DRAWING DIVISION: EMERGING AND DEVELOPING MULTIPLICATIVE STRUCTURE IN LOW-ATTAINING STUDENTS' REPRESENTATIONAL STRATEGIES

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ABSTRACT

This thesis examines the particular difficulties with multiplicative thinking experienced by students with very low attainment in school mathematics, and the representational strategies they use for multiplication and division-based tasks.

Selected students in two mainstream secondary schools, all performing significantly below age-related expectations in mathematics, placed in ‘bottom sets’, and described by their teachers as having particularly weak numeracy, received a series of tuition sessions (individual or paired). These involved ongoing qualitative diagnosis of their arithmetical strengths and weaknesses, and personalised, flexible learning support, delivered by the author. Students engaged mainly in division-based scenario tasks designed to encourage their engagement in multiplicative thinking, and explored various visuospatial representational strategies tailored to their specific areas of conceptual and procedural difficulty.

Multimodal audiovisual data collected from tuition sessions was analysed qualitatively across multiple analytic dimensions using a microgenetic approach. This led to the development of an adaptable framework for the analysis of nonstandard visuospatial representations of arithmetical structures and relationships. Analysis of changes in individual students’ strategies provided insight into some possible learning trajectories for multiplicative thinking. Parallel comparison of students’ varied representational strategies resulted in evidence for the psychological power of certain fundamental representation types, such as unit arrays and containers.

The main findings of this thesis concern: the fundamentally componential nature of the concept and practice of division, the potential difficulties this causes in understanding, and the importance of modelling and manipulating unitary multiplicative structures; and the relationship between representational strategies, economy and efficiency in carrying out multiplication and division-based tasks.

Conclusions are drawn on the relationship between the development of representational strategies and multiplicative thinking. Recommendations are given regarding learning and teaching practice for students with severe and milder difficulties in mathematics, and particularly the nature of 1:1 support provision for those considered to have Special Educational Needs.

I hereby declare that, except where explicit attribution is made, the work presented in this thesis is entirely my own.

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1 INTRODUCTION

1.1 Motivations

1.1.1 Students with difficulties in mathematics

My research interests stem from professional experience gained during ten years as a secondary school mathematics teacher. From the very start of my career I developed a particular interest in the students who found mathematics the most difficult, those consigned to the 'bottom set', following 'foundation' or 'remedial' programmes but still struggling with the material. I followed this professional interest with a move to the Special Education system, initially Pupil Referral Units, where I encountered students with a huge range of difficulties in mathematics, at least some of which resulted from the significant disruption in their prior schooling. I then held the post of head of mathematics at a specialist school for students with Specific Learning Difficulties, assuming responsibility for the mathematical development of individuals with diagnoses of dyslexia, dyspraxia, dyscalculia, autistic spectrum disorders, attention disorders and more. Although, of course, there are many people with these conditions who enjoy great academic success, the vast majority of mine were not at that point doing so; however, I have also taught a great number and variety of students whose prior attainment (according to conventional testing) was significantly below age-related expectations, but who had confused, conflicting or no formal diagnoses at all. Many of these students perceived themselves as failures at mathematics, and fear and hatred were often a debilitating factor in their relationship with the subject. As a teacher, I experienced powerful personal motivation to help these students to a better relationship with, and better understanding of mathematics.

I have chosen to open with this brief autobiography, and to write in first person, because this is a personal project that has required both my active participation and reflexive analysis. My prior and ongoing teaching experiences have not only motivated the research I do, but shaped the form it takes. I make no pretence (and nor, I believe, should one have to) of authorial invisibility or detachment from my material. As a researcher, while my previous motivation to help students of the kinds described above did not cease, it coexisted with an equally powerful motivation to observe, analyse, understand and communicate to others, in a formal and structured way, the precise nature of these students' struggles. I do not hide the fact that when encountering children like Paula – one of my case studies, aged fourteen and still struggling with single-digit arithmetic – I am fascinated to find out more about her mathematical experience, keen to share my findings with the research community, inspired to try to find some way to help her, and impelled to share pedagogical implications with the teaching community. This is the foundation of my dual role as researcher-practitioner; not a methodologically unproblematic role, certainly, but one which was a necessary feature of this work from its inception.

My previous research in this field – a Masters' project, smaller-scale but with similar methodology – focused on the representational strategies of students with a diagnosis of dyslexia, with attainment in the low to normal range. While retaining many elements from this study, I decided to move away from such diagnostic categories, and work with low-attaining students with a variety of learning difficulties, whatever the wording of their formal diagnosis (should they have one). Although it can be argued that research which sheds a light on the learning of very low-attaining students has value in itself, regardless of the proportion of the school population who would be considered to fall within such a category, I additionally argue that it provides particular information which has a wider application. The problems experienced by these students in certain aspects of school mathematics are more acute, more pronounced, and thus more visible to the observer than those of their typically-progressing peers; to take a simple example, a progression for unitary to grouped counting is obviously more clearly discernible when the counting is aloud and slow, and accompanied by determined hand gestures. It is also reasonable to assume that at least some of the differences between the problems experienced by students at the low extreme of a supposed 'attainment' spectrum and those nearer the middle of the spectrum are differences of degree rather than of kind. Thus, insights gained in solving the difficult pedagogical problems of teaching particular mathematical concepts and/or procedures to students who find them very hard will have applications to the less-difficult pedagogical problems relating to students who find them somewhat hard.

I have referred to students being 'low-attaining' and/or having 'difficulties' in mathematics, arithmetic or numeracy. Issues around the categorical terminology used to describe learners are addressed in 2.2, but I note here that I use ‘low-attaining’ to mean that within the confines of the National Curriculum for Mathematics, assessed in the classroom, the student has been performing significantly below age-related expectations. It carries no implications regarding the reasons why this might currently be the case for that individual, or any assumptions about their future attainment in or out of school mathematics. 'Difficulties in mathematics' refers to children or adults who struggle or fail to cope with some of the aspects of arithmetic that are necessary or desirable for educational or practical purposes (Dowker, 2005) and, likewise, does not imply a pathological problem (such as brain damage), a formal classification of learning difficulties/disability, or a stance on the potential genetic and/or environmental causes of the difficulties. People with mathematics difficulties are highly heterogeneous, but it seems that most of their difficulties (in the numerical/arithmetical part of mathematics, at least; other areas, e.g. geometry, are outside the scope of this study) lie on a 'normal' continuum between extreme talent and extreme weakness (ibid.). Another question is of exactly what does or does not count as having 'difficulties in mathematics', as most students will at some point have to study something that they find difficult, especially when high prior attainment is rewarded with more challenging 'extension work'. As generally used, the term must be to some extent context-dependent, as different schools, classes and teachers will create different expectations of student progress, and a student considered in one environment as 'struggling' may be perceived as typical (or better) elsewhere. (This was a factor in my selection of certain schools over alternatives for research setting.) The heterogeneous and componential nature of mathematical difficulties assumed in this study would make it inappropriate to set some arbitrary quantitative level of attainment below which a student is classified as having mathematics difficulties, although for now we may safely say that individuals such as Einstein, despite the (commonly attributed) claim "Do not worry about your difficulties in mathematics; I assure you that mine are greater", do not fall into this category!

1.1.2 Visuospatial representation

A methodological relationship links the two main characteristics of the chosen research participants (secondary-age, low-attaining) with the research focus. While in early schooling, visuospatial representations are commonplace, as students are encouraged to model arithmetical tasks by representing the quantities involved, in order to employ counting-based strategies; as the expected arithmetical strategies evolve, cubes, counters and pictures start to disappear, and the move from primary to secondary school marks quite a cut-off point regarding representational expectations of students. In individual tuition sessions, removed from class, my students frequently seized upon my bag of small plastic cubes with marked relief; whether their current classroom contained such things or not was of limited relevance, as they would not allow themselves to be seen using such things by their peers, and in the case of older participants, were aware these things are not available in examinations. There is also the relationship between (low) mathematical attainment and representational strategies: students who cannot manage to solve arithmetical problems in 'standard' ways (i.e. memorised algorithms) must try to find other, apparently less efficient, ways to work through them – assuming that they are motivated to do so. (Of course, there is evidence that expert mathematicians also make use of nonstandard and imaginative representations for problems, but in the case of arithmetic, this is more likely to take the form of finding a fast, elegant solution than coming up with any way at all, however cumbersome, of making soluble an initially incomprehensible task – as will be the case here.)

My interest in the use of visual and/or kinaesthetic modes and media for generating and enhancing students' understanding of arithmetical concepts and processes is also a long-standing one. Having personally always made use of internal and external visual imagery for thinking and problem-solving, my natural teaching style reflected this. Many of my past students (mentioned above) voiced clear approval, in some cases helpfully giving feedback on the exact representations that had helped them understand a concept or procedure for the first time. During my years teaching, and observing other teachers (as head of department, mentor and PGCE tutor), I came to the opinion that, notwithstanding the recent growth in books and courses on *Learning Styles* and *Multiple Intelligences*, the visual and kinaesthetic modes were generally undervalued and/or ineffectively used in secondary mathematics classrooms. This should not be read as an attack on the competence of teachers: the issue of visuospatial representations in teaching is an extremely complex one, and the questions of what, when, how and for whom, are not easily answered. Research has provided some answers for <u>what</u> representational strategies are helpful, for some students in some aspects of mathematics, <u>when</u> it is helpful to introduce them (with the same caveats) and <u>how</u> one might perhaps do so. However, there is a great deal still to be understood about how children's representational strategies develop, to which this thesis contributes.

The representation of numerical properties, structures and processes is central to mathematics. (Note that 'representation' as a concept requires considerable unpicking – see Chapter 4.) Many researchers consider visual representation in particular to be a fundamental system of cognitive representation for problem solving, and this representation comes in many forms. Schoolchildren spend a considerable amount of time listening to their teachers talk about mathematics, and (sometimes) engage in mathematical discussion themselves. They read texts about mathematics, along with many written tasks, and are required to 'explain their answers' in writing during examinations – something many students find deeply unappealing. They read and write not only in standard language but using specialised mathematical symbols, and they are required to interpret and create specialised geometrical and statistical diagrams. The representational modes mentioned above are those with which all schoolchildren are expected to become familiar, and in which they are formally tested. However, they are by no means the only forms of representation to be found in school mathematics. A brief examination of the exercise books of a class of students is also likely to show a selection of informal, nonstandard markings created by students during their mathematical endeavours, while observation of them working shows the co-opting of physical objects (including fingers) to temporarily embody the numbers and relationships needed. Individuals engaging in mathematical problem-solving are also frequently observed apparently 'staring into space', focusing not on the page and markings in front of them but on internal (mental) images. While these behaviours may be found almost everywhere that mathematical activity is taking place, they are not everywhere acknowledged, valued, or, outside of the educational research community, scrutinised. I make it my purpose to seek out the models students make and the images they draw, to design learning situations where these kinds of representations are foregrounded, and to create an environment where they can experiment with and discuss different representational strategies.

In current UK education systems, many students, particularly the lower-attaining ones, and including some of the participants in this study, come to believe that only certain types/uses of representation are 'legitimate' in school mathematics, and even when they have some awareness or experience of a visual representational strategy that might help them solve a problem, do not employ it. Many teachers also believe and express this view, but even when teachers give express 'permission' to represent problems in alternative ways, the students may not have the metacognitive skills to make appropriate representational choices. This situation can lead to an erroneous assumption that the child is 'not a visualiser' (as opposed to those children who create and use informal representations without being directed to do so), with lack of experience and/or initiative being mistaken for lack of ability. I do not suggest that every single child benefits from visualising mathematical problems, but I do suggest that many children are being denied the opportunity and/or encouragement to learn to do this. While the National Numeracy Strategy has explicitly required teachers to acknowledge and value a variety of arithmetical (and, by way of this, representational) strategies, in a classroom situation it is clearly logistically difficult to find out about the representational inclinations and needs of each individual student. In-class support tends to involve providing specific assistance on the curriculum topic being taught, and even specialist teachers withdrawing students from class have reported suffering time pressure from being required to cover a certain amount of curriculum 'ground' in a given time frame. One of the main aspects of this research, then, is temporarily to slow down the pace as much as necessary – in a way not usually possible for teaching or support staff – to properly explore individual students' developing representational behaviour within a particular subject area.

1.2 Focusing

1.2.1 Secondary-age students

As the great majority of my teaching experience has been with the 11-16-year-olds, this was the obvious choice for research. However, this age group also has several unique features which make it particularly interesting for research purposes – notwithstanding the fact that it has been considerably less popular for research in mathematics education than the primary school age group (itself a good reason for study). The move from primary to secondary school is a time of great change in children's lives, and there is a great deal of growing evidence (Dowker, 2005) that many students, particularly those with atypical educational requirements, find this change distressing; more specifically, it is the point at which they report starting to experience negative feelings towards mathematics (which is not to suggest that there are not also examples of the converse). This affective charge cannot be ignored when addressing issues of students with mathematics difficulties. Cognitively the 11-16 age group is also of particular interest, as recent neurocognitive research suggests that the onset of adolescence is thought to be a time of “particularly dramatic brain reorganization” and “a major opportunity for teaching” (Blakemore and Frith, 2005).

1.2.2 Division and multiplicative structures

Mathematics is generally assumed to be more hierarchical in structure than other school subjects, and the majority of curricula, including our own spiral-form system, are designed with the assumption that certain lower-level knowledge is necessary for the understanding of more advanced topics. The most prominent of this 'necessary' knowledge is basic arithmetic (meaning, effectively, competence in the 'four operations'). Many research studies on low-attaining students have focused on 'clean slate' curricular material, i.e. children being introduced to a mathematical topic for the first time (which is another reason for the preponderance of data on younger children). This will not be the case here, as, being of age 11 upward, all of my students will have experienced teaching in arithmetic, and racked up many hours – although in some cases apparently to little avail – 'practising' addition, subtraction, multiplication, division, number bonds, times tables, etc.

My experience in the classroom and in previous research suggested it likely that the students I encountered in 'bottom sets' at mainstream schools would be comfortable with the concepts of addition and subtraction (though might have difficulties carrying out the operations); these are also the most densely-researched of the arithmetical operations. Division, on the other hand, is generally perceived to be the most difficult by children and teachers, given weight by it being traditionally the last of the four operations to be taught, and tends to inspire a particular fear and dislike amongst those with a difficult relationship with mathematics. It is true that multiplication can also be unpopular, but children (and adults) who struggle with it are more likely to complain of being unable to 'learn their tables', or remember the order of steps in multi-digit multiplication procedures, than of not understanding it (although, in fact, incomplete understanding may well underpin the difficulties they report). There is, then, a good ethical reason for attempting to demystify and make comprehensible one of the most hated, yet necessary, arithmetical operations; it is also the least researched of the four. Because all participants would have previously encountered division at primary school, they could be expected to have some familiarity from their previous teaching, but with individuals displaying a range of levels and kinds of understanding of the division concept. My aim now became to qualitatively diagnose their understandings of division, such as they were, and then use certain carefully-chosen flexible representations to 'nudge' students to better understanding and competence. However, on considering the nature of division, and the metaphors by which children come to grasp it, it was clear that division could not and should not be addressed in isolation from multiplication. Therefore, my subject matter of choice must broaden to include *multiplicative structures* – considered to play a “pivotal role in the Key Stage 3 curriculum” (Brown et al., 2010).

1.2.3 Methodological directions

There has been a substantial and increasing body of work on the role of the visual in mathematics teaching and learning, much of it experimental or quasi-experimental; this has provided a degree of statistical generalisation, usually with either conclusions about what the ‘average’ student does, or comparisons between broad categories such as high- and low-attaining students. Some conclude that particular representational approaches work better than others (on average). However, statistical conclusions deal in probabilities; they do not speak for individuals, and to examine individual cases in greater detail provides a necessary complement. The children struggling at the bottom of the bottom mathematics set are individuals, not categories, and neither do they fit neatly into pedagogical or analytical categories. Their individual learning trajectories, however, contain aspects which may be compared, contrasted, theorised, and generalised. The complex nature of their understandings (and misunderstandings) of numerical structures and relationships is of great interest and importance, and with any quantification comes the loss of this complexity. This is by no means to dismiss the traditional, experimental-statistical approach to research in this field, which has elicited a great deal of valuable knowledge; it is simply a claim that qualitative approaches are of equal value in shedding light on the complex cognitive processes of learning. Likewise, studying a small number of participants in fine detail tells us different, but equally justifiable, things from studying a great number of participants in broad detail.

1.2.4 Initial questions

At this early stage in my research, I formulated three questions. These acted as a starting point, although with the kinds of qualitative methodology and data-first analysis I planned to employ, I expected to refine them at later stages (as indeed happened).

- What representations do the students utilise during arithmetical activity?
- Why do the students use the representational strategies they do?
- How do different representational forms interact with the development of the students' numerical understanding?

1.3 Visuospatial representations as curriculum entities

State schools in England devise their programmes of study based on a number of key government documents. I survey what the National Curriculum, Numeracy Framework, and Special Educational Needs Code of Practice have to say about the use of visuospatial representation in mathematics.

