SYMMETRICALLY APPROXIMATELY CONTINUOUS FUNCTIONS, CONSISTENT DENSITY THEOREMS, AND FUBINI TYPE INEQUALITIES

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ABSTRACT. Using the continuum hypothesis, Sierpiński constructed a non-measurable function f such that $\{h : f(x + h) \neq f(x - h)\}$ is countable for every x. Clearly, such a function is symmetrically approximately continuous everywhere.

Here we to show that Sierpiński's example cannot be constructed in ZFC. Moreover, we show it is consistent with ZFC that if a function is symmetrically approximately continuous almost everywhere, then it is measurable.

1. INTRODUCTION

The problem we address in this paper concerns the measurability of symmetrically approximately continuous functions.

It is well known that every approximately continuous function is measurable; in fact, a function is measurable if and only if it is approximately continuous almost everywhere.

A real function f is called symmetrically continuous at the point x if
$$\lim_{h \to 0} [f(x + h) - f(x - h)] = 0.$$

It was proved by I. N. Pesin [9] and, independently, by D. Preiss [10] (see also [12, Theorem 2.26]) that if the function f is symmetrically continuous everywhere, then f is measurable (even continuous almost everywhere). This result was improved by J. Uher [13] who showed that if f is symmetrically continuous almost everywhere, then f is necessarily continuous almost everywhere (and hence, measurable).

We call the function f symmetrically approximately continuous at the point x if $\lim \mathrm{app}_{h \to 0}[f(x + h) - f(x - h)] = 0$; that is, if there exists a measurable set H having a density point at zero such that $\lim_{h \to 0, \ h \in H}[f(x + h) - f(x - h)] = 0$. Now the results above would suggest that if a function f is symmetrically approximately continuous everywhere (or even almost everywhere), then f is necessarily measurable. However, as Sierpiński showed in [11], this is not the case. Using the continuum hypothesis, Sierpiński constructed a nonmeasurable function f such that $\{h : f(x + h) \neq f(x - h)\}$ is countable for every x. Clearly, such a function is

symmetrically approximately continuous (even symmetrically approximately differentiable) everywhere. (For related results and problems also involving symmetric approximate differentiability, see [1] and [12].)

Our aim is to show that Sierpiński's example cannot be constructed in ZFC. Moreover, it is consistent with ZFC that if a function is symmetrically approximately continuous almost everywhere, then it is measurable. What we actually prove is that the inequality $\operatorname{shr}(\mathcal{N}) < \operatorname{cov}(\mathcal{N})$ implies our statement (Theorem 12). Then the consistency result follows from the consistency of $\operatorname{shr}(\mathcal{N}) < \operatorname{cov}(\mathcal{N})$, which is well known (see [8] or [6]).

Every argument proving the continuity or measurability of a symmetrically (approximately) continuous function f must show that there is a large set on which the oscillation of f is small. Usually this is done as follows: we fix an $\varepsilon > 0$ and define $S = \{(x, x+h) : |f(x+h) - f(x-h)| < \varepsilon\}$. If f is symmetrically continuous at x, then $S_x = \{y : (x, y) \in S\}$ is a neighbourhood of x; if f is symmetrically approximately continuous at x, then S_x has density 1 at x. Then we show that there is a large set E such that any two points x, $y \in E$ can be joined by a k-chain in the following sense: there are points $x = x_0, x_1, \ldots, x_k = y$ and $p_1, \ldots, p_k$ such that $x_{i-1} + x_i = 2p_i$ and $(p_i, x_{i-1}) \in S$ for every $i = 1, \ldots, k$. It is clear that the existence of these k-chains implies that $|f(x) - f(y)| < k \cdot \varepsilon$ for every x, $y \in E$, and thus the oscillation of f on E is at most $2k \cdot \varepsilon$. Each of the papers [9], [10], [13], and [5] contains an argument of this kind. (See also [12], where these arguments are discussed as 'covering theorems'.) Our proof also follows this line: we show, under $\operatorname{shr}(\mathcal{N}) < \operatorname{cov}(\mathcal{N})$, that if S_x has density 1 for almost every x, then there is a measurable set F of positive measure such that every two points of F can be joined by an 8-chain (Theorem 11). A similar but weaker statement was proved by C. Freiling in a more general setting in [3].[1] Freiling's proof shows, also under $\operatorname{shr}(\mathcal{N}) < \operatorname{cov}(\mathcal{N})$, that if S_x has density 1 for every x, then there is an interval I such that every two points of I can be joined by a finite chain. However, his proof yields no uniform bound on the lengths of the chains.

One of the tools of our proof is a consistent inequality between some iterated integrals of bounded functions of two variables. Since these results might have some interest in themselves, we separated them in Section 3.

2. Notation

We denote by $\mathcal{N}$ the ideal of null sets in $\mathbb{R}$. We shall need the following classical cardinal numbers connected to $\mathcal{N}$.

$\operatorname{non}(\mathcal{N}) = \min\{|A| : A \subset \mathbb{R}, \ A \notin \mathcal{N}\}$;

$\operatorname{cov}(\mathcal{N}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{N} \ \& \ \bigcup \mathcal{F} = \mathbb{R}\}$;

$\operatorname{shr}(\mathcal{N}) = \min\{\kappa : \forall A \subset \mathbb{R} \ (A \notin \mathcal{N} \implies \exists B \subset A, \ |B| \leq \kappa, \ B \notin \mathcal{N})\}$.

In other words, $\operatorname{shr}(\mathcal{N})$ is the smallest cardinal κ such that any subset A of $\mathbb{R}$ not belonging to $\mathcal{N}$ contains a subset B such that $|B| \leq \kappa$ and $B \notin \mathcal{N}$.

The Lebesgue outer measure is denoted by λ^*. The Lebesgue inner measure of a set $A \subset \mathbb{R}$ is defined as $\lambda_*(A) = \max\{\lambda^*(H) : H \subset A \text{ is measurable}\}$. As usual, $\lambda(A)$ means the measure of the Lebesgue measurable set, A.

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[1]Freiling credits the techniques used to prove his theorem to Erdős, McGrotty, and Sierpiński. See [3] for details.

