Department of Mathematics
University College London

Stably free modules over infinite group algebras

Pouya Kamali

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Supervisor:
Professor
F.E.A. Johnson

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I, Pouya Kamali, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

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Abstract

We study finitely generated *stably-free* modules over infinite integral group algebras by using the language of cyclic algebras and relating it to well-known results in K-theory.

For G a free or free abelian group and Q_{8n}, the quaternionic group of order $8n$, we show that there exist infinitely many isomorphically distinct stably-free modules of rank 1 over the integral group algebra of the group $\Gamma = Q_{8n} \times G$ whenever n admits an odd divisor.

This result implies that the stable class of the augmentation ideal $\Omega_1(\mathbb{Z})$ displays infinite splitting at minimal level whenever G is the free abelian group on at least 2 generators. This is of relevance to low dimensional topology, in particular when computing homotopy modules of a cell complex with fundamental group Γ.

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Chapter 1

Introduction

1.1 Motivation

Given a weakly finite ring Λ (c.f. Chapter 3.1), we say a (right) Λ-module S is *stably-free* of rank N whenever

$$S \oplus \Lambda^b = \Lambda^a$$

and $a - b = N$, for integers $a, b, N \geq 0$. Clearly, a stably-free module is projective. Moreover, we say a group G is of type $\mathcal{F}$, if it is free or free abelian. The main result of this thesis is given by (c.f. Theorem 6.3.1):

Theorem A. *Let G be a group of type $\mathcal{F}$. Moreover, let Q_{8n} be the quaternionic group with $8n$ elements; then for n with at least one odd prime divisor, there is an infinite collection $\{\mathcal{S}_m\}_{m\geq 1}$ of isomorphically distinct stably-free modules of rank one over the group-algebra $\mathbb{Z}[Q_{8n} \times G]$.*

There are two motivations for this result, the first being of algebraic, and the second of topological nature. We start with the former: it is an important and natural question, to ask what the projective modules over a given ring R are. Thus it is not surprising that one of the most important algebraic problems of the last century, posed by Serre in his famous conjecture (c.f. [26]), was asking precisely that question, in the case where R is the polynomial ring in n variables with coefficients in a field. This conjecture prompted research by Sheshadri [27], Bass [2], Quillen [25] and others, implying the following statement (c.f. Chapter 3.3):

Theorem. *Let G be a group of type $\mathcal{F}$ and Λ an abelian principal ideal domain; then all finitely generated projective modules over $\Lambda[G]$ are free.*

On the contrary, our result shows that in the case when $\Lambda = \mathbb{Z}[Q_{8n}]$, with n as in the statement of Theorem A, we can produce infinitely many finitely generated (non-free) projective $\Lambda[G]$-modules. Here we make the identification $\mathbb{Z}[Q_{8n} \times G] \cong \mathbb{Z}[Q_{8n}][G]$. The fact that we have chosen to examine stably-free modules, which are particularly well-behaved projective modules, underlines the complexity of the ring $\mathbb{Z}[Q_{8n} \times G]$.

It is well known that the ring $\mathbb{Z}[Q_8]$ already has non-trivial projective modules (c.f. e.g. [29]). However, as Swan shows in his important paper [28], it is not until $n = 6$ that $\mathbb{Z}[Q_{4n}]$ possesses nontrivial stably-free modules (in fact this is the case for all $n \geq 6$). Thus it was a natural choice to start the study of stably-free modules over $\Lambda[G]$ by letting $\Lambda = \mathbb{Z}[Q_{24}]$, and then extend the result to quaternionic groups of order $8n$, where n admits at least one odd prime divisor. It should be added that in August 2009 Johnson proved that the analogue of Theorem A also holds in the case $\Lambda = \mathbb{Z}[Q_8]$ (c.f. [14]).

Particular emphasis should be placed on the point that the ring $\mathbb{Z}[Q_{8n} \times G]$ is not a generic algebra, constructed solely to illustrate a particularly complex case of a ring with stably-free (non-free) modules, but rather, and this is where we move on to topological considerations, that it appears naturally. Namely, let

$$\mathcal{P} =< x_1, \dots, x_g | W_1, \dots, W_r >$$

be a presentation for the group $\pi := Q_{8n} \times G$. Then π can be interpreted as the fundamental group of a two dimensional CW complex $X_{\mathcal{P}}$, the so called Cayley complex of $\mathcal{P}$ (see [10] p. 183 for further details). The chain complex of the universal cover $\tilde{X}_{\mathcal{P}}$ gives rise to a complex of $\mathbb{Z}[\pi]$ modules thus:

$$C_*(\tilde{X}_{\mathcal{P}}) = \left(0 \longrightarrow ker(\delta_2) \longrightarrow \mathbb{Z}[\pi]^r \xrightarrow{\delta_1} \mathbb{Z}[\pi]^g \xrightarrow{\delta_1} \mathbb{Z}[\pi] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \right).$$

Here δ_1, δ_2 are completely determined by the presentation $\mathcal{P}$, and $\mathbb{Z}$ denotes the trivial $\mathbb{Z}[\pi]$-module. More generally, by an algebraic 2-complex over π we mean any exact sequence of (right) $\mathbb{Z}[\pi]$-modules of the form

$$\mathcal{E} = \left(0 \longrightarrow ker(\delta_2) \longrightarrow F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \right),$$

where each F_i is finitely generated free and $\mathbb{Z}$ denotes the trivial $\mathbb{Z}$-module. Given two algebraic complexes $\mathcal{E}$, $\mathcal{E}'$, it follows by Schanuel's Lemma that

$$ker(\delta_2) \oplus F \cong ker(\delta_2') \oplus F',$$

for finitely generated free modules F, F'. We shall say that $ker(\delta_2)$ and $ker(\delta_2')$ are *stably equivalent*. We denote the class of all modules stably equivalent to $ker(\delta_2)$ by $\Omega_3(\mathbb{Z})$.

This relates to two unanswered questions in low dimensional topology, the first of which, known as the *Realisation Problem*, asks:

Given a group Γ, is every algebraic two complex chain homotopy equivalent to a complex arising from a two dimensional CW complex, i.e. the Cayley complex of some presentation of Γ?

Clearly, the Realisation Problem is parametrised by the class $\Omega_3(\mathbb{Z})$. The second question, originally phrased by Wall in [30], is called the *$D(2)$-Problem* (initially it was called the $D(n)$-Problem which Wall solved in the same paper for all $n \neq 2$). It asks:

Given a geometrically 3-dimensional CW-complex with zero homology and cohomology in dimensions higher than two, over all possible coefficient systems; is it necessary that this complex is homotopy equivalent to one of geometric dimension two?

It is not hard to see that the $D(2)$-Problem is parametrised by the fundamental group of the CW-complex in question. Thus, we say the $D(2)$-property holds for a fundamental group, if the above question can be answered affirmatively for it.

Johnson ([10] p. 256) has shown that the $D(2)$-Problem is equivalent to the Realisation Problem for all fundamental groups satisfying certain conditions. This result was then extended by Mannan [19] to hold for all finitely presented fundamental groups. To date there is no known fundamental group for which the $D(2)$-property does not hold. To see why Theorem A above is motivated by, and of potential relevance to the $D(2)$-problem we state the following theorem by Johnson [11] (here $\Omega_1(\mathbb{Z})$ is defined analogously to $\Omega_3(\mathbb{Z})$ above and $\mathcal{SF}_+$ denotes the set of isomorphism classes of non-zero stably-free $\mathbb{Z}[\Gamma]$-modules):

Theorem. *Let Γ be a finitely generated group satisfying $Ext^1(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$; then the duality map $J \mapsto J^*$ induces a level preserving mapping of trees*

$$\delta : \Omega_1(\mathbb{Z}) \to \mathcal{SF}_+;$$

moreover, δ induces a $1-1$ correspondence $\delta : \Omega_1^{min}(\mathbb{Z}) \to \mathcal{SF}_1$ between the minimal level of $\Omega_1(\mathbb{Z})$ and the isomorphism classes of stably-free modules of rank 1.

Groups of the form $\Gamma = Q_{8n} \times G$, where G is free abelian of rank at least 2, satisfy $Ext^1(\mathbb{Z}, \mathbb{Z}[\Gamma]) = 0$ by [13] and Shapiro's Lemma. So the above theorem implies a surjective correspondence $\delta : \Omega_1(\mathbb{Z}) \to \mathcal{SF}_+$. But, by Theorem A, $\mathcal{SF}_1$ is infinite when n admits an odd divisor. Therefore, the minimal level of $\Omega_1(\mathbb{Z})$ is also infinite. One then hopes for a relationship between $\Omega_1(\mathbb{Z})$ and $\Omega_3(\mathbb{Z})$, as, for example, in the case of finite groups of cohomological period 4 where there exists an isomorphism of trees between $\Omega_1(\mathbb{Z})$ and $\Omega_3(\mathbb{Z})$ (c.f. [10] p. 153).

1.2 Statement of results

As already mentioned in the previous section, Theorem A is the main result of this thesis. In order to prove it we use the language of cyclic algebras and fibre squares. Thus in Chapter 2 we first review properties of cyclic algebras over general commutative rings, and then restrict ourselves to the types of rings we are interested in, namely quotients of integral polynomial rings in one variable, by products of cyclotomic polynomials. Two cyclic algebras of particular interest are (here p denotes an odd prime, and $k > 1$ an odd integer):

$$\mathcal{C}(\Lambda_{(p,n)}) := \mathcal{C}_2\left(\mathbb{F}_p[x]/(x^{2n}+1), \gamma, -1\right)$$

$$\mathcal{C}(\Lambda_{(2,k)}) := \mathcal{C}_2\left(\mathbb{F}_2[x]/(x^{k-1}+\ldots+1), \gamma, -1\right).$$

We refer the reader to Chapter 2 for details on the notation. Decomposing $\mathcal{C}(\Lambda_{(2,k)})$ as a product of local rings and matrix algebras yields

Theorem B. *Let D_{2k} be the dihedral group of order $2k$, where $k > 1$ is odd. Then*

$$\mathbb{F}_2[D_{2k}] \cong \mathbb{F}_2[x]/(x^2-1) \times \prod_{i=1}^{l} M_2\left(\mathbb{F}_{2^{d_i}}\right).$$

It should be noted that Theorem B can be proved using fairly basic concepts in modular representation theory or character theory (c.f. e.g. [1]). However, we give a different proof, as we compute an explicit isomorphism. Identifying $\mathcal{C}(\Lambda_{(p,n)})$ as a quaternion algebra we are then able to give its decomposition as a product of matrix algebras thus:

Theorem C.
$$\mathcal{C}(\Lambda_{(p,n)}) \cong \prod_{i=1}^{m} M_2\left(\mathbb{F}_p[x]/q_i^{d_i}(x)\right)$$
for distinct monic irreducible polynomials $q_i(x) \in \mathbb{F}_p[x]$ *and natural numbers* d_i.

In Chapter 3 we first review functors from the module category of one ring to that of another. Here we are particularly interested in functors which preserve *stably-free cancellation*, i.e. the property of having no non-trivial stably-free modules. As already mentioned in the previous section, the work of Sheshadri [27], Bass [2], Quillen [25] and others, gives a good source of group algebras of type $\mathcal{F}$ groups with stably-free cancellation. Via the functors developed in this chapter we shall transfer the property of having stably-free cancellation to the algebras we are interested in:

Theorem D. $\mathbb{F}_2[D_{2n} \times G]$ *has stably-free cancellation when* G *is of type* $\mathcal{F}$.

Theorem E. $\mathcal{C}(\Lambda_{(p,n)})[G]$ *has stably-free cancellation when* G *is of type* $\mathcal{F}$.

Another necessary step in our proof of the main theorem is to show that both $\mathbb{F}_2[D_{2n} \times G]$ and $\mathcal{C}(\Lambda_{(p,n)})[G]$ are *weakly Euclidean.* Thus in Chapter 4 we first review in depth what it means to be weakly Euclidean, in essence a condition which makes the the general linear group of a ring particularly simple. Then, in a similar fashion to Chapter 3, we transfer the property from rings for which it is already known to hold to the ones we are interested in. Thus we review recognition criteria by Cohn and Johnson, but also develop new criteria in the case of matrix algebras and algebras over finite fields. The main results of Chapter 4 are:

Theorem F. $\mathbb{F}_2[D_{2n} \times G]$ *is weakly Euclidean whenever* G *is a free group.*

Theorem G. *$\mathcal{C}(\Lambda_{(p,n)})[G]$ is weakly Euclidean whenever G is a free group.*

In Chapter 5 we first review how fibre squares are constructed and how they cohere with the cyclic algebra construction. Next we lay the foundations for the proof of Theorem A by constructing several fibre squares which connect the algebra $\mathbb{Z}[Q_{8n} \times G]$ to the above mentioned algebras $\mathbb{F}_2[D_{2n} \times G]$ and $\mathcal{C}(\Lambda_{(p,n)})[G]$. Finally, we give a brief recapitulation of Milnor's method of patching projective modules over certain fibre squares. We end the chapter by showing that all fibre squares constructed here satisfy the necessary conditions to apply Milnor's method.

Finally, we prove Theorem A in Chapter 6. In order to do so we need to construct, and lift stably-free modules. For this we employ a method, involving Milnor patching, based on an approach by Johnson. We can then apply this method to the fibre squares constructed in Chapter 5. This firstly results in (here $\mathcal{SF}_1$ stands for the isomorphism classes of rank one stably free modules of a ring):

Theorem H. *Let G be a group of type $\mathcal{F}$. Moreover, let Q_{4n}, D_{2n} denote the quaternionic and dihedral groups, of order $4n$, $2n$, respectively and $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[G]$ be the algebra introduced in Proposition 5.2.3, Chapter 5. There exists a surjective correspondence*

$$\mathcal{SF}_1\left(\mathbb{Z}[Q_{4n} \times G]\right) \to \mathcal{SF}_1\left(\mathbb{Z}[D_{2n} \times G]\right) \times \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[G]\right).$$

We then restrict ourselves to the setting of Theorem A, by choosing $n = 2^s k$ for an odd number $k \geq 3$. We prove (for a definition of two unique product groups, see Chapter 6.1):

Theorem I. *Let G be a two unique product group, p an odd prime and $s \geq 1$ an integer. There exists an infinite set, $\{\mathcal{S}_m\}_{m=1}^{\infty}$, of isomorphically* *distinct stably-free modules of rank 1 over the the group-algebra* $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1), \gamma, -1\right)$ *of Proposition 5.2.5, Chapter 5.*

We prove Theorem A by constructing a chain of surjective correspondences

$$\mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[C_\infty]\right) \to \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1), \gamma, -1\right)[C_\infty]\right).$$

The result then follows by Theorem H and Theorem I.

Chapter 2

Cyclic algebras and related concepts

2.1 Basic definitions

All group algebras, as well as their building blocks, considered in this thesis are constructed as *cyclic algebras*. We begin by defining the concept (for a comparison, see [10] p. 43). Given

- *i)* a commutative ring Λ,
- *ii)* an automorphism $s : \Lambda \to \Lambda$ such that $s^n = Id$ for some natural number $n \geq 2$,
- *iii)* an element a in Λ such that $s(a) = a$;

we define the *cyclic algebra* $\mathcal{C}_n(\Lambda, s, a)$ to be the free two-sided Λ-module of rank n, with basis $\mathcal{B} = \{1, y, \ldots, y^{n-1}\}$ (here we make the identification $y^0 = 1$), satisfying the following relation:

$$y^i \lambda = s^i(\lambda) y \quad \text{for } 0 \leq i \leq n-1.$$

We note that the set of all elements in Λ which are fixed under s, $\Lambda^s := \{\lambda \in \Lambda : s(\lambda) = \lambda\}$, forms a subring of Λ, called the *fixed point ring*. Moreover, if we introduce the following rule of multiplication:

$$\begin{aligned} y.y^i &= y^{i+1} \quad \text{for } 0 \le i \le n-2 \\ y^n &= a.1, \end{aligned}$$

$\mathcal{C}_n(\Lambda, s, a)$ may equivalently be regarded as an algebra over the fixed point ring. Given two cyclic algebras of the same rank, say $\mathcal{C}_n(\Lambda_1, s_1, a_1)$, $\mathcal{C}_n(\Lambda_2, s_2, a_2)$, there is an isomorphism of rings $\Lambda_1^{s_1} \times \Lambda_2^{s_2} \cong (\Lambda_1 \times \Lambda_2)^{s_1 \times s_2}$, since the involutions act componentwise. Thus, there is a natural algebra isomorphism of crossproducts of cyclic algebras and cyclic algebras of crossproducts, given by $(\lambda_1 y_1^i, \lambda_2 y_2^i) \mapsto (\lambda_1, \lambda_2) y^i$, i. e.

Proposition (c. f. e. g. [10] p. 44) 2.1.1.

$$\mathcal{C}_n(\Lambda_1 \times \Lambda_2, s_1 \times s_2, a_1 \times a_2) \cong \mathcal{C}_n(\Lambda_1, s_1, a_1) \times \mathcal{C}_n(\Lambda_2, s_2, a_2),$$

as $\Lambda_1^{s_1} \times \Lambda_2^{s_2}$*-algebras.*

Given two cyclic algebras of rank n, say $\mathcal{C}_n(\Lambda_1, s_1, a_1)$, $\mathcal{C}_n(\Lambda_2, s_2, a_2)$ and a ring-homomorphism

$$p : \Lambda_1 \to \Lambda_2,$$

such that $p \circ s_1 = s_2 \circ p$, and $p(a_1) = a_2$, we say p is a *cyclic ring-homomorphism.*

Proposition 2.1.2. *Cyclic ring-homomorphisms induce ring-homomorphisms of the associated cyclic algebras. Moreover, they preserve exact sequences.*

Proof. Given two cyclic algebras of rank n, say $\mathcal{C}_n(\Lambda_1, s_1, a_1)$, $\mathcal{C}_n(\Lambda_2, s_2, a_2)$, together with a cyclic ring-homomorphism

$$p : \Lambda_1 \to \Lambda_2,$$

define

$$\pi : \mathcal{C}_n\left(\Lambda_1, s_1, a_1\right) \to \mathcal{C}_n\left(\Lambda_2, s_2, a_2\right),$$

by

$$\sum_{i=1}^{n} y_1^i \lambda_i \mapsto \sum_{i=1}^{n} y_2^i p(\lambda_i), \tag{2.1}$$

for $\lambda_i \in \Lambda_1$. By (2.1) it is clear that π preserves sums, and $\pi(\epsilon) = \epsilon$, for $\epsilon = 0, 1$. Moreover, the rules $p \circ s_1 = s_2 \circ p$, $p(a_1) = a_2$, ensure that π preserves products, and thus it is indeed a ring-homomorphism. Furthermore, let

$$\Lambda_1 \xrightarrow{p} \Lambda_2 \xrightarrow{p'} \Lambda_3\ , \tag{2.2}$$

be a sequence of cyclic ring-homomorphisms which is exact at Λ_2. We remind ourselves that $\mathcal{C}_n\left(\Lambda_i, s_i, a_i\right)$ is free as a Λ_i-module, for $i = 1, 2, 3$, respectively. Therefore, since (2.2) is exact at Λ_2, by the construction defined in (2.1),

$$\mathcal{C}_n\left(\Lambda_1, s_1, a_1\right) \xrightarrow{\pi} \mathcal{C}_n\left(\Lambda_2, s_2, a_2\right) \xrightarrow{\pi'} \mathcal{C}_n\left(\Lambda_3, s_3, a_3\right) \tag{2.3}$$

is exact each component y_2^i, for $i = 0, \ldots n-1$, i. e. it is exact at $\mathcal{C}_n\left(\Lambda_2, s_2, a_2\right)$. For a more general proof of this statement see [4], p. 147. □

Two basic examples of cyclic algebras are $\mathbb{Z}[D_{2n}]$, the integral group-algebra of the dihedral group

$$D_{2n} = \langle\, x, y \quad | \quad x^n = y^2 = 1, \quad yx = x^{-1}y\,\rangle, \tag{2.4}$$

and $\mathbb{Z}[Q_{4n}]$, the integral group algebra of the quaternionic group

$$Q_{4n} = \langle\, x, y \quad | \quad x^n = y^2, \quad yx = x^{-1}y\,\rangle. \tag{2.5}$$

Indeed, let C_n denote the cyclic group of order n ($C_n = \langle x | x^n = 1\rangle$). Consider its integral group algebra $\mathbb{Z}[C_n]$ together with the involution γ, given by

$x \mapsto x^{-1}$; then

$$\mathbb{Z}[D_{2n}] = \mathcal{C}_2 \left(\mathbb{Z}[C_n], \gamma, 1\right). \tag{2.6}$$

