Equipartitioning by a convex 3-fan

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Abstract

We show that for a given planar convex set K of positive area there exist three pairwise internally disjoint convex sets whose union is K such that they have equal area and equal perimeter.

1 Introduction and main result

The following interesting and annoyingly resistant question has been recently asked by R. Nandakumar and N. Ramana Rao [NR06] and [NR08]. A *convex k-partition* of the plane $\mathbb{R}^2$ is, quite naturally, a family of k internally disjoint convex sets $P_1, \dots, P_k$ with $\mathbb{R}^2 = \cup_1^k P_i$. The question is whether, given a convex set K of positive area and an integer $k \geq 2$, there exists a convex k-partition of $\mathbb{R}^2$ such that all parts $K \cap P_i$ have equal area and equal perimeter. For $k = 2$ the answer is, quite trivially, yes. The main result of this paper implies that the answer is also yes when $k = 3$. This is contained in Theorem 1.1 below.

The solution of the problem relies on the methods from equivariant topology and can be considered as a continuation of [BM1] and [BM2] whose notation and terminology are used here without much change. A point x in the plane and three halflines, ℓ_1, ℓ_2, ℓ_3, starting from x form a 3-fan. The halflines are in anticlockwise order around x. They determine three angular sectors $\sigma_1, \sigma_2, \sigma_3$ with σ_i between ℓ_i and ℓ_{i+1}. The 3-fan is *convex* if each of the sectors σ_i is convex.

Theorem 1.1. *Assume μ is an absolutely continuous (with respect to the Lebesgue measure) Borel probability measure on $\mathbb{R}^2$, and f is a continuous function defined on the sectors in $\mathbb{R}^2$. Then there is a convex 3-fan $(x; \ell_1, \ell_2, \ell_3)$ with*

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_3) = \tfrac{1}{3} \qquad \textit{and} \qquad f(\sigma_1) = f(\sigma_2) = f(\sigma_3).$$

The case $k = 3$ of the Nandakumar–Rao conjecture follows from the theorem by taking $f(\sigma)$ to be the perimeter of $K \cap \sigma$. Also, the Lebesgue measure restricted to K has to be approximated by absolutely continuous measures which is no problem. The same way Theorem 1.1 implies the existence of a convex 3-partition of K where the pieces have equal diameter, or equal width, etc. We mention that every convex 3-partition of $\mathbb{R}^2$ come from a convex 3-fan, including the convex partition by two parallel lines when the center of the 3-fan is at infinity. One of the difficulties in the case of $k > 3$ is the lack of nice or natural description of convex k-partitions.

About ten years ago Kaneko and Kano [KK] raised a question which is similar to that of Nandakumar and Ramana Rao, and which was solved, independently, by Bespanyatnikh et al. [BKS] and by Sakai [Sak]. They showed that, given an integer $k \geq 2$ and two absolutely continuous probability measures μ_1 and μ_2 in the plane, there is a convex k-partition, $P_1, \ldots, P_k$ of the plane with with $\mu_i(P_j) = \frac{1}{k}$ for all $i = 1, 2$ and $j = 1, \ldots, k$. Neither this result, nor its proof seem to help with the problem raised by Nandakumar and Ramana Rao because the perimeter is not a measure.

It is more convenient to lift the measure and the 3-fans from $\mathbb{R}^2$ to the 2-sphere S^2 mainly because S^2 is compact. So let S^2 be the unit sphere of $\mathbb{R}^3$ and let $\mathbb{R}^2$ be embedded in $\mathbb{R}^3$ as the horizontal plane tangent to S^2 (at the North Pole). Denote by ρ the central projection from the upper hemisphere to the embedded $\mathbb{R}^2$. Clearly, ρ^{-1} lifts any Borel measure on $\mathbb{R}^2$ to a Borel measure on the upper hemisphere of S^2. A 3-fan in $\mathbb{R}^2$ is lifted to a 3-fan in S^2 in a natural way: a spherical 3-fan $(x, \ell_1, \ell_2, \ell_3)$ is a point $x \in S^2$ and three great half circles ℓ_1, ℓ_2, ℓ_3 starting at x (and ending at $-x$) that are ordered anticlockwise when viewed from x. The angular sector between ℓ_i and ℓ_{i+1} is σ_i. It is clear that a spherical 3-fan is projected by ρ to a 3-fan in $\mathbb{R}^2$, and conversely, a 3-fan in $\mathbb{R}^2$ is mapped by ρ^{-1} to a spherical 3-fan on S^2. A spherical 3-fan is convex if the angle of each sector is at most π. It is also evident that a spherical 3-fan is convex if and only if the corresponding planar 3-fan is convex. We will prove Theorem 1.1 in a slightly stronger form:

Theorem 1.2. *Assume μ is an absolutely continuous (with respect to the Lebesgue measure) Borel probability measures on S^2 and f is a continuous function on the sectors in S^2. Then there is a convex 3-fan $(x, \ell_1, \ell_2, \ell_3)$ such that*

$$\mu(\sigma_1) = \mu(\sigma_2) = \mu(\sigma_3) = \tfrac{1}{3} \qquad \textit{and} \qquad f(\sigma_1) = f(\sigma_2) = f(\sigma_3).$$

In fact, this theorem holds under the weaker assumption that μ is not positive on any great circle. This follows from a routine compactness argument.

A measure on the sphere S^2 will be called *nice* if it is a probability measure that has a continuous density function which is positive on S^2. We will prove Theorem 1.2 assuming that μ nice. This will suffice for the general case by the same compactness argument. By the same token it is enough to prove the theorem for a dense set of nice measures, and we will assume, in case of need, that our measure satisfies certain extra properties.

We are going to give two proofs of Theorem 1.2. Both use equivariant topology, whose basic phase space/test map method, applied in our case, will be described in the next section, without considering convexity. The phase space V is given in Section 2, and its restriction to the so called *convex part* V^{conv} in Section 4. The topology of V^{conv} is needed in both proofs of Theorem 1.2. The first uses basic algebraic topology: degree, linking number, homology (Section 4), while the second applies the Serre spectral sequence (Section 6).

Besides Theorems 1.1 and 1.2, the main novelty of this paper is the description of the convex part and understanding its topology. In geometric applications of equivariant topology the phase space is usually given, but in our case it depends on the measure on S^2. The description of the convex part and of its topology is accomplished here by combining methods from convexity, measure theory, and topology.

