

# Strong Contagion with Weak Spillovers\*

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## Abstract

In this paper, we develop an explanation for why events in one market may trigger similar events in other markets, even though at first sight the markets appear to be only weakly related. We allow for escape dynamics in each market, and show that an escape in one market is contagious because it more than doubles the probability of a similar escape in another market. We claim that contagion is strong since escapes become highly synchronised across markets. Spillovers are weak because the instantaneous spillover of events from one market to another is small. To illustrate our result, we demonstrate how a currency crisis may be contagious with only weak links between countries. Other examples where weak spillovers would create strong contagion are various models of monetary policy, imperfect competition and endogenous growth.

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# 1 Introduction

On many occasions it is observed that developments in one market appear to follow those in another, despite the fact that markets seem to be only weakly related. One of the most obvious examples of this is the contagious nature of currency crises, since a crisis in one country is often followed by a crisis in another country, even though the two countries only have weak trade or financial linkages. Existing theories find such phenomena hard to explain and typically resort to the idea of correlated sunspots to explain the contagion of events from one market to another. However, the question then remains of why sunspots would be correlated across markets.

In this paper, we offer an explanation for why developments in separate markets may be synchronised even if there are no sunspots and spillovers between markets are weak. Our proposed explanation is based on the learning processes which determine the dynamics of each market. We characterise markets as having escape dynamics, with escapes occurring endogenously through learning as in Sargent (1999), and show that an escape in one market significantly increases the probability of an escape in the other market. The mechanism is not one in which agents observe an escape in the other market and this directly induces an escape in their own market, since we would interpret this as a strong spillover between markets. Instead, we restrict agents to only observe events in their own market, in which case the model is self-referential and weak spillovers are the only possible source of contagion. In other words, we demonstrate that weak spillovers create a channel by which an escape in one market is likely to trigger an escape in the other market.

Our preferred example to illustrate contagious escapes is the model of endogenous currency crises of Cho and Kasa (2005), which itself is derived by simplifying and adding learning to the third generation currency crisis model of Aghion, Bacchetta and Banerjee (2000). We analyse a model of two small open economies and one large economy, in which international spillovers between the two small economies are weak. Whilst our example is drawn from the currency crisis literature, our results are applicable to a more general class of self-referential models with escapes occurring through learning. This class includes models of monetary policy, imperfect competition, growth, and alternative models of currency crises, as discussed by Cho, Williams

and Sargent (2002), Bullard and Cho (2005), Primiceri (2005), Williams (2004) and Kasa (2004). Escapes have the potential to be contagious in all these models if there is another similar market and weak spillover of events from one market to another.

Consistent definitions of the terms *contagion* and *spillover* are yet to emerge in the literature. We take contagion to mean that an escape in one market leads to a significant increase in the correlation of events across similar markets. Our understanding of the term spillover follows Masson (1999), who writes that “Spillover effects explain why a crisis in one country may affect other emerging markets through linkages operating through trade, economy activity, or competitiveness”. We argue that even if such linkages are weak, an escape in one market is likely to trigger an escape in the other, resulting in an increase in the correlation between the two markets during the period of the escapes.

The remainder of the paper is organised as follows: Section 2 describes a version of the Cho and Kasa (2005) model of endogenous currency crises in which two small countries interact with a large country, and events in the first small country spill over to the second small country. In Section 3 we examine the behaviour of the first small country, which acts as the source of the spillover. Section 4 focuses on the second small country, the subject of the spillover, and shows how an escape in the first small country is likely to trigger an escape in the second small country. A final section concludes.

## 2 Model

Our model consists of the following ingredients: true structural relationships linking output in each small country to exchange rates; a description of each central bank’s perception of its own economy; a derivation of optimal exchange rate policy for each central bank; and a definition of equilibrium.

### 2.1 Structural relationships

The structure of the first small economy is represented by the open-economy expectations-augmented Phillips curve (1), in which output  $y_t$  is determined by its natural rate  $y_0$ , unex-

pected movements in the country's exchange rate relative to the large country,  $s_t - E_{t-1}s_t$ , and a Gaussian output disturbance  $v_{1t}$  with variance  $\sigma_1^2$ . Cho and Kasa (2005) derive equation (1) by assuming that firms set prices in advance, are credit constrained, and hold some unhedged debt denominated in the currency of the large country. Unexpected depreciations have a potentially ambiguous impact on output in the model. The *terms of trade* effect is positive as an unexpected depreciation temporarily improves the competitive position of firms in the first small economy relative to firms in the large economy. The *balance sheet* effect is negative because an unexpected depreciation increases the value of foreign-denominated debt held by firms in the first small country, tightening credit constraints and contracting output. We assume strong balance sheet effects dominate so  $\theta < 0$ .

$$y_t = y_0 + \theta(s_t - E_{t-1}s_t) + v_{1t} \quad (1)$$

$$y_t^* = y_0 + \theta(1 - \delta)(s_t^* - E_{t-1}^*s_t^*) - \theta\delta(s_t - s_t^*) + v_{1t}^* \quad (2)$$

Equation (2) is the analogous Phillips curve for the second small country, with variables and expectations identified by the \* superscript. To introduce a weak unilateral spillover from the first small country to the second, we assume that firms in the second small country hold a small but fixed quantity of their unhedged foreign debt in the currency of the first small country. The holdings are assumed to be fixed because they are too small to warrant continuous attention in debt portfolio management. As the quantity of debts denominated in the currency of the first small country is fixed, any movement (whether expected or unexpected) in the exchange rate of the first small country has balance sheet effects on firms in the second small country. Our spillover therefore acts through depreciations in the first small country's currency having a small expansionary effect on output in the second small country. In equation (2),  $\delta > 0$  is the (small) proportion of foreign-denominated debt held in the currency of the first country.  $s_t^*$  is the second small country's exchange rate against the large country, so  $s_t - s_t^*$  is the bilateral exchange rate between the two small countries.

Equations (3) and (4) state that the exchange rate in each small country is a function of the level set by the respective central bank,  $\hat{s}_t$  or  $\hat{s}_t^*$ , plus a Gaussian control error  $v_{2t}$  or  $v_{2t}^*$  with variance  $\sigma_2^2$ . We refer to  $\hat{s}_t$  and  $\hat{s}_t^*$  as intended exchange rates. Since private agents are assumed to have rational expectations, the expected exchange rates in equations (1) and

(2) will be equal to the intended exchange rates. Unexpected exchange rate movements are caused by the control errors  $v_{2t}$  and  $v_{2t}^*$ .

$$s_t = \hat{s}_t + v_{2t} \quad (3)$$

$$s_t^* = \hat{s}_t^* + v_{2t}^* \quad (4)$$

## 2.2 Central bank perceptions of the economy

Following Cho and Kasa (2005), we assume that each central bank does not know the true structure of its economy. Instead, they have approximating models which allow for the possibility that there might be a relationship between output and the exchange rate. The approximating models are subtly misspecified because they describe a relationship between output and the level of the exchange rate, when in reality it is only unexpected exchange rate movements that matter for output. Following the convention of Evans and Honkapohja (2001), we write the perceived law of motion (PLM) for each central bank as equations (5) and (6).  $\epsilon_t$  and  $\epsilon_t^*$  are approximation errors: the components of output movements that each central bank fails to explain by its model with a trade-off between output and the exchange rate.

$$y_t = \gamma_{0t} + \gamma_{1t}s_t + \epsilon_t \quad (5)$$

$$y_t^* = \gamma_{0t}^* + \gamma_{1t}^*s_t^* + \epsilon_t^* \quad (6)$$

The central banks are assumed to be unaware of the presence of spillovers. This introduces another subtle misspecification in the perceived law of motion (6) because the central bank in the second small country interprets output movements induced by spillovers as simple approximation errors, rather than due to events in the first small country. We justify this additional misspecification by appealing to the weak level of spillovers in the model. Since  $\delta$  is small in the Phillips curve (2), spillovers only account for a very small fraction of output fluctuations and it is difficult for the central banks to detect their presence.

