HELGE TVERBERG IS EIGHTY: A PERSONAL TRIBUTE

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1. Helge Tverberg is eighty

Helge Arnulf Tverberg was born on March 6, 1935 in Bergen, Norway. Throughout his carrier he has been associated with the University of Bergen. His 80th birthday was celebrated on a special day at the Eurocomb 2015 conference that was held between August 30 and September 4 in Bergen. Another reason for the celebration was that his most famous result, the so called Tverberg theorem, was 50 years old in 2015. To be more precise, the paper [8] containing the theorem appeared in 1966 but he found its proof in 1964. And here is, in his own words, how he found it: "I recall that the weather was bitterly cold in Manchester. I awoke very early one morning shivering, as the electric heater in the hotel room had gone off, and I did not have an extra shilling to feed the meter. So, instead of falling back to sleep, I reviewed the problem once more, and then the solution dawned on me!"

Helge Tverberg

My first encounter with him, or rather with his theorem, happened in 1976 when I had to present his result and its proof in a seminar. It was a demanding task but I succeeded. In fact I found a slightly different proof because I could not quite comprehend Lemma 1 in his paper. And more than that, I came up with a question that became known as the topological Tverberg theorem. It was in fact only a conjecture then as I could only prove a special case, the topological Radon theorem [1]. So in 1976 I wrote him a letter, handwritten and sent by ordinary mail, asking if he knew anything about it. And he answered, handwritten letter again, in two weeks time. I was thrilled! I am a beginner and Helge Tverberg, the great mathematician responds immediately and in person to my query. This was a tremendous boost to my confidence. I sent him my colourful Carathéodory theorem and he answered again. I learned later that he had presented my question in Oberwolfach in 1978.

I met him in person first time in 1980 in Oberwolfach. He is tall, very Scandinavian, has a good sense of humour, and is an excellent storyteller. He is interested in all kinds of mathematics. I am impressed. He is an extraordinarily nice, unassuming person apart from being a fine mathematician. He invites me to Bergen and three years later I enjoy his warm hospitality for two weeks there. I get to know his family (wife Sonja and four children). We go for an excursion around Bergen, to the mountains and to the fjords. We take a train, then a bus, then we go by boat, and train again plus plenty of walking and mathematical discussions. I learned later that he prefers train rather than plane as a mode of transportation. On the train he has plenty of time to ponder a mathematical problem. That's why several of his proofs are associated with specific train routes.

Helge Tverberg is friendly and generous, he loves intriguing questions and puzzles. He had been running a mathematical puzzle column in the newspaper Bergen Tidende for many years. He knows and loves art and literature and music, jazz in particular. He spent a couple of months in Reading in 1966 and mentions in his recollection [14] that "simply being abroad was great too, and I often went into London on weekends, taking much enjoyment in live performances by Duke Ellingtons Orchestra, Ella Fitzgerald, and others, which would not have been so common in Bergen in those days."

2. His famous theorem

Tverberg's theorem is the most important result in combinatorial convexity in the last half century. Its impact and influence is only comparable with the famous theorems of Carathéodory and Helly. Here is the statement.

Theorem 2.1. *Given positive integers d and r and a set $X \subset \mathbb{R}^d$ of $(r-1)(d+1)+1$ points, there is a partition of X into r sets $X_1, \dots, X_r$ such that $\bigcap_1^r \text{conv} X_i$ is nonempty.*

This innocent looking statement is a generalization of Radon's theorem which deals with the case $r = 2$ and for which there is an easy proof. Several proofs of Tverberg's theorem are known but even the simplest ones by Sarkaria [7] and Roudneff [6] are not easy. Tverberg himself gave a second proof [11]. It has many applications in discrete geometry, in computational geometry, in abstract convexity, in quantum error correcting codes. There is a whole industry on what kind of Tverberg partitions exist. It also has connections to van Kampen Flores type results. Here is one neat application from combinatorics [9], originally a theorem by his Swedish friend Bernt Lindström [5]: If $A_1, \dots, A_{(r-1)n+1}$ is a sequence of non-empty subsets of an n element set, then there are non-empty disjoint subsets of $\{1, \dots, (r-1)n+1\}$, say $J_1, \dots, J_r$ such that $\bigcup_{i \in J_1} A_i = \dots = \bigcup_{i \in J_r} A_i$.

