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The formation of shear and density layers in stably stratified turbulent flows: linear processes

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The initial evolution of the momentum and buoyancy fluxes in a freely decaying, stably stratified homogeneous turbulent flow with r.m.s. velocity u_0' and integral lengthscale l_0 is calculated using a weakly inhomogeneous and unsteady form of the rapid distortion theory (RDT) in order to study the growth of small temporal and spatial perturbations in the large-scale mean stratification $N(z,t)$ and mean velocity profile $\bar{u}(z,t)$ (here N is the local Brunt–Väisälä frequency and $\bar{u}$ is the local velocity of the horizontal mean flow) when the ratio of buoyancy forces to inertial forces is large, i.e. $Nl_0/u_0' \gg 1$. The lengthscale L of the perturbations in the mean profiles of stratification and shear is assumed to be large compared to l_0 and the presence of a uniform background mean shear can be taken into account in the model provided that the inertial shear forces are still weaker than the buoyancy forces, i.e. when the Richardson number $Ri = (N/\partial_z\bar{u})^2 \gg 1$ at each height.

When a mean shear perturbation is introduced initially with no uniform background mean shear and uniform stratification, the analysis shows that the perturbations in the mean flow profile grow on a timescale of order N^{-1}. When the mean density profile is perturbed initially in the absence of a background mean shear, layers with significant density gradient fluctuations grow on a timescale of order N_0^{-1} (where N_0 is the order of magnitude of the initial Brunt–Väisälä frequency) without any associated mean velocity gradients in the layers. These results are in good agreement with the direct numerical simulations performed by Galmiche *et al.* (2002) and are consistent with the earlier physically based conjectures made by Phillips (1972) and Posmentier (1977). The model also shows that when there is a background mean shear in combination with perturbations in the mean stratification, negative shear stresses develop which cause the mean velocity gradient to grow in the density layers. The linear analysis for short times indicates that the scale on which the mean perturbations grow fastest is of order u_0'/N_0, which is consistent with the experiments of Park *et al.* (1994).

We conclude that linear mechanisms are widely involved in the formation of shear and density layers in stratified flows as is observed in some laboratory experiments and geophysical flows, but note that the layers are also significantly influenced by nonlinear and dissipative processes at large times.

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1. Introduction

Although the main effect of stable stratification (characterized by the Brunt–Väisälä frequency $N = [-(g/\rho_r)\,\mathrm{d}\bar{\rho}/\mathrm{d}z]^{1/2}$, where g is the acceleration due to gravity, $\bar{\rho}(z)$ is the mean density vertical profile and ρ_r is a suitable reference density) is to damp the vertical component of turbulence, and therefore to reduce the rate of mixing at large and small scales, there are subtle ways in which locally the diffusivity of mass and momentum is increased. Observations in natural and laboratory flows show that this generally occurs in stratified turbulent flows characterized by sharp vertical variations in the velocity and density fields, associated with the presence of horizontal 'layers'. In some cases, these may become part of the local mean state of the flow field, or decay slowly, or may be relatively transitory. This layering phenomenon has a significant effect on the net fluxes of momentum, heat and mass (Linden 1979).

Layering processes in geophysical turbulent flows involve various mechanisms, such as turbulence–mean flow interactions, wave–turbulence interactions, vortex–vortex interactions and wave–mean flow interactions (reviewed by Hunt & Galmiche 2000). Experimental and theoretical evidence of layer formation has been given by Billant & Chomaz (2000) in terms of columnar vortex pair instability. Wave–mean flow interactions involved in various systems have been extensively studied (e.g. Müller 1976; Yang 1990; Manin & Nazarenko 1994). Homogeneous, stratified turbulence has been widely investigated by Godeferd & Cambon (1994) using EDQNM methods to show that non-linear energy transfers force the tendency to anistropy and the formation of a random distribution of horizontal structures with long but finite timescales. This tendency has also been observed in direct numerical simulations (e.g. Métais & Herring 1989) and in recent stratospheric measurements (Alisse & Sidi 2000). The anistropic features of homogeneous turbulence can be explained by an energy transfer to Fourier modes with mainly vertical wave-vectors, a mechanism which is controlled by the vortex–vortex interactions (Godeferd & Cambon 1994).

On the other hand, the numerical simulations performed by Galmiche, Thual & Bonneton (2002) demonstrate that the turbulence–mean field interactions are widely involved in the layering processes. These simulations show that the presence of a stable stratification modifies the turbulence–mean field interactions considerably in such a way that layers tend to form associated with vertical variations in the mean stratification and shear profiles. For instance, horizontal mean flow modes can grow in strongly stratified turbulence, which causes shear layers to develop. This is the situation we are concerned with here, where the mean field is either a horizontal mean flow profile or the mean density profile. We study theoretically the processes involved in these interactions and discuss their consequences for layering. There are some strong connections between the present approach for vertically inhomogeneous stratified turbulence and the approach used by Godeferd & Cambon (1994) to study the energy drain into the modes with vertical wave-vectors in stratified turbulence, as these modes can be associated with vertical inhomogeneity of turbulence.

The problem of layer formation was addressed in general terms by Phillips (1972) who asked 'Turbulence in a strongly stratified fluid – is it unstable?' Of course, this question is related to the problem of stability of stratified shear flows addressed in the early 1960s (Miles 1961; Howard 1961). More recently, Majda & Shefter (1998) have used a linear stability analysis to further investigate the case of strongly stratified shear flows (see also the review by Cambon & Scott 1999). However, many aspects of layer formation in stratified turbulent flows cannot be accounted for in the framework of these linear stability calculations, and need further investigation. In particular, it is

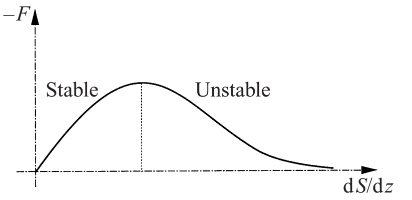

FIGURE 1. Vertical buoyancy flux F as a function of the salinity vertical gradient. This plot was discussed by Posmentier (1977) to explain the formation of salinity layers in the ocean.

important to understand how a small perturbation in the mean flow profile evolves in the presence of an active turbulent field. This problem has been addressed by Moffatt (1967) for non-stratified turbulent flows with large vertical background shear.

In most previous theoretical studies of stratified turbulence, the stratification and the mean shear are assumed to be uniform at any time. The question we address here is how initially homogeneous turbulence in the presence of a stable stratification affects perturbations in the mean density and velocity fields. It is necessary to take these vertical and temporal variations into account in order to explain the growth of shear and density layers as seen in the laboratory experiments of Park, Whitehead & Gnanadeskian (1994) for instance. Whereas consideration of the large-scale fluctuations that span the lengthscale of any perturbation in the mean fields is essential in non-stratified flows (Trefethen *et al.* 1993; Liu 1989), it is assumed here that the stratification is strong enough that the lengthscale L of the perturbations in the mean flow and stratification profiles is large compared to the typical lengthscale of turbulence l_0. This provides a method for calculating how perturbations to the mean flow profile distort the turbulence in such a way that the Reynolds stresses may amplify the perturbations.

It is interesting to compare this detailed approach with the global heuristic arguments of Posmentier (1977) (and independently Puttock 1976) for the formation of density layers, which in oceanography are called salinity fine structures. Let us consider the equation for conservation of the mean salinity $S(z,t)$ (where z is the vertical coordinate and t is time) in a horizontally homogeneous but vertically varying profile,

$$\partial_t S = -\partial_z F, \tag{1.1}$$

where F is the flux associated with turbulent fluctuations and microscale mixing. Posmentier (1977) pointed out that equation (1.1) may be written as

$$\partial_t S = -F^* \partial_{zz} S, \tag{1.2}$$

where * denotes differentiation with respect to $\partial_z S$ and F^* has the dimension of a diffusivity.

The sign of F^* is a key point which determines the stability of the solutions for S (see figure 1, after Posmentier 1977). Equation (1.2) has stable solutions if $F^* < 0$ and unstable solutions if $F^* > 0$. In the unstable case, any perturbation in the mean density profile is amplified leading to the formation of layers; however the problem is mathematically ill-posed since the growth rate diverges for small-scale perturbations. Furthermore, this theoretical discussion did not differentiate between stratified turbulent flows with and without mean shear, and did not account for the energy supply to the flow.

