ANALYSIS OF DYNAMIC SYSTEM OPTIMUM AND EXTERNALITIES WITH DEPARTURE TIME CHOICE

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Abstract

This paper aims to analyse the dynamic system optimal assignment with departure time choice, which is an important, yet underdeveloped area. The main contribution of this paper is the necessary conditions and the sensitivity analysis for dynamic system optimizing flow. Following this, we revisit the issue of dynamic externality in a more plausible way. We showed that how the externality can be derived and interpreted from the control theoretic formulation and the sensitivity analysis of traffic flow. To solve the system optimal assignment, we propose a dynamic programming solution approach. We present numerical calculations and discuss the characteristics of the results. In particular, we contrast the system optimal assignment with its equilibrium counterpart in terms of the amount of travel generated, flow profiles, and travel costs.

1 Introduction

This paper aims to analyse the dynamic system optimal assignment with departure time choice, which is an important, yet underdeveloped area. The dynamic system optimal assignment process suggests that there is a central "system manager" to distribute network traffic over time within a fixed horizon. Consequently, the total, rather than individual, travel cost of all travellers through the network is minimised.

The travel cost incurred by each traveller is considered to have three distinct components: time-specific costs associated with the departure time of the traveller from the origin, and the arrival time at the destination respectively; and a cost related to the travel time *en route*. Given the assigned network flow, the associated travel times through the network are determined by a traffic model. This paper uses the linear whole-link traffic model proposed by Friesz et al. (1993), who considered the travel time on each link to be a linear non-decreasing function of whole link traffic. This traffic model satisfies the principles of flow conservation, proper flow propagation (i.e. consistency between flows and travel times), non-negativity of flow, first-in-first-out (FIFO), and causality. Detailed discussion of this traffic model can be referred to Mun (2001).

This paper starts with introducing the formulation of dynamic system optimal assignment, which is an optimal control problem with state-dependent response. Following Friesz et al. (2001), the optimality conditions for this special kind of control problem can be derived using the calculus of variations technique. At optimality, traffic is assigned such that the total system travel cost is minimized. To solve the dynamic system optimization, information on the sensitivity of the value of the objective function with respect to the control variable is necessary. Section three illustrates a novel sensitivity analysis of travel time and travel cost with respect to perturbations in inflow. Section four then shows a dynamic-programme algorithm for solving dynamic system optimal assignment. Numerical calculations are given in section five. Finally, concluding remarks are given in section six.

2 Formulation of dynamic system optimal assignment

The system optimal assignment with departure time choice for fixed travel demand can be formulated as the following optimal control problem. The optimization problem (1) minimizes the total system travel cost within the planning horizon given a predefined amount of total throughput:

$$\min_{e_a^*(s)} Z = \sum_{\forall a} \int_0^T C_a(s) e_a(s) ds \tag{1a}$$

subject to:

$$g_a[\tau_a(s)]\frac{d\tau_a(s)}{ds} = e_a(s) \quad ,\forall a, \forall s \tag{1b}$$

$$\frac{dx_a(s)}{ds} = e_a(s) - g_a(s) \quad ,\forall a, \forall s \tag{1c}$$

$$\frac{dE_a(s)}{ds} = e_a(s) \quad ,\forall a ,\forall s \tag{1d}$$

$$\sum_{\forall a} E_a(T) = J_{od} \tag{1e}$$

$$e_a(s) \geq 0 \quad ,\forall \text{a}, \forall s \tag{1f}$$

We consider the total travel cost $C_a(s)$ encountered by each traveller on the travel link has three distinct components. The first component is the time spent on travelling along the link, which is determined by the travel time model embedded. In addition to the travel time, we add a time-specific cost $f[\tau_a(s)]$ associated with arrival time $\tau_a(s)$ at the destination. Finally, we add a time-specific cost $h(s)$ associated with departure from the origin at time *s*. Possible choices of these time-specific cost functions are investigated by Heydecker and Addison (2005). Consequently, the total travel cost $C_a(s)$ associated with departure on link *a* at time s is determined as a linear combination of these costs as

$$C_a(s) = h(s) + [\tau_a(s) - s] + f[\tau_a(s)]. \tag{2}$$

The notation $\tau_a(s)$ denotes the exit time from the link *a* for traffic which enters at time *s*. For Friesz's (1993) linear whole-link traffic model, $\tau_a(s)$ takes the following functional form:

$$\tau_a(s) = s + \phi_a + \frac{x_a(s)}{Q_a}, \tag{3}$$

where the amount of whole link traffic at time *s* is represented by $x_a(s)$. The free flow travel time and the capacity of the travel link are denoted by ϕ_a and Q_a respectively.