2 The proof without convexity

Write $V = \{(x, y) \in S^2 \times S^2 : x \perp y\}$; V is the Stiefel manifold of all orthogonal 2-frames in $\mathbb{R}^3$, which is homeomorphic to $SO(3)$ and to the 3-dimensional projective space $\mathbb{R}P^3$.

To every $(x, y) \in V$ we assign the 3-fan $(x; \ell_1, \ell_2, \ell_3)$ as follows: y is the midpoint of the half great circle ℓ_1 whose endpoints are x and $-x$, and ℓ_2, ℓ_3 are defined by the condition $\mu(\sigma_i) = \frac{1}{3}$ for all i. As μ is nice, the half great circles ℓ_i and the sectors σ_i are determined uniquely. Thus the mapping $(x, y) \to (x; \ell_1, \ell_2, \ell_3)$ is well-defined (see Figure 1). We will simply write ℓ_i or σ_i for $\ell_i(x, y)$ and $\sigma_i(x, y)$. This should not cause any confusion.

We are going to use equivariant topology. Write y^i for the midpoint of the great half circle ℓ_i. So $y = y^1$. Define the homeomorphism $\omega : V \to V$ via

$$\omega(x, y) = \omega(x, y^1) = (x, y^2).$$

This homeomorphism is in fact determined by the measure μ. Further, $\omega^2(x, y) = (x, y^3)$ and $\omega^3 = \mathrm{id}_V$. Thus the cyclic groups $\mathbb{Z}_3$ acts on V and ω is the action of its generator. Further, ω has no fixed point and is a $V \to V$ homeomorphism that keeps the orientation of V since $\omega^3 = \mathrm{id}$.

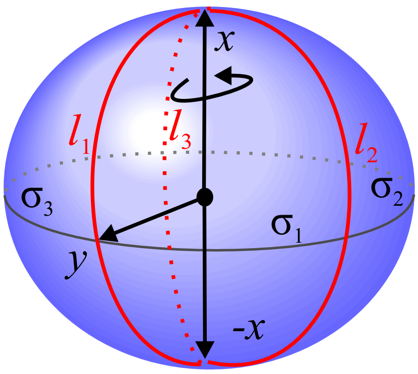

Figure 1: The sectors

We wish to show the existence of a (convex) 3-fan equipartitioning μ such that $f(\sigma_1) = f(\sigma_2) = f(\sigma_3)$. Define a continuous map $\overline{f} : V \to \mathbb{R}^3$ by

$$\overline{f} = (f(\sigma_1), f(\sigma_2), f(\sigma_3)) \in \mathbb{R}^3.$$

The group $\mathbb{Z}_3$ acts on $\mathbb{R}^3$ by shifting the coordinates cyclicly. That is, writing ω for the action of its generator,

$$\omega(t_1, t_2, t_3) = (t_2, t_3, t_1).$$

It is clear that the just defined $\overline{f}$ is a $\mathbb{Z}_3$-equivariant map, that is,

$$\overline{f} \circ \omega = \omega \circ \overline{f}.$$

Here the first ω acts on V while the second ω acts on $\mathbb{R}^3$.

We put aside the convexity condition for this section and prove the existence of an $(x, y) \in V$ with f equal on the three sectors. The proof is from [BM1] but the statement is slightly more general since here f does not come from a measure.

Proposition 2.1. *Under the above conditions there is $(x, y) \in V$ such that $f(\sigma_1) = f(\sigma_2) = f(\sigma_3)$.*

Proof. We assume the contrary which means that $\overline{f}$ avoids the diagonal $\triangle = \{(t, t, t) \in \mathbb{R}^3\}$. This gives rise to a chain of maps

$$V \to \mathbb{R}^3 \to \triangle^{\perp} \to S^1$$

where the first arrow is $\overline{f}$, the second is the orthogonal projection onto $\triangle^{\perp}$ (the orthogonal complement of $\triangle$), and the last arrow maps $v \in \triangle^{\perp}$, $(v \neq 0)$ to $v/|v| \in S^1$ (the unit circle in $\triangle^{\perp}$). Let g denote composition map $V \to S^1$. On this $S^1 \subset \mathbb{R}^3$, ω acts as a rotation by $2\pi/3$. It follows that g is a $\mathbb{Z}_3$-equivariant map, again:

$$g \circ \omega = \omega \circ g.$$

The set $C = \{(e_3, y) \in V : y \perp e_3\}$ is invariant under ω, that is, $C = \omega C$. Further, C is homeomorphic to the circle S^1, so $g|_C : C \to S^1$ is an $S^1 \to S^1$ $\mathbb{Z}_3$-map. By a theorem of Krasnoselskii and Zabreiko [KZ] (cf. [BSS] and [Dold] as well), the degree of $g|_C$ is $1 \mod 3$. (For this the orientations of C and S^1 have to be chosen properly.)

Next, let $2C$ denote the cycle in V obtained as the composition of the (orientation preserving) standard double cover $S^1 \to S^1$ and the homeomorphism $S^1 \to C$. It follows that the degree of $g|_{2C}$ is $2 \mod 3$. The fundamental group of V is $\mathbb{Z}_2$, and so $2C$ is homotopic to 0 implying that the degree of $g|_{2C}$ is 0. A contradiction. $\square$

Remark. This proof does not go through when the 3-fan is required to be convex because the fundamental group of the "convex part" of V does not have to be (and is not) $\mathbb{Z}_2$. We mention further that the set C, which is a (geometric) circle and a (topological) cycle in V, is going to play an important role in what follows.

3 Preparations

In this section we introduce the necessary definitions to handle the condition of convexity in Theorem 1.2. We start by a variant of the Hopf fibration. The map $p : V \to S^2$ is defined for $(x, y) \in V$ as

$$p(x, y) = x \times y,$$

so $z = p(x, y)$ is the cross product of x and y. Since $z \in S^2$, p is indeed a map $V \to S^2$, see Figure 2. The following fact is well-known.

Fact 3.1. *The map $p : V \to S^2$ is a fibration, and every fibre $p^{-1}(z)$ is an S^1.*

We will often encounter the situation when $S \subset S^2$ is a circle, i.e., a homeomorphic image of S^1. Then $S^2 \setminus S$ consists of two connected components, Ω and Ω', each homeomorphic to the 2-dimensional open (topological) disk. Set $U = p^{-1}(\Omega)$, and restrict the fibration p to U. The base of this fibration $p : U \to \Omega$ is a disk which is, of course, contractible. By Feldbau's theorem (cf [FFG]), the fibration is trivial in the sense that U is homeomorphic to the product of the fibre, S^1, and the base Ω. Thus U is an open solid torus, and so is $U' = p^{-1}(\Omega')$.