If we consider the cyclic group of order $2n$, C_{2n}, again with the involution γ, then $x^n = x^{-n}$ is fixed, and

$$\mathbb{Z}[Q_{4n}] = \mathcal{C}_2 \left(\mathbb{Z}[C_{2n}], \gamma, x^n\right). \tag{2.7}$$

We now specify the rings and the respective automorphisms over which we take cyclic algebras. Let ζ_n denote the n-th primitive root of unity. It is well known that its minimal polynomial is given by

$$\Phi_n(x) = \prod_{\substack{(j,n)=1 \\ j<n}} (x - \zeta_n^j).$$

Furthermore, $\Phi_n(x)$ has integral coefficients, that is $\Phi_n(x) \in \mathbb{Z}[x]$. We denote by $\bar{\zeta_n}$ the complex conjugate of ζ_n. It is not hard to see that $\bar{\zeta_n} = \zeta_n^{n-1}$. Moreover, since $\Phi_n(x)$ has real coefficients $\Phi_n(\bar{\zeta_n}) = 0$. Thus, complex conjugation induces a ring automorphism on $\mathbb{Z}[\zeta_n] = \mathbb{Z}[x]/\Phi_n(x)$. We generalise this concept. Given a sequence $\mathcal{D} = (d_1, \cdots, d_k)$ of distinct integers $d_i \geq 1$ we denote by $\Phi_{\mathcal{D}}(x)$ the polynomial $\Phi_{\mathcal{D}}(x) = \Phi_{d_1}(x) \cdots \Phi_{d_k}(x) \in \mathbb{Z}[x]$. Furthermore, let R be a commutative ring. Mostly in the scope of this thesis R will either be the integers or $\mathbb{F}_p$, i. e. the finite field with p elements for some prime p. Consider the ring $R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$ (we shall always use the tensor symbol without subscript, when we take tensor products over $\mathbb{Z}$):

Proposition 2.1.3. *Given a sequence $\mathcal{D} = (d_1, \cdots, d_k)$ of distinct integers $d_i \geq 1$, let $n = \prod_{i=1}^k d_i$. The involution γ, given by $x \mapsto x^{n-1}$, on $R \otimes \mathbb{Z}[x]/(x^n - 1)$ induces an involution $\gamma_{\mathcal{D}}$ on $R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$. Moreover, given a sequence $\mathcal{D}' \subseteq \mathcal{D}$,*

$$p \circ \gamma_{\mathcal{D}} = \gamma_{\mathcal{D}'} \circ p,$$

where p denotes the canonical projection

$$p : R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x) \to R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}'}(x).$$

Proof. We prove the result for $R = \mathbb{Z}$, the general result then follows by tensoring with an arbitrary commutative ring R. It is well known, that

$$x^n - 1 = \prod_{d|n} \Phi_d(x).$$

This implies that $\mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$ is naturally a quotient ring of $\mathbb{Z}[x]/(x^n - 1)$. This allows us to construct $\gamma_{\mathcal{D}}$ as follows: Let $f(x) \in \mathbb{Z}[x]/(x^n - 1)$, and $f(x) + (\Phi_{\mathcal{D}})$ its class modulo $(\Phi_{\mathcal{D}})$. Define

$$f(x) + (\Phi_{\mathcal{D}}) \mapsto \gamma\left(f(x)\right) + (\Phi_{\mathcal{D}})\,.$$

Now, this map is well-defined, because $\gamma\left(\Phi_{\mathcal{D}}(x)\right) = \pm x^{n-d}\Phi_{\mathcal{D}}(x) \in (\Phi_{\mathcal{D}})$, where $d = deg\left(\Phi_{\mathcal{D}}(x)\right)$. Moreover,

$$p \circ \gamma_{\mathcal{D}}\left(f(x) + (\Phi_{\mathcal{D}})\right) = p\left(\gamma\left(f(x)\right) + (\Phi_{\mathcal{D}})\right) = \gamma\left(f(x)\right) + (\Phi_{\mathcal{D}'})\,.$$

Conversely,

$$\gamma_{\mathcal{D}'} \circ p\left(f(x) + (\Phi_{\mathcal{D}})\right) = \gamma_{\mathcal{D}'}\left(f(x) + (\Phi_{\mathcal{D}'})\right) = \gamma\left(f(x)\right) + (\Phi_{\mathcal{D}'})\,.$$

Thus $p \circ \gamma_{\mathcal{D}} = \gamma_{\mathcal{D}'} \circ p$, as claimed. □

Remark. Note that $\gamma_{\mathcal{D}}$ is an automorphism, since $R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$ is commutative. Furthermore, $\gamma_{\mathcal{D}}^2 = Id$, since $\gamma^2 = Id$. In most cases when we actually have to work out the action of $\gamma_{\mathcal{D}}$, it will be straightforward, namely it will easily be recognisable as the usual $x \mapsto x^{-1}$, where x^{-1} means the class of x^{n-1} modulo $(\Phi_{\mathcal{D}})$. Thus we shall usually drop the subscript $\mathcal{D}$, and just write γ, unless specifically needed. Finally, note that Proposition 2.1.3

also ensures that canonical projections onto rings of the form $R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$ are cyclic ring-homomorphisms with respect to γ, and thus induce homomorphisms of cyclic algebras.

In the remainder of this thesis we will only deal with cyclic algebras over rings of the form $R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$ with the associated involution γ, i. e. cyclic algebras of the form $\mathcal{C}_2(R \otimes \mathbb{Z}[x]/\Phi_{\mathcal{D}}(x), \gamma, a)$, where a is a non-zero element fixed under γ. For example, as we have seen in equations (2.6), (2.7), $\mathbb{Z}[Q_{4n}]$, and $\mathbb{Z}[D_{2n}]$ are of this form for $R = \mathbb{Z}$, but now we note that these equations hold for any commutative ring, e. g. for $R = \mathbb{F}_2$

$$\mathbb{F}_2[D_{2n}] = \mathcal{C}_2\left(\mathbb{F}_2[C_n], \gamma, 1\right). \tag{2.8}$$

We are particularly interested in the following two rings:

$$\Lambda_{(p,n)} := \mathbb{F}_p[x]/(x^{2n}+1), \tag{2.9}$$

$$\Lambda_{(2,k)} := \mathbb{F}_2[x]/(x^{k-1}+\ldots+1), \tag{2.10}$$

for an odd prime p, a natural number n and an odd number $k \geq 3$. The fixed point rings are denoted as usual by $\Lambda^{\gamma}_{(p,n)}$, and $\Lambda^{\gamma}_{(2,k)}$ respectively. Over these, we have the cyclic algebras:

$$\mathcal{C}(\Lambda_{(p,n)}) := \mathcal{C}_2\left(\mathbb{F}_p[x]/(x^{2n}+1), \gamma, -1\right) \tag{2.11}$$

$$\mathcal{C}(\Lambda_{(2,k)}) := \mathcal{C}_2\left(\mathbb{F}_2[x]/(x^{k-1}+\ldots+1), \gamma, 1\right). \tag{2.12}$$

2.2 The fixed point rings $\Lambda^{\gamma}_{(p,n)}$ and $\Lambda^{\gamma}_{(2,k)}$

As explained in Section 2.1, $\mathcal{C}(\Lambda_{(p,n)})$ and $\mathcal{C}(\Lambda_{(2,k)})$ are algebras over the fixed point rings $\Lambda^{\gamma}_{(p,n)}$ and $\Lambda^{\gamma}_{(2,k)}$, respectively. We shall now examine $\Lambda^{\gamma}_{(p,n)}$ and $\Lambda^{\gamma}_{(2,k)}$ in detail.

Proposition 2.2.1. *Let R be a commutative ring. In the ring $R\otimes\mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$ let k denote a natural number, such that $k \leq deg\,(\Phi_{\mathcal{D}}(x))$. Then any element of the kind $x^k + x^{-k}$ can be expressed as a linear combination*

$$x^k + x^{-k} = \theta^k + \sum_{i=0}^{k-1} a_i\theta^i,$$

where $\theta := x + x^{-1}$, and $a_i \in R$.

Proof. We prove the result for the ring $\mathbb{Z}[x]/\Phi_{\mathcal{D}}(x)$. The general result then follows by tensoring with an arbitrary commutative ring R. Note that the statement holds trivially for $k = 1$, but also in the case $k = 2$ as $\theta^2 = x^2 + x^{-2} + 2$. Now assume, inductively, that the statement holds for $k - 1$, and note

$$\theta^k = (x + x^{-1})^k = \sum_{i=0}^{k} \binom{k}{i} x^{2i-k} = x^k + x^{-k} + X, \tag{2.13}$$

$$\text{where } X = \begin{cases} \binom{k}{k/2} + \sum_{i=1}^{k/2-1}\binom{k}{i}\left(x^{k-2i} + x^{-(k-2i)}\right) & \text{if } k \text{ is even,} \\ \sum_{i=1}^{\lfloor k/2 \rfloor}\binom{k}{i}\left(x^{k-2i} + x^{-(k-2i)}\right) & \text{if } k \text{ is odd.} \end{cases}$$

Note, by the inductive hypothesis, X is a linear combination in $span_{\mathbb{Z}}\{1, \theta, \ldots \theta^{k-1}\}$, but then by (2.13) $x^k + x^{-k}$ is in the desired form. □

Proposition 2.2.2. *Let θ_p denote the element $x + x^{-1}$ in the fixed point ring $\Lambda^{\gamma}_{(p,n)}$. Then*

$$\mathcal{B}'_{(p,n)} = \{1, (x + x^{-1}), (x^2 + x^{-2}), \cdots, (x^{n-1} + x^{-(n-1)})\},$$

$$\mathcal{B}_{(p,n)} = \{1, \theta_p, \theta_p^2, \ldots, \theta_p^{n-1}\},$$

are both bases for $\Lambda^{\gamma}_{(p,n)}$ as a $\mathbb{F}_p$-algebra.

Proof. We know that the action of γ as defined above takes x to x^{-1}, so in particular on a typical element λ in $\Lambda_{(p,n)}$, say $\lambda = \sum_{k=-n+1}^{n} a_k x^k$, where

$a_k \in \mathbb{F}_p$, γ acts as follows:

$$\sum_{k=-n+1}^{n} a_k x^k \mapsto -a_n x^n + \sum_{k=-n+1}^{n-1} a_{-k} x^k.$$

From this it follows that a general element fixed under γ is of the type

$$a_0 + \sum_{k=1}^{n-1} a_k (x^k + x^{-k})$$

where each a_k is an element of $\mathbb{F}_p$. In particular, $\mathcal{B}'_{(p,n)}$ is a basis for $\Lambda^{\gamma}_{(p,n)}$ over $\mathbb{F}_p$. Thus, applying Proposition 2.2.1, we see that in fact $\mathcal{B}_{(p,n)}$ is also a basis. □

The analogue of this result for $\Lambda^{\gamma}_{(2,k)}$ is

Proposition 2.2.3. *Let θ_2 denote the element $x + x^{-1}$ in the fixed point ring $\Lambda^{\gamma}_{(2,k)}$. Then*

$$\mathcal{B}'_{(2,k)} = \{1, (x + x^{-1}), \cdots, (x^{(k-3)/2} + x^{(k+3)/2})\},$$

$$\mathcal{B}_{(2,k)} = \{1, \theta_2, \theta_2^2, \ldots, \theta_2^{(k-3)/2}\},$$

are both bases for $\Lambda^{\gamma}_{(2,k)}$ as a $\mathbb{F}_2$-algebra.

Proof. The result is proven by the same method used to prove Proposition 2.2.2. □

Remark. We know that $1 + x + \ldots + x^{k-1} = 0$ in $\Lambda_{(2,k)}$, or equivalently in $\Lambda^{\gamma}_{(2,k)}$. Thus

$$x^{(k-1)/2} + x^{(k+1)/2} = 1 + \sum_{i=1}^{(k-3)/2} \left(x^i + x^{-i}\right).$$

Consequently, $x^{(k-1)/2} + x^{(k+1)/2}$ does not appear as a basis element in $\mathcal{B}'_{(2,k)}$. In fact, when using the basis $\mathcal{B}'_{(2,k)}$, we shall always mean the element

$1 + \sum_{i=2}^{(k-3)/2} (x^i + x^{-i})$, when we write $x^{(k-1)/2} + x^{(k+1)/2}$. This is solely for notational simplicity, as after all they denote the same element.

We may now give a complete characterisation of both $\Lambda^\gamma_{(p,n)}$, and $\Lambda^\gamma_{(2,k)}$.

Theorem 2.2.4. *$\Lambda^\gamma_{(p,n)}$ is isomorphic to a direct product of finite local rings or fields of characteristic p.*

Proof. Since $\mathcal{B}_{(p,n)}$ is a basis, every element in $\Lambda^\gamma_{(p,n)}$ can be expressed as a sum $\sum_{i=0}^{n-1} a_i\theta_p^i$, where $a_i \in \mathbb{F}_p$. Moreover, by Proposition 2.2.1 there exists a polynomial $P(x) \in \mathbb{F}_p[x]$ of degree n, such that $x^n + x^{-n} = P(\theta_p)$. But as $x^{2n} = -1$, we know that $x^{-n} = -x^n$, and so $P(\theta_p) = 0$, i. e. $\Lambda^\gamma_{(p,n)}$ can be identified as

$$\Lambda^\gamma_{(p,n)} = \mathbb{F}_p[\theta_p]/P(\theta_p).$$

By taking a formal isomorphism $\theta_p \mapsto x$, we may write

$$\Lambda^\gamma_{(p,n)} \cong \mathbb{F}_p[x]/P(x).$$

Then, since is $\mathbb{F}_p[x]$ a unique factorisation domain, let

$$P(x) = c\prod_{i=1}^{m} q_i^{d_i}(x)$$

be the unique factorisation of $P(x)$, into a constant c times powers $d_i \in \mathbb{N}$ of distinct monic irreducible polynomials $q_i(x) \in \mathbb{F}_p[x]$. As $\mathbb{F}_p[x]$ is also a Euclidean domain we may apply the Chinese Remainder Theorem which gives

$$\Lambda^\gamma_{(p,n)} \cong \prod_{i=1}^{m} \mathbb{F}_p[x]/q_i^{d_i}(x),$$

where each factor $\mathbb{F}_p[x]/q_i^{d_i}(x)$ is a finite local ring or field of characteristic p whenever $d_i = 1$. □

In the case of $\Lambda^\gamma_{(2,k)}$ we have a stronger result, since 2 and k are coprime

(k is defined to be odd). We remind ourselves of some fundamental results on semisimple algebras. Recall that for a field k a finitely generated k-algebra $\mathcal{A}$ is said to be semisimple if and only if its unique two sided maximal nilpotent ideal, denoted by $rad(\mathcal{A})$, is equal to zero. Furthermore, we remind ourselves that all finitely generated, semisimple k-algebras are characterised as follows (c. f. e. g. [10] p. 20-21):

Theorem (Wedderburn) 2.2.5. *Let $\mathcal{A}$ be a semisimple, finitely generated k-algebra. Then there exists an isomorphism of k-algebras*

$$\mathcal{A} \cong M_{n_1}(D_1) \times \ldots M_{n_m}(D_m),$$

for some natural numbers m, n_i and division algebras D_i over k, determined uniquely up to order and isomorphism.

Finally, we shall also need (c. f. e. g. [10] p. 41):

Theorem (Maschke) 2.2.6. *Let G be a finite group and k a field with characteristic coprime to the order of G. Then the group-algebra $k[G]$ is semisimple.*

Theorem 2.2.7. *$\Lambda^{\gamma}_{(2,k)}$ is isomorphic to a direct product of finite fields of characteristic 2.*

Proof. We note that $x^k - 1 = (x-1)(x^{k-1} + \ldots + 1)$, and in $\mathbb{F}_2[x]$, $x - 1$, and $x^{k-1} + \ldots + 1$ are coprime. Therefore,

$$\begin{aligned} \mathbb{F}_2[x]/(x^k - 1) &\cong \mathbb{F}_2[x]/(x-1) \times \mathbb{F}_2[x]/(x^{k-1} + \ldots + 1) \qquad (2.14) \\ &\cong \mathbb{F}_2 \times \Lambda_{(2,k)}. \end{aligned}$$

We may consider $\mathbb{F}_2[x]/(x^k - 1)$ as a group-algebra, since it is isomorphic to $\mathbb{F}_2[C_k]$. Theorem 2.2.6 implies that $\mathbb{F}_2[x]/(x^k - 1)$ is semisimple, and so, by definition $rad\left(\mathbb{F}_2[x]/(x^k-1)\right) = \{0\}$. Consequently, by (2.14) $rad(\Lambda_{(2,k)}) = \{0\}$. But then, $\Lambda^{\gamma}_{(2,k)}$, being a subring, has the same property, i.e. $rad(\Lambda^{\gamma}_{(2,k)}) =$

$\{0\}$. Thus it is semisimple by definition. Since it is commutative, Theorem 2.2.5 implies that it is a direct product of finite fields of characteristic 2. □

Remark. It is no coincidence that we have used the notation $rad(\mathcal{A})$, which is normally used to denote the Jacobson radical. In fact, in an Artinian ring (in our case, $\mathcal{A}$ is Artinian, since it is finitely generated) the Jacobson radical is nilpotent, but it also contains all nilpotent ideals ((c. f. e. g. [4] p. 181). Therefore, in such rings, the Jacobson radical is precisely the unique two-sided maximal nilpotent ideal.

2.3 $\mathcal{C}(\Lambda_{(2,k)})$ and its applications

Having classified $\Lambda^{\gamma}_{(2,k)}$ in the previous section, we now consider the cyclic algebra $\mathcal{C}(\Lambda_{(2,k)})$. We first show that it is semisimple. Then, we shall use this result to decompose $\mathbb{F}_2[D_{2k}]$, for an odd number $k \geq 3$, into a product of local rings and simple algebras. Finally, we use this decomposition to express $\mathbb{F}_2[D_{2k} \times G]$, for an arbitrary group G, in a more manageable form.

Theorem 2.3.1.

$$\mathcal{C}(\Lambda_{(2,k)}) \cong \prod_{i=1}^{l} M_2\left(\mathbb{F}_{2^{d_i}}\right),$$

where $\mathbb{F}_{2^{d_i}}$ denotes the unique finite field with 2^{d_i} elements, for some $d_i \in \mathbb{N}$.