The central banks estimate the coefficients of their perceived law of motion independently. To isolate the causal effects of spillovers, we restrict each central bank to only use data from their own country in estimation. This means that central banks are self-referential in nature and so precludes any contagion of escapes that occurs because a central bank in one country

observes an escape in the other country. Conditional on this data restriction, the central banks use discounted least squares techniques to estimate the coefficients of their perceived law of motion, as in Cho and Kasa (2005). Equations (7)-(8) are standard recursive formulae for discounted least squares estimation by the central bank in the first small country, with the matrix of regressors defined as  $X_t = (1 \ s_t)'$ . The current estimates of the coefficients are collected in the vector  $\gamma_t = (\gamma_{0t} \ \gamma_{1t})$ , with  $R_t$  a  $2 \times 2$  matrix measuring the precision of the estimates.

$$\gamma_{t+1} = \gamma_t + gR_t^{-1}X_t(y_t - \gamma_t X_t) \quad (7)$$

$$R_{t+1} = R_t + g(X_t X_t' - R_t) \quad (8)$$

In discounting past data, central banks allow for the possibility of structural breaks, even though such breaks are not explicitly present in our model. Under such circumstances, it is reasonable for the central bank to place more emphasis on recent data than data from the distant past. Discounting at the rate  $g$  gives a weight of  $(1 - g)^{n-1}$  to observations from  $n$  periods ago. Equations (9)-(10) are the corresponding estimation formulae for the central bank in the second small country, with  $X_t^* = (1 \ s_t^*)'$ .

$$\gamma_{t+1}^* = \gamma_t^* + gR_t^{*-1}X_t^*(y_t^* - \gamma_t^* X_t^*) \quad (9)$$

$$R_{t+1}^* = R_t^* + g(X_t^* X_t^{*'} - R_t^*) \quad (10)$$

### 2.3 Optimal exchange rate policy

The objective of each central bank, following Cho and Kasa (2005), is to minimise the extent to which its output and exchange rate deviate from target values  $\bar{y}$  and  $\bar{s}$ . The central bank loss functions (11) and (12) place equal quadratic penalties on output and exchange rate deviations from target. The exchange rate target is normalised to zero and we assume central banks target output above its natural rate so  $\bar{y} > y_0$ .

$$\mathcal{L}_t = (y_t - \bar{y})^2 + s_t^2 \quad (11)$$

$$\mathcal{L}_t^* = (y_t^* - \bar{y})^2 + s_t^{*2} \quad (12)$$

Optimal policy requires a central bank to set the intended exchange rate to minimise expected losses, subject to the perceived law of motion of the economy. As in Sargent (1999) and Cho and Kasa (2005), we assume that the central bank displays anticipated utility behaviour, following Kreps (1998). This implies that a central bank takes its best estimate of the coefficients in the perceived law of motion as being the true values, fixed now and into the infinite future.<sup>1</sup> The policy problem is static and the first order conditions for expected loss minimisation under anticipated utility behaviour give policy rules (13) and (14).

$$\hat{s}_t = -\frac{\gamma_{1t}(\gamma_{0t} - \bar{y})}{1 + \gamma_{1t}^2} \quad (13)$$

$$\hat{s}_t^* = -\frac{\gamma_{1t}^*(\gamma_{0t}^* - \bar{y})}{1 + \gamma_{1t}^{*2}} \quad (14)$$

## 2.4 Equilibrium

The dynamics of the model are determined by the structure of the economies (1)-(4), optimal exchange rate policies (13)-(14), and the recursive schemes (7)-(10) by which central banks update their estimates of the coefficients in their perceived models of the economy. To characterise equilibrium, we follow Cho, Williams and Sargent (2002) and use stochastic approximation techniques to analyse the continuous time analogue of the model. Full details appear in Appendix A. Applying these techniques, we identify a unique stable equilibrium at beliefs given by equation (15).

$$\bar{\gamma} = \bar{\gamma}^* = \begin{pmatrix} y_0 + \theta^2(y_0 - \bar{y}) \\ \theta \end{pmatrix} \quad (15)$$

The unique stable equilibrium has a clear intuitive economic interpretation. It has beliefs  $(\gamma_0, \gamma_1) = (y_0 + \theta^2(y_0 - \bar{y}), \theta)$ , with the central banks setting intended exchange rates  $\hat{s} = \hat{s}^* = -\theta(y_0 - \bar{y})$  and output at its natural rate  $y = y^* = y_0$ . We term this the *Nash outcome*, since it is the same as the equilibrium that would prevail under discretionary policy if the central banks in the small countries know the true structure of their economies (1)-(4). The equilibrium in our model has the same features as the self-confirming equilibrium (SCE) in

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<sup>1</sup>Tetlow and von zur Muehlen (2004) show that escapes are still possible when a central bank treats its coefficient estimates as uncertain.

Cho, Williams and Sargent (2002). The central bank believes that a strong exchange rate is effective in boosting output ( $\gamma_1 = \theta$ ), and believes output would be low if the exchange rate was to depreciate ( $\gamma_0 < y_0$ ). Taken together, these beliefs delude the central bank into continuing with its strong exchange rate policy in equilibrium.

### 3 Source of the spillover

The behaviour of the first small country (as the source of the spillover) is a natural starting point for analysis. Since the spillover in our model is unilateral, the first small country can be analysed independently of the second small country. Even so, it is difficult to obtain further analytical results. To proceed, we therefore parameterise the model and analyse its behaviour by a combination of simulation and numerical techniques. Our parameterisation in Table 1 is based on the parameter values chosen by Cho, Williams and Sargent (2002) for a closed economy.

Parameter	Value
$\bar{y}$	5
$y_0$	0
$\theta$	-1
$\delta$	0.1
$\sigma_{v1}^2$	0.3
$\sigma_{v2}^2$	0.3
$g$	0.025

Table 1: Parameter values

In words, our parameterisation implies that the central bank targets output at a level 5% above a zero natural rate; surprise devaluations translate one for one into output; firms in the second small country hold 10% of their foreign-denominated debt in the currency of the first small country; the variance of both shocks is 0.3; and policymakers place a weight of  $0.975^{n-1}$  on data from  $n$  periods ago.



### 3.1 Simulation results

The first set of results we report are for a dynamic simulation of the first small country for 1600 periods. The top two panels of Figure 1 show the behaviour of the exchange rate and output respectively. According to the top left panel, the exchange rate has a tendency to appreciate towards the equilibrium Nash outcome level  $s_t = -5$ , but occasionally depreciates rapidly to a level close to  $s_t = 0$ . In the top right panel, output has no clear trend and fluctuates around its zero natural rate. The bottom two panels of Figure 1 plot the evolution of central bank beliefs about the economy. They illustrate that large depreciations in the exchange rate are associated with a rapid realignment of beliefs  $(\gamma_{0t}, \gamma_{1t})$  from close to  $(-5, -1)$  to close to  $(0, 0)$ .

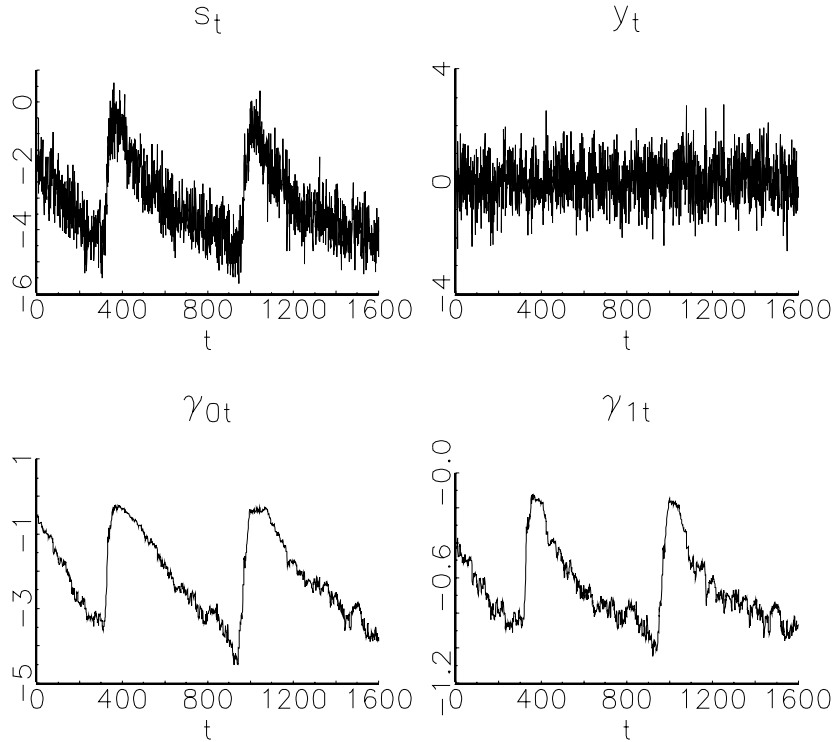


Figure 1: Simulation of the model with no spillovers

The exchange rate in the first small country exhibits escape dynamics. The attractor for escapes appears to be the belief pair  $(\gamma_0, \gamma_1) = (0, 0)$ , at which point the exchange rate and output are  $s_t = 0$  and  $y_t = 0$ . We denote this the *Ramsey outcome*, as it is equivalent to the equilibrium that arises under commitment policy if the central bank knows the structures of

the economy (1)-(2). An insight into the circumstances that trigger escapes can be obtained by looking at the shocks hitting the economy around the time of large exchange rate devaluations. Figure 2 summarises the distribution of output shocks  $v_1$  and exchange rate control errors  $v_2$  in the periods immediately preceding and during an exchange rate devaluation. In each panel, the most likely combinations of  $v_1$  and  $v_2$  are marked with a dot and surrounded by a one standard deviation confidence region. The top left panel is for shocks immediately preceding a devaluation; in the top right hand panel the devaluation has already begun. The bottom two panels are for shocks occurring as the devaluation progresses.