Equations (1b) ensure the proper flow propagation along each route. Equations (1c) are the state equations that govern the evolution of link traffic. Equations (1d) define the cumulative inflow $E_a(s)$. Equation (1e) specifies the amount of total throughput J_{od} generated in the system within the time horizon *T*. Conditions (1f) ensure the positivity of the control variable. Since Friesz's (1993) traffic model has been shown to satisfy FIFO structurally (Mun, 2001), we do not need to add any explicit constraint for this.

The optimality conditions for the optimization problem (1) can be derived as

$$e_a(s)\begin{cases} > 0 \Rightarrow \left.\dfrac{\partial Z}{\partial u}\right|_s + \lambda_a(s) - \lambda_a[\tau_a(s)] = \mu_a(s) = \nu \\ = 0 \Rightarrow \left.\dfrac{\partial Z}{\partial u}\right|_s + \lambda_a(s) - \lambda_a[\tau_a(s)] \geq \mu_a(s) = \nu \end{cases} ,\forall s \in [0,T], \tag{4}$$

where $\mu_a(s) = \nu$ is a constant of time and its magnitude is determined by the predefined amount of throughput. The derivative $\left.\frac{\partial Z}{\partial u}\right|_s$ represents the sensitivity of the value of the objective function with respect to a perturbation u in the profile of inflow at time s, where

$$\begin{aligned} \left.\frac{\partial Z}{\partial u}\right|_s &= \frac{\partial}{\partial u}\left[\int_0^T C_a(t)e_a(t)dt\right]\Bigg|_s \\ &= C_a(s) + \int_0^T \left.\frac{\partial C_a}{\partial u}\right|_s e_a(t)dt \end{aligned} . \tag{5}$$

The quantity $\left.\frac{\partial Z}{\partial u}\right|_s$ indeed can also be interpreted as the marginal contribution of adding an additional traffic to the link to the total travel cost on this link. It is the sum of two components: $C_a(s)$ is the travel time experienced by that additional traveller given the current traffic condition; $\int_0^T \left.\frac{\partial C_a}{\partial u}\right|_s e_a(t)dt$ is the additional travel cost, which is also known as externality, added by this traveller to each of the existing travellers. Understanding the nature of this externality is important in managing dynamic network, and it requires knowing $\left.\frac{\partial C_a}{\partial u}\right|_s$, which is analysed in Section 3.

Furthermore, the costate variable $\lambda_a(s)$ is determined as:

$$\lambda_a(s) = \frac{1}{Q_a}\int_{t=s}^{T}\left(1 + f'[\tau_a(t)]\right)e_a(t)dt . \tag{6}$$

This costate variable $\lambda_a(s)$ represents the sensitivity of the value of the objective function wih respect to the changes in state variable $x_a(s)$. In other words, the costate variable $\lambda_a(s)$ is interpreted as the marginal travel cost of increasing the link traffic volume by one unit. The details of derivation of this set of optimality conditions can be found in the full version (Chow, 2006).

3 Sensitivity analysis

In this section, we start with establishing an expression for the derivatives of the time of exit from a link with respect to a parameter of the inflow profile. Following this, the externality with respect to additional traffic can be derived.