It is clear that the angle of at most one of the sectors $\sigma_1, \sigma_2, \sigma_3$ can be larger than π. There is a simple and useful reformulation of the fact that for some $(x, y) \in V$ the sector $\sigma_3(x, y)$ is non-convex. We need a few definitions. For $z \in S^2$ let

$$H(z) = \{v \in S^2 : vz \leq 0\}$$

where vz stands for the scalar product of vectors v, z. Thus $H(z)$ is a half-sphere, see Figure 2. Define $h(z)$ as the μ-content of $H(z)$, that is, $h : S^2 \to \mathbb{R}$ is the function

$$h(z) = \mu(H(z)).$$

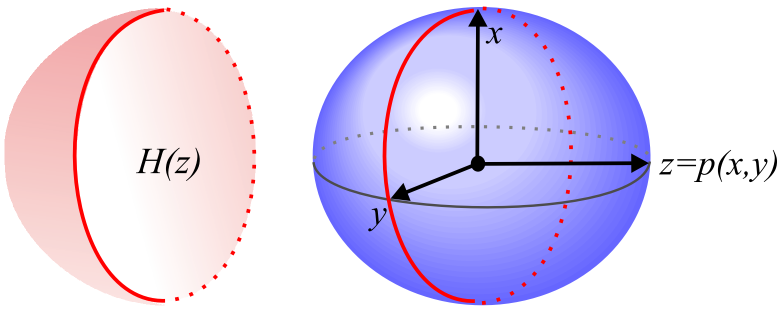

Figure 2: The hemisphere $H(z)$, its measure $h(z)$ and the fibration p

Lemma 3.2. *Assume $(x, y) \in V$ and $z = p(x, y)$. Then $\sigma_3(x, y)$ is not convex if and only if $h(z) < 1/3$.*

Proof. This is very simple: ℓ_1 is a great half circle on the boundary of $H(z)$ and ℓ_1 bounds the sector σ_3. Now $h(z) < 1/3$ if and only if σ_3 properly contains $H(z)$, which is the same as σ_3 is not convex. □

In the proofs to come we need to establish the existence of a cycle $C \subset V$ that is invariant under ω, that is, $\omega C = C$ and has the extra property that for each $(x, y) \in C$ the corresponding 3-fan is convex. The existence of such a cycle follows from the following result, proved independently by Dolnikov [Doln] and Živaljević, Vrecića [ŽV].

Theorem 3.3. *Given $k \leq d$ probability measures in $\mathbb{R}^d$, there is a $k-1$-dimensional affine subspace such that the measure of every halfspace containing this affine subspace is at least $1/(d + 2 - k)$ in every one of the k measures.*

We apply this theorem with $d = 3$ and $k = 2$: the first measure is μ and the second is concentrated at the origin. The affine subspace is a line, passing through the origin. We now fix the coordinate system in $\mathbb{R}^3$ so that this line passes through the points $\pm e_3$. Then $h(z) \geq 1/3$ for every $z \in S^2$ whose e_3 component is zero. By adding a little extra measure at e_3 we can achieve that $h(z) > 1/3$ for every such z. So we have the following

Corollary 3.4. *With the coordinata system fixed as above, the cycle $C = \{(e_3, y) \in V : y \perp e_3\}$ is invariant under ω and each point $(e_3, y) \in C$ defines a convex 3-fan.* $\square$

We need the following lemma saying that every nice measure μ can be approximated by another nice measure ν for which the set $\{z \in S^2 : \nu(H(z)) = 1/3\}$ is a nice 1-manifold. The technical proof of the lemma is given in the last section.

Lemma 3.5. *For every $\varepsilon > 0$ and every nice measure μ on S^2 there is a nice measure ν such that*

(i) $|\mu(\sigma) - \nu(\sigma)| < \varepsilon$ *for every sector* $\sigma \subset S^2$, *and*

(ii) $\{z \in S^2 : \nu(H(z)) = 1/3\}$ *is a piecewise smooth* 1*-manifold (without boundary) in* S^2.

4 The convex part of V

In this section we describe a particular partition of V into two pieces, the *convex part* V^{conv}, and the *non-convex part* $V^{\text{n-conv}}$, and establish some of their properties. This partition will be only given at the end of this section.

Lemma 3.5 implies, via a routine compactness argument, that it suffices to prove Theorem 1.2 for nice measures μ for which $h^{-1}(1/3)$ is a piecewise smooth 1-manifold in S^2. From now on we assume that μ is such a nice measure. We suppose further that there is a $z \in S^2$ with $h(z) < 1/3$ as otherwise Theorem 1.2 follows from Proposition 2.1. Then $h^{-1}(1/3)$ is nonempty and is the union of disjoint cycles S_i, $i \in [m_1]$ for some positive integer m_1, where for a positive integer k we denote the set $\{1, 2, \ldots, k\}$ by $[k]$.

Observe now that $p(C)$ is exactly the equator of S^2, and $h(z) > 1/3$ for every $z \in p(C)$. Then each S_i is disjoint from $p(C)$, so it is contained either in the upper or in the lower hemisphere. Each S_i splits S^2 into two connected components, and both are homeomorphic to a disk, and one of them contains the equator. Let Ω_i denote the other one.

As we have seen the set $U_i = p^{-1}(\Omega_i)$ is an open solid torus and $T_i = p^{-1}(S_i)$ is an ordinary torus. Since $\omega : V \to V$ is a homeomorphism, $\omega^\alpha U_i$ is an open solid torus, and $\omega^\alpha T_i$ is an ordinary torus for each $i = [m_1]$ and every $\alpha = 0, 1, 2$. A few properties of these tori are established next. The first one is very simple.