1.3.1 National Curriculum

The National Curriculum for England (QCA, 1999) explicitly mentions representation in Mathematics on many occasions; however, the great majority of these refer to parts of the subject with extra-numerical content and/or outside the scope of this study, e.g. geometric diagrams (in Ma3: Shape, space and measures), statistical charts (in Ma4: Handling data), etc. However, there are some mentions to be found regarding numerical relationships and calculations (i.e., in Ma2: Number in KS1-2; Number and algebra in KS3-4). The Key Stage 1 Programme of Study states explicitly that pupils should be taught to "communicate in spoken, pictorial and written form, at first using informal language and recording, then mathematical language and symbols" (Ma2 1(f), p.p.16), with the knowledge and skills being developed through "practical activity, exploration and discussion; using mathematical ideas in practical activities, then recording these using objects, pictures, diagrams, words, numbers and symbols; using mental images of numbers and their relationships to support the development of mental calculation strategies" (Breadth of study 1(a-c), p.p.20). These aims will likely seem appropriate to any educationalist who believes in the value of constructivist, connectionist, relational and/or understanding-based learning; however, the National Curriculum considers these aims appropriate specifically for children of 8 and under. While the representation of number relationships does not go entirely unmentioned in the later Programmes of Study – e.g." recognise, represent and interpret simple number relationships" (KS2: Ma2 4(d), p.p.24) and "move from one form of representation to another to get different perspectives on the problem" (KS3: Ma2 1(f), p.p.29) – it is clear from context that by these later points, the representations envisaged by the authors are considerably more formal, efficient and consistent than those produced either by children in KS1 or by the participants in this study. Meanwhile, children "represent[ing] their work with objects or pictures" is given as a descriptor for NC Level 1 (in Attainment target 1: Using and applying mathematics).

1.3.2 National Numeracy Strategy

Meanwhile, the National Numeracy Strategy Framework (DfEE, 1999) reminds teachers that "mathematics has a strong visual element" and to "capitalise on this wherever you can to illuminate meaning" with "frequent use of a number line, 100 square, number apparatus, pictures, diagrams, graphs . . ." (p.p.21), although this advice appears under the heading 'How do we cater for pupils with particular needs?' rather than as a general principle for teaching mathematics. Various other occurrences of the term 'visual' also refer to number lines and 100-squares, so it seems that these particular representations of the number system – and the particular aspects of it relating to the ordering and recognition of numerals – are considered highly important. It is noted that "visual interest, involvement and interaction" are required for mathematics lessons in a special school (p.p.24), and that classroom assistants might be asked to "use and make available to children . . . visual or practical aids", the aids mentioned being, unsurprisingly, "a number line and/or 100 square" (p.p.25). Regarding children's own nonstandard visuospatial representations, they are deemed age-appropriate for Reception classes, where "It is expected that children . . . will receive some direct teaching and talk about mathematical ideas, and will explore those ideas . . . sometimes recording informally what they have done with objects or drawings" (p.p.28). The use of drawing in arithmetical tasks could possibly be included in the "informal pencil and paper notes, recording some or all of their solution" (p.p.7) that the National Numeracy

Strategy suggests are appropriate for when students progress to more complex problems with larger numbers; it is unclear why these "personal jottings" are not also recommended for students having difficulties with simpler problems and smaller numbers.

1.3.3 SEN Code of Practice

The material from the KS1 and KS2 sections of the National Curriculum and from the Numeracy Strategy Framework is not actually denied to students in KS3. The Special Educational Needs (SEN) Code of Practice (DfES, 2001) states "Children in the primary sector will have had access to the National Literacy and Numeracy Strategy Frameworks alongside the National Curriculum; and for some children it may be appropriate to continue to adopt these strategies at key stage 3" (6:17, p.p.62), and reinforces this statement shortly after, with "For some pupils it will be necessary to choose work from earlier key stages so they are able to progress and demonstrate attainment" (6:19). Similarly-worded text appears in the National Curriculum's Principles for Inclusion (p.p.74). However, the implication is that these "some pupils" referred to will be those with confirmed and tested SEN, either placed in special schools or making up a very small proportion of the mainstream population. In recent years, the proportion of students entering KS3 at Level 4 or above has been around 80% (DfE, 2011), leaving a sizeable quantity of students below that (somewhat arbitrary and very simplistic) marker. Even if this larger group are taken to be the "some pupils" referred to by the SEN Code of Practice, a suggestion that they follow schemes of work intended for primary school children is not in itself necessarily helpful, especially if it may merely mean repeating the same strategies that have not been successful for them in the past.

1.3.4 Pictorial representations

The positioning of picture- and object-based representations for number work at NC Level 1 gives the distinct impression that they are only useful at ages 4-7 and/or abnormally low levels of attainment. The curriculum documents in general imply that the visuospatial representation of numbers, number relationships and processes is something children are expected to do while at the stage of learning to count, and perhaps add and subtract, but that once they have grasped the basic unary operations they should be moved swiftly into working with symbols alone, and leave pictorial and concrete representations behind. Of course, the fact that the National Curriculum does not suggest visuospatial representations for learning the commutative, associative and distributive principles of multiplicative structures (for example) does not mean that individual teachers may not decide to teach them that way; however, there will also be teachers who consider it sufficient to state the principles in words or symbols and instruct their students to memorise them, which is an inappropriate approach for many.

1.3.5 Arrays: A special case

The rectangular array representation is a standard one for teaching the concept of multiplication, recommended in current versions of both the Primary Mathematics Framework, e.g. for understanding "the operation of multiplication as repeated addition or as describing an array", and in the National Curriculum, e.g. for understanding "why the commutative, associative and distributive laws apply to addition and multiplication". It may generally be assumed that by the end of primary school, students of all levels of attainment will be familiar with rectangular arrays, both discrete (i.e. dot arrays) and continuous (i.e. grids), and, if current teaching guidelines have been followed, that these representations have some link with multiplication. Indeed, the students in this study should have followed the teaching progression in the National Numeracy Strategy, with arrays introduced in Year 2 (emphasising binary and commutative aspects) and reinforced in Year 3. However, such assumptions require modification when considering my particular set of participants: while the assumption may be made that they had seen arrays during previous mathematics/numeracy lessons, their relevance, usage, and the arithmetical structures they embody may not have been made clear.

In fact, it will be seen that many of my students independently introduced array forms into their working, and these representations became an important part of this study. This is in stark contrast to number lines – also mentioned several times in curriculum materials, and used extensively in the primary classroom, but which my students never once chose to employ.

1.3.6 Are representations limited too soon?

The struggling students in KS3 and KS4 that I encountered during my years of teaching responded very positively to increased visuospatial representation while working on subject matter significantly beyond Level 1, and my suspicion that many of them had suffered educationally from being ‘weaned’ too soon from the enactive and iconic to the symbolic domain was one of the factors leading to this project. It was thus no surprise (nor a particularly interesting finding) that the participants in this study also generally responded positively; however, what proved of great interest was the rich variety of responses, and the relationship between these adopted (or re-adopted) visuospatial representational strategies and their developing numeracy.

1.4 Overview

In Chapters 2, 3, and 4, I provide a critical appraisal of key literature relating to my field of study. Firstly, I ask what it means (and has meant in the past) to have difficulties in learning mathematics, and summarise some of the ways people involved in education have attempted to address the issue of low-attaining students. I then engage with research literature in the areas of arithmetical strategies and visuospatial representation.

Chapter 5 describes my methodology for this study, including methodological influences, details of the research settings and my role in relation to them, and the tasks, prompts, and other interactions I used with students.

Chapters 6, 7 and 8 are devoted to the fieldwork data and its analysis. In each I take a subset of the data collected during my time in the participating schools. The first of these focuses on two particular stand-alone tasks, the second on the learning trajectories of two individual students, and the third on four key representation types, used by various students across various tasks, in making sense of multiplicative structures.

In Chapter 9 I discuss my findings from the three previous chapters in the context of the existing literature addressed previously, drawing out patterns and relationships. I address three particular overarching themes which emerged during analysis: Conceptualising division, Tasks and time, and Efficiency in representational strategies.

Chapter 10 consists of my concluding remarks, in which I reflect on the study as a whole, its achievements, limitations and implications regarding learning and teaching practice for students with severe and milder difficulties in mathematics, with particular recommendations regarding the provision of 1:1 numeracy support.

INTRODUCTION TO LITERATURE

In this part I address the literature which provides the theoretical background to this study, and has helped form my thinking about methodology and analysis. It has required engagement with a wide variety of sources from a number of different academic disciplines, which are broadly organised under three headings which relate to the major foci defining the project.

Firstly, I have stated my intention to work specifically within the population of students experiencing significant difficulties in school mathematics; thus, in Chapter 2 I begin by considering historical and current understandings of, and school- and state-level reactions to, the issue of low attainment in mathematics/numeracy, drawing on political and pedagogical sources. My subject matter falls under the pedagogical area of arithmetic, so Chapter 3 engages with theoretical literature on the nature of natural-number arithmetic, and practical research on children's developing strategies for multiplication and division. My data consists primarily of the visuospatial representations created by students while performing arithmetical tasks, and so Chapter 4 completes the literature review, by presenting and critiquing some of the main stances on issues of representation in this context.

2 DIFFICULTIES IN MATHEMATICS

In this study I refer on many occasions to students with difficulties in mathematics, and/or 'low-attaining'. These are obviously not sharp-edged, easily-defined groups, so I first ask: who are we talking about when we talk about students with difficulties, and how do we think about levels of attainment? I begin with a necessarily brief overview of the historical context to this issue (in England), from which it appears that there have been both some great changes in the perception of these students and their education, and some constants. I then address and critique the terminology used in writing about these students, followed by a dissection of the concept of 'ability' in the context of learning mathematics.

2.1 Historical context

Organised Special Education provision for those with learning difficulties might be said to have begun in 1847 with the Highgate Asylum for Idiots, prior to which those not cared for privately would most likely have been placed in workhouses or infirmaries (Warnock, 1978). During the latter part of the 19th century, increasing philanthropic tendencies regarding the life prospects of the disabled and disadvantaged resulted in a number of local school boards choosing to establish classes for "feeble-minded pupils", or schools for "mentally defective children" (Leicester and London's respective School Boards in 1892, with five more joining them in the next few years). By 1897 London alone had 27 centres catering for some 1000 children judged "capable of learning elementary subjects at some rate, however slow" – although excluding "idiots" (Hurt, 1988, p.p.127).

Until this period, developments had been primarily driven by charity, but government involvement followed, for example, the Education Department Committee's Report on Defective and Epileptic Children (1898). Legislative changes affecting education were not confined to those with disabilities; this was part of a much broader movement towards the principle of national state education for all. This social change was not without opposition from those with strong beliefs about keeping to one's 'place in society' – and receiving an education appropriate to that end; however, the school-educated population grew significantly, as did theories and practices for its stratification and segregation. The 1898 report's recommendation for "idiots and imbeciles" was "seclusion for life in institutions . . . in the interests of society as well as in their own" (Hurt, 1988, p.p.129), while the "feeble-minded or defective" would be trainable for some manual occupation. Expectations of children (and schools) were, then, conceptualised in terms of both social class and of academic aptitude (as well as other factors, such as gender), and while a correlation between the two was certainly noticed, beliefs about differences in innate mental capacity of socioeconomic groups were prevalent and relatively unchallenged. There was further conflation of poor mental capacity with physical and moral degeneracy, and hereditarian/eugenicist views (e.g. "mentally defective parents will produce only mentally defective offspring" (Davenport 1911, in Hurt 1988)) were common. These all fed into and drew upon widespread presumptions about the limited learning potential of children, with little opportunity for them to prove different.

Legislating on how, where, and whether different categories of child should be educated necessitated those children to be diagnosed and assigned a category – and an admission on the part of educationalists that many were borderline cases. The early 20th century saw diagnostic change, in a rising emphasis on quantifiability and the popularity of Intelligence Quotient testing, due in particular to the work of psychologists Alfred Binet and Cyril Burt. Initially, the idea of actually testing children's intelligence had considerable appeal, as an improvement on the presumptive expectations discussed above. However, while formally asserting academic ability to be on a spectrum – as opposed to a have/have-not situation – was also an improvement, the idea of 'innate intelligence' as a fixed, unitary personal attribute nevertheless still resulted in the pinning of permanent labels on children after performance in a single test (e.g. 'moron' – now assigned to the IQ range 55-70). The 'Hadow Reports' (Hadow and Burt, 1924; Hadow, 1926; Hadow et al., 1931) did acknowledge this at the time, both praising the usefulness of the Binet-Simon scale, while warning of its limitations and the dangers of rigid classification. These reports are also notable for their social progressiveness: where "educational retardation" is noted as related to poverty levels, the implication was not that the poorer social classes were intellectually inferior by nature, but experienced environmental effects underestimated by the eugenic and biometric psychologists, who overemphasised heredity (Hadow et al., 1931).

Where all previous legislation had treated the education of the ‘handicapped’ as an entirely separate issue, the landmark Education Act of 1944 included specialist provision within its list of requirements for all local education authorities. The principle was also enshrined that any child considered educable had the legal right of access to schooling. The Act also brought in a new heterogeneous category of “educationally subnormal” (children of limited ability, retarded by more than 20% for their age by disability or other conditions such as irregular attendance, ill-health, lack of continuity in their education or unsatisfactory school conditions), believed to amount to approximately 10% of the school population, and made detailed suggestions regarding their school experience. The acceptance of multiple etiologies for underachievement, and the proposal, for example, that

> They should be taught in small groups, in attractive accommodation and by sympathetic teachers. They should not however be isolated, but should be regarded as full members of the ordinary school and should share in general activities. (Warnock, 1978, p.p.20)

appear strikingly contemporary. While intentions were good, there continued to be tensions over discrepancies in assessment, and some students deemed ineducable. However, rising levels of parental dissatisfaction and appeals against LEA decisions lead to further refinements of the law, and since the implementation of the Education (Handicapped Children) Act (1970), all children with disabilities, however severe, have been included in the framework of special education.

A second landmark event of the 20th century for children with educational difficulties was the publication of the Warnock Report (1978), from a committee appointed to review educational provision for children and young people ‘handicapped by disabilities of body or mind’. Addressing the tangled nature of concepts of handicap, disability, incapacity and disadvantage, and stating bluntly the impossibility of establishing precise criteria for what constitutes educational handicap, it introduced the formal term Special Educational Needs (SEN), and the idea of using a multidisciplinary approach to look at the complex picture of abilities, disabilities, and all factors bearing on an individual's educational progress. The report also included statistical information, which at the time was probably quite shocking: that at any time an estimated one in six children requires some form of special educational provision, and that one in five children are likely to qualify as having SEN at some time during their school career (Warnock, 1978). The dissemination of this finding dealt a considerable blow to the historical 'othering’ of those who do not succeed in traditional educational models. The idea that in an ordinary class of 30 students one might reasonably expect five requiring special assistance of some sort is a powerful one, the acceptance of which affects not only how teachers and parents perceive students with difficulties, but equally importantly, their peers. This gains weight when combined with the ‘seven year difference’ in mathematics attainment at age 11 highlighted in Cockcroft (1982). While I certainly do not suggest that all stigma is banished to the past, to be on an urban mainstream school's SEN Register is now comparatively unremarkable, and accepted as part of the diversity of strengths, weaknesses and atypicalities inherent in a slice of society.

While many of the principles, and even details, from the major education Acts and Reports discussed above are still very relevant, attitudes have not stood still. Warnock’s 2005 update ‘A New Look’ (republished 2010 as part of a larger volume of debate) points out some of the ways in which her previous work was misunderstood and misapplied in practice, and in some cases, also misguided in theory. The SEN Framework, as it currently stands, is compatible with what is sometimes described as an *interactionist* model of causation of SEN, which assumes that environmental and child factors interact over time to result in the difficulties that give rise to special educational needs (Wedell 2008, in Warnock et al., 2010, p.p.70). However, despite the various refinements it has undergone, it is still seen by many as applied in discriminatory and 'labelling' ways, which continue to emphasise individual differences as 'deficits' in a way which is ethically unacceptable (Warnock et al., 2010), and demanding of re-conceptualisation. One example of this is Terzi’s (among others) adoption, and application to education, of the *capability approach* philosophy of Amartya Sen. For example, a student’s specific learning difficulty is seen as a *limitation in particular functionings* resulting from the interaction of the personal characteristics of the child with the schooling environment; where the latter is not appropriately designed and/or the individual is not receptive, there is a limitation of capabilities, and thus of opportunities (Warnock et al., 2010).