Several phenomenological models (see for instance Barenblatt *et al.* 1993 and

Balmforth, Llewellyn Smith & Young 1998) have been used to estimate the buoyancy fluxes as a function of the mean density gradient in order to simulate layering processes. Various physical arguments, such as the existence of a finite mixing length or a finite adjustment time of turbulence, have been included in these models in order to avoid singularities in the solutions and to predict the layer formation. In these models, the turbulence is assumed to be in a state that is quasi-steady, and developing only as slowly as the mean gradients. This is consistent with the conditions of the stirred tank experiment of Park *et al.* (1994). There are other situations where the turbulence is changing rapidly, for example decaying, and then the model (1.2) is not necessarily applicable. In the early stages of homogeneous stratified grid turbulence (Rottman & Britter 1986), no maximum in the curve of the buoyancy flux against the mean density gradient and no layering were observed. On the other hand, through different mechanisms to those proposed by Phillips (1972), layers were observed in the final stage of decay of turbulence by Pearson & Linden (1983), who developed a theory where viscous rather than turbulent shear stresses balanced the buoyancy forces.

To overcome these uncertainties, a quantitative study is proposed for the evolution of the mean density profile $\bar{\rho}$ and the horizontal mean flow profile $\bar{u}$ as they vary initially with height z and time (here the bar denotes either the ensemble average or the average in a horizontal plane). In the previous studies, the effects of the coupling with the mean velocity gradients had not been considered despite the fact that in many environmental flows the mean shear plays an active dynamic role in the processes. In this case, the momentum and buoyancy fluxes, and therefore the time evolution of $\bar{u}$ and $\bar{\rho}$, are functions of the local mean shear and mean density gradient. In the absence of a mean horizontal pressure gradient, the mean flow and mean density fields evolve as

$$\left.\begin{aligned}\partial_t\bar{u} &= -\frac{\partial F_u}{\partial(\partial_z\bar{u})}\partial_{zz}\bar{u} - \frac{\partial F_u}{\partial(\partial_z\bar{\rho})}\partial_{zz}\bar{\rho},\\ \partial_t\bar{\rho} &= -\frac{\partial F_\rho}{\partial(\partial_z\bar{u})}\partial_{zz}\bar{u} - \frac{\partial F_\rho}{\partial(\partial_z\bar{\rho})}\partial_{zz}\bar{\rho},\end{aligned}\right\} \tag{1.3}$$

where F_u is the turbulent momentum flux and F_ρ is the turbulent buoyancy flux.

The aim of the present paper is to study this system in the first stage of decay of turbulence. We focus on the initial development over a time t of turbulence generated at $t = 0$ with r.m.s. velocity u'_0 in the presence of a strong stratification N and a horizontal mean flow $\bar{u}(z)$, in particular the evolution of momentum and buoyancy fluxes using the linearized method of rapid distortion theory (RDT). In this analysis, the assumptions are not based on a comparison between the amplitudes of the mean and fluctuating fields (as is usually the case in linear stability analysis), but instead are based on a comparison between the various timescales of the flow, namely the buoyancy timescale, the shear timescale and the typical timescale associated with the initial turbulent motions. In extended use of RDT (see also Nazarenko, Kevlahan & Dubrulle 1999), the nonlinear effects in the equations of motion are small as long as $t < l_0/u'_0$ (the integral timescale), so that the linearized equations of motion accurately describe the changes of the energy-containing eddies in the turbulence. In general, RDT describes the most-amplified elements of the flow even for $t > l/u'_0$ but is qualitatively incorrect for the components that are damped, because they are then susceptible to nonlinearity (Kevlahan & Hunt 1997). However, if the linear distortion effects are large enough compared to the typical nonlinear inertial forces, the linear theory over $t > l_0/u_0'$ describes the main structural features of the large-scale turbulence. The methodology of RDT has already been applied by Hunt, Stretch & Britter (1988), van Haren (1993) and Hanazaki & Hunt (1996, 1999) to sheared and unsheared turbulence in the presence of a uniform stratification. When strong shear layers develop, the turbulence outside the stratified shear layers and instabilities within them (e.g. Kelvin–Helmholtz billows) may increase or decrease the intensity of the shearing processes (e.g. Caulfield 1994).

The case of uniform and constant shear α and stratification N is considered first as a preliminary to the new calculation for the effects of variations of α and N. The case of time-dependent mean shear and stratification is considered in § 2.2. In § 2.3, the z-dependence of the mean shear and stratification is introduced and the coupled evolution equations for $\bar{u}(z,t)$ and $\bar{\rho}(z,t)$ are derived. In § 3, the behaviour of the coupled system is studied for different sets of initial conditions and the short-time results are compared to the direct numerical simulations performed by Galmiche *et al.* (2002).

2. The rapid distortion model

2.1. *Uniform shear and stratification*

We first consider a turbulent flow evolving in the presence of a uniform, vertical mean shear stress and a uniform stable stratification. The three-dimensional velocity field is $\boldsymbol{u}(\boldsymbol{x},t) = (u, v, w)$ in a reference frame $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ where $\boldsymbol{e}_3$ is antiparallel to the gravitational acceleration $\boldsymbol{g}$ and $\boldsymbol{x} = (x, y, z)$. Index notations (u_1, u_2, u_3) and (x_1, x_2, x_3) are used, where convenient, instead of (u, v, w) and (x, y, z). The velocity field may be decomposed as

$$\boldsymbol{u}(\boldsymbol{x},t) = \bar{u}(z,t)\,\boldsymbol{e}_1 + \boldsymbol{u}'(\boldsymbol{x},t), \tag{2.1}$$

where an overbar denotes either the ensemble average or the average over a horizontal plane. Here, $\boldsymbol{u}'$ is an initially isotropic and homogeneous turbulent velocity field and $\bar{u}$ is a horizontal mean flow in direction $\boldsymbol{e}_1$ associated with the vertical mean shear α $(\bar{u}(z) = \alpha z)$. Similarly, the fluid density is

$$\rho(\boldsymbol{x},t) = \bar{\rho}(z,t) + \rho'(\boldsymbol{x},t), \tag{2.2}$$

where $\bar{\rho}$ is the background density profile and ρ' is the density fluctuation from the mean profile. We define the Brunt–Väisälä frequency associated with the background stratification by

$$N^2 = -(g/\rho_r)\frac{\mathrm{d}\bar{\rho}(z)}{\mathrm{d}z}, \tag{2.3}$$

where g is the norm of $\boldsymbol{g}$ and ρ_r a suitable reference density. For convenience, we use the variable n defined by $n = N^2$.

With fully periodic boundary conditions, RDT uses a Fourier decomposition of the velocity and density fluctuations:

$$u_i'(\boldsymbol{x},t) = \sum_{\boldsymbol{k}_0} \hat{u}_i'(\boldsymbol{k}_0,t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}}, \tag{2.4}$$

$$\rho'(\boldsymbol{x},t) = \sum_{\boldsymbol{k}_0} \hat{\rho}'(\boldsymbol{k}_0,t)\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}}, \tag{2.5}$$

where the wavevector $\boldsymbol{k} = (k_1, k_2, k_3)$ varies with time according to

$$\frac{dk_1}{dt} = 0, \quad \frac{dk_2}{dt} = 0, \quad \frac{dk_3}{dt} = -k_1\alpha, \tag{2.6}$$

and $\boldsymbol{k}_0 = \boldsymbol{k}(0) = (k_{10}, k_{20}, k_{30})$ (or (k_1, k_2, k_{30}) as k_1 and k_2 are constant). The time-dependence of the vertical wavenumber can be seen as the distortion of the wave-number space by the background vertical shear. We also use the notation $k(t) = (k_1^2 + k_2^2 + k_3^2(t))^{1/2}$ and $k_0 = (k_1^2 + k_2^2 + k_{30}^2)^{1/2}$.

As with all nonlinear systems, the Navier–Stokes equations can also be linearized to calculate the solution near any point in the phase space. In turbulent flows, this approximation is better for the larger scales of motion. The derivation of the locally linearized equations of RDT under the Boussinesq approximation may be found in Townsend (1976):

$$\left.\begin{aligned} \frac{d\hat{u}_i'}{dt} &= \alpha\left(\frac{2k_ik_1}{k^2} - \delta_{i1}\right)\hat{u}_3' + \left(\frac{k_ik_3}{k^2} - \delta_{i3}\right)\frac{g}{\rho_r}\hat{\rho}', \\ \frac{d\hat{\rho}'}{dt} &= \frac{\rho_r}{g}n\hat{u}_3', \end{aligned}\right\} \tag{2.7}$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. Here, the effect molecular dissipation is not taken into account.