Claim 4.1. *The cycle C is disjoint from all $\omega^\alpha U_i$.* $\square$

Claim 4.2. *For all $i, j \in [m_1]$ and $\alpha, \beta = 0, 1, 2$ the sets $\omega^\alpha T_i$ and $\omega^\beta T_j$ are disjoint unless $i = j$ and $\alpha = \beta$.*

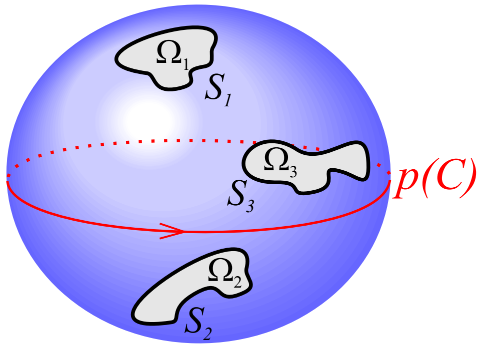

Figure 3: $h^{-1}(1/3)$

Proof. Assume the contrary, then $\omega^\alpha T_i \cap \omega^\beta T_j \neq \emptyset$. We can assume, by symmetry, that $\alpha \leq \beta$. If $\alpha = \beta$, then $T_i = p^{-1}(S_i)$ and $T_j = p^{-1}(S_j)$ intersect, yet S_i and S_j are disjoint. Simplifying by ω once or twice if necessary we can assume that $\alpha = 0$ and $\beta = 1$ or 2. Suppose $\beta = 1$. Then there is $(x, y) \in T_i \cap \omega^1 T_j$, implying $(x, y) = (x, y^1) \in T_i$ and $\omega^{-1}(x, y) = \omega^2(x, y) = (x, y^3) \in T_j$. Thus $\sigma_3(x, y)$ is a hemisphere, and so is $\sigma_2(x, y)$, which is impossible. The assumption $\beta = 2$ implies, the same way, that $\sigma_3(x, y)$ and $\sigma_1(x, y)$ are both hemispheres. □

Claim 4.3. *For all $i \in [m_1]$ the sets U_i, ωU_i and $\omega^2 U_i$ are disjoint.*

Proof. The key fact here is that each $\omega^\alpha T_i$ is a torus and so it splits V into two disjoint components.

Assume the statement is false. The condition $\omega^\alpha U_i \cap \omega^\beta U_i \neq \emptyset$ implies (via simplifying by ω or ω^2) that $U_i \cap \omega U_i \neq \emptyset$. It follows easily from $T_i \cap \omega T_i = \emptyset$ and from $H_2(V; \mathbb{Z}) = 0$ that $V \setminus (T_i \cup \omega T_i)$ consists of 3 connected components. Clearly, $U_i \cap \omega U_i$ is one of them. Its boundary is either T_i or ωT_i or $T_i \cup \omega T_i$. In the first case $U_i \subset \omega U_i$ which implies $U_i \subset \omega U_i \subset \omega^2 U_i \subset \omega^3 U_i = U_i$ showing that $U_i = \omega U_i$ and then $T_i = \omega T_i$, contradicting Claim 4.2. The second case implies $\omega U_i \subset U_i$ which leads to the same contradiction.

We show finally that the third case cannot come up. If it did, then $T_i \subset \omega U_i$ and $\omega T_i \subset U_i$, and so $U_i \cup \omega U_i = V$. But this is impossible since C is disjoint from both U and ωU_i. □

Recall that the cycles S_i are pairwise disjoint. Then, for distinct $i, j \in [m_1]$, Ω_i and Ω_j are either disjoint or one is contained in the other. To have simpler notation, let $[m_2]$ be the set of those $i \in [m_1]$ for which U_i not contained in any other U_j. Of course, $1 \leq m_2 \leq m_1$, and the disks Ω_i, $i \in [m_2]$ are pairwise disjoint.

The orbit of U_i is simply $O(U_i) = U_i \cup \omega U_i \cup \omega^2 U_i$.

Claim 4.4. *For distinct $i, j \in [m_2]$, the orbits $O(U_i)$ and $O(U_j)$ are either disjoint or one is contained in the other.*

Proof. This proof is almost identical with the previous one. Assume that $O(U_i)$ and $O(U_j)$ are not disjoint: $\omega^\alpha U_i \cap \omega^\beta U_j \neq \emptyset$. We can suppose again that $\alpha \leq \beta$ and $\alpha = 0$. In case $\beta = 0$, U_i and U_j would have a common point which is excluded since $i, j \in [m_2]$.

Thus $\beta = 1$ or 2. Consider the case $\beta = 1$; the other one is analogous. The tori T_i and ωT_j are disjoint so their union splits V into three connected components. Clearly, $U_i \cap \omega U_j$ is one of them. Its boundary is either T_i or ωT_j or $T_i \cup \omega T_j$. In the first case $\omega U_j \subset U_i$ which implies $\omega^2 U_j \subset \omega U_i$ and $U_j = \omega^3 U_j \subset \omega^2 U_i$ showing that $O(U_j) \subset O(U_i)$, indeed. In the second case $\omega U_i \subset U_j$ which implies, the same way, that $O(U_i) \subset O(U_j)$.

Again, the third case cannot come up. If it did, then $T_i \subset \omega U_j$ and $\omega T_j \subset U_i$, and so $U_i \cup \omega U_j = V$. But this is impossible since C is disjoint from both U_i and ωU_j. $\square$

Now we define the convex part V^{conv}. To keep notation simple let $[m]$ be the set of those subscripts $i \in [m_2]$ for which the orbit $O(U_i)$ is not contained in any other $O(U_j)$. Set

$$V^{\text{n-conv}} = \bigcup_{i \in [m]} \bigcup_{\alpha=0,1,2} \omega^\alpha U_i \qquad \text{and} \qquad V^{\text{conv}} = V \setminus V^{\text{n-conv}}.$$

The above definitions and results are summarized as follows.

Theorem 4.5. *The sets $\omega^\alpha U_i$ ($\alpha = 0, 1, 2$ and $i \in [m]$) are pairwise disjoint open solid tori. Moreover, for every point (x, y) of the set V^{conv} the corresponding 3-fan is convex. Further, $C \subset V^{\text{conv}}$, and both V^{conv} and $V^{\text{n-conv}}$ are invariant under ω.* $\square$

Remark. It is not hard to construct a nice probability measure on S^2 so that some disk U_i contains another disk U_j. This shows that that there may be a point $(x, y) \in V^{n-\text{conv}}$ such that all $\sigma_i(x, y)$ are convex. So the name "convex part" is slightly misleading. This should not cause any confusion, though.

The proof of Theorem 1.2 starts the same way as in Proposition 2.1, just replace V by the ω-invariant subset V^{conv}. We get the same chain of maps $V^{\text{conv}} \to \mathbb{R}^3 \to D^\perp \to S^1$ and the composition $\mathbb{Z}_3$-equivariant map $V^{\text{conv}} \to S^1$. Thus Theorem 1.2 is a consequence of the following Borsuk–Ulam type result.

Theorem 4.6. *There is no $\mathbb{Z}_3$-equivariant map $F : V^{\text{conv}} \to S^1$.*

In the next two sections we are going to give two different proof of Theorem 4.6.