2.2 Terminology

> Feeble-minded are classified by grade as idiot, imbecile, and moron; by form as endogenous and exogenous; and by type as familial, mongoloid,

> cretin, etc., showing that feeble-mindedness is an aggregate of various clinical syndromes. The clinical appraisal of mental deficiency requires evaluation in terms of a number of criteria, such as social inadequacy, mental inadequacy, of which intelligence is only one phase, developmental retardation, educational inaptitude as distinguished from special educational deficiencies, and somatic infantility. (Doll, 1940, p.p.395)

The different theoretical and professional fields (e.g. medicine, psychology, sociology) have their own verbal conventions for discussing and categorising academic ability, and these have all influenced educational literature. There have been significant changes over time in the language used – for example, the term 'idiot': now a commonplace playground insult, but recorded in a statute of 1325 relating to the lands belonging to those of unsound mind (*De Praerogativa Regis*, in Race, 1995, p.p.13), carefully distinguished from 'imbecile' or 'lunatic' in the Idiots Act of 1886 (ibid), and equated with the specific IQ range <25 (Doll, 1936). These changes have reflected the changing concerns and priorities of cultures in addressing the nonstandard needs of these individuals – as discussed above. During the 20th century the conceptualisation based on defect ('mentally defective') gave way to that of abnormality ('educationally subnormal'), which was replaced with that of exceptionality ('special needs') and perhaps-temporary obstacle ('learning difficulties'). There is also fine detail in the interdisciplinary usage of terms, which carry great weight; while 'retardation' is considered an appropriate descriptive term within current psychiatry discourse, to call an individual 'retarded' is widely considered highly offensive. The difference in this case is mainly based on whether the label is being applied to one particular measurable aspect of attainment currently presenting as chronologically delayed, or permanently and grossly to the person.

I do not propose to address individually the long list of terms that have been used in the literature on this subject. In research, pedagogical and legislative discourse past and present, as well as in the informal conversations of teachers, parents, and students themselves, the need (real or perceived) to express ability-based judgements involves certain issues overarching the details of terminology common to the different contexts. Whether an example is couched in terms of (low or under-) achievement, attainment or performance, of disability, handicap, difficulties, struggles or special needs, of delay, subnormality, (risk of) failure, or any of the other variants that can be found, the following issues arise. (Note that I do not imply this list is exhaustive, or that the issues are independent of each other.)

***Specificity*:** Is a judgement being made of a specific aspect of a student's educational performance, or is it assumed to be an overall attribute affecting all (or most) aspects? In the case of "Educationally Subnormal" (introduced in the 1944 Education Act, rejected in the 1978 Warnock Report), the implication was certainly the latter. Many research studies – increasingly during the 1980s-90s – have compared experimental results from students categorised as "low-achieving" across the board with those whose mathematics scores are discrepant with (usually) their reading scores. While it might be said that this model is still a very blunt instrument, it indicates increasing acknowledgement of students' patterns of strengths and weaknesses in different areas of the curriculum; this complemented the general tendency of (UK) schools to move from 'streamed' to 'setted' differentiation structures (discussed in more detail in 2.3).

***Cohort*:** Who are students being judged against? While in research literature there is generally effort put into clarifying parameters of achievement/attainment (frequently relying on norm-comparison against sets of standardised tests), teachers' judgements on this tend to be highly context-dependent, and based on the range of students at their own school, rather than more general measures. There is also the curious concept of the 'underachieving' student, in which a student is compared not directly with an external cohort, but with some imagined version of themself who fails to perform as the teachers (or parents) expect and think they 'ought to'. While the application, or not, of an 'underachieving' label is affected by gender, ethnicity and particularly socioeconomic status (Gillborn and Mirza, 2000) – although teachers may especially deny the latter (Dunne and Gazeley, 2008) – the way the term is used carries an implication that the 'underachieving' student is at least partially responsible for their own lack of achievement (perhaps through inapplication), in the way that the 'low-achieving' student is not.

***Permanence*:** Is an ability-judgement made about a student (specific or general) assumed to be an unchanging quality they possess or a temporary status which may improve (or worsen) in future? As discussed above, there has been a trend in research and legislative discourse from assumptions of permanence toward the possibility of change; it is unclear whether this is widely reflected in mainstream education. Informal conversation with teachers involved in special education indicates that they consciously perceive themselves as having the potential to make significant changes in their students' attainment – although it is possible that unconscious beliefs about the potential of actual individuals may not be in line with stated beliefs.

Aims**:** As stated in the governmental acts and reports cited above, disability does not in itself imply handicap. How important a factor a limitation or delay in performing a particular task or acquiring a particular skill set is depends on culture-specific educational goals, expectations and teaching practices, which have varied considerably across time and place. In terms of school mathematics, the effects of physical disabilities are clear, and catered for in appropriate ways. However, the effects of cognitive and neurological differences are less obvious. Tests requiring high-speed calculation are likely to produce a different set of poorly-performing students to tests with generous time limits; teaching with the intention of students reproducing procedures for solving certain standard problems will favour a different group to teaching with the intention that students develop general strategies for unfamiliar and open problems. Greater or lesser emphasis on calculating mentally, with pen and paper, with calculators or computer software will suit individual students better or worse. Judgements of ability do not always take these into account, and frequently treat contemporary, local, cultural assumptions about educational aims as if they were timeless and global. The role of rote memorisation, for example, is a case of this particularly relevant to the learning of multiplication and division.

Normality**:** For various practical purposes it has been necessary for educational systems to identify categories of students with difficulties, but in few cases are the boundaries non-arbitrary. With certain disabilities there is a genetic or other medical test which unambiguously identifies a specific condition (although as already seen, the educational implications of this will vary considerably), but in most cases, 'low achievement' (etc.) judgements rely on arbitrary statistical measures – standard deviations below the mean on a test, for example, or failure to attain a certain proportion of age-referenced criteria. Arbitrary cut-off points may be appropriate in selecting study samples, but are rarely helpful in real life; nevertheless, there are today still major consequences in terms of access, for example to support services, for scoring above or below a certain IQ – despite ongoing doubts about the tests' reliability. There is also a tendency, partly dependent on views on *specificity* and *permanence* (see above) for attributes that lie on a continuum across the population to be perceived as categorical – an 'us and them' situation which has been (perhaps grudgingly) allowed to continue for the sake of guaranteeing appropriate support for the ‘them’ that need it most.

A final point on terminology, particularly highlighted by the growing neurodiversity movement, concerns whether particular attributes are conceptualised as ‘difficulties’ or ‘differences’, and their role in identity. Although I have argued against many past and present labelling practices, certain labels, for example ‘autistic’ or ‘dyslexic’, have become inseparable aspects of many individuals’ personal identity (Kapp et al., 2012), signifying membership of a group to which they are proud to belong, and implying patterns of difference (from neurotypicality) rather than deficit or disorder. Unsurprisingly, there is strong objection within and around these communities to suggestions that their differences are something to be ‘cured’, and discourses around conformity, intervention, adaptation and acceptance are changing (Rosqvist, 2012). On this subject, a minor point of interest is that none of the students participating in this study ever used such terminology in my hearing, to refer to themselves or others (although several had diagnoses of dyslexia and one of autism), whereas those in my previous study (Finesilver, 2006) did so freely. This may reflect a cultural difference between students at mainstream and special schools.

2.3 From ability to abilities

I have mentioned the latter 20th century trend from streaming to setting, as part of a growing acknowledgement that children may show discrepant performance in different areas of the curriculum. In similar fashion, it has become steadily clearer that individuals’ performance <u>within</u> mathematics also shows significant discrepancies and irregularities. In curricula which treat various mathematical disciplines as separately-examined courses, it is unsurprising to find some students struggling with numerical tasks but excelling in geometry, and vice versa, but the subdivision of mathematical ability has continued further: there has been increasing – now extensive – evidence that there is not actually such a thing as arithmetical ability, only arithmetical abilitie*s* (Dowker, 2005). Theoreticians have sub-classified aspects of arithmetic in various ways, based on subject content structure, sensory involvement (e.g. Clausen-May’s Learning Styles pedagogy (2005), or on cognitive processes involved (e.g. Levine et al.'s ‘16 interactive subcomponents’(1992)). Other categorisations have emerged from large-scale factor analytic studies (such as a basic distinction between *numerical facility* and *mathematical reasoning*) and detailed studies of the varied impairments and functioning of patients with brain damage (Dowker, 2005). Where cultural aspects have been considered, alternative divisions have appeared, for example between ‘street mathematics and school mathematics’ (Nunes et al., 1993).

From the various different disciplines taking an interest in individual differences have sprung several ways of looking at those students whose differences present educational difficulties. There is the language of specific cognitive dysfunction: dyslexia, dyspraxia, dyscalculia, etc. (the latter of which is still, frustratingly, used inconsistently in research literature and sloppily in schools and pedagogical texts). Although the many identifying characteristics of these conditions clearly occur, to varying degrees, across spectra of severity, and exist to a lesser extent in the population in general, the arbitrary diagnostic boundaries required (as discussed above) for ensuring SEN support give the impression not only of non-existent clear categorical distinctions, but of an erroneous homogeneity in groups given the labels ‘dyslexic’, ‘autistic’, etc.

This tendency to oversimplification of individual differences is widespread. For example, learners have their own, often quite strongly-held, preferences regarding arithmetical strategies; some, on learning a 'short cut' one day, will want to know why it works, the range of situations in which it will (and will not) work, and how it might be extended, while others are content that right now it gives the right answer for the particular type of task currently in front of them. Influential mathematics educator Steve Chinn (2004) used the terms “inchworm” (formulaic, procedural, sequential, needing to document working) and “grasshopper” (holistic, intuitive, disliking to document working) to label two different thinking styles in mathematics; however he states clearly that these labels indicate two extremes of a continuum, with most learners falling in between. This has not prevented a vast number of lesser educators, on whom this distinction has been lost, instructing teachers and parents to discover in which of two distinct categories their children lie (as others advocate deciding if a child is unilaterally a ‘visual’ or ‘verbal’ learner). This is not to say that the terms are completely useless; they serve a reasonable purpose in describing a student’s position in relation to a particular area of mathematics at a particular time. However, as is clear both from my teaching experience and from research on individual differences, students have different relationships and attitudes at different times and with different areas of mathematics (Dowker et al., 1996), so to sort them into two or more 'either-or' global learning patterns is not only simplistic but potentially detrimental.

2.4 Summary

There have been significant changes in how students with education-related difficulties are perceived, described, assessed, assigned to certain types of educational establishment, and supported once there. While a range of increasingly specific diagnostic labels have been used, there are persisting problems with categorisation, in particular regarding spectra of capabilities, lack of acknowledgement of intra-category heterogeneity, and over-assumptions of permanence. However, ethical considerations relating to how society perceives and labels difference have come further to the fore, along with some significant shifts in viewpoint – for example, in increased criticism of educational models themselves as opposed to individuals whose needs they fit poorly. The emphasis is no longer squarely on the child's 'deficit', nor transferred completely to social causes, but on their interconnection.

The design of my research does not require students to be set formal tests and have their scores compared, and there is no need to identify their overall level in school mathematics any more precisely than 'significantly below average' (which should, of course, not be taken to imply a unitary model of mathematical ability). Although some of the (contemporary) terminology discussed above was used by teachers in conversation with me and each other, and in individuals' SEN documentation, most are unnecessary to my analysis, and so I take a stance of avoiding such labelling practices. When I do use the terms 'difficulty' and 'ability', it is in reference to the specific arithmetical concepts and processes under discussion, and without unwarranted assumptions about what else the individual may or may not be able to do, or have difficulties with. Additionally, these terms are used to describe performances I observed (or, to some extent, from students' own accounts of their experiences in mathematics); if a student was not able to perform a certain task on a given occasion, it cannot be assumed with certainty that they do not possess that ability. Similarly, when I use 'low-attaining', I take the ethical stance of assuming non-permanence, i.e. that there is potential for improvement in all students with difficulties; that at least some of their current limitations are surmountable, given the right conditions. This is not the same as assuming all students <u>must</u> improve in ways that are observable; merely that it is indefensible to assume that any individual <u>cannot</u>.

3 ARITHMETICAL STRATEGIES

3.1 What is meant by a 'strategy'?

3.1.1 Definitions

The term ‘strategy’ is generally used in education and educational research in its common meaning of an action or set of actions designed to achieve a particular goal. This does not mean the term is used equivalently throughout the literature. Strategies are considered by some authors to apply only to the conscious decision-making that goes into planning and solving a nonroutine task (e.g. Pólya, 1945), or in other cases may include all the numerical relationships and procedures drawn upon to carry out a straightforward arithmetical calculation (e.g. Anghileri, 2001). In writings on the subject of arithmetical tasks, there is also considerable overlap in usage of the terms ‘strategy’ and ‘method’. However, ‘method’ is more closely associated with the latter of the two above meanings, and also, I suggest, carries linguistic connotations with the rote reproduction of taught procedures. It is more common to talk of one method, or <u>the</u> method when discussing particular types of (closed) question, but of having a choice between multiple possible strategies when discussing more open-ended tasks.

Within the study of strategies for mathematical problem-solving, there are different aspects of strategy on which to focus: computational strategy, representational strategy, heuristic strategy, etc., and within each of these are also different possible levels of detail on which to focus. For a particular analysis, it might be enough to describe a participant’s actions when working on a particular task as ‘counting-based’, or to specify ‘upward step-counting’, or a highly detailed ‘upward counting with unit counts represented on fingers and the cardinal number of each group spoken aloud’, depending on context. As seen in that last example, the different aspects and levels of detail which may be described do not exist in isolation but affect each other multidirectionally, and descriptions and analyses which focus on one aspect and level of strategy – while often necessary for clarity – tend to carry a host of implicit assumptions. In this study I need to describe various aspects of strategy use, and at various levels of detail, but for an overall definition, look to that favoured by psychology-oriented researchers such as

Bjorklund (and Harnishfeger, 1990; and Hubert and Reubens, 2004, both in Voutsina, 2012, p.p.367) of "strategies are goal-oriented operations employed to facilitate task performance" – in colloquial terms, a means to an end.

3.2 Assumptions about arithmetical strategies

3.2.1 What strategies individuals are likely to use

From the level of national curricula, through mathematics textbooks, down to individual teachers and classroom assistants, there is considerable variation in the emphasis placed on choosing between multiple potential arithmetical strategies, and in the encouragement (or discouragement) of creativity and nonstandard strategies in problem-solving. For different generations, there have been strategies which were considered the single standard method for almost any given problem type, and as such, necessary for all students to master. Current English/Welsh curriculum documentation includes a swing towards the acceptance and encouragement of a variety of arithmetical strategies – although from the statements of teachers, assistants, parents, and indeed government ministers, it is clear that this is not unanimously believed to be a positive state of affairs.

Amongst adults who were taught at a certain age to perform arithmetic a certain way, there are often assumptions that all adults (and children above the relevant age) do it the same way – or ought to do so. A particularly prevalent example is the assumption that after a given period of mathematics education, people must use retrieval from memory of a particular set of number facts to answer single-operation arithmetic questions. This view appears to have been first queried in print by Browne (1906), but nevertheless persisted throughout the twentieth century, with even many recent researchers assuming that calculations involving smaller numbers, e.g. single-digit multiplicands, were so trivial that adults would not need to use any other strategy or procedure (e.g. Ashcraft amd Battaglia, 1978; Campbell and Graham, 1985; Siegler, 1988a; all in LeFevre et al., 2003, p.p.204). However, this view has been challenged, particularly by Dowker (1992; et al. 1996; 2005), who provided strong evidence of adults maintaining the availability of, and using, multiple arithmetical strategies for a given task type. Indeed, in Siegler's later work (Siegler and Shipley, 1995; Shrager and Siegler, 1998) he adapted his previous model of individual strategy development to include 'overlapping waves', thus allowing for a variety of strategies at any given point, and even with different strategies dominant in different contexts (see 3.5). It is now, one hopes, generally considered unacceptably biased in either research or teaching to make prior assumptions that individuals will necessarily go about a given task in a particular way.

3.2.2 What strategies an observed individual is using

With what confidence can a teacher or researcher claim to know what occurred in the passage between setting an individual a task and receiving an answer? In some cases, it would initially appear that the strategy is clear: a formally-written calculation in columns, with all decimal exchanges noted, for example, or a student step-counting aloud and in rhythm; in other cases, the participant's working appears to exist in a 'black box' state, with a single number – correct or incorrect – emerging at the end. However, neither of these cases is quite as clear-cut as it may initially seem. In the imagined example of a student performing a formally-written calculation, there is still much that is unknown, particularly with multi-stage calculations. Within the separate stages of a multi-digit addition, might they be unit-counting? For the stages of a subtraction, are they counting backwards or counting on ('shopkeeper' strategy)? Are individual multiplication facts retrieved from memory, or do they have to be calculated in sub-steps, or does it perhaps vary depending on the numbers concerned? On the other hand, in what are superficially 'black box' cases, there is usually some data with which to build an informed hypothesis about the kind of thought processes occurring. For example, when reporting children's division strategies, Anghileri (e.g. 1997) describes their speech patterns (such as the numbers which are spoken aloud, and the vocal emphasis they are given), their gestures (such as the order and rhythm of their finger movements), and their various interactions with representations external to themselves. Similarly, in studying children's conceptualisation of multiplicative structures, Battista (1999) observes the order in which they point to different parts of a cuboid, and by combining it with verbal data, builds sturdy hypotheses of the details of their enumeration strategies. Recent technological developments have also allowed for the tracking of an individual's eye movements while engaged in screen-based tasks (e.g. Lindstrom et al., 2009). While it would obviously be inappropriate to claim certainty of another person's mental processes, by collecting this kind of multimodal data, and triangulating participants' words, gestures, and interactions with external visuospatial representations, it is in many cases possible to infer arithmetical strategies with a high degree of confidence.

