HOUSEHOLD NASH EQUILIBRIUM WITH VOLUNTARILY CONTRIBUTED PUBLIC GOODS

Valérie Lechene
Ian Preston

THE INSTITUTE FOR FISCAL STUDIES
WP05/06

Household Nash Equilibrium with Voluntarily Contributed Public Goods*

Valérie Lechene †
Wadham College, Oxford and Institute for Fiscal Studies

Ian Preston‡
University College London and Institute for Fiscal Studies

March 1, 2005

Abstract

We study noncooperative models with two agents and several voluntarily contributed public goods. We focus on interior equilibria in which neither agent is bound by non negativity constraints, establishing the conditions for existence and uniqueness of the equilibrium. While adding-up and homogeneity hold, negativity and symmetry properties are generally violated. We derive the counterpart to the Slutsky matrix, and show that it can be decomposed into the sum of a symmetric and negative semidefinite matrix and another the rank of which never exceeds the number of public goods plus one. Under separability of the public goods the deviation from symmetry is at most rank two.

JEL: D11, C72

Keywords: Nash equilibrium, Intra-household allocation, Slutsky symmetry.

*We are grateful for comments from Martin Browning and Pierre-André Chiappori. This research has been partly funded by the ESRC Centre for the Microeconomic Analysis of Fiscal Policy at the Institute for Fiscal Studies and by the Leverhulme Trust.

†Address: Valérie Lechene, Wadham College, Oxford OX1 3PN, UK. Email: valerie.lechene@economics.ox.ac.uk

‡Address: Ian Preston, Department of Economics, University College London, Gower Street, London WC1E 6BT, UK. Email: i.preston@ucl.ac.uk

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1 Introduction

Maximisation of utility by a single consumer subject to a linear budget constraint implies strong testable restrictions on the properties of demand functions. Empirical applications to data on households however often reject these restrictions. In particular, such data frequently show a failure of Slutsky symmetry - the restriction of symmetry on the matrix of compensated price responses (see for example Deaton (1990), Browning and Meghir (1991), Banks, Blundell and Lewbel (1997) and Browning and Chiappori (1998)).

From the theoretical point of view, the inadequacy of the single consumer model as a description of decision making for households with more than one member has also long been recognised. Attempts to reconcile this model with the existence of several sets of individual preferences have been made for instance by Samuelson (1956) and Becker (1974, 1991) but rely upon restrictive assumptions about preferences or within-household decision mechanisms (see Bergstrom, 1989; Cornes and Silva, 1999).

A large body of recent research has investigated alternative models accommodating more realistic descriptions of within-household decision-making processes. Efficiency of household decisions holds in a number of models of household behaviour which have been suggested: for instance in the Nash bargaining models of Manser and Brown (1980), McElroy and Horney (1981) and McElroy (1990), and in Browning, Bourguignon, Chiappori and Lechene (1994) and Bourguignon and Chiappori (1994). However, it is not a property of noncooperative models such as those of Ulph (1988) and Chen and Woolley (2001).

An important advance is made by Browning and Chiappori (1998), who show that under the assumption of efficient within-household decision making, the counterpart to the Slutsky matrix for demands from a k member household is the sum of a symmetric matrix and a matrix of rank $k-1$. Tests on Canadian data are found to reject symmetry for couples, but not for single individual households. The hypothesis that the departure from symmetry for the sample of couples has rank 1, as implied by the assumption of efficiency, is also not rejected. This work is important not only in filling a gap in our theoretical understanding of demand behaviour but also in the prospect which it presents of reconciling demand theory and data on consumer behaviour.

While the inability of Browning and Chiappori to reject the symmetry and rank condition for couples is intriguing, it is not clear what power it has, if any, as a test of efficiency of intrahousehold decisions, unless one understands the nature of the departure from symmetry under the principal alternative models of household decision making.

If noncooperative models give rise to a departure of similar rank as that obtained under the assumption of efficiency, this would obviously not be a feature of demand behaviour which would be of use in discriminating between these alternative assumptions. On the other hand, if the departure from symmetry under noncooperative behaviour is of greater rank, then the Browning-Chiappori result not only reconciles assumptions of optimising behaviour with demand data, but also provides evidence in favour of the collectively rational model against other descriptions of within-household decision making.[1]

In this paper, we establish properties of demands in noncooperative models with several voluntarily contributed public goods. Models of this type warrant attention in their own right as marking an opposite extreme to fully efficient models of the sort described above. They are also interesting in so far as the equilibria in such models can be considered as the fallback position in bargaining models as suggested, for example, in Chen and Woolley (2001).

Models of voluntarily contributed public goods have relevance beyond analysis of household demand. When they involve more than two players, these models can be used to represent a variety of situations involving private contributions to public goods either in the national or international context. What distinguishes what we have termed the "household Nash equilibrium model" from the general Nash equilibrium model is the number of agents, which is two in the case considered here.

We concentrate attention on interior equilibria in which each partner contributes to all the public goods.[2] Such equilibria have important income pooling properties which help render the description of demand properties tractable. In section 2, we establish conditions under which there can be no more than one such equilibrium and derive conditions on preferences and income shares allowing for existence. In section 3, we show that equilibrium quantities vary with prices and household income in ways compatible with

[1] In general, Nash bargaining and other specific cooperative models should not give rise to a departure from symmetry of a lower rank than that of the collective model. See McElroy and Horney (1981, 1990) and Chiappori (1988,1991) for a discussion of price effects in the Nash bargaining model.