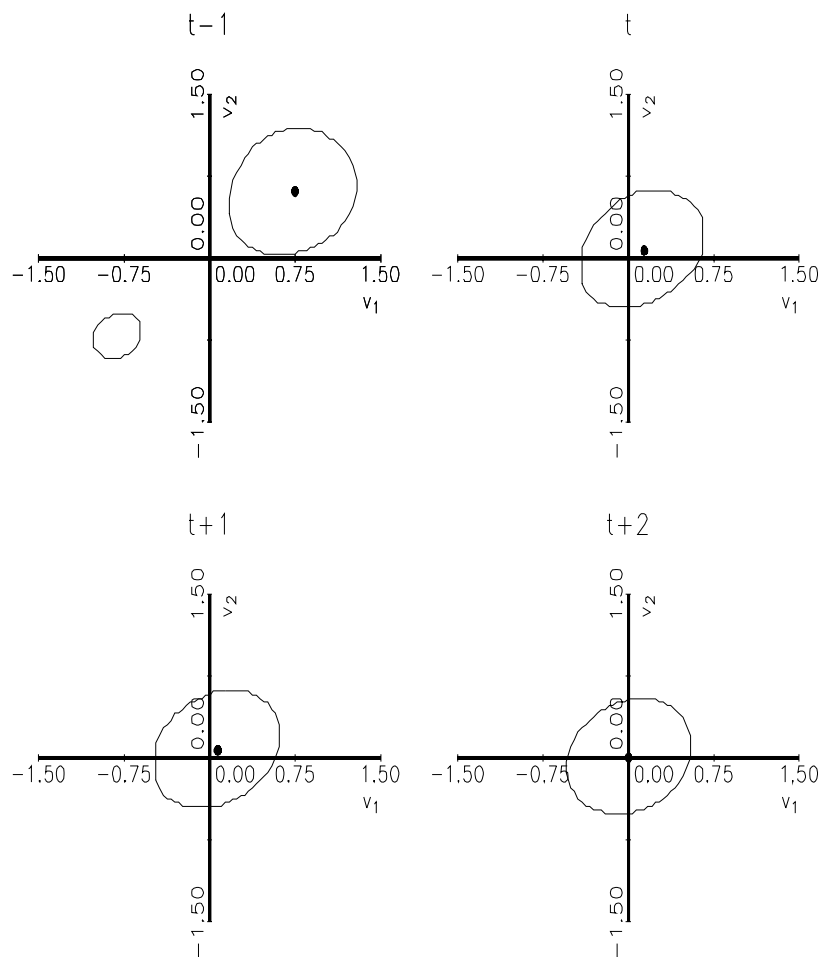


Figure 2: Distributions of shocks at the time of large devaluations

The panels in Figure 2 show considerable regularities in the distribution of shocks around the time of a large devaluation. If the shocks  $v_1$  and  $v_2$  were completely random then the most

likely combination in each period would be  $(0, 0)$  and the confidence region would be a perfect circle cutting the axes at  $\pm 0.548$ , one standard deviation of each shock. Instead, the shocks appear to be both positive and positively correlated in each period. This is reflected in the most likely combination of shocks lying in the positive-positive quadrant and the confidence regions being skewed towards the positive-positive and negative-negative quadrants. The pattern is particularly clear in the period immediately preceding a devaluation. The salient features of the shocks are summarised in Proposition 1.

**Proposition 1** *Large devaluations in the exchange rate tend to be preceded by output shocks and exchange rate control errors that are (i) positive and (ii) positively correlated.*

The basic intuition for why shocks preceding an escape have these properties can be understood by noting that an escape is synonymous with a realignment of beliefs from their Nash to Ramsey levels. The fact that shocks are (i) positive promotes the escape in  $\gamma_0$ . Positive shocks create unexpectedly high output ( $y_t - \gamma_t X_t > 0$ ), prompting the central bank to revise upwards its belief about what would happen to output after a devaluation. Beliefs escape from the Nash level  $\gamma_0 = -5$  to the Ramsey level  $\gamma_0 = 0$ . Property (ii) that shocks tend to be positively correlated promotes the escape in  $\gamma_1$ . Under normal circumstances, control errors  $v_2$  lead to unexpected exchange rate changes and observable movements in output, which reinforce the equilibrium Nash outcome. However, if  $v_2$  is positively correlated with the output shock  $v_1$  then the movement induced in output will be offset by an output shock, and output does not appear to react to the exchange rate change. In such circumstances, the central bank starts to discount the possibility of a relationship between output and the exchange rate. Beliefs escape from the Nash level  $\gamma_1 = -1$  to the Ramsey level  $\gamma_1 = 0$ .

### 3.2 Dominant escape path

The second result we report is a calculation of the dominant escape path in the model. The dominant escape path is a useful indicator of the model's dynamics as it describes the most likely path the country will follow in an escape episode. Calculation of the dominant escape path involves solving an optimal control problem to find the most likely way the country will escape from the stable equilibrium in the continuous-time analogue of our model. The seminal contribution of Williams (2004) describes the technicalities.<sup>2</sup> Figure 3 shows the dominant escape path of the exchange rate in the first small country of our model. Full details of our calculations are presented in Appendix B.

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<sup>2</sup>Williams (2004) applies a result from Worms (1999) to the general analysis of Dupuis and Kushner (1989) to derive a simple deterministic control problem whose solution characterises escape dynamics in linear-quadratic Gaussian models such as ours.

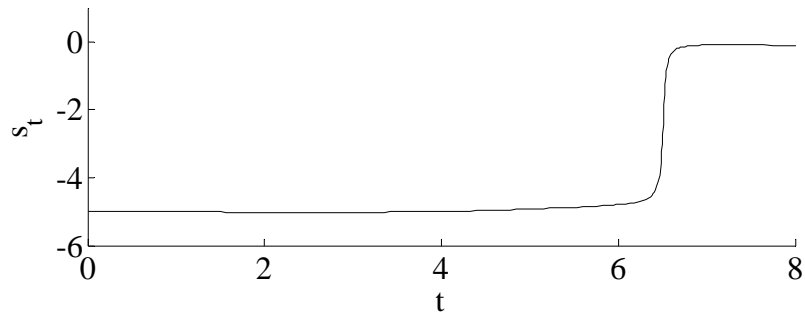


Figure 3: Dominant escape path in the first small country

The dominant escape path confirms the Ramsey outcome as an attractor for escape dynamics. Figure 3 shows that the most likely escape path takes the exchange rate from its initial Nash level  $s_t = -5$  to close to its Ramsey level  $s_t = 0$ . The beliefs behind the exchange rate in Figure 3 also escape to the neighbourhood of levels  $(\gamma_0, \gamma_1) = (0, 0)$ , consistent with the Ramsey outcome.

## 4 Subject of spillover

The second small country is the subject of the spillover. To bring out the implications of spillovers, we compare its behaviour when there are spillovers ( $\delta = 0.1$ ) to when spillovers are absent ( $\delta = 0$ ). We apply the same simulation and numerical techniques to analyse the second small country as we did in the previous section for the first small country.

### 4.1 Simulation results

Our first results are based on comparing long simulations of the model with and without spillovers. Figure 4 shows one such comparison.