The physical implication of the linearization of the equations is that immediately after the turbulence is initiated, the large-scale energetic eddies do not interact with one another; therefore the cascade of energy from large to small scales is neglected for short times. Typically, this assumption is relevant for eddies of size l and characteristic velocity u' as long as $t \ll l/u'$. In the case of strongly stratified flows for which the turbulent Froude number $Fr = u_0'/Nl_0$ (where l_0 and u_0' are the initial integral lengthscale and r.m.s. velocity associated with turbulence) is small, the timescale of nonlinear transfers l_0/u_0' is large compared to the timescale of stratification N^{-1} and RDT is an accurate approximation for the energy-containing scales when $t = O(N^{-1}) \ll l_0/u_0'$.

When a uniform vertical mean shear is present, $\alpha^{-1} = (d\bar{u}/dz)^{-1}$ is another characteristic timescale of the problem. We assume here that the stratification is strong enough that the Richardson number $Ri = N^2/\alpha^2$ is large. This means that $N^{-1} \ll \alpha^{-1}$ and as we are concerned with timescales of order N^{-1}, this implies that $\alpha t \ll 1$.

Our assumptions may be summarized as follows:

$$Fr = u_0'/Nl_0 \ll 1, \tag{2.8}$$

$$Ri = N^2/\alpha^2 \gg 1, \tag{2.9}$$

$$t = O(N^{-1}). \tag{2.10}$$

Note that when $\alpha = 0$, these assumptions reduce to (2.8) and (2.10), and the crucial assumption of RDT $t \ll l_0/u_0'$ is still satisfied (see e.g. Townsend 1976 for the unsheared case). When $\alpha \neq 0$, the strain αt is a small parameter provided that conditions (2.9) and (2.10) are satisfied. RDT equations (2.7) have already been evaluated numerically by Komari *et al.* (1983) and Hunt *et al.* (1988) with constant mean shear and stratification for different values of the Richardson number. More recently, the case of constant shear and stratification has also been investigated analytically by Hanazaki & Hunt (1999). Here our aim here is to study analytically in detail the behaviour of this dynamical system for short times, first when N and

α are fixed, and subsequently when they vary with time. We particularly focus on the covariances $\overline{u'w'}$ and $\overline{\rho'w'}$, which determine the momemtum and mass transport respectively. Only the outline of the analysis is presented here, because much of it follows previous treatments. Note that if the condition (2.10) is not satisfied so that $t >\sim N^{-1}$ and $\alpha t \sim 1$, the results of the RDT may still provide a useful approximation to the statistical and instantaneous eddy structure for large-scale turbulence, because for some situations the dominant nonlinear terms tend to be suppressed by the linear distortion processes (Kevlahan & Hunt 1997).

The Fourier components of the velocity and density fields will be expanded as follows:

$$\hat{u}'_i(\boldsymbol{k}_0, t) = \sum_{p=0}^{\infty} \hat{u}'_{ip}(\boldsymbol{k}_0)(\alpha t)^p, \quad \hat{\rho}'(\boldsymbol{k}_0, t) = \sum_{p=0}^{\infty} \hat{\rho}'_p(\boldsymbol{k}_0)(\alpha t)^p. \tag{2.11}$$

Solving (2.7) at order 3 in αt, we have analytically computed the $\hat{u}'_{ip}$ and $\hat{\rho}'_p$ for $p = 1$, 2 and 3. This allows us to calculate the spectrum tensors Φ_{ij} and $\Phi_j^{(\rho)}$ as time develops, where

$$\Phi_{ij}(\boldsymbol{k}_0, t) = \tfrac{1}{2}\overline{\hat{u}'^*_i \hat{u}'_j + \hat{u}'_i \hat{u}'^*_j} \quad \text{and} \quad \Phi_j^{(\rho)}(\boldsymbol{k}_0, t) = \tfrac{1}{2}\overline{\hat{\rho}'^* \hat{u}'_j + \hat{\rho}' \hat{u}'^*_j}, \tag{2.12}$$

and an asterisk denotes a complex conjugate.

For computational convenience, both the initial velocity and density perturbations are assumed to be intially isotropic. A slight anisotropy does not greatly affect the results as shown by Hunt & Carruthers (1990). A discussion on the effect of strong anisotropy may be found in Cambon & Scott (1999).

The initial conditions are given by

$$\Phi_{ij}(\boldsymbol{k}_0, 0) = \frac{E(k_0, 0)}{4\pi k_0^2}\left(\delta_{ij} - \frac{k_{i0}k_{j0}}{k_0^2}\right) \quad \text{and} \quad \Phi^{(\rho)}(\boldsymbol{k}_0, 0) = 2N^2 \frac{S(k_0, 0)}{4\pi k_0^2}. \tag{2.13}$$

However, the results are sensitive to the ratio (r) of the initial turbulent kinetic and potential energies, namely

$$KE_0 = \int_0^{\infty} E(k_0, 0)\, \mathrm{d}k_0 = \tfrac{3}{2}u_0'^2, \quad PE_0 = \int_0^{\infty} S(k_0, 0)\, \mathrm{d}k_0 \tag{2.14}$$

$$r = PE_0/KE_0. \tag{2.15}$$

We assume that the density and velocity perturbations are initially uncorrelated:

$$\Phi_j^{(\rho)}(\boldsymbol{k}_0, 0) = 0, \tag{2.16}$$

but the effect of non-zero $\Phi_j^{(\rho)}(\boldsymbol{k}, 0)$ could be taken into account by using the same method.

Writing the wave-vector in spherical coordinates as

$$k_1 = k_0 \sin\theta \cos\phi, \quad k_2 = k_0 \sin\theta \sin\phi, \quad k_{30} = k_0 \cos\theta, \tag{2.17}$$

correlations $\overline{u'w'}$ and $\overline{\rho'w'}$ are obtained as

$$\overline{u'w'}(t) = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \Phi_{13}(\boldsymbol{k}_0, t) k_0^2 \sin\theta\, \mathrm{d}k_0\, \mathrm{d}\theta\, \mathrm{d}\phi \tag{2.18}$$

and

$$\overline{\rho'w'}(t) = \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \Phi_3^{(\rho)}(\boldsymbol{k}_0, t) k_0^2 \sin\theta\, \mathrm{d}k_0\, \mathrm{d}\theta\, \mathrm{d}\phi. \tag{2.19}$$

We find at leading order in Ri, when $Ri \gg 1$, that

$$\frac{-\overline{u'w'}(t)}{u_0'^2} = A(\alpha t) + BRi(\alpha t)^3 + O((\alpha t)^4) \tag{2.20}$$

and

$$\frac{-\overline{\rho' w'}(t)\alpha}{u_0'^2\,\mathrm{d}\bar{\rho}/\mathrm{d}z} = H(\alpha t) + IRi(\alpha t)^3 + O((\alpha t)^4), \tag{2.21}$$

where $A = 2/5$, $B = -2/15$, $H = 1 - 2r$ and $I = -(8/15)(1 - 2r)$.

When $\alpha = 0$, the results are still valid provided that assumptions (2.8) and (2.10) are satisfied. Then, no momentun flux develops and the buoyancy flux may be rewritten as

$$\frac{-\overline{\rho' w'}(t)}{(u_0'^2/N)\,\mathrm{d}\bar{\rho}/\mathrm{d}z} = H(Nt) + I(Nt)^3 + o((Nt)^4). \tag{2.22}$$

These results are to be compared with the solutions for the variances:

$$\left.\begin{aligned}\overline{u'^2}/u_0'^2 &= 1 - \tfrac{1}{10}(1-2r)(Nt)^2 + o((Nt)^3),\\ \overline{w'^2}/u_0'^2 &= 1 - \tfrac{4}{5}(1-2r)(Nt)^2 + o((Nt)^3),\\ \overline{\rho'^2}/u_0'^2N^2 &= 3r + (1-2r)(Nt)^2 + o((Nt)^3).\end{aligned}\right\} \tag{2.23}$$

$A = 2/5$ is a classical result of RDT (Townsend 1976). $H = (1 - 2r)$ is in agreement with the RDT analysis of Hanazaki & Hunt (1996) with no mean shear. Equation (2.20) shows that, since the mean shear affects the pressure fluctuations at $t = 0$, it affects the mean shear stress when $t \sim \alpha^{-1}$, and that, although the stratification affects the vertical motion when $t \sim N^{-1}$, it only affects the shear stress at $O(t^3)$, when $t \sim \alpha^{-1}Ri^{-1/3}$. On the other hand, the solution shows that in the limit of high Richardson number (i.e. strong stratification and weak shear), the buoyancy flux $-\overline{\rho'w'}$ and variances $\overline{u'^2}$, $\overline{w'^2}$ and $\overline{\rho'^2}$ are not affected by the mean shear but start oscillating under the effect of the restoring buoyancy forces. Notice that their evolution is controlled by the value of $r = PE_0/KE_0$ but oscillations are expected provided that $r \neq 1/2$, i.e. $PE_0/w_0'^2 \neq 2/3$ (the case $r = 1/2$ is addressed in Godeferd & Cambon 1994). On the contrary, the short-time evolution of the momentum flux does not depend on the initial amount of potential energy. Note how $\overline{u'^2}$ is reduced as well as $\overline{w'^2}$ because of the strong effect of buoyancy pressure gradients.