[2] Specifically, what we mean is that neither partner would like to reduce the household's spending on any public good but cannot because they do not contribute to it.

the adding up and homogeneity properties of unitary demands and that negativity and symmetry properties will generally be violated, as in the collective model. We derive the counterpart to the Slutsky matrix, and show that it can be decomposed into the sum of a symmetric matrix and another the rank of which never exceeds one plus the number of public goods. The departure from symmetry therefore typically falls considerably short of full rank. Section 4 is devoted to the properties of demands for specific forms of preferences. Indeed, additional restrictions on preferences reduce the rank further, but it falls to the rank one departure seen in the collective setting only under very restrictive assumptions - for example, separability of public goods *and* identical preferences. These results imply that the Browning-Chiappori assumption of efficiency can be tested against other models within the class of those based on individual optimisation. Section 5 concludes.

2 Household Nash equilibrium with voluntary contributions to the public goods

2.1 The model

Consider a household with two individuals, A and B. The household spends on a set of m private goods $q \in \mathbb{R}^m$ and n public goods $Q \in \mathbb{R}^n$. The quantities purchased by the individuals are q^A, q^B, Q^A and Q^B with total household quantities $q \equiv q^A + q^B$ and $Q \equiv Q^A + Q^B$. Individual utility functions are $u^A(q^A, Q)$ and $u^B(q^B, Q)$, assumed increasing and differentiable in all arguments, so that individual preferences are defined over the sum of contributions to the public goods. The partners have incomes of y^A and y^B.

Household income is denoted $y \equiv y^A + y^B$. Prices of private and public goods respectively are the vectors p and P.

Each person decides on the purchases made from their income so as to maximise their utility subject to the spending decisions of their partner. We can write the agents' problems as

$$\max_{q^A, Q^A} u^A(q^A, Q) \text{ s. t. } p'q^A + P'Q^A \leq y^A,\ Q^A \geq 0,\ q^A \geq 0$$

and

$$\max_{q^B, Q^B} u^B(q^B, Q) \text{ s. t. } p'q^B + P'Q^B \leq y^B,\ Q^B \geq 0,\ q^B \geq 0$$

where inequalities should be read where appropriate as applying to each element of the relevant vector. We assume at least one private good for each partner and at least one public good are consumed in positive quantities in all equilibria considered below.

This problem can be considered as one where each agent has to choose the level of the public goods for the household, subject to the constraint that this level is greater than or equal to the contribution of the other agent. Given that $y^A = y - p'q^B - P'Q^B$, and similarly for B, the agents'problems can be re-written as:

$$\max_{q^A, Q} u^A(q^A, Q) \text{ s. t. } p'q^A + P'Q \leq y - p'q^B,\ Q \geq Q^B,\ q^A \geq 0$$

and

$$\max_{q^B, Q} u^B(q^B, Q) \text{ s. t. } p'q^B + P'Q \leq y - p'q^A,\ Q \geq Q^A\ q^B \geq 0$$

A *household Nash equilibrium* consists of a set of quantities (q^A, q^B, Q) simultaneously solving these two problems. We call such an equilibrium an *interior household Nash equilibrium* if non-negativity constraints on public good contributions $Q \geq Q^B$ and $Q \geq Q^A$ are binding on neither partner. In such equilibria, quantities purchased will satisfy

$$q^A = f^A(y - p'q^B, p, P) \tag{1}$$

$$q^B = f^B(y - p'q^A, p, P) \tag{2}$$

and

$$Q = F^A(y - p'q^B, p, P) \tag{3}$$

$$= F^B(y - p'q^A, p, P). \tag{4}$$

where $f^A(.)$ and $F^A(.)$ are Marshallian demand functions corresponding to A's preferences and together satisfying the usual demand properties and $f^B(.)$ and $F^B(.)$ are demand functions corresponding to B's preferences, of which the same is true[3]. We use subscripts to denote derivatives of these demand functions: f^i_y, f^i_p, f^i_P and F^i_y, F^i_p, F^i_P for $i = A, B$, with respect to income y and price vectors p and P respectively.

Note that it is natural to consider the equilibria in terms of quantities of public goods Q rather than in terms of levels of individual contributions Q^A and Q^B since the equilibrium in terms of the latter is indeterminate when the number of public goods n is greater than one. Since individuals care only about the level of public goods and not about the level of their individual contributions, it makes no difference to either of them whether, say, A pays for the heating and B for the housing or *vice versa* given the quantities Q.

[3] While f^A and f^B are not themselves reaction functions, note that (1) and (2) can be used to define reaction functions for q^A and q^B given y, p and P if regarded as defining responses of each as a function of the other.

2.2 Existence and uniqueness of the equilibrium

Equilibria of this type have important properties.