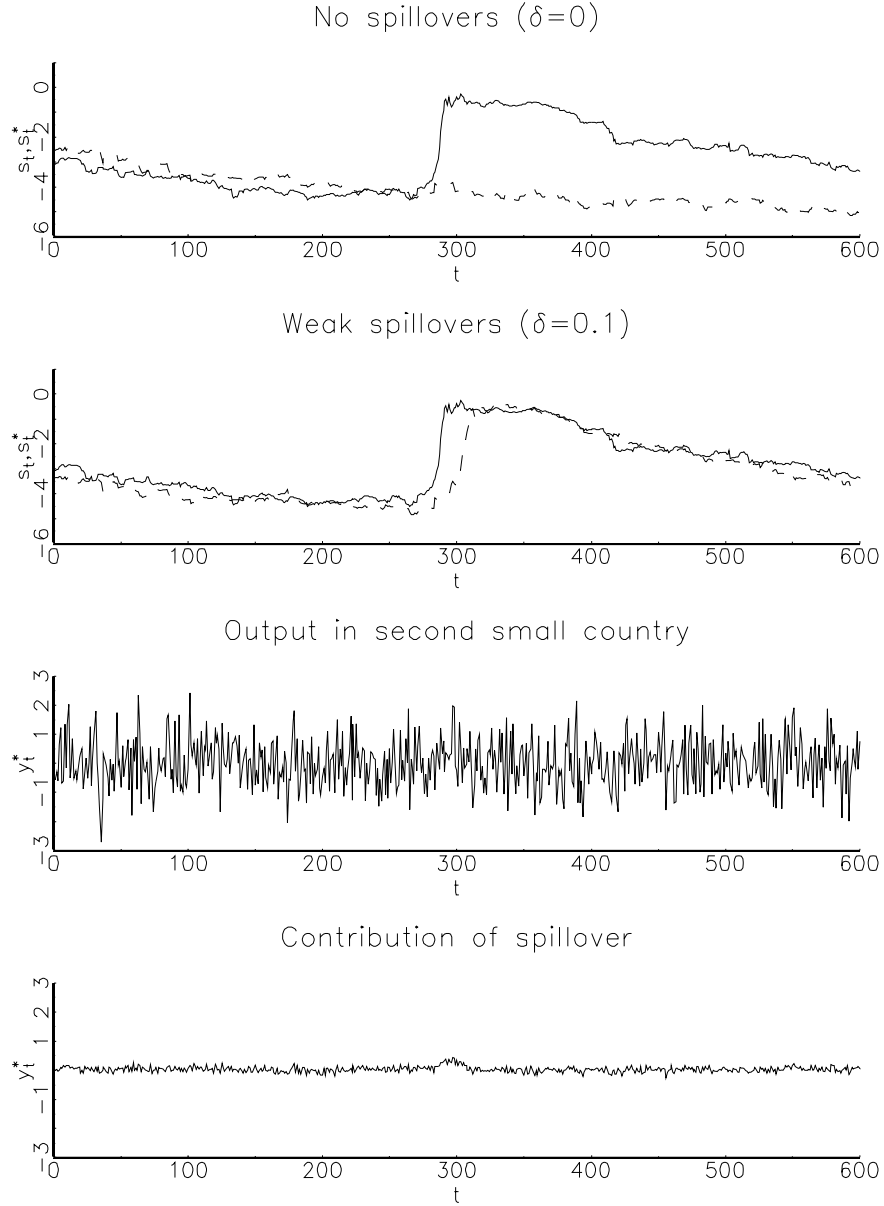


Figure 4: Simulation of the model with and without spillovers

The first panel of Figure 4 shows the behaviour of the intended exchange rate in the no spillovers case ( $\delta = 0$ ): the solid line is the intended exchange rate  $\hat{s}_t$  for the first small country; the dashed line is the intended exchange rate  $\hat{s}_t^*$  for the second small country. Since there are no spillovers, the intended exchange rates are independent and there is no interaction between the small countries. Even though  $\hat{s}_t$  escapes around period  $t = 290$ , there is no effect on  $\hat{s}_t^*$ . In sufficiently long simulations, the correlation between  $\hat{s}_t$  and  $\hat{s}_t^*$  is zero.

The second panel of Figure 4 shows simulated paths for intended exchange rates when there is a weak spillover ( $\delta = 0.1$ ) from the first small country to the second. As the spillover is unilateral, it has no effect on the first small country and the solid line in the second panel is the same as in the upper panel for the no spillovers case. In contrast, the dashed line for the intended exchange rate in the second small country is very different. Rather than the gradual appreciation seen with no spillovers,  $\hat{s}_t^*$  escapes soon after the escape in  $\hat{s}_t$ . The large devaluation in the first small country appears to have triggered a similar large devaluation in the second small country. We interpret this as evidence that escapes are strongly contagious. The contagion leads to a positive correlation between the intended exchange rates in the two small countries. In long simulations the correlation coefficient is approximately 0.15.

To gauge the weakness of spillovers, the third panel of Figure 4 plots the simulated path of output in the second small country. There are three factors in the model which can cause output to deviate from its natural rate: output shocks  $v_{1t}^*$ , exchange rate control errors  $v_{2t}^*$ , and spillovers from the first small country  $\delta(s_t - s_t^*)$ . The fourth panel shows the deviation in output that is due to the third factor. The role of spillovers is apparently small, with deviations due to spillovers barely discernible compared to the much larger deviations caused by output shocks and unexpected exchange rate movements. In variance decomposition terms, only 5% of the variance in output in the second small country is attributable to spillovers from the first small country. It is in this sense that we claim to have only weak spillovers in the model.

An intuitive understanding of why weak spillovers are sufficient to cause strong contagion can be obtained by recalling the pattern of shocks that typically precedes an escape. According to Proposition 1, escapes tend to be triggered by a series of output shocks and exchange rate control errors that are (i) positive and (ii) positively correlated. Weak spillovers cause strong contagion because an escape in the first small country spills over into unexpectedly high output in the second small country. From the second small country's point of view, it is as if there is a run of positive  $v_{1t}^*$  output shocks. Condition (i) of Proposition 1 is more likely to be satisfied and the probability of an escape in the second small country increases in the aftermath of an

escape in the first small country.<sup>3</sup>

The strong contagion is confirmed by Table 2, which reports how the presence of weak spillovers increases the probability of an escape in the second small country occurring within a given number of periods of an escape in the first small country. The increase in probability is dependent on the level of the exchange rate in the second small country at the time the first small country escapes, ranging from no change when  $\hat{s}_t^* > -2$  to almost doubling when  $\hat{s}_t^* < -4$ . The dependency arises because conditions have to be ripe for an escape in the first small country to trigger an escape in the second small country. An exchange rate in the second small country close to the Nash level  $\hat{s}_t^* = -5$  puts the country in the “danger zone” and makes it more susceptible to escapes and contagion. Conversely, if the exchange rate in the second small country is already close to its Ramsey level  $\hat{s}_t^* = 0$  then escapes and contagion are highly unlikely.

Probability of escape in second small country within $k$ periods of escape in first small country				
	$k = 5$	$k = 10$	$k = 20$	$k = 50$
<i>No spillovers</i> ( $\delta = 0$ )				
$s_t^* > -2$	0.000	0.000	0.000	0.000
$-2 > s_t^* > -4$	0.008	0.011	0.013	0.017
$-4 > s_t^*$	0.010	0.021	0.045	0.121
<i>Weak spillovers</i> ( $\delta = 0.1$ )				
$s_t^* > -2$	0.000	0.000	0.000	0.000
$-2 > s_t^* > -4$	0.006	0.006	0.010	0.010
$-4 > s_t^*$	0.014	0.033	0.101	0.226

Table 2: Probability of escapes in second small country

<sup>3</sup>A similar mechanism operates in the model of McGough (2005), where a permanent but unobservable increase in the natural rate of output is perceived as a series of positive output shocks, thereby increasing the probability of an escape.



Further evidence that contagion is strong is provided in Table 3, which gives summary statistics for the relationship between the exchange rates of the two small countries in the frequency domain. Coherence measures the correlation between the exchange rates at a given frequency, whereas group delay can be interpreted as the extent to which the exchange rate in the first small country lags or leads that in the second small country at a given frequency.<sup>4</sup> If there are no spillovers then coherence and group delay are zero by definition. According to Table 3, weak spillovers create significant coherence between exchange rates at very low frequencies, with the group delay statistic indicating that the first small country leads the second small country by just short of 20 periods. There is very little coherence at higher frequencies. The results are consistent with contagion acting through escapes in the first small country triggering escapes in the second: coherence at low frequency matches the long period between escapes in Figure 1; a group delay of about 20 periods confirms the lead of the first country over the second seen in the simulation of Figure 4.

Frequency	Equivalent number of periods in cycle	Coherence	Group delay
0.005	200	0.316	18.5
0.010	100	0.278	18.3
0.020	50	0.191	17.4
0.040	25	0.092	14.9
0.080	12.5	0.038	10.5

Table 3: Frequency domain properties of exchange rates with weak spillovers

Our contention that spillovers are weak is based on the observation that they only make a small contribution to the variance of output in the second small country. The robustness of this result is examined in Figure 5, which shows how the share of output variance attributed to spillovers depends on the strength of the spillover  $\delta$ . As a comparison, we also show the

<sup>4</sup>See Hannan and Thomson (1971) for more details on the interpretation of coherence and group delay.

degree of contagion for each  $\delta$  by plotting the corresponding coherence and group delay between exchange rates at low frequency (cycles of period 200). At low levels of  $\delta$ , the share of variance is low and spillovers are weak. In contrast, coherence is higher and contagion is strong with a short group delay. Figure 5 provides the evidence for the central claim of our paper: weak spillovers are sufficient to cause strong contagion.