The momentum and buoyancy fluxes are plotted on figures 2(a) and 3(a) as a function of αt for $Ri = 10$ and $Ri = 100$. The solution for the momentum flux does not depend on r and the buoyancy flux has been plotted for $r = 0$ and $r = 1$.

The behaviour of the solution may be physically interpreted by considering the motion of the fluid particles for $t > 0$. Because of buoyancy forces, they immediately oscillate in the vertical plane in a time period of order π/N (see figures 2b and 3b). For the first quarter-period, $\overline{\rho'w'} > 0$ and $\overline{u'w'} < 0$, but for the second quarter-period, when the particles are driven back to their initial level, the associated momentum and buoyancy fluxes change sign. This is a more significant effect than the effect of stratification on the variances, which is small and does not change the sign of the fluxes. Such oscillations of turbulent fluxes were also observed in direct numerical simulations of the full nonlinear Navier–Stokes equations by Métais & Herring (1989) and in laboratory experiments by Piccirillo & Van Atta (1997) for instance. Our solutions, which confirm the numerical results of Hunt *et al.* (1988), show that these oscillations are largely linear processes.

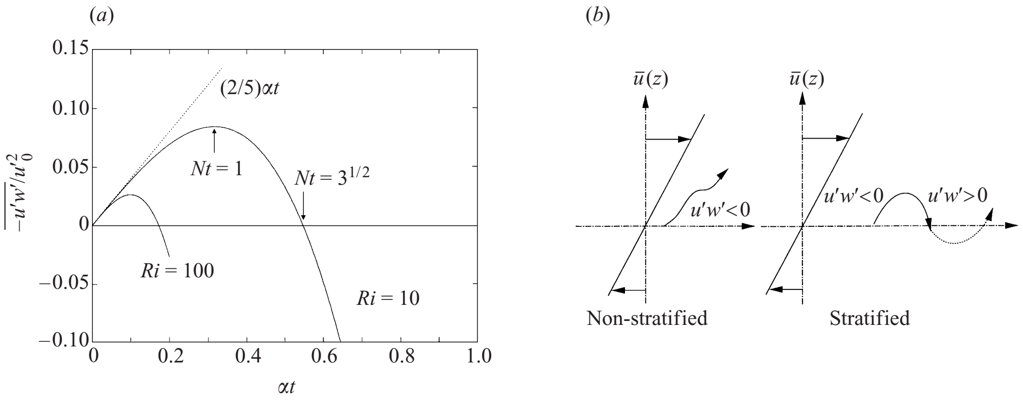

FIGURE 2. (a) Short-time evolution of the momentum flux in a strongly stratified shear flow for Ri = 10 and Ri = 100 computed using RDT. The turbulent momentum flux changes sign at $Nt = \sqrt{3}$, as a result of the restoring effect of the buoyancy forces. (b) Sketch of the particle motion in the initial stage of decay of a turbulent shear flow, and the induced momentum flux. Left: non-stratified flow. Right: stratified flow.

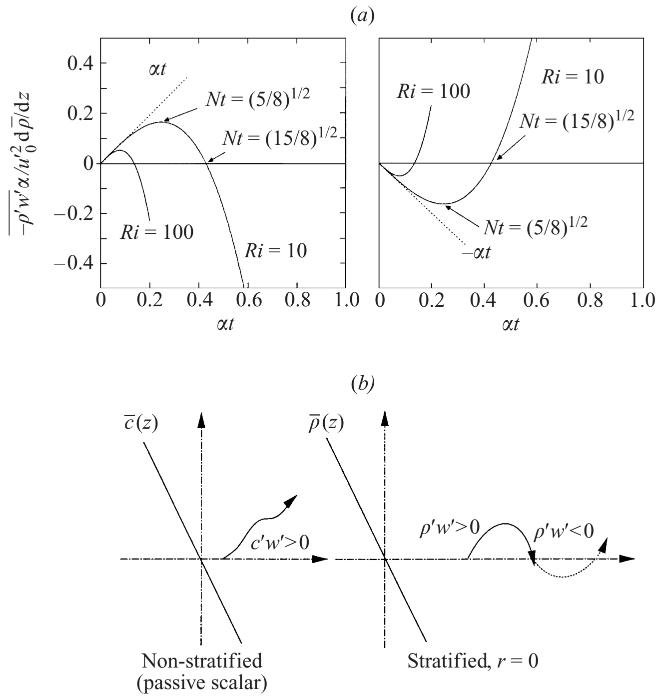

FIGURE 3. (a) Short-time evolution of the buoyancy flux in a strongly-stratified shear flow for Ri = 10 and Ri = 100 computed using RDT. Here two values of the potential to kinetic energy ratio are considered: $r = 0$ (left) and $r = 1$ (right). The turbulent buoyancy flux changes sign at $Nt = \sqrt{15/8}$, as a result of the restoring effect of the buoyancy forces. (b) Sketch of the particle motion in the initial stage of decay of turbulence, and the induced mass flux. Left: non-stratified flow (c is the concentration of a passive scalar). Right: stratified flow.

By writing the fluxes $-\overline{u'w'}$ and $-\overline{\rho'w'}$ in terms of the mean gradients of velocity and density as

$$-\overline{u'w'}\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}z}, t\right) = \tfrac{2}{5}u_0'^2 t[1 - \tfrac{1}{3}(Nt)^2]\frac{\mathrm{d}\bar{u}}{\mathrm{d}z} \tag{2.24}$$

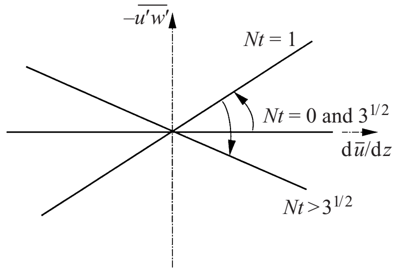

FIGURE 4. Short-time plot of the momemtum flux against the mean flow gradient in a strongly stratified turbulent shear flow, computed using RDT for any r. For $Nt > \sqrt{3}$, the momentum flux becomes counter-gradient.

and

$$-\overline{\rho' w'}\left(\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}, t\right) = -u_0'^2(1-2r)t\left[1 - \frac{8}{15}\frac{g}{\rho_r}(Nt)^2\left(\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}\right)^2\right]\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z}, \qquad (2.25)$$

it is clear how the above explanation of oscillation leads to a change in the signs of the fluxes.

Also, following Phillips (1972) and Posmentier (1977), this allows us to plot the turbulent fluxes as functions of the mean gradients (see figures 4 and 5) at different times. These graphs must be treated with caution because they are instantaneous curves during a rapidly changing flow.

2.2. *Time-dependent shear and stratification*

The previous results are easily extended to the case of a time-dependent mean shear and stratification: α and N have initial values α_0 and N_0 (associated with the initial mean flow and mean density linear profiles $\bar{u}_0(z)$ and $\bar{\rho}_0(z)$) and are slowly varying as a function of time (on a time scale N_0^{-1}). Thus, the assumptions (2.8), (2.9) and (2.10) are still valid, based on α_0 and N_0.

For short times, time-series expansions may be used:

$$\alpha(t) = \sum_{p=0}^{\infty} \alpha_p(\alpha_0 t)^p, \quad n(t) = N^2(t) = \sum_{p=0}^{\infty} n_p(\alpha_0 t)^p = \sum_{p=0}^{\infty} \tilde{n}_p(N_0 t)^p, \qquad (2.26)$$

$$\hat{u}_i'(\boldsymbol{k}_0, t) = \sum_{p=0}^{\infty} \hat{u}_{ip}'(\boldsymbol{k}_0)(\alpha_0 t)^p, \quad \hat{\rho}'(\boldsymbol{k}_0, t) = \sum_{p=0}^{\infty} \hat{\rho}_p'(\boldsymbol{k}_0)(\alpha_0 t)^p, \qquad (2.27)$$

where $\tilde{n}_p = n_p Ri^{-p/2}$. Therefore, the changes in $\alpha(t)$ and $N(t)$ are small $(O(Ri^{-1/2}))$ over the period N_0^{-1}, which means that the flow is subject to a *slowly varying rapid distortion*. Nazarenko *et al.* (1999) have recently studied similar effects of a slowly varying strain using WKB methods.