Lemma 1 *If both public and private goods are normal for both partners, in the sense that*

$$a \leq p' f_y^i \leq 1 - a \qquad \textit{for some } a > 0,\ i = A, B, \tag{5}$$

then

1. *(Existence and uniqueness for private goods) there exist unique q^A, q^B satisfying (1) and (2) and*
2. *(Income pooling for private goods) these quantities depend only on (y, p, P).*

Proof. By substitution between (1) and (2), we have the equilibrium condition for q^A

$$q^A = f^A(y - p' f^B(y - p' q^A, p, P), p, P). \tag{6}$$

Defining $x^A = p' q^A$ and $x^B = p' q^B$ we have

$$x^A = p' f^A(y - p' f^B(y - x^A, p, P), p, P) = \xi^A(x^A) \tag{7}$$

so that the value of x^A in household Nash equilibrium is a fixed point of $\xi^A(\cdot)$. Given (5) then $\xi^A(\cdot)$ is a contraction mapping and therefore has a unique fixed point by the Banach fixed point theorem. This uniquely determines values for q^B from (2) and then for q^A from (1). Given uniqueness, it is immediate from (6) that these equilibrium values depend only on (y, p, P). ■

In what follows we assume normality of public and private goods in the sense of (5). Let us denote the mappings from (y, p, P) to these unique interior equilibrium private good vectors by $\theta^A(y, p, P)$ and $\theta^B(y, p, P)$.

Lemma 2 1. *(Existence and uniqueness for public goods) There exists a unique interior household Nash equilibrium if and only if*

$$F^A(y - p'\theta^B(y, p, P), p, P) \; = \; F^B(y - p'\theta^A(y, p, P), p, P) \qquad (8)$$

and

$$p'\theta^A(y, p, P) \leq y_A \leq y - p'\theta^B(y, p, P), \qquad (9)$$

in which case

2. *(Income pooling for public goods) equilibrium public good quantities depend only on (y, p, P).*

Proof. Given that private good quantities in interior equilibrium are determined uniquely by Lemma 1, (3) and (4) provide alternative equations for equilibrium quantities of the public goods and (8) is a necessary and sufficient condition for these to coincide. If they do coincide, then income pooling for public goods follows immediately from the income pooling result for private goods. These quantities constitute an interior household equilibrium if the nonnegativity constraints bind on neither household member, which will be the case if each partner has income sufficient to purchase the interior equilibrium private goods quantities $\theta^A(y, p, P)$ and $\theta^B(y, p, P)$ as stated in (9). ■

Existence and uniqueness of equilibrium are considered for the case of one public good and k contributors by Bergstrom, Blume and Varian (1986, 1992) and Fraser (1992). We are not aware of a previous proof of uniqueness of the interior equilibrium with many goods. The income pooling result shows that provided that a redistribution of income between the partners does not take the household out of interior equilibrium, quantities consumed in equilibrium are invariant. This result is well known and has been discussed by many authors. Warr (1983) established income pooling for the case of a single public good and Kemp (1984) extended the claim to the case of multiple public goods, assuming interior equilibrium. Kemp's proof is queried by Bergstrom, Blume and Varian (1986) who offer an alternative proof.

Though often found surprising, the source of the result is easily illustrated for the three good case in Figure 1[4]. Any allocation of total household income y across the three goods q^A, q^B and Q can be represented as a point in the triangular area ADO with the shares of household income spent on private goods represented by the distances along the axes and the remaining share allocated to the public good given by the perpendicular distance to the boundary AD. Given any amount spent on the private good of individual A, the remainder of household income is spent on goods of interest to individual B and the line AEB represents B's preferred allocation between q^B and Q. Correspondingly, the line DEC represents A's preferred allocation between q^A and Q given any amount spent on q^B. The line AEB and DEC represent graphically the reaction functions implied by equations (1) and (2). The intersection at E shows an allocation over the three goods with which each partner is content given the spending decisions of the other. This point is

[4] Ley (1996) presents several diagrammatic representations of the Bergstrom, Blume and Varian (1986) model though not that of Figure 1.

clearly unique if the slope of AEB is always more negative than -1 and that of DEC always between 0 and -1 which will be the case if Q and q are normal in the preferences of A and B. This point will be an interior household Nash equilibrium if it involves neither partner spending more than their private income on their own private good. Private income shares y^A/y and y^B/y are shown on the diagram and in this case exceed household budget shares for the private goods at E so that E is the unique household Nash equilibrium. Furthermore it is clear that small changes in the income shares will not alter the location of this equilibrium, which is the income pooling result. Noninterior equilibria will pertain in cases of sufficiently extreme income shares and the locus of all equilibria is given by the line CEB.

Existence depends critically on compatibility of public good demands in a way that requires a certain coincidence of individual preferences, which may be more or less thoroughgoing depending upon assumptions made about separability of public goods in individual preferences.

Let us denote the mapping from (y, p, P) to interior equilibrium public goods quantities, if these exist, by $\Theta(y, p, P)$ and assume that the equilibrium mappings are all differentiable. One implication of interior equilibrium that we make use of below is a requirement that public good Engel curves be proportional in the sense that each partner should wish to spend household funds allocated to the public goods in similar proportions at the margin. To state this result, define[5] $\alpha \equiv 1 - P'F_y^A = p'f_y^A$ and $\beta \equiv 1 - P'F_y^B = p'f_y^B$.

