this function in terms of arbitrary complex coefficients, subject to algebraic relationships, for the same reasons discussed in the context of (3.181). Moreover, by imposing $\beta = \gamma = 0$, $\delta = 1$, we do not reduce the number of equations, as happens in the case of (3.152). For these reasons, we are not able to establish that the solution (4.24-4.25) generically possesses two or more degrees of parameteric freedom. However, we are able to establish the existence of such solutions in particular special cases of (4.23) explored in the subsequent examples section.

Case II: $x_0^{22} = x_0^{33} = x_0^{44} = x_0^{41} = x_0^{42} = x_0^{43} = 0$

The rank of the Jacobian vanishes when all but the x_0^{ij} appearing in B_4 (and B_1) are zero. This leads to the matrix

$$x_0^{23} X_0^6 + x_0^{32} X_0^7 = \begin{bmatrix} 0 & x_0^{23} \frac{m_0 s_0^4 - 1}{m_0 s_0^2 c_0 d_0} - x_0^{32} \frac{1}{m_0 s_0^2} & 0 & 0 \\ x_0^{23} \frac{1}{m_0 s_0^2} + x_0^{32} \frac{c_0 d_0}{m_0 s_0^2} & 0 & x_0^{23} & 0 \\ 0 & x_0^{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.26)$$

and discrete equation

$$\begin{aligned} \left(x_0^{23} \frac{1}{m_0 s_0^2} + x_0^{32} \frac{c_0 d_0}{m_0 s_0^2}\right) u + \frac{1}{2} \left(x_0^{23} \frac{m_0 s_0^4 - 1}{m_0 s_0^2 c_0 d_0} - x_0^{32} \frac{1}{m_0 s_0^2}\right) (\underline{u} + \bar{u}) \\ + \frac{1}{2} x_0^{32} u^2 (\bar{u} + \underline{u}) + x_0^{23} u \bar{u} \underline{u} = 0. \end{aligned} \quad (4.27)$$

We redefine parameters:

$$\begin{aligned} b_1 &= \frac{1}{2} \left(x_0^{23} \frac{m_0 s_0^4 - 1}{m_0 s_0^2 c_0 d_0} - x_0^{32} \frac{1}{m_0 s_0^2}\right) (\underline{u} + \bar{u}), & b_2 &= -\frac{1}{2} \left(x_0^{23} \frac{m_0 s_0^4 - 1}{m_0 s_0^2 c_0 d_0} x_0^{32} \frac{1}{m_0 s_0^2}\right) \\ b_3 &= -\frac{1}{2} x_0^{32}, & b_4 &= x_0^{23}, \end{aligned} \quad (4.28)$$

to obtain

$$b_1 u - (b_2 + b_3 u^2) (\bar{u} + \underline{u}) + b_4 u \bar{u} \underline{u} = 0. \quad (4.29)$$

Rather than demonstrate that this transformation of parameters is surjective, we will work with (4.29), which is at least as general as (4.27), and show that it admits a multiparameter family of order-two elliptic solutions. The solution of (4.29) is

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m), \quad (4.30)$$

together with the constraints

$$b_1 m c d s^2 = \alpha^2 [2b_3 (1 - m s^4) - b_4 c d] \quad (4.31a)$$

$$2b_2 m s^2 = \alpha^2 (2b_3 - b_4 c d). \quad (4.31b)$$

Let us first assume that one of the coefficients of α^2 appearing in (4.31) is zero. If the coefficient of α^2 in (4.31a) is zero, we must have $b_1 = 0$. We choose $m \in \mathbb{C} \setminus \{0, 1\}$ so that $f(\Omega) = 2b_3 (1 - m s^4) - b_4 c d$ is a non-constant elliptic function of Ω when b_3 and b_4 are not simultaneously zero. In this case $f(\Omega) = 0$ can be solved for Ω . Inserting our chosen m into (4.31b) leads to possible values for α . In the case where instead $2b_3 - b_4 c d = 0$, a nearly identical argument applies and provides a solution when $b_2 = 0$.

Now assume that neither coefficient of α^2 in (4.31) is zero. Then we may solve each equation for α^2 and equate the results to obtain:

$$\frac{b_1 m c d s^2}{2b_3(1 - ms^4) - b_4 c d} - \frac{2b_2 m s^2}{2b_3 - b_4 c d} = 0. \quad (4.32)$$

If we choose $m \in \mathbb{C} \setminus \{0, 1\}$, the RHS of (4.32) is again an elliptic function of Ω for suitable values of b_1 and b_2 and so may be solved for Ω . This value, along with our chosen value for m , may be substituted into either condition in (4.31) so that α may be determined.

Two discrete equations with multiparameter families of elliptic solutions are obtained when (4.20) is satisfied (and other x_0^{ij} are zero).

Case III: $x_0^{23} = x_0^{32} = x_0^{33} = x_0^{44} = x_0^{41} = x_0^{42} = x_0^{43} = 0$

Here, x_0^{22} is arbitrary and all other parameters are zero. This leads to

$$X_0^5 = \begin{bmatrix} \frac{s_0^2 c_0 d_0}{m_0 s_0^4 - 1} & 0 & \frac{c_0 d_0}{m_0 s_0^4 - 1} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4.33)$$

and

$$\frac{s_0^2 c_0 d_0}{m_0 s_0^4 - 1} + \frac{1}{2} u(\bar{u} + \underline{u}) + \frac{c_0 d_0}{m_0 s_0^4 - 1} \bar{u} \underline{u} = 0. \quad (4.34)$$

We multiply this equation by two and make the redefinitions

$$b_1 = \frac{2s_0^2 c_0 d_0}{m_0 s_0^4 - 1}, \quad b_2 = \frac{2c_0 d_0}{m_0 s_0^4 - 1} \quad (4.35)$$

to obtain

$$b_1 + u(\bar{u} + \underline{u}) + b_2 \bar{u} \underline{u} = 0. \quad (4.36)$$

The solution of this equation is

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m), \quad (4.37)$$

subject to

$$b_1 = \frac{\alpha^2 s^2 cd}{ms^4 - 1} \quad (4.38a)$$

$$b_2 = \frac{cd}{ms^4 - 1}. \quad (4.38b)$$

Given b_2 , we fix $m \in \mathbb{C} \setminus \{0, 1\}$ so that the RHS of (4.38b) is an elliptic function of Ω . Thus it may be solved for Ω . The result may then be substituted into (4.38a) to determine α . Thus, (4.37-4.38) generically provides a two-parameter family of solutions to (4.36).

Case IV: $x_0^{23} = x_0^{32} = x_0^{22} = x_0^{44} = x_0^{41} = x_0^{42} = x_0^{43} = 0$

The second discrete equation arising from (4.20) has arbitrary x_0^{33} and other parameters zero:

$$X_0^8 = \begin{bmatrix} \frac{1}{m_0} & 0 & -\frac{1}{m_0 s_0^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m_0 s_0^2} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (4.39)$$

or

$$\frac{1}{m_0} + \frac{u^2}{ms_0^2} + \left(u^2 - \frac{1}{m_0 s_0^2} \right) \bar{u}u = 0. \quad (4.40)$$

Making the redefinitions

$$b_1 = \frac{1}{m_0}, \quad b_2 = \frac{1}{m_0 s_0^2}, \quad (4.41)$$

we arrive at

$$b_1 + b_2 u^2 + (u^2 - b_2) \bar{u}u = 0. \quad (4.42)$$

The solution of this equation is

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0 | m), \quad (4.43)$$

subject to

$$b_1 = -\frac{\alpha^4}{m} \quad (4.44a)$$

$$b_2 = \frac{\alpha^2}{ms^2}. \quad (4.44b)$$

Suppose b_1 and b_2 are nonzero, and that $m \in \mathbb{C} \setminus \{0, 1\}$ is given. Then α may be determined from (4.44a) and the result can be substituted into (4.44b). The RHS of (4.44b) is an elliptic function of Ω and may be solved for any value of b_2 . We conclude that (4.43-4.44) generically provides a multiparameter family of solutions to (4.42).

Case V: $x_0^{22} = -\frac{\Omega_0(c_0^2+d_0^2)}{s_0c_0d_0}$, $x_0^{44} = 1$, $x_0^{23} = x_0^{32} = x_0^{41} = x_0^{42} = x_0^{43} = 0$

Lastly, we find a genuine delay-differential equation when (4.18) is satisfied.

Here, we have

$$X_0^1 - \frac{\Omega_0(c_0^2+d_0^2)}{s_0c_0d_0} X_0^5 = \begin{bmatrix} 0 & 0 & \frac{\Omega_0}{s_0} & 0 \\ 0 & -\frac{\Omega_0(c_0^2+d_0^2)}{s_0c_0d_0} & 0 & 0 \\ \frac{\Omega_0}{s_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (4.45)$$

and

$$u'(\bar{u} - \underline{u}) = \frac{\Omega_0}{s_0} \left[-2u^2 + \frac{c_0^2+d_0^2}{c_0d_0} u(\bar{u} + \underline{u}) - 2\bar{u}\underline{u} \right]. \quad (4.46)$$

This equation is very similar to the bi-Riccati equation (3.181) and will be treated similarly. The equation (4.46) possesses an obvious scaling freedom. If we redefine parameters according to

$$b_1 = -2\frac{\Omega_0}{s_0}, \quad b_2 = \frac{\Omega_0}{s_0} \frac{c_0^2+d_0^2}{c_0d_0}, \quad (4.47)$$

we obtain the equation

$$u'(\bar{u} - \underline{u}) = b_1 (u^2 + \bar{u}\underline{u}) + b_2 u(\bar{u} + \underline{u}). \quad (4.48)$$

From (4.47) it follows that

$$b_2^2 (b_1^2 - 4\Omega_0^2) (b_1^2 - 4\Omega_0^2 m_0) = b_1^2 [b_1^2 - 2\Omega_0^2(1 + m_0)]^2. \quad (4.49)$$

Given b_1 and b_2 , (4.49) yields an algebraic relationship between Ω_0 and m_0 . Demanding that this relationship is consistent with (4.47) leads, in principle, to explicit values for Ω_0 and m_0 . The solution of (4.48) is then

$$u(z) = \alpha \operatorname{sn}(\Omega_0 z + z_0 | m_0) \quad (4.50)$$

where Ω_0 and m_0 are determined from (4.47) and (4.49) and α and z_0 are free. Alternatively, starting from (4.46) with known Ω_0 and m_0 we obtain the same solution.