3.2.3 Self-report

Researchers have also asked their participants directly about their arithmetical strategies, particularly in studies relating to recall of number facts, featuring one conventional formal procedure versus various other potential informal strategies. Unsurprisingly, children can be unreliable in their self-reporting, for example insisting that they “just remembered” a multiplication fact after being observed finger-counting. There are various reasons a child might give such misinformation, such as having inadequate language skills to understand what they are being asked, inability to remember their recent actions in sufficient detail, undeveloped metacognitive skills (i.e. inexperienced at thinking about thinking), or – particularly undesirous – a wish to give the adult whatever answer the child thinks will please them. With adult and secondary-age participants, there is greater likelihood they will comprehend what they are being asked; however, researchers, teachers and psychologists who spend a great deal of time thinking about thinking must be wary of significantly overestimating participants' familiarity with metacognition – they may be unused to this kind of self-analysis.

These issues suggest not that self-report should not be used, but that individual participant statements should be treated with caution unless substantiated by confirmatory evidence of another form (e.g. observed actions). A study by Robinson (2001) found evidence for the validity of 6-10 year olds’ verbal reports of their subtraction strategies. Children’s self-reporting can also give insight into unexpected and puzzling findings, such as when Baroody et al (1983, in Baroody and Ginsburg 1986, p.p.102) were studying addition and subtraction strategies, and found an efficient ‘shortcut’ being used far more frequently by younger children than older ones (with similar results from Bisanz, LeFevre, Scott, and Champion (1984), cited in the same publication). One girl’s comment that "I cheated on that one; I looked at the [previously computed sum]" led to the plausible explanation that after greater exposure to school mathematics lessons, children might come to believe that when set an arithmetic problem, they are supposed to calculate it ‘properly’ (i.e. in the standard method their teacher had demonstrated), rather than making use of any patterns they might notice, and that not going through the full calculation procedure was tantamount to cheating!

(Baroody and Ginsburg, 1986, p.p.102) Such a belief can only be reinforced by the oft-heard “Show all your working for each one”.

3.3 Types of strategies

All tasks featured in this study involve multiplicative structures of some kind, and involve some form of multiplication, partitive division (sharing) or quotitive division (grouping). Although my tuition sessions were designed with intervention in mind – all the selected students were at that point performing very poorly in mathematics lessons – the intention was not that they should simply become faster and more accurate at answering standard division (and multiplication) questions – although this could be a possible secondary outcome – but that they should have a more thorough conceptual understanding of multiplicative structures, and, in consequence, be able to form calculation strategies for division (and multiplication) based on their understanding and on any number facts and relationships they were able to recall at that moment. The kind of learning that involves deliberate commitment to memory of ‘times tables’ and formal procedures is entirely outside the scope of this project. The focus is on how one might go about an arithmetical task when simple recall of the required number facts and standard procedures is unavailable or unreliable, the options being (1) to count, or (2) to build on other number relationships given or known. These two options are somewhat consistent with the idea of *procedural* vs *deductive* strategies (Gray, 1991), although it should be pointed out that Gray also emphasises that knowing how to solve a task by counting can be “a significant cog within the . . . mathematical entity” and “a link in the conceptual chain which leads towards the growth in relational understanding” (p.p.552). In 3.3.1 and 3.3.2, respectively, I discuss a selection of theory and evidence relating to these two broad strategy types: counting-based and derived-fact.

3.3.1 Counting-based strategies

Strategies based on counting are used for arithmetical tasks involving all four operations, and include the simple upward counting of the natural number sequence (from 1 or another point), reverse, grouped, rhythmic, step/skip and double counting. Exhibited by many pre-school children, to count is the first requirement listed in the ‘Number’ category of the National Curriculum. Counting in ones is a skill widely assumed to be something all can do (save those with severe or profound learning difficulties), although there have in fact been various studies providing evidence of significant individual differences in the speed and accuracy of counting in upper primary-age children (e.g. Houssart, 2001) and older, sometimes with comparisons between participants with different levels of academic attainment (e.g. Geary and Brown, 1991). Indeed, many of the secondary mainstream students in this study displayed errors, in some cases quite startling ones, despite the frequency with which they chose counting as a calculation strategy (see Chapters 6-8). Here, I address previous research on when, why and how students use counting-based strategies for arithmetical tasks.

3.3.1.1 Counting in additive structures

There are well-established, flexibly hierarchical taxonomies for basic addition and subtraction strategies that use counting. For example, the most common counting strategies used for addition, in order of development, are generally deemed to be:

Addition example: 3 + 6

> **Count all** Count out a set of 3, count out a set of 6, count the union set.
> **Count on** Start from the first number (3) and count on 6.
> **Count min** As above, but starting with the larger number (6).

The above processes (and the various others) may be performed using fingers, physical objects, marks on a page, gestures or words, in both external (e.g. speaking aloud, moving sets of objects, extending fingers) and internal forms (e.g. running through counting words in head, visualising sets of objects, looking at fingers and focusing attention on them without movement). Note that while equivalent in enumerative terms, these representational media have different affordances and should not be considered cognitively equivalent.

Although it is uncommon to find an older child or adult (again, excepting those with severe or profound learning difficulties) using ‘count all’, it is unexceptional to find them sometimes using counting-based strategies for mental addition or subtraction calculations. Counting (verbally, with fingers, or both) is retained into adulthood as a supplementary or backup strategy called upon for various reasons, including but not restricted to when (a) normal cognitive activity has been reduced, perhaps through tiredness, illness, etc., (b) sensory distraction is present to disrupt fact recall and calculation processes, (c) concurrently performing another task, (d) in detrimental affective states, and (e) the answer is particularly important and a higher than usual level of confidence is required. Of course, temporary circumstances such as these could affect any person’s performance on a given day, but regarding general arithmetical behaviour, there are studies (Geary, Bow-Thomas, and Yao, 1992; Gray and Tall, 1994; Ostad, 1997; 1998; Siegler, 1988; all in Dowker 2005) indicating that children with arithmetical difficulties are significantly more likely than their typically-attaining peers to rely on counting-based strategies (as compared with, e.g. recalled-fact or derived-fact strategies). This is unsurprising. What is perhaps less obvious is that the causation in this relationship runs both ways: over-attachment to counting strategies may contribute to arithmetical difficulties by inhibiting procedural and/or conceptual development, and in these cases a 'reasoning habit' must be deliberately fostered (Yeo, 2003).

3.3.1.2 Counting in multiplicative structures

Counting-based strategies are also important in multiplication and division tasks, but research on them is considerably less exhaustive than that on addition and subtraction – although this deficit is decreasing. This situation may have been partly due to the international emphasis in schools on memorisation and deployment of multiplication facts (or ‘times tables’), leading to a great many studies focusing on whether participants of a given age demonstrate successful direct recall of these facts or not. Unlike with additive structures, there is no one widely-accepted taxonomy of multiplication or division strategies, and neither can they be as easily fitted into a hierarchy (due in part to the parallel partitive and quotitive models for division); however, strategies which do not involve fact retrieval have been addressed by authors with a particular interest in the foundations of multiplication and division (e.g. Anghileri) or in concretely-represented multiplicative structures (e.g. Battista), and discussed in some detail within more specialist SEN-oriented literature. One complication in is that commonly-used classifications such as those in Anghileri's (2001) study (on Year 5 students' division) mix aspects of representation (e.g. "Using tally marks or some symbol for each unit" – where those marks could be used in a variety of ways) with aspects of numerical process (e.g. "Repeated subtraction of the divisor from the dividend" – where that subtraction could be accomplished via a variety of representations). While the visual representation chosen with which to work on a task is certainly part of the solution strategy, it is a different dimension of strategy from the arithmetical structure aspect, e.g. unit counting vs. repeated addition (and thus is addressed separately here, in Chapter 4). On the other hand, Kouba (1989) proposed classifications of multiplication and division strategies to separate out aspects of arithmetical strategy (including various types of counting) and representational strategy, but does not include derived fact strategies (see 3.3.2), which have since come to be considered very important.

3.3.1.3 In the early stages of learning multiplication and division

It is in the literature on Early Years education that detailed discussion of counting-based methods is found, and Anghileri has contributed a great deal to the study of children's developing understanding of multiplication and division. It has been shown that children as young as four years can perform certain grouping and sharing division procedures with concrete objects (Carpenter et al, 1993; Kouba and Franklin, 1995; in Anghileri 2006), after which a move to finger counting is frequently seen. What is sometimes inaccurately described as a single ‘finger counting stage' may actually comprise a progression from ungrouped unitary counting, through grouped unitary counting, rhythmic grouped counting with increasing emphasis on the terminal numbers of each count, and then step- or skip-counting (i.e. using a recalled number pattern) (Anghileri, 1995). However, as noted above, although in this case she describes this progression in terms of finger-counting, it could be applied equally well to different representational forms – for example, the counting of tally- or other drawn marks, or verbal counting with no external visual or kinaesthetic representation. The advantages and disadvantages of different representations of number for counting strategies will be discussed later, but it is important briefly to note that while certain representational forms may be particularly helpful or appealing to particular students, the forms that work best for them when dealing with additive structures cannot be assumed equally successful for multiplicative structures.

Nunes and Bryant (1996), among many others stretching all the way back to Piaget and colleagues, suggest that to understand multiplication and division represents a significant qualitative change in children's thinking (compared to understanding addition and subtraction). As an illustration of this, Anghileri (1997) points out (a) that a counting strategy for a multiplicative task requires three distinct counts: the number in each set, the number of sets, and the total number of items; and (b) there is transfer between counting-string and cardinal uses of number words. For the development of such multiplicative understanding, Nunes and Bryant, among others, have recommended the use of a *replications* model of multiplication, which is highly relevant to counting-based strategies. The use of a replications model is sometimes incorrectly assumed equivalent to the use of a repeated-addition strategy, but while these two things do often go together in practice, there is theoretical separation. In working out, for example, the total number of wheels on a given number of cars (a popular task which I also use), the internal or external visualisation of three cars may prompt the calculation 4 + 4 + 4; however, this is not cognitively equivalent to activating the count sequence 1, 2, 3, <u>4</u>; 5, 6, 7, <u>8</u>; 9, 10, 11, <u>12</u>. Hence, although addition is presumed to be generally easier than multiplication, and usually mastered at an earlier age, it is perfectly possible for an individual successfully to solve many multiplicative tasks without the repeated-addition model, or in fact any addition skills at all. Similarly, while unlikely given the progressions in the majority of curricula, there is no theoretical reason why division could not be the first type of arithmetical calculation in which a learner succeeds.

3.3.1.4 Continuing use of counting-based strategies

Having observed and documented, in her various studies, a wide variety of counting-based strategies used by primary-age children, Anghileri observes "[I]t is quite worrying when these methods are used by Year 6 pupils about to move on to secondary school" (Anghileri and Beishuizen, 1998, p.p.3). How much more concerning, then, is it to observe – as did I – these methods in use by a Year 10 student, soon to move on <u>from</u> secondary school? A single observed occurrence of unit-counting in an adult (or near-adult), as I have said, tells us very little; however, when repeated observations imply that counting is the only strategy the individual in question feels they can rely on, there is a case for concern. There is, however, a dearth of research which even mentions the use of counting-based arithmetical strategies by students in secondary mainstream education, and my searches have revealed none where it is the main focus of investigation. Perhaps this is to be expected: those older students who are still counting-reliant form a small proportion of the population (although perhaps not as small as is generally assumed). While I criticise the use of pedagogical materials intended for normally-attaining younger children with lower-attaining older ones, in my analysis it will thus be necessary to compare the counting-based arithmetical strategies of my struggling KS3-4 students against those of the normally-attaining KS1-2 children in the studies of Anghileri, Nunes and the others.

3.3.2 Derived fact strategies

3.3.2.1 Recalling and deriving

Educational literature is full of references to 'recalling' and 'retrieving' arithmetic facts. Here, in a practical sense, it may be taken to mean that on being asked one of a given set of arithmetic questions (e.g. “how many sixes go into 18?”), the person 'just knows' the answer without engaging in any counting or calculation processes, perhaps thanks to having memorised the verbal string “three sixes are eighteen”. I have already discussed one option of what a person may do in response to an arithmetic question, when no answer automatically appears in their mind: count. A second, more advanced, option is the use of Derived Fact Strategies (DFS). I begin with two anecdotal examples.

> **Example 1**
>
> Rob was an 11-year-old boy I taught (prior to the current study), who at the time I first encountered him displayed almost no recall of any number bonds (even 5 + 5 = 10, which had to be worked out via finger-counting). For addition and subtraction calculations involving totals of up to around 20, he unit-counted, supported by fingers, counters, tally marks, or the like; for multiplication calculations (when expressed in equal-sets form) he could also use unit-counting, this time with rhythmic grouping, again supported by some external visual or kinaesthetic representation. Although he did generally have difficulty committing information to memory, a significant problem regarding the information involved in primary school mathematics appears to have been that it had not been presented to him in a structured, connected or hierarchical manner. He was aware that other children had mental access to 'times tables' but, unaware that some of these were more useful than others, and that some could be worked out from others, rejected the entire undertaking as impossible. Likewise, it had not occurred to him that instant recall of number bonds for 10 would increase efficiency of calculation any more than would learning the answers to any arbitrarily-chosen number combinations – in which context there seemed little sense in trying to learn any of them. Like others described by Anghileri et al

(above), he was highly attached to the most basic of counting strategies. After intensive tuition emphasising patterns and the connectedness of numbers, explicitly prioritising some numerical relationships and the way they could be used to find out others, he was able to make use of those number facts he could recall, and those he had recently worked out, in the calculation of further answers. However, it is telling that he referred to this – with some delight – as 'cheating' (cf. Baroody and Ginsburg, 1986, above).

Example 2

The other example is from my own education. With obvious caveats regarding the accuracy of childhood memories, I have a clear memory of, at the age of 8 or so, enjoying writing or reciting number patterns (e.g. 3, 6, 9, 12, ...) but disliking being asked to write out or recite the multiplication facts 'in full' (e.g. 1 × 3 = 3, 2 × 3 = 6 and "one three is three, two threes are six", etc.), as I saw no advantage to the extra time it took. I was considered by my then teacher to be very good at mathematics, apart from anomalous low scores in the weekly quick-fire 'tables tests' popular at the time. When questioned about this, I said that I didn't know how the other children were <u>working out</u> the answers so quickly – I had been quietly doubling, reversing, adding and subtracting, step-counting and reciting bits of pattern, sometimes quickly enough and sometimes not. In fact I could retrieve a fair number of the multiplication facts from memory – in the sense that sometimes the answer to a question would 'pop up' before I had begun to calculate it – but I had perceived this as a fortunate accident that tended to occur with numbers I liked, such as the squares, rather than being the (teacher's) actual desired outcome. On being informed that the other children were memorising a particular set of multiplication facts – which was genuinely surprising – it became clear what all the tedious writing and reciting had been about (although the overall aim of this memorisation exercise was still never directly clarified). From then on, I looked at the required answers for a given week's 'table' and contrived to hold them in my head just long enough to pass the test, which pleased my teacher. However, to me, this was the strategy that felt like ‘cheating’.

An assumption made frequently in non-academic educational literature is that a person either knows something or does not know it – a simple categorical situation – and for a student to 'know' something carries implications about what new knowledge may be built upon that foundation. As with most binary views (here and elsewhere), this is a massive and problematic oversimplification. To return to the subject matter of multiplicative relationships: even in the historic case of times tables being memorised one at a time, in ascending sequence, the process would be repeated or the information otherwise revisited, because it was understood that without practice and revision, it would not be retained. Chinn and Ashcroft (1998, p.p.65) suggested the teacher introduce "learning-check charts with headings 'Taught, Revised, Learnt' . . . [with] a fourth column for use with dyslexics, 'Forgotten' " (although forgetting one's times tables is hardly unique to people with dyslexia!) (Yeo, 2003, p.p.74) suggested that teachers should know whether their students "are able to quickly access individual tables facts, or whether they always step-count from the beginning to work out a given tables fact", and it is the "always" that is of particular interest here, for it carries an entirely different and important assumption – that not only can information be learned and forgotten, but it may be recalled and used on some occasions but not others. To talk of what arithmetical facts, concepts, procedures and strategies a student 'knows' – particularly in the case of those, like my research participants, with significant difficulties in mathematics – does not make sense. Between knowing (i.e. consistent and reliable recall) and not-knowing (i.e. complete unfamiliarity or consistent non-recall), there is considerable grey area, a spectrum of partial knowledge and potentially complex interconnections.

The concept of DFS relies on two assumptions, one trivial: that a student cannot memorise and perfectly recall the infinite number of arithmetic facts that they might potentially have use for in their lives – and one non-trivial but which may be reasonably assumed for all students in mainstream education: that a student can memorise and recall some arithmetic facts, some of the time. The issues then become of which numerical relationships to prioritise committing to memory, and how to make the best use of the facts that one has. Curricula through history and across cultures have reflected different opinions regarding the first of these, and often contained surprisingly little on the second. They also do not tend to acknowledge that some children (such as those in my examples above) require a meta-level understanding about the utility of recallable number bonds, without which they may not be willing to commit to the time-consuming task of memorisation.