Lemma 3 *(Engel curve proportionality for public goods) Individual Engel*

[5]Note that α and β are scalars and are defined by slopes of demand functions rather than slopes of equilibrium conditions.

curves for public goods are proportional in the sense that

$$F_y^A/(1-\alpha) = F_y^B/(1-\beta). \tag{10}$$

Proof. Differentiating (3) and (4) gives

$$F_y^A(1-p'\theta_y^B) = F_y^B(1-p'\theta_y^A).$$

Differentiating (1) and (2) gives

$$\begin{aligned} p'\theta_y^A &= p'f_y^A(1-p'\theta_y^B) \\ p'\theta_y^B &= p'f_y^B(1-p'\theta_y^A). \end{aligned}$$

Substituting and solving simultaneously gives

$$\begin{aligned} p'\theta_y^A &= \alpha(1-\beta)/(1-\alpha\beta) \\ p'\theta_y^B &= \beta(1-\alpha)/(1-\alpha\beta) \end{aligned}$$

and therefore

$$F_y^A(1-\beta)/(1-\alpha\beta) = F_y^B(1-\alpha)/(1-\alpha\beta).$$

from which (10) follows immediately. ■

This is clearly a strong restriction but does not exhaust the implications of existence of such an equilibrium. Interior equilibria exist only if public goods choices made by the partners are compatible in the sense of (8). Since both partners face the same relative prices, the same total quantities of the jointly contributed public goods can constitute solutions to the individual optimisation problems only if marginal rates of substitution between these public goods are the same for both partners. Equivalently, we require the existence of public goods quantities Q solving[6]

$$\frac{1}{\lambda^A}\, u_Q^A(\theta^B(y,p,P),Q) \;=\; \frac{1}{\lambda^B}\, u_Q^B(\theta^A(y,p,P),Q) \;=\; P \qquad (11)$$

for some $\lambda^A,\ \lambda^B > 0$ and subject to $P'Q = y - p'\theta^A(y,p,P) - p'\theta^B(y,p,P)$. This is a system of equations which is clearly typically insoluble (since there are $n+2$ unknowns with which to satisfy $2n+1$ equations including the budget constraint). Nonetheless restrictions on preferences which either tie together the equilibrium private good quantities in a useful way or which enforce some sort of separability between public and private goods choices can make an interior equilibrium possible.

In particular, we may note that interior equilibria exist if private goods can be partitioned, $q^i = (q_0^i, q_1^i),\ i = A, B$, in such a way that individual preferences take the weakly separable form

$$u^i(q^i, Q) = \upsilon^i(q_0^i, \nu(q_1^i, Q)) \qquad i = A, B \qquad (12)$$

for some $\upsilon^i(.,.),\ i = A, B$ and some common subutility function $\nu(.,.)$.

Lemma 4 *If individual preferences take the form (12) then there exists a range of income shares over which there exists an interior household Nash equilibrium.*

Proof. Let the price vector for private goods be similarly partitioned $p = (p_0, p_1)$. Given weak separability of (q_1^i, Q) in each partner's preferences and the common subutility function, we know that two stage budgetting

[6] We consider only public goods which are bought in positive quantities in equilibrium. Relaxing this would raise no interesting issues.

will hold (Blackorby, Primont and Russell 1978) with a common lower stage. Therefore there exists a function $g(.)$ such that

$$\begin{aligned} q_1^A &= g(y - p'q^B - p_0'q_0^A, p_1, P) \\ &= g(y - p'q_1^B - p_0'q_0, p_1, P) \\ q_1^B &= g(y - p'q^A - p_0'q_0^B, p_1, P) \\ &= g(y - p'q_1^A - p_0'q_0, p_1, P). \end{aligned}$$

Given uniqueness of private goods quantities in interior equilibrium, by Lemma 1, and the symmetry of these equations it must be that $\theta_1^A(y, p, P) = \theta_1^B(y, p, P) = \theta_1^*(y, p, P)$ for some common $\theta_1^*(y, p, P)$ so equilibrium quantities of the separable private goods are equal for the two partners. Furthermore, using weak separability and the fact that $\nu(.)$ is common again, (11) simplifies to

$$\nu_Q(\theta_1^*(y, p, P), Q) = \frac{\lambda^A}{u_\nu^A(.)}P = \frac{\lambda^B}{u_\nu^B(.)}P$$

which is generically soluble since it involves only $n + 2$ equations. Hence (8) can be satisfied and an interior household Nash equilibrium exists for income shares satisfying (9). ■

This class of preferences covers all particular cases in which we are aware that interior equilibria exist. It covers, for example, the extreme case of identical preferences

$$u^i(q^i, Q) = \upsilon^i(\nu(q^i, Q)) \qquad i = A, B \tag{13}$$

for some common $\nu(.,.)$. In this case, since there exists a unique interior equilibrium (given appropriate income shares) defined by equations which are fully symmetric, the partners consume identical quantities in equilibrium: $\theta^A(y, p, P) = \theta^B(y, p, P) = \theta^*(y, p, P)$ for some common $\theta^*(y, p, P)$.

It also covers the case in which public goods are separable with a common public goods subutility function

$$u^i(q^i, Q) = \upsilon^i(q^i, \nu(Q)) \qquad i = A, B \tag{14}$$

and therefore, trivially, also the case usually considered in which there is only one public good, $n = 1$. Both (13) and (14) can be shown to imply particularly special properties for equilibrium demands and we pay particular attention to them below in section (4).