4.2 Examples

Similarly to the previous chapter, we now consider the relationship between the equations we have identified and a number of known equations. In particular, we construct a number of new solutions to known models.

4.2.1 Korteweg-de Vries type equations

We again begin with the lattice potential mKdV equation in this setting:

$$\frac{w'_n}{w_n} = \lambda \frac{w_{n+1} - w_{n-1}}{w_{n-1} + w_{n+1}}. \quad (4.51)$$

From here, we take a traveling wave reduction to obtain

$$p \frac{u'}{u} = \lambda \frac{\bar{u} - \underline{u}}{\underline{u} + \bar{u}} \quad (4.52)$$

with associated matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\lambda}{p} \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (4.53)$$

and we observe that $X = X_0^2$ when

$$\frac{\lambda}{p} = -\frac{\Omega_0 c_0 d_0}{s_0}. \quad (4.54)$$

From (4.24-4.25), we find that

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta} \quad (4.55)$$

solves (4.52) when the parameters satisfy

$$(\alpha\delta + \beta\gamma)(p\Omega cd - \lambda s) = 0 \quad (4.56a)$$

$$p\Omega(\beta\delta - \alpha\gamma s^2) - \lambda\beta\delta s = 0 \quad (4.56b)$$

$$\lambda\alpha\gamma s - p\Omega(\alpha\gamma - \beta\delta ms^2) = 0. \quad (4.56c)$$

It is worth remarking that the equation (4.52) is easily obtained from the addition law (A.58a). Setting $x_1 = \Omega z + z_0$ and $x_2 = \Omega h$ we find that

$$\frac{\operatorname{sn}(\Omega z + \Omega h + z_0|m) - \operatorname{sn}(\Omega z - \Omega h + z_0|m)}{\operatorname{sn}(\Omega z - \Omega h + z_0|m) + \operatorname{sn}(\Omega z + \Omega h + z_0|m)} = \frac{\operatorname{scn}(\Omega z + z_0|m)\operatorname{dn}(\Omega z + z_0|m)}{\operatorname{cd}\operatorname{sn}(\Omega z + z_0|m)}, \quad (4.57)$$

it follows that

$$u(z) = \alpha \operatorname{sn}(\Omega z + z_0|m) \quad (4.58)$$

solves (4.52) when the parameters satisfy

$$p\Omega cd - \lambda s = 0. \quad (4.59)$$

In fact, by taking $\beta = \gamma = 0$ and $\delta = 1$ in (4.56) we obtain the same solution. Clearly, (4.58-4.59) is a multiparameter family of solutions to (4.52): α and z_0 are free while the remaining parameters are subject to (4.59). By arguments similar to before, there will generically be a third free parameter.

A Schwarzian KdV equation in our class of equations is

$$w'_n(w_{n+1} - w_{n-1}) = \lambda \left[w_n(w_{n-1} + w_{n+1}) - w_n^2 - w_{n-1}w_{n+1} \right] \quad (4.60)$$

with traveling wave reduction

$$pu'(\bar{u} - \underline{u}) = \lambda \left[u(\underline{u} + \bar{u}) - u^2 - \underline{u}\bar{u} \right]. \quad (4.61)$$

The matrix corresponding to this equation is

$$X = \begin{bmatrix} 0 & 0 & \frac{\lambda}{p} & 0 \\ 0 & -\frac{2\lambda}{p} & 0 & 0 \\ \frac{\lambda}{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (4.62)$$

and we see that X is equivalent to (4.45) when

$$\frac{\lambda}{p} = \frac{\Omega_0}{s_0} = \frac{\Omega_0 [c_0^2 d_0^2 - (c_0^2 + d_0^2)]}{s_0 (m_0 s_0^4 - 1)} = \frac{\Omega_0 (c_0^2 + d_0^2)}{2s_0 c_0 d_0}. \quad (4.63)$$

Thus, we obtain the solution (because (4.61) is Schwarzian)

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0 | m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0 | m) + \delta}, \quad (4.64)$$

subject to

$$\lambda s - p\Omega = 0 \quad (4.65a)$$

$$2\lambda c d s - p\Omega (c^2 + d^2) = 0 \quad (4.65b)$$

$$\lambda s (1 - 2cd - ms^4) + p\Omega c^2 d^2 = 0. \quad (4.65c)$$

As in the bi-Riccati case, these conditions lead to a degenerate solution. Substituting the first condition into the second and third, we obtain $c = d$ (twice) and consequently $m = 1$. Therefore, following the same logic as before, the

corresponding solution to (4.60) is

$$u(z) = \frac{\alpha \exp \Omega(nh + pt + z_0) + \beta}{\gamma \exp \Omega(nh + pt + z_0) + \delta} \quad (4.66)$$

subject to

$$\Omega \sinh^2 \Omega h = \frac{\lambda}{p}. \quad (4.67)$$

We lastly consider a semidiscrete KdV equation

$$\lambda w'_n - w'_n(w_{n+1} - w_{n-1}) = w_{n+1} - w_{n-1}, \quad (4.68)$$

which becomes

$$\lambda p u' - p u'(\bar{u} - \underline{u}) = \underline{u} - \bar{u} \quad (4.69)$$

under a traveling wave reduction. This equation does not admit order-two elliptic solutions. However, like its bi-Riccati counterpart (3.241), it does have a solution in terms of the Weierstrass ζ -function. From the addition law for this function (A.27), we compute

$$\begin{aligned} & \zeta(\Omega z + \Omega h + z_0; g_2, g_3) - \zeta(\Omega z - \Omega h + z_0; g_2, g_3) = \\ & 2\zeta(\Omega h + z_0; g_2, g_3) - \frac{\zeta''(\Omega h; g_2, g_3)}{\zeta'(\Omega z + z_0; g_2, g_3) - \zeta'(\Omega h; g_2, g_3)}. \end{aligned} \quad (4.70)$$

from which it follows (observing the translational freedom in the dependent variables in (2.99) and (4.69)) that

$$u(z) = \zeta(\Omega z + z_0; g_2, g_3) + \beta \quad (4.71)$$

solves (4.69) and

$$w_n(t) = \zeta(\Omega(nh + pt) + z_0; g_2, g_3) + \beta \quad (4.72)$$

solves (4.68), both subject to

$$\Omega\zeta(\Omega h; g_2, g_3) = -\frac{1}{p} \quad (4.73a)$$

$$\frac{2\zeta(\Omega h; g_2, g_3)}{\Omega\zeta'(\Omega h; g_2, g_3)} = -\lambda p \quad (4.73b)$$

$$2\zeta(\Omega h; g_2, g_3)\zeta'(\Omega h; g_2, g_3) - \zeta''(\Omega h; g_2, g_3) = 0. \quad (4.73c)$$

Generically, there are no free parameters beyond z_0 and β (and p) in the solutions (4.71-4.72). We remark that there is a Miura transformation from (4.68) to the equation

$$v'_n = \lambda v_n^2 (v_{n+1} - v_{n-1}); \quad (4.74)$$

if w_n solves (4.68), then

$$v_n = -\frac{1}{\lambda + w_{n-1} - w_{n+1}} \quad (4.75)$$

solves (4.74) [45]. Solutions to (4.74) are constructed directly in the subsequent section. However, from (4.72), we obtain an order-two elliptic solution:

$$v_n(t) = -\frac{1}{\lambda + \zeta(\Omega[(n+1)h + pt] + z_0; g_2, g_3) - \zeta(\Omega[(n-1)h + pt] + z_0; g_2, g_3)}, \quad (4.76)$$

or, in standard form (3.4) (with $g'_2 := \Omega^4 g_2$ and $g'_3 := \Omega^6 g_3$):

$$v_n(t) = \frac{\Omega}{\lambda\Omega + 2\zeta(h; g'_2, g'_3)} \times \left[\frac{\wp'(h; g'_2, g'_3)}{[\lambda\Omega + 2\zeta(h; g'_2, g'_3)][\wp(nh + pt + z_0; g'_2, g'_3) - \wp(h; g'_2, g'_3)] - \wp'(\Omega h; g'_2, g'_3)} - 1 \right], \quad (4.77)$$

either subject to (4.73). This solution is, however, less general than one (4.100-4.101 with $p = \lambda^{-1}$) constructed subsequently.

4.2.2 Wadati lattice solutions

Wadati introduced the equation

$$w'_n = (a_0 + a_1 w_n + a_2 w_n^2)(w_{n+1} - w_{n-1}) \quad (4.78)$$

in his study of Miura-type transformations between differential-difference systems [80]. Sometimes (4.78) is referred to as the hybrid lattice equation; it contains the Volterra lattice ($a_0 = a_2 = 0$) [47, 53], the self-dual nonlinear network equation ($a_1 = 0$) [41], and the semidiscrete KdV equation ($a_0 = a_1 = 0$) [42]. Some examples of Jacobi elliptic solutions to the Wadati lattice (4.78) were found in [15, 74]. Here we will show that if a_1 and a_2 are not both zero (i.e. the equation is nonlinear), then (4.78) admits an order-two elliptic solution (the form of which depends on the parameters) with at least two degrees of parametric freedom. We proceed by cases, making use of the discriminant of the quadratic appearing in (4.78):

$$\Delta = a_1^2 - 4a_0a_2. \quad (4.79)$$

We begin with the fully nondegenerate case.