The upper and lower outer density of a set $A \subset \mathbb{R}$ at the point x are defined by

$$\overline{d}^* (A, x) = \limsup_{h \to 0+} \lambda^*(A \cap [x - h, x + h])/(2h)$$

and

$$\underline{d}^* (A, x) = \liminf_{h \to 0+} \lambda^*(A \cap [x - h, x + h])/(2h).$$

If, in these definitions, the Lebesgue outer measure λ^* is replaced by the Lebesgue inner measure λ_*, then we obtain the upper and lower inner density of $A \subset \mathbb{R}$, denoted by $\overline{d}_*(A, x)$ and $\underline{d}_*(A, x)$. If $\overline{d}^* (A, x) = \underline{d}^* (A, x)$, then we say that A has outer density at x, and define $d^* (A, x)$ as the common value of $\overline{d}^* (A, x)$ and $\underline{d}^* (A, x)$. We say that x is an outer density point of A if $d^* (A, x) = 1$. If $\overline{d}_*(A, x) = \underline{d}_*(A, x)$ holds, then we say that A has inner density at x, and define $d_*(A, x)$ as the common value of $\overline{d}_*(A, x)$ and $\underline{d}_*(A, x)$. If $d_*(A, x) = 1$, then we say that x is a density point of A.

If (X, Σ, μ) is a measure space, we shall denote by μ^* the outer measure generated by μ. That is, for each $H \subset X$, $\mu^*(H) = \inf\{\mu(A) : H \subset A, \ A \in \Sigma\}$.

Let $A \subset X$ be measurable. If $f : A \to \overline{\mathbb{R}}$, then we shall denote by $\underline{\int}_A f \, d\mu(x)$ the supremum of the integrals $\int_A g \, d\mu(x)$, where $g : A \to \overline{\mathbb{R}}$ is an arbitrary summable function such that $g \leq f$ everywhere on A. If there is no such g, then we put $\underline{\int}_A f \, d\mu(x) = -\infty$. The upper integral $\overline{\int}_A f \, d\mu(x)$ is defined analogously.

The interval $[0, 1]$ will be denoted by I. When integrating over I, we shall omit the subscript I; that is, $\underline{\int}$, $\overline{\int}$, $\int$ stand for $\underline{\int}_I$, $\overline{\int}_I$, $\int_I$.

Let Ω denote the measure space $I^{\mathbb{N}} = I \times I \times \ldots$ with the product measure ν. The generic element of Ω will be denoted by $\overline{y} = (y_1, y_2, \ldots)$, where each y_i belongs to I. Note that the measure space (Ω, ν) is isomorphic to (I, λ).

The characteristic function of a set H is denoted by χ_H. If $H \subset X \times Y$, then the sections of H are defined as $H_x = \{y \in Y : (x, y) \in H\}$ and $H^y = \{x \in X : (x, y) \in H\}$ for every $x \in X$ and $y \in Y$.

A set $H \subset I \times I$ is a *Sierpiński set* if $\lambda(H_x) = 1$ for every $x \in I$ and $\lambda(H^y) = 0$ for every $y \in I$.

Finally, a set $H \subset I \times I$ is a *weak Sierpiński set* if $\lambda(H_x) = 1$ for every $x \in I$ and

$$\lambda^*(\{y \in I : \lambda(H^y) = 0\}) > 0.$$

3. Consistent Fubini type inequalities

By a Fubini type inequality we shall mean an inequality of the form

$$(1) \qquad \int_1 \left(\int_2 f(x, y) \, dy \right) dx \leq \int_3 \left(\int_4 f(x, y) \, dx \right) dy,$$

where each of the signs $\int_i$ $(i = 1, \ldots, 4)$ stands for either $\underline{\int}$ or $\overline{\int}$. There are 16 inequalities of this form. In this section our aim is to show that some of them hold consistently for every bounded $f : I^2 \to \mathbb{R}$. More precisely, we shall prove that the following statement is consistent with ZFC: *For every bounded function*

$f : (I \times I) \to \mathbb{R}$ *the following inequalities hold:*

(2)
$$\underline{\int} \left(\underline{\int} f(x,y)\,dy \right) dx \leq \overline{\int} \left(\overline{\int} f(x,y)\,dx \right) dy,$$

(3)
$$\underline{\int} \left(\underline{\int} f(x,y)\,dy \right) dx \leq \underline{\int} \left(\overline{\int} f(x,y)\,dx \right) dy,$$

and

(4)
$$\overline{\int} \left(\underline{\int} f(x,y)\,dy \right) dx \leq \overline{\int} \left(\underline{\int} f(x,y)\,dx \right) dy.$$

Clearly, (2) is a consequence of each of (3) and (4). The consistency of (2) is closely related to a theorem of C. Freiling [2] stating that the following two statements are equivalent to each other:

(i)

(5)
$$\int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx = \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy$$

holds for every bounded function $f : I^2 \to \mathbb{R}$ for which the integrals involved in (5) exist;

(ii) *there is no Sierpiński set.*

Now the argument proving Freiling's theorem can be easily modified in order to prove that (ii) is also equivalent to the statement that (2) holds for every bounded $f : I^2 \to \mathbb{R}$. Since the nonexistence of Sierpiński sets is consistent with ZFC (see, e.g., [7]), it follows that (2) holds consistently for every bounded f.

We remark that no Fubini type inequality can be proved in ZFC. Indeed, if S is a Sierpiński set and $f = \chi_S$, then

$$1 = \int_0^1 \left(\int_0^1 f(x,y)\,dy \right) dx > \int_0^1 \left(\int_0^1 f(x,y)\,dx \right) dy = 0,$$

and thus

$$\int_1 \left(\int_2 f(x,y)\,dy \right) dx > \int_3 \left(\int_4 f(x,y)\,dx \right) dy,$$

for every choice of $\int_i$ $(i = 1, \dots, 4)$.

Before turning to the proof of the consistency of the stronger inequalities (3) and (4) we show that apart from (2), (3), and (4) there are no other consistent Fubini type inequalities.

