Is it mathematics or is it school mathematics?

Presidential address to The Mathematical Association, April 2017

In my professional life, I have repeatedly had cause to return to the question of

> 'What relationship does the 'mathematics' experienced in classrooms have to the mathematics studied in universities, or used outside academia? What relationship should it, and could it, have, and how do we get there?'

It's time I began to frame some responses! In doing so, I am very aware that membership of The Mathematical Association is broad church, ranging from those with an amateur interest in mathematics (and mathematics education), through non-specialist teachers, to professional mathematicians. As such it is a rich community, but one in which it is challenging to address such questions in a way which is meaningful to all. However, I shall attempt to do so and I hope that the balance will in a small way also prove illuminating to the reader. In particular, I have a hope that it will point to the breadth and depth of what we expect from our teachers of mathematics, at all levels, and to a valuing of that.

In order to answer the questions posed, I need first to address some supplementary issues. First, what do I mean by mathematics? The answer would probably be different for each of us. But in the light of *my* answer, why should (some) mathematics be learnt at school? So precisely *what* mathematics should we aspire for young people to learn? I shall then ask why that is not happening at a systemic scale, what are the related opportunities and challenges, and how should we set about making progress with those?

In order to understand my responses, the reader should appreciate the lenses I bring, and how those might complement the reader's own. For many years, I have had the enormous joy and privilege of working as a co-learner of mathematics with learners from 3 to 93, and in several cultures. For much of that time, I have in parallel worked in (beginner or experienced) teacher development – again, across phases and cultures. But I also had a brief career as an academic mathematician, and for the last five years have worked entirely in Higher Education, again in teacher development but also doing research in mathematics education, and in particular researching how mathematics education policy impacts on teachers and students in the classroom. The motivation for the last arises from the 25 years, thanks initially to MA activity, I've in a small way contributed to the policy/practitioner interface, via QCA (or similar) committees, DfE expert groups, ACME, APPG…. The themes of this address will inevitably therefore be informed by the policy contexts in which we currently find ourselves.

My experience is that academic mathematics, school mathematics, and HE mathematics education departments, operate as different but overlapping cultures. Politician and policy-makers' understanding of these cultures is usually limited to their own experiences as young people in schools – but serves to remind us that there is also a world out there which is peopled not, in the main, by mathematicians, but by those who, whether they appreciate it or not, draw on mathematics to a greater or lesser extent in their daily lives and work. I shall come back to that as I develop my argument – which will divert us into some (school) classrooms and into our broader cultural heritage, though inevitably both informed and coloured by my own professional background. What follows arises in many ways from thinking about the common denominators of those professional experiences.

What do I mean by mathematics? Is it discovered or invented?

In common with many, but by no means all, mathematicians, I understand mathematics to be a human endeavour, concerned with the exploration of, and connections between, patterns, their abstraction and relationships as established through reasoning. I think it's important to note that every human society we know about has developed a mathematical lens on the world, with similarities and differences in those lenses across different cultures. For an inspirational treatment of this theme from a philosophical point of view, I recommend Francis Su's Presidential address to the AMS [1].

Mathematics is therefore part of our cultural heritage. It is often concerned with describing, illuminating and harnessing natural phenomena, with this analysis then often codified and further developed for a variety of purposes within and beyond mathematics. The Noble Prize-winning Wigner [2], in his paper 'The unreasonable effectiveness of mathematics in the natural sciences', says

> "The miracle of the appropriateness of the language of mathematics to the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve".

A different response, advocated by Tegmark in [3], is that physics is so well modelled by mathematics because the physical world *is* completely mathematical, isomorphic to a mathematical structure, and that we are simply uncovering this bit by bit. And perhaps given the primeval origin of much mathematics, there is an argument that this extraordinary power is not so unreasonable, yet it remains the case that mathematically 'ordinary' fourteen year olds can, for example, feel 'mind-blown' (in teen speak) by the notion that without imaginary numbers, our understanding of the very real fields of electricity and magnetism would not have developed to give us their computers and iphones – just how overwhelming is that?