Case I: $a_2 \neq 0$ and $\Delta \neq 0$

Under the first assumption, (4.78) can be written as

$$w'_n = \left[a_2 \left(w_n + \frac{a_1}{2a_2} \right)^2 - \frac{\Delta}{4a_2} \right] (w_{n+1} - w_{n-1}). \quad (4.80)$$

Adding the second assumption, we see that the change of variables

$$w_n(t) = \frac{1}{2a_2} [-b_1 \pm v_n(T)], \quad T = -\frac{\Delta}{4a_2} t \quad (4.81)$$

is well-defined and renders (4.80) equivalent to a self-dual nonlinear network equation:

$$v'_n = (1 - \Delta^{-1} v_n^2)(v_{n+1} - v_{n-1}). \quad (4.82)$$

We take a traveling wave reduction

$$v_n(T) = u(z), \quad z = nh + pT + z_0, \quad (4.83)$$

to arrive at a differential-delay equation

$$pw' = (1 - \Delta^{-1}w^2)(\bar{w} - w). \quad (4.84)$$

The matrix representation of this equation is

$$X = \begin{bmatrix} 0 & 0 & 0 & -\frac{2}{p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{p\Delta} \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad (4.85)$$

and $X = X_0^5$ if the conditions

$$p = \frac{2s_0}{\Omega_0}, \quad \Delta = \frac{1}{m_0s_0^2} \quad (4.86)$$

are satisfied. Substituting these conditions and $a_1 = 0$, $a_2 = a_3 = 0$ into (4.25), we see that the solution of (4.84) is

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta} \quad (4.87)$$

subject to

$$2s(\delta^2\Delta - \beta^2) + p\Delta\Omega(\gamma^2s^2 - \delta^2) = 0 \quad (4.88a)$$

$$2s(\alpha\beta - \gamma\delta\Delta) + p\Delta\gamma cd = 0 \quad (4.88b)$$

$$2s(\gamma^2\Delta - \alpha^2) + p\Delta\Omega(\gamma^2 - \delta^2ms^2) = 0. \quad (4.88c)$$

These conditions reduce considerably when $\beta = \gamma = 0$:

$$\frac{\Omega}{s} = \frac{2}{p}, \quad (4.89a)$$

$$\Omega ms = \frac{2\alpha^2}{p\Delta}, \quad (4.89b)$$

leading to the solution

$$u(z) = \pm\sqrt{\Delta ms} \operatorname{sn}(\Omega z|m), \quad (4.90)$$

subject to (4.89a). It is clear that this solution generically possesses two degrees of parametric freedom: upon fixing $m \in \mathbb{C} \setminus \{0, 1\}$, the LHS of (4.89a) is a non-constant meromorphic function of Ω and thus (4.89a) may be solved for Ω for all but at most two values of p . The solution (4.89a-4.90) can also be obtained simply using only the addition law for the Jacobi sine function; setting $x = \Omega z$ and $y = \Omega h$ in (A.58a) we compute

$$\operatorname{sn}(\Omega z + \Omega h|m) - \operatorname{sn}(\Omega z - \Omega h|m) = \frac{2s \operatorname{cn}(\Omega z|m) \operatorname{dn}(\Omega z|m)}{1 - ms^2 \operatorname{sn}^2(\Omega z|m)}, \quad (4.91)$$

at which point it is clear that (4.90) solves (4.84) when (4.89a) holds.

We now return to (4.80), for which the solution corresponding to (4.87) is

$$w_n(t) = \frac{1}{2a_2} \left[-b_1 + \pm \frac{\alpha \operatorname{sn} \left(\Omega \left(nh - \frac{c\Delta}{4p^2} t \right) + z_0 \middle| m \right) + \beta}{\gamma \operatorname{sn} \left(\Omega \left(nh - \frac{c\Delta}{4p^2} t \right) + z_0 \middle| m \right) + 1} \right], \quad (4.92)$$

subject to (4.88). We remark that this solution, which generically has at least two degrees of parametric freedom, is more general than that given in [15, 74]. In fact, even when $\beta = \gamma = 0$, the solution is more general; we have

$$w_n(t) = -\frac{a_1}{2a_2} \pm \sqrt{\Delta ms} \operatorname{sn} \left(h\Omega n - \frac{s\Delta}{2a_2} t + z_0 \middle| m \right), \quad (4.93)$$

where z_0 is free and Ω and m are constrained by (4.89a). Starting from either (4.92) or (4.93), analogues (again with additional parametric freedom) of the

other 11 Jacobian elliptic solutions given in [74] can be obtained simply using argument-modulus transformations between the twelve Jacobi functions [84].

Case II: $a_2 \neq 0$ and $\Delta = 0$

This first degenerate case takes the form

$$w'_n = a_2 \left(w_n + \frac{a_1}{2a_2} \right)^2 (w_{n+1} - w_{n-1}). \quad (4.94)$$

Under the change of dependent variable

$$w_n = \pm \frac{1}{\sqrt{a_2}} \left(v_n - \frac{a_1}{2} \right), \quad (4.95)$$

we recover Hirota's discrete KdV equation:

$$v'_n = v_n^2 (v_{n+1} - v_{n-1}), \quad (4.96)$$

from which a traveling wave reduction

$$v_n(t) = u(z), \quad z = nh + pt + z_0, \quad (4.97)$$

leads to the delay-differential equation

$$pu' = u^2(\bar{u} - \underline{u}). \quad (4.98)$$

The matrix corresponding to this equation is

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{p} \\ 1 & 0 & 0 & 0 \end{bmatrix}; \quad (4.99)$$

we find the solution of (4.98) is

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta}, \quad (4.100)$$

subject to

$$2\beta^2 s - p\Omega(\gamma^2 s^2 - \delta^2) = 0 \quad (4.101a)$$

$$2\alpha\beta s - \gamma\delta p\Omega c d = 0 \quad (4.101b)$$

$$2\alpha^2 s - p\Omega(\gamma^2 - \delta^2 m s^2) = 0. \quad (4.101c)$$

A subfamily of the order-two elliptic solutions of (4.98) can be obtained simply from the addition law for the Weierstrass \wp -function (A.26): we take $x = z$ and $y = h$ to obtain

$$\wp(z+h; g_2, g_3) - \wp(z-h; g_2, g_3) = -\frac{\wp'(z; g_2, g_3)\wp'(h; g_2, g_3)}{[\wp(z; g_2, g_3) - \wp(h; g_2, g_3)]^2}, \quad (4.102)$$

which shows that

$$u(z) = \pm \sqrt{\frac{-p}{\wp'(h; g_2, g_3)}} [\wp(z+z_0; g_2, g_3) - \wp(h; g_2, g_3)] \quad (4.103)$$

satisfies (5.26) without constraints on the parameters. This family of solutions is a proper subclass of (4.100-4.101); in particular, (4.103) possesses only double poles while (4.100) may have either only simple poles or only double poles, depending on the Möbius parameters. It follows that (4.100-4.101) is generically a multiparameter family of solutions to (4.98). Applying these results, we find that

$$v_n(t) = \frac{\alpha \operatorname{sn}(\Omega(nh+ct) + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega(nh+qt)z + z_0|m) + \delta} \quad (4.104)$$

satisfies (4.96) and

$$w_n(t) = \pm \frac{1}{2\sqrt{a_2}} \frac{(2\alpha - a_1) \operatorname{sn}(\Omega(nh+pt) + z_0|m) + 2\beta - a_1}{\gamma \operatorname{sn}(\Omega(nh+pt)z + z_0|m) + \delta} \quad (4.105)$$

satisfies (4.94). In both cases the parameters are subject to (4.101).

Case III: $a_2 = 0$ and $a_1 \neq 0$

In this second degenerate case the assumptions reduce (4.78) to

$$w'_n = (a_0 + a_1 w_n)(w_{n+1} - w_{n-1}). \quad (4.106)$$

and by changing the dependent variable

$$w_n = \frac{1}{a_1}(v_n - a_0), \quad (4.107)$$

we arrive at the Volterra lattice

$$v'_n = v_n(v_{n+1} - v_{n-1}). \quad (4.108)$$

Taking the same reduction (4.97) as before, we have a differential-delay form of the Volterra lattice:

$$pu' = u(\bar{u} - \underline{u}), \quad (4.109)$$

corresponding to the matrix

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{p} \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (4.110)$$

The solution of (4.108) is

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta}, \quad (4.111)$$

subject to

$$2\alpha\gamma s + p\Omega(\delta^2 m s^2 - \gamma^2) = 0 \quad (4.112a)$$

$$2\beta\delta s + p\Omega(\gamma^2 s^2 - \delta^2) = 0 \quad (4.112b)$$

$$(\alpha\delta + \beta\gamma)s - p\gamma\delta\Omega cd = 0. \quad (4.112c)$$

It is clear that (4.111-4.112) represents, generically, a three parameter family of solutions. Choosing $p \in \mathbb{C}$, $\Omega \in \mathbb{C} \setminus \{0\}$, $m \in \mathbb{C} \setminus \{0, 1\}$, and $\delta \in \{0, 1\}$, (4.112a) and can be sequentially solved for α and β , respectively. The results can be substituted into to give a cubic equation for γ . Back-substitution into 4.112a-4.112b) yields values for α and β .

The corresponding solutions to (4.108) and (4.106) are, respectively,

$$v_n(t) = \frac{\alpha \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta} \quad (4.113)$$

and

$$w_n(t) = \frac{1}{a_1} \frac{(\alpha - \gamma a_0) \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta - a_0}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta}, \quad (4.114)$$

both subject to (4.112).

We can construct higher-order elliptic solutions to (4.108), and consequently (4.106), by means of Miura transformations. In [13], Chandre compares two mKdV-type equations

$$\phi'_n = (1 + \phi_n^2)(\phi_{n+1} - \phi_{n-1}) \quad (4.115a)$$

$$\phi'_n = (1 + \phi_n)^2(\phi_{n+1} - \phi_{n-1}) \quad (4.115b)$$

and provides continuum limits, Lax pairs, and Miura transformations to (4.108) for each. Using results from above, we can easily write down order-two elliptic solutions to both equations in (4.115). In particular, (4.115a) is equivalent to (4.78) with $a_1 = 0$, $a_0 = a_2 = 1$. Because we have a nonzero discriminant $\Delta = -4$, the solution is given by (4.92):

$$\phi_n(t) = \pm \frac{1}{2} \frac{\alpha \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta} \quad (4.116)$$

if the parameters satisfy (4.88) with $\Delta = -4$. On the other hand, (4.115b) has

a vanishing discriminant and so is equivalent to (4.94) with $a_1 = 2$ and $a_2 = 1$. Using (4.105) we obtain

$$\phi_n(t) = \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta}, \quad (4.117)$$

subject to (4.101).