Equations (2.7) are then solved for short times using exactly the same method as previously described, but now the change in the vertical wavenumber is affected by the varying shear as

$$k_3(t) = k_{30} - k_1 \sum_{p=0}^{\infty} \frac{\alpha_p}{p+1}(\alpha_0 t)^{p+1}. \qquad (2.28)$$

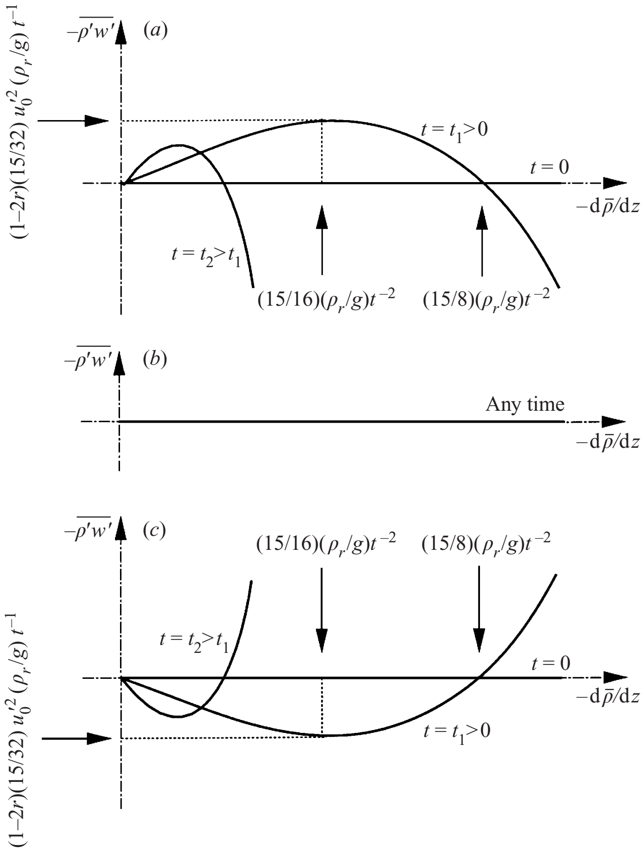

FIGURE 5. Short-time plot of the buoyancy flux against the mean density gradient in a strongly stratified turbulent flow, computed using RDT for three values of the potential to kinetic energy ratio: (*a*) $r < 1/2$, (*b*) $r = 1/2$ and (*c*) $r > 1/2$.

The new solutions for the momentum and buoyancy fluxes when $Ri \gg 1$ are modified as follows:

$$\frac{-\overline{u'w'}(t)}{u_0'^2} = A(\alpha_0 t) + C\frac{\alpha_1}{\alpha_0}(\alpha_0 t)^2 + \left(BRi_0 + D\frac{\alpha_2}{\alpha_0}\right)(\alpha_0 t)^3 + o((\alpha_0 t)^4) \tag{2.29}$$

and

$$\frac{-\overline{\rho'w'}(t)\alpha_0}{u_0'^2\,\mathrm{d}\bar{\rho}_0/\mathrm{d}z} = H(\alpha_0 t) + J\frac{n_1}{n_0}(\alpha_0 t)^2 + \left(IRi_0 + K\frac{n_2}{n_0}\right)(\alpha_0 t)^3 + o((\alpha_0 t)^4), \tag{2.30}$$

or

$$\frac{-\overline{\rho'w'}(t)}{(u_0'^2/N_0)\,\mathrm{d}\bar{\rho}_0/\mathrm{d}z} = H(N_0 t) + J\frac{\tilde{n}_1}{n_0}(N_0 t)^2 + \left(I + K\frac{\tilde{n}_2}{n_0}\right)(N_0 t)^3 + o((N_0 t)^4), \tag{2.31}$$

where A, B, H and I are unchanged. The new coefficients are: $C = 1/5$, $D = 2/15$, $J = 1/2$ and $K = 1/3$. Thus, the time variation of α and N does not produce any new effect linking the shear and stratification parameters. Effectively, (2.29) and (2.31) show that the turbulence adjusts quasi-steadily.

2.3. *Non-uniform shear and stratification*

When the initial shear and stratification (and thus the initial Richardson number) are uniform, the previous results show that the momentum and buoyancy fluxes are also uniform. Thus, unless as in the previous section α and N are changed by an external forcing, the mean flow and stratification remain unchanged as turbulence decays. This was also clear in (1.3) which shows that when $\partial_{zz}\bar{u} = 0$ and $\partial_{zz}\bar{\rho} = 0$, the mean flow and mean density profiles remain unaffected even if turbulent fluxes develop. But suppose that initially the mean flow and mean density gradients are not uniform, then the turbulence causes these gradients to change, and we have to consider the equations for the changes of the mean flow $\bar{u}(z,t)$ and mean density $\bar{\rho}(z,t)$ caused by the energy-containing motion. For these scales of high Reynolds number, molecular processes may be neglected, namely:

$$\partial_t\bar{u} = \partial_z(-\overline{u'w'}), \quad \partial_t\bar{\rho} = \partial_z(-\overline{\rho'w'}). \tag{2.32}$$

Equivalently, one can write the equations for $\alpha(z,t)$ and $n(z,t)$ $(= N^2(z,t))$ as

$$\partial_t\alpha = \partial_{zz}(-\overline{u'w'}), \quad \partial_t n = \frac{g}{\rho_r}\partial_{zz}(\overline{\rho'w'}). \tag{2.33}$$

As we do not consider the effect of rigid boundaries, we may impose periodic boundary conditions for α and n.

If α and n vary with z on a lengthscale L such that

$$L \gg l_0, \tag{2.34}$$

then eddies of size l_0 are locally distorted at each level, z, by a uniform background shear and stratification. If, in addition, the perturbations of the initial shear profile $\alpha_0(z)$ and stratification profile $N_0(z)$ are small, assumptions (2.8), (2.9) and (2.10) are valid at each height and the RDT analysis continues to be valid in the non-uniform and changing conditions of these new calculations. Given the expansions (2.26) and (2.27), and the results (2.29) to (2.31) for the unsteady, uniform shear and stratification problem, together with the assumption (2.34), we find that $\alpha(z,t)$ and $n(z,t)$ are governed by

$$\left.\begin{aligned}\partial_t\alpha(z,t) &= u_0'^2\partial_{zz}[A\alpha_0 t + C\alpha_1\alpha_0 t^2 + Bn_0\alpha_0 t^3 + D\alpha_2\alpha_0^2 t^3],\\ \partial_t n(z) &= u_0'^2\partial_{zz}[Hn_0 t + Jn_1\alpha_0 t^2 + In_0^2 t^3 + Kn_2\alpha_0^2 t^3].\end{aligned}\right\} \tag{2.35}$$

Identifying powers of t in this system, we can express functions $\alpha_p(z)$ and $n_p(z)$ $(p \neq 0)$ as functions of the initial profiles $\alpha_0(z)$ and $n_0(z)$ and their vertical derivatives. Details of the solution to (2.35) are given in the Appendix. Given the assumptions that $Ri \gg 1$ and $L \gg l_0$, it is found that the mean velocity and density perturbations $\Delta\bar{u}(z,t) = \bar{u}(z,t) - \bar{u}_0(z)$ and $\Delta\bar{\rho}(z,t) = \bar{\rho}(z,t) - \bar{\rho}_0(z)$ up to $O(t^4)$ are in fact given by $-\int_0^t \partial_z\overline{u'w'_{ss}}\,\mathrm{d}t$ and $-\int_0^t \partial_z\overline{\rho'w'_{ss}}\,\mathrm{d}t$, where $\overline{u'w'_{ss}}$ and $\overline{\rho'w'_{ss}}$ are the steady-state values of the fluxes given by (2.24) and (2.25), namely:

$$\left.\begin{aligned}\Delta\bar{u}(z,t) &= \frac{u_0'^2}{5}\left(\frac{\mathrm{d}^2\bar{u}_0}{\mathrm{d}z^2}\right)t^2 + \frac{u_0'^2}{30}\frac{g}{\rho_r}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\mathrm{d}\bar{u}_0}{\mathrm{d}z}\frac{\mathrm{d}\bar{\rho}_0}{\mathrm{d}z}\right)t^4,\\ \Delta\bar{\rho}(z,t) &= (1-2r)\left[\frac{u_0'^2}{2}\left(\frac{\mathrm{d}^2\bar{\rho}_0}{\mathrm{d}z^2}\right)t^2 + \frac{4u_0'^2}{15}\frac{g}{\rho_r}\left(\frac{\mathrm{d}\bar{\rho}_0}{\mathrm{d}z}\right)\left(\frac{\mathrm{d}^2\bar{\rho}_0}{\mathrm{d}z^2}\right)t^4\right].\end{aligned}\right\} \tag{2.36}$$