It is well known that $[0,1]$ can be decomposed into continuum many pairwise disjoint sets of outer measure 1. Let $[0,1] = \bigcup_{x \in [0,1]} H_x$ be such a decomposition, and put $H = \{(x,y) \in I^2 : y \in H_x\}$ and $f = \chi_H$. Then $\lambda^*(H_x) = 1$ for every $x \in I$ and $\lambda^*(H^y) = 0$ $(y \in I)$, as H^y is a singleton for every $y \in I$. Thus $\overline{\int}\overline{\int} f\,dx\,dy = 0$ and $\underline{\int}\overline{\int} f\,dy\,dx = 1$. Consequently, the inequality $\underline{\int}\overline{\int} f\,dy\,dx > \overline{\int}\overline{\int} f\,dx\,dy$ holds for some bounded f in ZFC. Then, *a fortiori,* neither of the following inequalities is true for every bounded f :

(6)
$$\underline{\int}\overline{\int} f\,dy\,dx \leq \int_1 \int_2 f\,dx\,dy, \quad \overline{\int}\overline{\int} f\,dy\,dx \leq \int_1 \int_2 f\,dx\,dy,$$

for any choice of $\int_1$ and $\int_2$. This example shows that eight of the possible Fubini type inequalities are false in ZFC. Moreover, we can find other false inequalities by taking duals.

If a function f satisfies (1), then the function $g(x, y) = -f(y, x)$ satisfies

$$(7) \qquad \int_3' \left(\int_4' g(x, y)\, dy \right) dx \leq \int_1' \left(\int_2' g(x, y)\, dx \right) dy,$$

where $\int_i' = \overline{\int}$ if $\int_i = \underline{\int}$ and $\int_i' = \underline{\int}$ if $\int_i = \overline{\int}$. (That is, (7) is obtained from (1) by switching its sides and by changing $\overline{\int}$ into $\underline{\int}$ and $\underline{\int}$ into $\overline{\int}$ in every occurrence.) Let us call (7) the *dual* of (1). (Thus (4) is the dual of (3).) Clearly, if (1) holds for every bounded f, then so does its dual. It is easy to check that the inequalities (6) together with their duals exclude all but four Fubini type inequalities that can be consistent with ZFC. The remaining inequalities are (2), (3), (4), and

$$(8) \qquad \overline{\int} \left(\underline{\int} f(x, y)\, dy \right) dx \leq \underline{\int} \left(\overline{\int} f(x, y)\, dx \right) dy.$$

We show that (8) is also false in ZFC. Let non $(\mathcal{N}) = \kappa$, and let $A \subset I$ be a set such that $\lambda^*(A) > 0$ and $|A| = \kappa$. Replacing A by $I \cap \bigcup_{q \in \mathbb{Q}}(A + q)$ if necessary, we may assume that $\lambda^*(A) = 1$. Let $A = \{x_\alpha : \alpha < \kappa\}$ be an enumeration of A, and put

$$H = \{(x_\alpha, x_\beta) : \alpha < \beta < \kappa\} \cup (A \times (I \setminus A)).$$

If $y = x_\beta \in A$, then $H^y = \{x_\alpha : \alpha < \beta\}$. Consequently, $|H^y| < \kappa$ and $\lambda^*(H^y) = 0$ by the definition of κ. Since $\lambda^*(A) = 1$, this implies $\underline{\int}\,\overline{\int} \chi_H\, dx\, dy = 0$. On the other hand, if $x = x_\alpha \in A$, then $I \setminus H_x = \{x_\beta : \beta \leq \alpha\}$. Thus $\lambda^*(I \setminus H_x) = 0$ and $\lambda_*(H_x) = 1$ for every $x \in A$. Therefore $\overline{\int}\,\underline{\int} \chi_H\, dy\, dx = 1$ showing that (8) is not satisfied by χ_H. Summing up: no Fubini type inequalities can be consistent with ZFC apart from (2), (3), and (4). The rest of this section will be devoted to the proof of consistency of these inequalities.

In the next three lemmas we assume that (X, Σ, μ) is a measure space, $A \subset X$ is measurable, and $f : A \to \overline{\mathbb{R}}$ is arbitrary.

Lemma 1. *If $\underline{\int}_A f\, d\mu(x)$ is finite, then there is a summable function $g : A \to \overline{\mathbb{R}}$ such that $g \leq f$ on A, $\int_A g\, d\mu(x) = \underline{\int}_A f\, d\mu(x)$, and for every $\varepsilon > 0$, $\mu^*(\{x \in A : f(x) < g(x) + \varepsilon\}) = \mu(A)$.*

Proof. Let $\underline{\int}_A f\, d\mu(x) = a$, and choose summable functions g_n such that $g_n \leq f$ and $\int_A g_n d\mu(x) > a - (1/n)$. Put $g = \lim_n \max(g_1, \ldots, g_n)$. Then $g \leq f$, and it is easy to see by the monotone convergence theorem that g is summable and $\int_A g\, d\mu(x) = a$. Let $\varepsilon > 0$, and suppose that $\mu^*(B) < \mu(A)$, where $B = \{x \in A : f(x) < g(x) + \varepsilon\}$. Then there is a measurable set $C \subset A$ of positive measure such that $f \geq g + \varepsilon$ on C. Putting $h = g + \varepsilon \cdot \chi_C$, we find $h \leq f$ and $\int_A h\, d\mu(x) > a$, which contradicts the definition of a. $\qquad\square$

Lemma 2. *For every $f : A \to \overline{\mathbb{R}}$ and $g : A \to \overline{\mathbb{R}}$ we have*
   (1) $\underline{\int}_A (f + g)\, d\mu(x) \geq \underline{\int}_A f\, d\mu(x) + \underline{\int}_A g\, d\mu(x)$ *and*
   (2) $\overline{\int}_A (f + g)\, d\mu(x) \leq \overline{\int}_A f\, d\mu(x) + \overline{\int}_A g\, d\mu(x)$,
whenever the right-hand sides make sense.