So mathematics is concerned with concepts, but works with culturally-developed tools and processes. I often ask beginner teachers whether they think mathematics is discovered or invented, since I want them to reflect on the nature of the subject they will teach. We are by nature both curious and inventive – how do the two interact with mathematics? For Tegmark it is clearly the former; for Adler and Sfard [4], mathematical objects are understood as discursive constructs, created for the sake of communication about the world rather than as self-sustained entities, existing independently of humans, and it is in that sense that mathematical truth is always provisional, open-ended and relational.

Then how is mathematical knowledge established? Many would argue that what we have in western society is one, or some, possible mathematical accounts, and in Lakatos' [5] terms I, for one, am a fallibilist, or relativist, not an absolutist. Answers to such questions have implications for the way mathematics should be learned and taught. For mathematics to be a shared endeavour, there need to be shared ways of working and of establishing new knowledge in the field: a shared epistemology and syntax. How those are applied when working with novice mathematicians will also be influenced by one's beliefs about how young people (and others) learn, so I also implicitly draw on my espoused theories of learning.

Figure 1: Babylonian mathematical tablet ca 1600 BC

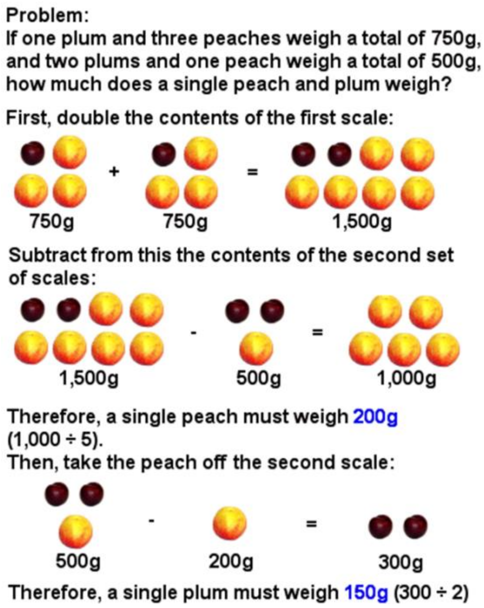
For the next part of my argument, I cite two examples I often use with early or middle teenagers to draw them into an awareness of both the historical legacy of the sub-culture in which I am endeavouring to induct them, and away from what I perceive to be an unhealthy Euro-centricity of much English education. Both problems also have the merit of inducing an unusual degree of persistence in teenagers!

The first of these comes from a Babylonian clay tablet dating from about 1600 B.C., written in cuneiform with reeds and similar to that in Figure 1 (check source). Such mathematical tablets, so far as we know, were usually one of two types: accounting tablets (grain, perhaps, or slaves), or problem tablets, as here:

I found a stone but I did not weigh it; after I weighed out 6 times its weight, added 2 gin, then added one-third of one-seventh of that multiplied by 24. I weighed it: 1 ma-na (60 gins). What was the weight of the stone?

This is one of 22 such problems on one tablet, each involving a stone weighing one mana! See, for example, [6].

The second example is of a similar antiquity, but this time originates from China. In translation (and with a change of units of weight), it reads [7]:



The described method of solution is of course one that is commonly taught in secondary classrooms today.

Note that in both cases, the problems is artificial: if one wanted to know the weight of the stone, or a peach, or a plum, one would weigh it. So the purpose of the puzzles is educational, or recreational (or both). Such problems, for their own sakes, have been valued for many years and across cultures. I shall return to that fact, but first I ask

Why should (some) mathematics be learnt at school?

I would argue that the purpose of socially-sanctioned (and funded) school education is to nurture young people's constructive potential and induct them into the culture of the society, so that they both mature into fulfilled and well-rounded adults, and are able to contribute to and constructively critique society. Again, if you think school education is for different purposes, your thoughts about what that education should comprise will be different.

Secondly, and here we return to my stated beliefs, mathematics is an integral part both of young people's potential and of that culture (at any scale, whether family, school or work, local area, national or global).