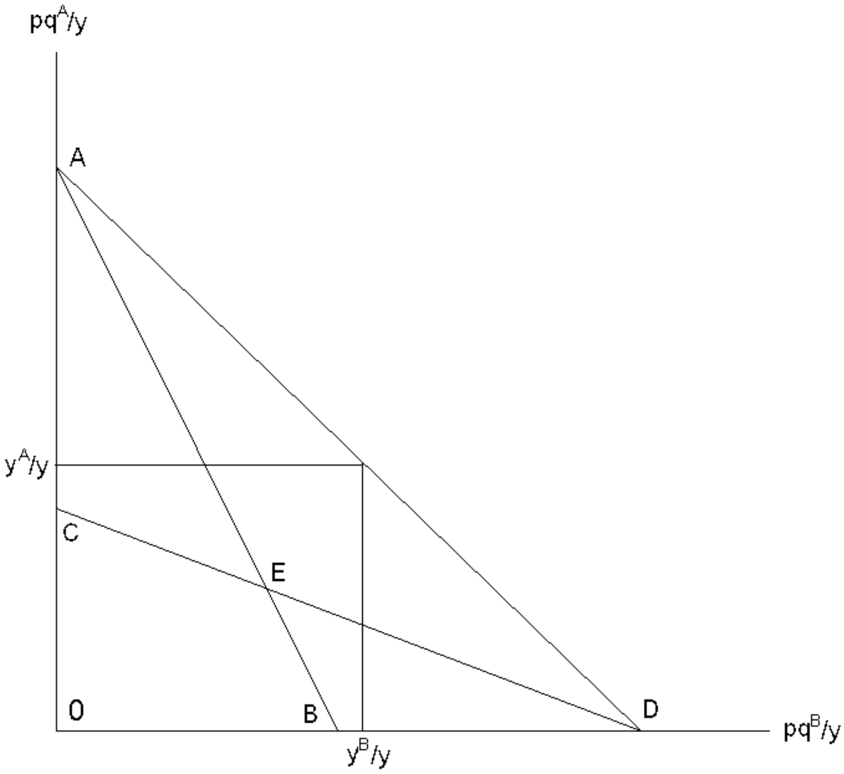

Figure 1: Household Nash equilibrium

3 Properties of equilibrium demands in the general case

Lemmas 1 and 2 establish that in interior equilibrium, quantities are uniquely determined as functions

$$q = q^A + q^B = \theta(y, p, P) \qquad Q = \Theta(y, p, P)$$

of the same economic determinants y, p and P as would be the case under the "unitary" model where the household maximises a household utility function given the household budget constraint. Distinguishing unitary and noncooperative household behaviour therefore requires that we establish whether these equilibrium quantities have properties dissimilar to demands in unitary households. Browning and Chiappori (1998) provide such an analysis for the collective model, and Browning, Chiappori and Lechene (2004) examine the relationship between collective and unitary models.

The properties of unitary demands are the standard Hurwicz-Uzawa (1971) integrability requirements of adding up, homogeneity, negativity and symmetry.

3.1 Adding up and homogeneity

It is easy to establish that the household Nash equilibrium quantities satisfy adding-up and homogeneity.

Theorem 1 *Household Nash equilibrium demands satisfy*

1. *(Adding up)* $p'\theta(y, p, P) + P'\Theta(y, p, P) = y$

2. *(Homogeneity)* $\theta(\lambda y, \lambda p, \lambda P) = \theta(y, p, P)$ *and* $\Theta(\lambda y, \lambda p, \lambda P) = \Theta(y, p, P)$ *for any* $\lambda > 0$.

Proof.

1. Adding up of demands in household Nash equilibrium follows from the fact that the partners are on their individual budget constraints and the sum of their demands therefore satisfies the household budget constraint.

2. Equilibrium quantities satisfying (1), (2) and (3) and (4) will satisfy homogeneity given homogeneity of the individual demand functions.

■

3.2 Negativity and symmetry

Negativity and symmetry are less simply dealt with. These are concerned in the case of the unitary model with the properties of the Slutsky matrix, the matrix of price responses at fixed household utility. Since household utility is undefined in a noncooperative setting, no such matrix is defined but we can adopt the Browning and Chiappori (1998) notion of the "pseudo-Slutsky matrix". This in the current context is the matrix

$$\Psi \equiv \begin{pmatrix} \theta_p & \theta_P \\ \Theta_p & \Theta_P \end{pmatrix} + \begin{pmatrix} \theta_y \\ \Theta_y \end{pmatrix} \begin{pmatrix} \theta' & \Theta' \end{pmatrix} \tag{15}$$

composed in a comparable way from derivatives of the equilibrium household quantities with respect to prices and income. This is what would be calculated as the Slutsky matrix if the household were treated as behaving according to the unitary model. The properties of the Pseudo-Slutsky matrix can then be examined by relating its terms to the "true" compensated price effects on the functions f^A, f^B, F^A and F^B, which correspond to the individual utility functions assumed to have given rise to the observed behaviour of the household.