Proof. This follows directly from the definitions. $\qquad\square$

Lemma 3. *For every sequence of nonnegative functions $f_n : A \to [0, \infty]$ we have*

$$\overline{\int}_A \liminf_n f_n \, d\mu(x) \leq \liminf_n \overline{\int}_A f_n \, d\mu(x).$$

Proof. By the dual of Lemma 1 there are measurable functions $g_n : A \to [0, \infty]$ such that $f_n \leq g_n$ and $\overline{\int}_A f_n \, d\mu(x) = \int_A g_n \, d\mu(x)$ for every n (if $\overline{\int}_A f_n \, d\mu(x) = \infty$, then we take $g_n \equiv \infty$). Then by Fatou's lemma we obtain

$$\overline{\int}_A \liminf_n f_n \, d\mu(x) \leq \overline{\int}_A \liminf_n g_n \, d\mu(x) = \int_A \liminf_n g_n \, d\mu(x)$$

$$\leq \liminf_n \int_A g_n \, d\mu(x) = \liminf_n \overline{\int}_A f_n \, d\mu(x). \qquad \square$$

Theorem 4. *For every probability space (X, Σ, μ) exactly one of the following statements is true.*

  (i) *There is a measurable set of positive measure $A \subset X$ and a set $S \subset A \times I$ such that $\lambda(S_x) = 1$ for every $x \in A$ and $\lambda^*(\{y \in I : \mu^*(S^y) < \mu(A)\}) > 0$.*
  (ii) *For every function $f : X \times I \to [0, \infty]$ we have*

(9)
$$\underline{\int}_X \left(\underline{\int} f(x, y) \, dy \right) d\mu(x) \leq \underline{\int} \left(\overline{\int}_X f(x, y) d\mu(x) \right) dy.$$

Proof. Suppose (i), and put $f = \chi_S$. Then the left-hand side of (9) equals $\mu(A)$. Let $C_n = \{y \in I : \mu^*(S^y) < \mu(A) - \frac{1}{n}\}$. If n is large enough, then $\lambda^*(C_n) > 0$, and thus the right-hand side of (9) is at most

$$(1 - \lambda^*(C_n)) \cdot \mu(A) + \lambda^*(C_n) \cdot (\mu(A) - (1/n)) < \mu(A),$$

violating (ii).

Now suppose that (i) does not hold, and let $f : X \times I \to [0, \infty]$ be arbitrary. We prove (9). Suppose it is not true; that is,

(10)
$$\underline{\int}_X \left(\underline{\int} f(x, y) \, dy \right) d\mu(x) > \underline{\int} \left(\overline{\int}_X f(x, y) \, d\mu(x) \right) dy.$$

We shall construct two sets, $A \subset X$ and $S \subset A \times I$, satisfying the conditions of (i). Since we assumed that (i) is false, this will complete the proof. We separate this part of the argument in the next lemma stating slightly more than what we actually need.

Lemma 5. *Let $f : X \times I \to [0, \infty]$ and suppose that*

(11)
$$\underline{\int}_X \left(\underline{\int} f(x, y) \, dy \right) d\mu(x) > K \cdot \underline{\int} \left(\overline{\int}_X f(x, y) \, d\mu(x) \right) dy,$$

where $K > 1$. Then there is a measurable set of positive measure $A \subset X$ and a set $S \subset A \times I$ such that $\lambda(S_x) = 1$ for every $x \in A$ and $\lambda^*(\{y \in I : \mu^*(S^y) < \mu(A)/K\}) > 0$.

Proof. Let M denote the right-hand side of (10), and for each $x \in X$ put $g(x) = \underline{\int} f(x, y) \, dy$. Since $\underline{\int}_X g \, d\mu(x) > K \cdot M$, there is a summable function $h : X \to \overline{\mathbb{R}}$ such that $0 \leq h(x) \leq g(x)$ for every $x \in X$, and $\int_X h \, d\mu(x) > K \cdot M$. Since $\int_X h \, d\mu(x)$ equals the supremum of the integrals of all step functions less than h, we may assume that h itself is a step function; that is, $h = \sum_{i=1}^N c_i \chi_{A_i}$, where

$c_1, \ldots, c_N \geq 0$ and $A_1 \cup \ldots \cup A_N$ is a partition of X into measurable sets. Note that

$$(12) \qquad \underline{\int} f(x, y)\, dy = g(x) \geq h(x) = c_i \ \text{ for every } \ x \in A_i \ (i = 1, \ldots, N).$$

It is easy to check, using the measurability of the sets A_i, that

$$\sum_{i=1}^{N} \overline{\int}_{A_i} f(x, y)\, d\mu(x) = \overline{\int}_{X} f(x, y)\, d\mu(x)$$

for every y. Therefore, by Lemma 2 and by (11) we obtain

$$\sum_{i=1}^{N} K \cdot \underline{\int} \left(\overline{\int}_{A_i} f\, d\mu(x) \right) dy \leq K \cdot \underline{\int} \left(\overline{\int}_{X} f\, d\mu(x) \right) dy$$

$$= K \cdot M < \int_{X} h\, d\mu(x) = \sum_{i=1}^{N} c_i \mu(A_i).$$

Hence there is an index j such that

$$(13) \qquad K \cdot \underline{\int} \left(\overline{\int}_{A_j} f\, d\mu(x) \right) dy < c_j \mu(A_j),$$

and since $f \geq 0$, we have $c_j > 0$ and $\mu(A_j) > 0$. Furthermore, by replacing f by f/c_j, we may assume that $c_j = 1$. Fix an $\varepsilon > 0$ such that

$$K \cdot \underline{\int} \left(\overline{\int}_{A_j} f(x, y)\, d\mu(x) \right) dy < (1 - \varepsilon) \cdot \mu(A_j),$$

and define $u : I \to \overline{\mathbb{R}}$ by $u(y) = \overline{\int}_{A_j} f(x, y)\, d\mu(x)$. By Lemma 1, there is a non-negative summable function $k : I \to \overline{\mathbb{R}}$ such that $K \cdot \int k\, dy < (1 - \varepsilon) \cdot \mu(A_j)$ and $\lambda^*(D) = 1$, where $D = \{ y \in I : u(y) < k(y) + (\varepsilon \cdot \mu(A_j))/K \}$.

Now we define $F : A_j \times \Omega \to \overline{\mathbb{R}}$ by

$$F(x, \overline{y}) = \liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=1}^{n} f(x, y_i)$$

and put

$$U = \{ (x, \overline{y}) \in A_j \times \Omega : F(x, \overline{y}) \geq 1 \}.$$