Consider the expression of the whole link traffic, $x_a(s)$, it can be written alternatively as

$$x_a(s) = E_a(s) - G_a(s) = E_a(s) - E_a[\sigma_a(s)] = \int_{t=\sigma_a(s)}^{s} e_a(t)dt , \tag{7}$$

in which $\sigma_a(s)$ is the time of entry to the link that leads to exit at time s. The expression for the time of exit in (3) then becomes

$$\tau_a(s) = s + \phi_a + \frac{1}{Q_a}\int_{t=\sigma_a(s)}^{s} e_a(t)dt\,. \tag{8}$$

A perturbation *u* in the profile of inflow $e_a(s)$ induces a change in the time of exit as

$$\begin{aligned} \left.\frac{d\tau_a}{du}\right|_s &= \frac{d}{du}\left(s + \phi_a + \frac{1}{Q_a}\int_{t=\sigma_a(s)}^{s} e_a(t)dt\right) \\ &= \frac{1}{Q_a}\frac{d}{du}\left(\int_{t=\sigma_a(s)}^{s} e_a(t)dt\right) \\ &= \frac{1}{Q_a}\left\{\int_{t=\sigma_a(s)}^{s} \frac{de_a(t)}{du}dt - \frac{d\sigma_a(s)}{du}e_a[\sigma_a(s)]\right\} \end{aligned}\,. \tag{9}$$

The first term is the bracket can be calculated directly. To determine the second term in (9), we first apply the definitional relationship,

$$\tau_a[\sigma_a(s)] = s\,, \tag{10}$$

and using chain rule implies

$$\left.\frac{d\tau_a}{du}\right|_{\sigma_a(s)} = \left.\frac{\partial\tau_a}{\partial u}\right|_{\sigma_a(s)} + \frac{\partial\tau_a[\sigma_a(s)]}{\partial\sigma_a(s)}\frac{\partial\sigma_a(s)}{\partial u}\,. \tag{11}$$

However, at the same time we note that

$$\left.\frac{d\tau_a}{du}\right|_{\sigma_a(s)} = \frac{ds}{du} = 0\,, \tag{12}$$

since *s* is fixed with respect to perturbation *u*.

Hence,

$$\left.\frac{\partial\tau_a}{\partial u}\right|_{\sigma_a(s)} + \frac{\partial\tau_a[\sigma_a(s)]}{\partial\sigma_a(s)}\frac{\partial\sigma_a(s)}{\partial u} = 0\,. \tag{13}$$

Furthermore,

$$\begin{aligned} &\frac{d\tau_a[\sigma_a(s)]}{ds} = \frac{\partial\tau_a[\sigma_a(s)]}{\partial\sigma_a(s)}\frac{\partial\sigma_a(s)}{\partial s} = \frac{ds}{ds} = 1 \\ &\Rightarrow \frac{\partial\tau_a[\sigma_a(s)]}{\partial\sigma_a(s)} = \frac{1}{\dfrac{\partial\sigma_a(s)}{\partial s}} \end{aligned}\,. \tag{14}$$

Therefore,

$$\frac{\partial\sigma_a(s)}{\partial u}=-\left(\frac{\partial\tau_a[\sigma_a(s)]}{\partial\sigma_a(s)}\right)^{-1}\frac{\partial\tau_a}{\partial u}\bigg|_{\sigma_a(s)}=-\frac{d\sigma_a(s)}{ds}\frac{\partial\tau_a}{\partial u}\bigg|_{\sigma_a(s)}. \tag{15}$$

Thus,

$$\begin{aligned}\frac{\partial\tau_a}{\partial u}\bigg|_s&=\frac{1}{Q_a}\left\{\int_{t=\sigma_a(s)}^{s}\frac{de_a(t)}{du}dt-\frac{d\sigma_a(s)}{du}e_a[\sigma_a(s)]\right\}\\&=\frac{1}{Q_a}\left\{\int_{t=\sigma_a(s)}^{s}\frac{de_a(t)}{du}dt+e_a[\sigma_a(s)]\frac{d\sigma_a(s)}{ds}\frac{\partial\tau_a}{\partial u}\bigg|_{\sigma_a(s)}\right\}.\\&=\frac{1}{Q_a}\left\{\int_{t=\sigma_a(s)}^{s}\frac{de_a(t)}{du}dt+g_a(s)\frac{\partial\tau_a}{\partial u}\bigg|_{\sigma_a(s)}\right\}\end{aligned} \tag{16}$$

The derivative of exit time with respect to the perturbation *u* is then expressed in terms of the dependence of the inflow profile $e_a(s)$ in which *s* lies between *s* and $\sigma_a(s)$, the current outflow $g_a(s)$, and the derivative of exit time at the time of entry, $\sigma_a(s)$.