Given that justification, young people need, for *utilitarian purposes,* concepts, facts, skills and processes that equip them to be mathematically functional now and in the future, as well as those that give them a foundation for later mathematical development in work or further study.

They should have a *broader mathematics education* that equips them less directly, but *powerfully*: skills of mathematical problem posing and solving, and a critical appreciation of the use of mathematical approaches in society (leading to social empowerment through mathematics). In this way they can experience mathematics not only as discovered and transmitted, but also as invented.

Further, and I would argue this is often marginalised in educational planning, they need the *affective resources* to harness that education confidently (self-efficacy in relation to an appropriate degree of mathematical functioning, resilience, collaborative and learning dispositions…) – see for example [8].

But young people also, as a particular and expanded interpretation of the second, benefit from a *cultural appreciation of mathematics* – an understanding of mathematics as part of their cultural heritage, of its nature (the mathematical landscape and its development and functioning) and of the nature of mathematical activity – the syntax and epistemology of the discipline, the valued ways of working in the mathematics sub-culture, their surprises, frustrations and joys – at an appropriate level. I would therefore argue, in common with Ernest [9]:

> *Learners should gain a qualitative or intuitive understanding of some of the big ideas of mathematics such as pattern, symmetry, structure, proof, paradox, recursion, randomness, chaos, infinity. Mathematics contains many of the deepest, most powerful and exciting ideas created by humankind. These extend our thinking and imaging power, as well as providing the scientific equivalent of poetry, offering noble, aesthetic, and even spiritual experiences.*

I can already hear the sceptical labelling this a naïve and idealistic agenda. Yes if we are to progress beyond a mathematics education that fits young people for the utilitarian, why should we not aim high? To whom should we ration that deep satisfaction and sense of wonder (and that constructive frustration) that most readers will have experienced?

Figure 3: Beginnings of a Sierpinski gasket

I wouldn't claim that all teachers of mathematics work in a context where that is achievable with all young people, but it is my experience that it is accessible to many 'mathematically ordinary' young people in this country and elsewhere in the world, and need not be highly dependent on physical resources. A commitment to that vision and mathematical and pedagogical knowledge on the part of the teacher is what determines, in Adler and Ronda's words [10], 'the mathematics that is made available to learn'. Just two examples: Figure 2 shows the beginnings of a Sierpinski gasket constructed by a class of fairly streetwise 14 year olds, sent to me by a beginner teacher of no particular mathematical distinction. (When the caretaker took exception to it at the end of term, she took it home on the underground).

Figure 2: Truncated icosahedron meets football (or C60)

Her students had chosen to stay after school to finish it. Figure 3 shows a pair of poorly-attaining sixteen year olds, for once both in school on the same day, proudly exhibiting their approximation to a football – or, as they described it to their peers who were dragged in from the corridor at lunchtime, their 'truncated icosa-what, Miss?' Their chemistry teacher suggested it might also be a model for a carbon 60 molecule 'Nah', they said 'you've gotta be joking? Really? Really and truly?' of such can the life of a teacher be made – on occasion, with many young people, in many contexts.

Then if mathematics is part of our cultural heritage, and so the birthright of young people, how are they to learn its nature? I give just one avenue I've used for some years with the range of 11 year olds new to secondary school. I'm both ashamed, and delighted, to say that to the best of my knowledge I didn't encounter this conjecture until I had already been teaching for at least 25 years – such is the breadth of elementary mathematics.

Figure 4: 1742, correspondence between Goldbach and Euler

In 1842 a letter from Goldbach to Euler (Figure 4, taken from https://en.wikipedia.org/wiki/File:Letter_Goldbach-Euler.jpg) shows that Goldbach had formulated a conjecture equivalent to the following:

Every even number greater than two is the sum of two primes.

You might like to test this conjecture with, for example, 24, 48, 132, 2088. You'll notice there's no claim to uniqueness of partition into two primes, as these examples show (and I include 2088 since one possible partition is of significance this year and students particularly enjoy such resonances).