For every $x \in A_j$ we have $\underline{\int} f(x, y)\, dy \geq 1$ by (12) and thus, by the strong law of large numbers, $F(x, \overline{y}) \geq 1$ for a.e. $\overline{y} \in \Omega$. Therefore $\nu(U_x) = 1$ (i.e. U_x is measurable and of full measure in Ω) for every $x \in A_j$. Applying the strong law of large numbers again, we find that the set

$$E = \left\{ \overline{y} \in \Omega : \lim_{n} \frac{1}{n} \cdot \sum_{i=1}^{n} k(y_i) = \int k\, dy < \frac{(1 - \varepsilon) \cdot \mu(A_j)}{K} \right\}$$

is a set of full measure in Ω. Since D is of full outer measure in I, it follows from [4, 254 L Theorem, p. 249] that $D^{\mathbb{N}} = D \times D \times \ldots$ and hence, $E \cap D^{\mathbb{N}}$ are of full

outer measure in Ω. If $\overline{y} \in E \cap D^{\mathbb{N}}$, then, by Lemmas 2 and 3, we have

$$
\overline{\int}_{A_j} F(x, \overline{y}) \, d\mu(x) = \overline{\int}_{A_j} \left(\liminf_n \frac{1}{n} \cdot \sum_{i=1}^n f(x, y_i) \right) d\mu(x)
$$

$$
\leq \liminf_n \overline{\int}_{A_j} \left(\frac{1}{n} \cdot \sum_{i=1}^n f(x, y_i) \right) d\mu(x)
$$

$$
\leq \liminf_n \frac{1}{n} \cdot \sum_{i=1}^n \overline{\int}_{A_j} f(x, y_i) \, d\mu(x)
$$

$$
\leq \liminf_n \frac{1}{n} \cdot \sum_{i=1}^n (k(y_i) + (\varepsilon/K) \cdot \mu(A_j))
$$

$$
< (1 - \varepsilon) \cdot (\mu(A_j)/K) + (\varepsilon/K) \cdot \mu(A_j) = \mu(A_j)/K.
$$

Hence, for every $\overline{y} \in E \cap D^{\mathbb{N}}$

$$
\mu^*(U^{\overline{y}}) = \mu^*(\{x \in A_j : F(x, \overline{y}) \geq 1\}) < \mu(A_j)/K.
$$

That is, U has the desired property but in the space $X \times \Omega$.

It now follows from the fact that Ω is isomorphic to I that there is a set $S \subset X \times I$ such that $\lambda(S_x) = 1$ for every $x \in A_j$, and $\mu^*(S^y) < \mu(A_j)/K$ for every y belonging to a set of positive outer measure. This then completes the proof of Lemma 5 and hence the proof of Theorem 4. $\qquad \square$

Theorem 6. *The following statements are equivalent.*

(i) *There is no weak Sierpiński set.*

(ii) *For every nonnegative function $f : (I \times I) \to [0, \infty]$ we have*

$$
\underline{\int} \left(\underline{\int} f(x, y) \, dy \right) dx \leq \underline{\int} \left(\overline{\int} f(x, y) dx \right) dy.
$$

Proof. (ii)$\Longrightarrow$ (i): Obvious.

(i)$\Longrightarrow$ (ii): Suppose (ii) is not true. Then, by Theorem 4, there is a measurable set of positive measure $A \subset I$ and a set $U \subset A \times I$ such that $\lambda(U_x) = 1$ for every $x \in A$ and $\lambda^*(\{y \in I : \lambda^*(U^y) < \lambda(A)\}) > 0$. Since the measure space A equipped with the normalized measure $\lambda/\lambda(A)$ is isomorphic to (I, λ), we can find a set $T \subset I \times I$ such that $\lambda(T_x) = 1$ for every $x \in I$ and $\lambda^*(\{y \in I : \lambda^*(T^y) < 1\}) > 0$.

Now we define $S = \{(x, y) \in I \times I : (x + r, y) \in T$ for every $r \in [-x, 1 - x] \cap \mathbb{Q}\}$. Then $\lambda(S_x) = 1$ for every $x \in I$. Moreover, if $\lambda^*(T^y) < 1$, then $\lambda_*(I \setminus T^y) > 0$ so that $I \setminus T^y$ contains a measurable set of positive measure. Hence for each such y, $\lambda(S^y) = 0$. Thus S is a weak Sierpiński set. This, however, contradicts our assumption, which completes the proof. $\qquad \square$

Corollary 7. *It is consistent with ZFC that* (2), (3), *and* (4) *hold true for every bounded $f : I^2 \to \mathbb{R}$.*

Proof. Since the nonexistence of weak Sierpiński sets is consistent with ZFC (see [7]), it follows from the previous theorem that (3) holds consistently for every bounded f. Then, by duality, the same is true for (4), while (2) is a consequence of (3). $\qquad \square$

4. Consistent density theorems

Lemma 8. *Let (X, Σ, μ) be a measure space. Then exactly one of the following two statements holds.*

   (i) *There exists a set $F \subset X \times [0,1]$ such that $\lambda(F_x) = 1$ for every $x \in X$, and $\lambda^*(\{y \in [0,1] : \mu(F^y) = 0\}) > 0$.*
   (ii) *Whenever $H \subset X \times \mathbb{R}$ is such that $d_*(H_x, 0) = 1$ for every $x \in X$, then there is a $\delta > 0$ such that*

$$\underline{d}_*(\{y \in \mathbb{R} : \mu^*(H^y) > \delta\}, 0) > 0.$$

Proof. Suppose $F \subset X \times [0,1]$ is such that $\lambda(F_x) = 1$ for every $x \in X$, and $\lambda^*(\{y : \mu(F^y) = 0\}) > 0$. Let $y_0 \in (0,1)$ be an outer density point of $\{y : \mu(F^y) = 0\}$, and put $H = \{(x, y - y_0) : (x,y) \in F\}$. It is clear that H violates (ii), and thus (i) $\Longrightarrow \neg$(ii).

Next suppose that statement (ii) is false. We prove that in this case (i) must hold. Let $H \subset X \times \mathbb{R}$ be such that $d_*(H_x, 0) = 1$ for every $x \in X$, and let $\underline{d}_*(\{y \in \mathbb{R} : \mu^*(H^y) > 1/n\}, 0) = 0$ for every $n = 1, 2, \ldots$. Then we can select $0 < h_n < 1/n$ such that

$$\lambda_*(\{y \in [0, h_n] : \mu^*(H^y) > 1/n\}) < h_n/2^n.$$

Put $B_n = \{y \in [0,1] : \mu^*(H^{h_n y}) \leq 1/n\}$; then $\lambda^*(B_n) > 1 - 2^{-n}$ for every n. Recall that $\Omega = I \times I \times \ldots$, and ν is the product measure on Ω. If $B = B_1 \times B_2 \times \ldots$, then it follows from [4, 254 L Theorem, p. 249] that

$$\nu^*(B) = \prod_{n=1}^{\infty} \left(1 - \frac{1}{2^n}\right) > 0.$$

We define $f_n : X \times I \to \{0, 1\}$ by

$$f_n(x, y) = \begin{cases} 1 & \text{if } (x, h_n y) \in H, \\ 0 & \text{otherwise}, \end{cases}$$

and define $\Phi : X \times \Omega \to [0, +\infty]$ by

$$\Phi(x, \overline{y}) = \liminf_{n \to \infty} \frac{1}{n} \cdot \sum_{i=1}^{n} f_i(x, y_i).$$