When the analysis is implemented in computer, calculating $\frac{d\tau_a(s)}{du}$ requires knowing the value of $\frac{d\tau_a}{du}\Big|_{\sigma_a(s)}$, in which $\sigma_a(s)$ usually is not an integer. Therefore, a linear interpolation is needed to determine the value of $\frac{d\tau_a}{du}\Big|_{\sigma_a(s)}$ as

$$\begin{aligned}&\frac{\frac{d\tau_a}{du}\Big|_{\sigma_a(s)}-\frac{d\tau_a}{du}\Big|_{\lfloor\sigma_a(s)\rfloor}}{\sigma_a(s)-\lfloor\sigma_a(s)\rfloor}=\frac{\frac{d\tau_a}{du}\Big|_{\lceil\sigma_a(s)\rceil}-\frac{d\tau_a}{du}\Big|_{\lfloor\sigma_a(s)\rfloor}}{\lceil\sigma_a(s)\rceil-\lfloor\sigma_a(s)\rfloor}\\&\Rightarrow\frac{d\tau_a}{du}\bigg|_{\sigma_a(s)}=\frac{d\tau_a}{du}\bigg|_{\lfloor\sigma_a(s)\rfloor}+\left(\frac{d\tau_a}{du}\bigg|_{\lceil\sigma_a(s)\rceil}-\frac{d\tau_a}{du}\bigg|_{\lfloor\sigma_a(s)\rfloor}\right)(\sigma_a(s)-\lfloor\sigma_a(s)\rfloor)\end{aligned}, \tag{17}$$

where the notation $\lceil\sigma_a(s)\rceil$ represent the smallest integer not smaller than $\sigma_a(s)$, and $\lfloor\sigma_a(s)\rfloor$ is the greatest integer not larger than $\sigma_a(s)$.

After deriving the sensitivity of travel time with respect to inflow, the sensitivity of the objective function with respect to inflow can be deduced as

$$\left.\frac{\partial Z}{\partial u}\right|_s = \frac{\partial}{\partial u}\left[\int_0^T C_a(t)e_a(t)dt\right]\Bigg|_s$$
$$= C_a(s) + \int_0^T \left.\frac{\partial C_a}{\partial u}\right|_s e_a(t)dt \quad . \tag{18}$$
$$= C_a(s) + \int_0^T \left(1 + f'[\tau_a(t)]\right)\left.\frac{\partial \tau_a}{\partial u}\right|_s e_a(t)dt$$

4 Solving dynamic system optimum

We propose the following procedure to solve for the dynamic system optimal assignment with fixed travel demand:

Step 0: Initialisation

0.1. Guess an initial equilibrium cost C_{od}^*;

0.2 set the overall iteration counter $n := 1$;

0.3 set $e_a(k) := 0$ for all links *a*, $a \in [1, A]$, and all times *k*, $k \in [1, K]$. The notation $e_a(k)$ represents the assigned inflow to link *a* between times $k\Delta s$ and $(k+1)\Delta s$. The total number of simulated time steps is denoted as $K = T/\Delta s$ and the total number of parallel links is denoted by *A*; set time index $k := 0$;

0.4 set costates $\lambda_a(k) := 0$ for all times $k \in [1, K]$;

0.5 set the link index $a := 1$;

0.6 set the time index $k := 0$;

0.7 set the overall iteration counter $n^i := 1$.

Step 1: network loading

Find $\tau_a(k+1)$ by loading the travel link using the inflow $e_a(k)$ at the current iteration.