Noting that at the equilibrium, $\theta^A(y,p,P) = f^A(y - p'q^B, p, P)$ and similarly for B and for the public goods, equilibrium quantities responses follow from

$$M \begin{pmatrix} d\theta^A \\ d\theta^B \\ d\Theta \end{pmatrix} = N_1 dy + N_2 \begin{pmatrix} dp \\ dP \end{pmatrix}$$

where

$$M \equiv \begin{pmatrix} I_m & f_y^A p' & 0 \\ f_y^B p' & I_m & 0 \\ 0 & F_y^A p' & I_n \end{pmatrix}$$

$$N_1 \equiv \begin{pmatrix} f_y^A \\ f_y^B \\ F_y^A \end{pmatrix} \qquad \text{and} \qquad N_2 \equiv \begin{pmatrix} f_p^A - f_Y^A q^{B\prime} & f_P^A \\ f_p^B - f_y^B q^{A\prime} & f_P^B \\ F_p^A - F_y^A q^{B\prime} & F_P^A \end{pmatrix}$$

Hence

$$\begin{pmatrix} d\theta^A \\ d\theta^B \\ d\Theta \end{pmatrix} = M^{-1} N_1 dy + M^{-1} N_2 \begin{pmatrix} dp \\ dP \end{pmatrix}$$

Since we work in terms of household purchases q and Q, we have therefore

$$\begin{pmatrix} d\theta \\ d\Theta \end{pmatrix} = E M^{-1} \left(N_1 dy + N_2 \begin{pmatrix} dp \\ dP \end{pmatrix} \right)$$

where

$$E \equiv \begin{pmatrix} I_m & I_m & 0 \\ 0 & 0 & I_n \end{pmatrix}$$

is an appropriate aggregating matrix.

The matrix M has a block lower triangular structure which makes it readily invertible. In a convenient representation

$$M^{-1} = \begin{pmatrix} I_m + \frac{\beta}{(1-\alpha\beta)} f_y^A p' & -\frac{1}{(1-\alpha\beta)} f_y^A p' & 0 \\ -\frac{1}{(1-\alpha\beta)} f_y^B p' & I_m + \frac{\alpha}{(1-\alpha\beta)} f_y^B p' & 0 \\ \frac{\beta}{(1-\alpha\beta)} F_y^A p' & -\frac{1}{(1-\alpha\beta)} F_y^A p' & I_n \end{pmatrix}$$

$$EM^{-1} = \begin{pmatrix} I_m - \frac{1}{(1-\alpha\beta)} (f_y^B - \beta f_y^A) p' & I_m - \frac{1}{(1-\alpha\beta)} (f_y^A - \alpha f_y^B) p' & 0 \\ \frac{\beta}{(1-\alpha\beta)} F_y^A p' & -\frac{1}{(1-\alpha\beta)} F_y^A p' & I_n \end{pmatrix}.$$

The pseudo-Slutsky matrix now follows from:

$$\Psi = EM^{-1} \left(N_2 + N_1 \begin{pmatrix} q \\ Q \end{pmatrix}' \right) = EM^{-1} \Phi$$

where

$$\begin{aligned} \Phi &= \begin{pmatrix} f_p^A + f_y^A q^{A\prime} & f_P^A + f_y^A Q' \\ f_p^B + f_y^B q^{B\prime} & f_P^B + f_y^B Q' \\ F_p^A + F_y^A q^{A\prime} & F_P^A + F_y^A Q' \end{pmatrix} \\ &= \begin{pmatrix} \Psi_{11}^A & \Psi_{12}^A \\ \Psi_{11}^B & \Psi_{12}^B \\ \Psi_{21}^A & \Psi_{22}^A \end{pmatrix}. \end{aligned}$$

Note that the terms in Φ are all elements of the underlying true individual Slutsky matrices corresponding to the individual decision problems

$$\Psi^A \equiv \begin{pmatrix} \Psi_{11}^A & \Psi_{12}^A \\ \Psi_{21}^A & \Psi_{22}^A \end{pmatrix} \quad \text{and} \quad \Psi^B \equiv \begin{pmatrix} \Psi_{11}^B & \Psi_{12}^B \\ \Psi_{21}^B & \Psi_{22}^B \end{pmatrix}$$

Lemma 5 *The pseudo Slutsky matrix admits the decomposition*

$$\Psi = \Psi^A + \Psi^B - \Lambda + \Delta$$

where

$$\begin{aligned}
\Lambda &= \begin{pmatrix} 0 \\ R^A \end{pmatrix} \begin{pmatrix} \Psi_{21}^A & \Psi_{22}^A \end{pmatrix} = \begin{pmatrix} 0 \\ R^B \end{pmatrix} \begin{pmatrix} \Psi_{21}^B & \Psi_{22}^B \end{pmatrix} \\
\Delta &= \frac{1}{1-\alpha\beta} \begin{pmatrix} f_y^B - \beta f_y^A \\ -\beta F_y^A \end{pmatrix} P' \begin{pmatrix} \Psi_{21}^A & \Psi_{22}^A \end{pmatrix} \\
&\quad + \frac{1}{1-\alpha\beta} \begin{pmatrix} (f_y^A - \alpha f_y^B) \\ -\alpha F_y^B \end{pmatrix} P' \begin{pmatrix} \Psi_{21}^B & \Psi_{22}^B \end{pmatrix}.
\end{aligned}$$

and $R^A = I_n - F_y^A P'/P'F_y^A$, $R^B = I_n - F_y^B P'/P'F_y^B$. *Furthermore* $R^A = R^B$ *is an idempotent matrix of rank* $n-1$.