Fix $\overline{y} \in B$. Then for every $n \in \mathbb{N}$, $y_n \in B_n$ and thus

$$\overline{\int} f_n(x, y_n) \, d\mu(x) = \mu^* \left(H^{h_n \cdot y_n}\right) \leq \frac{1}{n}.$$

Therefore, by Lemmas 2 and 3, we have

$$\overline{\int} \Phi(x, \overline{y}) \, d\mu(x) \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \overline{\int} f_i(x, y_i) \, d\mu(x)$$

$$\leq \liminf_{n \to \infty} \frac{1}{n} \left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right) = 0.$$

Thus, for every $\overline{y} \in B$ we have $\Phi(x, \overline{y}) = 0$ for μ-a.e. $x \in X$.

On the other hand, for every $x \in X$ we have $\Phi(x, \overline{y}) = 1$ for ν-a.e. $\overline{y} \in \Omega$. Indeed, from $d_*(H_x, 0) = 1$ it follows that

$$\underline{\int} f_n(x, y)\, dy = \frac{1}{h_n} \cdot \lambda_*(H_x \cap [0, h_n]) \to 1 \quad (n \to \infty).$$

Thus by the strong law of large numbers we obtain that

$$\lim_{n \to \infty} \frac{f_1(x, y_1) + \ldots + f_n(x, y_n)}{n} = 1$$

for ν-a.e. $\overline{y} \in \Omega$; that is, $\Phi(x, \overline{y}) = 1$ for ν-a.e. $\overline{y}$. Now we put $F = \{(x, \overline{y}) \in X \times \Omega : \Phi(x, \overline{y}) = 1\}$. Then $\nu(\Omega \setminus F_x) = 0$ for every $x \in X$, and $\mu^*(F^{\overline{y}}) = 0$ for every $\overline{y} \in B$. Since $\nu^*(B) > 0$, we obtain the statement of (i) with Ω in place of I. As the measure spaces (Ω, ν) and (I, λ) are isomorphic, we find that (i) holds, which completes the proof. $\qquad\square$

Theorem 9. *Suppose that there is no weak Sierpiński set. Then, whenever $H \subset I \times \mathbb{R}$ is such that $d_*(H_x, 0) = 1$ for every $x \in I$, we have*

$$(14) \qquad\qquad d_*(\{y : \lambda^*(H^y) > 1 - \delta\}, 0) = 1$$

for every $\delta > 0$.

Proof. Let H be as in the theorem, and let $\delta > 0$ be arbitrary. Suppose (14) is false. Then there is an $\varepsilon > 0$ and there is a sequence $h_n > 0$ such that $h_n \to 0$ and

$$\lambda_*(\{y \in [0, h_n] : \lambda^*(H^y) > 1 - \delta\}) < (1 - \varepsilon) \cdot h_n$$

for every n. Put $B_n = \{y \in I : \lambda^*(H^{h_n y}) \leq 1 - \delta\}$; then $\lambda^*(B_n) \geq \varepsilon$ for every n. We define the functions f_n and Φ as in the proof of Lemma 8 (with I in place of X). Then as in the proof of Lemma 8, it can be shown that for every $x \in I$, we have $\Phi(x, \overline{y}) = 1$ for ν-a.e. $\overline{y} \in \Omega$. If there is no weak Sierpiński set, then, by Theorem 6, we have

$$(15) \qquad \underline{\int} \left(\underline{\int}_\Omega \Phi(x, \overline{y})\, d\nu(\overline{y}) \right) dx \leq \underline{\int}_\Omega \left(\overline{\int} \Phi(x, \overline{y})\, dx \right) d\nu(\overline{y}).$$

Since $\Phi(x, \overline{y}) = 1$ for ν-a.e. $\overline{y} \in \Omega$, the left-hand side of (15) equals 1. Then the right-hand side also equals 1, and thus we have $\overline{\int} \Phi(x, \overline{y})\, dx = 1$ for ν-a.e. $\overline{y}$. Then, by Lemma 3, we obtain

$$(16) \qquad 1 = \overline{\int} \Phi(x, \overline{y})\, dx \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \overline{\int} f_i(x, y_i)\, dx = \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \lambda^*(H^{h_i y_i})$$

for ν-a.e. $\overline{y}$. Now we define $b_i = \chi_{B_i}$, and set

$$(17) \qquad\qquad D = \{\overline{y} \in \Omega : \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} b_i(y_i) \geq \varepsilon\}.$$

We claim that $\nu^*(D) = 1$. If the sets B_i were measurable, then this would follow from the strong law of large numbers. In the general case we can argue as follows. Let M_i be a measurable hull of B_i, and consider the measure spaces (X_i, Σ_i, λ), where $X_i = (I \setminus M_i) \cup B_i$, and Σ_i is the σ-algebra of all sets of the form $X_i \cap A$, where A is Lebesgue measurable. Then $B_i \in \Sigma_i$. Let $\Omega_1 = X_1 \times X_2 \times \ldots$, and let ν_1 denote the product measure on Ω_1. It is easy to check that $\nu^*(A) = \nu_1(A)$ for

every ν_1-measurable $A \subset \Omega_1$. Since, by the strong law of large numbers, $\nu_1(D) = 1$, it follows that $\nu^*(D) = 1$.