Step 2: equilibrating

2.1 Calculate $C_a(k+1) = h(k+1) + [\tau_a(k+1) - (k+1)] + f[\tau_a(k+1)] + \lambda_a(k+1) - \lambda_a[\tau_a(k+1)]$;

2.2 calculate $\Omega = \dfrac{C_a(k+1) - C_a(k)}{\Delta s}$ and $\Omega' = \dfrac{\partial \Omega}{\partial e_a(k)} = \left(1 + f'[\tau_a(k+1)]\right)\dfrac{1}{Q_a}$, in which $f'[\tau_a(k)] = \dfrac{f[\tau_a(k+1)] - f[\tau_a(k)]}{\tau_a(k+1) - \tau_a(k)}$;

2.3 update $e_a(k) := e_a(k) - \pi d$ with the second-order searching direction $d = \Omega / \Omega'$ and the step size π, which is interpolated linearly as

$$\pi = \frac{C_{od}^* - C_a^0(k+1)}{C_a^1(k+1) - C_a^0(k+1)},$$

where $C_a^1(k+1)$ and $C_a^0(k+1)$ represent the corresponding values of $C_a(k+1)$ when $e_a(k)$ is being updated with π is taken as 1 and 0 respectively. To determine π, two network loadings are required.

Step 3: Calculating costate variables

3.1 Compute $\lambda_a(k) = \lambda_a(k+1) + \left(1 + f'[\tau_a(t)]\right)\dfrac{e_a(t)}{Q_a}\Delta s$;

3.2 calculate $\lambda_a[\tau_a(k)]$ from $\lambda_a(k)$ and $\tau_a(k)$.

Step 4: Convergence verification

4.1. Check if $\left|C_a(s) - C_{od}^*\right| \le \varepsilon$ or n^i is greater than the predefined maximum number of inner iterations, then go to step 3.2; otherwise, set $n^i := n^i + 1$ and go to step 1.1.

4.2. if $k = K$, then go to step 3.3; otherwise k:= k + 1 and go to step 1.1;

4.3. if $a = A$, then go to step 3.4; otherwise a:= a + 1 and go to step 0.5;

4.4 define $\xi = \dfrac{\sum_{k\in K}\sum_{a\in A} e_a(k)\left|C_a(k+1) - C_{od}^*\right|}{\sum_{k\in K}\sum_{a\in A} e_a(k) C_{od}^*}$ as the measure of disequilibrium, which is equal to zero at equilibrium. If n is greater than the predefined maximum number of overall iterations or ξ is sufficiently small, i.e. $\xi \le \varepsilon$ where ε is an arbitrarily small number, then go to step 3.2; otherwise set n:=n+1 and go to step 1.2;

4.5. check if the total throughput $E_{od} = \sum_{\forall a}\sum_{\forall k} e_a(k)$ from the system is equal to the predefined total demand J_{od} for the *o-d* pair. If yes, then *terminate* the algorithm; otherwise update $C^* := C^* + \left[\dfrac{J_{od} - E_{od}}{\frac{dE_{od}}{dC^*}}\right]$, and go back to step 0.3. The derivative $\dfrac{dE_{od}}{dC^*}$ is derived by Heydecker (2002) as

$$\frac{\partial E_{od}}{\partial C^*} = \sum_{a\in A}\left(\frac{\left[h'(s_a^0) + f'[\tau_a(s_a^0)]\right] - \left[h'(s_a^1) + f'[\tau_a(s_a^1)]\right]}{\left[h'(s_a^0) + f'[\tau_a(s_a^0)]\right]\left[h'(s_a^1) + f'[\tau_a(s_a^1)]\right]}\right) Q_a .$$

5 Numerical calculations

To illustrate the analyses above, we demonstrate some numerical calculations. We consider a single link, which has a free flow time 3 mins and a capacity 20 vehs/min, connecting a single origin-destination pair. The origin-specific cost is considered to be a monotone linear function of time with a slope -0.4. The destination cost function is piecewise linear, which has no penalty for arrivals before the preferred arrival time $t^* = 50$, and increases with a rate 2 afterwards. The size of discretized time interval Δs is taken as 1 min. The length of the planning horizon $[0, T]$, where T=100, is set such that that all traffic can be cleared by time T. The total amount of traffic J_{od} is taken as 390 vehs. Figure 1 plots the inflow, the outflow, and the total cost at dynamic user equilibrium. The traffic is assigned to the link during times 18 and 49.