3.3.2.2 DFS and the curriculum

Much of the discussion of DFS in educational literature focuses on additive number relationships, but the principles behind some of the most common addition/subtraction strategies apply equally well to multiplication and/or division. Some examples are:

> **Reordering** Using the commutative principle to put the terms of a multiplication into an order which makes retrieval, counting, or any other strategy easier.
> **Transforming operations** Recognising division as the inverse of multiplication, transforming a multiplication or division into repeated addition, etc.
> **Step-counting from known fact** E.g. 13 × 4 counted as 40, 44, 48, 52
> **Decomposition** Breaking up one or more of the terms: e.g. in a multiplication, treating 13 × 4 as (10 × 4) + (3 × 4) (distributive principle).

In 3.2 I discussed some of the assumptions about arithmetical strategies. There is one which seems to arise particularly regarding the use of DFS: that those students for whom the standard, taught methods are not a good ‘fit’ will work out alternatives for themselves. For multiplicative relationships, this could mean that all students are instructed and encouraged to commit the prescribed set of multiplication facts to memory, with the expectation that those who cannot reliably recall the correct text string on cue will independently find some other way of achieving similar results (as in my Example 2 above). It is a comparatively recent development to explicitly teach children – particularly those with a history of low attainment – that they are not restricted to the options of either ‘recall’ or ‘guess’ (or avoid!)

Although it is rarely stated explicitly in curriculum documents, there is a relationship between the calculation strategies it is considered desirable for students to be able to use and the number facts it is considered desirable for them to be able to recall – as, if an answer cannot be recalled, it must be calculated. From this follows that, explicitly or implicitly, there must be a hierarchy of importance for both facts and strategies emphasised in class. This was perhaps less the case in the past, where, to take the example of multiplicative structures, a student might (as was my experience in the 1980s) work through rote memorisation of the 'times tables' in ascending order from 1 through to 12, followed by rote memorisation of the procedures for short then long multiplication and division, after which they were said to be able to ‘do’ natural-number arithmetic. It is striking to consider that, at least from a student’s point of view, each part of this body of knowledge was separate and given equal weight, which contrasts starkly with more recent connectionist models of learning, where relationships and pattern recognition are at the fore. Chinn and Ashcroft's (1998) programme of study, for example, while still placing an emphasis on memorisation of the 'times tables', re-ordered them according to basic principles (e.g. identity), their importance within the structure of the decimal system (e.g. prioritising 10, 2, and 5), and their relationship with the patterns already encountered – while also introducing the commutative principle at an early stage. Yeo's (2003) system was a further adaptation of this, which, while retaining memorised facts as an ideal end-point, placed even greater emphasis on the relationships (e.g. between 2, 4 and 8). Of course, some students recognise and use these relationships without being taught to do so, but for others – for example, many of the dyslexic students for whom Chinn and Ashcroft, and Yeo, originally devised their systems – an active emphasis both introduces new information in a connected way and encourages the development of not only strategies based on those connections, but meta-strategies for obtaining as yet unknown arithmetical information.

The National Numeracy Strategy was generous in its acknowledgement of alternative strategies for mental and written arithmetic, stating, for example, “Through a process of regular explanation and discussion of their own and other people’s methods they will begin to acquire a repertoire of mental calculation strategies” (DfEE, 1999, p.p.7), including “personal jottings” and “part written, part mental methods”. Specifically, the National Curriculum states that KS1 students should “develop a range of mental methods for finding, from known facts, those that they cannot recall” (QCA, 1999, p.p.16). However, there have been many issues regarding classroom implementation. A highly enthusiastic, highly mathematically able teacher who introduces their class to a rich panoply of strategic ideas may leave certain students "totally strategied out" *[sic]* (words of a 9-year-old dyslexic girl, in Yeo, 2003), while another teacher, mathematically capable but inflexible, may very willingly present to their class all the different strategies listed on the curriculum, but in a way such as they are learned by rote by a significant proportion of students; they may also state connectionist beliefs about learning while displaying *transmission*-oriented practices (Askew et al., 1997). This issue is, in theory, addressed later, when KS2 students are supposed to “understand why the commutative, associative and distributive laws apply to addition and multiplication and how they can be used to do mental and written calculations more efficiently” (QCA, 1999, p.p.23). This is undoubtedly a good thing, but there is an assumption that the strategies taught to students will be immediately connected up to previous knowledge and transferred to other appropriate tasks and contexts. Although some will do this, the students who are most desperately in need of flexible understanding-based strategies are often those who find them the most difficult to pick up in a short space of time, and least able to connect them to other knowledge (Gray, 1991; Yeo, 2003). This could be viewed as an ongoing self-fuelling lack of metacognitive skills (Karsenty et al., 2007 ; Kroesbergen and Van Luit, 2003 ; Zohar and Peled, 2008).

3.3.2.3 DFS and the individual

Dowker states that "[p]erhaps one of the most crucial aspects of arithmetical reasoning is the ability to derive and predict unknown arithmetical facts from known facts, by using arithmetical principles" (2005, p.p.123). This is a stance with which it is difficult to argue, as without any ability in using the interrelationships between numbers and processes, one would be utterly dependent on an insufficient pool of rote-learned information. In actuality, a wide variation has repeatedly been shown from person to person in both the quantity and frequency of DFS used in arithmetic, and the range of different strategies used. There is a growing body of qualitative accounts of individual children and adults using DFS to compensate for various arithmetical issues (e.g. dyscalculia, in Hittmair-Delazer, Sailer and Berke (1995; in Dowker 2005); dyslexia and dyspraxia, in Yeo (2003); a low boredom threshold for memorisation, as in Example 2 (above), or the various labelled and un-labelled difficulties experienced by the students in this study). However, quantitative studies on strategy use (ages 5-9, in Dowker (1998), ages 6-8, in Canobi et al. (1998), ages 16-18, in Hope and Sherrill (1987)) indicate that, in general, it is the more competent calculators who show the greatest use of DFS. This is not a contradiction, as the more number facts and principles one has in place, the wider the scope of potential strategies that might make use of them.

According to Dowker, there are three areas of understanding that a student must have in order to use DFS: (1) the underlying principles on which the strategies are based; (2) strategy selection, implementation, and appropriate and flexible usage; and (3) the capacity for unknown fact derivation, i.e. "the ability to cope with uncertainty and to realise that the absence of a memorised fact or well-learned procedure does not imply a lack of any knowledge at all about the arithmetical problem" (2005, p.p.138–9). I fully agree with these, but will comment additionally on each.

(1) Note that observed use of a given arithmetical strategy does not necessarily imply conceptual understanding, but some strategies carry greater implication than others. For example, a student successfully using a commutativity-based strategy (e.g. converting the multiplication ‘five sevens’ to ‘seven fives’) may well have simply learned the trick that you can switch the order of a multiplication, as opposed to understanding that the two calculations are structurally equivalent so must necessarily represent the same total. On the other hand, strategies such as using powers of two (i.e. repeated doubling or halving), or step-counting on/back from a known multiplication fact, are based on some conception of multiplicative structures and relationships. Likewise, while there are various ‘tricks’ for nines (e.g. holding out ten fingers and tucking under the nth finger to display $9 \times n$), a student who can see $9 \times n$ as $(10 \times n) - (1 \times n)$, can not only use it beyond the range of the ‘times tables’, but soon also multiply by 99, 999, 19, etc.

(2) As suggested above, I consider it a part of any proposed ‘flexibility’ with DFS that students should not only be able to select and implement appropriate strategies, but to extend them further. Students should be able to test known strategies under new conditions or in new scenarios, and judge whether they still work or not (e.g. commutativity and division).

(3) This aspect is a little different, as, in addition to requiring certain cognitive/metacognitive understanding on the part of the student, it is an affective issue, touching on their relationship with mathematics as a school subject, and perhaps their more general attitudes to education. Uncertainty can be highly unsettling, particularly for children lacking in confidence in a subject area. A popular perception of mathematics is that it is rife with right-or-wrong, tick-or-cross certainties – not only in terms of the answer, but “the” way to work it out. To be put in a position of not having clear rules for what to do may provoke considerable fear and distrust in some, and these reactions must be overcome for flexible strategic ability to develop.

These points implicitly include the understanding that there is frequently more than one possible strategy to use for a task: I suggest it is worth stating this explicitly.

Additionally, students must be willing to discard strategies which are not working, return to known facts and try alternatives. This may seem obvious, but when any task requires considerable effort from a student, they are prone to sunk cost bias: not wanting any of the hard work already done to ‘go to waste’, so persisting with a strategy they know is unreliable or inefficient.

There has been considerable research on children’s use of DFS. However, as with research on counting, the majority of participants have been primary-age children working on enumeration in unary additive structures including single-digit addition/subtraction (e.g. Carpenter and Moser, 1984; Carpenter et al., 1988), additive commutativity (e.g. Cowan and Renton, 1996), and decomposition in multi-digit addition/subtraction (e.g. Carpenter et al., 1998), while one study of Dowker’s (1998) focused specifically on derived fact usage. As yet there have been few studies covering DFS in secondary students or adults, despite the ongoing usage and the probability of extensive individual differences in the extent, nature and variety of strategies used (Dowker, 2005). An early study by Baroody on “mentally handicapped” participants (1987) had an age range going up to 21, but again, focused only on addition tasks. Another study (Hope and Sherrill, 1987) looked at mental multiplication strategies in 16-18-year-old “skilled” and “unskilled” participants. This is of particular interest, as the two groups used significantly different kinds of strategy, with the “skilled” students making greater use of number relationships, and in more flexible ways. Additive, subtractive, quadratic distribution and factoring strategies were reported; in contrast, “unskilled” students often attempted a mental analogue of traditional pencil-and-paper methods.

Of the studies of primary-age children working on enumeration in multiplicative structures, Mulligan's (1992) longitudinal study used a two-dimensional framework of representational and arithmetical strategies, based on those of Kouba (1989) and Carpenter (and Moser, 1984; et al., 1988). However, unlike Kouba, she included DFS as a separate category, and identified it as one of the final stages of a general progression through increasingly advanced types of counting to greater reliance on recalled number relationships (see also Mulligan and Mitchelmore, 1997). Sherin and Fuson point out that children acquire a great deal of knowledge about certain specific numbers and their relationships, e.g. 4, 12, and 32 (2005, p.p.348) – my own students have been known to ask why I “love the number 24 so much” – and a situation where certain multiplicative relationships are well known, while others are not, is ideal for developing derived fact strategies. These authors also contributed to the theoretical frameworks for multiplication by including ‘hybrid’ strategies; as will be seen in Chapter 6, this kind of mixing of strategies, which I term *inconsistency of enumeration*, is important in analysing students’ arithmetical thinking.

Carpenter et al.'s (1993) study of kindergarten students (i.e. age 5-6, with <1 year of formal schooling) demonstrated that they could carry out a wider range of multiplication and division tasks, with greater success, than had formerly been realised – provided the tasks were presented in the form of scenarios which could be directly modelled. Furthermore, they argued that many older students abandon their fundamentally sound and powerful general problem-solving approaches for the mechanical application of formal arithmetic procedures (Carpenter et al., 1993), and would make fewer errors if they applied some of the intuitive, analytic modelling skills of their younger counterparts. In contrast, more recently, Brissiaud and Sander (2009) have suggested that when children have intuitive, scenario-based strategies (e.g. mental representations of sharing and grouping for division) in place for solving arithmetical problems, they may continue to use them as number sizes increase, even after mastering formal calculation strategies, for as long as they remain efficient. When doing so, they are missing out on the opportunity to practise and strengthen strategies based on derived facts and/or principles such as commutativity. (Of course, this cautionary note could be – but rarely is – applied to a student who is able to memorise a great quantity of poorly-interconnected number facts!)

Two quantitative studies from 2006 address multiplication and division strategies in older children, with age groups (8-12 and 11-14, respectively) that overlap with my participants. The first (Robinson, Arbuthnott, et al., 2006) reported that the proportion of direct retrieval strategies was surprisingly low (<20% of problems) and fairly constant across the whole age range. Use of addition strategies was high in younger students and steadily decreased, while replacing the division with its inverse (multiplication) steadily rose. Student self-report is used to triangulate script and observation/timing data, and while the reporting appears reliable for the level of analysis found in the paper, it cannot differentiate in fine enough detail to tell whether a response categorised as ‘multiplication’ has itself been achieved by retrieval of a memorised multiplication fact, or by fast DFS calculation – and while it is true that these are both ‘not-division’, they are quite different cognitive processes. While there is an argument that as the most recently learned, division facts will be less well-recalled, I suggest the reported results imply instead that many of those students who are confident retrieving multiplication facts, and understand the idea of inverse operations, do not see the need to memorise a separate set of facts for division, but make sensible use of the knowledge they already have.

The second paper (Robinson, Ninowski, et al., 2006) reports on, among other things, the use of the multiplicative inversion principle, in questions such as (9 × 6 ÷ 6). The authors state “Overall, 63.8% of Grade 6 students and 76.7% of Grade 8 students reported using the Inversion strategy at least once on the addition/subtraction inversion problems” (p.p.354) and comment on the greater understanding of inversion shown by older students. However, given that they state that in (Canadian) schools “Children are taught division . . . typically as the inverse, or reverse, of multiplication” (p.p.359), it is startling that such a significant proportion of the study participants did not spot (or ignored) the replicated number and inverse signs, but took the much harder route of working through the calculations. As proposed in 3.3.2.1, I suggest that the students wished to impress the observers with their skill at calculation, rather than use ‘cheating’ short cuts. Another possibility is that even if children have been allowed or encouraged in school to “solve simple division problems . . . by using related multiplication facts and rarely by retrieving division facts directly” (p.p.359), what they may have learned from more procedural-oriented teachers is not a conceptual understanding of inverse operations, but a non-transferrable ‘trick’ for turning a division into a multiplication-with-missing-multiplier/multiplicand.

3.4 Scenarios and strategies

3.4.1 Task terminology

The genre of task used most frequently in my fieldwork bears a resemblance to those referred to in both research and pedagogical literature most frequently as ‘word problems’ (also ‘story/real-life problems/questions’). These expressions lack clarity: the term 'word(ed)' could refer to any task expressed in verbal form, whether it involves extra-mathematical content or simply differentiates the worded “What is twelve plus two?” from the symbolic “12 + 2”. 'Story' implies the existence of some basic narrative involving a starting situation which is altered by some action, producing an end situation which provides the 'answer'; this is an inappropriate term for tasks based in static settings (i.e. not involving change over time). The descriptor ‘real-life’ is particularly problematic (although this did not prevent it appearing in the National Numeracy Strategy, which suggested students practice “solving problems involving numbers in context: ‘real life’, money, measures” (DfEE, 1999, p.p.39)) and is critiqued from a methodological viewpoint in 5.4.2.1. The term ‘context(ual)’ problem is also sometimes used. I have no semantic objection to this, but use *context* to refer to the actual surroundings in which a participant is engaging with a task (i.e. in or out of class; with or without teacher support, in a 1:1 or paired situation, etc.), and *scenario* to refer to the world described in a particular task, e.g. sharing biscuits, catching buses. I also prefer *task* over 'question' or 'problem'. 'Question' will be used in its grammatical sense, as a particular type of utterance which occurs during a task, intended to prompt thought or verbal response. 'Problem', with its negative connotations, I use to refer to an obstacle or difficulty which occurs during work on a task.

For discussing my own data I will be using the terms *scenario tasks*, which include extra-mathematical objects and activities (e.g. buses, biscuits, parking, packing) and *bare tasks*, which refer only to numbers and operations (although some scenario-based terms are so commonly used in otherwise bare mathematical activity (e.g. takeaway, share) that the categorisation is necessarily a little blurred at the boundary). Content and presentation of tasks will be discussed in detail in 5.4, but briefly, the main differences between my scenario tasks and traditional word problems are in their emphases on being flexible, extendable, individually negotiated, and relatively straightforward to represent in a manipulable visuospatial format.

3.4.2 ‘Word problem’ strategies

Scenario tasks are central to this study, as they lend themselves particularly well to visuospatial representation, with even my extremely basic visual/kinaesthetic materials affording considerable variety of options, and opportunities to observe the close relationship and interaction between representational and arithmetical strategies. While there is very little previous research using what I class as true scenario tasks, there is a considerable body of studies involving observation of participants undertaking traditional textbook-style ‘word problems’. As with the DFS literature discussed above, secondary-age students are an under-represented group. In those studies which do include participants of 11 upwards (e.g. Fischbein et al., 1985; Bell et al., 1989), tasks are frequently more complex, involving more than one stage, using larger, rational numbers, and also requiring a greater degree of extra-mathematical knowledge to comprehend the presented information, so those with younger participants again form a more apt model and comparison.

A recent longitudinal study series by Brissiaud and Sander (2010) focused on a cohort of primary-aged children working on subtraction, multiplication and (partitive and quotitive) division-based tasks. All the tasks were set in text form, involving so-called ‘real-life’ settings, and the key aspect of interest is that the numbers were carefully chosen and matched so that in each case the most obvious, intuitive solution strategy was either easy to carry out or difficult (e.g. a multiplicative structure prompting the repeated addition of 3 lots of 50 versus 50 lots of 3). Results provided support for a framework the authors call *Situation Strategy First*, in which children first build up some kind of model of the situation, then assess it for efficiency (e.g. does it call for performing 49 additions, as opposed to two), and only if this initial representation seems too “high-cost” do they make use of their knowledge of arithmetical rules which would enable them to transform the calculation into one more efficient and reliable. While this effect appears yet to be tested with older children (or adults), it is a plausible hypothesis which has pedagogical and methodological implications: careful manipulation of the numbers in scenario tasks to which students became accustomed could ‘nudge’ them into making use of, for example, the commutative principle, while helping loosen their grip on certain enumeration strategies to which they had become over-attached.