We will use the following two propositions to prove Theorem 2.3.1:

Proposition 2.3.2.

$$\mathcal{C}(\Lambda_{(2,k)}) = \Lambda^{\gamma}_{(2,k)} \dotplus \Lambda^{\gamma}_{(2,k)}x \dotplus \Lambda^{\gamma}_{(2,k)}y \dotplus \Lambda^{\gamma}_{(2,k)}xy,$$

with the following rules of multiplication:

i) $yx = \gamma(x)y = x^{-1}y = (x + x^{-1})y + xy;$
ii) $x^2 = 1 + (x + x^{-1})x;$
iii) $y^2 = 1.$

Proof. By Proposition 2.2.3

$$\mathcal{B}'_{(2,k)} = \{1, (x + x^{-1}), \cdots, (x^{(k-3)/2} + x^{(k+3)/2})\},$$

is a $\mathbb{F}_2$-basis for $\Lambda^{\gamma}_{(2,k)}$. First, reminding ourselves of the remark on page 22, we show by induction that $x^l \in \Lambda^{\gamma}_{(2,k)} + \Lambda^{\gamma}_{(2,k)}x$ for all l in $\{0, 1, \dots, k-2\}$. Clearly 1, x are in $\Lambda^{\gamma}_{(2,k)} + \Lambda^{\gamma}_{(2,k)}x$, thus also $x^{-1} = (x + x^{-1}) + x$. Now assume x^i, x^{k-i} are in $\Lambda^{\gamma}_{(2,k)} + \Lambda^{\gamma}_{(2,k)}x$ for all $i \leq r < (k-1)/2$, and note that $x^{r+1} = (x^r + x^{k-r})x + x^{k-(r-1)}$. This implies that x^{r+1} as well as $x^{k-(r+1)} = (x^{r+1} + x^{k-(r+1)}) + x^{r+1}$ are in $\Lambda^{\gamma}_{(2,k)} + \Lambda^{\gamma}_{(2,k)}x$, which proves the claim. But then as

$$\mathcal{B} = \{1, x, \dots, x^{k-2}\}$$

is a basis for $\Lambda_{(2,k)}$ as an $\mathbb{F}_2$-algebra, we have shown that $\Lambda^{\gamma}_{(2,k)} + \Lambda^{\gamma}_{(2,k)}x$ spans $\Lambda_{(2,k)}$. Also, since $\mathcal{B}_{(2,k)} \cap \mathcal{B}_{(2,k)}x = \{0\}$, we have $\Lambda^{\gamma}_{(2,k)} \cap \Lambda^{\gamma}_{(2,k)}x = \{0\}$. Therefore, $\Lambda_{(2,k)} = \Lambda^{\gamma}_{((2,k)} \dot{+} \Lambda^{\gamma}_{(2,k)}x$, and the action of γ on $\Lambda_{(2,k)}$ is completely described by its action on the element x. It follows that $\mathcal{C}(\Lambda_{(2,k)})$ can indeed be expressed in the form:

$$\mathcal{C}(\Lambda_{(2,k)}) = \Lambda^{\gamma}_{(2,k)} \dot{+} \Lambda^{\gamma}_{(2,k)}x \dot{+} \Lambda^{\gamma}_{(2,k)}y \dot{+} \Lambda^{\gamma}_{(2,k)}xy.$$

Finally, note that the relations *i)*-*iii)* are self-evident. □

Proposition 2.3.3. *$x + x^{-1}$ is a unit in $\Lambda^{\gamma}_{(2,k)}$.*

Proof. The inverse of $x + x^{-1}$ is described as follows: We write

$$\begin{aligned} \mu(1) &:= x + x^{-1}, \\ \mu(r) &:= x^r + x^{k-r} \end{aligned}$$

for $2 \leq r \leq (k-1)/2$. And

$$\begin{aligned} A &:= \sum_{4r+2 \leq (k-1)/2} \mu(4r+2), \\ B &:= \sum_{4r+3 \leq (k-1)/2} \mu(4r+3), \\ C &:= \sum_{4r+1 \leq (k-1)/2} \mu(4r+1). \end{aligned}$$

Then if $k \equiv 1 \ mod\ (4)$, $(\mu(1))^{-1} = A + B$, and if $k \equiv 3 \ mod\ (4)$, $(\mu(1))^{-1} = A + C$. □

We prove Theorem 2.3.1:

Proof. We use the description of $\mathcal{C}(\Lambda_{(2,k)})$ given by Proposition 2.3.2, and define a map

$$\varphi : \mathcal{C}(\Lambda_{(2,k)}) \to M_2(\Lambda^{\gamma}_{(2,k)})$$

as follows:

$$\begin{aligned} 1 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ x &\mapsto \begin{pmatrix} 1 & x^{(k-1)/2} + x^{(k+1)/2} \\ x^{(k-1)/2} + x^{(k+1)/2} & 1 + x + x^{-1} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} y &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ xy &\mapsto \varphi(x)\varphi(y) \end{aligned}$$

Now it can easily be checked that

$i)$ $\varphi(yx) = \varphi(y)\varphi(x)$;
$ii)$ $\varphi(x^2) = (\varphi(x))^2$;
$iii)$ $(\varphi(y))^2 = Id.$

Whence φ is indeed a ring homomorphism. Now, since both $\mathcal{C}(\Lambda_{(2,k)})$ and $M_2(\Lambda^{\gamma}_{(2,k)})$ have (the same) finite cardinality, bijectivity follows directly from surjectivity. To see that φ is indeed surjective, it suffices to show that $\varphi(1)$, $\varphi(x)$, $\varphi(y)$, $\varphi(xy)$ span $M_2(R^{\gamma})$. We note the following identities

$$\begin{aligned} \varphi\left((1+x+x^{-1})+x+(x^{(k-1)/2}+x^{(k+1)/2})y\right) &= \begin{pmatrix} x+x^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \\ \varphi\left(1+x+(x^{(k-1)/2}+x^{(k+1)/2})y\right) &= \begin{pmatrix} 0 & 0 \\ 0 & x+x^{-1} \end{pmatrix}, \\ \varphi\left((x^{(k-1)/2}+x^{(k+1)/2})+x+(1+x+x^{-1})y+xy\right) &= \begin{pmatrix} 0 & x+x^{-1} \\ 0 & 0 \end{pmatrix}, \\ \varphi\left((x^{(k-1)/2}+x^{(k+1)/2})+x+y+xy\right) &= \begin{pmatrix} 0 & 0 \\ x+x^{-1} & 0 \end{pmatrix}. \end{aligned}$$

By Proposition 2.3.3, $x+x^{-1}$ is a unit in $\Lambda^{\gamma}_{(2,k)}$. So the identities above imply that $\varphi(1)$, $\varphi(x)$, $\varphi(y)$, $\varphi(xy)$ do, in fact, span $M_2(\Lambda^{\gamma}_{(2,k)})$. Note, by Theorem 2.2.7, $\Lambda^{\gamma}_{(2,k)}$ is isomorphic to a product of finite fields of characteristic 2, say $\Lambda^{\gamma}_{(2,k)} \cong \prod_{i=1}^{l} \mathbb{F}_{2^{d_i}}$. Thus the result follows. □

We turn to $\mathbb{F}_2[D_{2k}]$:

Theorem 2.3.4.

$$\mathbb{F}_2[D_{2k}] \cong \mathbb{F}_2[x]/(x^2-1) \times \prod_{i=1}^{l} M_2\left(\mathbb{F}_{2^{d_i}}\right),$$

for an odd number $k \geq 3$.

Proof. We know by (2.8) that $\mathbb{F}_2[D_{2k}] = \mathcal{C}_2(\mathbb{F}_2[x]/x^k - 1, \gamma, 1)$. Furthermore, (2.14) states

$$\mathbb{F}_2[x]/(x^k - 1) \cong \mathbb{F}_2 \times \Lambda_{(2,k)}.$$

Our usual involution γ splits as $\gamma = Id \times \gamma'$, where γ' is just the restriction of γ to $\Lambda_{(2,k)}$. Indeed, the idempotents for the isomorphism in (2.14) are $i_1 = x^{k-1} + \ldots + 1$, and $i_2 = x^{k-1} + \ldots + x$, but $\gamma(i_1) = i_1$, and $\gamma(i_2) = i_2$. Now γ restricted to $\mathbb{F}_2[x]/(x-1) \cong \mathbb{F}_2$ is just the identity, hence the result. We apply Proposition 2.1.1 to get

$$\mathbb{F}_2[D_{2k}] \cong \mathcal{C}_2(\mathbb{F}_2, Id, 1) \times \mathcal{C}(\Lambda_{(2,k)}).$$

We note $\mathcal{C}_2(\mathbb{F}_2, Id, 1) \cong \mathbb{F}_2[x]/x^2-1$. The result follows by Theorem 2.3.1 □

As a corollary we have

Theorem 2.3.5. *For any group G,*

$$\mathbb{F}_2[D_{2k} \times G] \cong \left(\mathbb{F}_2[x]/(x^2-1)\right)[G] \times \prod_{i=1}^{l} M_2\left(\mathbb{F}_{2^{d_i}}[G]\right).$$

Proof. The result follows by applying the functor $-\otimes \mathbb{Z}[G]$ on both sides of the isomorphism in Theorem 2.3.4. □

2.4 Quaternion algebras and $\mathcal{C}(\Lambda_{(p,n)})$

We start by defining quaternion algebras. Let R be a commutative ring. Furthermore, let α, β be units in R. We call the algebra, with R-basis

$\{1,\, x,\, y,\, xy\}$ and rules of multiplication

$$x^2 = \alpha.1, \quad y^2 = \beta.1, \quad yx = -xy,$$

a quaternion algebra, and denote it by

$$\left(\frac{\alpha, \beta}{R}\right).$$

The basic example of a quaternion algebra is the Hamiltonian integers,

$$\mathbb{H}_{\mathbb{Z}} = \left(\frac{-1, -1}{\mathbb{Z}}\right).$$

Remark. O'Meara defines quaternion algebras over fields, only (c. f. [23] p. 142). The stricter definition bears better-behaved algebras. For example, we will see that quaternion algebras over fields are often division algebras. However, in the context of classifying $\mathcal{C}(\Lambda_{(p,n)})$ it makes sense to relax the definition.

Proposition 2.4.1. *Let* $\Lambda = \left(\frac{\alpha,\beta}{k}\right)$ *be a quaternion algebra over a field* k*. Then the element* $\mathbf{x} = \alpha_0.1 + \alpha_1.x + \alpha_2.y + \alpha_3.xy$*, is a unit in* Λ *if and only if*

$$\alpha_0^2 - \alpha_1^2\alpha - \alpha_2^2\beta + \alpha_3^2\alpha\beta \neq 0,$$

for $\alpha_i \in k$*.*

Proof. See [23] p. 143. □

As an immediate corollary we have

Proposition 2.4.2. *Let* $\Lambda = \left(\frac{\alpha,\beta}{k}\right)$ *be a quaternion algebra over a field* k*.* Λ *is a division algebra if and only if the equation*

$$\alpha_0^2 - \alpha_1^2\alpha - \alpha_2^2\beta + \alpha_3^2\alpha\beta = 0,$$

has no non-zero solution for all $\alpha_i \in k$*.*

As already explained in Section 2.1, the involution γ reduces to complex conjugation on $\mathbb{Q}(\zeta_d) \cong \mathbb{Q}[x]/(\Phi_d)$, and therefore $\mathbb{Q}(\zeta_d)^\gamma = \mathbb{Q}(\zeta_d + \bar{\zeta}_d) = \mathbb{Q}(\zeta_d) \cap \mathbb{R}$. Similarly, $\mathbb{Z}[\zeta_d]^\gamma = \mathbb{Z}[\zeta_d + \bar{\zeta}_d] = \mathbb{Z}[\zeta_d] \cap \mathbb{R}$, which is precisely the ring of integers of $\mathbb{Q}(\zeta_d + \bar{\zeta}_d)$ (see for example [31] p. 15-17 for more details).

Proposition 2.4.3.

$$\mathcal{C}_2\left(\mathbb{Q}(\zeta_d), \gamma, -1\right) \cong \left(\frac{(\zeta_d - \bar{\zeta}_d)^2, -1}{\mathbb{Q}(\zeta_d + \bar{\zeta}_d)}\right),$$

for $d \geq 3$.

Proof. See [10] p. 51. □

Proposition 2.4.4. $\mathcal{C}_2\left(\mathbb{Z}[\zeta_d], \gamma, -1\right)$ *is an integral domain, for* $d \geq 3$.

Proof. Clearly, $\mathcal{C}_2\left(\mathbb{Q}(\zeta_d), \gamma, -1\right) \supseteq \mathcal{C}_2\left(\mathbb{Z}[\zeta_d], \gamma, -1\right)$. But we know by Proposition 2.4.3 that $\mathcal{C}_2\left(\mathbb{Q}(\zeta_d), \gamma, -1\right)$ is a quaternion algebra over the field $\mathbb{Q}(\zeta_d + \bar{\zeta}_d)$. Now as $(\zeta_d - \bar{\zeta}_d)^2$ is a negative real number and $\mathbb{Q}(\zeta_d + \bar{\zeta}_d)$ is a subfield of the real numbers, it follows that the equation

$$\alpha_0^2 - \alpha_1^2(\zeta_- \bar{\zeta}_d)^2 + \alpha_2^2 - \alpha_3^2(\zeta_- \bar{\zeta}_d)^2 = 0,$$

has no non-zero solution for all $\alpha_i \in \mathbb{Q}(\zeta_d + \bar{\zeta}_d)$. Therefore, by Proposition 2.4.2 $\mathcal{C}_2\left(\mathbb{Q}(\zeta_d), \gamma, -1\right)$ is a division algebra. This makes $\mathcal{C}_2\left(\mathbb{Z}[\zeta_d], \gamma, -1\right)$ a subring of a division algebra, and therefore an integral domain. □

We now move on to classify $\mathcal{C}(\Lambda_{(p,n)})$. Our aim is to show that it is a matrix algebra over the fixed point ring $\Lambda_{(p,n)}^\gamma$. Thus we shall need some preliminary results on $\mathbb{F}_p$-algebras, for an odd prime p. The following is well known (c. f. e. g. [23] p. 145, 147, 158):

Proposition 2.4.5. *Let* $\mathbb{F}$ *denote a finite field of characteristic* p*, an odd prime, and* α, β *non-zero elements in* $\mathbb{F}$*. Then*

$$\left(\frac{\alpha, \beta}{\mathbb{F}}\right) \cong \left(\frac{1, -1}{\mathbb{F}}\right) \cong M_2(\mathbb{F}).$$

Now, this result may be generalised to $\mathbb{F}_p$-algebras in the obvious manner: Proposition 2.4.5 holds, in particular, for $\mathbb{F}_p$, the finite field with p elements. Let $\mathcal{A}$ be any $\mathbb{F}_p$-algebra, and α, β non-zero elements in $\mathbb{F}_p$. Then

$$\left(\frac{\alpha, \beta}{\mathcal{A}}\right) \cong \left(\frac{\alpha, \beta}{\mathbb{F}_p}\right) \otimes_{\mathbb{F}_p} \mathcal{A} \cong M_2(\mathbb{F}_p) \otimes \mathcal{A} \cong M_2(\mathcal{A}).$$

Thus we have

Proposition 2.4.6. *Let $\mathcal{A}$ denote an $\mathbb{F}_p$-algebra, an odd prime, and α, β non-zero elements in $\mathbb{F}_p$. Then*

$$\left(\frac{\alpha, \beta}{\mathcal{A}}\right) \cong M_2(\mathcal{A}).$$

Proposition 2.4.6 is general enough for our purposes, but for the sake of completeness we give a criterion for quaternion algebras and matrix algebras over general commutative rings to coincide. This is achieved by generalising O'Meara's proof of Proposition 2.4.5, which hinges on the fact that in a finite field $\mathbb{F}$ there exist elements ξ, η, such that $\alpha = \xi^2 - \beta\eta^2$ (c. f. e. g. [23] p.147).

Proposition 2.4.7. *Let R be a commutative ring, such that 2 is invertible. Furthermore, let α, β be invertible elements in R. If there exist elements $\xi, \eta \in R$, such that $\xi^2 - \beta\eta^2 = \alpha$. Then*

$$\left(\frac{\alpha, \beta}{R}\right) \cong M_2(R).$$

Proof. Note, the map $\varphi : \left(\frac{\alpha,\beta}{R}\right) \to M_2(R)$ defined by

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(x) = \begin{pmatrix} \xi & \eta \\ -\beta\eta & -\xi \end{pmatrix},$$

$$\varphi(y) = \begin{pmatrix} 0 & 1 \\ \beta & 0 \end{pmatrix}, \quad \varphi(xy) = \begin{pmatrix} \beta\eta & \xi \\ \beta\xi & -\beta\eta \end{pmatrix},$$

is an algebra-homomorphism. Moreover, a standard calculation bears that φ is an isomorphism whenever α, β, 2 are invertible. $\square$

Finally, we classify $\mathcal{C}(\Lambda_{(p,n)})$.

Theorem 2.4.8.

$$\mathcal{C}(\Lambda_{(p,n)}) \cong M_2(\Lambda^{\gamma}_{(p,n)}).$$

In particular,

$$\mathcal{C}(\Lambda_{(p,n)}) \cong \prod_{i=1}^{m} M_2\left(\mathbb{F}_p[x]/q_i^{d_i}(x)\right)$$

for distinct monic irreducible polynomials $q_i(x) \in \mathbb{F}_p[x]$, and natural numbers d_i.

Proof. As we know from Proposition 2.2.2

$$\mathcal{B}'_{(p,n)} = \{1, (x + x^{-1}), (x^2 + x^{-2}), \cdots, (x^{n-1} + x^{-(n-1)})\},$$

is a basis for $\Lambda^{\gamma}_{(p,n)}$ as a $\mathbb{F}_p$-algebra. Since 2 is invertible in $\mathbb{F}_p$, we may extend this to

$$\mathcal{B} = \{1, (x + x^{-1}), \cdots, (x^{n-1} + x^{-(n-1)}), x^n, (x - x^{-1}), \cdots, (x^{n-1} - x^{-(n-1)})\},$$

which is a basis for $\Lambda_{(p,n)}$ over $\mathbb{F}_p$, and make the following observation: Defining $\omega := x^n$, we calculate

$$(x^k + x^{-k})\omega = (x^k + x^{-k})x^n = x^{n+k} + x^{n-k} = x^{n-k} - x^{-(n-k)},$$

where $1 \leq k \leq n-1$. So

$$\mathcal{B} = \{1, (x+x^{-1}), \cdots, (x^{n-1}+x^{-(n-1)}), \omega, (x+x^{-1})\omega, \cdots, (x^{n-1}+x^{-(n-1)})\omega\},$$

in the new symbols. This allows us to identify $\Lambda_{(p,n)}$ as $\Lambda_{(p,n)} = \Lambda^{\gamma}_{(p,n)} \dot{+} \Lambda^{\gamma}_{(p,n)}\omega$. So the action of γ on $\Lambda_{(p,n)}$ is entirely described by its action on ω, which is $\gamma(\omega) = \gamma(x^n) = x^{-n} = -x^n = -\omega$. Thus, by definition,

$$\mathcal{C}(\Lambda_{(p,n)}) = \left(\frac{-1,-1}{\Lambda^{\gamma}_{(p,n)}}\right).$$

But $\Lambda^{\gamma}_{(p,n)}$ is a $\mathbb{F}_p$-algebra, so by Proportion 2.4.6

$$\mathcal{C}(\Lambda_{(p,n)}) \cong M_2(\Lambda^{\gamma}_{(p,n)}).$$

Furthermore, by Theorem 2.2.4, we know that

$$\Lambda^{\gamma}_{(p,n)} \cong \prod_{i=1}^{m} \mathbb{F}_p[x]/q_i^{d_i}(x),$$

for distinct monic irreducible polynomials $q_i(x) \in \mathbb{F}_p[x]$, and natural numbers d_i. This proves the claim.

□

The following is an immediate corollary:

Theorem 2.4.9. *For any group G,*

$$\mathcal{C}(\Lambda_{(p,n)})[G] \cong \prod_{i=1}^{m} M_2\left(\mathbb{F}_p[x]/q_i^{d_i}(x)\right)[G].$$

Proof. The result follows by applying the functor $-\otimes\mathbb{Z}[G]$ on both sides of the isomorphism in Theorem 2.4.8. □

Chapter 3

Stably-free cancellation

3.1 Projective modules

Given a ring R, we let $\mathfrak{M}(R)$ be the category whose objects are finitely generated (right) R-modules together with R-linear maps as morphisms. We remind ourselves that an object $F \in \mathfrak{M}(R)$ is said to be *free* when it has an N-element basis, for some natural number N or equivalently when $M \cong R^N$, that is the N-ary Cartesian product of R (c.f. e.g. [18] p. 20). We denote the subcategory of $\mathfrak{M}(R)$, consisting of free modules, by $\mathfrak{F}(R)$. Moreover, an object $P \in \mathfrak{M}(R)$ is said to be *projective*, if there exists a $Q \in \mathfrak{M}(R)$, such that

$$P \oplus Q \cong R^N,$$

for some natural number N. We denote the subcategory of $\mathfrak{M}(R)$, consisting of projective modules, by $\mathfrak{P}(R)$. We are interested in a particular type of projective module, the so called *stably-free module*, that is an an object $S \in \mathfrak{M}(R)$, such that

$$S \oplus R^{N_1} \cong R^{N_2}, \tag{3.1}$$

for some natural numbers N_1, N_2. The subcategory of $\mathfrak{M}(R)$, consisting of stably-free modules, is denoted by $\mathfrak{SF}(R)$. Clearly, we have

$$\mathfrak{F}(R) \subseteq \mathfrak{SF}(R) \subseteq \mathfrak{P}(R) \subseteq \mathfrak{M}(R).$$

Remark. The notions projective, free and stably-free can be defined irrespective of whether a module is finitely generated (c. f. e. g. [18] p. 17), but as in the realm of this work we only consider finitely generated modules, we have made this choice at a categorical level. It should be added, the properties which we shall prove for finitely generated projective modules are not automatic for their infinitely generated equivalents.

We define the *rank* of a free and a stably-free module, respectively. The rank of a free module is defined to be the number of its basis elements. This number is not necessarily unique. We say a ring R has *invariant basis number*, or IBN, if every free module has a unique rank. Most known rings possess the IBN condition. Furthermore, given $S \in \mathfrak{SF}(R)$, say S satisfies (3.1); then we may define the rank of S by

$$rk(S) = N_2 - N_1.$$

If R has IBN, $rk(S)$ is a unique integer. But we need a stronger condition to ensure that $rk(S)$ is always positive. We say a ring R is weakly finite whenever for any $\alpha, \beta \in M_n(R)$,

$$\alpha\beta = I_n \Longrightarrow \beta\alpha = I_n$$

It is well known that (non-trivial) weakly finite rings have IBN (c.f. e.g. [4] p. 143). Moreover, stably-free modules over a weakly finite ring necessarily have a positive rank (c.f. e.g. [4] p. 143). We note, for any group G, the integral group-algebra $\mathbb{Z}[G]$ is weakly finite. This follows from [22] and the fact that subrings of weakly finite rings are again weakly finite (c.f. e.g. [4] p. 144).