This solution may be rewritten as

$$\left.\begin{aligned}\Delta\bar{u}(z,t) &\sim u_0'\frac{(\alpha_0 t)^2}{Sh}[1+\lambda_u(N_0 t)^2],\\ \Delta\bar{\rho}(z,t) &\sim L\left(\frac{\mathrm{d}\bar{\rho}_0}{\mathrm{d}z}\right)\frac{(\alpha_0 t)^2}{Sh^2}[1-\lambda_\rho(N_0 t)^2],\end{aligned}\right\} \tag{2.37}$$

where α_0 and N_0 are the order of magnitude of the initial mean shear and mean Brunt–Väisälä frequency and $Sh = \alpha_0 L/u_0'$ is a measure of the shear flow velocity compared to the r.m.s. velocity of the turbulence. The parameters λ_u and λ_ρ are coefficients of order unity that depend on the form of the initial profiles $\bar{u}_0(z)$ and $\bar{\rho}_0(z)$; λ_u may be positive or negative, whereas λ_ρ is positive. Note that in the absence of stratification, there would be the usual diffusive terms proportional to $u_0'^4(\mathrm{d}^4\bar{u}_0/\mathrm{d}z^4)t^4$ and $u_0'^4(\mathrm{d}^4\bar{\rho}_0/\mathrm{d}z^4)t^4$; these are smaller than the t^4 terms in (2.36), since the assumptions (2.8) and (2.34) imply that $u_0'^4/N_0^4L^4 \ll u_0'^4/N_0^4 l_0^4 \ll 1$ (see the Appendix).

This solution now allows us to calculate the short-time evolution of the mean flow and stratification profiles. As with other RDT analyses of locally homogeneous turbulence, the variation of the one-point moments (i.e. covariances and variances) does not depend on the form of the initial spectrum of turbulence and therefore the variations of the mean profiles also do not depend on the spectra; but they do depend on:

(i) the lengthscale of the initial mean flow and stratification profiles and the initial intensity of turbulence $u_0'^2$, characterized by $Sh = \alpha_0 L/u_0'$;

(ii) the initial ratio of potential to kinetic turbulent energy r; and

(iii) the degree of anisotropy (which in these calculations is taken as zero).

In the next section, we discuss the behaviour of the solution for different initial conditions.

3. Growth of density and velocity profile perturbations

3.1. *Shear-free mean density profile perturbation*

We first consider the small perturbation $\Delta\bar{\rho}(z,t)$ of an initial density profile $\bar{\rho}_0(z)$ in the absence of a mean shear. The buoyancy frequency varies slowly with space and time and has the initial mean value N_0. The analytical results of (2.36) reduce to

$$\left.\begin{aligned}\Delta\bar{u}(z,t) &= 0,\\ \Delta\bar{\rho}(z,t) &= \frac{(1-2r)u_0'^2}{2}t^2[1-\tfrac{8}{15}(N_0 t)^2]\frac{\mathrm{d}^2\bar{\rho}_0}{\mathrm{d}z^2}.\end{aligned}\right\} \tag{3.1}$$

Thus, a perturbation in the mean density profile has no effect on the mean flow for short times when no mean shear is imposed at $t=0$, whereas the evolution of the perturbation may be described by defining a time-dependent *eddy diffusivity* as

$$\kappa_e(t) = \frac{\partial\bar{\rho}/\partial t}{\partial^2\bar{\rho}/\partial z^2} \simeq \frac{\partial\bar{\rho}/\partial t}{\mathrm{d}^2\bar{\rho}_0/\mathrm{d}z^2} = (1-2r)u_0'^2 t[1-\tfrac{16}{15}(N_0 t)^2], \tag{3.2}$$

which may be written in non-dimensional terms as

$$\kappa_e(t)/\kappa = (1-2r)Fr\,Re\,P\,N_0 t[1-\tfrac{16}{15}(N_0 t)^2], \tag{3.3}$$

where $P = \nu/\kappa$ is the Prandtl number, κ is the molecular diffusivity of the fluid, ν is the viscosity of the fluid, $Re = u_0' l_0/\nu$ is the initial Reynolds number and $Fr = u_0'/N_0 l_0$. Thus, comparing (2.25), (3.1) and (3.3), we note that the density flux $\overline{\rho' w'}$, the mean

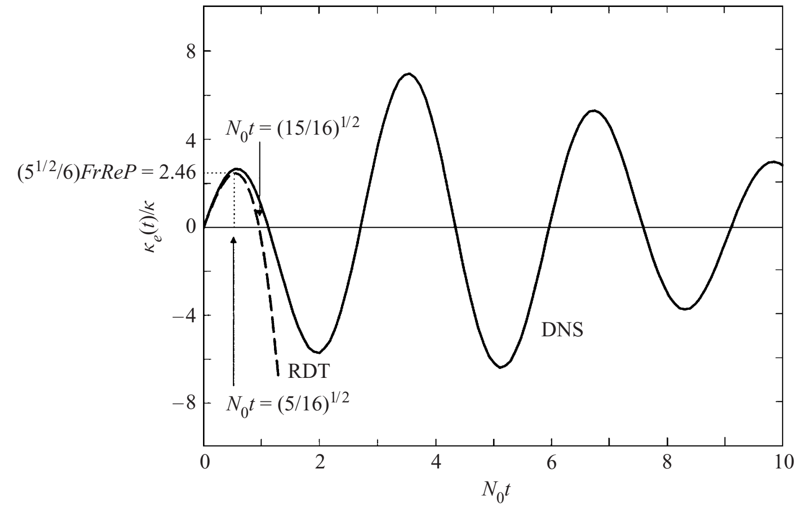

FIGURE 6. Short-time evolution of the eddy diffusivity in a strongly stratified turbulent flow computed using RDT and direct numerical simulation (from Galmiche *et al.* 2002). Here the time unit is the Brunt–Väisälä period.

density perturbation $\Delta\bar{\rho}$ and the eddy diffusivity do not change sign until $t \sim N_0^{-1}$ as a result of the correlated oscillations of the variances and other statistics of the flow field, as noted in previous direct numerical simulations and RDT calculations for $N_0 t >\sim 1$ (reviewed by Hanazaki & Hunt 1996), and in Reynolds stress transport models (e.g. Launder 1996) calculations. In the previous studies the density gradient was uniform.

For non-uniform density gradients, we note on figure 6 that the density flux and κ_e/κ are found to reach a maximum value according to (3.3) of $(\sqrt{5}/6) Fr\, Re\, P = 2.46$. In dimensional terms, the maximum value of κ_e is about $2.5u_0'^2/N$, which is the same order as that observed for the diffusivity in a steady-state turbulent shear flow (Hunt, Kaimal & Gaynor 1985). Figure 6 also shows that for $t <\sim N_0^{-1}$ (as anticipated by the order of magnitude analysis justifying the use of RDT), there is a close agreement between the RDT solution for short times and the temporal oscillations of an initial z-periodic perturbation ($\sim \cos z$) of the mean density profile computed in direct numerical simulations by Galmiche *et al.* (2002) when $Fr = 0.12$, $Re = 55$, $P = 1$ and $r = 0$. In particular, the maximum value of κ_e/κ agrees to within 1%. The eddy diffusivity starts decreasing and becomes negative at $N_0 t \simeq 1.1$ in these direct simulations, whereas the value provided by (3.3) is $\sqrt{15/16}$.

Physically, (3.3) is simple to interpret: where the density gradient is larger the turbulence is damped and therefore the gradient is locally diffused less by the turbulence than in regions where the gradient is weaker. This is Phillips's (1972) mechanism and leads to 'layering' of the vertical density gradient. This occurs when the stratification is strong (low Fr) but notice that according to (3.3), the phenomenon is enhanced when Re or P are increased, and for small r.