3.4.3 Unpopularity of word problems

One attribute of interest regarding scenario tasks is the ‘conventional wisdom’ that word problems are particularly unpopular among children – for example, Askew describes them as "the castor oil of the mathematics curriculum: fairly unpleasant but possibly good for you" (2003, p.p.78). McLeod describes these negative attitudes as “well-established” (1992, p.p.589), citing other studies (e.g. Marshall, 1989, ibid.) which describe some children’s distress during “story problems” (although in fact other children in that study reported positive affect). Leaving aside those children and adults who enjoy and excel at mathematics and puzzles for their own sake, but focusing on low-attaining students, is it actually the case that students particularly dislike tasks that are couched in some kind of extra-mathematical setting? It does not fit with my teaching experience, even amongst severely dyslexic students, whom one might have expected to have a grudge against tasks involving more words. I suggest that the commonly-accepted discourse around ‘word problems’ (a) rests on a shared non-intergenerational experience of now-outdated styles of mathematics teaching which have nevertheless become entrenched in popular (English-speaking) culture, (b) focuses mainly on the experience of neurotypical, normally-attaining children, while ignoring the experience of those with significantly underdeveloped numeracy levels, and (c) confuses content with presentation, when the problem is frequently the poor choice or presentation of tasks rather than the presence of extra-mathematical content.

Regarding the first point, I can suggest several possible reasons. People who did not enjoy their experience of school mathematics may, decades later, more readily recall superficial aspects of the lessons than actual mathematical concepts: boards covered with Euclidean geometric figures, worksheets full of sums, and apparently pointless problems about trains (velocities, distances and times), baths (rates of flow from taps and plugs) and suchlike. This is not the place for an analysis of mathematics teaching materials over the last 50 years, but there have certainly been great changes in linguistic and visual presentation: children’s experience of school mathematics is not as it was. Another potential major factor is changes in teaching methods, particularly classroom discourse around problem solving. The recent increased emphasis (in England/Wales) on teaching children that the same task may be represented and solved in different ways, and a range of strategies are acceptable (DfEE, 1999), stands in contrast to traditional emphases on mastery of standard calculations, or in the case of word problems, a similarly procedural approach involving extracting the numbers and deciding which operation(s) to apply to them.

The second aspect relates to the relationship between scenarios, enactive strategies, and intuitive models of arithmetical operations. For children with a firm grasp on basic arithmetic, who know and can carry out standard calculations expressed symbolically and in bare form, scenario-based tasks expressed as text can cause negative emotional responses such as annoyance (because they require an extra stage of translation into the preferred format for calculation, so require more time and effort), or fear (because they may be nonstandard in arithmetical structure, and not translate directly to well-practised calculations). These students can carry out a multiplication or division, for example, once they recognise that as the necessary operation. Hughes (1991) challenged assumptions about young children's actual understanding of the operator symbols they appeared to be familiar with, while a study by Stallard (cited in same volume) showed difficulties translating symbolic representations (of additive structures) persisting among lower-attaining 10-year-olds, and I see no reason to expect such problems would immediately disappear at secondary school. For students whose understanding of arithmetical operations is weak, with insecure links between formal and informal representations, and further difficulties in recalling and/or carrying out standard calculations, preferences may in fact be reversed from the 'normal' pattern. For these students, a bare task may be initially incomprehensible, and require translation of the number and operator symbols into some enactive and/or visuospatial format based on the manipulation of equal sets, whereas a well-chosen scenario naturally triggers more secure intuitive models for the numerical structures. Of course, the issue of children's intuitive models for operations, and especially the misconceptions that can go with them (e.g. multiplication makes things bigger, division makes them smaller), has been rightly pointed out by many as potentially interfering with their identifying appropriate operations in tasks involving non-natural numbers (e.g. non-integer multipliers and divisors, in Fischbein et al., 1985). However, when focusing on students whose conceptions of natural number arithmetical structures are weak, incomplete and at a level some years behind the great majority of their peers, this is not an immediate concern.

3.4.4 'Translation'

The idea of 'translation' arises frequently in the literature, particularly regarding whether it is problematic or not for children to 'translate' between verbal, numeric/symbolic, visuospatial and concrete/kinaesthetic task representations. I do not consider it an adequate term for the re-representational process in students with difficulties in mathematics; it implies a kind of neat, discrete one-to-one correspondence between, e.g., words and symbols, probably in well-practised standard-format tasks. The reality of how (these) students think – and how they need to think to gain better understanding – is generally a theoretically messier, multimodal process which is difficult to discern, describe and categorise. During the early observation period of my fieldwork, on several occasions I observed classroom support staff working on word problem worksheets with my participants and other students with SEN; a favourite strategy (of staff rather than students) appeared to be first to highlight all the numbers in the text, then anything that “looked like a maths word” (e.g. ‘share’), to try and remember what operation matched up with those word(s), and write down a calculation in symbols. This I would describe as ‘translation’, and I do not consider it an appropriate strategy. While some authors also use ‘translation’ in a critical sense (e.g. Mason and Davis, 1991), when others (e.g. Ainsworth, Bibby, and Wood, 2002; Brna, Cox, and Good, 2001; Cox, 1999; Dowker, 2001; Superfine, Canty, and Marshall, 2009) have tested and reported on ‘translation ability’, they generally use it the way I and others (e.g. Scaife and Rogers, 1996; Voutsina, 2012) use ‘re-representation’.

Research from the last three decades suggests that children are relatively good at re-representing simple word problems with concrete objects, and many can re-represent them in standard symbols (some cited above; more summarised in Dowker, 2005, p.p.112) – with the caveat that, as the reader may by now expect, the great majority of studies limit themselves to primary-age children and additive structures. Dowker’s suggestion is that the difficulties students do experience with these word problems are related to semantic structure, i.e. a failure to understand the meaning of the task as described. In this case, individual performance and group results could vary greatly depending on not only the size and type of numbers involved, and the arithmetical structures, but linguistic and semantic factors. This would apply particularly in studies where participants have been presented with tasks in text format only (as opposed to having them read aloud, as in some smaller-scale studies) and do not have the opportunity to ask a teacher or researcher for clarification of terms they do not understand.

3.5 Multiplicity of strategies

A central theme of Dowker's work is the very wide variety of strategies used by children engaged in arithmetical tasks. She states " There appears to be no form of arithmetic – from counting to complex arithmetical reasoning – for which people fail to use a remarkable variety of strategies" (2005, p.p.21). It has taken some time for a multiple-strategy viewpoint to be accepted; however, there is now a substantial body of research demonstrating older strategies persisting alongside new ones in individuals' arithmetic in both bare and scenario tasks. The majority of evidence cited by Dowker (2005, p.p.22–23) for multiplicity of strategy use comes from either young children – sometimes very young, i.e. the equivalent of KS1 or below (Baroody, Siegler, etc.) or from adults (Dowker, Lefevre, etc.). Baroody (1988) used participants of a wide range of ages, but all with very low IQ scores (and, with no working shown, verbal explanations, or even hand movements, had to deduce participants' strategies from their 'answers'). The Fletcher, Huffman, Bray, and Grupe (1998) paper describes a microgenetic study comparing the addition strategies of normally-attaining 'kindergarten' children with those of older children (mean age 8.9) with "mental retardation", and demonstrates participants discovering an effective strategy (e.g. *'min'*) but using it with increasing frequency rather than simply adopting it as a replacement for older strategies. One of the main benefits of having more than one strategy – being able to use one to solve a problem and another to check one's first answer – does not appear to have been tested.

3.5.1 U-shaped curve

In an earlier paper on estimation strategies, Dowker (et al., 1996) suggested that the relationship between amount of expertise acquired and variability of strategies used may follow a U-shaped curve, where

> Novices may use many strategies, often inappropriate or inefficient, because they have not yet fixed on any particular strategy or set of strategies. Experts may use many strategies, mostly appropriate, partly because they have access to more strategies, but mainly because they have a sufficiently good 'cognitive map' of the territory that they do not fear becoming irretrievably lost if they stray from a known path. People at intermediate levels of expertise are more likely to confine themselves to a small set of strategies that they have learned and with which they feel safe. (p.p.23).

This seems a highly plausible pattern for typically-developing children and adults to follow (although may not be appropriate to describe the mathematical behaviours of my participants, with their atypical attainment patterns). Additionally, it is not a question of cognition alone: external factors play a role in a person's strategy choices. As has been mentioned, there is also a certain amount of peer pressure to use certain context-approved strategies over others, and I hypothesise that this peer pressure is, as a generalisation, felt the most by the various people at the lowest point of the variability curve, and little by either novices or experts. Note also that some of the people Dowker would class as 'intermediate', i.e. on the lowest point of the U, are primary school teachers, classroom assistants, SEN support staff, and parents 'helping' with homework, and they, consciously or unconsciously, impose their beliefs about strategies on the struggling students under their care.

3.5.2 Strategy choice

Assuming that an individual has multiple strategies (i.e. more than one) at their disposal, how do they choose which to use (first) for a given task? Various cognitive hierarchies of strategies have been proposed to describe students' increasing efficiency both within and between counting strategies and DFS (generally with recalled-fact placed at the apex). Perhaps less familiar is Gray's model of *preferential* hierarchy and 'route of regression' (1991), an attempt to map students' addition and subtraction strategies in descending order of preference – i.e. Known fact → Derived fact → Count on → Count all (this example being for addition; subtraction is a little more complex). Gray concluded that students of above and below average abilities followed different routes of strategy preference, and while his model dealt with additive structures, it would be fairly straightforward to propose multiplicative equivalents. However, while all my students would fall within Gray's 'below average' category, because of individual differences I consider it highly doubtful they would all share the same strategy preference route for multiplication and division tasks, and more likely exhibit different and changeable strategy preferences.

Some studies have looked at strategy choice across different tasks, and have found no obvious correspondence between the structure of a task and the strategy chosen (Kouba, 1989), or reported children selecting a strategy to suit the numbers provided, independent of task type (Clark and Kamii, 1996). Others (e.g. Newton et al., 2010) list 'form of the problem' as a definite factor – although given the latter's bald opening statement "Textbooks present multiplication as merely a faster way of doing repeated addition" (p.p.41), it is possible that if that is so, the narrowness of their cohort's representational experience may have influenced results. It may safely be assumed only that there are a variety of factors influencing which strategy a given individual will use for a given task on a given day.

3.5.3 Discovering multiple strategy usage

Let us say a participant from one of the many studies cited so far is reported as successful in using a particular strategy on a given arithmetical task. This may be the only strategy they have for that type of task, the strategy their teacher has informed them is ‘best’ or age-appropriate, the first strategy that happens to come to mind, or a strategy chosen because of specific aspects of the set task (e.g. numbers involve sets of five; scenario has psychological commutativity, etc.) which would not work, or be less efficient, for other superficially similar tasks. How is it possible to know of other strategies a participant may have at their disposal, but not demonstrated? This is not a problem when studying the strategies of professional mathematicians or other adults competent at mathematical problem-solving (e.g. Dowker 1992; Dowker et al. 1996); these participants often make repetitive calculation tasks more amusing by deliberately varying their strategies, and happily discuss and compare alternatives. However, for those outside these groups and not involved in education, it is not particularly usual to think about all the different ways a task might have been accomplished; it is also possible that the act of coming up with one successful strategy actually makes it more difficult to then think of alternatives. Despite the encouragement of the researcher, participants may be embarrassed to share strategies that they consider immature, or may have successful strategies based on conceptual relationships which, despite being well-understood, are difficult for them to verbalise. One methodological response to this issue is the use of near-replication of tasks with participants, with variations in number magnitudes and relationships, scenarios and presentation formats (see 5.4).

3.6 Summary

It seems obvious now that while an individual’s preferred arithmetical strategies change over the years of schooling (and sometimes beyond) as a result of both external and internal mechanisms, old strategies are not completely discarded, and there are clear benefits to having multiple strategies available. It has in the past been (and within some educational cultures still is) considered good pedagogy to drill students in a single method for each of several classes of calculation, with the chosen method being that which was considered most efficient for the average student. Depending on educational context, spotting ‘short cuts’ that worked for particular examples may have been approved or frowned upon, but was rarely explicitly encouraged or practised. Currently, however, our curricula acknowledge multiple possible strategies, and encourage teachers to discuss these with their students, and allow them to pick the best for a given task.

Thus in recent years a new generation of teachers began their careers teaching from a national curriculum and numeracy framework that – however comfortable or not they were with this – specifically encourages the concurrent use of multiple child-led strategies in arithmetic. Assuming that variety of arithmetical strategies is indeed now generally encouraged in primary schools, is this also the case at secondary level? Although the move from primary to secondary is an artificial cultural milestone as opposed to (say) a Piagetian or Vygotskian one, the change of environment nevertheless carries significant assumptions about the arithmetical strategies that are appropriate for use in the classroom, and thus which will be adopted and which discarded (or, perhaps, used surreptitiously). Peer pressure intensifies, which may have a detrimental effect on students’ ability and willingness to express individual tendencies and requirements regarding arithmetical strategies – particularly if these still involve heavy use of counting, and to a lesser extent, derived fact strategies. Moreover, secondary mathematics teachers may lack knowledge of how children actually learn the elementary concepts drawn on at the later stage, which affects the support they are able to offer students who need to revisit those elementary concepts (Watson et al., 2013).

Previous studies on primary- or special-school children, quite understandably, select a simple and comparatively clearly-defined area of arithmetic/numeracy (e.g. single-digit addition) and generally a group of participants encountering the topic for the first time – hence the prevalence of research on five-year-olds and younger, and the comparative dearth of studies on 11-16-year-olds. There are obvious methodological complications in my researching secondary mainstream students’ multiplicative thinking: it is certain that they will have previously encountered multiplication- and division-based tasks at school and so have a prior relationship with the topic. It is arguable whether any child could really be described as a "clean slate" as Baroody does (1988, p.p.375), but were there such a thing, my students would be some degree further from it than the participants in most of the studies discussed above. Regarding the subject matter, even if I were testing students' strategies for abstract symbolically-presented 'sums', there is not a clear hierarchy of strategies in the same way that in addition *min* could be called superior to *count-on*, which is superior to *count-all*, so strategic progression is more difficult to define.

While the arithmetical and representational strategies employed for a task are deeply interlinked in a complex way, it is theoretically necessary to prise them apart somewhat; unfortunately several otherwise helpful analytical frameworks muddle aspects of the two. However, a single arithmetical strategy can be represented (visually or otherwise) in a variety of different ways; similarly, a single visuospatial representation can derive from or be used for a variety of different calculations, and any detailed analysis of problem-solving must take this into account. This section has focused on the ways numeric quantities and relationships are manipulated when calculating; the following looks at how these numbers and relationships are represented.

4 VISUOSPATIAL REPRESENTATION

4.1 What is meant by a 'representation'?

Authors on the subject of education have been interested in visuospatial representation for some time. For example, a stance taken by various recent researchers, that children respond well to being exposed simultaneously to different modes of representation, and the links between them (e.g. Fuson and Burghardt, 2003; Ainsworth et al., 2002; Ainsworth, 2006) was suggested by Maria Edgeworth back in 1798. The scientific study of the role of mental imagery in cognition is generally traced back to the work of Francis Galton in the late 19th century (Burbridge, 1994), in particular his research of 1879-80, asking scientists and non-scientists about the strength and nature of their mental imagery. Galton's opinion at this point was that "abstract thought is best carried on without the aid of this concrete imagery" (Galton, 1881, p.p.85), and that "the group of men who have vivid imagery differ from those who do not have it by including a large share of a certain flightiness or oddity of disposition" (unpublished paper, in Burbridge 1994, p.p.459). Perhaps because of this general perception, and almost certainly because of a lack of clearly-defined shared terminology, many of Galton's 'men of science' apparently protested that mental imagery was "entirely unknown" to them (Galton, 1880, p.p.302). However, the topic of "Thoughts without words" was of enough interest to spark considerable discussion, with contributions by several authors, in the 36th volume of Nature (1887).

In the 20th and 21st centuries, the subject of visuospatial representation arises in a great many disciplines, each of which define and treat it differently. In neurocognitive research, for example, the focus may be on the neural mechanisms involved in the acquisition of representations of numerical magnitude (e.g. Ansari, 2008), or on investigating the relation between representations of number and space through functional magnetic resonance imaging (e.g. Tang et al., 2011). In design research, the emphasis might be on the role of visual representations as epistemic objects (Ewenstein and Whyte, 2009) or the meanings of symbols as established through custom within a given culture (Ware, 2004b). The visualisation of information is also highly relevant to the fields of statistics, modelling, communications technology and interfaces – among many others. This study cannot and does not attempt to address theories of representation in all fields; the literature with which I engage is concentrated on the visuospatial representation of number in the teaching and learning of arithmetic. Of course, in addition to 'representation' there are also references to other related constructs such as imagery, visualisation, modelling, metaphor, spatial ability, figural information processing, etc., themselves used somewhat differently by authors; these will be addressed as and where they occur in cited works.