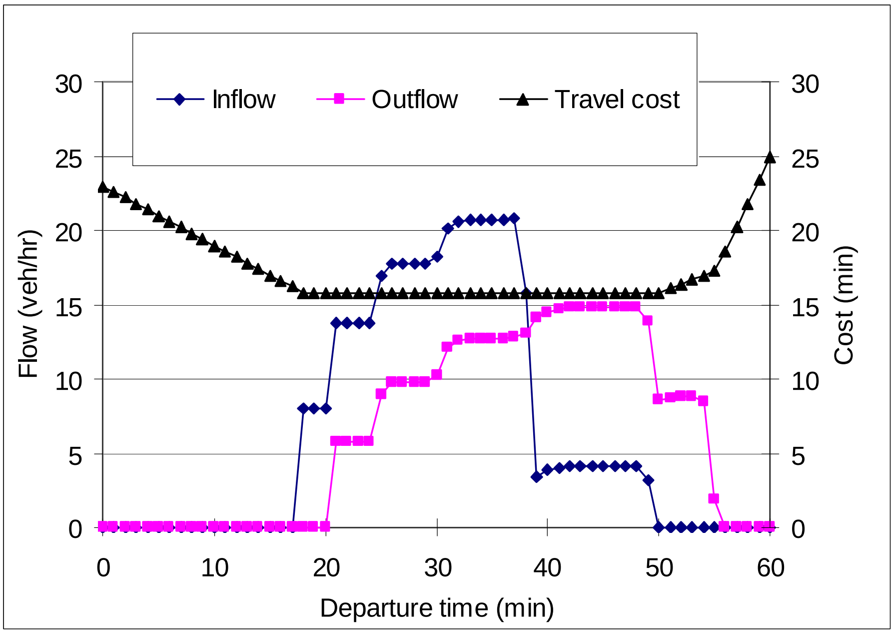

Figure 1 Dynamic user equilibrium assignment

To investigate the accuracy of the sensitivity analysis in Section 3, we suppose the inflow is perturbed at time 18, and plot the associated variations in travel time in Figure 2. The variations are calculated according to (16). In the same figure, we also plot the variations determined by using numerical finite difference method. To calculate the finite difference, we first increase the inflow at time 18 by one unit, and keep the values of other inflows at other times unchanged. The variations in travel times are then calculated by repeated link loading with the original inflow profile versus the perturbed inflow profile. The result shows that the analytical variations given by (16) can represent the true variations in travel time reasonably well. It can be observed that the variations take the value of $1/Q_a$ during time s and $\tau_a(s)$ (i.e during times 19 and 21), and then depend on the profile of outflow and previous variations after $\tau_a(s)$.

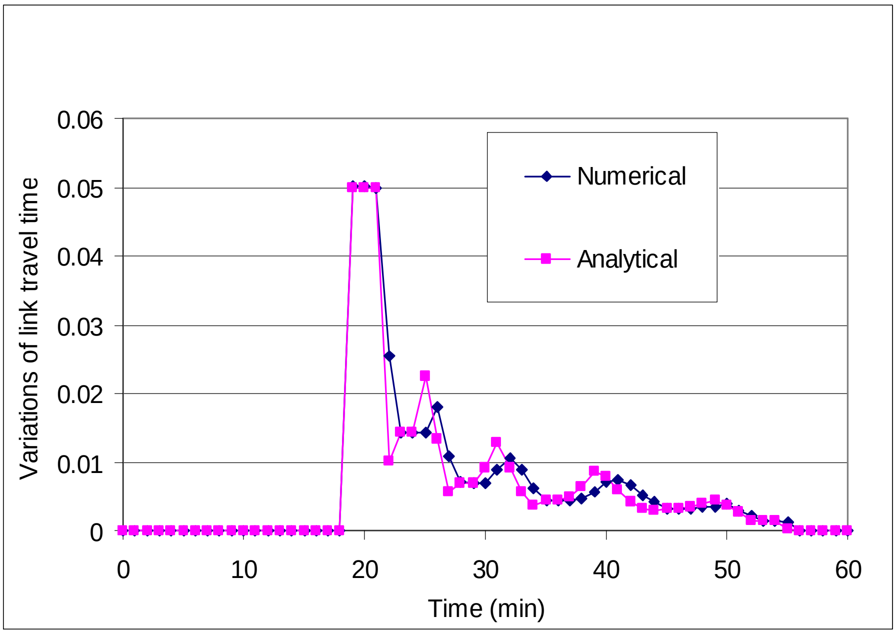

Figure 2 Sensitivity of travel time with respect to a perturbation in inflow

Using the derivative of travel time with respect to inflow, the externality, i.e.