The solution (3.1) also shows that the perturbation evolves faster when $\mathrm{d}^2\bar{\rho}_0/\mathrm{d}z^2$ is increased. This means that the small-scale modes grow faster than the large-scale modes. This is consistent with the experiments of mixing in salt water carried out by Park *et al.* (1994), in which small steps in the mean density profile are formed first. However, relations (2.8) and (2.34) impose the lower limit u_0'/N_0 on the lengthscale

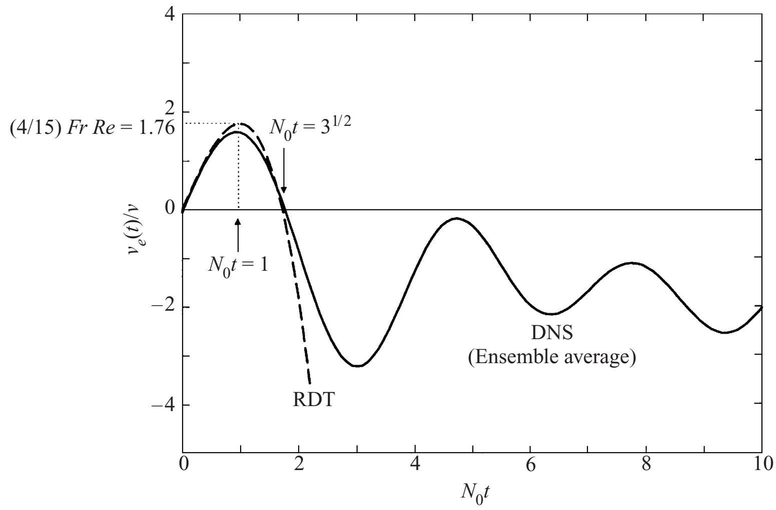

FIGURE 7. Short-time evolution of the eddy viscosity in a strongly stratified turbulent shear flow computed using RDT and direct numerical simulation (ensemble average over six realizations, from Galmiche *et al.* 2002). Here the time unit is the Brunt–Väisälä period.

of the growing perturbation. Thus, we expect from our short-time results that density layers may be formed with a thickness of order u_0'/N_0. This result is also in agreement with the observations of Park *et al.* (1994).

3.2. *Mean shear perturbations with uniform density gradient*

We now consider the mean perturbation $\Delta\bar{u}(z,t)$ to an initial mean profile $\bar{u}_0(z)$ with uniform Brunt–Väisälä frequency N_0. Equations (2.36) then lead to

$$\left.\begin{aligned} \Delta\bar{u}(z,t) &= \frac{u_0'^2}{5}t^2[1-\tfrac{1}{6}(N_0t)^2]\frac{\mathrm{d}^2\bar{u}_0}{\mathrm{d}z^2},\\ \Delta\bar{\rho}(z,t) &= 0. \end{aligned}\right\} \tag{3.4}$$

Thus, the mean shear does not affect the density profile for short times, whereas the mean flow is subject to the effect of a time-dependent *eddy viscosity*:

$$\nu_e(t) = \frac{\partial\bar{u}/\partial t}{\partial^2\bar{u}/\partial z^2} \simeq \frac{\partial\bar{u}/\partial t}{\mathrm{d}^2\bar{u}_0/\mathrm{d}z^2} = \tfrac{2}{5}u_0'^2 t[1-\tfrac{1}{3}(N_0t)^2], \tag{3.5}$$

which in non-dimensional terms is

$$\nu_e(t)/\nu = \tfrac{2}{5}Fr\,Re\,N_0t[1-\tfrac{1}{3}(N_0t)^2]. \tag{3.6}$$

Comparing (3.4) and (3.6) with (3.1) and (3.3) shows that in this case, the perturbations to the mean velocity profile and eddy viscosity caused by the buoyancy-driven oscillations are about one half to one third as great as those of the density profile and diffusivity.

Figure 7 shows that this RDT short-time solution is in agreement with the direct numerical simulations of Galmiche *et al.* (2002) when $Fr = 0.12$, $Re = 55$ and $P = 1$. In these simulations, the initial mean flow profile is z-periodic ($\sim \cos z$) with an initial z-periodic Richardson number varying between 100 and ∞ (which satisfies the assumption (2.9)). The ratio ν_e/ν is found to reach the maximum value of about 1.7 in the direct simulations, whereas the maximum value is $(4/15)Fr Re = 1.76$ in our RDT

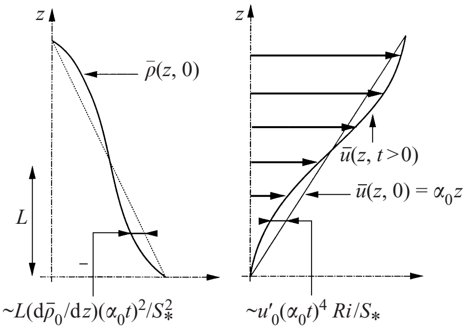

FIGURE 8. Sketch of the short-time evolution of the mean flow profile in a strongly stratified turbulent flow with uniform initial shear and non-uniform initial stratification, as predicted by the RDT model.

model. Then, the eddy viscosity starts decreasing and becomes negative at $N_0 t \simeq 1.7$, which is close to $\sqrt{3}$ as predicted by (3.6), and causes the mean current velocity to increase. Note that these are results for strongly stratified flows (low Fr and high Ri) but do not depend on the initial amount of potential energy r. Furthermore, (3.6) shows that the amplitude of the eddy viscosity variations are larger when the Reynolds number is increased.

The solution (3.4) also shows that the mean flow perturbation evolves faster the greater the initial curvature of the profile, $\mathrm{d}^2\bar{u}_0/\mathrm{d}z^2$. For the density layers, this result together with relations (2.8) and (2.34) suggest that shear layers may develop with a thickness of order u'_0/N_0 when $Ri \gg 1$.

3.3. *Coupled mean shear and mean stratification perturbation*

The evolution of $\Delta\bar{\rho}(z,t)$ up to $O((N_0 t)^4)$ when $Ri \gg 1$ (see system (2.36)) is uncoupled from the mean velocity gradient. This was already clear in (2.22) which shows that the momentum flux is not affected by the mean shear for short times. However, the mean perturbation $\Delta\bar{u}(z,t)$ to the velocity gradient is coupled to the non-uniformity in the initial density gradient through the term $(\mathrm{d}\bar{u}_0/\mathrm{d}z)(\mathrm{d}^2\bar{\rho}_0/\mathrm{d}z^2)$. In the simplest case in which this coupling effect appears there is a uniform mean shear $\alpha_0 = \mathrm{d}\bar{u}_0/\mathrm{d}z$ and an initially non-uniform stratification $N_0(z)$ (its mean value being again denoted by N_0). Then, the density profile evolves as in (3.1) in proportion to t^2, whereas the mean velocity perturbation $\Delta\bar{u}$ develops slowly in proportion to t^4 as given by

$$\Delta\bar{u}(z,t) = \frac{u_0'^2}{30}\frac{g}{\rho_r}\alpha_0\frac{\mathrm{d}^2\bar{\rho}_0}{\mathrm{d}z^2}t^4 \sim u'_0(\alpha_0 t)^4\frac{Ri}{Sh} \sim \frac{u'_0(N_0 t)^4}{RiSh}, \tag{3.7}$$

where again $S = \alpha_0 L/u'_0$ and $Ri = N_0^2/\alpha_0^2$. Thus, whatever the curvature of the density profile, the mean velocity profile develops a curvature and may tend to form layers (see figure 8).

The coupled evolution of $\bar{u}$ and $\bar{\rho}$ shows that the shear and density layers tend to develop spatially in phase. To our knowledge, there is no available direct numerical simulation to confirm this tendency. However, the coexistence of such layers was observed experimentally by Pearson & Linden (1983) in the final stage of decay of turbulence. Notice that this result is valid for any value of r, although it must be emphasized that the initial turbulence is isotropic.

4. Discussion and conclusion

We have used the rapid distortion theory to compute the short-time development of momentum and buoyancy fluxes when either or both strong stratification and mean shear are suddenly imposed on an initially isotropic turbulence, and when the mean stratification and velocity profiles vary slowly with time. The results (3.1)–(3.7), which all demonstrate the tendency of such flows to form layers in the initial stages, may be summarized as follows.