5 The first proof of Theorem 4.6

Assume that such a map F exists, and consider, again, the cycle $C \subset V^{\text{conv}}$. The restriction of F to C is clearly well-defined and is, again, an $S^1 \to S^1$ $\mathbb{Z}_3$-map. As we have seen in the proof of Proposition 2.1, the degree of $F|_C$ is 1 mod 3. We show, however, that its degree is divisible by 3. This contradiction will prove the theorem.

Theorem 5.1. *The restriction $F|_C \colon C \to S^1$ has degree zero mod 3.*

Proof. Note that the $\mathbb{Z}_3$-action on $V = RP^3$ can be lifted to that on S^3 using the standard double covering map $\pi : S^3 \to RP^3$. Let us denote by S^{conv} the preimage $\pi^{-1}(V^{\text{conv}})$. It is easy to check that the preimage $\pi^{-1}(U_i)$ is an open solid torus, to be denoted by W_i. Then S^{conv} is the complement of the union of open solid tori embedded in S^3 :

$$S^{\text{conv}} = S^3 \setminus (\cup_{i=1}^m W_i \cup_{i=1}^m W_i' \cup_{i=1}^m W_i'')$$

where W_i' and W_i'' are the images of the solid torus W_i under the $\mathbb{Z}_3$-action on S^3, if $\omega \in \mathbb{Z}_3$ is a selected generator, then $W_i' = \omega W_i$ and $W_i'' = \omega^2 W_i$. Of course, $W_i' = \pi^{-1}(\omega U_i)$ but we do not really need this.

Let γ_i be an embedded closed curve on the torus surface ∂W_i null-homologous in W_i but not in ∂W_i. Then there is a 2-dimensional disc D_i in W_i such that $\partial D_i = \gamma_i$. Let us denote by γ_i' and γ_i'' the images of γ_i under ω and ω^2 respectively.

Remark. Note that the $3m$ curves $\gamma_i, \gamma_i', \gamma_i''$ for $i = 1, 2, \dots, m$, form a minimal set of generators in the group $\mathbb{Z}^{3m} \approx H_1(S^{conv}; \mathbb{Z})$.

The last isomorphism follows from the Alexander duality. In the Lemma after the present proof we give an elementary proof for the statement of the Remark.

Let $L = \pi^{-1}(C)$ be the preimage of C. Clearly, L is a cycle in S^3 which is invariant under ω. Its homology class in $H_1(S^{conv}; \mathbb{Z})$ can be expressed in a unique way as a linear combination of the classes of $\gamma_i, \gamma_i', \gamma_i''$, $i = 1, 2, \dots, m.$:

$$L \cong \sum \alpha_i \gamma_i + \sum \alpha_i' \gamma_i' + \sum \alpha_i'' \gamma_i''.$$

Then

$$\omega L \cong \sum \alpha_i \gamma_i' + \sum \alpha_i' \gamma_i'' + \alpha_i'' \gamma_i$$

The coefficients in these decompositions are unique. But $L = \omega L$, so we have $\alpha_i = \alpha_i' = \alpha_i''$. Let G denote the map $S^{conv} \to S^1$, obtained as the composition $F \circ \pi : S^{conv} \to V^{conv} \to S^1$.

For any closed curve c we denote by $[c]$ its homology class. The classes $G_*[\gamma_i]$, $G_*[\gamma_i']$, $G_*[\gamma_i'']$ in $H_1(S^1; \mathbb{Z})$ coincide, because $\omega \in \mathbb{Z}_3$ acts on $H_1(S^1; \mathbb{Z})$ trivially.

Hence

$$G_*[L] = \sum \alpha_i (G_*[\gamma_i] + G_*[\gamma_i'] + G_*[\gamma_i'']) = 3 \sum \alpha_i G_*[\gamma_i].$$

So the class $G_*[L] \in H_1(S^1; \mathbb{Z}) = \mathbb{Z}$ is divisible by 3. Since π gives a double cover $L \to C$ we have $G_*[L] = 2F_*[C]$, and so $F_*[C]$ is divisible by 3, and this means that the degree of the map $f|_C : C \to S^1$ is divisible by 3. $\square$

The proof of the Theorem is now complete except for the promised Lemma.

Lemma 5.2. *Let K be an oriented link in S^3, i.e. a set of disjoint embedded oriented closed curves $K_1, \dots K_n$, and let γ_i be closed curves such that $lk(\gamma_i, K_j) = \delta_{ij}$, where lk denotes the linking number, and δ_{ij} the Kronecker δ. Then the curves γ_i form a minimal set of generators in $H_1(S^3 \setminus K; \mathbb{Z})$. Moreover if we denote by φ the map $H_1(S^3 \setminus K; \mathbb{Z}) \to \mathbb{Z}^n$ associating to the homology class of a curve c in $S^3 \setminus K$ the vector of linking numbers $\varphi_i([c]) = lk(c, \gamma_i)$, $\varphi([c]) = (\varphi_1([c]), \dots, \varphi_n([c]))$ then φ is an isomorphism.*

Proof. By definition $\varphi([\gamma_i]) = (0, \dots, 0, 1, 0 \dots, 0)$ (digit 1 is at the i-th place) and so φ is surjective. If D is a compact surface in S^3 such that its boundary is c, and D is transverse to each K_i, then there is another surface D' with the same boundary and disjoint from K. Therefore $[c]$ is zero in $H_1(S^3 \setminus K; \mathbb{Z})$ and so φ is injective. The construction of the surface D' goes by the following procedure. Take two (transverse or even orthogonal) intersection points of D with a K_i of opposite signs and neighbouring in the sense that (at least) one of the arcs of K_i between these two intersection points does not contain any more intersection points. Call this arc "empty". Now omit small disks of radius ε centered at these two intersection points from D and add a tube of radius ε along the "empty" arc of K_i. Thus we have a new surface having fewer intersection points with K. Repeating this procedure until we have no intersection point we get the surface D'. $\square$

6 The second proof

This proof is obtained by studying the homomorphism of the Serre spectral sequence associated with the Borel construction of S^1 (equipped with the standard $\mathbb{Z}_3$ action) to that of V^{conv}. We denote the cohomology of the group $\mathbb{Z}_3$ with $\mathbb{F}_3$ coefficients by $H^*(\mathbb{Z}_3;\mathbb{F}_3)$. It is well known that

$$H^*(\mathbb{Z}_3;\mathbb{F}_3) = \mathbb{F}_3[t] \otimes \left(\mathbb{F}_3[e]/e^2\right)$$

where $\deg t = 2$ and $\deg e = 1$, see [Hat, page 251].