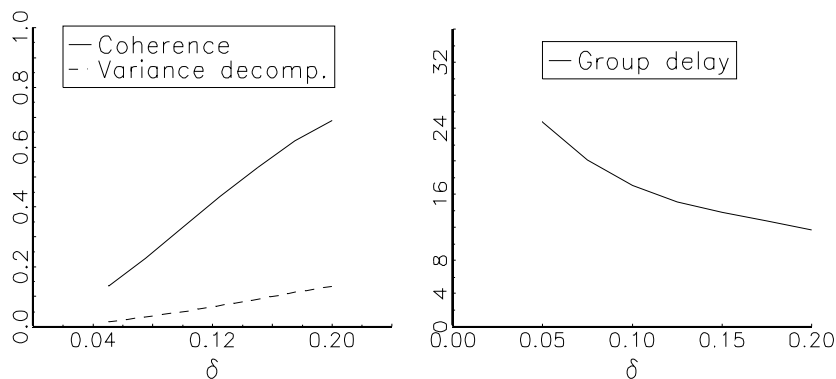


Figure 5: Coherence, variance decomposition and group delay

## 4.2 Dominant escape path

To complement the simulation results, we analyse how the dominant escape path in the second small country is affected by the presence of spillovers. The question we pose is a conditional one: what is the most likely escape path for the second small country if the first small country is following its own most likely escape path? Answering the question involves calculating the dominant escape path of the second small country *conditional* on spillover of the dominant escape path from the first small country. Technically, the dominant escape path of the first small country (which itself is the outcome of an optimal control problem) enters as an exogenous constraint in the optimal control problem defining the dominant escape path for the second small country. Appendix C gives full details.

The results of our calculations are presented in Figure 6. To highlight contagion, we start the second small country at its stable equilibrium and the first small country at some point on its dominant escape path. The top panel shows the dominant escape path in the first small

country. The path is identical to that in Figure 3, except we engineer an early escape by initialising beliefs at their values on the dominant escape path at time  $t = 6$ . The dominant escape path for the first small country is therefore a left-truncated version of the dominant escape path in Figure 3. The bottom panel of Figure 6 shows the dominant escape paths in the second small country with and without spillovers. Without spillovers ( $\delta = 0$ ), the dominant escape path mirrors the results from Section 3.2 and is the same as in Figure 3. The fact that the first small country has an early escape has no implications for the dominant escape path of the second small country. With spillovers ( $\delta = 0.1$ ), the early escape in the first small country does matter for the second small country. In the bottom panel, the presence of spillovers brings forward the escape in the second small country. It is in this sense we claim that the early escape in the first small country triggers an escape in the second small country.

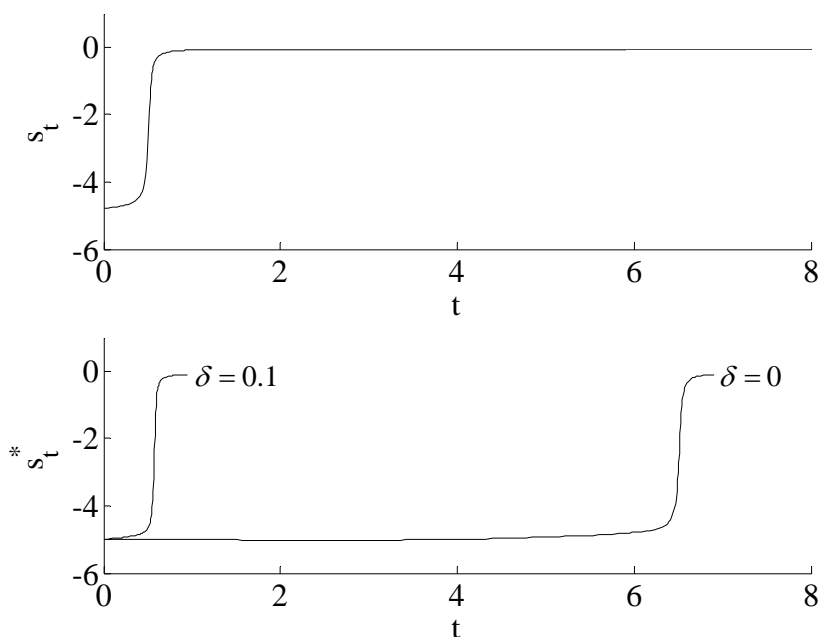


Figure 6: Dominant escape paths with ( $\delta = 0.1$ ) and without ( $\delta = 0$ ) spillovers

The dependency of the dominant escape path of the second small country on the dominant escape path of the first small country reinforces our conclusion that escapes are strongly contagious in the model. In terms of currency crises, a crisis in the first small country makes it most likely that a crisis in the second small country will happen sooner rather than later.

## 5 Conclusions

The central claim of this paper is that developments in one market may have a profound effect on other markets, even though at first sight the other markets appear to be only weakly related. To obtain our result, we constructed a simple model of two markets with weak spillovers from one market to another. Following Sargent (1999), individual markets were characterised by learning dynamics that occasionally caused them to escape from equilibrium. Our analysis showed that weak spillovers are sufficient to make an escape in one market significantly increase the probability of a similar escape in the other market. We therefore concluded that escapes are strongly contagious in the model. Contagion occurs because an escape in one market spills over and creates conditions that are conducive to an escape in the other market.

The claim that weak spillovers create strong contagion applies to a class of models with weakly related markets and escape dynamics. In our preferred example, we extended the model of Cho and Kasa (2005) to show that currency crises may be contagious with only weak financial links between countries. In the model, escapes were equated to currency crises and spillovers were weak because they only accounted for a small proportion of output fluctuations in the second small country. Our simulations indicated that a currency crisis in one small country almost doubles the probability of a currency crisis in another small country. Currency crises are therefore strongly contagious, with a crisis in one country highly likely to trigger crises in other countries.

Other situations in which we expect to observe strong contagion with weak spillovers are suggested by the models of Cho, Williams and Sargent (2002) and Williams (2004). In monetary policy, a rapid disinflation in one country may trigger a rapid disinflation in another country. With imperfect competition, an outbreak of collusion in one market makes collusion more likely in another market, even when the cross-price elasticity between the two markets is low. For endogenous growth models, a growth spurt in one country may create growth spurts in other countries.

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## A Model in continuous time

The dynamics of the model in discrete time are completely described by the structure of the economies (1)-(4), optimal exchange rate policies (13)-(14), and the recursive schemes (7)-(10) by which the central banks update their estimates of the coefficients in their perceived models of the economy. The starting point for derivation of the continuous time analogue of our model is to re-write equations (7)-(10) as (A.1)-(A.4).

$$\frac{\gamma_{t+1} - \gamma_t}{g} = R_t^{-1} X_t (y_t - \gamma_t X_t) \quad (\text{A.1})$$

$$\frac{\gamma_{t+1}^* - \gamma_t^*}{g} = R_t^{*-1} X_t^* (y_t^* - \gamma_t^* X_t^*) \quad (\text{A.2})$$

$$\frac{R_{t+1} - R_t}{g} = X_t X_t' - R_t \quad (\text{A.3})$$

$$\frac{R_{t+1}^* - R_t^*}{g} = X_t^* X_t^{*'} - R_t^* \quad (\text{A.4})$$

Equations (A.1)-(A.4) can be interpreted as a discrete-time approximation of an underlying continuous time process perturbed by shocks. Taking the limit as  $g \rightarrow 0$ , the approximation error tends to zero and a weak law of large numbers ensures that the stochastic elements become negligible. In the limit, the dynamics of the model are therefore represented by a deterministic system of ordinary differential equations (A.5)-(A.12). A proof of this is

presented in Cho, Williams and Sargent (2002).

$$\dot{\gamma} = R^{-1}\bar{g}(\gamma) \quad (\text{A.5})$$

$$\dot{\gamma}^* = R^{*-1}\bar{g}^*(\gamma^*, \gamma) \quad (\text{A.6})$$

$$\dot{R} = \bar{M}(\gamma) - R \quad (\text{A.7})$$

$$\dot{R}^* = \bar{M}^*(\gamma^*) - R^* \quad (\text{A.8})$$

$$\bar{g}(\gamma) = \begin{pmatrix} y_0 - \gamma_0 - \gamma_1 \hat{s} \\ (y_0 - \gamma_0 - \gamma_1 \hat{s}) \hat{s} + (\theta - \gamma_1) \sigma_2^2 \end{pmatrix} \quad (\text{A.9})$$

$$\bar{g}^*(\gamma^*, \gamma) = \begin{pmatrix} y_0 - \theta \delta(\hat{s} - \hat{s}^*) - \gamma_0^* - \gamma_1^* \hat{s}^* \\ (y_0 - \theta \delta(\hat{s} - \hat{s}^*) - \gamma_0^* - \gamma_1^* \hat{s}^*) \hat{s}^* + (\theta - \gamma_1^*) \sigma_2^2 \end{pmatrix} \quad (\text{A.10})$$

$$\bar{M}(\gamma) = \begin{pmatrix} 1 & \hat{s} \\ \hat{s} & \hat{s}^2 + \sigma_2^2 \end{pmatrix} \quad (\text{A.11})$$

$$\bar{M}^*(\gamma^*) = \begin{pmatrix} 1 & \hat{s}^* \\ \hat{s}^* & \hat{s}^{*2} + \sigma_2^2 \end{pmatrix} \quad (\text{A.12})$$