Given a ring R, the set of isomorphism classes of free modules of rank N only contains one element, that of R^N. We write $\mathcal{SF}_N(R)$, for the set of isomorphism classes of stably-free modules of rank N over R, and say R has *stably-free cancellation* whenever $|\mathcal{SF}_N(R)| = 1$, for all $N \in \mathbb{N}$. Moreover, the set of isomorphism classes of projective modules, albeit without a notion of rank, is of course also non-trivial in general. Denote the set of isomorphism classes of projective modules over R by $\mathcal{P}(R)$. We say R is *projective free* whenever any given projective R-module is free. Note, a projective free ring has stably-free cancellation. Thus, given a ring R, the central question of this chapter, as the name suggests, is whether $\mathcal{SF}_N(R)$ is trivial. More precisely: The aim is to show that $\mathcal{C}(\Lambda_{(p,n)})[G]$ and $\mathbb{F}_2[D_{2n} \times G]$, for various infinite groups G, have stably-free cancellation.

3.2 Preserving projective modules

We would like to be able to carry properties, such as stably-free cancellation or being projective free, from one ring over to another. The first step is to construct a functor between the categories of finitely generated modules over the rings in question which preserves projective, stably-free or free modules, respectively. Thus, consider the following:

Proposition (c. f. e. g. [18] p. 162) 3.2.1. *Given two rings R, and S, let $F : \mathfrak{M}(R) \to \mathfrak{M}(S)$, be a functor satisfying:*

i) $F(A \oplus B) = F(A) \oplus F(B)$, *for any* A, $B \in \mathfrak{M}(R)$;

ii) $F(R) \in \mathfrak{P}(S)$.

Then F restricts to a functor $\mathfrak{P}(R) \to \mathfrak{P}(S)$. If, in fact, $F(R) \in \mathfrak{F}(S)$, then F also restricts to functors $\mathfrak{S}\mathfrak{F}(R) \to \mathfrak{S}\mathfrak{F}(S)$ and $\mathfrak{F}(R) \to \mathfrak{F}(S)$, respectively.

Proof. Given $P \in \mathfrak{P}(R)$, there exists $Q \in \mathfrak{P}(R)$, such that $P \oplus Q \cong R^n$ for some natural number n, but then by condition i)

$$F(P) \oplus F(Q) \cong F(R)^n.$$

By condition ii) we have that $F(R)^n \in \mathfrak{P}(S)$. Thus $F(P) \in \mathfrak{P}(S)$. In the special case $F(R) \in \mathfrak{F}(S)$, we have $F(R)^n \in \mathfrak{F}(S)$ which yields the stronger result. □

Remark. Note, any additive functor (for a definition see [21] p. 49) satisfies condition i) above. In fact, additive functors preserve general co-products in abelian categories (c. f. e. g. [5] p. 78).

Proposition 3.2.2. *The following four functors satisfy Proposition 3.2.1.*

i) Given a ring homomorphism $\varphi : R \mapsto R'$, *then there is a functor* $\varphi_* : \mathfrak{M}(R) \to \mathfrak{M}(R')$ *defined as follows: For any object* $M \in \mathfrak{M}(R)$, *and* $m \in M$, *define* $\varphi_*(M)$ *by*

$$m \mapsto m \otimes_\varphi 1_{R'}.$$

For any two objects M,$N \in \mathfrak{M}(R)$, *and* $f \in Hom_R(M, N)$

$$\varphi_*(f) = f \otimes_\varphi Id_{R'}.$$

ii) Let $\times_{i=1}^l R_i$ *be a cartesian product of rings. Then, for* $1 \leq i \leq l$ *the canonical projection* $\pi_i : \times_{i=1}^l R_i \mapsto R_i$ *induces a functor* $\imath_i : \mathfrak{M}(R_i) \mapsto \mathfrak{M}(\times_{i=1}^l R_i)$*: For any object* $M_i \in \mathfrak{M}(R_i)$, $m \in M_i$ *and* $r \in \times_{i=1}^l R_i$ *define an* $\times_{i=1}^l R_i$ *action on* M_i *by*

$$m.r = m.\pi_i(r).$$

We write $\imath_i(M_i)$ *when we interpret* M_i *as a right* R*-module. Then for any two objects* M,$N \in \mathfrak{M}(R_i)$ *and* $f \in Hom_{R_i}(M, N)$

$$\imath_i(f) = f.$$

iii) Let R be a ring and $M_n(R)$ the ring of $n \times n$ matrices with entries in R. Further, let $R^{1\times n}$ and $R^{n\times 1}$ be the usual row and column vectors, respectively. There are functors $F_{1\times n} : \mathfrak{M}(R) \to \mathfrak{M}(M_n(R))$, and $F_{n\times 1} : \mathfrak{M}(M_n(R)) \to \mathfrak{M}(R)$, defined as follows: For any object $M \in \mathfrak{M}(R)$,

$$F_{1\times n}(M) = M \otimes_R R^{1\times n}.$$

For any two objects M,$N \in \mathfrak{M}(R)$, and $f \in Hom_R(M, N)$,

$$F_{1\times n}(f) = f \otimes_R Id_{R^{1\times n}}.$$

Conversely, given an object $M \in \mathfrak{M}(M_n(R))$,

$$F_{n\times 1}(M) = M \otimes_{M_n(R)} R^{n\times 1}.$$

For any two objects M,$N \in \mathfrak{M}(M_n(R))$, and $f \in Hom_{M_n(R)}(M, N)$,

$$F_{1\times n}(f) = f \otimes_{M_n(R)} Id_{R^{n\times 1}}.$$

Proof. It is well known that tensors distribute over direct sums, thus the functors in *i*) and *iii*) satisfy condition *i*) of Proposition 3.2.1. To see that the same holds for $\imath_i : \mathfrak{M}(R_i) \mapsto \mathfrak{M}(\times_{i=1}^{l} R_i)$, note that the $\times_{i=1}^{l} R_i$-action is defined component-wise on $M \oplus N$, for arbitrary objects M,$N \in \mathfrak{M}(R_i)$. Moreover, we have

$$\varphi^*(R) = S \in \mathfrak{F}(S), \tag{3.2}$$

$$\imath_i(R_i) = R_i \in \mathfrak{P}(\times_{i=1}^{l} R_i), \tag{3.3}$$

$$F_{1\times n}(R) = R^{1\times n} \in \mathfrak{P}(M_n(R)), \tag{3.4}$$

$$F_{n\times 1}(M_n(R)) = R^{n\times 1} \in \mathfrak{F}(R). \tag{3.5}$$

Thus all functors satisfy condition *ii*), as well. □

Remark. All functors in Proposition 3.2.2 are special cases of the functors defined in [18] p. 162-163.

Proposition 3.2.2 *ii*) equips us with a functor precisely tailored to compare modules over cartesian products, with those over the individual terms in the product. Similarly, the functors given in *iii*) compare modules over a ring with those over the respective matrix algebra. However, the functor given in *i*) is more general, in the sense that it is induced by any ring-homomorphism. We start with an application on ring-isomorphisms.

Proposition 3.2.3. *Let $\varphi : R \cong R'$ be a ring-isomorphism; then R has stably-free cancellation (is projective free), if and only if R' has stably-free cancellation (is projective free).*

Proof. Suppose R' has stably-free cancellation. Note, $\varphi^{-1} \circ \varphi = Id_R$. Thus for a stably-free module S over R we have

$$S = (\varphi^{-1} \circ \varphi)_*(S) = {\varphi^{-1}}_* \left(\varphi_*(S)\right).$$

By Proposition 3.2.2 $\varphi_*(S) \in \mathfrak{S}\mathfrak{F}(R')$, so by assumption $\varphi_*(S)$ is free. But then, again by Proposition 3.2.2, ${\varphi^{-1}}_* \left(\varphi_*(S)\right) = S$ is free. A similar argument applied to the map $\varphi \circ \varphi^{-1} = Id_{R'}$ yields the converse. Moreover, the statement for projective free rings is proven analogously . $\square$

Given a surjective ring-homomorphism $R \twoheadrightarrow R'$, we ask: Can $\mathfrak{M}(R)$ be parametrised by $\mathfrak{M}(R')$? A a partial answer can be given, using the following proposition

Proposition 3.2.4. *If I is a radical ideal of a ring R, then the functor $p_* : \mathfrak{M}(R) \to \mathfrak{M}(R/I)$, induced by the canonical surjection $p : R \to R/I$, gives an injective map*

$$\mathcal{P}(R) \hookrightarrow \mathcal{P}(R/I).$$

That is, given two projective modules $P, Q \in \mathfrak{P}(R)$, such that $P/IP \cong Q/IQ$ as R/I-modules, then $P \cong Q$ as R-modules.

Proof. See [18] p. 182. □

As a corollary we have

Proposition 3.2.5. *Let $\varphi : R \to R'$ be a surjective ring-homomorphism with nilpotent kernel. Then if R' has stably-free cancellation (is projective free), the same holds true for R.*

Proof. First, note that any nilpotent (right) ideal is automatically radical (c.f. e.g. [18] p. 180). Moreover, $R' \cong R/ker(\varphi)$, thus, by Proposition 3.2.3, R' having stably-free cancellation (being projective free) implies $R/ker(\varphi)$ has stably-free cancellation (is projective free). First, assume R' has stably-free cancellation, then, by Proposition 3.2.4, $1 \leq |\mathcal{SF}_N(R)| \leq |\mathcal{SF}_N(R/ker(\varphi))| = 1$, i.e. R has stably-free cancellation. If moreover, R' is projective free, then for any $P \in \mathfrak{P}(R)$ there exists a natural number N, such that

$$P/IP \cong (R/ker(\varphi))^N \cong R^N/ker(\varphi)R^N,$$

and, again by Proposition 3.2.4, $P \cong R^N$. □

Next, we consider cartesian products.

Proposition 3.2.6. *Let $R \cong \times_{i=1}^{l} R_i$ be a ring-isomorphism. Then, R has stably-free cancellation, if each R_i has stably-free cancellation for $1 \leq i \leq l$.*

Proof. Let S be a stably-free module of rank N over $\times_{i=1}^{l} R_i$, by Proposition 3.2.2, $\pi_{i*}(S) \in \mathfrak{SF}(R_i)$. Thus, by assumption it is free for all $1 \leq i \leq l$, i.e.

$$\pi_{i*}(S) \cong R_i^N.$$

Now, note that every module M over $\times_{i=1}^{l} R_i$ can be written as

$$M = \oplus_{i=1}^{l} \imath_i \left(\pi_{i*}(M)\right).$$

So $\oplus_{i=1}^{l} (\imath_i \circ \pi_{i*})$ is the identity morphism on $\mathfrak{M}(R)$. But then,

$$S = \oplus_{i=1}^{l} (\imath_i \circ \pi_{i*}) (S) \cong \oplus_{i=1}^{l} R_i^N \cong \left(\oplus_{i=1}^{l} R_i\right)^N ,$$

i. e. S is free of rank N. Finally, by Proposition 3.2.3 , $\times_{i=1}^{l} R_i$ has stably-free cancellation if and only if so has R. □

We move on to consider matrix rings. We shall need the original version of Morita's theorem, i.e.

Proposition 3.2.7. *Let R be a ring and m, n natural numbers. There exists an isomorphism of $M_m(R), M_m(R)$-bimodules*

$$R^{m\times n} \otimes_{M_n(R)} R^{n\times m} \cong M_m(R),$$

given by the map

$$M \otimes N \mapsto MN,$$

for any $M \in \mathfrak{M}(R^{m\times n})$, and $N \in \mathfrak{M}(R^{n\times m})$.

Proof. See [18] p. 166. □

Recall, two rings are said to be Morita equivalent whenever there exists an equivalence of categories between the category of finitely generated (right) modules of one ring and that of the other (c. f. e. g. [4] p. 139). Note, Morita's theorem (Proposition 3.2.7) shows that, for any ring R, $\mathfrak{M}(R)$ and $\mathfrak{M}(M_n(R))$ are equivalent, since

$$(F_{n\times 1} \circ F_{1\times n})(M) = M \otimes_R R^{1\times n} \otimes_{M_n(R)} R^{n\times 1} \cong M \otimes_R R \cong M.$$

Similarly, given $N \in \mathfrak{M}(M_n(R))$, we get $(F_{1\times n} \circ F_{n\times 1})(N) \cong N$. As we have already shown in Proposition 3.2.2 the functors $F_{1\times n}$, $F_{n\times 1}$ restrict to the categories of finitely generated projective modules, so the above statement yields that also $\mathfrak{P}(R)$ and $\mathfrak{P}(M_n(R))$ are equivalent. In fact, by the general version of Morita's theorem (c. f. e. g. [5] p. 231) the same holds true for any pair of Morita equivalent rings. However, the equivalence of the categories $\mathfrak{P}(R)$, and $\mathfrak{P}(M_n(R))$ is not strong enough for our purposes. For example, assume R is projective free, then it is not true that $M_n(R)$ is also projective free, as $R^{1\times n}$ is a non-free projective, right $M_n(R)$-module. Nevertheless, the functor $F_{n\times 1}$ restricts to a functor on free modules, as we have seen in (3.5). It is this intrinsic property of $F_{n\times 1}$ which allows us to prove the following

Proposition 3.2.8. *Let n be a natural number, R a ring, and $M_n(R)$ the $n \times n$ matrix-algebra over it. If R has stably-free cancellation, then so does also $M_n(R)$.*

Proof. Let S be a stably-free module of rank N over $M_n(R)$. Note, by (3.5), $F_{n\times 1}(M_n(R)) \cong R^n$, thus $F_{n\times 1}(S)$ is a stably-free R-module of rank nN. But, by assumption, R has stably-free cancellation, i. e. $F_{n\times 1}(S) \cong R^{nN}$. Now, by Proposition 3.2.7, we know that $(F_{1\times n} \circ F_{n\times 1})(S) \cong S$. Moreover, $F_{1\times n}(R) = R^{1\times n}$, as we have already seen in (3.4) and clearly $\left(R^{1\times n}\right)^n \cong M_n(R)$, as right $M_n(R)$-modules. Therefore, we have

$$S \cong (F_{1\times n} \circ F_{n\times 1})(S) \cong F_{1\times n}(R^{nN}) = \left(R^{1\times n}\right)^{nN} \cong (M_n(R))^N ,$$

i. e. S is free of rank N. □

3.3 Rings with stably-free cancellation

We say a group G is of type $\mathcal{F}$, if it is free on $m \geq 2$ generators or free abelian on $n \geq 1$ generators. Our main aim in this section is to prove that both $\mathcal{C}(\Lambda_{(p,n)})[G]$ and $\mathbb{F}_2[D_{2n} \times G]$, with G a group of type $\mathcal{F}$, have stably-free cancellation. In order to do so, we need a source of simpler algebras over type $\mathcal{F}$ groups, with stably free cancellation. Thus we first review some important and famous examples from the literature.

Theorem (Bass, 1964) 3.3.1. *Let G be a free group (or monoid), let R be a principal ideal domain. Then finitely generated projective right (or left) $R[G]$-modules are free.*

Proof. See [2]. □

Remark. It should be added that Sheshadri, in 1958, motivated by the famous conjecture of Serre ([26] p. 243), proved a special case of the above statement ([27]), assuming that G is a monoid on one generator (i.e. R[G], the polynomial ring in one indeterminate, with coefficients in R). Moreover, in 1963, Cohn proved for a commutative field R and a free group G that $R[G]$ is a free ideal ring ([6], p. 68). Now it is known that free ideal rings, i. e. integral domains in which every right ideal is free as a module, are projective free (c.f. e.g. [6] p. 49).

An analogous version of Theorem 3.3.1 for free abelian groups can deduced using Quillen's solution ([25]) to Serre's Conjecture, that is

Theorem 3.3.2. *Let G be a free abelian group (or monoid), let R be a commutative principal ideal domain. Then finitely generated projective right (or left) $R[G]$-modules are free.*

Proof. See [16] p. 147. □

By theorems 3.3.1 and 3.3.2 it is clear that $R[G]$ is projective free whenever G is a group of type $\mathcal{F}$ and R a commutative principal ideal domain. Before proving the main theorems of this chapter we shall need the following lemma

Lemma 3.3.3. *Let $\mathbb{F}$ be a field, $m(x)$ and irreducible polynomial in $\mathbb{F}[x]$ and $d \geq 1$ a natural number. Then $\mathbb{F}[x]/\left(m(x)^d\right)[G]$ is projective free whenever G is of type $\mathcal{F}$.*

Proof. First consider the case $d = 1$. Since $m(x)$ is irreducible, $\mathbb{F}[x]/\left(m(x)\right)$ is a field, and by theorems 3.3.1, 3.3.2 the statement holds. If $d > 1$, let

$$p : \mathbb{F}[x]/\left(m^d(x)\right) \to \mathbb{F}[x]/\left(m(x)\right)$$

be the canonical surjection. Note, $ker(p) = (m(x))$, i. e. the principal ideal generated by $m(x)$. Evidently, in $\mathbb{F}[x]/\left(m^d(x)\right)$ we have $m^d(x) = 0$. Therefore $ker(p)^d = 0$, i. e. it is nilpotent. We induce a map

$$p^* : \mathbb{F}[x]/\left(m^d(x)\right)[G] \to \mathbb{F}[x]/\left(m(x)\right)[G]$$

by putting $p^* = p \otimes_{\mathbb{Z}} Id_{\mathbb{Z}[G]}$. Then p^* is surjective, since tensoring preserves surjective maps. Moreover,

$$ker(p^*) = ker(p) \otimes \mathbb{Z}[G] = (m(x)) \otimes \mathbb{Z}[G] \tag{3.6}$$

Now, since $ker(p)^d = 0$, (3.6) implies $ker(p^*)^d = 0$. So p^* is a surjective map, with nilpotent kernel, onto a ring which is projective free, as the case $d = 1$ establishes. By Proposition 3.2.5 $\mathbb{F}[x]/\left(m(x)^d\right)[G]$ is projective free. □

Theorem 3.3.4. *$\mathcal{C}(\Lambda_{(p,n)})[G]$ has stably-free cancellation whenever G is of type $\mathcal{F}$.*

Proof. By Theorem 2.4.9, there exist natural numbers m, d_i and irreducible polynomials $q_i(x)$ in $\mathbb{F}_p[x]$, such that

$$\mathcal{C}(\Lambda_{(p,n)})[G] \cong \prod_{i=1}^{m} M_2\left(\mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G]\right),$$

for any group G. Now by Lemma 3.3.3 each $\mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G]$ is projective free, and therefore has stably-free cancellation. Proposition 3.2.8 implies that each factor $M_2\left(\mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G]\right)$ has stably-free cancellation. But then, by Proposition 3.2.6, so does also $\mathcal{C}(\Lambda_{(p,n)})[G]$, being isomorphic to a product of rings with stably free cancellation. □

Next we consider $\mathbb{F}_2[D_{2n} \times G]$, with G of type $\mathcal{F}$. We shall need the following

Lemma 3.3.5. *Let $k \geq 3$ be an odd number and $s \geq 1$ an integer. There exists a surjective map*

$$\pi : \mathbb{F}_2[D_{2^{s+1}k}] \to \mathbb{F}_2[D_{2^s k}],$$

with nilpotent kernel.