The relevant Miura transformations are

$$w_n = (1 + i\phi_n)(1 - i\phi_{n+1}) \quad (4.118a)$$

$$w_n = (1 + \phi_n)(1 + \phi_{n+1}), \quad (4.118b)$$

which map solutions of (4.115a) and (4.115b), respectively, into solutions of (4.108). Applying the first Miura transformation to (4.116), we obtain

$$w_n(t) = \left[1 \pm \frac{i}{2} \frac{\alpha \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega(nh + qt) + z_0|m) + \delta} \right] \times \left[1 \mp \frac{i}{2} \frac{\alpha \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \delta} \right], \quad (4.119)$$

subject to (4.88), as a solution to (4.108). Similarly, applying (4.118b) to (4.117) leads to

$$w_n(t) = \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta} \right] \times \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \delta} \right], \quad (4.120)$$

subject to (4.101). (4.119) and (4.120) are order-four elliptic solutions; each of the factors in each of the solutions is an order-two elliptic function, and both factors within a single solution have identical lattices. Regardless of whether or not $h \in \Lambda$, it follows that the solutions have four poles (counting multiplicity) within each period parallelogram. The corresponding order-four solutions to

(4.106) are

$$w_n(t) = -\frac{a_0}{a_1} + \frac{1}{a_1} \left[1 \pm \frac{i \alpha \operatorname{sn}(\Omega(nh + qt) + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega(nh + qt) + z_0|m) + \delta} \right] \\ \times \left[1 \mp \frac{i \alpha \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \delta} \right], \quad (4.121)$$

and

$$w_n(t) = -\frac{a_0}{a_1} + \frac{1}{a_1} \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(nh + pt) + z_0|m) + \delta} \right] \\ \times \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega[(n+1)h + pt] + z_0|m) + \delta} \right], \quad (4.122)$$

subject to (4.88) and (4.101), respectively. To summarize, we have shown by explicit construction of solutions that any nonlinear Wadati lattice can be solved in terms of order-two elliptic functions. We have also shown that the case $a_2 = 0$ admits higher-order elliptic solutions.

4.2.3 Toda lattice solutions

In this section, we will construct some new explicit solutions to the Toda lattice (2.107). To do this, it will be convenient for us to use modified Flaschka variables

$$u_n = \exp(w_{n-1} - w_n) \quad (4.123a)$$

$$v_n = -Q'_{n-1} \quad (4.123b)$$

so that the resulting first-order system that is separately linear in each dependent variable:

$$u'_n = u_n (v_{n+1} - v_n) \quad (4.124a)$$

$$v'_n = u_n - u_{n-1}. \quad (4.124b)$$

This form has the advantage of possessing a simple (absent of branching) Miura transformation from the Volterra lattice (4.108). We will construct solutions

to the Toda lattice in this form by means of the Miura map given in [25]. Given a solution $\phi_n = \phi_n(t)$ of the Volterra lattice, (4.108),

$$u_n = \phi_{2n}\phi_{2n+1} \quad (4.125a)$$

$$v_n = \phi_{2n-1} + \phi_{2n} \quad (4.125b)$$

provides a solution to (4.124). Thus, using (4.119) and (4.120), we obtain, respectively,

$$\begin{aligned} u_n(t) &= \left[1 \pm \frac{i \alpha \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \\ &\quad \times \left[1 + \frac{1}{4} \left(\frac{\alpha \operatorname{sn}(\Omega[(2n+1)h + pt] + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega[(2n+1)h + pt] + z_0|m) + 1} \right)^2 \right] \\ &\quad \times \left[1 \mp \frac{i \alpha \operatorname{sn}(\Omega[(2n+2)h + pt] + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega[(2n+2)h + pt] + z_0|m) + \delta} \right] \end{aligned} \quad (4.126a)$$

$$\begin{aligned} v_n(t) &= \left[1 \pm \frac{i \alpha \operatorname{sn}(\Omega[(2n-1)h + pt] + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega[(2n-1)h + pt] + z_0|m) + \delta} \right] \\ &\quad \times \left[1 \pm \frac{i \alpha \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \\ &\quad + \left[1 \pm \frac{i \alpha \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \\ &\quad \times \left[1 \pm \frac{i \alpha \operatorname{sn}(\Omega[(2n+1)h + pt] + z_0|m) + \beta}{2 \gamma \operatorname{sn}(\Omega[(2n+1)h + pt] + z_0|m) + \delta} \right] \end{aligned} \quad (4.126b)$$

and

$$\begin{aligned} u_n(t) &= \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \\ &\quad \times \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega[(2n+1)h + pt] + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega[(2n+1)h + pt] + z_0|m) + \delta} \right]^2 \\ &\quad \times \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega[(2n+2)h + pt] + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega[(2n+2)h + pt] + z_0|m) + \delta} \right] \end{aligned} \quad (4.127a)$$

$$\begin{aligned} v_n(t) &= \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega[(2n-1)h + pt] + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \\ &\quad \times \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \end{aligned}$$

$$\begin{aligned}
& + \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega(2nh + pt) + z_0|m) + \delta} \right] \\
& \times \left[1 \pm \frac{(\alpha - 1) \operatorname{sn}(\Omega[(2n + 1)h + pt] + z_0|m) + \beta - 1}{\gamma \operatorname{sn}(\Omega[(2n + 1)h + pt] + z_0|m) + \delta} \right]. \quad (4.127b)
\end{aligned}$$

4.2.4 Quispel-Roberts-Thompson maps

In this section we relate discrete equations we have identified to special cases of the (symmetric) Quispel-Roberts-Thompson maps. We recall that three discrete equations were identified in the previous section:

$$b_1 u - (b_2 + b_3 u^2)(\bar{u} + \underline{u}) + b_4 u \bar{u} \underline{u} = 0 \quad (4.128)$$

$$b_1 + u(\bar{u} + \underline{u}) + b_2 \bar{u} \underline{u} = 0 \quad (4.129)$$

$$b_1 + b_2 u^2 + (u^2 - b_2) \bar{u} \underline{u} = 0. \quad (4.130)$$

None of these equations fit directly into the symmetric QRT class of maps (2.38), but they can be embedded if we multiply the equations by particular polynomials in u . The first equation (4.128) has previously been obtained in [40] as a discretization of the differential equation for an anharmonic oscillator. A biquadratic conserved quantity was computed:

$$I(u, \bar{u}) = \frac{(b_1 b_3 - b_2 b_4) u^2 \bar{u}^2 + b_1 b_2 (u^2 + \bar{u}^2) - b_1^2 u \bar{u}}{b_4^2 u^2 \bar{u}^2 - b_1 b_4 (u^2 + \bar{u}^2) + b_1^2}. \quad (4.131)$$

From this invariant, one constructs a symmetric QRT mapping equivalent to (4.128):

$$A_1 = \begin{bmatrix} \frac{b_3 \delta_0 - b_4 \gamma_0}{b_1} & 0 & \gamma_0 \\ 0 & \delta_0 & 0 \\ \gamma_0 & 0 & -\frac{b_2 \delta_0 - b_1 \gamma_0}{b_1} \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{b_3 \delta_1 - b_4 \gamma_1}{b_1} & 0 & \gamma_1 \\ 0 & \delta_1 & 0 \\ \gamma_1 & 0 & -\frac{b_2 \delta_1 - b_1 \gamma_1}{b_1} \end{bmatrix}, \quad (4.132)$$

where γ_0 , γ_1 , δ_0 , and δ_1 are arbitrary complex parameters, together with (2.34).

We have found the QRT matrices for the remaining equations:

$$A_1 = \begin{bmatrix} \frac{\delta_0 - \gamma_0 b_2}{b_1} & 0 & \gamma_0 \\ 0 & \delta_0 & 0 \\ \gamma_0 & 0 & -\frac{\gamma_0 b_1}{b_2} \end{bmatrix}, \quad A_1 = \begin{bmatrix} \frac{\delta_1 - \gamma_1 b_2}{b_1} & 0 & \gamma_1 \\ 0 & \delta_1 & 0 \\ \gamma_1 & 0 & -\frac{\gamma_1 b_1}{b_2} \end{bmatrix} \quad (4.133)$$

for (4.129) and

$$A_0 = \begin{bmatrix} \alpha_0 & 0 & -\alpha_0 b_2 \\ 0 & \delta_0 & 0 \\ -\alpha_0 b_2 & 0 & -\alpha_0 b_1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} \alpha_1 & 0 & -\alpha_1 b_2 \\ 0 & \delta_1 & 0 \\ -\alpha_1 b_2 & 0 & -\alpha_1 b_1 \end{bmatrix} \quad (4.134)$$

for (4.130). Again these matrices contain arbitrary complex parameters.

Chapter 5

Equations from Bäcklund transformations

In this short chapter we will, starting from auto-Bäcklund transformations for the classical Painlevé equations, identify a number of delay-differential equations with continuum limits to the first Painlevé equation. Our approach here is very similar to that of Fokas, Grammaticos, and Ramani [28], who obtained discrete Painlevé equations from the same starting point. An example of their method is described in the second chapter of this thesis.

There is no general algorithm for deriving a delay-differential equation from a Painlevé auto-Bäcklund transformation. The approach we use here is, apparently, only applicable to two different auto-Bäcklund transformations for (special cases of) P_{III} (2.17c). Interestingly, we obtain delay-differential equations in both of the classes we have studied in this thesis: bi-Riccati equations (3.1) and three-point equations (4.1).