Proof. The decomposition is established in the Appendix. That $R^A = R^B$ follows from the Engel curve proportionality result in Lemma 3. Idempotency follows from $R^i F_y^i = 0,\ i = A, B$. The rank of an idempotent matrix is equal to its trace (Magnus and Neudecker, 1988, p.20) and the trace of R^i is $n - P'F_y^i/P'F_y^i = n-1$ for $i = A, B$. ■

Both Ψ^A and Ψ^B are individual Slutsky matrices and therefore negative semidefinite and symmetric. The departure from negativity and symmetry in the pseudo-Slutsky matrix Ψ therefore depends on the properties of the matrices Λ and Δ.

Theorem 2 *The pseudo-Slutsky matrix* Ψ *is the sum of a negative semidefinite symmetric matrix and a matrix of rank no greater than* $n+1$.

Proof. $\Psi^A + \Psi^B$ is symmetric and negative semidefinite since Ψ^A and Ψ^B both are. The rank of Λ is no greater than the rank of R^i which is established in Lemma 5 to be $n-1$. Δ, being the sum of two outer products of vectors, is of rank no greater than 2 in general. ■

4 Specific preferences

Theorem 2 establishes an upper bound for the rank of the departure from symmetry and negativity. Particular restrictions on preferences will however reduce this bound further. Two specific special cases have been identified above - those of identical preferences (13) and separability of public goods (14).

4.1 Identical preferences

Under identical preferences, interior equilibria are symmetric in the sense that private good quantities are identical for the two partners. This reduces the rank of the departure from symmetry and negativity by one.

Theorem 3 *Given identical preferences (13), the pseudo-Slutsky matrix* Ψ *is the sum of a negative semidefinite symmetric matrix and a matrix of rank no greater than* n*.*

Proof. Since private goods quantities are identical, the two outer products of vectors in Δ are identical and therefore Δ has rank n ■

4.2 Separability of public goods

Separability of public goods with a common public goods subutility as in (14) offers an interesting case. In particular we know that the separable structure allows us to write

$$\begin{aligned} F^A(y - p'q^B, p, P) &= G(P'F^A(y - p'q^B, p, P), P) \\ F^B(y - p'q^A, p, P) &= G(P'F^B(y - p'q^A, p, P), P) \end{aligned}$$

for some common function $G(.)$.

Hence

$$\begin{aligned}
F_y^i &= G_X P' F_y^i, \\
F_p^i &= G_X P' F_p^i, \\
F_P^i &= G_X (P' F_P^i + Q') + G_P, \qquad i = A, B
\end{aligned}$$

where G_X is the partial derivative of $G(.)$ with respect to its first argument. Substituting the first of these expressions into the definition of R^i and noting that $P'G_X = 1$, gives the simplified formula $R^i = I_m - G_X P',\ i = A, B$.

Thus

$$\begin{aligned}
R^i F_y^i &= 0, \\
R^i F_p^i &= F_p^i - G_X P' F_p^i = 0, \\
R^i F_P^i &= F_P^i - G_X (P' F_P^i) = G_X Q' + G_P, \qquad i = A, B
\end{aligned}$$

and therefore

$$R^i \Psi_{21}^i = 0 \tag{16}$$

$$R^i \Psi_{22}^i = G_X Q' + G_P \equiv \psi, \qquad i = A, B \tag{17}$$

where we may note that ψ, being the Slutsky matrix corresponding to the preferences $\nu(Q)$, is itself symmetric and negative semidefinite.

Theorem 4 *Given separable and identical preferences over public goods (14), the pseudo-Slutsky matrix Ψ is the sum of a symmetric matrix and a matrix of rank no greater than 2.*

Proof. By (16) and (17), Λ is itself symmetric and negative semidefinite therefore Ψ is the sum of a symmetric matrix $\Psi^A + \Psi^B - \Lambda$ and a matrix Δ of rank no greater than 2. ■

Combining the assumptions of fully identical preferences and separability of public goods reduces the rank of the departure to unity, that established for the collective model by Browning and Chiappori (1998). It is interesting to ask whether there are other conditions under which this is so in a noncooperative setting. Given separability, the rank of the deviation reduces to 1 if either $\begin{pmatrix} f_y^B - \beta f_y^A \\ -\beta F_y^A \end{pmatrix}$ is proportional to $\begin{pmatrix} f_y^A - \alpha f_y^B \\ -\alpha F_y^B \end{pmatrix}$ or $P' \begin{pmatrix} \Psi_{21}^A & \Psi_{22}^A \end{pmatrix}$ is proportional to $P' \begin{pmatrix} \Psi_{21}^B & \Psi_{22}^B \end{pmatrix}$. Remembering that $F_y^A/(1-\alpha) = F_y^B/(1-\beta)$, it can be shown that the former, for instance, is true only if $f_y^A/\alpha = f_y^B/\beta$, which is to say that Engel curves for private goods are also proportional.

5 Conclusion

In this paper, we establish properties of demands in the Nash equilibrium with two agents and several voluntarily contributed public goods. This noncooperative model is the polar case to the cooperative model of Browning and Chiappori (1998). In reality, neither the assumption of fully efficient cooperation nor of complete absence of collaboration is likely to be an entirely accurate description of typical household spending behaviour and analysis of such extreme cases can be seen as a first step towards understanding of a more adequate model.