I have applied Tverberg's theorem frequently. Quite often, the idea of a line of attack or of the proof have come from his theorem. For instance, a combination of colourful Carathéodory and Tverberg's theorem gives the *selection lemma* [2]: Given a finite set $X \subset \mathbb{R}^d$ points in general position, there is a point common to the convex hull of a positive fraction of the $d+1$-tuples of X. On another occasion, working with Füredi and Lovász [3] on the halving plane problem in $\mathbb{R}^3$ we encountered the following question: Given a set $X \subset \mathbb{R}^2$ of n points in general position, a *crossing* is the intersection of the lines spanned by x, y and by u, v where x, y, u, v are distinct points from X. It is evident that there are $\frac{1}{2}\binom{n}{2}\binom{n-2}{2} \sim n^4$ crossings. How many of them are contained in a typical triangle with vertices $a, b, c \in X$? A direct application of Tverberg's theorem shows that the number of crossings is again of order n^4. This was a key step in establishing an $O(n^{3-\varepsilon})$ bound on the number of halving planes. It also lead to what is now called the colourful Tverberg theorem. The moral is that when working on a question in combinatorial convexity it is always good to check what Tverberg's theorem says in the given situation.

3. Beautiful pieces

Helge Tverberg loves all kinds of mathematics and has eclectic interest in algebra, analysis, number theory, combinatorics, convexity, and geometry. Besides his famous theorem in convexity, he has several remarkable and beautiful results, some of them so clean and elegant that I cannot resist writing about them. One is his short and clever proof of the Jordan curve theorem [10], probably the best ever. Another gem is his proof [13] of Grünbaum's conjecture on common transversals of translates, a very lucid and transparent argument cracking a 31 year old conjecture.

Here comes my favourite, the splendid proof of a theorem of Graham and Pollack [4] about the decomposition into complete bipartite graph of the complete graph on n vertices. Precisely, let K_n be the complete graph on vertex set $\{1, \dots, n\}$ and assume that subgraphs $G_1, \dots, G_r$

decompose its edge set such that (1) they are edge-disjoint and (2) contain all edges of K_n. The Graham–Pollack **theorem** says that if every G_i is a complete bipartite graph and they decompose K_n then $r \geq n-1$.

Proof by Tverberg. Associate with vertex i of K_n a variable x_i. With A_j and B_j denoting the bipartition classes of G_j, define $L_j = \sum_{i \in A_j} x_i$ and $M_j = \sum_{i \in B_j} x_i$. Then $\sum_{i<k} x_i x_k = \sum_{j=1}^r L_j M_j$. Consider the system of linear equations

$$L_1 = L_2 = \ldots = L_r = \sum_{i=1}^n x_i = 0.$$

If $r < n-1$, then this linear system has a non-trivial solution $x_1 = a_1, x_2 = a_2, \ldots, x_n = a_n$, and

$$\begin{aligned} 0 &< a_1^2 + \ldots + a_n^2 = (a_1 + \ldots + a_n)^2 - 2\sum_{i<k} a_i a_k \\ &= 0 - 2\sum_{j=1}^r L_j M_j = 0, \end{aligned}$$

a contradiction showing that $r \geq n-1$. □

What a beautiful and elegant proof, indeed. I'd like to finish with a quote from his student Andreas Holmsen: "Tverberg once told me that if he hadn't start to study mathematics he would probably study law. What he found appealing about both fields i the logical deduction from a fixed set of axioms/laws. I think that many people, me in particular, are very grateful that his choice landed on mathematics!"

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