5.1 Bi-Riccati equations

We recall, from the second chapter of this thesis, that the equation¹

$$w'' = \frac{(w')^2}{w} - \frac{w'}{x} + \frac{b-w^2}{x} - \frac{1}{w} \quad (5.1)$$

¹We have relabeled the variables in (2.22) and (2.23) ($u \rightarrow w$ and $z \rightarrow x$) so that the delay-differential equations we obtain have dependent variable u and independent variable z , which is consistent with the notation throughout the thesis.

admits the auto-Bäcklund transformations

$$w(x; b+2) = \frac{x[1+w'(x; b)]}{w(x; b)^2} - \frac{b+1}{w(x; b)} \quad (5.2a)$$

$$w(x; b-2) = \frac{x[1-w'(x; b)]}{w(x; b)^2} - \frac{b-1}{w(x; b)}. \quad (5.2b)$$

If we make the replacement $b \rightarrow b+2$ in (5.2b) we obtain

$$w(x; b) = \frac{x[1-w'(x; b+2)]}{w(x; b+2)^2} - \frac{b+1}{w(x; b+2)}. \quad (5.3)$$

Then from (5.2a) and (5.3) we obtain

$$w(x; b)w(x; b+2) = \frac{x[1+w'(x; b)]}{w(x; b)} - \beta - 1 \quad (5.4a)$$

$$w(x; b)w(x; b+2) = \frac{x[1-w'(x; b+2)]}{w(x; b+2)} - \beta - 1. \quad (5.4b)$$

Equating (5.4a) and (5.4b) leads to

$$w'(x; b)w(x; b+2) + w(x; b)w'(x; b+2) + w(x; b+2) - w(x; b) = 0. \quad (5.5)$$

Viewing this as a differential-difference equation we take a reduction

$$w(x; b) = \frac{q}{2}xu(z), \quad z = \frac{bh}{2} + \frac{2p}{q} \quad (5.6)$$

and obtain a differential-delay equation:

$$p(u'\bar{u} + u\bar{u}') + qu\bar{u} + \bar{u} - u = 0. \quad (5.7)$$

This equation has a continuum limit to a differentiated first Painlevé equation: under the transformations

$$u(z) = -\frac{h}{2p} + \frac{h^3}{2p}y(z) + O(h^4), \quad \frac{q}{p} = \frac{1}{6}h^4 + O(h^5) \quad (5.8)$$

we obtain

$$y''' = 12yy' + 1 \quad (5.9)$$

in the limit $h \rightarrow 0$. Integrating, we obtain a translated P_I (2.17a) equation

$$y'' = 6y^2 + z + z_0. \quad (5.10)$$

Because (5.7) has a continuum limit to P_I , it cannot generally be solved in terms of elementary transcendents. However, when $q = 0$, the equation has a family of order-two elliptic solutions. Together with (3.1), the equation (5.7) with $q = 0$ can be expressed as

$$X = \begin{bmatrix} 0 & \frac{1}{p} & 0 & 0 \\ -\frac{1}{p} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (5.11)$$

which is Möbius equivalent to $X_0^2 + X_0^6$ in (3.72). The solution of (5.7) with $q = 0$ is

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta}, \quad (5.12)$$

subject to

$$\gamma \delta s + p \Omega (\alpha \delta + \beta \gamma) = 0 \quad (5.13a)$$

$$p \alpha \gamma \Omega s^2 + p \beta \delta \Omega (1 + \operatorname{cd}) + \delta^2 s = 0 \quad (5.13b)$$

$$\gamma^2 s + p \alpha \gamma \Omega (1 + \operatorname{cd}) + p \beta \delta \Omega m s^2 = 0. \quad (5.13c)$$

We now recall the second auto-Bäcklund transformation given in the second chapter. This transformation applies to

$$w'' = \frac{1}{w} (w')^2 - \frac{1}{x} w' + \frac{aw^2 + b}{x} + w^3 - \frac{1}{w}; \quad (5.14)$$

it is given by

$$w(x; -a, -b) = -w(x; a, b) \quad (5.15a)$$

$$w(x; -b, -a) = \frac{1}{w(x; a, b)} \quad (5.15b)$$

$$w(x; -b-2, -a-2) = w(x; a, b) + \frac{(2+a+b)w(x; a, b)^2}{x[w'(x; a, b) + w(x; a, b)^2 + 1] - (1+b)w(x; a, b)}. \quad (5.15c)$$

Using (5.15a) and (5.15b) to compute $w(x; a+2, b+2)$ in (5.15c) leads to

$$\begin{aligned} \frac{w'(x; a+2, b+2)}{w(x; a+2, b+2)} + \frac{w'(x; a, b)}{w(x; a, b)} = \\ w(x; a+2, b+2) - w(x; a, b) + \frac{1}{w(x; a+2, b+2)} - \frac{1}{w(x; a, b)}. \end{aligned} \quad (5.16)$$

Again, we view this as a differential-difference equation (now with two discrete variables a and b) and take the reduction

$$w(x; a, b) = u(z), \quad z = \frac{(a+b)h}{4} - px. \quad (5.17)$$

The result is the delay-differential equation

$$p(u'\bar{u} + u\bar{u}') + (u\bar{u} - 1)(\bar{u} - u) = 0. \quad (5.18)$$

Its corresponding matrix,

$$X = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 1 & p \\ 0 & -1 & 0 & 0 \\ 0 & p & 0 & 0 \end{bmatrix}, \quad (5.19)$$

is also Möbius equivalent to $X_0^2 + X_0^6$ (and so is equivalent to (5.7) with $q = 0$).

The equation (5.1) is solved by

$$u(z) = \frac{\alpha \operatorname{sn}(\Omega z + z_0|m) + \beta}{\gamma \operatorname{sn}(\Omega z + z_0|m) + \delta}, \quad (5.20)$$

together with

$$(\beta^2 - \delta^2)s + p\alpha\gamma\Omega s^2 + p\beta\delta\Omega(1 + cd) = 0 \quad (5.21a)$$

$$(\alpha^2 - \gamma^2)s + p\beta\delta\Omega ms^2 + p\alpha\gamma\Omega(1 + cd) = 0 \quad (5.21b)$$

$$(\alpha\beta - \gamma\delta)s + p\Omega(\alpha\delta + \beta\gamma) = 0. \quad (5.21c)$$

5.2 Three-point equations

We are able to manipulate (5.2) to obtain a three-point delay-differential equation, in addition to the bi-Riccati equation obtained in the previous section.

We simply take the difference of (5.2a) and (5.2b) to obtain

$$w(x; b+2) - w(x; b-2) = \frac{2xw'(x; b)}{w(x; b)^2} - \frac{2}{w(x; b)}. \quad (5.22)$$

Again, we view this as a differential-difference equation and take a reduction

$$w(x; b) = \pm \sqrt{\frac{2}{q}} u(z), \quad z = \frac{bh}{2} + \frac{p}{q} \log x \quad (5.23)$$

to obtain a delay-differential equation:

$$pu' - qu = u^2(\bar{u} - \underline{u}). \quad (5.24)$$

This is a known equation, first obtained in [35], where a continuum limit to a differentiated P_I (2.17a) equation is given. Under the transformation

$$u(z) = h^{-1} - hy(z), \quad p = 2, \quad q = h^4 \quad (5.25)$$

we again obtain (5.9) and consequently (5.10). Because (5.24) has a continuum limit to the first Painlevé equation, it does not admit solutions in terms of elementary transcendents. However, in the case $q = 0$, we obtain

$$pu' = u^2(\bar{u} - \underline{u}). \quad (5.26)$$

We have studied this equation previously. It was obtained in the previous chapter as a special case of the Wadati lattice (4.78). It admits a multiparameter family of order-two elliptic solutions whose precise form is given by (4.100-4.101).

Chapter 6

General Conclusions

This thesis considers two classes of delay-differential equations: two-point bi-Riccati equations (3.1) and a class of three-point equations (4.1) that extends the McMillan family of discrete maps (2.50). Within each of these classes, we have identified analogues of symmetric QRT maps and classical Painlevé equations. This chapter serves to summarize and contextualize these results.

6.1 Bi-Riccati equations

We have identified five equations within the bi-Riccati class that admit multi-parameter families of order-two elliptic solutions. They are:

$$a_1(u' + \bar{u}') + a_2(u^2\bar{u}' + u'\bar{u}^2) = b_1(\bar{u} - u) + b_2(u\bar{u}^2 - u^2\bar{u}) \quad (6.1)$$

$$a_1(\bar{u}' - u') + a_2(u'\bar{u}^2 - u^2\bar{u}') = b_1(u + \bar{u}) + b_2(u^2\bar{u} + u\bar{u}^2) \quad (6.2)$$

$$u'\bar{u} - u\bar{u}' = b_1 + b_2u^2\bar{u}^2 \quad (6.3)$$

$$u'\bar{u} + u\bar{u}' = b(\bar{u}^2 - u^2) \quad (6.4)$$

$$a_1(u'\bar{u} + u\bar{u}') + a_2u'\bar{u}' = \frac{\Omega_0}{s_0}a_1(\bar{u}^2 - u^2) + \frac{\Omega_0^2c_0d_0}{s_0^2}a_2(u^2 + \bar{u}^2) \quad (6.5)$$
$$- \frac{\Omega_0^2(2 - s_0^2 - m_0s_0^2)}{s_0^2}a_2u\bar{u}.$$

The first four equations (6.1-6.4) are related to the sine-Gordon equation (2.90) through its Bäcklund transformation (2.91). This relationship is discussed extensively in the third chapter of this thesis, where new elliptic solutions

to semidiscrete sine-Gordon equations are constructed. The fifth equation we identify (6.5) differs from the first four in two principal ways. Firstly, the parameters appearing in the equations are themselves parameterized by elliptic functions (and it does not appear possible to reparametrize this equation in a simpler way, as discussed in the third chapter). Secondly, the solution of (6.5) does not involve constraints between Möbius parameters and internal elliptic parameters (Ω, m) , as in the case of the symmetric QRT map and the other equations we have identified. The two degrees of parametric freedom in the solution to (6.5) are obvious from the equation: a scaling freedom as the equation is homogenous in u, u', \bar{u}, \bar{u}' and a translational freedom because the equation is autonomous. For these reasons, we do not anticipate that (6.5) is an important equation relative to (6.1-6.4).