$$\int_0^T \frac{\partial C_a}{\partial u}\bigg|_s e_a(t)dt = \int_0^T (1+f'[\tau_a(t)])\frac{\partial \tau_a}{\partial u}\bigg|_s e_a(t)dt\,,$$

induced by adding an additional inflow at all times can then be calculated accordingly. Figure 3 shows the profile of the externality for inflow under dynamic user equilibrium.

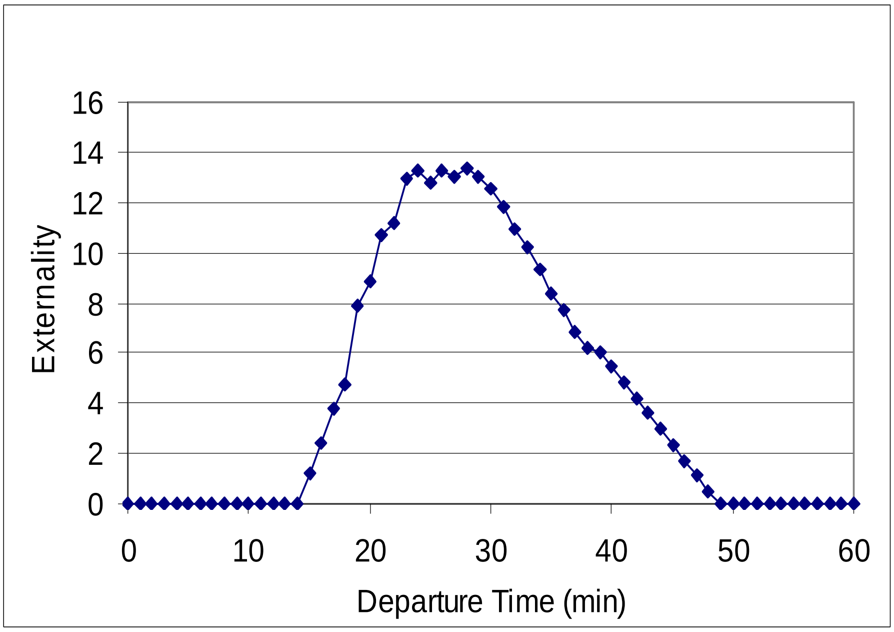

Figure 3 Externality at dynamic user equilibrium

Calculating the dynamic system optimal assignment is still in progress. Figure 4 shows the inflow, the outflow, and the travel cost after one iteration of optimization from the dynamic user equilibrium. With the same total throughput J_{od}, the period of assignment shifts from times [18, 49] to times [9, 50]. It is also observed that this assignment, on the one hand, encourages late departures. On the other hand, it also has to maintain a certain amount of early departures to induce a high service rate for the departures at later times. The total system travel cost is decreased from 6,143.45 veh-hr in user equilibrium to 5,777.60 veh-hr. This assignment profile is still subject to further revision.