But if $y \in B_i$, then $\lambda^*\left(H^{h_i y}\right) \leq 1 - \delta$, and therefore

$$(18) \qquad \lambda^*\left(H^{h_i y}\right) \leq 1 - \delta \cdot b_i(y)$$

for every $y \in I$ and $i = 1, 2, \ldots$. Let $\overline{y} \in D$ be arbitrary. By (18) and (17) we have

$$\liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \lambda^*\left(H^{h_i y_i}\right) \leq \left[1 - \delta \cdot \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} b_i(y_i)\right] \leq 1 - \delta \cdot \varepsilon.$$

Since $\nu^*(D) = 1$, this contradicts the fact that (16) holds for ν-a.e. $\overline{y}$, which completes the proof. $\qquad \square$

Theorem 10. *Suppose* $\mathrm{shr}\,(\mathcal{N}) < \mathrm{cov}\,(\mathcal{N})$. *Let* $E_1, E_2 \subset \mathbb{R}$ *be measurable sets of positive measure, and let* $H \subset E_1 \times E_2$ *be such that* $\lambda_*(H_x) > 0$ *for every* $x \in E_1$. *Then there is a* $y \in E_2$ *and there is a* $\delta > 0$ *such that*

$$(19) \qquad \underline{d}_* \left(\{z \in E_2 : \lambda^*(H^y \cap H^z) > \delta\}, y\right) > 0.$$

Proof. Let E_1, E_2, H be as in the theorem. We may assume, by deleting a set of measure zero from E_2, that E_2 is nonempty and d-open; i.e. that every point of E_2 is a density point of E_2. Also, by taking a suitable subset of H, we may assume that H_x is nonempty and d-open for every $x \in E_1$. Suppose that the conclusion of the theorem is not true; that is,

$$(20) \qquad \underline{d}_* \left(\{z \in E_2 : \lambda^*(H^y \cap H^z) > \delta\}, y\right) = 0$$

for every $y \in E_2$ and $\delta > 0$.

First we show that, for every $y \in E_2$, H^y can be covered by $\mathrm{shr}\,(\mathcal{N})$-many null sets. This is obvious if $\lambda^*(H^y) = 0$. Let $y \in E_2$ be fixed such that $\lambda^*(H^y) > 0$, and put $X = H^y$. Since H_x is d-open for every $x \in X$, we have $d(H_x, y) = 1$ for every $x \in X$.

We apply Lemma 8 to the measure space $(X, \lambda^*|X)$ (the restriction of λ^* to the subsets of X) and for the set $\{(x, z) : x \in X, \ (x, z + y) \in H\}$. Since, by (20), the statement (ii) of Lemma 8 is not true, we obtain a set $F \subset X \times I$ such that $\lambda(F_x) = 1$ for every $x \in X$, and

$$\lambda^* \left(\{z \in I : \lambda^*(F^z) = 0\}\right) > 0.$$

It is the existence of such a set that implies that X can be covered by $\mathrm{shr}\,(\mathcal{N})$-many null sets. To see this, let $A \subset \{z \in I : \lambda^*(F^z) = 0\}$ be a set such that $|A| \leq \mathrm{shr}\,(\mathcal{N})$ and $\lambda^*(A) > 0$. Then we claim that $X = \bigcup_{z \in A} F^z$. Indeed, if $x \in X$, then $\lambda(F_x) = 1$ and since $\lambda^*(A) > 0$, $F_x \cap A \neq \emptyset$. But if $z \in F_x \cap A$, then $x \in F^z$ so that $x \in \bigcup_{z \in A} F^z$.

Summing up, assuming the theorem is false, we have proved that for every $y \in E_2$, H^y can be covered by $\mathrm{shr}\,(\mathcal{N})$-many null sets. Now, define

$$K = \{(x, y + r) : (x, y) \in H, \ r \in \mathbb{Q}\}.$$

Then K_x is a measurable set and of full measure for every $x \in E_1$. Moreover, for every $y \in \mathbb{R}$, K^y is the union of countably many horizontal sections of H and so for every $y \in \mathbb{R}$, K^y can be covered by $\mathrm{shr}\,(\mathcal{N})$-many null sets. Let $B \subset \mathbb{R}$ be a set such that $|B| \leq \mathrm{shr}\,(\mathcal{N})$ and $\lambda^*(B) > 0$. Then $E_1 = \bigcup_{z \in B} K^z$, and since each K^z can be covered by $\mathrm{shr}\,(\mathcal{N})$-many null sets, the same is true for E_1. It follows that $\mathbb{R}$ can be covered by $\mathrm{shr}\,(\mathcal{N})$ many null sets, since we can cover $\mathbb{R}$ by a null set and by

countably many translated copies of E_1. Thus we obtain $\mathrm{cov}\,(\mathcal{N}) \le \mathrm{shr}\,(\mathcal{N})$, which contradicts our assumption. $\qquad\square$

5. Symmetrically approximately continuous functions

Let $S \subset \mathbb{R}^2$ be arbitrary. We say that the points $x, y \in \mathbb{R}$ can be joined by a k-chain using the centers $p_1, \ldots, p_k$, if there are points $x = x_0, x_1, \ldots, x_k = y$ such that $x_{i-1} + x_i = 2p_i$ and $(p_i, x_{i-1}) \in S$ for every $i = 1, \ldots, k$.

Theorem 11. *Suppose* $\mathrm{shr}\,(\mathcal{N}) < \mathrm{cov}\,(\mathcal{N})$. *Let* $E \subset \mathbb{R}$ *be a measurable set of positive measure, and let* $S \subset \mathbb{R}^2$ *be a set such that* $d_*(S_x, x) = 1$ *for a.e.* $x \in E$. *Then there is a point* $x_0 \in E$ *and there is a measurable set* $F \subset E$ *of positive measure such that every point of* F *can be joined to* x_0 *by a suitable 4-chain in* E.

Proof. We may assume that E is a nonempty d-open subset of the interval $\left[\frac{1}{4}, \frac{1}{2}\right]$. We may also suppose that $d_*(S_x, x) = 1$ for a.e. $x \in \mathbb{R}$. Indeed, let V denote the d-interior of $\mathbb{R} \setminus E$, and replace S by the set $S' = (S \cap (E \times \mathbb{R})) \cup (V \times V)$. Suppose we can prove that there is a point $x_0 \in E$ and a measurable set $F \subset E$ of positive measure such that every point of F can be joined to x_0 by a suitable 4-chain using S' instead of S. Then, in every chain consisting of elements of E, the centers must also belong to E since for $p \notin E$, $S'_p \cap E = V \cap E = \emptyset$. Therefore, if two points of E can be connected by a chain in E using the set S', then they also can be connected using S. Taking a suitable subset of S we may also assume that S has the following properties:

    (i) for every x the section S_x is d-open;
    (ii) for a.e. x the section S_x is nonempty;
    (iii) for every x, if $S_x \ne \emptyset$, then $d_*(S_x, x) = 1$;
    (iv) if $x \in E$, then $S_x \subset E$ and if $x \notin E$, then $S_x \cap E = \emptyset$; and
    (v) for every x, the section S_x is symmetric about x; that is, $2x - S_x = S_x$.