We have also identified a Painlevé type equation in the bi-Riccati class through a Bäcklund transformation for the third classical Painlevé equation:

$$a(u'\bar{u} + u\bar{u}') = b\bar{u} + u - \bar{u}. \quad (6.6)$$

When $b = 0$, the equation is Möbius equivalent to (6.4), and so possesses a multiparameter family of elliptic solutions.

6.2 Three-point equations

Within the three-point class, we also identified five equations with multiparameter order-two elliptic solutions. However, only two of these are delay-differential equations:

$$\frac{u'}{\bar{u} - \underline{u}} = \frac{\frac{\Omega_0}{s_0} b_1 + \Omega_0 s_0 b_2 + u \frac{\Omega_0 c_0 d_0}{s_0} b_2 - u^2 \left(\Omega_0 m_0 s_0 b_1 - \frac{\Omega_0}{s_0} b_2 \right)}{2b_1 + b_2(\bar{u} - \underline{u}) + 2b_3 \underline{u} \bar{u}}. \quad (6.7)$$

and

$$u'(\bar{u} - \underline{u}) = b_1(u^2 + \bar{u}\underline{u}) + b_2 u(\bar{u} + \underline{u}). \quad (6.8)$$

The first of these (6.7) generalizes the Wadati lattice, which always admits order-two elliptic solutions (as shown in chapter 4). Despite the difficulty in

parameterizing (6.7) without elliptic functions, we regard it as an important equation since it reduces to known equations in particular cases. The second equation we identified (6.8) is similar to (6.5) in that the free parameters in the solution are somewhat trivial: the scaling and translational freedoms follow from the equation while the internal elliptic parameters are fixed. For this reason, we regard (6.8) as less interesting than (6.7). We also identified three (4.128-4.130) discrete equations and showed that each can be embedded into the symmetric QRT map (2.38). This underscores the efficacy of our approach in identifying interesting equations, even though we have not achieved a full classification of equations with order-two elliptic solutions.

As in the bi-Riccati class, we have found a Painlevé type equation from a Bäcklund transformation for the classical third Painlevé equation:

$$au' = bu + u^2(\bar{u} - \underline{u}). \quad (6.9)$$

This equation is not new, but we have been able to solve it in a particular limit ($b = 0$) and relate it to our results. In particular, (6.9) with $b = 0$ is a special case of the Wadati lattice (4.78) and is Möbius-equivalent to a special case of (6.7), and so admits order-two elliptic solutions (4.100-4.101).

6.3 Future directions

This work reported in this thesis was motivated by a desire to understand the structure of delay-differential Painlevé equations from the perspective of discrete integrable systems. We have identified a number of symmetric QRT analogues and Painlevé type equations in two large classes of delay-differential equations. Our results are expected to be useful in constructing further Painlevé type delay-differential equations. This can be done by considering singularity structure (in the sense of [32]) generalizations of the QRT analogues (6.1-6.4,6.7-6.8) we have identified with continuum limits to classical Painlevé equations.

Appendix A

Elliptic functions

In this appendix we collect the essential results and formulae from the theory of elliptic functions. There are a number of number of ways to define the standard classes of elliptic functions; elliptic integrals, the defining differential equations, or elliptic curves are all suitable starting points. Here we take a constructive approach, defining the Weierstrass \wp function as a sum over a lattice in the complex plane. From the Weierstrass functions and their properties, the Jacobi elliptic functions can be constructed by means of the Halphen elliptic functions. Elliptic integrals are briefly considered at the conclusion of this appendix. We begin with the definition of an elliptic function.

Definition A.0.1 (Elliptic function). *A function $f : \mathbb{C} \rightarrow \mathbb{C} \cup \{\infty\}$ is called elliptic if f is meromorphic and there exist complex numbers ω_1 and ω_2 , where*

$$\operatorname{Im} \frac{\omega_1}{\omega_2} \neq 0 \tag{A.1}$$

and

$$f(z + m\omega_1 + n\omega_2) = f(z), \quad z \in \mathbb{C}, \quad (m, n) \in \mathbb{Z}^2. \tag{A.2}$$

Any elliptic function is naturally associated with a lattice

$$\Lambda = \{m\omega_1 + n\omega_2 : (m, n) \in \mathbb{Z}^2\}. \tag{A.3}$$

Given the generators of the lattice and a point $z \in \mathbb{C}$ we can construct the

period parallelogram

$$P(z) = \{z + t_1\omega_1 + t_2\omega_2 : 0 \leq t_1, t_2 < 1\}. \quad (\text{A.4})$$

Specification of the values of the elliptic function on the any period parallelogram defines the elliptic function on the entire complex plane through (A.2). We now state a number of important results for elliptic functions.

Theorem A.0.1. *If an elliptic function has no poles, it is a constant.*

Theorem A.0.2. *If f is an elliptic function, the sum of the residues of f or $1/f$ in any period parallelogram is zero.*

Theorem A.0.3. *If f is a nonconstant elliptic function and $c \in \mathbb{C} \cup \{\infty\}$, the number of solutions, counting multiplicity, to the equation $f(z) = c$ in any period parallelogram is a constant.*

The number of solutions, counting multiplicity, to $f(z) = c$ within a period parallelogram is called the order of f . From (A.0.2) it follows that the order of any nonconstant elliptic function is at least two. We will now look at an example of an order-two elliptic function.

A.1 Weierstrass functions

Given a lattice $\Lambda \subset \mathbb{C}$ with generators satisfying (A.1), the Weierstrass \wp -function is defined as

$$\wp(z|\Lambda) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{(z + \omega)^2} - \frac{1}{\omega^2} \right]. \quad (\text{A.5})$$

It can be shown that this sum converges uniformly on any compact set. The resulting function is even with a single double pole at $z = 0 \pmod{\Lambda}$ in each period parallelogram. The function is commonly denoted by

$$\wp(z; g_2, g_3) = \wp(z|\Lambda) \quad (\text{A.6})$$

where

$$g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^4}, \quad g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{\omega^6} \quad (\text{A.7})$$

are called lattice invariants. Up to translation in z , $\wp(z; g_2, g_3)$ is the unique solution of the differential equation

$$(u')^2 = 4u^3 - g_2u - g_3. \quad (\text{A.8})$$

From here it is possible to generate series expansions for \wp . About any $z_0 \in \Lambda$ we have the Laurent series

$$\wp(z; g_2, g_3) = \frac{1}{(z - z_0)^2} + \frac{g_2}{20}(z - z_0)^2 + \frac{g_3}{28}(z - z_0)^4 + O((z - z_0)^6), \quad (\text{A.9})$$

valid in a punctured disk of radius

$$r = \min\{|\omega - z_0| : \omega \in \Lambda \setminus \{z_0\}\}. \quad (\text{A.10})$$

About a regular point $z_0 \notin \Lambda$, we find

$$\begin{aligned} \wp(z; g_2, g_3) &= \wp(z_0; g_2, g_3) + \wp'(z_0; g_2, g_3)(z - z_0) \\ &\quad + \left[3\wp(z_0; g_2, g_3)^2 - \frac{g_2}{4} \right] (z - z_0)^2 + O((z - z_0)^3), \end{aligned} \quad (\text{A.11})$$

valid in a disk centered at z_0 with radius

$$r = \min\{|\omega - z_0| : \omega \in \Lambda\}. \quad (\text{A.12})$$

Useful information about the zeros of \wp can be extracted from (A.11). Suppose $\wp(z_0; g_2, g_3) = 0$. Then from (A.8) it follows that $\wp'(z_0; g_2, g_3) = \pm\sqrt{-g_3}$, i.e. the Weierstrass \wp -function has two simple zeros with the same coefficients up to a sign, unless $g_3 = 0$. In this case we have a double zero with coefficient $-g_2/4$.

The derivative of the Weierstrass \wp -function, appearing in (A.11), is and

odd elliptic function of order three. In this thesis we will not find much use for this function (in solutions to delay-differential equations), but it does appear in a number of formulas associated with the \wp -function. On the other hand, we will make extensive use of the Weierstrass ζ -function. This function is defined by

$$\zeta'(z; g_2, g_3) = -\wp(z; g_2, g_3) \quad (\text{A.13})$$

together with the condition

$$\lim_{z \rightarrow 0} \left[\zeta(z; g_2, g_3) - \frac{1}{z} \right] = 0 \quad (\text{A.14})$$

so that ζ is odd with respect to its first argument. Term by term integration of (A.5) yields

$$\zeta(z; g_2, g_3) = \frac{1}{z} + \sum_{\omega \in \Lambda \setminus \{0\}} \left[\frac{1}{z - \omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right]. \quad (\text{A.15})$$

From here, we see that the ζ -function possesses only simple poles of residue one, and so by (A.0.2) it cannot be an elliptic function. The function is only quasi-periodic, satisfying

$$\zeta(z + \omega; g_2, g_3) = \zeta(z; g_2, g_3) + 2\zeta(\omega; g_2, g_3), \quad \omega \in \Lambda. \quad (\text{A.16})$$

About $z = 0$ we have the Laurent series

$$\zeta(z; g_2, g_3) = \frac{1}{z} - \frac{g_2}{60} z^2 - \frac{g_3}{140} z^5 + O(z^7), \quad (\text{A.17})$$

convergent in an origin-centered punctured disk of radius (A.10). Expansions about other lattice points follow from (A.15). About a regular point $z_0 \notin \Lambda$ we have the expansion

$$\begin{aligned} \zeta(z; g_2, g_3) = & \zeta(z_0; g_2, g_3) - \wp(z_0; g_2, g_3)(z - z_0) - \frac{1}{2} \wp'(z_0; g_2, g_3)(z - z_0)^2 \\ & + O((z - z_0)^3), \end{aligned} \quad (\text{A.18})$$

convergent in a disk of radius (A.12) centered at z_0 .