4.1.1 Terminology

Not only does the concept of representation vary considerably between the different fields of study touched on above, but even within narrower genres, such as empirical studies of representations used in mathematical problem-solving. While there have been various taxonomies produced for different types of representation, a definition of the term itself is not always given, and where it is, the articulation tends to be cumbersome. A notable feature of definitions for the term "representation" in the literature surveyed is their recursivity, i.e. their inclusion of the terms 'represent/-ing/-ed', etc. An example of this is "representation involves a relation between two (or more) configurations, with one representing another in a sense to be specified" (Goldin, 2002, p.p.196) – although it does at least emphasise the relationship aspect of representation. Another linguistically recursive example is Palmer's concept of a representation as consisting of

> (1) the represented world, (2) the representing world, (3) what aspects of the represented world are being represented, (4) what aspects of the representing world are doing the modelling and (5) the correspondence between the two worlds" (1977, in Ainsworth, 2006, p.p.2)

This provides a more detailed structure of components, and was used by Ainsworth as the basis for an expanded version for use with multiple-representation systems. An equivalent five-part articulation was also used by Kaput and others, although with different terms:

> [A] representational entity; the entity that it represents; particular aspects of the representational entity; the particular aspects of the entity it represents that form the representation; and finally, the correspondence between the two entities. (Kaput, 1987, in Presmeg, 2006, p.p.3).

For this study, which focuses on the drawings, symbols, models and gestures used in students' mathematical activity, the latter of these (Kaput) seems the most appropriate to adopt.

4.1.2 Internal and external

One particular cause of disagreement in the literature is the issue of internal and external representation. Those from a behaviourist tradition rejected on first principles any discussion of internal mental representation, visuospatial or otherwise, while radical constructivists, on their own *a priori* grounds, rejected direct knowledge of the 'real world' and representations external to individual experience (Goldin, 1998). Those from the cognitive tradition have also frequently assumed that representations are exclusively in the mind, with external objects merely peripheral aids (Zhang and Norman, 1994), while much empirical research relies exclusively on these external objects. In the last twenty years, there has been a trend toward cognitive models which include interaction between the internal and external, between "knowledge in the head" and "knowledge in the world" (Norman, 1988; 1993; in Scaife and Rogers, 1996, p.p.188). This led to the concept of *external cognition,* and analytical frameworks for it such as that of Scaife and Rogers, the three central characteristics of which are: *computational offloading* (the extent to which differential external representations reduce the amount of cognitive effort required to solve informationally equivalent problems), *re-representation* (how different external representations, that have the same abstract structure, make problem-solving easier or more difficult) and *graphical constraining* (the way graphical elements in a graphical representation are able to constrain the kinds of inferences that can be made about the underlying represented world) (Scaife and Rogers, 1996).

4.1.3 Definitions and assumptions

For the purposes of this study:

- *Representation* can refer to both a process and the product of such a process.
- The act of representation involves a relationship between: a representational entity; the entity that it represents; particular aspects of the representational entity; and the particular aspects of the entity it represents that form the representation.

- Representational entities may be predominantly internal or external, or a combination of the two.
    - An external representation is the observable communication of some thought or idea to others, or a record for one's own use, through some visuospatial/kinaesthetic mode(s) and media.
    - Internal representations are assumed to exist in some form, and while they are not (yet) directly observable, approximations are sometimes available through either verbal description or recreation in observable media.
- Represented information may be re-represented in another form, which may affect how it is understood.
- Similarities, parallels and isomorphisms may be recognised between different representational entities, which enable discussion of soft-edged and overlapping *representational types*, but not necessarily clearly-demarcated categories.

4.1.4 Role of representations in this study

This study is concerned with representation of numbers and numerical relationships at a basic level, i.e. the representational strategies which support individual students' understandings of the concepts and processes involved in multiplicative thinking with natural numbers. The development from a position of non-understanding to understanding may take place in leaps or in tiny steps, and the progression may not be linear. My intention is to throw light on the roles played by visuospatial representation in triggering leaps and supporting steps students make while working on this material. However, while doing this kind of microscopic observation and analysis of change during a comparatively short space of time, it is important to consider the longer-term educational aims from a teacher's point of view. For example, while it is a significant achievement for a student previously unable to solve a particular type of multiplication- or division-based tasks to do so via pictorial depiction, ideally they should eventually be able to do so using standard symbolic notation. Similarly, it is uncontroversial to suggest that teachers wish students to be able to make use of prior experience in tackling tasks with the same arithmetical content but of a different format, or to understand the relationships between different arithmetical operations. Thus, the tasks I set students were designed to allow for the observation of (a) how, after solving a given task, students' representations change (or do not) in working on subsequent tasks of the same type, and (b) similarities and differences in their representations of isomorphic and non-isomorphic tasks.

To explain and exemplify points of discussion raised by the research literature on visuospatial representations, it is helpful to include some visuospatial representations. For this purpose I have selected relevant images from data collected during fieldwork. Though functioning here as illustrations, they are also theoretically significant, as the relationship between theory and data in this study was bidirectional; my data influenced my interrogation of the literature as well as vice versa (see also 5.2).

4.2 Presented, created and co-created representations

One of the most important aspects of research into the use of visuospatial representations in arithmetical problem-solving contexts is the origin of those representations. In some studies, participants are presented with tasks which have a particular representation provided alongside (e.g. picture, diagram, model, etc.), in order to compare their performance against participants provided with a different kind of image, or who have the information in verbal form only. In others, participants are provided with some media to work with (e.g. paper/pen, cubes) but use them to create their own representations, which is a fundamentally different paradigm. There are also situations which fall somewhere between the two; computer microworlds, for example, often provide more structure for a task than simply directing participants to a particular media, but through their interactive nature allow for more creativity than putting a static teacher-provided image in front of them.

My work with students contained only a tiny proportion of presented representations (i.e. created by me), a large proportion of representations created by students from media provided (usually paper, coloured pens and cubes), and a substantial proportion which fall between the two, which I term *co-created*. By this, I mean representations in which both the student and I participated – for example, a drawing which I began and to which they added, or vice versa. In Figure 4-a, Wendy required support on a multiplicative task, and I drew a partial image (purple ink) conveying the number relationships within the task scenario (bottles packed into boxes, packed into vans); she took over the representation (black ink) where I left off, subsequently discarding the visual aspects that she found superfluous (van and box boundaries) but keeping that which she found necessary (all individual units, which she then counted). This example involves both *container* and *array* forms, which are of particular interest (see 4.3).

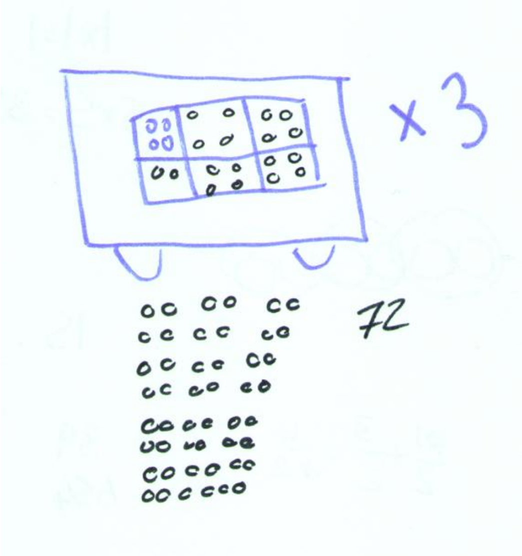


Figure 4-a: 4 bottles per box, 6 boxes per van, 3 vans (CF and Wendy)

4.2.1 Presented representations

While the mental processes of students working with representations provided by someone else are almost certainly quite different from those working with self- or co-created ones, this question of how these representations (or aspects of them) function within arithmetical tasks is still relevant, and the typologies described in the following sections have been analytically influential, to greater and lesser degrees.

4.2.1.1 Classifying presented images

There has been interest from the educational psychology community in the uses of visual representation. Research on students using representations provided for use on a particular task (usually via worksheets) has shown that different kinds of image vary in their effect on students' learning. Carney and Levin's (2002) meta-analytic review has proved particularly influential in research on the function of images in mathematics tasks, despite the fact that it does not focus specifically on that area. The authors differentiate five functions for images presented as text adjuncts: the more common *decorative, representational, organizational, interpretational,* and the less conventional *transformational*. Here, *decorational* pictures simply decorate the page, bearing little or no relationship to the text content, while *representational* pictures mirror part or all of the text content. *Organizational* pictures provide a structural framework for the text content, *interpretational* pictures help to clarify difficult text, and *transformational* pictures include systematic mnemonic components designed to improve recall of text information. A series of studies by researchers at the University of Cyprus (e.g. Theodoulou, Gagatsis and Theodoulou, 2003, in Agathangelou et al., 2008, p.p.1–2; Elia and Philippou, 2004 ; Elia et al., 2007) took this model and applied it (with minor changes) specifically to images presented to students in arithmetical tasks, drawing various conclusions about the relative usefulness of the types.

The problem here is a subtle one, and pertains to whether the researchers are discussing the potentially multiple functions or roles that presented visuospatial images (or models) have the potential to fill (as I believe Carney and Levin intended), i.e. their affordances, or if they are assigning a single assumed function or role to particular images (as the more recent papers appear to do). These may effectively be the same thing when one is dealing with an experimental condition with a small number of contrasting images carefully chosen by researchers to exemplify the representational categories, in closed tasks, with groups of neurotypical and fairly homogeneous participants. However, as soon as one moves out of this kind of highly controlled case, it becomes clear that individuals cannot be assumed all to use the same images in the same way (and that being the way that the researcher envisaged). While certain aspects of certain images or models may fulfil the roles of representing part or all of the task content, supporting organization of task content, or making the page more visually appealing, it is impossible to claim with certainty that a given image clearly falls into a single one of these categories. To do so would imply that different individuals look at a representation and take notice of exactly the same aspects of it, make the same mental connections, and draw the same conclusions. While it may be possible to say that a given representation was intended by its creator as organisational (for example), that does not guarantee a student will use it in that way, or that a different individual may not use it another way. Riley et al. (1983) have suggested that the representation of a problem can affect the kinds of procedures required for solution, as well as the ability to solve related problems: I suggest that these will be further affected by the way a given student reads and uses a given representation. I also suggest that assumptions about the usefulness or otherwise of particular types of representation – e.g. the 'decorative' pictures generally dismissed by studies such as those cited above – may be considerably less applicable to very low-attaining and/or neurodivergent students. They may well also be of limited application outside of highly-controlled conditions.

4.2.1.2 Negative and positive views

While the various ways presented images can support students' mathematical problem-solving are debated, some authors have highlighted possible obstacles. To support understanding, representations must have salient features that suggest the correct meaning, and not have misleading features (Fuson and Burghardt, 2003); moreover, some representations can divert attention to irrelevant details and highlight certain aspects of the problem at the expense of other more relevant ones (Presmeg, 1986). Students may fail to understand how a representation is to be used in relation to the set task (Verschaffel and De Corte, 1996), or the cognitive demands of reading it efficiently may be too great (Scaife and Rogers, 1996). These findings should not dissuade teachers from using combinations of text and image in the tasks they set to students, but be considered when designing tasks and making decisions about what information and relationships should be included, and in what form. There have been clear positive effects reported of tasks presented in multiple representative modes, usually text and an image (e.g. Eilam and Poyas, 2008), but more commonly are seen positive statements of the potential benefits, qualified by concerns that the most appropriate representations for purpose are chosen (Ainsworth et al., 2002; Ainsworth, 2006; Brna et al., 2001), that the prior knowledge (Superfine et al., 2009) and visualisation abilities (Pitta-Pantazi and Christou, 2010) of the individual students are considered.

4.2.1.3 Spoken versus printed text

In previous research involving mathematical tasks presented with verbal and visuospatial components, the worded information has generally appeared in print – due, for the most part, to the numbers of participants involved, and the intention to provide them with identically-presented tasks. Clearly, in these cases, the participants' abilities in reading and processing sentences is involved, and studies where some or all of the participants were considered low-attaining for their age group have addressed this, e.g. Jordan et al.'s (2003) comparison of performance on mathematical thinking tasks in children with mathematical difficulties, some of whom were good readers and some poor readers. As reading ability was a factor for neither selection nor analysis in my study (although I note that a significant proportion of my participants had diagnoses of dyslexia or other learning difficulties), the complications inherent in studies requiring reading of task information and instructions influenced my methodological decision to use minimal text, and, as with research on young children, make greater use of spoken instructions (see 5.4.3).

4.2.1.4 Internal representational response

I have discussed above the fact that individuals may react differently to the same presented visuospatial representation, and also stated as one of the assumptions of this study that internal representations such as ‘mental images’ exist in some form. To this I add the representational experience (reported informally by myself and others) of looking at an external mathematical image and simultaneously ‘seeing’ an additional internally-generated image superimposed over it – to the extent that it may be described in great detail in terms of colour, style, etc. (This might happen, for example, when looking at a geometrical figure and thinking about the spatial relationships within.) Zhang and Norman (1994) described the interwoven processing of information from internal and external representations as *distributed cognition*, and proposed a theoretical framework for *distributed representations*, demanding the consideration of the internal and external representations of a task as one *representational system*. While their system of representational analysis is not appropriate for my data, the idea of someone working on a mathematical task through a representational system with both internal and external components is a helpful one, and one of the explanations for why students may see the same external image and interact quite differently with it. In practical terms, it may be possible to infer something of their internal representation through external interactions, for example, drawing over a presented image, or gesturing above it; certain individuals may also be able to describe internal imagery which is invisible to the observer, and recent technological developments have included software which tracks eye movements. In such a sense, all representations used in tasks involve some individual representational response from the task performer; however, this response is likely different to that in personal representations created ‘from scratch’ (although it can hardly fail to be influenced by the task representations they have seen before). Also somewhat different are the *ad hoc* task representations produced or participated in by a teacher/researcher for the sole use of that particular individual (which I describe as co-created). These other representational situations are now considered.

4.2.2 Student-created representations

4.2.2.1 Why students don’t draw

Mason and Davis (1991) mention repeatedly the issue of students’ poor responses to tasks expressed in word form, asserting that “Many pupils have no idea where diagrams come from” (p.p.35) and “it often seems to pupils that mathematical symbols spring miraculously from pencil, pen or chalk” (p.p.33) – meaning that students expect to be able immediately to translate the words into the symbols for formal solution, without much independent reflection or decision-making. This is not an isolated finding, but rather an expression of a phenomenon known to a great proportion of teachers of mathematics. For students with mathematical difficulties, the leap from a paragraph of text to a formal, or even informal, calculation can seem insurmountable. Pedagogical literature has various suggestions on the subject of decoding the text of questions (e.g. searching for and highlighting certain key words) which do not fall under the remit of this study, but one commonly-seen piece of advice is of particular interest: the instruction to ‘draw something’. For example, “Draw a figure” is one of the first suggestions in Pólya's 1945 classic *‘How to solve it’*, and it is notable that the suggestion appears under the first of his four stages of problem solving, ‘Understanding the problem’, as opposed to later on in the process. However, again, experienced teachers will observe that simply telling students to draw will not do: the students ask “Draw what?”. Mason and Davis (1991) report that “many pupils are unaware that the story is intended to evoke mental images, a sense of 'being in the situation', and that it is the situation and one's experience in similar situations (whether actual or vicarious) that enables you to read relationships and operations that need to be carried out” (p.p.34).

There are also powerful disincentives to use informal drawing in mathematics tasks, in the form of teachers’, peers’, and one’s own expectations of the representational strategies appropriate at a given age. Ben-Yehuda et al. (2005) have criticised teachers for failing to recognise mathematical potential in the case of students who do not perform formal procedures well; likewise, when students use informal visuospatial representations it may incline some teachers toward low academic expectations. This situation is well-known as a potentially self-fulfilling prophecy (Hoge and Coladarci, 1989), although more recent research (e.g. Smith et al., 1998; Madon et al., 1997) suggests the effect is not as strong as originally thought.

4.2.2.2 When students do draw

From a constructivist/constructionist viewpoint, the distinction between reasoning with one's own representation versus those created by others is vital (e.g. Papert, 1993). The representational skills and knowledge which a given student can access at a certain point in their education vary considerably, leading some to suggest a domain-independent 'graphics curriculum', including direct teaching of generalised heuristics and principles for choosing representations with appropriate levels of expressiveness, and how to match the type of information that can easily be drawn from a particular representation to the requirements of the task (Cox, 1999). Others, however, advocate a more open-ended educational approach, with more emphasis on children creating their own, original and often highly effective, representational forms (examples in DiSessa, 2002; 2004). DiSessa defines the term *metarepresentational competence* (MRC) as a set of abilities for dealing with representational issues, in particular the abilities to create one's own representations for a given purpose, and to critique their adequacy and suitability for that or other purposes. Of course, the mathematical task representations an individual creates are affected by their past representational experiences, and so also within the scope of MRC is the ability to learn to use new kinds of representation quickly and appreciate their properties. I note that studies such as DiSessa's have often been carried out with mid- to high-attaining students, and rarely with children facing the kind of struggle with mathematics, and/or schoolwork in general, as are my participants; it may be the case that some kinds of students require the more directly instructive approach. The concept of MRC is important to this study, both in terms of assessing the representations created by students, and in terms of the 'teacher input' I provided through both verbal feedback and co-creating images or models.