By applying Theorem 10 for the set $S \cap (E \times E)$ we obtain a point $x_0 \in E$ and a $\delta > 0$ such that

$$(21) \qquad \underline{d}_*(\{y \in E : \lambda^*(S^{x_0} \cap S^y \cap E) > \delta\}, x_0) > 0.$$

Let $T = \{(x, y) : x \in I,\ (x, y + x) \in S\}$. Then $T_x = S_x - x$, and thus $d_*(T_x, 0) = 1$ for a.e. $x \in I$. Since $\mathrm{shr}\,(\mathcal{N}) < \mathrm{cov}\,(\mathcal{N})$ implies that there is no weak Sierpiński set, we may apply Theorem 9 to obtain

$$(22) \qquad d_*(\{y \in \mathbb{R} : \lambda^*(T^y) > 1 - \delta\}, 0) = 1.$$

It now follows from (21) and (22) that there is a measurable set F of positive measure such that

$$(23) \qquad \begin{aligned} &F \subset E \cap \{y \in [x_0 - \delta, x_0 + \delta] : \\ &\lambda^*(S^{x_0} \cap S^y \cap E) > \delta,\ \lambda^*\left(T^{(y - x_0)/2}\right) > 1 - \delta\}. \end{aligned}$$

(In fact, we can find such a measurable set F with $\underline{d}_*(F, x_0) > 0$.) We complete the proof by showing that every point of F can be joined to x_0 by a 4-chain.

Let $y \in F$ be arbitrary. By (23), we have $\lambda^*(S^{x_0} \cap S^y \cap E) > \delta$. Let D denote the set of outer density points of $S^{x_0} \cap S^y \cap E$. Then D is measurable and $\lambda(D) > \delta$. As $E \subset \left[\frac{1}{4}, \frac{1}{2}\right]$, it follows that $x_0 \in \left[\frac{1}{4}, \frac{1}{2}\right]$ and both $\ S^{x_0} \cap S^y \cap E \subset \left[\frac{1}{4}, \frac{1}{2}\right]$ and $\ D \subset \left[\frac{1}{4}, \frac{1}{2}\right]$. Hence, $2D - x_0 \subset I$ is measurable and $\lambda(2D - x_0) > 2\delta$. Since $T^{(y - x_0)/2} \subset I$, $\lambda^*(T^{(y - x_0)/2}) > 1 - \delta$ and $|y - x_0| \le \delta$, it follows that

$\lambda^*([2D - x_0] \cap [T^{(y-x_0)/2} + (y - x_0)/2]) > 0$. Fix an element $q \in [2D - x_0] \cap [T^{(y-x_0)/2} + (y - x_0)/2])$ such that $S_q \neq \emptyset$. We claim that the set

$$U = (S^{x_0} \cap S^y) \cap \frac{1}{2}[x_0 + S_q] \cap \frac{1}{2}\left[(2q + x_0) - S_{q-(y-x_0)/2}\right]$$

is nonempty. Indeed, as $q \in 2D - x_0$, then $(q + x_0)/2$ is an outer density point of $S^{x_0} \cap S^y$. Since q is a density point of S_q, it follows that $(q + x_0)/2$ is a density point of $\frac{1}{2}[x_0 + S_q]$. The condition $q \in T^{(y-x_0)/2} + (y - x_0)/2$ implies

$$(q - (y - x_0)/2, (y - x_0)/2) \in T \quad \text{and} \quad (q - (y - x_0)/2, q) \in S;$$

that is, $q \in S_{q-(y-x_0)/2}$. Therefore q is a density point of $S_{q-(y-x_0)/2}$, and thus $(q + x_0)/2$ is a density point of $\frac{1}{2}\left[(2q + x_0) - S_{q-(y-x_0)/2}\right]$. Therefore, $(q + x_0)/2$ is an outer density point of U, and thus $U \neq \emptyset$. Let $p \in U$.

We claim that x_0, $2p - x_0$, $2q - 2p + x_0$, $2p - y$, y is a 4-chain, using the centers p, q, $q - (y - x_0)/2$, and p. All that we have to check is that the relations

$$x_0 \in S_p, \; 2p - x_0 \in S_q, \; 2q - 2p + x_0 \in S_{q-(y-x_0)/2}, \text{ and } 2p - y \in S_p$$

hold. The first three of these relations follow from $p \in U$ (note that if $p \in S^{x_0}$, then $x_0 \in S_p$). To see that $2p - y \in S_p$ it is enough to check that $y \in S_p$ since S_p is symmetric about p. But, $p \in S^y$ so $y \in S_p$ and this completes the proof. $\qquad\square$

Theorem 12. *Suppose* $\mathrm{shr}(\mathcal{N}) < \mathrm{cov}(\mathcal{N})$. *If* $E \subset \mathbb{R}$ *is measurable and* $f : E \to \mathbb{R}$ *is symmetrically approximatively continuous at a.e. point of* E, *then* f *is measurable.*

Proof. It is enough to show that for every $\varepsilon > 0$ there is a partition $E = \bigcup_{n=1}^{\infty} E_n$ such that each E_n is measurable, and for every n, the oscillation of $f|E_n$ is at most ε. By using a standard exhaustion argument, it is enough to prove that there exists a measurable set $F \subset E$ of positive measure such that the oscillation of $f|F$ is at most ε.

Since f is symmetrically approximatively continuous at a.e. point of E, for a.e. $x \in E$ there is a set S_x such that $d_*(S_x, x) = 1$, and $|f(x + h) - f(x - h)| < \varepsilon/8$ whenever $x + h$, $x - h \in S_x$. Put $S = \{(x, y) : y \in S_x\}$. By Theorem 11, there is a point $x_0 \in E$ and a measurable set $F \subset E$ of positive measure such that every point of F can be joined to x_0 by a suitable 4-chain. It is clear that if the points x_0 and y can be joined by a 4-chain, then $|f(x_0) - f(y)| < \varepsilon/2$. Therefore $|f(x_0) - f(y)| < \varepsilon/2$ holds for every $y \in F$, and thus the oscillation of $f|F$ is at most ε. $\qquad\square$

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