An order-two elliptic function is easily constructed from the difference of ζ -functions:

$$\eta(z, w; g_2, g_3) := \zeta(z; g_2, g_3) - \zeta(z - w; g_2, g_3); \quad (\text{A.19})$$

for any $w \notin \Lambda$, the above function is order-two elliptic with two simple poles of opposite unit residue in each period parallelogram. In particular, if $z_0 \in \Lambda$ we have

$$\begin{aligned} \eta(z, w; g_2, g_3) &= \frac{1}{z - z_0} + \zeta(w; g_2, g_3) + \wp(w; g_2, g_3)(z - z_0) \\ &\quad - \frac{1}{2}\wp'(w; g_2, g_3)(z - z_0)^2 + O\left((z - z_0)^3\right) \end{aligned} \quad (\text{A.20})$$

in a punctured disk of radius

$$r = \min\{|\omega - z_0| : \omega \in \Lambda \cup (w + \Lambda) \setminus \{z_0\}\} \quad (\text{A.21})$$

centered at z_0 , and if instead z_0 is an element of the coset $w + \Lambda$, we have

$$\begin{aligned} \eta(z, w; g_2, g_3) &= -\frac{1}{z - z_0} - \zeta(w; g_2, g_3) - \wp(w; g_2, g_3)(z - z_0) \\ &\quad - \frac{1}{2}\wp'(w; g_2, g_3)(z - z_0)^2 + O\left((z - z_0)^3\right) \end{aligned} \quad (\text{A.22})$$

in a punctured disk of radius (A.21) centered at z_0 . Lastly, about a regular point $z_0 \notin \Lambda \cup (w + \Lambda)$,

$$\begin{aligned} \eta(z, w; g_2, g_3) &= \zeta(z_0; g_2, g_3) - \zeta(z_0 - w; g_2, g_3) \\ &\quad - [\wp(z_0; g_2, g_3) - \wp(z_0 - w; g_2, g_3)](z - z_0) \\ &\quad + O\left((z - z_0)^2\right), \end{aligned} \quad (\text{A.23})$$

which converges in a disk of radius

$$r = \min\{|\omega - z_0| : \omega \in \Lambda \cup (w + \Lambda)\} \quad (\text{A.24})$$

centered at z_0 . It turns out that, up to affine transformations of the dependent variable and translations of the independent variable, \wp and η are the only possibilities for order-two elliptic functions. This is an immediate corollary of the following theorem [6].

Theorem A.1.1. *Any elliptic function possessing poles at $\{a_1, \dots, a_N\}$ with corresponding orders $\{M_1, \dots, M_N\}$ in a period parallelogram can be represented as:*

$$f(z) = A_0^0 + \sum_{n=1}^N A_n^1 \zeta(z - a_n; g_2, g_3) + \sum_{n=1}^N \sum_{m=2}^{M_n} A_n^m \wp^{(m-2)}(z - a_n; g_2, g_3), \quad (\text{A.25})$$

for some choice of coefficients A_n^m .

Here we list a number of identities for the Weierstrass functions introduced in the previous section. Of particular importance are the so-called addition theorems for the Weierstrass functions, which relate values of a function and its upshift. Proofs of these properties rely on the differential equation (A.8) and are found in any standard reference on elliptic functions.

The addition laws for the Weierstrass \wp and ζ functions are, respectively,

$$\wp(x_1 \pm x_2; g_2, g_3) = \frac{1}{4} \left[\frac{\wp'(x_1; g_2, g_3) \mp \wp'(x_2; g_2, g_3)}{\wp(x_1; g_2, g_3) - \wp(x_2; g_2, g_3)} \right]^2 - \wp(x_1; g_2, g_3) - \wp(x_2; g_2, g_3) \quad (\text{A.26})$$

and

$$\zeta(z_1 + z_2; g_2, g_3) = \zeta(z_1; g_2, g_3) + \zeta(z_2; g_2, g_3) + \frac{1}{2} \frac{\zeta''(z_1; g_2, g_3) - \zeta''(z_2; g_2, g_3)}{\zeta'(z_1; g_2, g_3) - \zeta'(z_2; g_2, g_3)}, \quad (\text{A.27})$$

for any $x_1, x_2 \in \mathbb{C}$. There are also three point identities for these functions.

When $x_1, x_2, x_3 \in \mathbb{C}$ satisfy $x_1 + x_2 + x_3 = 0$, we have

$$\det \begin{bmatrix} 1 & 1 & 1 \\ \wp(x_1, g_2, g_3) & \wp(x_2, g_2, g_3) & \wp(x_3, g_2, g_3) \\ \wp'(x_1, g_2, g_3) & \wp'(x_2, g_2, g_3) & \wp'(x_3, g_2, g_3) \end{bmatrix} = 0 \quad (\text{A.28})$$

and

$$\begin{aligned} & [\zeta(x_1; g_2, g_3) + \zeta(x_2; g_2, g_3) + \zeta(x_3; g_2, g_3)]^2 \\ & + \zeta'(x_1; g_2, g_3) + \zeta'(x_2; g_2, g_3) + \zeta'(x_3; g_2, g_3) = 0. \end{aligned} \quad (\text{A.29})$$

It is sometimes convenient to write (A.27) and (A.29) in terms of the \wp -function according to $\zeta'(z; g_2, g_3) = -\wp(z; g_2, g_3)$ and $\zeta''(z; g_2, g_3) = -\wp'(z; g_2, g_3)$.

We find occasional use for the following scaling properties of the Weierstrass functions:

$$\wp(\lambda z; \lambda^{-4}g_2, \lambda^{-6}g_3) = \lambda^{-2}\wp(z; g_2, g_3) \quad (\text{A.30})$$

$$\zeta(\lambda z; \lambda^{-4}g_2, \lambda^{-6}g_3) = \lambda^{-1}\zeta(z; g_2, g_3). \quad (\text{A.31})$$

These properties follow directly from the definition (A.5) and (A.7).

A.2 Halphen functions

We begin with the differential equation for the Weierstrass \wp -function (A.8), which can be written as

$$(\wp')^2 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3) \quad (\text{A.32})$$

where

$$2(e_1^2 + e_2^2 + e_3^2) = g_2, \quad 4e_1e_2e_3 = g_3, \quad (\text{A.33})$$

and

$$e_1 + e_2 + e_3 = 0. \quad (\text{A.34})$$

We see then that the zeroes of the cubic polynomial appearing in (A.32) correspond to zeroes of \wp' , which we will now characterize. Exploiting the parity and periodicity of \wp' , we have

$$\wp'(z; g_2, g_3) = -\wp'(\omega - z; g_2, g_3), \quad \omega \in \Lambda, \quad (\text{A.35})$$

and we see that $\wp'(z; g_2, g_3) = 0$ when

$$z \in \left(\frac{\omega_1}{2} + \Lambda\right) \cup \left(\frac{\omega_2}{2} + \Lambda\right) \cup \left(\frac{\omega_1 + \omega_2}{2} + \Lambda\right). \quad (\text{A.36})$$

In particular,

$$\wp'\left(\frac{\omega_1}{2}; g_2, g_3\right) = \wp'\left(\frac{\omega_2}{2}; g_2, g_3\right) = \wp'\left(\frac{\omega_3}{2}; g_2, g_3\right) \quad (\text{A.37})$$

where

$$\omega_3 := -\omega_1 - \omega_2 \quad (\text{A.38})$$

and thus

$$e_i = \wp\left(\frac{\omega_i}{2}; g_2, g_3\right), \quad i = 1, 2, 3. \quad (\text{A.39})$$

We will now construct a new set of functions by appealing to the following theorem [23].

Theorem A.2.1. *Suppose f is a meromorphic function on a simple connected, open set $\Omega \subset \mathbb{C}$. If the order of each zero and pole of f in Ω is even, there exists a meromorphic function $g: \Omega \rightarrow \mathbb{C} \cup \{\infty\}$ satisfying $g(z)^2 = f(z)$ for each $z \in \Omega$.*

We can apply this theorem to the set of functions defined by

$$H_i(z; e_1, e_2)^2 = \wp(z; g_2, g_3) - e_i, \quad i = 1, 2, 3, \quad (\text{A.40})$$

where the H_i are known as Halphen functions. Clearly the RHS of (A.40) possesses only double poles. From (A.11) and (A.39) all zeroes are double as well. Therefore the theorem (A.2.1) applies; we choose the square root so that

$$\lim_{z \rightarrow 0} z H_i(z; e_1, e_2) = 1, \quad i = 1, 2, 3. \quad (\text{A.41})$$

From here and (A.40), it follows that the Halphen functions are odd functions

of their first argument. Then using the periodicity of \wp , it is easily seen that

$$H_i(z + \omega_i; e_1, e_2) = H_i(z, e_1, e_2) \quad (\text{A.42})$$

and

$$H_i(z + \omega_j; e_1, e_2) = -H_i(z, e_1, e_2), \quad i \neq j, \quad (\text{A.43})$$

i.e. the Halphen functions are elliptic with associated lattices

$$\Lambda_1 = \langle \{\omega_1, 2\omega_2\} \rangle, \quad \Lambda_2 = \langle \{2\omega_1, \omega_2\} \rangle, \quad \Lambda_3 = \langle \{2\omega_1, 2\omega_2\} \rangle. \quad (\text{A.44})$$

We can now establish a number of properties for the Halphen functions. In particular, these will be useful in establishing analogous properties for the Jacobi functions. Directly from the definition (A.40) we obtain differential relations between the Halphen functions:

$$H'_i(z; e_1, e_2) = -H_j(z; e_1, e_2)H_k(z; e_1, e_2), \quad i \neq j \neq k. \quad (\text{A.45})$$

and also the algebraic relations between the functions:

$$H_i(z; e_1, e_2)^2 + e_i = H_j(z; e_1, e_2)^2 + e_j. \quad (\text{A.46})$$

Combining (A.45) and (A.46) leads to the differential equations

$$H'_i(z; e_1, e_2)^2 = [H_i(z; e_1, e_2)^2 + e_j - e_i] [H_i(z; e_1, e_2)^2 + e_j - e_k], \quad i \neq j \neq k. \quad (\text{A.47})$$

A.3 Jacobi functions

The Jacobian functions are constructed in terms of the Halphen function with a change of independent variable:

$$\text{sn}(x|m) = \frac{\sqrt{e_1 - e_2}}{H_2(z; e_1, e_2)} \quad (\text{A.48})$$

$$\operatorname{cn}(x|m) = \frac{H_1(z; e_1, e_2)}{H_3(z; e_1, e_2)} \quad (\text{A.49})$$

$$\operatorname{dn}(x|m) = \frac{H_3(z; e_1, e_2)}{H_2(z; e_1, e_2)}, \quad (\text{A.50})$$

where

$$x = \sqrt{e_1 - e_2} z, \quad m = \frac{e_3 - e_2}{e_1 - e_2}. \quad (\text{A.51})$$

Often these functions are written with an auxiliary parameter k , called the elliptic modulus. We will instead use the auxiliary parameter $m = k^2$ because this simplifies some formulae for these functions.