(i) The oscillations of the energy and the fluxes of the turbulence can amplify initial perturbations to the mean density profile (on lengthscales that are much larger than that of the turbulence). This is equivalent to an oscillation in the *eddy diffusivity*, such that it reaches a maximum value of order $u_0'^2/N_0$ and changes sign after a time of order N_0^{-1}. This result is in agreement with the direct numerical simulations performed by Galmiche *et al.* (2002) for short times.

(ii) Similarly, the effect of stable stratification on turbulence can cause amplification to perturbations to the mean velocity profile. This is mathematically equivalent to oscillations in the value of the *eddy viscosity* which reaches a maximum value of order $u_0'^2/N_0$ and then decreases and becomes negative. The timescale for the mean flow oscillations is about twice that for the mean density profile oscillations. This result is also in agreement with the direct numerical simulations of Galmiche *et al.* (2002) for short times.

(iii) The mean velocity perturbations are coupled to the perturbations in the mean density profile when a uniform mean shear is initially present, whereas the evolution of the mean density profile is uncoupled from the mean shear profile (these results are valid for $Fr \ll 1$, $Ri \gg 1$ and $t \sim N_0^{-1}$). The solution shows how this coupling leads to the formation of shear layers in a turbulent shear flow subject to a non-uniform stratification. Direct numerical simulation of such flows or laboratory observations are still needed to confirm this tendency in the first stage of decay of turbulence.

For long times (when $N_0 t >\sim 1$), the growth of the perturbations in $\bar{u}$ and N may depend on the mean shear and the value of Ri, as suggested by the linear calculations of the buoyancy flux in stratified turbulent shear flows undertaken by Hanazaki & Hunt (1999).

Another important result is that the perturbation of the mean density and mean flow profiles are expected to grow if its scale $L > u_0'/N_0$, but that the growth is faster as L decreases (i.e. $\mathrm{d}^2\bar{\rho}_0/\mathrm{d}z^2$ or $\mathrm{d}^2\bar{u}_0/\mathrm{d}z^2$ increases). This suggests a theoretical reason why the characteristic thickness of layers is of order u_0'/N_0, as observed by Park *et al.* (1994). Other quasi-steady-state arguments, such as those invoked by Balmforth *et al.* (1998) have also been used to address this question. Of course, this scale is the natural scale for particle displacements and determines density fluctuations measured in the environment (Hunt *et al.* 1985).

From these linear calculations it is not clear whether the fluxes and mean perturbations of the profiles lead to permanent layering or whether their evolution is merely oscillatory. Also it is not clear whether mean density perturbations can grow that are independent of mean velocity perturbations. Calculations incorporating nonlinear and molecular effects are necessary to address these two questions. In the direct numerical simulations of the fully nonlinear equations of motion performed by Galmiche *et al.* (2002), it is observed that the eddy viscosity acting on a mean flow profile in the presence of a strong uniform stratification not only oscillates but also remains persistently negative, which leads to the formation of permanent shear layers. In laboratory experiments where grids or obstacles with wakes have been towed through stably stratified tanks at very low values of Froude number, the velocity profiles in the wakes tend to be sharply diffused and to persist downwind. This is consistent with (3.7) which shows that the eddy viscosity tends to be reduced at the top and bottom of the wake where $|\mathrm{d}^2\bar{\rho}_0/\mathrm{d}z^2|$ is largest. On the other hand, in the numerical simulation of Galmiche *et al.* (2002) in the absence of a mean shear, a perturbation of the mean density profile was found to oscillate in time but no permanent growth was observed. One possible reason is the low values of the Reynolds and Prandtl numbers used in these simulations. Furthermore, the laboratory experiments of Pearson & Linden (1983) and Park *et al.* (1994) on decaying stratified turbulence without any imposed shear show that density layers form and slowly decay. No oscillations were reported. Perhaps one reason for the formation of semi-permanent layers is that the mean velocity and mean density perturbations are coupled, but apparently tend to vary on different timescales. This precludes any coupled oscillation. Further study of this point is needed.

In conclusion, this analysis is in good agreement with the direct simulations performed by Galmiche *et al.* (2002) for short times, and also provides us with a better understanding of the unsteady formation of layering in geophysical flows and laboratory experiments when the layering involves both density and mean velocity perturbations. It is important to emphasize that steady and unsteady stably stratified turbulent flows have some quite distinct characteristics (Fernando & Hunt 1996). However, further investigations are still needed to take nonlinear mechanisms into account. In strongly stratified flows at high Reynolds number, nonlinear and wave interaction at levels of large vertical mean velocity and density gradients invalidate the local eddy diffusivity and eddy viscosity concepts, partly because wave/wave interactions and local critical layers are expected to generate small scales in the mean profiles as described by Galmiche, Thual & Bonneton (2000) for instance.

More generally, it seems that there is much interesting work still to be done on layering processes in stratified turbulence. As mentioned in the introduction, a variety of theoretical approaches have been proposed by different authors, suggesting that layers may be described either in terms of tendency to anisotropy or turbulence–mean field interactions, or in the framework of the stability theory. They may also be seen as a balanced state of stratified turbulence. Although different approaches lead to similar conclusions, the connection between them is not obvious. For example, the typical lengthscale of layers u_0'/N_0 derived in the present paper also appears as a typical lengthscale in the study of Billant & Chomaz (2000), where it is defined as a vertical decorrelation lengthscale. In realistic oceanic or atmospheric flows the definition of the best indicator for layering is not straightforward. The approach of the problem still needs to be unified, and special attention has to be paid to the difference between anisotropy and vertical inhomogeneity of stratified turbulence.

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Appendix

The resolution of (2.35) leads to

$$\alpha(z,t) = \alpha_0 + \frac{Au_0'^2}{2}\frac{\mathrm{d}^2\alpha_0}{\mathrm{d}z^2}t^2 + \left[\frac{Bu_0'^2}{4}\frac{\mathrm{d}^2(\alpha_0 n_0)}{\mathrm{d}z^2} + \frac{ADu_0'^4}{8}\frac{\mathrm{d}^4\alpha_0}{\mathrm{d}z^4}\right]t^4 \tag{A 1}$$

and

$$n(z,t) = n_0 + \frac{Hu_0'^2}{2}\frac{\mathrm{d}^2 n_0}{\mathrm{d}z^2}t^2 + \left[\frac{Iu_0'^2}{4}\frac{\mathrm{d}^2 n_0^2}{\mathrm{d}z^2} + \frac{HKu_0'^4}{8}\frac{\mathrm{d}^4 n_0}{\mathrm{d}z^4}\right]t^4. \tag{A 2}$$

Using vertical lengthscale L, we have (when $t \sim N_0^{-1}$)

$$\frac{Bu_0'^2}{4}\frac{\mathrm{d}^2(\alpha_0 n_0)}{\mathrm{d}z^2}t^4 = O\left[\left(\frac{u_0'}{N_0 L}\right)^2\alpha_0\right], \quad \frac{ADu_0'^4}{8}\frac{\mathrm{d}^4\alpha_0}{\mathrm{d}z^4}t^4 = O\left[\left(\frac{u_0'}{N_0 L}\right)^4\alpha_0\right],$$

$$\frac{Iu_0'^2}{4}\frac{\mathrm{d}^2 n_0^2}{\mathrm{d}z^2}t^4 = O\left[\left(\frac{u_0'}{N_0 L}\right)^2 n_0\right], \quad \frac{HKu_0'^4}{8}\frac{\mathrm{d}^4 n_0}{\mathrm{d}z^4}t^4 = O\left[\left(\frac{u_0'}{N_0 L}\right)^4 n_0\right].$$

Provided that assumptions (2.8) and (2.34) are satisfied, we have

$$\frac{u_0'}{N_0 L} = \frac{u_0'}{N_0 l_0}\frac{l_0}{L} \ll 1, \tag{A 3}$$

and the solution thus reads at leading order:

$$\alpha(z,t) = \alpha_0 + \frac{Au_0'^2}{2}\frac{\mathrm{d}^2\alpha_0}{\mathrm{d}z^2}t^2 + \frac{Bu_0'^2}{4}\frac{\mathrm{d}^2(\alpha_0 n_0)}{\mathrm{d}z^2}t^4, \tag{A 4}$$

$$n(z,t) = n_0 + \frac{Hu_0'^2}{2}\frac{\mathrm{d}^2 n_0}{\mathrm{d}z^2}t^2 + \frac{Iu_0'^2}{4}\frac{\mathrm{d}^2 n_0^2}{\mathrm{d}z^2}t^4, \tag{A 5}$$

which leads rapidly to the solution for $\Delta\bar{u}(z,t)$ and $\Delta\bar{\rho}(z,t)$.

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