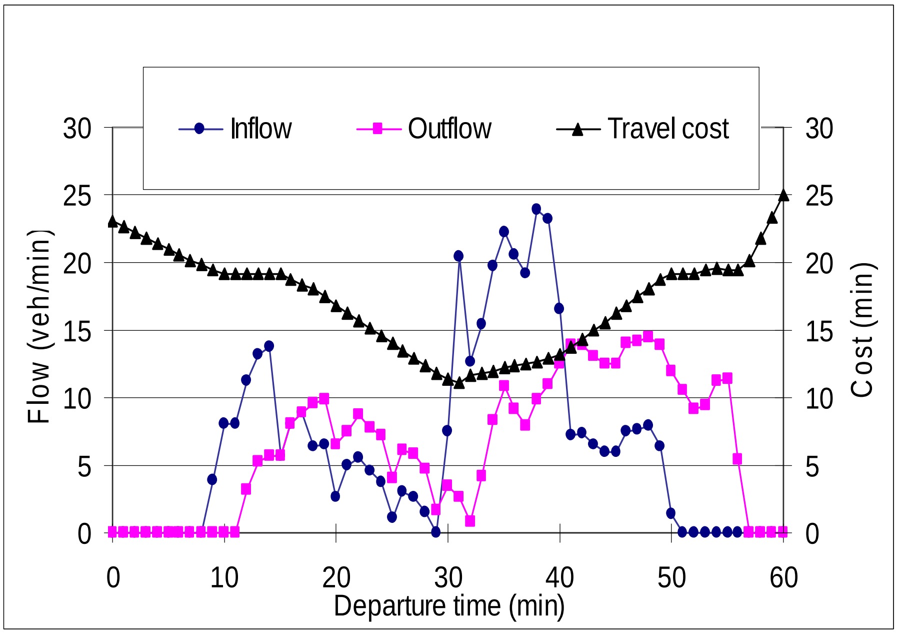

Figure 4 Solving dynamic system optimal assignment

6 Concluding remarks

This paper analyses the dynamic system optimizing flow along a single travel link. We propose a novel sensitivity analysis of travel time and travel cost with respect to perturbations in inflow. We also presented a solution method using the dynamic programming approach and applied it to the numerical example. The characteristics of the results were discussed.

The main contribution of this paper is the necessary conditions and the sensitivity analysis for dynamic system optimizing flow. The investigation also gives us a deeper understanding of the nature of system optimal assignment problems. In addition to analyzing and solving the system optimizing flow, we also note that each additional traveller, who enters the system at a certain time, imposes an additional travel cost on the others who enter the system at that time and thereafter. We regard this additional cost as "externality". Understanding the nature of the externality is important in managing dynamic network. However, previous research is implausible due to the underlying traffic model adopted (Carey and Srinvasan, 1993; Yang and Huang, 1997). This paper revisited the dynamic externality in a more plausible way. We also showed that how the externality can be derived and interpreted from the control theoretic formulation and the sensitivity analysis. This paper considered single-link networks in which only the departure time choices of travellers are considered. We are currently extending the present analysis and discussion to multi-route and multi-destination networks in which the joint choices of departure time and travel route of travellers are investigated.

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References

Astarita, A (1996) A continuous time link model for dynamic network loading based on travel time functions. J-B Lesort, ed. *Transportation and Traffic Theory*. Pergamon, Oxford, 79-102.

Carey M. and Srinivasan, A. (1993) Externalities, average and marginal costs, and tolls on congested networks with time-varying flows. *Operations Research* **41**, 217-231.

Chow, AHF (2006) Analysis of dynamic system optimum and externalities with departure time choice. *Paper to be presented at the 1st International Symposium on Dynamic Traffic Assignment* (full version), 21 – 23 June. Leeds, England.

Friesz, TL, Bernstein, D, Smith, TE, Tobin, RL and Wie BW (1993) A variational inequality formulation of the dynamic network user equilibrium problem. *Operations Research* **41**(1), 179-91.

Friesz, TL, Bernstein, D, Suo, Z and Tobin, RL (2001) Dynamic network user equilibrium with state-dependent time lags. *Network and Spatial Economics*, **1**(3/4), 319-347.

Heydecker, BG (2002) Dynamic equilibrium network design. MAP Taylor, ed. *Transportation and Traffic Theory*. Pergamon, Oxford, 349-70.

Heydecker, BG and Addison, JD (2005) Analysis of dynamic traffic equilibrium with departure time choice. *Transportation Science*, **39**(1), 39–57.

Mun, JS (2001) A divided linear travel time model for dynamic traffic assignment. *9th World Conference on Transport Research*, Seoul, S.Korea.

Yang, H. and Huang, H.J. (1997) Analysis of the time-varying pricing of a bottleneck with elastic demand using optimal control theory. *Transportation Research* **31B**, 425-440.