Lemma 6.1. (a) $H^0(V^{conv};\mathbb{F}_3) = \mathbb{F}_3$
(b) $H^1(V^{conv};\mathbb{F}_3) = \bigoplus_{i=1}^{m} \mathbb{F}_3[\mathbb{Z}_3]$

Proof. Recall that V^{n-conv} is a set of solid tori that are permuted by the $\mathbb{Z}_3$-action, each orbit consists of three tori, their total number is denoted by $3m$. Part (a) is clear since V^{conv} is connected. Part (b) follows by the sequence of isomorphisms:

$$H^1(V^{\text{conv}};\mathbb{F}_3) \cong H_2(V, V^{\text{n-conv}};\mathbb{F}_3) \cong H_1(V^{\text{n-conv}};\mathbb{F}_3) \cong \bigoplus_{i=1}^{m} \mathbb{F}_3[\mathbb{Z}_3].$$

Here the first isomorphism holds by the Poincaré-Lefschetz duality [Mun, Theorem 70.2, page 415] , the second comes from the homology exact sequence of the pair $(V, V^{\text{n-conv}})$ since $H_1(V,\mathbb{F}_3) = 0$ and $H_2(V,\mathbb{F}_3) = 0$. The third isomorphism is clear since $V^{\text{n-conv}}$ is homotopy equivalent to the disjoint union of $3m$ circles (the notation indicates the $\mathbb{Z}_3$-action as well). $\square$

Let us consider the Serre spectral sequence of the fibration $V^{\text{conv}} \times_{\mathbb{Z}_3} E\mathbb{Z}_3 \to B\mathbb{Z}_3$. The E_2-term of this sequence is the following: $E_2^{p,q} = H^p(\mathbb{Z}_3, H^q(V^{\text{conv}},\mathbb{F}_3))$. We shall need only the first two rows of this spectral sequence, i.e. the groups $E_2^{p,0}$ and $E_2^{p,1}$. Clearly $E_2^{p,0} = \mathbb{F}_3$ by part (a) of the Lemma above.

From part (b) of the Lemma we obtain that

$$E_2^{p,1} = H^p(\mathbb{Z}_3; \bigoplus_{i=1}^{m} \mathbb{F}_3[\mathbb{Z}_3]) = \begin{cases} \bigoplus_{i=1}^{m} \mathbb{F}_3, & p = 0 \\ 0, & p \neq 0 \end{cases}.$$

Here for $p > 0$ we used the fact that

$$H^p(\mathbb{Z}_3; \mathbb{F}[\mathbb{Z}_3]) = H^p(E\mathbb{Z}_3;\mathbb{F}_3) = 0,$$

see [Hat, Proposition 3.55, page 321].

Since the differentials in the spectral sequence are $H^*(\mathbb{Z}_3;\mathbb{F}_3)$-module maps [Bre, page 247], we have $d_2^{0,1}=0$. (Indeed, if $d_2^{0,1}\neq 0$, then there exist $x\in E_2^{0,1}$ and $\alpha\in\mathbb{F}_3\backslash\{0\}$ such that $d_2^{0,1}(x)=\alpha t$. Denoting by a dot the $H^*(\mathbb{Z};\mathbb{F}_3)$-module action one has that $0\neq t\cdot(\alpha t)=t\cdot d_2^{0,1}(x)=d_2^{2,1}(t\cdot x)=d_2^{2,1}(0)=0$.) In particular the element $t\in H^2(\mathbb{Z}_3;\mathbb{F}_3)=E_2^{2,0}$ survives to the E_∞-term (left hand side in Figures 4 and 5).

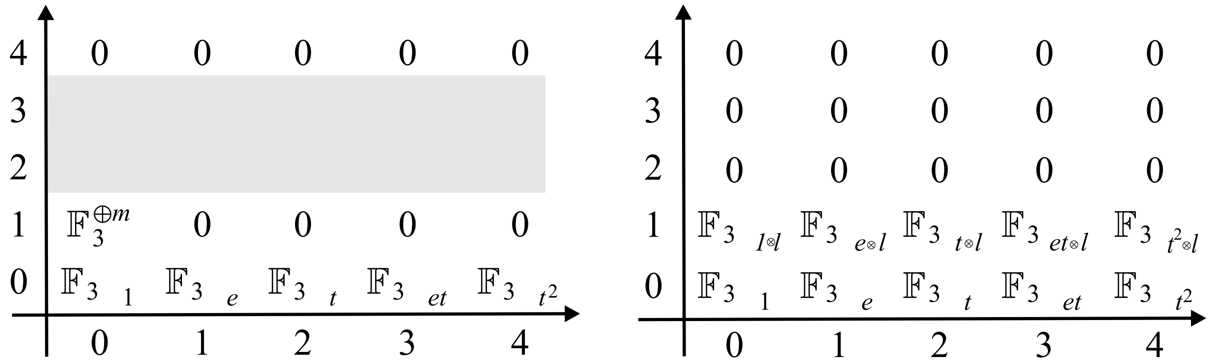

Figure 4: E_2-terms of $V^{\mathrm{conv}}\times_{\mathbb{Z}_3}E\mathbb{Z}_3$ and $S^1\times_{\mathbb{Z}_3}E\mathbb{Z}_3$ spectral sequence

Next we consider the Serre spectral sequence of the fibration $S^1\times_{\mathbb{Z}_3}E\mathbb{Z}_3\to B\mathbb{Z}_3$. The E_2-term of this sequence is

$$E_2^{p,q}=H^p(\mathbb{Z}_3;H^q(S^1;\mathbb{F}_3))=H^p(\mathbb{Z}_3;\mathbb{F}_3)\otimes H^q(S^1;\mathbb{F}_3)=\begin{cases}H^p(\mathbb{Z}_3;\mathbb{F}_3), & q=0,1\\ 0, & \text{otherwise}\end{cases}.$$

Here apriori the coefficients should be twisted, but a $\mathbb{Z}_3$-action on $H^*(S^1;\mathbb{F}_3)$ is clearly trivial, hence the coefficients are untwisted. The action of $\mathbb{Z}_3$ on S^1 is free and therefore $S^1\times_{\mathbb{Z}_3}E\mathbb{Z}_3\simeq S^1/\mathbb{Z}_3$ Hence this spectral sequence converges to $H^*(S^1\times_{\mathbb{Z}_3}E\mathbb{Z}_3;\mathbb{F}_3)=H^*(S^1;\mathbb{F}_3)$ and so all the groups the $E_\infty^{p,q}$ for $p+q>1$ must vanish. The only possibly non-zero differential is $d_2^{0,1}$, therefore $d_2^{0,1}(1\otimes l)=t\in H^*(\mathbb{Z}_3;\mathbb{F}_3)=E_2^{2,0}$. Here $l\in H^1(S^1;\mathbb{F}_3)$ denotes a generator. Thus the element $t\in H^*(\mathbb{Z}_3;\mathbb{F}_3)=E_2^{2,0}$ vanishes in the E_3-term (right hand side in Figures 4 and 5).