Proof. In $\mathbb{F}_2[x]$, $(x^{2^s k} - 1) = (x^{2^{s-l}k} - 1)^{2^l}$, for any integer l. In particular, $(x^{2^s k} - 1) = (x^{2^{s-1}k} - 1)^2$, and there exists a canonical surjection

$$p : \mathbb{F}_2[x]/(x^{2^s k} - 1) \to \mathbb{F}_2[x]/(x^{2^{s-1}k} - 1),$$

with kernel $ker(p) = (x^{2^{s-1}k} - 1)$. Now by (2.8) in Chapter 2, we know $\mathbb{F}_2[D_{2^s k}] = \mathcal{C}_2\left(\mathbb{F}_2[C_{2^{s-1}k}], \gamma, 1\right)$. Furthermore, since $p \circ \gamma = \gamma \circ p$ and $p(1) = 1$, we see that p is a cyclic ring-homomorphism. Therefore, by Proposition 2.1.2, p induces a surjective map

$$\pi : \mathbb{F}_2[D_{2^{s+1}k}] \to \mathbb{F}_2[D_{2^s k}],$$

with kernel

$$ker(\pi) = \ker p \dotplus y \ker p = (x^{2^{s-1}k} - 1) \dotplus y(x^{2^{s-1}k} - 1).$$

We claim $ker(\pi)^2 = 0$, i. e. $ker(\pi)$ is nilpotent. Given any two elements in $k_1,\, k_2 \in ker(\pi)$, say

$$k_1 = (x^{2^{s-1}k} - 1)f_1(x) + y(x^{2^{s-1}k} - 1)g_1(x),$$

$$k_2 = (x^{2^{s-1}k} - 1)f_2(x) + y(x^{2^{s-1}k} - 1)g_2(x),$$

with $f_i(x), g_i(x) \in \mathbb{F}_2[x]/(x^{2^s k}-1)$. Then, as $(x^{2^{s-1}k})^2 = 1$ in $\mathbb{F}_2[x]/(x^{2^s k}-1)$,

$$(x^{2^{s-1}k} - 1)^{-1} = (x^{2^{s-1}k} - 1) \Leftrightarrow \gamma(x^{2^{s-1}k} - 1) = (x^{2^{s-1}k} - 1). \tag{3.7}$$

In other words, the elements $(x^{2^{s-1}k} - 1)$ and y commute in $\mathbb{F}_2[D_{2^{s+1}k}]$. We compute thus

$$k_1.k_2 = F(x)(x^{2^{s-1}k} - 1)^2 = F(x).0 = 0,$$

where $F(x) \in \mathbb{F}_2[D_{2^{s+1}k}]$ denotes the element

$$F(x) = f_1(x)f_2(x) + f_1(x)yg_2(x) + yg_1(x)f_2(x) + yg_1(x)yg_2(x).$$

□

Theorem 3.3.6. $\mathbb{F}_2[D_{2n} \times G]$ *has stably-free cancellation whenever* G *is of type* $\mathcal{F}$.

Proof. Write $2n = 2^s k$, for a natural number $s \geq 1$ and odd number $k \geq 3$. We prove the statement by induction on s. Thus consider the case $s = 1$, by Theorem 2.3.5

$$\mathbb{F}_2[D_{2k} \times G] \cong \big(\mathbb{F}_2[x]/(x^2 - 1)\big)[G] \times \prod_{i=1}^{l} M_2\left(\mathbb{F}_{2^{d_i}}[G]\right),$$

for any group G. But then, observing that $(x^2 - 1) = (x - 1)^2$ in $\mathbb{F}_2[x]$, a similar proof as in Theorem 3.3.4 shows that $\mathbb{F}_2[D_{2k} \times G]$ has stably free cancellation. Next assume the statement holds for $s = \sigma$, and consider the case $s = \sigma + 1$: By Lemma 3.3.5 we have a surjective map

$$\pi : \mathbb{F}_2[D_{2^{\sigma+1}k}] \to \mathbb{F}_2[D_{2^{\sigma}k}],$$

with nilpotent kernel. As in Lemma 3.3.3, we induce a ring homomorphism

$$\pi^* : \mathbb{F}_2[D_{2^{s+1}k} \times G] \to \mathbb{F}_2[D_{2^s k} \times G],$$

by putting $\pi^* = \pi \otimes_{\mathbb{Z}} Id_{\mathbb{Z}[G]}$. Then π^* is surjective. Moreover,

$$ker(\pi^*) = ker(\pi) \otimes \mathbb{Z}[G] \tag{3.8}$$

Now, since $ker(\pi)$ is nilpotent, (3.8) implies $ker(\pi^*)$ is nilpotent. So π^* is a surjective map, with nilpotent kernel, onto $\mathbb{F}_2[D_{2^s k} \times G]$, a ring which has stably-free cancellation by the inductive hypothesis. By Proposition 3.2.5 $\mathbb{F}_2[D_{2^{s+1}k} \times G]$ also has stably-free cancellation. □

Chapter 4

Weakly Euclidean rings

4.1 Basic definitions

In this section we discuss so called *weakly Euclidean rings* which sometimes in the literature are referred to as *generalised Euclidean rings* (e. g. see [7]). We start by giving a few elementary definitions (for a comparison see [18] p. 319-321). Given a ring R and a natural number n, denote by $M_n(R)$ the ring of $n \times n$ matrices with entries in R. Let ϵ_{ij} be the $n \times n$ matrix over R with the only non-zero entry at the i, j-th position, where it is 1_R. It is well known that these ϵ_{ij}'s, usually called *matrix units*, form a basis for $M_n(R)$ as a right (or left) module over R . It can easily be calculated that matrix units multiply according to the rule:

$$\epsilon_{ij}\epsilon_{kl} = \begin{cases} 0 & \text{if} \quad j \neq k \\ \epsilon_{il} & \text{if} \quad j = k \end{cases} \tag{4.1}$$

Furthermore, the group of units under matrix multiplication of $M_n(R)$ is called the *general linear group* of R, abbreviated to $GL_n(R)$. We are interested in certain subgroups of $GL_n(R)$. Firstly, the *elementary matrices* $E_n(R)$, i. e. the subgroup consisting of all finite words $t_1 \dots t_k$, where each t_i

is an *elementary transvection.* Here, by an elementary transvection we mean a matrix of the form $e_{ij}(r) = I_n + r\epsilon_{ij}$, where I_n as usual denotes the $n \times n$ identity matrix, r is any element in R and $i \neq j$ for some $1 \leq i, j \leq n$. Indeed, $E_n(R)$ is a subgroup of $GL_n(R)$, as each elementary transvection $e_{ij}(r)$ has inverse $e_{ij}(r)^{-1} = e_{ij}(-r)$. Next we consider the subgroup of *diagonal matrices* denoted by $D_n(R)$. This subgroup consists of all products $d_1(u_1) \ldots d_n(u_n)$, where $d_i(u_i) = I_n + (u_i - 1)\epsilon_{ii}$ with $u_i \in R^*$ (as usual R^* denotes the group of units in R). Writing $diag(u_1, \ldots, u_n)$ for the diagonal matrix with unit entry u_i at i,i-th position, we may equivalently describe $D_n(R)$ as the subgroup of $GL_n(R)$ which consists of all matrices of the form $diag(u_1, \ldots, u_n)$, as

$$diag(u_1, \ldots, u_n) = d_1(u_1) \ldots d_n(u_n). \tag{4.2}$$

Finally, we consider the set of all finite length products where each term is either of the form $e_{ij}(r)$ or $d_i(u)$, where $r \in R$ and $u \in R^*$. By the above discussion it is clear that these products also make up a subgroup of $Gl_n(R)$. We shall call this subgroup the *restricted linear group* and denote it by $GE_n(R)$. Observe that

$$d_k(u)e_{ij}(r)d_k(u)^{-1} = \begin{cases} e_{ij}(ur) & \text{if} \quad k = i \\ e_{ij}(ru^{-1}) & \text{if} \quad k = j \\ e_{ij}(r) & \text{if} \quad k \notin \{i, j\}. \end{cases} \tag{4.3}$$

Therefore, $D_n(R)$ normalizes $E_n(R)$, and we have

Proposition 4.1.1.

$$GE_n(R) = D_n(R)E_n(R) = E_n(R)D_n(R).$$

Furthermore, define

$$p_k(u) := e_{k1}(-u)e_{1k}(u^{-1})e_{k1}(-u)e_{k1}(1)e_{1k}(-1)e_{k1}(1).$$

Then for any $u \in R^*$ we have $d_k(u) = d_1(u)p_k(u)$. So for $u_1, \ldots, u_n$ in R^*, by (4.2),

$$diag(u_1, \ldots, u_n) = d_1(u_1)d_1(u_2)p_2(u_2)\ldots d_1(u_n)p_n(u_n). \tag{4.4}$$

Now by observation (4.3), given an element E in $E_n(R)$ and a unit $u \in R^*$ there exists $E' \in E_n(R)$ such that $E\, d_k(u) = d_k(u)E'$. Evidently, each $p_i(u_i)$ is an element in $E_n(R)$, and therefore the right-hand side of Equation (4.4) may be rewritten as

$$diag(u_1, \ldots, u_n) = d_1(u_1)\ldots d_1(u_n)E = d_1(u_1 \ldots u_n)E, \tag{4.5}$$

for some E in $E_n(R)$. Now by Proposition 4.1.1 every element in $GE_n(R)$ may be written as $diag(u_1, \ldots, u_n)E$, for some E in $E_n(R)$ and $u_1, \ldots, u_n$ in R^*. But by (4.5) there exists an E' in $E_n(R)$, such that $diag(u_1, \ldots, u_n)E = d_1(u_1 \ldots u_n)E'E$. We formulate this observation as

Proposition 4.1.2.

$$GE_n(R) = d_1(R^*)E_n(R).$$

Here $d_1(R^*)$ denotes the set of all elements $d_1(u)$, with u in R^*.

We say a ring R is *weakly Euclidean* if its the restricted linear group exhausts its general linear group, in other words, whenever $GL_n(R) = GE_n(R)$.

4.2 Examples of weakly Euclidean rings

Having defined the notion of a weakly Euclidean ring, the natural question to ask is: Which rings are weakly Euclidean? As a first result we have

Theorem 4.2.1. *Every Euclidean domain is weakly Euclidean.*

Proof. The proof is simply the existence proof of the Smith normal form in the case of a Euclidean domain (c.f. e.g. [9] p. 80). □

Dieudonné considers (possibly non-commutative) division rings in [8] p. 29:

Theorem 4.2.2. *Every division ring is weakly Euclidean.*

Klingenberg elaborates on Dieudonné's result in [15] p. 76, showing:

Theorem 4.2.3. *Every local ring, possibly non-commutative, is weakly Euclidean.*

The following result by Cohn (see [7] p.373) is particularly important, as it will enable us to recognise all relevant examples of weakly Euclidean domains.

Theorem 4.2.4. *Let G be a free group, and k a field. Then the group algebra $k[G]$ is weakly Euclidean.*

In fact, Cohn gives examples of several different classes of weakly Euclidean rings in [7]. However, these classes do not necessarily encompass the specific rings we encounter in the scope of this thesis.

4.3 Recognition criteria

We will now review a few recognition criteria particularly useful to us.

Proposition 4.3.1. *The direct product of a finite number of weakly Euclidean rings is again weakly Euclidean.*

Proof. See (3.1) in [7], p. 371. □

Proposition 4.3.2. *Let R be weakly Euclidean; then $M_k(R)$, the ring of $k \times k$ matrices over R, is again weakly Euclidean.*

Proof. Given $X \in GL_n(M_k(R))$ then clearly $X \in GL_{nk}(R)$. But R is weakly Euclidean, thus

$$X = DE,$$

for some $D \in D_{nk}(R)$, $E \in E_{nk}(R)$. Firstly, consider D. As an element in $GL_{nk}(R)$ it has the form

$$D = diag(u_1, \dots, u_{nk}),$$

for $u_1, \dots, u_{nk} \in R^*$. Equivalently, as an element in $GL_n(M_k(R))$,

$$D = (U_1, \dots, U_n),$$

where $U_i = diag(u_{(i-1)k+1}, \dots, u_{ik})$, and clearly $U_i \in GL_k(R)$. Therefore, D is an element in $D_n(M_k(R))$. Moreover, E, viewed as an element in $E_{nk}(R)$, is a finite product of elementary transvections $e_{i,j}(s)$, with $1 \le i,j \le nk$, $i \ne j$, $s \in R$. Now, view $e_{i,j}(s) = I_{nk} + s\epsilon_{i,j}$ as an element in $GL_n(M_k(R))$. First, suppose the (i,j)-th entry is inside one of the blocks in the $k \times k$-block-diagonal, say it is in the (λ, λ)-th block. But then the block containing the (i,j)-th entry is of the form $e_{\imath,\jmath}(s) \in GL_k(R)$, $1 \le \imath, \jmath \le k$, with $\imath \ne \jmath$. Therefore,

$$e_{i,j}(s) = diag(\underbrace{I_k, \dots, I_k}_{\lambda-1}, e_{\imath,\jmath}(s), I_k, \dots, I_k)$$

which is an element in $D_n(M_k(R))$. If, however, the (i,j)-th entry is not inside one of the blocks in the $k \times k$-block-diagonal, then it is in the (ι, κ)-th block, $1 \le \iota, \kappa \le n$, with $\iota \ne \kappa$. Denoting this block by B_s, it is clear that

$$e_{i,j}(s) = e_{\iota,\kappa}(B_s)$$

which is an element in $E_n(M_k(R))$. In either case E is a finite product in $GE_n(M_k(R))$, thus so is also $X = DE$. □

The next criterion, due to Johnson (c. f. [12]), requires some work to establish. We start with a definition. A ring homomorphism $\varphi : A \to B$ has the *lifting property for units* if *i*) the induced map $\varphi^* : A^* \to B^*$ on the unit groups A^*, B^* is surjective.

We say φ has the *strong lifting property for units*, if additionally the following holds

ii) given $a \in A$ such that $\varphi(a) \in B^*$, then $a \in A^*$.

Theorem 4.3.3. *Let $\varphi : R \to S$ be a surjective ring homomorphism with strong lifting for units; then if S is weakly Euclidean, so is also R.*

In order to prove this statement we shall need the following.

Lemma 4.3.4. *Let $\varphi : R \to S$ be a ring homomorphism with strong lifting for units and let X be an element in $GL_n(R)$, such that $\varphi(X) = I_n$. Then $X \in GE_n(R)$. Here $\varphi : M_n(R) \to M_n(S)$ denotes induced ring homomorphism by applying φ componentwise.*

Proof. The result is proved by induction on n. Note the case n=1 is trivial, thus we assume $n \geq 2$ in what follows. We write $X = (x_{ij})$. Since $\varphi(x_{nn}) = 1$, condition *ii*) of the strong lifting property implies that $x_{nn} \in R^*$, and we may define

$$\begin{aligned} D &:= D := diag(\underbrace{1, \ldots, 1}_{n-1}, x_{nn}^{-1}) \\ E_+ &:= \prod_{r=1}^{n-1} e_{rn}(-x_{rn}x_{nn}^{-1}) \\ E_- &:= \prod_{r=1}^{n-1} e_{nr}(-x_{nn}^{-1}x_{nr}), \end{aligned}$$

i. e. $D, E_+, E_- \in GE_n(R)$. A straightforward calculation and an abuse of the $diag(a_1, \ldots, a_n)$-notation yield

$$E_+XE_-D = diag(X', 1), \tag{4.6}$$

for some $X' \in GL_{n-1}(R)$. Furthermore, since $\varphi(x_{nr}) = \varphi(x_{rn}) = 0$ for $r < n$ and $\varphi(x_{nn}^{-1}) = \varphi(x_{nn}) = 1$, we have that

$$\varphi(E_+) = \varphi(E_-) = \varphi(D) = I_n.$$

This implies that $\varphi(E_+XE_-D) = I_n$, and hence that $\varphi(X') = I_{n-1}$. Now rewriting (4.6) gives

$$X = E_-^{-1} diag(X', 1) E_+^{-1} D^{-1},$$

with $E_-^{-1}, E_+^{-1}, D^{-1} \in GE_n(R)$. It remains to show that $diag(X', 1) \in GE_n(R)$. If $n = 2$ the statement clearly holds as $X' \in GL_1(R) = R^*$. Next consider the case $n = k$, i. e. $X' \in GL_{k-1}(R)$, such that $\varphi(X') = I_{k-1}$. Therefore, by the inductive hypothesis $X' \in GE_{k-1}$, but then clearly $diag(X', 1) \in GE_k(R)$. □

We prove Theorem 4.3.3.

Proof. Given $X \in GL_n(R)$, our hypothesis implies $\varphi(X) \in GL_n(S) = GE_n(S)$, and thus may be written as $\varphi(X) = diag(u_1, \dots, u_n)E$, for some $u_1, \dots, u_n \in S^*$ and $E \in E_n(S)$. Now condition $i)$ of the strong lifting property allows us to choose $u'_1, \dots u'_n \in R^*$, such that $\varphi(u'_i) = u_i$. Furthermore, E, by definition, is a finite product of elementary transvections

$$E = \prod_{r=1}^{m} e_{ij_r}(k_r),$$

with $k_r \in S$. By the surjectivity of φ, for each term $e_{ij_r}(k_r)$ in E we can find $e_{ij_r}(k'_r)$ with $k'_r \in R$ such that $\varphi(e_{ij_r}(k'_r)) = e_{ij_r}(k_r)$. Then clearly for

$$E' := \prod_{r=1}^{m} e_{ij_r}(k'_r),$$

we have $\varphi(E') = E$, and $E' \in E_n(R)$. Therefore, if we define

$$Y := diag(u'_1, \dots, u'_n)E',$$

then $\varphi(Y) = \varphi(X)$ as well as $Y \in GE_n(R)$. But then

$$\varphi(XY^{-1}) = \varphi(X)\varphi(Y)^{-1} = \varphi(X)\varphi(X)^{-1} = I_n,$$

and Lemma 4.2.1 implies

$$XY^{-1} = Z, \tag{4.7}$$

for some $Z \in GE_n(R)$. Multiplying (4.7) with Y on the right, we see that $X = ZY \in GE_n(R)$, as Z and Y are in $GE_n(R)$. □

Remark. Recognition criterion 4.3.3 allows us to give a simple proof of Klingenberg's result (Theorem 4.2.3): Let R be a local ring with maximal ideal $\mathfrak{m}$. Then the canonical map $p : R \to R/\mathfrak{m}$ has strong lifting (all non-units in R map to zero). Now $R/\mathfrak{m}$ is a division ring, thus by Theorem 4.2.2 weakly Euclidean, and the result follows by Theorem 4.3.3.

4.4 Further examples of weakly Euclidean rings

Let $\mathbb{F}_p$ denote the field with p elements, for a prime p. We consider a special case of strong lifting of units involving $\mathbb{F}_p$-algebras.