## A.1 Equilibrium

An equilibrium of the model is a fixed point of the ordinary differential equations (A.5)-(A.8). Imposing  $\dot{\gamma} = \dot{\gamma}^* = \dot{R} = \dot{R}^* = 0$  to identify fixed points, we obtain the unique equilibrium defined by equations (A.13) and (A.14). Equation (A.13) is equation (15) in the paper.

$$\bar{\gamma} = \bar{\gamma}^* = \begin{pmatrix} y_0 + \theta^2(y_0 - \bar{y}) \\ \theta \end{pmatrix} \quad (\text{A.13})$$

$$R = \bar{R}^* = \begin{pmatrix} 1 & -\theta(y_0 - \bar{y}) \\ -\theta(y_0 - \bar{y}) & \theta^2(y_0 - \bar{y})^2 + \sigma_2^2 \end{pmatrix} \quad (\text{A.14})$$

## A.2 Stability of equilibrium

A sufficient condition for local asymptotic stability is that all the eigenvalues of the Jacobian of equations (A.5)-(A.8) have negative real parts when evaluated at equilibrium.<sup>5</sup> The Jacobian

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<sup>5</sup>Proposition 5.6, Evans and Honkapohja (2001), p. 96.



is defined by (A.15).

$$J = \begin{pmatrix} \frac{\partial \dot{\gamma}}{\partial \gamma} & \frac{\partial \dot{\gamma}}{\partial \gamma^*} & \frac{\partial \dot{\gamma}}{\partial R} & \frac{\partial \dot{\gamma}}{\partial R^*} \\ \frac{\partial \dot{\gamma}^*}{\partial \gamma} & \frac{\partial \dot{\gamma}^*}{\partial \gamma^*} & \frac{\partial \dot{\gamma}^*}{\partial R} & \frac{\partial \dot{\gamma}^*}{\partial R^*} \\ \frac{\partial \dot{R}}{\partial \gamma} & \frac{\partial \dot{R}}{\partial \gamma^*} & \frac{\partial \dot{R}}{\partial R} & \frac{\partial \dot{R}}{\partial R^*} \\ \frac{\partial \dot{R}^*}{\partial \gamma} & \frac{\partial \dot{R}^*}{\partial \gamma^*} & \frac{\partial \dot{R}^*}{\partial R} & \frac{\partial \dot{R}^*}{\partial R^*} \end{pmatrix} \quad (\text{A.15})$$

When evaluated at equilibrium, the Jacobian is (A.16).

$$J|_{SCE} = \begin{pmatrix} \frac{\partial R^{-1} \bar{g}(\gamma)}{\partial \gamma} & 0 & 0 & 0 \\ \frac{\partial R^{*-1} \bar{g}^*(\gamma^*, \gamma)}{\partial \gamma} & \frac{\partial R^{*-1} \bar{g}^*(\gamma^*, \gamma)}{\partial \gamma^*} & 0 & 0 \\ \frac{\partial \bar{M}(\gamma)}{\partial \gamma} & 0 & -I & 0 \\ 0 & \frac{\partial \bar{M}^*(\gamma^*)}{\partial \gamma^*} & 0 & -I \end{pmatrix} \quad (\text{A.16})$$

The lower triangular structure of the Jacobian means its eigenvalues are equal to the eigenvalues of the matrices on the leading diagonal. The eigenvalues of  $-I$  necessarily have negative real parts, so the stability properties of the equilibrium depend on the eigenvalues of  $\partial R^{-1} \bar{g}(\gamma) / \partial \gamma$  and  $\partial R^{*-1} \bar{g}^*(\gamma^*, \gamma) / \partial \gamma^*$  evaluated at equilibrium. After simple but tedious calculations, these derivatives are given by equations (A.17) and (A.18).

$$\left. \frac{\partial R^{-1} \bar{g}(\gamma)}{\partial \gamma} \right|_{SCE} = \begin{pmatrix} -\frac{1}{1+\theta^2} & \frac{\theta(1-\theta^2)(y_0-\bar{y})}{1+\theta^2} \\ 0 & -1 \end{pmatrix} \quad (\text{A.17})$$

$$\left. \frac{\partial R^{*-1} \bar{g}^*(\gamma^*, \gamma)}{\partial \gamma^*} \right|_{SCE} = \begin{pmatrix} -\frac{1+\delta\theta^2}{1+\theta^2} & \frac{\theta(1-\delta)(1-\theta^2)(y_0-\bar{y})}{1+\theta^2} \\ 0 & -1 \end{pmatrix} \quad (\text{A.18})$$

The eigenvalues of the matrices defined in (A.17) and (A.18) are given by equations (A.19)-(A.22).

$$\lambda_1 = -\frac{1}{1+\theta^2} \quad (\text{A.19})$$

$$\lambda_2 = -1 \quad (\text{A.20})$$

$$\lambda_3 = -\frac{1+\delta\theta^2}{1+\theta^2} \quad (\text{A.21})$$

$$\lambda_4 = -1 \quad (\text{A.22})$$

As  $\delta > 0$ , all the eigenvalues of the system have negative real parts and the equilibrium is asymptotically stable.

## B Dominant escape path in source of spillover

The dominant escape path describes the most likely path for beliefs to take in the first small country if they deviate significantly (escape) from their mean dynamics described by equations (A.5) and (A.7). To characterise this, we need a way of selecting the most likely path amongst all candidate escape paths. A natural metric is the likelihood function of the shocks needed to drive beliefs along each escape path. The path that minimises this function is the dominant escape path, representing the path of least resistance for beliefs to escape.

The formal analysis of escape dynamics in economic models is laid out in the pioneering work of Williams (2004), where the dominant escape path is characterised by solving an optimal control problem. The method involves choosing the series of perturbations to mean dynamics that is most likely to cause beliefs to escape from a neighbourhood around the self-confirming equilibrium. Mathematically, the dominant escape path is given by the solution to optimal control problem (B.1).

$$\begin{aligned}
 \bar{\Psi} &= \inf_{\dot{v}} \int_0^t \dot{v}(\psi)' Q(\gamma(\psi), R(\psi))^{-1} \dot{v}(\psi) d\psi \\
 & \text{s.t.} \\
 \dot{\gamma} &= R^{-1} \bar{g}(\gamma) + \dot{v} \\
 \dot{R} &= \bar{M}(\gamma) - R \\
 \gamma(0) &= \bar{\gamma}, \quad M(0) = \bar{M}, \quad \gamma(t) \notin G \text{ for some } 0 < t < T
 \end{aligned} \tag{B.1}$$

The optimal control problem works by perturbing the mean dynamics of the model (A.5) by a factor  $\dot{v}$  and asking which series of perturbations is most likely to cause beliefs to escape. In the objective,  $Q(\gamma, R)$  is a weighting function that measures the likelihood of the shocks needed to perturb beliefs by  $\dot{v}$ .<sup>6</sup> We initialise beliefs at their self-confirming values and define a neighbourhood  $G$  around the self-confirming equilibrium that beliefs must escape from. The outcome of the optimal control problem is the series of belief perturbations that occur along the dominant escape path. To solve the optimal control problem (B.1) we define the Hamiltonian

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<sup>6</sup>An analytic expression for  $Q(\gamma, R)$  is given in Section B.1.

(B.2), where  $a$  and  $\lambda$  are co-state vectors for the evolution of  $\gamma$  and  $R$ .

$$\mathcal{H} = a \cdot R^{-1} \bar{g}(\gamma) - \frac{1}{2} a' Q(\gamma, R) a + \lambda \cdot (\bar{M}(\gamma) - R) \quad (\text{B.2})$$

The Hamiltonian is convex so first order conditions (B.3)-(B.6) necessarily hold along the dominant escape path.

$$\dot{\gamma} = R^{-1} \bar{g}(\gamma) - Q(\gamma, R) a \quad (\text{B.3})$$

$$\dot{R} = \bar{M}(\gamma) - R \quad (\text{B.4})$$

$$\dot{a} = -a R^{-1} \frac{\partial \bar{g}(\gamma)}{\partial \gamma} + \frac{1}{2} a' \frac{\partial Q(\gamma, R)}{\partial \gamma} a - \lambda \frac{\partial \bar{M}(\gamma)}{\partial \gamma} \quad (\text{B.5})$$

$$\dot{\lambda} = -\mathcal{H}_R \quad (\text{B.6})$$

The first order conditions form a system of ordinary differential equations. They characterise a family of escape paths, with each path being indexed by different initial values of the co-state vectors. The dominant escape path is the member of this family that achieves the escape with the most likely series of belief perturbations. A solution to the optimal control problem can therefore be obtained by searching over all possible initial values of  $a$  and  $\lambda$ , applying equations (B.3)-(B.6), and choosing the initial values that imply belief perturbations that are most likely in terms of the  $Q(\gamma, R)$  metric.