Having grasped the related ideas, including that of a counterexample, and beginning to deal with bigger even numbers, young people are often awed by the idea that a problem they can grasp, that uses ideas well within their experience (and potentially big numbers – that's always a winner), has been lying unsolved for over 270 years. (In my youth, it was the four-colour conjecture that caught my imagination, but that of course is no longer available). Despite being armed with that knowledge, significant numbers of them will go home that night, and for weeks to come, usually determined not to find a counterexample, since they recognise that if one hasn't been found using a computer, they're unlikely to find one, but to prove the conjecture. They return, day after day, to report their failure to do so – yet. I would argue those young people are beginning to grasp something of some important aspects of the nature of mathematics, and of mathematical activity.

And isn't it just indicative of the nature of this wonderful subject, that after many years, one is still meeting new ideas in elementary mathematics? The icing on the cake must be if those ideas emanate from one's 'students', as they did for me when half my (this time, fairly mathematically-alert) year 10 class had been working on the Koch snowflake (the other half had been bouncing balls – but best not go there). One lad had come to some very satisfactory conclusions, so I asked what would happen if he changed the situation slightly. He went away and started thinking about squares on the middle third of the edges of a square, rather than equilateral triangles on the middle third of the edges of an equilateral triangle, and didn't like his answer: 'it doesn't feel right', he said (at which my teacher's heart swelled…). He talked me through it, diagnosed his error, and…well. That would be spoiling the story.

If young people are to own the mathematics with which they're working, if elementary mathematics is to become part of their own culture, I would argue it's helpful for them to see their teachers engaged with it too – and essential for the sustained mathematical vigour and health of the teacher. I offer the following as a teaser (to me, presented the problem by a 13 year old, but also, I was glad to see, a challenge to more respected mathematicians I've shared it with (Figure 5):

How can you construct a single straight cut to replace the 'elbow' but maintain the two areas into which the outer quadrilateral is divided?

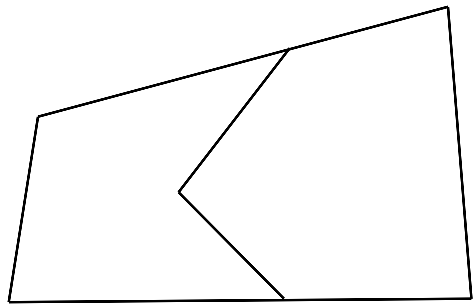


Figure 5

I would argue, then, that school mathematics can, and should, include authentic mathematics – for all young people. I also suggest that the mathematics practised by academic mathematicians should not be the sole arbiter of authenticity: many people function in mathematically authentic ways in their daily lives and work. So how should school mathematics relate to these?

I recently had the privilege of working with secondary teachers in Armenia, where the mathematics education is, in the Soviet tradition, very much 'top-down' from the university mathematics department. Teachers sought advice on teaching the area of a square to their fourteen year olds. In the UK we tackle general areas of squares inductively and informally; in Armenia they deal with sides of integer, then rational, then irrational lengths, leading to formal analytic consideration of the product of infinite sums. Unsurprisingly, most fourteen-year olds find this overwhelming. So I would suggest as a general curricular approach might be inappropriately top-down – although for a few students, it produces mathematical rigour and, often, elegance, of an impressive robustness rarely seen in most British schools.

On the other hand, in South Africa I worked with teachers educated under apartheid, who as black South Africans were only entitled to a 'Bantu curriculum' – essentially, basic arithmetic of a utilitarian nature. South Africa now works with an 'outcomes based' education system designed to promote equity of access both to modern science and technology and to a range of cultural heritages. This is an ambitious undertaking, of course, with its aims the mathematical empowerment of the range of South Africans. It is beset by challenges of both an historical and an economic nature – though supported by the enthusiastic commitment of many teachers and students.

I would argue, then, that school mathematics should engage young people in coming to understand fundamental mathematical concepts, and with mastering (sic) a range of tools, skills and processes, giving them the potential to sustain effective mathematical functioning in everyday lives and appropriate work or further study. For some young people that should include a foundation for further academic work in mathematics.