Proposition 4.4.1. *Let A be an $\mathbb{F}_p$-algebra, and let $\varphi : A \to B$ be a surjective ring homomorphism with nilpotent kernel. Then φ has the strong lifting property.*

Proof. Since φ is surjective, it is enough to prove that condition *ii)* of the strong lifting property holds. Thus assume $a \in A$ such that $\varphi(a) \in B^*$. Let $b \in B^*$ be its inverse, with preimage $b' \in A$ under φ, i. e. $\varphi(b') = b$. Then the equations $\varphi(a)b = 1$, and $b\varphi(a) = 1$, in B, may be lifted to equations

$$\begin{aligned} ab' &= 1 + k_1 \\ b'a &= 1 + k_2 \end{aligned}$$

in A, where $k_1, k_2 \in ker\varphi$. By assumption, there exists a natural number m with $k_1^m = k_2^m = 0$. Choose n such that $p^n \geq m$. But then, as A is an $\mathbb{F}_p$-algebra

$$(ab')^{p^n} = (1 + k_1)^{p^n} = 1^{p^n} + {k_1}^{p^n} = 1.$$

Similarly $(b'a)^{p^n} = 1$. Therefore, a has inverse $b'(ab')^{p^n-1} = (b'a)^{p^n-1}b'$, and $a \in A^*$. □

We turn to the cyclic group algebras already discussed in Chapter 2. First we consider $\mathcal{C}(\Lambda_{(p,n)})[G]$, for a free group G. Analogously to Theorem 3.3.4 we prove

Theorem 4.4.2. *$\mathcal{C}(\Lambda_{(p,n)})[G]$ is weakly Euclidean whenever G is a free group.*

Proof. By Theorem 2.4.9, there exist natural numbers m, d_i and irreducible polynomials $q_i(x)$ in $\mathbb{F}_p[x]$, such that

$$\mathcal{C}(\Lambda_{(p,n)})[G] \cong \prod_{i=1}^{m} M_2\left(\mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G]\right),$$

for any group G. Now by the proof of Lemma 3.3.3 there exist surjective maps

$$p_i^* : \mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G] \to \mathbb{F}_p[x]/\left(q_i(x)\right)[G],$$

with nilpotent kernel, for $1 \leq i \leq m$. By Proposition 4.4.1 all p_i^* have the strong lifting property. Also each $\mathbb{F}_p[x]/\left(q_i(x)\right)$ is a field, as $q_i(x)$ is irreducible. Therefore, by Theorem 4.2.4, each $\mathbb{F}_p[x]/\left(q_i(x)\right)[G]$ is weakly Euclidean whenever G is a free group. But then, by Theorem 4.3.3, so is also $\mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G]$, for $1 \leq i \leq m$ and G a free group. Furthermore, Proposition 4.3.2 implies that each factor $M_2\left(\mathbb{F}_p[x]/\left(q_i^{d_i}(x)\right)[G]\right)$ is weakly Euclidean. Finally, by Proposition 4.3.1, so is also $\mathcal{C}(\Lambda_{(p,n)})[G]$, being isomorphic to a finite product of weakly Euclidean rings. □

Finally, we consider the group algebra $\mathbb{F}_2[D_{2n} \times G]$, for a free group G

Theorem 4.4.3. *The group algebra $\mathbb{F}_2[D_{2n} \times G]$ is weakly Euclidean whenever G is a free group.*

Proof. Similarly to Theorem 3.3.6, this theorem is proven by induction on $s \geq 1$, where $2n = 2^s k$, for an odd number $k \geq 3$. Thus consider the case $s = 1$, by Theorem 2.3.5

$$\mathbb{F}_2[D_{2k} \times G] \cong \left(\mathbb{F}_2[x]/(x^2-1)\right)[G] \times \prod_{i=1}^{l} M_2\left(\mathbb{F}_{2^{d_i}}[G]\right),$$

for any group G. But then, observing that $(x^2-1) = (x-1)^2$ in $\mathbb{F}_2[x]$, a similar proof as in Theorem 4.4.2 shows that $\mathbb{F}_2[D_{2k} \times G]$ is weakly Euclidean whenever G is a free group. Next assume the statement hold for $s = \sigma$, and consider the case $s = \sigma + 1$: By the proof of Theorem 3.3.6 there exists a surjective map

$$\pi^* : \mathbb{F}_2[D_{2^{s+1}k} \times G] \to \mathbb{F}_2[D_{2^s k} \times G]$$

with nilpotent kernel. Then Proposition 4.4.1 implies that π^* has the strong lifting property. Now by the inductive hypothesis $\mathbb{F}_2[D_{2^s k} \times G]$ is weakly Euclidean. But then, by Theorem 4.3.3, so is also $\mathbb{F}_2[D_{2^{s+1}k} \times G]$. □

Chapter 5

Fibre squares

5.1 Fibre squares of cyclic algebras

We start by defining a *fibre square*: Let

$$\begin{array}{ccc} \Lambda & \xrightarrow{\rho_-} & \Lambda_- \\ {\scriptstyle \rho_+}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_-} \\ \Lambda_+ & \xrightarrow{\varphi_+} & \Lambda_0 \end{array} \tag{5.1}$$

be a commutative square of ring-homomorphisms. We say (5.1) is a fibre square, if for each pair $(\lambda_+, \lambda_-) \in \Lambda_+ \times \Lambda_-$ with $\varphi_+(\lambda_+) = \varphi_-(\lambda_-)$ there exists exactly one $\lambda \in \Lambda$, such that $\rho_+(\lambda) = \lambda_+$, $\rho_-(\lambda) = \lambda_-$. Moreover, we call Λ the *fibre product* of Λ_+ and Λ_- over Λ_0. There is an equivalent description of fibre squares given by

Proposition (c. f. e. g. [18] p. 435) 5.1.1. *The following are equivalent*

i) (5.1) *is a fibre square*

ii) *there exists an exact sequence of additive groups*

$$0 \longrightarrow \Lambda \xrightarrow{\binom{\rho_+}{\rho_-}} \Lambda_+ \oplus \Lambda_- \xrightarrow{(\varphi_+ - \varphi_-)} \Lambda_0 \tag{5.2}$$

Proof. Note, $im\binom{\rho_+}{\rho_-} \subseteq ker(\varphi_+ - \varphi_-)$, if and only if (5.1) commutes. Moreover, $ker(\varphi_+ - \varphi_-) \subseteq im\binom{\rho_+}{\rho_-}$, if and only if for each pair $(\lambda_+, \lambda_-) \in \Lambda_+ \times \Lambda_-$ with $\varphi_+(\lambda_+) = \varphi_-(\lambda_-)$ there exists one $\lambda \in \Lambda$, such that $\rho_+(\lambda) = \lambda_+$, $\rho_-(\lambda) = \lambda_-$. Finally, there exists exactly one such λ, if and only if $\binom{\rho_+}{\rho_-}$ is injective. □

Now assume we are given cyclic algebras $\mathcal{C}_n(\Lambda, s, a)$, $\mathcal{C}_n(\Lambda_+, s_+, a_+)$, $\mathcal{C}_n(\Lambda_-, s_-, a_-)$ and $\mathcal{C}_n(\Lambda_0, s_0, a_0)$ together with a fibre square of cyclic ring-homomorphisms

$$\begin{array}{ccc} \Lambda & \xrightarrow{r_-} & \Lambda_- \\ {\scriptstyle r_+}\downarrow & & \downarrow{\scriptstyle p_-} \\ \Lambda_+ & \xrightarrow{p_+} & \Lambda_0 \end{array} .$$

By Proposition 5.1.1 this is equivalent to saying there exists an exact sequence of additive groups

$$0 \longrightarrow \Lambda \xrightarrow{\binom{r_+}{r_-}} \Lambda_+ \oplus \Lambda_- \xrightarrow{(p_+ - p_-)} \Lambda .$$

But r_+,r_-, p_+ and p_- are cyclic ring-homomorphisms, thus by propositions 2.1.1 and 2.1.2 they induce an exact sequence of additive groups

$$0 \longrightarrow \mathcal{C}_n(\Lambda, s, a) \xrightarrow{\binom{\rho_+}{\rho_-}} \mathcal{C}_n(\Lambda_+, s_+, a_+) \oplus \mathcal{C}_n(\Lambda_-, s_-, a_-) \xrightarrow{(\pi_+ - \pi_-)} \mathcal{C}_n(\Lambda_0, s_0, a_0) .$$

Then applying Proposition 5.1.1 again we see that there exists a fibre square

$$\begin{array}{ccc} \mathcal{C}_n(\Lambda, s, a) & \xrightarrow{\rho_-} & \mathcal{C}_n(\Lambda_-, s_-, a_-) \\ {\scriptstyle \rho_+}\downarrow & & \downarrow{\scriptstyle \pi_-} \\ \mathcal{C}_n(\Lambda_+, s_+, a_+) & \xrightarrow{\pi_+} & \mathcal{C}_n(\Lambda_0, s_0, a_0) \end{array} .$$

Thus we have

Proposition 5.1.2. *A fibre square of cyclic ring-homomorphisms induces a* *fibre square of the associated cyclic algebras.*

Let G be a group then the algebra $\mathbb{Z}[G]$ is free as a two-sided $\mathbb{Z}$-module. It is well known (c. f. e. g. [17] p. 162) that the functor $- \otimes_R P$ is exact for any ring R and projective R-module P. Thus the functor $- \otimes \mathbb{Z}[G]$ is exact, and by Proposition 5.1.1 applying $- \otimes \mathbb{Z}[G]$ to (5.1) yields

$$\begin{array}{ccc} \Lambda[G] & \xrightarrow{\rho_-\otimes Id} & \Lambda_-[G] \\ {\scriptstyle \rho_+\otimes Id}\downarrow & & \downarrow{\scriptstyle \varphi_-\otimes Id} \\ \Lambda_+[G] & \xrightarrow{\varphi_+\otimes Id} & \Lambda_0[G] \end{array}, \tag{5.3}$$

i. e.

Proposition 5.1.3. *Given a fibre square of ring-homomorphisms such as* (5.1) *and a group G, the functor $- \otimes \mathbb{Z}[G]$ induces a fibre square of group-algebra homomorphisms such as* (5.3).

5.2 Examples of fibre squares

The following will be our source of fibre squares:

Proposition(c. f. e. g. [18] p. 435) 5.2.1. *If I and J are ideals of a ring Λ, the square of canonical maps*

$$\begin{array}{ccc} \Lambda/(I\cap J) & \longrightarrow & \Lambda/J \\ \downarrow & & \downarrow \\ \Lambda/I & \longrightarrow & \Lambda/(I+J) \end{array}$$

is a fibre square. In particular, all maps are automatically surjective.

We want to apply Proposition 5.1.1 to the ring $\mathbb{Z}[x]$, i. e. the ring of polynomials in one indeterminate, with integer coefficients. It is well known that $\mathbb{Z}[x]$

is a unique factorisation domain. Thus the following result on commutative unique factorisation domains will prove to be useful (c. f. e. g. [3] p. 502)

Proposition 5.2.2. *Let a_1, a_2 be two elements in a commutative unique factorisation domain R. Write $(a_1) \cap (a_2)$ for the intersection of the principal ideals generated by a_1 and a_2, respectively. Then*

$$(a_1) \cap (a_2) = (lcm(a_1, a_2)),$$

i. e. the principal ideal generated by the least common multiple of a_1 and a_2.

Consider the polynomial $x^{2n} - 1 \in \mathbb{Z}[x]$, for a natural number n. Evidently, $x^{2n} - 1 = (x^n - 1)(x^n + 1)$. Since, as we already noted in Chapter 2, $x^{2n} - 1 = \prod_{d|2n} \Phi_d$, it is a product of distinct irreducible polynomials. In particular, $x^n - 1$, $x^n + 1$ are co-prime, and therefore $lcm(x^n - 1, x^n + 1) = x^{2n} - 1$. Moreover, it can easily be verified that $(x^n - 1) + (x^n + 1) = (x^n - 1) + (2)$. Thus, by propositions 5.2.1 and 5.2.2 we have a fibre square

$$\begin{array}{ccc} \mathbb{Z}[x]/(x^{2n} - 1) & \longrightarrow & \mathbb{Z}[x]/(x^n + 1) \\ \downarrow & & \downarrow \\ \mathbb{Z}[x]/(x^n - 1) & \longrightarrow & \mathbb{F}_2[x]/(x^n - 1) \end{array} \tag{5.4}$$

of canonical surjections. Now, by Proposition 2.1.3, we may define the usual involution γ on each of the rings in (5.4). Moreover, they commute with the canonical surjections. Clearly, γ fixes the elements x^n, -1, 1, 1, in the rings $\mathbb{Z}[x]/(x^{2n} - 1)$, $\mathbb{Z}[x]/(x^n + 1)$, $\mathbb{Z}[x]/(x^n - 1)$, $\mathbb{F}_2[x]/(x^n - 1)$, respectively. By Proposition 5.1.2 we have a fibre square

$$\begin{array}{ccc} \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2n} - 1), \gamma, x^n\right) & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[x]/(x^n + 1), \gamma, -1\right) \\ \downarrow & & \downarrow \\ \mathcal{C}_2\left(\mathbb{Z}[x]/(x^n - 1), \gamma, 1\right) & \longrightarrow & \mathcal{C}_2\left(\mathbb{F}_2[x]/(x^n - 1), \gamma, 1\right) \end{array} . \tag{5.5}$$

We note that all maps are again surjective, since cyclic ring homomorphisms are exact functors (c. f. Proposition 2.1.2). Moreover, by the discussion in Chapter 2, we recognise $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2n}-1),\gamma,x^n\right)$, $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n-1),\gamma,1\right)$, and $\mathcal{C}_2\left(\mathbb{F}_2[x]/(x^n-1),\gamma,1\right)$, as $\mathbb{Z}[Q_{4n}]$, $\mathbb{Z}[D_{2n}]$ and $\mathbb{F}_2[D_{2n}]$, respectively. By Proposition 5.1.3, applying the exact functor $-\otimes\mathbb{Z}[G]$, for any group G, yields

Proposition 5.2.3. *For any group G there exists a fibre square of surjective ring-homomorphisms*

$$\begin{array}{ccc} \mathbb{Z}[Q_{4n}\times G] & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1),\gamma,-1\right)[G] \\ \downarrow & & \downarrow \\ \mathbb{Z}[D_{2n}\times G] & \longrightarrow & \mathbb{F}_2[D_{2n}\times G] \end{array} \quad . \tag{5.6}$$

Note, in the fibre square (5.6) the only unknown ring is $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1),\gamma,-1\right)[G]$. In order to understand this ring better, we shall factorise it as a fibre product. We make the following restriction:

$$n = 2^s k,$$

where $k>1$ is odd and $s\geq 1$. Choose k', a divisor of k, such that $k=pk'$, for some prime p, and consider the polynomial $x^{2^s k}+1\in\mathbb{Z}[x]$. Note $x^{2^s k}+1=\prod_{d|k}\Phi_{2^{s+1}d}(x)$ and $x^{2^s k'}+1=\prod_{d|k'}\Phi_{2^{s+1}d}(x)$, so $x^{2^s k'}+1$ divides $x^{2^s k}+1$. Moreover, if we define

$$\phi(x) := (x^{2^s k}+1)/(x^{2^s k'}+1),$$

then $\phi(x)$ and $x^{2^s k'}+1$ are co-prime, and therefore $lcm(\phi(x),x^{2^s k'}+1)=x^{2^s k}+1$. We make the following observation:

$$\phi(x)=\sum_{i=1}^{p}(-1)^{i+1}x^{2^s(k-ik')}=(x^{2^s k'}+1)\sum_{i=1}^{p-1}(-1)^{i+1}ix^{2^s(k-(1+i)k')}+p.$$

Thus $(\phi(x))+\left(x^{2^s k'}+1\right)=(p)+\left(x^{2^s k'}+1\right)$, and by propositions 5.2.1 and

5.2.2 we have a fibre square

$$\begin{array}{ccc} \mathbb{Z}[x]/(x^{2^s k}+1) & \longrightarrow & \mathbb{Z}[x]/(x^{2^s k'}+1) \\ \downarrow & & \downarrow \\ \mathbb{Z}[x]/\left(\phi(x)\right) & \longrightarrow & \mathbb{F}_p[x]/(x^{2^s k'}+1) \end{array} \tag{5.7}$$

of canonical surjections. Now, just as in the case of (5.4), we may define γ on each of the rings in (5.7), which in each case fixes the element -1. Thus the canonical surjections become cyclic ring-homomorphisms, and, by Proposition 5.1.2, induce a fibre square of cyclic algebras. Furthermore, for a given group G we apply the exact functor $-\otimes\mathbb{Z}[G]$ which by Proposition 5.1.3 gives

Proposition 5.2.4. *Let s, k, k', p and $\phi(x)$ be defined as above. For any group G, there exists a fibre square of surjective ring-homomorphisms*

$$\begin{array}{ccc} \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s k}+1),\gamma,-1\right)[G] & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s k'}+1),\gamma,-1\right)[G] \\ \downarrow & & \downarrow \\ \mathcal{C}_2\left(\mathbb{Z}[x]/\left(\phi(x)\right),\gamma,-1\right)[G] & \longrightarrow & \mathcal{C}\left(\Lambda_{(p,2^{s-1}k')}\right)[G] \end{array}, \tag{5.8}$$

where $\mathcal{C}\left(\Lambda_{(p,2^{s-1}k')}\right)[G] = \mathcal{C}_2\left(\mathbb{F}_p[x]/(x^{2^s k'}+1),\gamma,-1\right)[G]$*, as defined in* (2.11), *Chapter 2.*

A useful special case of Proposition 5.2.4 is given by:

Proposition 5.2.5. *For any odd prime p, there exists a fibre square of surjective ring-homomorphisms*

$$\begin{array}{ccc} \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1),\gamma,-1\right) & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[\zeta_{2^{s+1}}],\gamma,-1\right) \\ \downarrow & & \downarrow \\ \mathcal{C}_2\left(\mathbb{Z}[\zeta_{2^{s+1}p}],\gamma,-1\right) & \longrightarrow & \mathcal{C}\left(\Lambda_{(p,2^{s-1})}\right) \end{array} \tag{5.9}$$

Proof. Apply Proposition 5.2.4 with $G =< Id >$, the trivial group, $k = p$, and $k' = 1$. Moreover, note that $x^{2^s} + 1 = \Phi_{2^{s+1}}(x)$ and $x^{2^s p} + 1 = \Phi_{2^{s+1}}(x).\Phi_{2^{s+1}p}(x)$, thus $\phi(x) = \Phi_{2^{s+1}p}(x)$, by definition. □

5.3 Milnor patching

We close this chapter with a brief recapitulation on Milnor's method of constructing projective modules over fibre products, by 'patching' projective modules over the constituent factors (c. f. [20] p. 19-24). In our exposition we shall specifically focus on so called *locally free* modules and the sufficient condition, given by Milnor, for these to be projective. Thus, let

$$\begin{array}{ccc} \Lambda & \xrightarrow{\rho_-} & \Lambda_- \\ {\scriptstyle \rho_+}\downarrow & & \downarrow{\scriptstyle \varphi_-} \\ \Lambda_+ & \xrightarrow{\varphi_+} & \Lambda_0 \end{array} \tag{5.10}$$

be a fibre square of ring-homomorphisms. By Proposition 3.2.2 in Chapter 3, we know that each of the above ring-homomorphisms induces a functor of the respective module categories. In particular these functors restrict to the categories of finitely generated free, stably-free and projective modules, respectively. So, for example $\rho_- : \Lambda \to \Lambda_-$ induces a functor $\rho_{-*} : \mathfrak{M}(\Lambda) \to \mathfrak{M}(\Lambda_-)$, which preserves finitely generated free, stably-free and projective modules, respectively. Thus, given a projective module $P_- \in \mathfrak{P}(\Lambda_-)$ and a projective module $P_+ \in \mathfrak{P}(\Lambda_+)$, such that there exists an isomorphism

$$h : \varphi_{-*}(P_-) \to \varphi_{+*}(P_+),$$

as Λ_0-modules, define $M(P_-, P_+, h)$ to be the additive subgroup of $P_- \oplus P_+$ consisting of all pairs (p_-, p_+), such that $h\varphi_{-*}(p_-) = \varphi_{+*}(p_+)$. The corresponding fibre square of additive groups is given by

$$\begin{array}{ccc} M\left(P_{-}, P_{+}, h\right) & \longrightarrow & P_{-} \\ \downarrow & & \downarrow{\scriptstyle h\varphi_{-*}} \\ P_{+} & \xrightarrow{\varphi_{+*}} & \varphi_{+*}\left(P_{+}\right) \end{array}, \tag{5.11}$$

where the unlabelled arrows represent canonical maps. Moreover, $M\left(P_{-}, P_{+}, h\right)$ can be made into a right Λ-module via the following Λ-action

$$(p_{-}, p_{+}).\lambda = \left(p_{-}.\rho_{-}(\lambda), p_{+}.\rho_{+}(\lambda)\right).$$

We say $M\left(P_{-}, P_{+}, h\right)$ is finitely generated *locally free* of rank N whenever $P_{-} \cong \Lambda_{-}^{N}$, and $P_{+} \cong \Lambda_{+}^{N}$. In such a case $\varphi_{\sigma *}\left(P_{\sigma}\right) \cong {\Lambda_0}^{N}$, for $\sigma = +, -$, and thus h can be regarded as en element in $GL_N(\Lambda_0)$. So we define the set of finitely generated locally free Λ-modules of rank N to be

$$\mathfrak{LF}_N(\Lambda) = \left\{M\left({\Lambda_{-}}^{N}, {\Lambda_{+}}^{N}, h\right) : h \in GL_N(\Lambda_0)\right\}.$$

In particular, we note (c. f. e. g. [18] p. 441):

$$M\left(\Lambda_{-}^{N}, \Lambda_{+}^{N}, Id\right) \cong \Lambda^{N}. \tag{5.12}$$

The following gives a useful parametrisation of the isomorphism classes in $\mathfrak{LF}_N(\Lambda)$

Proposition (c. f. e. g. [18] p. 442) 5.3.1. *Let $h, h' \in GL_N(\Lambda_0)$. Then*

$$M\left(\Lambda_{-}^{N}, \Lambda_{+}^{N}, h\right) \cong M\left(\Lambda_{-}^{N}, \Lambda_{+}^{N}, h'\right)$$

if and only if there exist $\alpha_\sigma \in GL_N(\Lambda_\sigma)$, for $\sigma = +, -$, such that

$$h' = \varphi_{+}(\alpha_{+}^{-1}) h \varphi_{-}(\alpha_{-}),$$

where φ_σ *is applied entrywise. Thus writing* $\mathcal{LF}_N(\Lambda)$ *for the set isomorphism classes in* $\mathfrak{LF}_N(\Lambda)$*, we see that*

$$|\mathcal{LF}_N(\Lambda)| = |\varphi_+ \left(GL_N(\Lambda_+)\right) \backslash GL_N(\Lambda_0) / \varphi_- \left(GL_N(\Lambda_-)\right)|$$

We say the fibre square (5.10) satisfies the *patching condition* whenever $\mathfrak{LF}_N(\Lambda) \subseteq \mathfrak{P}(\Lambda)$. Moreover, let us call the fibre square (5.10) a *Milnor square* whenever at least one of φ_-, φ_+ is surjective. In [20], p. 21, Milnor first shows

Lemma 5.3.2. *Milnor squares satisfy the patching condition.*

He then continues to prove (c. f. [20] p. 23)

Theorem (Milnor) 5.3.3. *If* (5.10) *satisfies the patching condition, then* $M\left(P_-, P_+, h\right) \in \mathfrak{P}(\Lambda)$*. Moreover, it is finitely generated, and the modules* P_-, P_+ *are naturally isomorphic to* $\rho_{-*}\left(M\left(P_-, P_+, h\right)\right)$ *and* $\rho_{+*}\left(M\left(P_-, P_+, h\right)\right)$*, respectively. Conversely, given a projective module* $P \in \mathfrak{P}(\Lambda)$*, there exist* $P_- \in \mathfrak{P}\left(\Lambda_-\right)$*,* $P_+ \in \mathfrak{P}\left(\Lambda_+\right)$ *and* $h : \varphi_{-*}\left(P_-\right) \to \varphi_{+*}\left(P_+\right)$*, such that* $P \cong M\left(P_-, P_+, h\right)$*.*

Thus for the fibre squares discussed in the previous section we have

Proposition 5.3.4. *The fibre squares* (5.6)*,* (5.8) *and* (5.9) *are all Milnor squares, in particular they satisfy the patching condition, and therefore also Milnor's theorem (5.3.3, above).*

Remark. Swan points out in [28] p. 140 that there exist fibre squares which are not Milnor squares, but still satisfy the patching condition. In particular, he mentions *Karoubi squares*. These are fibre squares which satisfy *E-surjectivity*, a property which implies the patching condition. In fact, Johnson has used Karoubi squares to show there exist infinitely many (isomorphically distinct) stably-free modules over the group-algebra $\mathbb{Z}[Q_8 \times C_\infty]$ (c.f. [14]).