## B.1 Calculation of $a' Q(\gamma, R) a$

The cost function  $Q(\gamma, R)$  is used to weight belief perturbations along potential escape paths. It is equal to the variance-covariance matrix of belief dynamics  $\dot{\gamma}$ . As belief dynamics are quadratic forms of Gaussian variables,  $Q(\gamma, R)$  itself is a fourth moment matrix. In static models such as ours, Williams (2004) shows that  $Q$  reduces to the logarithm of a moment generating function, meaning the Hamiltonian (B.2) can be derived analytically. We begin by expressing the second term of the Hamiltonian by the corresponding moment generating function (B.7).

$$a' Q(\gamma, R) a = \log E \exp \langle a \cdot R^{-1} (g(\gamma, \xi) - \bar{g}(\gamma)) \rangle \quad (\text{B.7})$$

We obtain an explicit analytic expression for the right hand side of (B.7) by first using

equation (A.1) to define

$$g(\gamma, \xi) - \bar{g}(\gamma) = \begin{pmatrix} \sigma_1 v_1 + (\theta - \gamma_1) \sigma_2 v_2 \\ \hat{s} \sigma_1 v_1 + (\hat{s}(\theta - 2\gamma_1) + y_0 - \gamma_0) \sigma_2 v_2 + \sigma_1 \sigma_2 v_1 v_2 + (\theta - \gamma_1) \sigma_2^2 v_2^2 - (\theta - \gamma_1) \sigma_2^2 \end{pmatrix}$$

To economise on notation, let  $R^{-1}$  and  $a$  be defined by equations (B.8) and (B.9) respectively.

$$R^{-1} = \begin{pmatrix} R^1 & R^2 \\ R^2 & R^4 \end{pmatrix} \quad (\text{B.8})$$

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \quad (\text{B.9})$$

The right hand side of equation (B.7) can now be expressed in equation (B.10) in terms of the underlying shocks  $v_1$  and  $v_2$ .

$$\log E \exp \langle a \cdot R^{-1} (g(\gamma, \xi) - \bar{g}(\gamma)) \rangle = \log E [e^{d_0 + d_1 v_1 + d_2 v_2 + d_3 v_1 v_2 + d_4 v_2^2}] \quad (\text{B.10})$$

The constants  $d_0 \dots d_4$  are simple functions (B.11)-(B.15) of the structural parameters  $\{y_0, \theta, \sigma_1, \sigma_2\}$ , beliefs  $\gamma$ , the co-state vector  $a$  and the precision matrix  $R$ .

$$d_0 = -(a_1 R^2 + a_2 R^4) (\theta - \gamma_1) \sigma_2^2 \quad (\text{B.11})$$

$$d_1 = (a_1 R^1 + a_2 R^2) \sigma_1 + (a_1 R^2 + a_2 R^4) \hat{s} \sigma_1 \quad (\text{B.12})$$

$$d_2 = (a_1 R^1 + a_2 R^2) (\theta - \gamma_1) \sigma_2 \\ + (a_1 R^2 + a_2 R^4) (\hat{s}(\theta - 2\gamma_1) + y_0 - \gamma_0) \sigma_2 \quad (\text{B.13})$$

$$d_3 = (a_1 R^2 + a_2 R^4) \sigma_1 \sigma_2 \quad (\text{B.14})$$

$$d_4 = (a_1 R^2 + a_2 R^4) (\theta - \gamma_1) \sigma_2^2 \quad (\text{B.15})$$

The next step is to factorise  $v_1$  out from the right hand side of equation (B.10). The key stage in the factorisation below is the third line, where we exploit the fact that  $e^{v_1}$  is log-normally distributed, with expected value half the variance of  $v_1$ .

$$\begin{aligned} & \log E [e^{d_0 + d_1 v_1 + d_2 v_2 + d_3 v_1 v_2 + d_4 v_2^2}] \\ &= d_0 + \log E [E(e^{(d_1 + d_3 v_2) v_1} | v_2) e^{d_2 v_2 + d_4 v_2^2}] \\ &= d_0 + \log E [e^{.5(d_1 + d_3 v_2)^2} e^{d_2 v_2 + d_4 v_2^2}] \\ &= d_0 + .5 d_1^2 + \log E [e^{(d_2 + d_1 d_3) v_2 + (d_4 + .5 d_3^2) v_2^2}] \end{aligned}$$

The outcome of factorisation is an expression in only the  $v_2$  shock. The remaining expectation can be solved analytically by defining  $k_1 = d_2 + d_1d_3$  and  $k_2 = d_4 + .5d_3^2 - .5$  and completing the square of  $k_1x + k_2x^2$ . In expression (B.16) we have  $A = \sqrt{-2k_2}$ ,  $B = -k_1/A$  and  $C = -B^2/2$ .

$$\begin{aligned}
E[e^{(d_2+d_1d_3)v_2+(d_4+.5d_3^2)v_2^2}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{(k_1x+k_2x^2)} dx \\
&= \frac{e^{-C}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-.5(Ax+B)^2} dx \\
&= \frac{e^{-C}}{A}
\end{aligned} \tag{B.16}$$

The final analytic expression for  $a'Q(\gamma, R)a$  is then (B.17).

$$a'Q(\gamma, R)a = d_0 + .5d_1^2 - \log A - C \tag{B.17}$$

## C Dominant escape path in subject of spillover

The dominant escape path in the subject of the spillover describes the way beliefs are most likely to escape in the second small country *conditional* on the spillover from the first small country. Our interest is in how the dominant escape path changes with the strength of the spillover. We therefore set the first small country on its own dominant escape path, and derive the dominant escape path in the second small country conditional on the strength of the spillover. The optimal control problem to solve is presented in (C.1). The first two constraints are the mean dynamics (A.6) and (A.8) of the second small country perturbed by a factor  $\dot{v}^*$ . The next four constraints describe the - here exogenous - evolution of beliefs in the first small country. They are taken directly from the first-order conditions (B.3)-(B.6) that characterise the dominant escape path in the first small country. Beliefs are initialised in the second small country at the stable equilibrium  $(\bar{\gamma}^*, \bar{M}^*)$  and in the first small country at some point on the country's own dominant escape path  $(\gamma_{t_{dp}}, M_{t_{dp}})$ . The function  $Q^*(\gamma^*, R^*, \gamma)$  weights belief perturbations  $\dot{v}^*$  along potential escape paths, with  $\gamma$  reflecting the presence of the spillover. An analytic expression for  $Q^*(\gamma^*, R^*, \gamma)$  is derived in Section C.1.

$$\begin{aligned}
\bar{\Psi}^* &= \inf_{\dot{v}^*} \int_0^t \dot{v}^*(\psi^*)' Q^*(\gamma^*(\psi^*), R^*(\psi^*), \gamma)^{-1} \dot{v}^*(\psi^*) d\psi^* \\
s.t \quad & \\
\dot{\gamma}^* &= R^{*-1} \bar{g}^*(\gamma^*, \gamma) + \dot{v}^* \\
\dot{R}^* &= \bar{M}^*(\gamma^*) - R^* \\
\dot{\gamma} &= R^{-1} \bar{g}(\gamma) - Q(\gamma, R)a \tag{C.1} \\
\dot{R} &= \bar{M}(\gamma) - R \\
\dot{a} &= -aR^{-1} \frac{\partial \bar{g}(\gamma)}{\partial \gamma} + \frac{1}{2} a' \frac{\partial Q(\gamma, R)}{\partial \gamma} a - \lambda \frac{\partial \bar{M}(\gamma)}{\partial \gamma} \\
\dot{\lambda} &= -\mathcal{H}_R \\
\gamma^*(0) &= \bar{\gamma}^*, \quad M^*(0) = \bar{M}^*, \quad \gamma^*(t) \notin G \text{ for some } 0 < t < T \\
\gamma(0) &= \gamma_{t_{dp}}, \quad M(0) = M_{t_{dp}}
\end{aligned}$$

The Hamiltonian for control problem (C.1) is defined in equation (C.2). Additional terms associated with the evolution of beliefs in the first small country are omitted for clarity.

$$\mathcal{H}^* = a^* \cdot R^{*-1} \bar{g}^*(\gamma^*, \gamma) - \frac{1}{2} a^{*'} Q^*(\gamma^*, R^*, \gamma) a^* + \lambda^* \cdot (\bar{M}^*(\gamma^*) - R^*) + \dots \tag{C.2}$$

The Hamiltonian is convex, so the dominant escape path satisfies first order conditions (C.3)-(C.10).

$$\dot{\gamma}^* = R^{*-1} \bar{g}^*(\gamma^*, \gamma) - Q^*(\gamma^*, R^*, \gamma) a^* \tag{C.3}$$

$$\dot{R}^* = \bar{M}^*(\gamma^*) - R^* \tag{C.4}$$

$$\dot{a}^* = -a^* R^{*-1} \frac{\partial \bar{g}^*(\gamma^*, \gamma)}{\partial \gamma^*} + \frac{1}{2} a^{*'} \frac{\partial Q^*(\gamma^*, R^*, \gamma)}{\partial \gamma^*} a^* - \lambda^* \frac{\partial \bar{M}^*(\gamma^*)}{\partial \gamma^*} \tag{C.5}$$

$$\dot{\lambda}^* = -\mathcal{H}_{R^*}^* \tag{C.6}$$

$$\dot{\gamma} = R^{-1} \bar{g}(\gamma) - Q(\gamma, R)a \tag{C.7}$$

$$\dot{R} = \bar{M}(\gamma) - R \tag{C.8}$$

$$\dot{a} = -aR^{-1} \frac{\partial \bar{g}(\gamma)}{\partial \gamma} + \frac{1}{2} a' \frac{\partial Q(\gamma, R)}{\partial \gamma} a - \lambda \frac{\partial \bar{M}(\gamma)}{\partial \gamma} \tag{C.9}$$

$$\dot{\lambda} = -\mathcal{H}_R \tag{C.10}$$

The first order conditions form a system of ordinary differential equations with a block-recursive structure. The dominant escape path can be obtained by searching over all possible

initial values of  $a^*$  and  $\lambda^*$ , applying equations (C.3)-(C.10), and choosing the initial values that imply the most-likely belief perturbations in terms of the  $Q^*(\gamma^*, R^*, \gamma)$  metric.