But it is not just the content of the curriculum that is important: if young people are to function confidently and effectively mathematically, they need to experience, in sustained ways, valued mathematical ways of working.

Cuoco et al, in [11], frame these as 'mathematical habits of mind, saying mathematicians are 'Pattern sniffers, Experimenters, Describers, Tinkerers, Inventors, Visualisers, Conjecturers, Guessers'. While we might recognise those characteristics, Burton [12], rather more helpfully for classroom teachers I think, quotes a poster she had seen in a classroom, claiming that mathematicians 'have imaginative ideas, ask questions, make mistakes and use them to learn new things, are organised and systematic, describe, explain and discuss their work, look for patterns and connections, and keep going when it is difficult.' We might argue over the detail, but the point is made that authentic mathematical activity is a far cry from that seen in many classrooms [13], where in too many cases the teacher (or the textbook) is seen as the source of mathematical authority and the students' job is to reproduce demonstrated approaches to solving standard exercises.

So can school mathematics function as an authentic sub-discipline of mathematics? I would argue that's not completely achievable, because young people are novices – and that is perhaps consistent with part of Nick Gibb's somewhat over-stated argument [14], at least in the case of mathematics. However, the ways in which young people learn to work can, and should, be consistent with the values of the discipline. They should not be restricted to learning about concepts and facts, skills and processes, though they need all those, and need also to understand that the discipline itself is the authority. The messiness and debate and choices of the discipline are often hidden unnecessarily.

Anne Watson [15] suggests that school mathematics can never be a subset of the academic discipline since school mathematics has different warrants, authorities, forms of reasoning, and there are other priorities and structures in schools. Maybe, but I would argue teachers can work within those constraints to induct young people into an authentic and complementary sub-culture of the discipline.

Michael Young's [16] book begins to develop an argument that the school curriculum, whatever else it does, should endow young people with disciplinary 'powerful knowledge' – and that historically, they have been shortchanged by the rationing of school-engendered knowledge to that which preserves social hierarchies – and limits individual potential. Powerful knowledge is discipline-embedded, and I would argue in mathematics is focused on deep and robust conceptual understanding – and familiarity with mathematically valued ways of working with those. To harness those concepts young people of course also need to develop a repertoire of facts, skills, and processes.

But here we come to one of the biggest challenges associated with trying to establish principled approaches to mathematics education. Teaching for the building up of powerful knowledge is hard. It requires, among other things [17] understanding of the mathematical landscape, and 'unpeeling' [10] of one's own compressed understanding. Ma [18] talks about 'profound understanding of elementary mathematics'.

For example, a Primary teacher whose children have a reasonable grasp of integers has then to understand that ¾ is part of a whole (and which whole). It is also 3 lots of a quarter of the same whole, 3 wholes shared equally among 4, an operator (as in ¾ *of* a number of Smarties), a number in its own right, with its own position on a number line, a ratio, an equivalence class of fractions… All these concepts have to be planned for, introduced and worked with at appropriate times, with a range of appropriate representations, and in appropriate ways, and young people supported in making connections between these different conceptualisations, until, in the long term, they compress that understanding into a single, but multi-faceted, idea of '3/4'.

Contrast that with the academic mathematician defining the rationals from the integers, which might look something like Figure 6:

(A) Define in $Z \times Z^*$ the following equivalence relation: $(a,b) \sim (c,d)$ if and only if ad equals bc.

(B) Define Q as the set whose elements are the equivalence classes of the relation $\sim$.

(C) Define addition and multiplication in Q taking representatives of the equivalence classes, and then show that these definitions are "good enough" in the sense that the results are independent of the chosen representatives. Then one proves that those operations have their zero and identity elements, they satisfy the commutative, associative, distributive laws, and so on. The proofs are direct consequences of the correspondent properties already established for the integers, except for the existence of the multiplicative inverse, which follows immediately from the given definition of multiplication.

(D) Finally, define $Z_0 = \{\overline{(a,1)},\ a \in Z\}$ and a function $f: Z \rightarrow Z_0$ putting $f(x) = \overline{(x,1)}$. Then prove that f is an isomorphism (one to one correspondence preserving the relevant operations) that identifies the structure of Z with that of Z_0 ("inherited" from Q).