We focus on interior equilibria in which each partner contributes to all the public goods. Although this involves assuming nonbinding the constraint that neither partner should spend more than their private income on goods of private interest to themselves, this is anyway likely to be a fairly soft constraint in real circumstances where the distinction between partners' incomes is neither practically nor legally clearcut.

We derive the conditions for existence and uniqueness of the equilibrium and we show that adding-up and homogeneity hold. We show that the nature of the departure from unitary demand properties in household Nash equilibrium is qualitatively similar to that in collectively efficient models in that negativity and symmetry of compensated price responses is not guaranteed. The counterpart to the Slutsky matrix can be shown to depart from negativity and symmetry by a matrix of bounded rank but this rank typically exceeds that found in the collective model unless strong auxiliary restrictions are placed on preferences. In the Nash equilibrium, the deviation from symmetry falls to the rank one deviation seen in the collective setting only under very restrictive assumptions - for example, separability of public goods *and* identical preferences. However, under the sole assumption of separability of the public goods, the deviation is a matrix of rank 2. These results imply that the Browning-Chiappori assumption of efficiency can be tested against other models within the class of those based on individual optimisation.

Appendix

To derive the decomposition of Lemma 5 it is useful to note the adding up restrictions

$$\begin{aligned} p' f_y^i + P' F_y^i &= 1 \\ q^{i\prime} + p' f_p^i + P' F_p^i &= 0 \\ p' f_P^i + Q' + P' F_P^i &= 0 \\ \text{for } i &= A, B \end{aligned}$$

from which it follows that

$$\begin{aligned} p' \Psi_{11}^i + P' \Psi_{21}^i &= 0 \\ p' \Psi_{12}^i + P' \Psi_{22}^i &= 0 \\ \text{for } i &= A, B. \end{aligned}$$

We derive the elements of Ψ as follows. Dealing firstly with the upper submatrices, we have, from $\Psi = EM^{-1}\Phi$,

$$\begin{aligned} \Psi_{11} &= (I - \frac{1}{(1-\alpha\beta)}(f_y^B - \beta f_y^A)p')\Psi_{11}^A + (I - \frac{1}{(1-\alpha\beta)}(f_y^A - \alpha f_y^B)p')\Psi_{11}^B \\ &= \Psi_{11}^A + \Psi_{11}^B + \frac{1}{(1-\alpha\beta)}[(f_y^B - \beta f_y^A)P'\Psi_{21}^A + (f_y^A - \alpha f_y^B)P'\Psi_{21}^B] \end{aligned}$$

and

$$\begin{aligned} \Psi_{12} &= (I - \frac{1}{(1-\alpha\beta)}(f_y^B - \beta f_y^A)p')\Psi_{12}^A + (I - \frac{1}{(1-\alpha\beta)}(a - Ab)p')\Psi_{12}^B \\ &= \Psi_{12}^A + \Psi_{12}^B + \frac{1}{(1-\alpha\beta)}[(f_y^B - \beta f_y^A)P'\Psi_{22}^A + (f_y^A - \alpha f_y^B)P'\Psi_{22}^B] \end{aligned}$$

using the adding up restrictions above.

Turning to the lower submatrices, we have

$$\begin{aligned}\Psi_{21} &= \frac{1}{(1-\alpha\beta)}\beta F_y^A p' \Psi_{11}^A - \frac{1}{(1-\alpha\beta)} F_y^A p' \Psi_{11}^B + \Psi_{21}^A \\ &= \Psi_{21}^A + \Psi_{21}^B - \frac{1}{(1-\alpha\beta)}\beta F_y^A P' \Psi_{21}^A - (I - \frac{1}{(1-\alpha\beta)} F_y^A P')\Psi_{21}^B.\end{aligned}$$

However, from the Engel curve proportionality result in Lemma 3, $F_y^A P' = (1-\alpha)F_y^A P'/(1-\beta)$. Using this together with the identity $(1-\alpha)/(1-\alpha\beta)(1-\beta) = -\alpha/(1-\alpha\beta) + 1/(1-\beta)$ and the definition of R^B gives

$$\Psi_{21} = \Psi_{21}^A + \Psi_{21}^B - \frac{1}{(1-\alpha\beta)}[\beta F_y^A P' \Psi_{21}^A + \alpha F_y^B P' \Psi_{21}^B] - R^B \Psi_{21}^B$$

Finally, applying similar reasoning to the final submatrix,

$$\begin{aligned}\Psi_{22} &= \frac{1}{1-\alpha\beta}\beta F_y^A p' \Psi_{12}^A - \frac{1}{1-\alpha\beta} F_y^A p' \Psi_{12}^B + \Psi_{22}^A \\ &= \Psi_{22}^A + \Psi_{22}^B + \frac{1}{(1-\alpha\beta)}\beta F_y^A P' \Psi_{22}^A - (I - \frac{1}{(1-\alpha\beta)} F_y^A P')\Psi_{22}^B \\ &= \Psi_{22}^A + \Psi_{22}^B + \frac{1}{(1-\alpha\beta)}[\beta F_y^A P' \Psi_{22}^A - \alpha F_y^B P' \Psi_{22}^B] - R^B \Psi_{22}^B.\end{aligned}$$

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