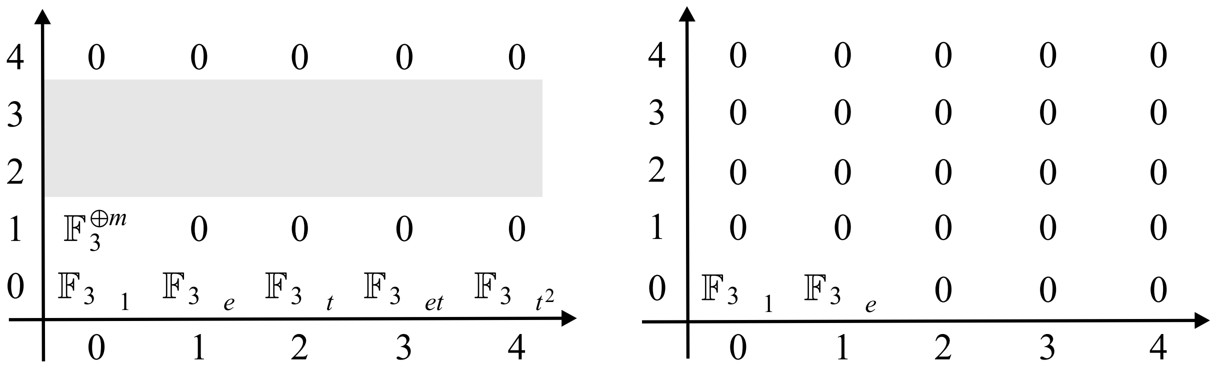

Figure 5: E_3-terms of $V^{\mathrm{conv}}\times_{\mathbb{Z}_3}E\mathbb{Z}_3$ and $S^1\times_{\mathbb{Z}_3}E\mathbb{Z}_3$ spectral sequence

Proof of Theorem 4.6. Let us assume that there is a $\mathbb{Z}_3$-map $f:V^{\mathrm{conv}}\to S^1$. Then f induces a map between

(1) Borel constructions $V^{\text{conv}} \times_{\mathbb{Z}_3} E\mathbb{Z}_3 \to S^1 \times_{\mathbb{Z}_3} E\mathbb{Z}_3$,
(2) equivariant cohomologies $f^* : H_{\mathbb{Z}_3}(S^1; \mathbb{F}_3) \to H_{\mathbb{Z}_3}(V^{\text{conv}}; \mathbb{F}_3)$, and
(3) associated Serre spectral sequences $E_r^{p,q}(f) : E_r^{p,q}(S^1; \mathbb{F}_3) \to E_r^{p,q}(V^{\text{conv}}; \mathbb{F}_3)$ such that in the 0-row

$$E_2^{p,0}(f) : \left(E_2^{p,0}(S^1; \mathbb{F}_3) = H^p(\mathbb{Z}_3; \mathbb{F}_3)\right) \to \left(E_2^{p,0}(V^{\text{conv}}; \mathbb{F}_3) = H^p(\mathbb{Z}_3; \mathbb{F}_3)\right)$$

it is the identity map.

The contradiction is obtained by tracking the behavior of the $E_r^{2,0}(f)$ images of $t \in H^2(\mathbb{Z}_3; \mathbb{F}_3)$ as r grows from 2 to 3 (see Figures 4 and 5). Explicitly,

$$E_2^{2,0}(S^1; \mathbb{F}_3) \ni t \overset{E_2^{2,0}(f)}{\longmapsto} t \in E_2^{2,0}(V^{\text{conv}}; \mathbb{F}_3),$$

and

$$E_3^{2,0}(S^1; \mathbb{F}_3) \ni 0 \overset{E_3^{2,0}(f)}{\longmapsto} t \in E_3^{2,0}(V^{\text{conv}}; \mathbb{F}_3).$$

Since the image of zero can not be different from zero we have reached a contradiction. Theorem 4.6 is proved. □

7 Proof of Lemma 3.5

We assume that μ is a nice probability measure on S^2 and ε is a small positive number. Let λ_0 denote the uniform probability measure on S^2.

We are going to construct the measure ν. We use a result of Vapnik and Chervonenkis [VC] (cf [Mat] as well) saying, in our case, that there is a finite set $X \subset S^2$ such that

$$\left| \mu(\sigma) - \frac{|\sigma \cap X|}{|X|} \right| < \frac{\varepsilon}{2} \tag{1}$$

for every sector $\sigma \subset S^2$. The proof shows that X is a random set points (of large enough size) chosen from S^2 according to μ. So we can assume that $|X| = 3n + 1$, where n is as large as we want, and further, that no three points of X are contained in a 2-dimensional plane through the origin. Now for each $x \in X$, let S_x denote the 2-dimensional sphere centered at x and of radius η. Here we choose $\eta > 0$ so small that no 2-plane through the origin intersects more than two small spheres S_x. Let λ_x denote the uniform probability measure on the small sphere S_x. We write $H^-(z)$ for the halfspace $\{v \in S^2 : zv \leq 0\}$, this is the halfspace with $H^-(z) \cap S^2 = H(z)$.

With this definition the computations will be easy since $\lambda_x(H^-(z) \cap S_x)$ is proportional to the width of $H^-(z) \cap S_x$. Precisely, $S_x \subset H^-(z)$ iff $xz \leq -\eta$ in which case, of course, $\lambda_x(H^-(z) \cap S_x) = 1$, and S_x is disjoint from $H^-(z)$ iff $xz > \eta$ and then $\lambda_x(H^-(z) \cap S_x) = 0$, and further,

$$\lambda_x(H^-(z) \cap S_x) = \frac{\eta - xz}{2\eta}, \text{ if } -\eta \leq xz \leq \eta, \tag{2}$$

Next we define a probability measure ν^* on $\mathbb{R}^3$ as

$$\nu^* = \delta \lambda_0 + \frac{1 - \delta}{3n + 1} \sum_{x \in X} \lambda_x.$$

here δ is a small positive number, for instance $\delta = n^{-2}$ will certainly do, as the reader can readily check. Finally, ν is the radial projection of ν^* onto S^2. We have to prove that ν has the required properties. Clearly, ν is a nice probability measure on S^2 since its density function is continuous and positive (that's why λ_0 is needed).