Figure 6: from [19]

The related two sub-disciplines require different kinds of knowledge. Further, the same Primary teacher in the melee of the classroom moment, has to comprehend, evaluate, and decide how to deal with young people's alternative understandings as they develop. For example, in Figure 7 the task is to identify the value of the number indicated.

Figure 7: Year 4 number line task

In one class I observed, pupils variously offered 0, ¼, 2/4, -2… They were not being capricious in doing so, but trying to make their sense of the situation – and in each case, were part way towards achieving our own shared sense-making. The teacher has to respond in the moment, and in ways which will build not only that child but the other (in this case, 32) young children's mathematical ways of functioning. That is no small task, even for a mathematics specialist, but our Primary teachers in this country typically teach across the curriculum.

Mathematical knowledge for teaching

Teaching for mathematics – at any level –therefore requires a great breadth and depth of subject-related knowledge if it is to be effective. Such knowledge has been theorised in ways the reader might find interesting, if (s)he has not met them before. One such development, widely referred to in western mathematics education, is Ball et al's 'egg' (Figure 8). This divides subject specific knowledge needed into subject knowledge and subject pedagogic knowledge. The former encompasses not only what the mathematically competent adult commonly needs, but deep knowledge of its different conceptualisations, where those might be encountered and the relationship between them, as in the example of 'knowing' ¾ above, as well as the links of each idea to others within or beyond mathematics. Pedagogically, the teacher also needs to know where those different conceptualisations sit within the curriculum, the progressions to and from those, how concepts might meaningfully and constructively be re-presented to children, in what contexts, how children, and particularly the children in this class, might typically understand them in 'different' ways, how to elicit children's related developing understanding, how to harness that constructively, and how to support construction of the valued connections.

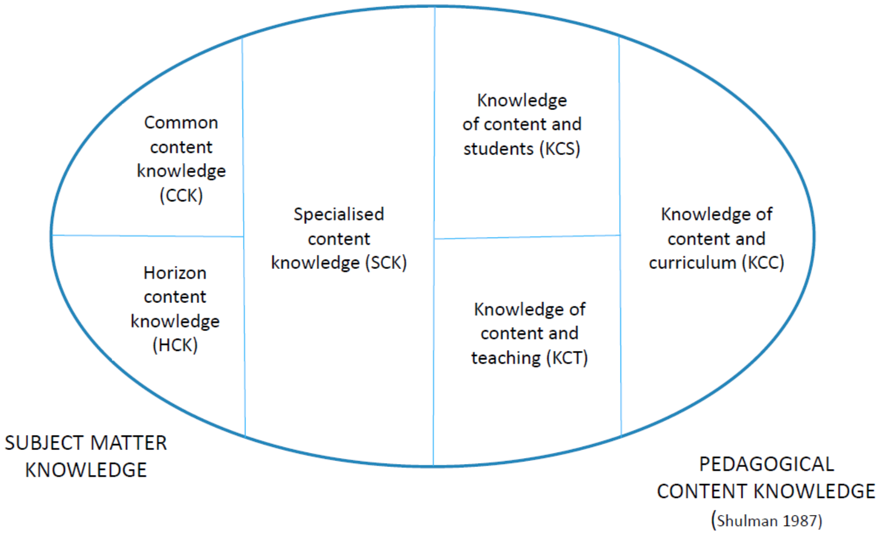


Figure 8 [20]

An alternative, higher-level theorisation comes from [21]. From classroom observations, it is claimed that in teaching mathematics, teachers draw on mathematical Foundation knowledge – of mathematics, students in relation to mathematics, curriculum, etc; on Transformation knowledge – of the mathematics to a form where students begin to see the underlying motivations, structures and warrants; Connection knowledge – within and beyond mathematics, of the authentic use of mathematical thinking and of utilitarian applications; and on Contingency knowledge of how to field questions, and responses such as those given above, in mathematically constructive ways. Again, what is seen as necessary is a rich and deeply connected discipline-embedded network.