### C.1 Calculation of $a^{*'}Q^*(\gamma^*, R^*, \gamma)a^*$

The cost function  $Q^*(\gamma^*, R^*, \gamma)$  weights belief perturbations along possible escape paths in the second small country. It can be calculated analytically using the same steps applied in Section B.1 to derive an analytical expression for  $Q(\gamma, R)$  in the first small country. Following Williams (2004),  $Q^*(\gamma^*, R^*, \gamma)$  is the logarithm of the moment generating function and we can define the quadratic form by equation (C.11).

$$a^{*'}Q^*(\psi^*), R^*(\psi^*), \gamma a^* = \log E \exp \langle a^* \cdot R^{*-1}(g^*(\gamma^*, \xi^*, \gamma) - \bar{g}^*(\gamma^*, \gamma)) \rangle \quad (\text{C.11})$$

Application of (A.2) means the stochastic element  $g^*(\gamma^*, \xi^*, \gamma) - \bar{g}^*(\gamma^*, \gamma)$  of the right hand side of (C.11) is equal to the expression .

$$\left( \begin{array}{c} -(\theta - \gamma_1^*)\sigma_2^2 + \hat{s}^*\sigma_1v_1^* \\ \hat{s}^*\sigma_1v_1^* + (\hat{s}^*(\theta - \gamma_1^*) + y_0 - \theta\delta(\hat{s} - \hat{s}^*) - \gamma_0^* - \gamma_1^*\hat{s}^*)\sigma_2v_2^* + \dots \\ \dots\sigma_1\sigma_2v_1^*v_2^* + (\theta - \gamma_1^*)\sigma_2^2v_2^{*2} - \hat{s}^*\theta\delta\sigma_2v_2 - \hat{s}^*\theta\delta + \sigma_2^2v_2^*v_2 - (\theta - \gamma_1^*)\sigma_2^2 \end{array} \right)$$

For notation reasons, let  $R^*$  and  $a^*$  be defined by equations (C.12) and (C.13).

$$R^{*-1} = \begin{pmatrix} R^{*1} & R^{*2} \\ R^{*2} & R^{*4} \end{pmatrix} \quad (\text{C.12})$$

$$a^* = \begin{pmatrix} a_1^* \\ a_2^* \end{pmatrix} \quad (\text{C.13})$$

The right hand side of equation (C.11) can now be expressed in equation (C.14) in terms of the underlying shocks  $v_1^*$  and  $v_2^*$ .

$$\log E[e^{d_0^* + d_1^*v_1^* + d_2^*v_2^* + d_3^*v_1^*v_2^* + d_4^*v_2^{*2} + n_1v_2 + n_2v_2^*v_2}] \quad (\text{C.14})$$

The constants  $d_0^* \dots d_4^*, n_1, n_2$  are functions (C.15)-(C.21) of the structural parameters

$\{y_0, \theta, \sigma_1, \sigma_2\}$ , beliefs  $(\gamma^*, \gamma)$ , the co-state vectors  $(a^*, a)$  and the precision matrices  $(R^*, R)$ .

$$d_0^* = -(a_1^* R^{*2} + a_2^* R^{*4})(\theta - \gamma_1^*)\sigma_2^2 \quad (\text{C.15})$$

$$d_1^* = (a_1^* R^{*1} + a_2^* R^{*2})\sigma_1 + (a_1^* R^{*2} + a_2^* R^{*4})\hat{s}^* \sigma_1 \quad (\text{C.16})$$

$$d_2^* = (a_1^* R^{*1} + a_2^* R^{*2})(\theta - \gamma_1^*)\sigma_2 \\ + (a_1^* R^{*2} + a_2^* R^{*4})(\hat{s}^*(\theta - \gamma_1^*) + y_0 - \theta\delta(\hat{s} - \hat{s}^*) - \gamma_0^* - \gamma_1^* \hat{s}^*)\sigma_2 \quad (\text{C.17})$$

$$d_3^* = (a_1^* R^{*2} + a_2^* R^{*4})\sigma_1\sigma_2 \quad (\text{C.18})$$

$$d_4^* = (a_1^* R^{*2} + a_2^* R^{*4})(\theta - \gamma_1^*)\sigma_2^2 \quad (\text{C.19})$$

$$n_1 = -(a_1^* R^{*1} + a_2^* R^{*2})\delta\theta\sigma_2 \quad (\text{C.20})$$

$$n_2 = -(a_1^* R^{*2} + a_2^* R^{*4})\hat{s}^*\theta\delta\sigma_2^2 \quad (\text{C.21})$$

We next factorise  $v_1^*$  out from the right hand side of equation (C.14). The key stages are the third and fifth line, where we sequentially exploit the log-normal distribution of  $e^{v_1^*}$  and  $e^{v_2}$ .

$$\begin{aligned} & \log E[e^{d_0^* + d_1^* v_1^* + d_2^* v_2^* + d_3^* v_1^* v_2^* + d_4^* v_2^{*2} + n_1 v_2 + n_2 v_2^* v_2}] \\ &= d_0^* + \log E[E(e^{(d_1^* + d_3^* v_2^*)v_1^*} | v_2^*) e^{d_2^* v_2^* + d_4^* v_2^{*2} + n_1 v_2 + n_2 v_2^* v_2}] \\ &= d_0^* + \log E[e^{.5(d_1^* + d_3^* v_2^*)^2} e^{d_2^* v_2^* + d_4^* v_2^{*2} + n_1 v_2 + n_2 v_2^* v_2}] \\ &= d_0^* + .5d_1^{*2} + \log E[E(e^{(n_1 + n_2 v_2^*)v_2} | v_2^*) e^{(d_2^* + d_1^* d_3^*)v_2^* + (d_4^* + .5d_3^{*2})v_2^{*2}}] \\ &= d_0^* + .5d_1^{*2} + \log E[e^{.5(n_1 + n_2 v_2^*)^2} e^{(d_2^* + d_1^* d_3^*)v_2^* + (d_4^* + .5d_3^{*2})v_2^{*2}}] \\ &= d_0^* + .5d_1^{*2} + .5n_1^2 + \log E[e^{(d_2^* + d_1^* d_3^* + n_1 n_2)v_2^* + (d_4^* + .5d_3^{*2} + .5n_2^2)v_2^{*2}}] \end{aligned}$$

The result of the factorisation is an expression in only the  $v_2^*$  shock. To solve the remaining expectation, define  $k_1^* = d_2^* + d_1^* d_3^* + n_1 n_2$ ,  $k_2^* = d_4^* + .5d_3^{*2} + .5n_2^2 - .5$ , and complete the square of  $k_1^* x + k_2^* x^2$ . In expression (C.22), we have  $A^* = \sqrt{-2k_2^*}$ ,  $B^* = -k_1^*/A^*$  and  $C^* = -B^{*2}/2$ .

$$\begin{aligned} E[e^{(d_2^* + d_1^* d_3^* + n_1 n_2)v_2^* + (d_4^* + .5d_3^{*2} + .5n_2^2)v_2^{*2}}] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{(k_1^* x + k_2^* x^2)} dx \\ &= \frac{e^{-C^*}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-.5(A^* x + B^*)^2} dx \\ &= \frac{e^{-C^*}}{A^*} \quad (\text{C.22}) \end{aligned}$$



The final analytic expression for  $a^{*'}Q^*(\gamma^*, R^*)a^*$  is (C.23).

$$a^{*'}Q^*(\gamma^*, R^*)a^* = d_0^* + .5d_1^{*2} + .5n_1^2 - \log A^* - C^* \quad (\text{C.23})$$