To establish properties (i) and (ii) we introduce some notation. Let $L(z)$ be the bounding hyperplane of $H^-(z)$. Set $X(z) = \{x \in X : S_x \subset H^-(z)\}$ and $m(z) = |X(z)|$. Define $\triangle(z) = \{x \in X : S_x \cap L(z) \neq \emptyset\}$. By the properties of X, $|\triangle(z)| \leq 2$ for every $z \in S^2$. Moreover, $h^*(z) = \nu(H(z)) = \nu^*(H^-(z))$ can be computed easily:

$$h^*(z) = \frac{1}{2}\delta + \frac{1-\delta}{3n+1}\left(m(z) + \sum_{x \in \triangle(z)} \frac{\eta - zx}{2\eta}\right). \tag{3}$$

We check condition (i) first. Every sector σ is the intersection or the union of two hemispheres $H(z_1)$ and $H(z_2)$. We check the case $\sigma = H(z_1) \cap H(z_2)$, and then (i) follows for unions as well by considering the complement of σ. It is evident that $X(z_1) \cap X(z_2) \subset X \cap \sigma$. Also, these sets differ by at most four elements because $L(z_i)$ intersects at most two small spheres. Consequently

$$\left|\nu(\sigma) - \frac{|X \cap \sigma|}{3n+1}\right| \leq \frac{1}{2}\delta + \frac{4\delta}{3n+1} < \frac{\varepsilon}{2},$$

if n is large enough and δ is small enough. This, together with inequality (1) implies condition (i).

Finally we go for condition (ii). We will show that $h^{*-1}(1/3)$ consists of circular arcs. With each arc we associate a pair $(Y, \triangle)$ where both Y and $\triangle$ are subsets of X. For different arcs, the associated pairs $(Y, \triangle)$ will be different. This will prove that there are finitely many circular arcs in $h^{*-1}(1/3)$. We will show further that these arcs are internally disjoint and that each endpoint of an arc coincides with a uniquely determined endpoint of another, also uniquely determined, circular arc. This is what is needed for condition (ii).

Suppose $h^*(z) = 1/3$. We claim that $\triangle(z)$ contains at least one element, a say, with $-\eta < az < \eta$. Indeed, otherwise equation (3) implies that

$$\frac{1}{2}\delta + \frac{1-\delta}{3n+1}m(z) = \frac{1}{3}$$

which has no solution with $m(z)$ an integer. Since $|\triangle(z)| \leq 2$ for every $z \in S^2$, $\triangle(z)$ has one or two elements.

Assume first that $h^*(z_0) = 1/3$ and $\triangle(z_0)$ consists of a single element $a \in X$. Of course, $-\eta < az_0 < \eta$. Then, in a small neighbourhood of z_0, $X(z) = X(z_0)$ and $\triangle(z) = \triangle(z_0)$. Thus equation (3) holds in this neighbourhood if and only if $az = az_0$. This is the intersection of S^2 with the plane $az = az_0$, which is clearly a circular arc. This circular arc belongs to $h^{*-1}(1/3)$ as long as $X(z)$ and $\triangle(z)$ and az remain the same. Let $A(Y, \triangle)$ denote this (open) arc where $Y = X(z_0)$ and $\triangle = \triangle(z_0)$, here $(Y, \triangle)$ is the pair associated with the arc under consideration. Of course, $Y = X(z)$ and $\triangle = \triangle(z)$ for every $z \in A(Y, \triangle)$. It is clear that for distinct arcs of the type $|\triangle(z)| = 1$, the associated pairs are also distinct. So there are finitely many such arcs. It is also clear that two such arcs have no point in common.

At an endpoint z of the arc $A(Y, \triangle)$ a small sphere, say S_b, becomes tangent to $L(z)$, and $\triangle(z)$ will have two elements. Note that $S_a \neq S_b$ since $az = az_0$ for all $z \in A(Y, \triangle)$ while $bz = \pm\eta$ when S_b is tangent to $L(z)$.

Assume, next, that $h^*(z_0) = 1/3$ and $\triangle(z_0)$ consists of two elements, a and b say, and $-\eta < az_0, bz_0 < \eta$. Again, for $z \in S^2$ in a small neighbourhood of z_0, $X(z) = X(z_0)$, $\triangle(z) = \triangle(z_0)$. Consequently equation (3) holds in this neighbourhood if and only if $(a+b)z = (a+b)z_0$. This is again a circular arc which belongs to $h^{*-1}(1/3)$ as long as $X(z)$ and $\triangle(z)$ and az remain the same. Let $A(Y, \triangle)$ denote this (open) arc where $Y = X(z_0)$ and $\triangle = \triangle(z_0)$, and let $(Y, \triangle)$ be the pair associated with this arc. Again, $Y = X(z)$ and $\triangle = \triangle(z)$ for every $z \in A(Y, \triangle)$. It is clear that for distinct arcs of the type $|\triangle(z)| = 2$, the associated pairs are also distinct. So there are finitely many such arcs. It is also clear that two arcs of this type have no point in common, and, further, that an arc of this type, and another one of type $|\triangle| = 1$ are disjoint.

At an endpoint z of the arc $A(Y, \triangle)$ some small sphere becomes tangent to $L(z)$. This sphere must be either S_a or S_b since otherwise $\triangle(z)$ would contain three elements of X. It is not hard to see that at one endpoint S_a, and at the other endpoint S_b, becomes tangent to the corresponding plane $L(z)$.

The remaining case is when $h^*(z_0) = 1/3$ and $\triangle(z_0) = \{a, b\}$ and for one element, say $b \in \triangle(z_0)$, S_b is tangent to $L(z_0)$. The reader will have no difficulty checking that such a z_0 is the endpoint of exactly two circular arcs: one of them is $A(Y_1, \{a\})$ and the other one is $A(Y_2, \{a, b\})$ where $Y_1 = Y_2 = X(z_0)$ if $b \notin X(z_0)$ and $Y_1 = X(z_0)$ and $Y_2 = X(z_0) \setminus \{b\}$ if $b \in X(z_0)$. $\square$

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