So, even if we re agreed on where we want to be, achieving an authentically mathematically well-equipped population, how do we get there? There are a number of well-intended initiatives in train, but for me, the key is the supply and development of sufficient mathematically knowledgeable and pedagogically effective teachers. As a succession of ACME and other UK reports have shown, there is no magic bullet, but I would argue that for individual and societal flourishing we need to reverse moves which since the introduction of the National Curriculum, have served to reduce much classroom-level 'mathematics' experience to the banal and performativity-driven. These have been the consequence of well-intentioned policy moves designed to ensure accountability to the taxpayer, minimum entitlements for all, and, in many places, a move away from complacency with 'what we have always done'. But they have had unintended, and often far-reaching, consequences – not least in moving towards fragmented and often mediocre initial and continuing teacher development that frequently pays only lip service to the discipline-specific needs of teachers – see e.g. [22]. We could start by looking across UK borders to see what we should learn.

As a community, we have often been insufficiently proactive in publicising a coherent alternative to party political/individual politician misuse of frequent (and often un-evaluated) policy hyperactivity that undermines attempts by, and for, teachers, to develop effective teaching and learning of mathematics in the classroom. Across political parties, we have seen ministers using discourses of 'evidence-based policy' to mean 'cherry-picking of any titbit of evidence that fits my preconceived ideas'. We are the professionals and subject experts here: we need to pull together to ensure our informed voice is heard. Our young people's education is too important to be left to party politics.

Similarly, we have a job to do in ensuring maintenance of teacher morale and self-efficacy (and their energy and motivation) in the light of ever-increasing expectations, frequent change (with a threat of 'policy fatigue') and negative messages from the public and sometimes, the media. Our teachers largely work long and hard to do the best they can for young people: as a subject community we have a duty to appreciate them, and to respect their professionalism by working with government to find ways of empowering them further mathematically, so enhancing both their effectiveness and their job satisfaction (and hence, retention).

We should not shrink from addressing hard issues in partnership with government. For example, current post-16 funding and accountability regimes in England serve only to undermine aspirations for greater mathematical participation post-16: we need to be proactive in addressing that, not only for economic reasons, but for the proper stewardship of individual and national mathematical potential.

Then what are the *opportunities* this ever-shifting landscape brings? Certainly, a variety of initial teacher education models and moves to bottom-up CPD challenge our assumptions about what development teachers need, when and how. That's healthy – provided it's also regularly and effectively evaluated, and changes made in response to that evaluation, so that we don't persist in perpetuating mediocrity wherever it occurs.

We now know far more than we did about how young people learn mathematics (though there is still a way to go), and what teachers need to be able to support that: we need to capitalise on that knowledge in a coherent and longterm way, and to recognise the need for discipline-specific expertise in both subject and subject pedagogical arenas. That's not an admission of failure – it's a mature response to a pressing need.

Current curricula at 5-16 and post-16 do, to a large extent, embody fundamental values of the maths education community, though not always in detail or reflected in 'expected' pace of 'coverage'. While not ideal, teachers will deal with any more revolution only at a cost to their effectiveness and their well-being (and in many cases, retention). As we acquire evidence of what is working well and why, we should find ways to support the *evolution* of those curricula and surrounding artefacts (assessment, CPD, resources) into a coherent whole.

Remarkably, given the above challenges, our recent 60-second survey, reported in the MA January 2017 newsletter, shows that teaching mathematics remains for many a satisfying career: we as a community need to find ways to support, nurture and spread that satisfaction.

In conclusion..

I have talked about us as a 'community' and we should nurture that, as well as work closely with other organisations and individuals concerned for the wellbeing of mathematics education in this country. We each of us have opportunities to work with others to celebrate that which is good, and to address some of the challenges. At a time of significant challenges for mathematics education in schools and colleges, and for teachers of mathematics at any level, what better time to learn to pull together rather better?

Our goal? I return to The MA strapline: the support and promotion of confidence and enjoyment in mathematics - for all.

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