Chapter 6

Constructing and lifting stably-free modules

6.1 Constructing stably free modules

As explained in the introduction, our main task is to produce infinitely many, isomorphically distinct stably-free modules of rank 1 over the group-algebra $\mathbb{Z}[Q_{8n} \times G]$, where n admits an odd divisor and G is of type $\mathcal{F}$. Thus given a Milnor square

$$\begin{array}{ccc} \Lambda & \xrightarrow{\rho_-} & \Lambda_- \\ {\scriptstyle\rho_+}\downarrow & & \downarrow{\scriptstyle\varphi_-} \\ \Lambda_+ & \xrightarrow{\varphi_+} & \Lambda_0 \end{array}, \tag{6.1}$$

we ask, what further conditions are necessary in order to construct infinitely many, isomorphically distinct stably-free modules of rank one over the ring Λ. Consider locally-free modules. We remind ourselves that the set of locally-free modules of rank one over Λ is defined as

$$\mathfrak{L}\mathfrak{F}_1(\Lambda) = \{M(\Lambda_-, \Lambda_+, h) : h \in U(\Lambda_0)\},$$

where $U(\Lambda_0)$ denotes the units in Λ_0. Moreover, by Proposition 5.3.1, the set of isomorphism classes $\mathcal{LF}_1(\Lambda)$ is parametrised by the bijective correspondence

$$\mathcal{LF}_1(\Lambda) \leftrightarrow \varphi_+ \left(U(\Lambda_+)\right) \backslash U(\Lambda_0) / \varphi_- \left(U(\Lambda_-)\right).$$

As we know from Theorem 5.3.3 in the previous chapter, (6.1) being a Milnor square ensures that $\mathfrak{LF}_1(\Lambda) \subseteq \mathfrak{P}(\Lambda)$. However, it is not automatically true that every element in $\mathfrak{LF}_1(\Lambda)$ is stably free. Thus the following conditions are sufficient to construct a set $\{\mathcal{S}_n\}_{n=1}^{\infty} \subseteq \mathfrak{LF}_1(\Lambda)$, say $\mathcal{S}_n = M\left(\Lambda_-, \Lambda_+, h_n\right)$, of isomorphically distinct stably-free modules of rank 1:

i) The set $\varphi_+ \left(U(\Lambda_+)\right) \backslash U(\Lambda_0) / \varphi_- \left(U(\Lambda_-)\right)$ contains an infinite subset $\{(h_i)\}_{i=1}^{\infty}$. Equivalently, it is infinite.

ii) $\mathcal{S}_n \oplus \Lambda \cong \Lambda^2$ for all $n \in \mathbb{N}$.

We follow an approach by Johnson to produce sufficient conditions on a Milnor square for *i)* and *ii)* to hold. Since we are interested in stably-free modules over infinite group algebras, we shall restrict ourselves to precisely that case. Thus let G be a group, and Λ a ring. As usual, an element x in $\Lambda[G]$ is a sum

$$x = \sum_{g \in G} \lambda_g g$$

with all, but finitely many $a_g \in \Lambda$ equal to zero. Let us define the support of x, or $Supp(x)$ for short, to be the finite subset of G consisting of elements g, such that $\lambda_g \neq 0$ in $x = \sum_{g \in G} \lambda_g g$. Furthermore, we define a map $\chi : \Lambda[G] \to \mathbb{N}$, by

$$x \mapsto |Supp(x)|.$$

We say a unit in $u \in U\left(\Lambda[G]\right)$ is *trivial* whenever $\chi(u) = 1$. Evidently, this is the case, if and only if $u = \lambda_u g$, where $\lambda_u \in U\left(\Lambda\right)$ and $g \in G$. We denote by $\mathbf{T}$ the subgroup of trivial units of $U\left(\Lambda[G]\right)$. Note, there is a two-sided action of $\mathbf{T}$ on $\Lambda[G]$, $\mathbf{T} \times \Lambda[G] \times \mathbf{T} \to \Lambda[G]$, given by

$$(u_1, x, u_2) \mapsto u_1 x u_2.$$

Given $u = \lambda_u g \in \mathbf{T}$ and $x \in \Lambda[G]$, we observe

$$Supp(ux) = gSupp(x); \tag{6.2}$$

$$Supp(xu) = Supp(x)g. \tag{6.3}$$

Therefore, we have

Proposition 6.1.1. *Given x and y in $\Lambda[G]$, such that they are equivalent elements in the set $\mathbf{T}\backslash\Lambda[G]/\mathbf{T}$ then $\chi(x) = \chi(y)$.*

Proof. If x and y are equivalent in $\mathbf{T}\backslash\Lambda[G]/\mathbf{T}$, then $x = u_1 y u_2$, for some $u_1, u_2 \in \mathbf{T}$. But then by observations (6.2) and (6.3)

$$\chi(x) = \chi(u_1 y u_2) = \chi(y).$$

□

Next we prove

Proposition 6.1.2. *Let G be an infinite group and Λ a ring which contains a non-zero, nilpotent element; then the set $\mathbf{T}\backslash U(\Lambda[G])/\mathbf{T}$ of double cosets is infinite. In particular, it has an infinite subset $\left\{\left(x_{(n,+)}\right) : n \in \mathbb{N}\right\}$, where*

$$x_{(n,+)} = 1 + \lambda \sum_{i=1}^{n-1} g_i,$$

for distinct elements $g_i \in G$ and λ a non-zero element in Λ, such that $\lambda^2 = 0$.

Proof. Since G is infinite we may choose a family of subsets $G_n = \{g_1, \ldots, g_{n-1}\} \subseteq G - \{Id\}$, such that $|G_n| = n - 1$, for arbitrary $n \in \mathbb{N}$.

Let us write x_n for the associated element

$$x_n = \sum_{i=1}^{n-1} g_i$$

in $\Lambda[G]$. Clearly, $Supp(x_n) = G_n$. Moreover, since Λ has a non-zero, nilpotent element it contains an element λ, such that $\lambda \neq 0$, but $\lambda^2 = 0$. Thus, writing $x_{(n,+)} = 1 + \lambda x_n$ and $x_{(n,-)} = 1 - \lambda x_n$, we see

$$x_{(n,+)} x_{(n,-)} = (1 + \lambda x_n)(1 - \lambda x_n) = 1 - \lambda^2 x_n^2 = 1$$

and similarly $x_{(n,-)} x_{(n,+)} = 1$. This shows that $x_{(n,+)}$ is a unit. Notice, since $Supp\left(x_{(n,+)}\right) = 1 \cup G_n$, we have $\chi\left(x_{(n,+)}\right) = n$. Writing $\left(x_{(n,+)}\right)$ for the class of $x_{(n,+)}$ in $\mathbf{T} \backslash U\left(\Lambda[G]\right)/\mathbf{T}$, suppose $\left(x_{(n,+)}\right) \sim \left(x_{(m,+)}\right)$, for some $m \in \mathbb{N}$. Then Proposition 6.1.1 implies $n = m$. Therefore, the set $\left\{\left(x_{(n,+)}\right) : n \in \mathbb{N}\right\} \subseteq \mathbf{T} \backslash U\left(\Lambda[G]\right)/\mathbf{T}$ is infinite. □

We now consider examples of group algebras with trivial units, only. A group G is said to be a *two unique products group*, or t.u.p.-group for short, if, given any two nonempty, finite subsets A and B of G with $|A| + |B| > 2$, there exist at least two distinct elements g and h in G which have unique representations in the form $g = ab$, $h = cd$, where $a, c \in A$ and $b, d \in B$. We note, every right ordered group is t.u.p. (c.f. e.g. [24], p. 588). Therefore, free abelian groups are t.u.p.-groups.

Proposition 6.1.3. *Let Λ be an integral domain, possibly non-commutative; then for any t.u.p.-group G, $\Lambda[G]$ has only trivial units.*

Note, a proof of Proposition 6.1.3, in the case when Λ is a field, is given in [24], p. 589. The proof can without modification be extended to that of Proposition 6.1.3. For the sake of completeness, we add a proof at this point.

Proof. Given $x, y \in \Lambda[G]$, with $xy = 1$ and $yx = 1$, note that assuming $\chi(x) = 1$ implies $\chi(y) = 1$, as Λ has no non-trivial zero-divisors. Similarly, if $\chi(y) = 1$, then $\chi(x) = 1$. Thus assume $\chi(x) \geq 2$, $\chi(y) \geq 2$. Put $X = Supp(x)$ and $Y = Supp(y)$. Since G is a t.u.p.-group there exist distinct $g_1, g_2 \in G$ which are uniquely represented in XY, say $g_1 = ab$, $g_2 = cd$, where $a, c \in X$ and $b, d \in Y$. Firstly, this implies that g_1, g_2 appear only once, respectively, in the product xy. Moreover, since a, c and b, d are in the support of x and y, respectively then $g_1 = ab$, $g_2 = cd$ have non-zero coefficients in xy, as Λ is an integral domain. Therefore, $g_1, g_2 \in Supp(xy)$, and $\chi(xy) \geq 2$ which is a contradiction, since $xy = 1$. □

As discussed in Proposition 5.1.2, Chapter 5, a ring homomorphism $\varphi : \Lambda \to \Lambda'$, and a group G, induce an algebra homomorphism $\varphi^* : \Lambda[G] \to \Lambda'[G]$, where $\varphi^* = \varphi \otimes Id_{\mathbb{Z}[G]}$. Thus, in the case of group algebras of t.u.p. groups, a set of sufficient conditions for $i)$ above to hold is given by

Theorem 6.1.4. *For $\sigma = +, -$ let $\varphi_\sigma : \Lambda_\sigma \to \Lambda_0$ be ring homomorphisms where Λ_σ is an integral domain, and Λ_0 contains a non-trivial nilpotent element; then for any t.u.p. group G the set $\varphi_+^* \left(U(\Lambda_+[G])\right) \backslash U(\Lambda_0[G]) / \varphi_-^* \left(U(\Lambda_-[G])\right)$ has quotient $\mathbf{T} \backslash U \left(\Lambda_0[G]\right) / \mathbf{T}$. In particular, it is infinite.*

Proof. As usual, let $\mathbf{T}$ denote the trivial units in $U(\Lambda_0[G])$. By Proposition 6.1.3 $U(\Lambda_\sigma[G])$ only has trivial units, and therefore $\varphi_\sigma^* \left(U(\Lambda_\sigma[G])\right) \subseteq \mathbf{T}$. This implies that $\mathbf{T} \backslash U \left(\Lambda_0[G]\right) / \mathbf{T}$ is indeed a quotient of $\varphi_+^* \left(U(\Lambda_+[G])\right) \backslash U(\Lambda_0[G]) / \varphi_-^* \left(U(\Lambda_-[G])\right)$. Moreover, by Proposition 6.1.2, $\mathbf{T} \backslash U \left(\Lambda_0[G]\right) / \mathbf{T}$ is infinite then by the above so is also $\varphi_+^* \left(U(\Lambda_+[G])\right) \backslash U(\Lambda_0[G]) / \varphi_-^* \left(U(\Lambda_-[G])\right)$. □

Next, we give sufficient conditions for $ii)$ above to hold

Theorem 6.1.5. *Given a Milnor square*

$$\begin{array}{ccc} A & \xrightarrow{\rho_-} & A_- \\ {\scriptstyle \rho_+}\downarrow & & \downarrow{\scriptstyle \varphi_-} \\ A_+ & \xrightarrow{\varphi_+} & A_0 \end{array} ,$$

suppose A_0 has a subring R, such that there exists a ring-isomorphism $A_0 \cong M_2(R)$, i.e. A_0 is isomorphic to the ring of 2×2 matrices over R. Then for any $r \in R$ there exists a locally free module of rank one, $S(r)$, over A, such that

$$S(r) \oplus \Lambda \cong \Lambda^2.$$

Proof. For notational simplicity, we make the explicit identification $A_0 = M_2(R)$. Thus note, for any $r \in R$, the element

$$1 + r\nu = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \in \Lambda_0, \quad \text{where} \quad \nu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

is a unit. We make the following definition

$$S(r) := M\left(A_-, A_+, 1 + r\nu\right).$$

Now, it is not hard to see (c.f. e.g. [18], p. 440) that

$$S(r) \oplus A \cong M\left(A_-^2, A_+^2, (1 + r\nu) \oplus 1\right).$$

By Proposition 5.3.1 and (5.12) in Chapter 5, it remains to show

$$(1 + r\nu) \oplus 1 = \begin{pmatrix} 1 + r\nu & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\varphi_+\left(GL_2(A_+)\right) \backslash GL_2(A_0) / \varphi_-\left(GL_2(A_-)\right)$. By assumption $A_0 = M_2(R)$, and therefore every 2×2 matrix over A_0 may equivalently be regarded as 4×4 matrix over R. In particular,

$$\begin{pmatrix} 1 & r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 + r\nu & 0 \\ 0 & 1 \end{pmatrix}.$$

Conversely the following invertible 4×4 matrices over R

$$\alpha = \begin{pmatrix} 1 & 0 & r & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

may be regarded as the 2×2 elementary transvections (c.f. Chapter 4) over A_0, i.e. $\alpha \equiv e_{12}(\eta)$, and $\beta \equiv e_{21}(\nu)$, where

$$\eta = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, \nu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in A_0.$$

A simple computation yields

$$e_{12}(\eta)e_{21}(\nu)e_{12}(-\eta)e_{21}(-\nu) \equiv \begin{pmatrix} 1 & r & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} 1+r\nu & 0 \\ 0 & 1 \end{pmatrix}.$$

Also, since (6.1) is a Milnor square we may assume, without loss of generality, that $\varphi_+ : A_+ \to A_0$ is surjective. Therefore, there exist elements $\tilde{\eta}, \tilde{\nu} \in A_+$, such that $\varphi_+(\tilde{\eta}) = \eta$, $\varphi_+(\tilde{\nu}) = \nu$. Thus applying φ_+ entrywise, we see that

$$\begin{aligned} \varphi_+ \left(e_{12}(\tilde{\eta})e_{21}(\tilde{\nu})e_{12}(-\tilde{\eta})e_{21}(-\tilde{\nu})\right) &= \\ \varphi_+(e_{12}(\tilde{\eta}))\varphi_+(e_{21}(\tilde{\nu}))\varphi_+(e_{12}(-\tilde{\eta}))\varphi_+(e_{21}(-\tilde{\nu})) &= \\ e_{12}(\eta)e_{21}(\nu)e_{12}(-\eta)e_{21}(-\nu) &= \\ \begin{pmatrix} 1+r\nu & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

i.e.

$$\begin{pmatrix} 1+r\nu & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\varphi_+\left(GL_2(A_+)\right)\backslash GL_2(A_0)/\,\varphi_-\left(GL_2(A_-)\right)$. $\square$

We say ring Λ is *constrained* whenever it factorises as a Milnor square such as (6.1), such that

(a) Λ_+, Λ_- are, possibly non-commutative, integral domains.

(b) Λ_0 has a subring R, such that there exists a ring isomorphism $\Lambda_0 \cong M_2(R)$, i.e. Λ_0 is isomorphic to the ring of 2×2 matrices over R.

Theorem 6.1.6. *Let Λ be a constrained ring and G a t.u.p group; then there exists an infinite set, $\{\mathcal{S}_n\}_{n=1}^{\infty}$, of isomorphically distinct stably-free modules of rank 1 over the group-algebra $\Lambda[G]$.*

Proof. Let us assume Λ has factorisation (6.1). By Proposition 5.1.3, Chapter 5, applying the functor $- \otimes \mathbb{Z}[G]$ to (6.1) induces a fibre square of group algebras

$$\begin{array}{ccc} \Lambda[G] & \xrightarrow{\rho_-\otimes Id} & \Lambda_-[G] \\ {\scriptstyle \rho_+\otimes Id}\downarrow & & \downarrow{\scriptstyle \varphi_-\otimes Id} \\ \Lambda_+[G] & \xrightarrow{\varphi_+\otimes Id} & \Lambda_0[G] \end{array} .$$

Then, since $\Lambda_0 \cong M_2(R)$ for some subring R, we see that $\Lambda_0[G] \cong M_2(R[G])$. Thus, by Theorem 6.1.5,

$$S(r) = M\left(\Lambda_-[G], \Lambda_+[G], 1 + r\nu\right)$$

is stably-free, of rank one. Here $\nu = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \Lambda_0[G]$ and $r \in R[G]$. Moreover, viewing Λ_0 as a subring of $\Lambda_0[G]$, we see that $\nu \in \Lambda_0$, and $\nu \neq 0$, $\nu^2 = 0$. Thus, by Proposition 6.1.2 the elements

$$x_{(n,+)} = 1 + \nu \sum_{i=1}^{n-1} g_i,$$

for distinct $g_i \in G$, give rise to an infinite subset $\left\{\left(x_{(n,+)}\right) : n \in \mathbb{N}\right\} \subseteq \mathbf{T}\backslash U\left(\Lambda[G]\right)/\mathbf{T}$. Now, by Theorem 6.1.4 $\mathbf{T}\backslash U\left(\Lambda_0[G]\right)/\mathbf{T}$ is a quotient of $\varphi_+^*\left(U(\Lambda_+[G])\right)\backslash U(\Lambda_0[G])/\varphi_-^*\left(U(\Lambda_-[G])\right)$ (here $\varphi_\sigma^* = \varphi_\sigma \otimes Id_{\mathbb{Z}[G]}$ for $\sigma = +, -$), and therefore $\left\{\left(x_{(n,+)}\right) : n \in \mathbb{N}\right\}$ may equally be regarded as an infinite subset of $\varphi_+^*\left(U(\Lambda_+[G])\right)\backslash U(\Lambda_0[G])/\varphi_-^*\left(U(\Lambda_-[G])\right)$. Consequently, defining

$$\mathcal{S}_n := M\left(\Lambda_-[G], \Lambda_+[G], x_{(n,+)}\right),$$

we see, either by 5.3.1, Chapter 5, or, equivalently, the discussion at the beginning of this section that $\mathcal{S}_n \cong \mathcal{S}_{n'}$, if and only if $n = n'$. Finally, we note that $\sum_{i=1}^{n-1} g_i \in R[G]$. This ensures that $\{\mathcal{S}_n\}_{n=1}^{\infty} \subseteq \{S(r) : r \in R[G]\}$, i.e. $\mathcal{S}_n$ is stably-free of rank one for all $n \in \mathbb{N}$. □

6.2 Lifting stably free modules

As we have seen in the previous section, constrained rings give rise to group algebras with infinitely many, isomorphically distinct stably free modules of rank one. However, many of the algebras which we are interested in, are too complex, to satisfy the conditions necessary to be constrained. Thus a feasible approach would be, to factor a given ring as a fibre product, and find sufficient conditions for the stably free modules over the constituent factors to be lifted to stably free modules over the original ring. The following result by Johnson (c.f. [14]), does precisely that.