The functions (A.48-A.50) are elliptic with associated lattices

$$\Lambda_{\operatorname{sn}} = \langle \{4K, 2iK'\} \rangle, \quad \Lambda_{\operatorname{cn}} = \langle \{4K, 2K + 2iK'\} \rangle, \quad \Lambda_{\operatorname{dn}} = \langle \{2K, 4iK'\} \rangle. \quad (\text{A.52})$$

where

$$2K = \sqrt{e_1 - e_2} \omega_1, \quad 2iK' = \sqrt{e_1 - e_2} \omega_2. \quad (\text{A.53})$$

The analogues of (A.45) are

$$\operatorname{sn}'(x|m) = \operatorname{cn}(x|m)\operatorname{cn}(x|m) \quad (\text{A.54a})$$

$$\operatorname{cn}'(x|m) = -\operatorname{sn}(x|m)\operatorname{dn}(x|m) \quad (\text{A.54b})$$

$$\operatorname{dn}'(x|m) = -m\operatorname{sn}(x|m)\operatorname{cn}(x|m) \quad (\text{A.54c})$$

and the analogues of (A.46) are

$$\operatorname{sn}^2(x|m) + \operatorname{cn}^2(x|m) = 1 \quad (\text{A.55a})$$

$$m\operatorname{sn}^2(x|m) + \operatorname{dn}^2(x|m) = 1 \quad (\text{A.55b})$$

$$m\operatorname{cn}^2(x|m) - \operatorname{dn}^2(x|m) = m - 1. \quad (\text{A.55c})$$

Then (A.55-A.54) imply the following differential equations:

$$\operatorname{sn}'(x|m)^2 = [1 - \operatorname{sn}(x|m)^2] [1 - m\operatorname{sn}(x|m)^2] \quad (\text{A.56a})$$

$$\operatorname{cn}'(x|m)^2 = [1 - \operatorname{cn}(x|m)^2] [1 - m + m \operatorname{cn}(x|m)^2] \quad (\text{A.56b})$$

$$\operatorname{dn}'(x|m)^2 = [\operatorname{dn}(x|m)^2 - 1] [1 - m - \operatorname{dn}(x|m)^2]. \quad (\text{A.56c})$$

The differentiated forms of these equations are occasionally useful:

$$\operatorname{sn}''(x|m) = -(1+m)\operatorname{sn}(x|m) + 2m\operatorname{sn}(x|m)^3 \quad (\text{A.57a})$$

$$\operatorname{cn}''(x|m) = -(1-2m)\operatorname{cn}(x|m) - 2m\operatorname{cn}(x|m)^3 \quad (\text{A.57b})$$

$$\operatorname{dn}''(x|m) = (2-m)\operatorname{dn}(x|m) - 2\operatorname{dn}(x|m)^3. \quad (\text{A.57c})$$

Like the Weierstrass and Halphen elliptic functions, the Jacobi elliptic functions obey addition laws:

$$\operatorname{sn}(x_1 \pm x_2|m) = \frac{\operatorname{sn}(x_1|m)\operatorname{cn}(x_2|m)\operatorname{dn}(x_2|m) \pm \operatorname{sn}(x_2|m)\operatorname{cn}(x_1|m)\operatorname{dn}(x_1|m)}{1 - m\operatorname{sn}^2(x_1|m)\operatorname{sn}^2(x_2|m)} \quad (\text{A.58a})$$

$$\operatorname{cn}(x_1 \pm x_2|m) = \frac{\operatorname{cn}(x_1|m)\operatorname{cn}(x_2|m) \pm \operatorname{sn}(x_1|m)\operatorname{dn}(x_1|m)\operatorname{cn}(x_2|m)\operatorname{dn}(x_2|m)}{1 - m\operatorname{sn}^2(x_1|m)\operatorname{sn}^2(x_2|m)} \quad (\text{A.58b})$$

$$\operatorname{dn}(x_1 \pm x_2|m) = \frac{\operatorname{dn}(x_1|m)\operatorname{dn}(x_2|m) \pm m\operatorname{sn}(x_1|m)\operatorname{cn}(x_1|m)\operatorname{sn}(x_2|m)\operatorname{cn}(x_2|m)}{1 - m\operatorname{sn}^2(x_1|m)\operatorname{sn}^2(x_2|m)}. \quad (\text{A.58c})$$

An important property of the Jacobian elliptic functions is that they degenerate to trigonometric functions in simple limiting cases. In particular, we have

$$\lim_{m \rightarrow 0} \operatorname{sn}(z|m) = \sin z, \quad \lim_{m \rightarrow 0} \operatorname{cn}(z|m) = \cos z, \quad \lim_{m \rightarrow 0} \operatorname{dn}(z|m) = 1 \quad (\text{A.59})$$

and

$$\lim_{m \rightarrow 1} \operatorname{sn}(z|m) = \tanh z, \quad \lim_{m \rightarrow 1} \operatorname{cn}(z|m) = \operatorname{sech} z, \quad \lim_{m \rightarrow 1} \operatorname{dn}(z|m) = \operatorname{sech} z \quad (\text{A.60})$$

as can be seen from the differential equations (A.56) or (A.57).

We remark that further order-two elliptic functions can be constructed using the three basic Jacobi functions (A.48-A.50) as a starting point. More specifically, inversions of and ratios between the basic Jacobi functions are elliptic and their properties can be easily deduced from the preceding discussion. We list these functions below, noting that only the first of these finds application in this thesis:

$$\operatorname{ns}(z|m) = \frac{1}{\operatorname{sn}(z|m)}, \quad \operatorname{nc}(z|m) = \frac{1}{\operatorname{cn}(z|m)}, \quad \operatorname{nd}(z|m) = \frac{1}{\operatorname{dn}(z|m)}, \quad (\text{A.61})$$

$$\operatorname{sc}(z|m) = \frac{\operatorname{sn}(z|m)}{\operatorname{cn}(z|m)}, \quad \operatorname{cs}(z|m) = \frac{\operatorname{cn}(z|m)}{\operatorname{sn}(z|m)}, \quad \operatorname{sd}(z|m) = \frac{\operatorname{sn}(z|m)}{\operatorname{dn}(z|m)}, \quad (\text{A.62})$$

$$\operatorname{ds}(z|m) = \frac{\operatorname{dn}(z|m)}{\operatorname{sn}(z|m)}, \quad \operatorname{cd}(z|m) = \frac{\operatorname{cn}(z|m)}{\operatorname{dn}(z|m)}, \quad \operatorname{dc}(z|m) = \frac{\operatorname{dn}(z|m)}{\operatorname{cn}(z|m)}. \quad (\text{A.63})$$

A.4 Elliptic integrals

The Jacobian elliptic functions can be constructed as the inverse functions of particular *elliptic integrals*. While we have not constructed the Jacobi functions in this way, some essential identities associated with these functions involve elliptic integrals. For sake of completeness we begin with a definition.

Definition A.4.1 (Elliptic integral). *An integral*

$$\int R(z; P(z)) dz \quad (\text{A.64})$$

is called an elliptic integral if R is rational in its arguments and $P(z)^2$ is either a cubic or quartic polynomial in z .

There are a number of canonical forms for elliptic integrals. We will only require the first two of these. The elliptic integral of the first kind is

$$F(\phi|m) = \int_0^\phi \frac{d\theta}{\sqrt{1 - m \sin^2 \theta}} \quad (\text{A.65})$$

and the elliptic integral of the second kind is

$$E(\phi|m) = \int_0^\phi \sqrt{1 - m \sin^2 \theta} \, d\theta. \quad (\text{A.66})$$

When the integration limit ϕ has the value $\pi/2$, these elliptic integrals are called complete. In particular, the complete elliptic integral of the first kind is

$$K(m) = F\left(\frac{\pi}{2} \middle| m\right) \quad (\text{A.67})$$

and the complete elliptic integral of the second kind is

$$E(m) = E\left(\frac{\pi}{2} \middle| m\right). \quad (\text{A.68})$$

The simplest appearance of an elliptic integral in the context of Jacobi elliptic functions is as formulas for the periods. In particular, in (A.52) we have $K = K(m)$ as in (A.67) and

$$K' = K(1 - m) = F\left(\frac{\pi}{2} \middle| 1 - m\right). \quad (\text{A.69})$$

Elliptic integrals also appear in formulas involving some auxiliary Jacobian functions. The Jacobi \mathcal{E} -function is defined as

$$\mathcal{E}(z|m) = \int_0^z \text{dn}^2(x|m) \, dx. \quad (\text{A.70})$$

This function obeys the addition law

$$\mathcal{E}(z_1 + z_2|m) = \mathcal{E}(z_1|m) + \mathcal{E}(z_2|m) - m \text{sn}(z_1|m) \text{sn}(z_2|m) \text{sn}(z_1 + z_2|m); \quad (\text{A.71})$$

when $z_2 = K$, this reduces to

$$\mathcal{E}(z + 2K|m) = \mathcal{E}(z|m) + 2E(m). \quad (\text{A.72})$$

The Jacobi Z -function is defined as

$$Z(z|m) = \mathcal{E}(z|m) - \frac{E(m)}{K(m)}z. \quad (\text{A.73})$$

This function satisfies the same addition law as $\mathcal{E}(z|m)$ (A.71), but in the special case $z_2 = 2K$, we have

$$Z(z + 2K|m) = Z(z|m). \quad (\text{A.74})$$

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