Theorem (Johnson) 6.2.1. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\rho_-} & A_- \\ {\scriptstyle\rho_+}\downarrow & & \downarrow{\scriptstyle\varphi_-} \\ A_+ & \xrightarrow{\varphi_+} & A_0 \end{array} \qquad (6.4)$$

be a Milnor square, such that A_0 is weakly Euclidean and has stably-free *cancellation. Then there exists a surjective correspondence*

$$\rho_{+*} \times \rho_{-*} : \mathcal{SF}_1(A) \to \mathcal{SF}_1(A_+) \times \mathcal{SF}_1(A_-),$$

given by the functors $\rho_{\sigma *} : \mathfrak{M}(A_\sigma) \to \mathfrak{M}(A_0)$, *where* $\sigma = +, -$.

The proof of this statement occupies the rest of this section. We start with a definition. Let Λ be a ring and $k \geq 1$ an integer. By $\mathcal{S}(k, 1)$ we mean the standard short exact sequence $0 \longrightarrow \Lambda^k \xrightarrow{\imath} \Lambda^{k+1} \xrightarrow{\pi} \Lambda \longrightarrow 0$, i.e. where

$$\imath \begin{pmatrix} y_1 \\ \vdots \\ y_k \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ \vdots \\ y_k \end{pmatrix}, \quad \text{and} \quad \pi \begin{pmatrix} x \\ y_1 \\ \vdots \\ y_k \end{pmatrix} = x.$$

We shall need

Lemma 6.2.2. *Let* (6.4) *be a fibre square satisfying the patching condition and let* S_+, S_- *be stably-free modules of rank 1 over* A_+, A_-, *respectively, such that*

$$S_+ \otimes_{\varphi_+} A_0 \cong S_- \otimes_{\varphi_-} A_0 \cong A_0.$$

Moreover, assume that the ring A_0 *is weakly Euclidean. Then there exist an* A_0*-isomorphism* $h : S_- \otimes A_0 \to S_+ \otimes A_0$ *and an element* $E \in E_{k+1}(A_0)$ *(c.f. Chapter 4),* $k \geq 1$, *such that we have an* A*-module isomorphism*

$$M(S_+, S_-, h) \oplus A^k \cong M(A_+^{k+1}, A_-^{k+1}, E).$$

Proof. Choose A_0-isomorphisms $\eta_\sigma : S_\sigma \otimes_{\varphi_\sigma} A_0 \to A_0$ for $\sigma = +, -$, respectively. Now, S_σ is stably-free of rank 1, i.e. $S_\sigma \oplus A_\sigma^{k_\sigma} \cong A_\sigma^{k_\sigma+1}$, for some $k_\sigma \geq 1$. Choose $k = max\{k_+, k_-\}$, thus $S_\sigma \oplus A_\sigma^k \cong A_\sigma^{k+1}$, and we may choose specific exact sequences $0 \longrightarrow A_\sigma^k \xrightarrow{i_\sigma} A_\sigma^{k+1} \xrightarrow{p_\sigma} S_\sigma \longrightarrow 0$ of A_σ-homomorphisms. Applying the exact functor $- \otimes_{\varphi_\sigma} A_0$ we get exact sequences:

$$\mathcal{F}_\sigma : \qquad 0 \longrightarrow A_0^k \xrightarrow{i_{\sigma *}} A_0^{k+1} \xrightarrow{p_{\sigma *}} A_0 \longrightarrow 0 \; ,$$

for $\sigma = +, -$. Here, $i_{\sigma *} = \varphi_{\sigma *}(i_\sigma) = i_\sigma \otimes_{\varphi_\sigma} Id$ and $p_{\sigma *} = \varphi_{\sigma *}(p_\sigma) = p_\sigma \otimes_{\varphi_\sigma} Id$. Let the A_0-linear map $j_\sigma : A_0^{k+1} \to A_0^k$ be a left splitting for $\mathcal{F}_\sigma$, i.e. $j_\sigma i_{\sigma *} = Id$, and define $X_\sigma : A_0^{k+1} \to A_0^{k+1}$ by $X_\sigma := \eta_\sigma p_{\sigma *} \oplus j_\sigma$. This gives the following isomorphism of short exact sequences

$$\begin{array}{c} \mathcal{F}_\sigma \\ {\scriptstyle X_\sigma}\downarrow \\ \mathcal{S}(k,1) \end{array} = \left(\begin{array}{ccccccccc} 0 & \longrightarrow & A_0^k & \xrightarrow{i_{\sigma *}} & A_0^{k+1} & \xrightarrow{p_{\sigma *}} & S_\sigma \otimes_{\varphi_\sigma} A_0 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle X_\sigma} & & \downarrow{\scriptstyle \eta_\sigma} & & \\ 0 & \longrightarrow & A_0^k & \xrightarrow{\imath} & A_0^{k+1} & \xrightarrow{\pi} & A_0 & \longrightarrow & 0 \end{array} \right).$$

Then by the Five Lemma X_σ is an A_0-isomorphism. By assumption A_0 is weakly Euclidean. Therefore, by Proposition 4.1.2, Chapter 4, $X_\sigma = d_1(u_\sigma)E_\sigma$, for a unit $u_\sigma \in A_0^*$ and product of elementary transvections $E_\sigma \in E_{k+1}(A_0)$. Thus let $u_\sigma^{-1} \in A_0^*$ be the inverse of u_σ. Note that $d_1(u_\sigma^{-1}) \in GL_{k+1}(A_0)$ gives rise to the following automorphism of $\mathcal{S}(k,1)$

$$\begin{array}{c} \mathcal{S}(k,1) \\ {\scriptstyle d_1(u_\sigma^{-1})}\downarrow \\ \mathcal{S}(k,1) \end{array} = \left(\begin{array}{ccccccccc} 0 & \longrightarrow & A_0^k & \xrightarrow{\imath} & A_0^{k+1} & \xrightarrow{\pi} & A_0 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle d_1(u_\sigma^{-1})} & & \downarrow{\scriptstyle u_\sigma^{-1}} & & \\ 0 & \longrightarrow & A_0^k & \xrightarrow{\imath} & A_0^{k+1} & \xrightarrow{\pi} & A_0 & \longrightarrow & 0 \end{array} \right).$$

Recall, $d_1(u_\sigma^{-1}) = d_1(u_\sigma)^{-1}$, and so the composition $d_1(u_\sigma^{-1}) \circ X_\sigma$, interpreted as an isomorphism $d_1(u_\sigma^{-1}) \circ X_\sigma : \mathcal{F}_\sigma \to \mathcal{S}(k,1)$, is of the form

$$\begin{array}{c} \mathcal{F}_\sigma \\ \downarrow \\ \mathcal{S}(k,1) \end{array} = \left(\begin{array}{ccccccccc} 0 & \longrightarrow & A_0^k & \xrightarrow{i_{\sigma *}} & A_0^{k+1} & \xrightarrow{p_{\sigma *}} & S_\sigma \otimes_{\varphi_\sigma} A_0 & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle E_\sigma} & & \downarrow{\scriptstyle u_\sigma^{-1}\eta_\sigma} & & \\ 0 & \longrightarrow & A_0^k & \xrightarrow{\imath} & A_0^{k+1} & \xrightarrow{\pi} & A_0 & \longrightarrow & 0 \end{array} \right).$$

Inverting the isomorphism for $\sigma = +$ and composing we get an isomorphism $X_+ d_1(u_+) d_1(u_-^{-1}) X_- : \mathcal{F}_- \to \mathcal{F}_+$, i.e.

$$\begin{array}{c} \mathcal{F}_- \\ \big\downarrow \\ \mathcal{F}_+ \end{array} = \left(\begin{array}{ccccccccc} 0 & \longrightarrow & A_0^k & \xrightarrow{i_{-*}} & A_0^{k+1} & \xrightarrow{p_{-*}} & S_- \otimes_{\varphi_-} A_0 & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow E & & \Big\downarrow h & & \\ 0 & \longrightarrow & A_0^k & \xrightarrow{i_{-*}} & A_0^{k+1} & \xrightarrow{p_{+*}} & S_+ \otimes_{\varphi_+} A_0 & \longrightarrow & 0 \end{array} \right), \tag{6.5}$$

where $E = E_+^{-1} E_-$, and $h = \eta_+^{-1} u_+ u_-^{-1} \eta_-$. Note, for $\sigma = +, -$ our original choices of exact sequences $0 \longrightarrow A_\sigma^k \xrightarrow{i_\sigma} A_\sigma^{k+1} \xrightarrow{p_\sigma} S_\sigma \longrightarrow 0$ are split exact, thus so are also $\mathcal{F}_\sigma$. But then (6.5) gives rise to a short exact sequence of A-modules thus

$$0 \longrightarrow M(A_+^k, A_-^k, Id) \xrightarrow{(i_+, i_-)} M(A_+^{k+1}, A_-^{k+1}, E) \xrightarrow{(p_+, p_-)} M(S_+, S_-, h) \longrightarrow 0\,.$$

Now, (6.4) satisfies the patching condition, and therefore, by Theorem 5.3.3, $M(S_+, S_-, h)$ is projective. Moreover, by (5.12), $M(A_+^k, A_-^k, Id) \cong A^k$. So that $M(S_+, S_-, h) \oplus A^k \cong M(A_+^{k+1}, A_-^{k+1}, E)$, as claimed. □

We prove Theorem 6.2.1.

Proof. Given stably free modules of rank one, S_+, S_-, over A_+, A_-, respectively. We see that

$$S_+ \otimes_{\varphi_+} A_0 \cong S_- \otimes_{\varphi_-} A_0 \cong A_0,$$

as A_0 has stably-free cancellation. Moreover, since 6.4 is a Milnor square it satisfies the patching condition (c.f. Lemma 5.3.2, Chapter 5). Thus, Lemma 6.2.2 gives an A_0-isomorphism $h : S_- \otimes A_0 \to S_+ \otimes A_0$ and an element $E \in E_{k+1}(A_0)$, $k \geq 1$, such that we have an A-module isomorphism

$$M(S_+, S_-, h) \oplus A^k \cong M(A_+^{k+1}, A_-^{k+1}, E).$$

Recall, $E \in E_{k+1}(A_0)$ is a finite length product of $k+1 \times k+1$ elementary tranvections. As already discussed in Chapter 4, in general such an element is of the form $e_{ij}(a_0) = I_{k+1} + a_0\epsilon_{ij}$, where $a_0 \in A_0$ and $1 \leq i, j \leq k+1$. Now, since (6.4) is a Milnor square, we may assume, without loss of generality, that $\varphi_+ : A_+ \to A_0$ is surjective. Therefore, there exists an element $a_+ \in A_+$, such that $\varphi_+(a_+) = a_0$. Thus applying φ_+ entrywise, we see that $\varphi_+(e_{ij}(a_+)) = e_{ij}(a_0)$, i.e. $e_{ij}(a_0) \sim I_{k+1}$ in $\varphi_+(GL_{k+1}(A_+)) \backslash GL_{k+1}(A_0) / \varphi_-(GL_{k+1}(A_-))$. But then since φ_+ applied entrywise gives a group-homomorphism $\varphi_+ : GL_{k+1}(A_+) \to GL_{k+1}(A_0)$, we see that the same holds true for any finite product of elementary transvections. In particular, $E \sim I_{k+1}$. So by Theorem 5.3.1 and (5.12) in Chapter 5

$$M(A_+^{k+1}, A_-^{k+1}, E) \cong M(A_+^{k+1}, A_-^{k+1}, I_{k+1}) \cong A^{k+1}.$$

Consequently, $M(S_+, S_-, h) \oplus A^k \cong A^{k+1}$, i.e. $M(S_+, S_-, h)$ is a stably-free A-module of rank one. Finally, Theorem 5.3.3 ensures that $\rho_{\sigma *}(M(S_+, S_-, h)) = S_\sigma$, for $\sigma = +, -$, which proves the claim. □

6.3 The main theorem

We state the main theorem thus

Theorem 6.3.1. *Let G be a group of type $\mathcal{F}$. Moreover, let Q_{8n} be the quaternionic group with $8n$ elements; then for n with at least one odd prime divisor, there is an infinite collection $\{\mathcal{S}_m\}_{m \geq 1}$ of isomorphically distinct stably-free modules of rank one over the group-algebra $\mathbb{Z}[Q_{8n} \times G]$.*

To prove this statement we apply the framework for constructing and lifting stably-free modules discussed in the previous two sections of this chapter. We shall use the fibre squares (5.6), (5.8) and (5.9), constructed in Chapter 5. The results established in chapters 2 to 4 show that the algebras making up these fibre squares do indeed have the necessary properties to apply theorems

6.1.6 and 6.2.1. We shall prove two important intermediary theorems which will imply Theorem 6.3.1. But first a technical proposition which shows that the essential case to consider is $G = C_\infty$, the infinite cyclic group.

Proposition 6.3.2. *Let Λ be a ring, G a group of type $\mathcal{F}$ and C_∞ the infinite cyclic group. Then the induced group-algebra epimorphism $\gamma : \Lambda[G] \to \Lambda[C_\infty]$ gives a surjective correspondence*

$$\gamma_* : \mathcal{SF}_N\left(\Lambda[G]\right) \to \mathcal{SF}_N(\Lambda[C_\infty])\,,$$

for any integer $N \geq 1$.

Proof. Let $\gamma_* : \mathfrak{M}(\Lambda[G]) \to \mathfrak{M}(\Lambda[C_\infty])$ be the functor which is given by $\gamma_* = - \otimes_\gamma \Lambda[C_\infty]$. Now since G is of type $\mathcal{F}$ and $\gamma : G \to C_\infty$ is surjective, there exists a group homomorphism $\delta : C_\infty \to G$, such that $\gamma \circ \delta = Id_{C_\infty}$. Moreover, the the same holds true for the induced map $\delta : \Lambda[C_\infty] \to \Lambda[G]$. But then the functor $\delta_* : \mathfrak{M}(\Lambda[C_\infty]) \to \mathfrak{M}(\Lambda[G])$, $\delta_* = - \oplus_\delta \Lambda[G]$, is a right inverse to γ_*. Let S_1, S_2 be two stably-free modules of rank N over $\Lambda[C_\infty]$, such that $S_1 \ncong S_2$. Firstly, note $\delta_*(S_1), \delta_*(S_2)$ are stably free of rank N (c.f. Proposition 3.2.2, Chapter 3). Finally, if $\delta_*(S_1) \cong \delta_*(S_2)$, then, since $\gamma_* \circ \delta_*$ is the identity functor on $\mathfrak{M}(\Lambda[C_\infty])$, we have $S_1 \cong S_2$ which is a contradiction. □

Recall the algebra $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[G]$ introduced in Proposition 5.2.3, Chapter 5.

Theorem 6.3.3. *Let G be a free group. Moreover, let Q_{4n}, D_{2n} denote the quaternionic, and dihedral groups, of order $4n$, $2n$, respectively. There exists a surjective correspondence*

$$\mathcal{SF}_1\left(\mathbb{Z}[Q_{4n} \times G]\right) \to \mathcal{SF}_1\left(\mathbb{Z}[D_{2n} \times G]\right) \times \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[G]\right).$$

Proof. Note, by Proposition 5.3.1,

$$\begin{array}{ccc} \mathbb{Z}[Q_{4n} \times G] & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[G] \\ \downarrow & & \downarrow \\ \mathbb{Z}[D_{2n} \times G] & \longrightarrow & \mathbb{F}_2[D_{2n} \times G] \end{array}.$$

is a Milnor square. Moreover, by propositions 4.4.3 and 3.3.6, $\mathbb{F}_2[D_{2n} \times G]$ is weakly Euclidean, and has stably-free cancellation. Therefore, we may apply Theorem 6.2.1 which yields the result. □

Theorem 6.3.4. *Let G be a t.u.p group, p an odd prime and $s \geq 1$ an integer. Recall, the algebra $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1), \gamma, -1\right)$ of Proposition 5.2.5, Chapter 5. There exists an infinite set, $\{\mathcal{S}_m\}_{m=1}^{\infty}$, of isomorphically distinct stably-free modules of rank 1 over the group-algebra $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1), \gamma, -1\right)[G]$.*

Proof. By Proposition 5.3.1,

$$\begin{array}{ccc} \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1), \gamma, -1\right) & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[\zeta_{2^{s+1}}], \gamma, -1\right) \\ \downarrow & & \downarrow \\ \mathcal{C}_2\left(\mathbb{Z}[\zeta_{2^{s+1}p}], \gamma, -1\right) & \longrightarrow & \mathcal{C}\left(\Lambda_{(p,2^{s-1})}\right) \end{array}$$

is a Milnor square. Also, by Proposition 2.4.4, $\mathcal{C}_2\left(\mathbb{Z}[\zeta_{2^{s+1}}], \gamma, -1\right)$ as well as $\mathcal{C}_2\left(\mathbb{Z}[\zeta_{2^{s+1}p}], \gamma, -1\right)$ are integral domains. Furthermore, by Theorem 2.4.8 $\mathcal{C}(\Lambda_{(p,2^{s-1})}) \cong M_2(\Lambda^{\gamma}_{(p,2^{s-1})})$ for the subring $\Lambda^{\gamma}_{(p,2^{s-1})} \subseteq \mathcal{C}(\Lambda_{(p,2^{s-1})})$. Therefore, $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p}+1), \gamma, -1\right)$ is constrained, and we may apply Theorem 6.1.6 which yields the result. □

We now prove the main theorem.

Proof. Firstly, note that, by Proposition 6.3.2, it is enough to prove the statement for $G = C_\infty$. Moreover, since C_∞ is free (of rank one), as well as a t.u.p. group, theorems 6.3.3 and 6.3.4 certainly apply for $G = C_\infty$. Thus consider Q_{4n}, and let $n = 2^s k$, where $s \geq 1$, is an integer and $k \geq 3$ an odd number. We write $k = \prod_{j=1}^{r} {p_j}^{d_j}$ for the prime decomposition of k into the odd primes p_j, $d_j > 0$. By renumbering we may write $k = \prod_{j=1}^{r'} p_j$. We make the following definitions:

$$\begin{array}{rcl} n_i := & 2^s \prod_{j=1}^{i} p_j & 1 \leq i \leq r' \\ \phi_i(x) := & (x^{n_{i+1}} + 1)/(x^{n_i} + 1) & 1 \leq i \leq r' - 1. \end{array}$$

We note that $n_{i+1}/n_i = p_{i+1}$. So for $1 \leq i \leq r' - 1$ Proposition 5.2.4 gives the following fibre square of ring homomorphisms

$$\begin{array}{ccc} \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{n_{i+1}}+1), \gamma, -1\right)[C_\infty] & \longrightarrow & \mathcal{C}_2\left(\mathbb{Z}[x]/(x^{n_i}+1), \gamma, -1\right)[C_\infty] \\ \downarrow & & \downarrow \\ \mathcal{C}_2\left(\mathbb{Z}[x]/\left(\phi_i(x)\right), \gamma, -1\right)[C_\infty] & \longrightarrow & \mathcal{C}\left(\Lambda_{(p_{i+1}, 2^{-1}n_i)}\right)[C_\infty] \end{array}$$

which, by Proposition 5.3.1, is a Milnor square. Now, by Theorem 3.3.4, $\mathcal{C}\left(\Lambda_{(p_{i+1}, 2^{-1}n_i)}\right)[C_\infty]$ has stably-free cancellation. Moreover, by Theorem 4.4.2, it is weakly Euclidean. So, Theorem 6.2.1 gives us a surjective map on isomorphism classes of stably free modules of rank 1

$$\psi_i : \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{n_{i+1}}+1), \gamma, -1\right)[C_\infty]\right) \to \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{n_i}+1), \gamma, -1\right)[C_\infty]\right),$$

for $1 \leq i \leq r' - 1$. Clearly, $n_{r'} = n$, and $n_1 = 2^s p_1$, so we have a surjective correspondence $\Psi := \psi_1 \circ \ldots \circ \psi_{r'-1}$

$$\Psi : \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{n}+1), \gamma, -1\right)[C_\infty]\right) \to \mathcal{SF}_1\left(\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p_1}+1), \gamma, -1\right)[C_\infty]\right).$$

But we know, by Theorem 6.3.4, that there exist infinitely many, isomorphically distinct stably-free modules of rank one over

$\mathcal{C}_2\left(\mathbb{Z}[x]/(x^{2^s p_1}+1), \gamma, -1\right)[C_\infty]$, thus by the above they have corresponding equivalents over $\mathcal{C}_2\left(\mathbb{Z}[x]/(x^n+1), \gamma, -1\right)[C_\infty]$. Finally, the result follows by Theorem 6.3.3.

□

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