4.2.2.3 Descriptive or taxonomic

Studies which involve collecting and analysing the images and models created during mathematical problem-solving have focused on different aspects. As one might expect, there are considerably fewer experimental-type comparison studies than with presented representations. There are many which report on whether or not participants were observed used any visuospatial strategies on tasks, some of which also note of what form the representations took (e.g. Presmeg and Balderas-Cañas, 2001). They sometimes focus on the mathematics which emerges from a particular scenario (e.g. Edo et al.'s 2009 study of young children engaged in structured play based on shopping), or on tasks designed to probe a particular topic from the mathematics curriculum. Of particular interest in this latter type are those focusing on multiplicative structures, such as Sherin and Fuson (2005), who collected many children's drawings in single-digit multiplication tasks to use in their categorisation of counting strategies, contrasting "situational" or "semi-situational drawings" with "math drawings" (the former corresponding to more informal/pictorial and the latter to more formal/symbolic markings). On a similar theme, Saundry and Nicol (2006) investigated the drawings young children used in simple and more complex divisional scenario tasks (using the ever-popular themes of biscuits and wheels). They describe how students manipulated their pictures on the page, moving, eliminating, sharing and distributing them, in some cases with patterns of movement resembling the use of physical manipulatives (e.g. counters), and in some cases with the video data showing clear indications of internal visualisation. They also looked at the style of the drawings, suggesting that a high level of detail in the pictures (e.g. car passengers' eyelashes!) could be to the detriment of solving the tasks. (As with decorative aspects to presented representations, this aspect may apply less well to very low-attaining and/or neurodivergent students.)

4.2.2.4 Change and comparison

While a significant proportion of the research on participants' drawings and models is mainly descriptive or taxonomic of the representations themselves, other authors have turned their attention more on the participants, looking for differences in representational behaviour that correlate with age, classroom experience, level of attainment in some prior form of assessment, or non-hierarchical categorisations such as "cognitive style". Unsurprisingly, studies of young children's mathematical problem-solving representations suggest stylistic change from more pictorial to more symbolic mark-making when influenced by the implicit (and explicit) expectations of the classroom, as demonstrated in, e.g., Deliyianni et al.'s (2009) comparison of children in pre-school and the first year of school, which they describe as 'visual creativeness' giving way to an obedience to 'didactical contract rules' regarding appropriate representational forms. Gray et al. (2000) compared groups of students selected to represent "high" and "low achievers", finding very different levels of abstraction/symbolism and of surface detail, while Mulligan (2011) highlights the lack of mathematical pattern, structure and coherence in representations by "low achievers".

Karsenty et al. (2007) also refer to low/high achievers in a rare longitudinal study of secondary-age students, concluding that "rushing into formal mathematical outcomes, without taking into consideration the intuitions and informal ideas of students, might weaken potential strengths of learners, especially low achievers" (p.p.175), and arguing for an enhanced role for visualisation in the teaching of these students. The findings of research on these students have resulted in specific pedagogical advice for them on representation, which is considerably more consistent than advice on other aspects of their learning (e.g. the role of rote memory, use of multiple arithmetical strategies).

4.2.2.5 'Visualisers'

In both educational research and pedagogical materials, and even in some student texts, is found the concept of the 'visualiser'. As discussed earlier, the idea of some people's mathematical experience being more visual than others' was present in academic discourse well over a century ago, but became of increased moment to educationalists in the 1970s and 80s, with the theory of two opposing cognitive styles, one primarily visuospatial and the other primarily linguistic. Krutetskii (1976) identified two factors in school mathematical performance, the first being the verbal, logical component of thinking, contributing to an individual's level of mathematical ability, the second a preference for visual or nonvisual methods of problem solving, contributing to the form of the individual's mathematical thinking. This dual model appears to have been influential in Presmeg's early work (e.g. 1986), in which she distinguishes two classes of people: visualisers (who prefer to use visual methods to solve mathematical problems) and 'nonvisualisers' (who prefer not to use visual methods for problems when there is an alternative). Others have differentiated between visualisers and 'verbalisers' (Riding and Douglas, 1993), 'verbal' and 'spatial reasoners' (Ford, 1995), 'diagrammatic' and 'non-diagrammatic reasoners' (Cox, Stenning and Oberlander, 1994; 1995a), etc. (all in Cox, 1999), on the basis of psychometric or study-specific ability tests. As with the historical ability-judgements described in Chapter 2, this kind of fixed two-state categorisation is seriously problematic, as it no longer describes particular examples of an observed or theorised behaviour, viewpoint or strategy, or even general trends, but crudely identifies the people themselves as one thing or another. To label a person as 'visualiser' or 'verbaliser' does not describe something that they do but something that they are, with the implication that this is how they are in all situations, permanently. Translated into pedagogy, this would imply that students should be labelled at an early age and provided with a certain specific kind of teaching/learning experience, with children led to believe that they 'have' certain abilities and 'do not have' others. This is all quite contrary to the ontological view of this study, of individuals using different strategies in different situations, having strengths and weaknesses in different areas of mathematics, their patterns of strengths and weakness changing over time and experience, and, moreover, changing from day to day (potential factors being cognitive load, environment, affect, etc.).

While some more recent research still identifies students as categorically 'visualisers' or not, and comments on the groups' relative mathematical abilities (e.g. Woolner, 2004; 2006), a visualiser-verbaliser spectrum has also been proposed, which, while it does not remove the problems above, is at least more sensitive to individuality than placing learners into one of two boxes. Also proposed have been: a third category, 'mixers' (Clements, 1982), who have no general tendency in either direction; different sub-categories of visualiser, e.g. high- vs. low-spatial visualisers (Kozhevnikov et al., 2002; 2005) and spatial vs. object visualisers (Pitta-Pantazi and Christou, 2010). While the need to empirically measure and categorise participants is necessary for the quantitative research methods of traditional psychology and neurocognition, there has been a clear historical trend from categories to continua, or to increasing numbers of categories within which to place participants.

The design of this research study does not require such measurement and categorisation of participant; however, I also take issue with a one-dimensional visualiser-verbaliser spectrum as being too simplistic. As Cox (1999) points out, the implicit assumptions made in many studies, based on participants' scores on tests of spatial visualisation, about their internal cognitive modality preferences, use of external representations, and the relationship between the two, are not justified. To describe and compare individuals' 'visualising' behaviour will require more than one dimension of analysis, and Cox suggests that an additional representational continuum might be characterised "at one extreme, by an ability to translate information between modalities (graphical and sentential) in both directions, and at the other by a tendency to habitually use only one modality" (p.p.358). This is a distinction which may reasonably be applied in the analysis of qualitative data such as mine.

4.2.3 Co-created representations

At time of writing, I have found no major studies which focus specifically on visuospatial representations created conjointly by student and teacher/researcher in arithmetical problem-solving.

4.3 Key representation types for multiplicative thinking

There are two fundamental representational forms which I theorised as of central importance in coming to conceptualise multiplicative structures: the *container* and the *array*. Tasks choices for this study will be discussed in 5.4, but a glimpse may be useful at this stage, the better to understand the relevance of these key representation types.

Two specific scenarios were chosen for use in tuition sessions: 'Biscuits', which situated partitive division in the action of sharing a given number of biscuits between a given number of people, and 'Taxis', which situated quotitive division in the action of fitting a given number of people into vehicles of a specified capacity. Although not all students used visuospatial representation for all tasks, drawing and modelling were both popular and successful. Most students required support at some point, and there were visuospatial as well as verbal teacher-student interactions.

4.3.1 Numbers as containers

When a student was completely stuck on a task and unable to create any kind of usable representation, the first type I would suggest was the *container*, i.e. cardinal number represented by a designated closed visual area with markers of some sort inside it. Students also used this representation type independently, and Figure 4-b and Figure 4-c show examples of numbers represented as containers, created by students; in each case the boundary and units within are embedded in familiar scenarios.

Figure 4-d is an example of a student choosing container forms to designate groups and subgroups of items (in this case, people), although there were no physical containers or boundaries (such as plate or bus) specified in the question.

Figure 4-b: 9 biscuits on a plate (Leo)

4.3.1.1 Object Collection grounding metaphor

Lakoff and Núñez's work on embodied arithmetic includes a system of *grounding metaphors*, of which the first is 'Arithmetic as Object Collection'.

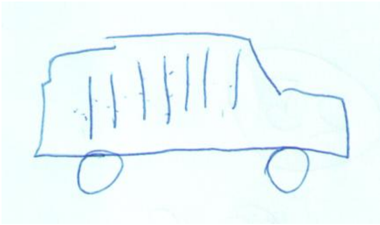


Figure 4-c:7 people on a bus (George)

> One of the major ways in which metaphor preserves inference is via the preservation of image-schema structure. For example, the formation of a collection or pile of objects requires conceptualizing that collection as a container -- that is, a bounded region of space with an interior, an exterior, and a boundary -- either physical or imagined. When we conceptualize numbers as collections, we project the logic of collections onto numbers. In this way, experiences like grouping that correlate with simple numbers give further logical structure to . . . notion of number. (Lakoff and Núñez, 2000, p.p.54)

Figure 4-d: 10 people split into 2 groups, each containing 2 girls and 3 boys (Tasha)

This metaphor allows the mapping of common actions and relationships within the domain of physical objects onto actions and relationships in the domain of numbers, so ‘Collections of objects (of the same size)’ maps to ‘Numbers’, ‘Putting collections together’ maps to ‘Addition’, etc. This is essentially a formalisation of the pre-symbolic arithmetic of children first learning about numbers. Although the term ‘container’ appears in discussions such as the above quotation, it does not appear in their formal statement of metaphor mapping terms. I, however, regard the container form as a vital element in the linking of numbers with collections of objects, as a boundary (whether physical or imagined) is necessary to separate off the objects being counted from all other potentially countable similar objects (e.g. when using collections of multilink cubes to represent numbers, there needs to be implicit or explicit clarification that the cubes relevant at any given time are the ones in a particular designated area (e.g. in a pot, or on a sheet of paper), and all other similar cubes that may be visibly present (e.g. lying around on the desk) are not of current concern.) This is not to say that there are not other ways of specifying which items are to be counted and which not (e.g. layout, colour, etc.), but I propose that the container image is a particularly powerful one.

4.3.1.2 Object Collection and multiplication

The entailments of the 'Arithmetic Is Object Collection' metaphor are fairly obvious with regard to one-dimensional arithmetic, the laws of closure, equality, commutativity, etc. mapping easily between performing additive actions on collections of objects and on numbers, and, importantly, it being possible to operate only in the domain of object collections or only with numbers. Clearly it is a little more complicated for two-dimensional multiplicative operations, and Lakoff and Núñez go on to state:

> But with multiplication, we do need to refer to numbers and collections simultaneously, since understanding multiplication in terms of collections requires performing operations on collections a certain number of times. This cannot be done in a domain with collections alone or numbers alone. In this respect, multiplication is cognitively more complex than addition or subtraction. (Lakoff and Núñez, 2000, p.p.60)

It is an obvious step to conceive of multiplicative operations in this way, by moving from adding collections to totalling a given number of identical collections; in effect, multiplication as repeated addition. However, the authors appear to miss the fact that using a container metaphor means that the two dimensions of multiplication could also be represented as a 'collection of collections', which can be perceived visually as items in containers which are themselves items in a larger container – without any such recourse to the 'number domain'. An example of this is shown in Figure 4-e.

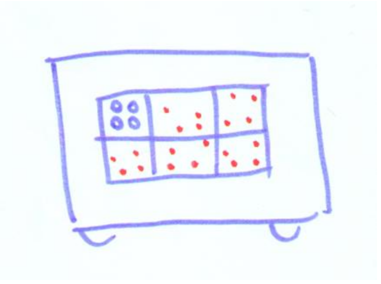


Figure 4-e: Van containing 6 boxes, each containing 4 bottles (CF and Jenny)

4.3.2 Multiplying and dividing with containers

Whereas additive structures require unary operations with each input representing the same kind of element – e.g. a total quantity made from a set of *x* cubes joined with a set of *y* cubes, students need to view multiplicative structures as binary operations with two distinctive inputs – e.g. a total quantity made from the number of cubes in a set and the number of replications of that set. Nunes and Bryant (1996) identified three main kinds of multiplicative situation: (1) those involving one-to-many correspondence; (2) those involving relationships between variables; and (3) those involving sharing and splitting. Of these, (1) and (3) are consistent with container representations. They also point out that while to an adult ‘one-to-one correspondence’ situations may seem very similar to ‘sharing’ situations, they may not seem so to children focusing on the enaction of replication or distribution rather than the underlying numerical structure. It is reasonable to suggest that the use of basic visuospatial representational elements such as containers may highlight the underlying relationships for students, and so help draw connections between different kinds of multiplicative situation.

The idea of multiplication as replication is consistent with that of multiplication as a collection of (equal) collections as described above, and lends itself well to visuospatial representation, as in Figure 4-e, where the van could be seen as holding 6 replications of a box of 4 bottles (a container of containers). One must be careful, though, of over-interpreting student- or co-created representations, imputing more sophisticated understanding and reasoning to students than is perhaps the case (as demonstrated by Thompson and Thompson, 1994; 1996). In Figure 4-e, it would be quite possible for the student to have carefully put four dots into each square, then counted the total number of dots, without being actively conscious of the replicatory structure. (In actuality, when posing this task, the replication element was referred to explicitly in the extension question which directly followed: What if there were three of these vans, all the same? The car park then acts as another level of enclosing container, together representing the 3-dimensional multiplicative structure 3 × 6 × 4.)

Barmby et al. (2009) criticise container representations, suggesting they encourage unary thinking and repeated addition, and do not illustrate commutativity or distributivity (in contrast to the authors' preferred array representations). However, the students on whom I focus were at a stage where, in the absence of reliable multiplication facts, repeated addition was the most appropriate calculation process for a multiplicative enumeration situation (and indeed, was more advanced than the unstructured counting-based strategies used in some cases). While it is true that drawn container representations do not well illustrate the commutative property (although fairly straightforward with movable concrete units), they may still play a role, as in the *array-container blend* (4.3.5).While the distributive property was not a focus of my study, I would argue that, in fact, container representations may illustrate it very well, as in Figure 4-d and Figure 4-f.

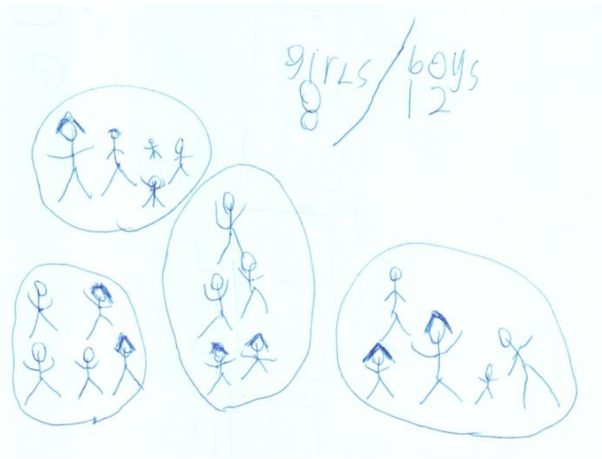


Figure 4-f: 4 × (2 girls) + 4 × (3 boys) = 4 × (5 children) (Leo)

4.3.2.1 The dynamic/temporal aspect

The issue of interpreting students' use of visuospatial representations as accurately as possible leads to another methodological distinction: is the dynamic/temporal aspect of representation considered? Collecting students' inscriptions tells us something about how they tackled tasks, but even knowing the tasks involved (with the exact wording used by the teacher/researcher) and looking at the images in their original context on the page, there is still ambiguity in the static final images, as they may have been arrived at in different ways. One reason the container representation type is so

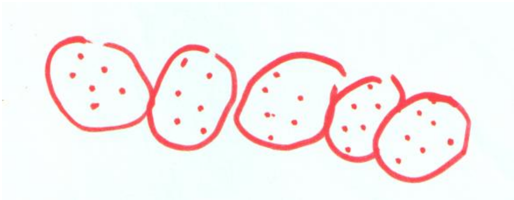


Figure 4-g: An arithmetical relationship between 5, 6 and 30 (Jenny)

powerful is that it is useful for both partitive and quotitive division, and also multiplication. (As will be demonstrated, it is thus also very useful for conceptually linking these operations.) For example, the containers image in Figure 4-g could have resulted from (a) a partitive action, i.e. drawing 5 containers and 'dealing out' 30 units, resulting in 6 in each group, (b) a quotitive action, i.e. drawing a container to hold each group of 6, until all 30 units had been grouped, resulting in 5 groups, (c) a multiplicative action, i.e. creating 5 groups of 6, ending up with 30 units altogether, or a number of other arithmetical actions.

4.3.3 Numbers as arrays

The second fundamental representation type for multiplicative structures is the rectangular array, usually in drawn form. One of the tuition sessions was designed around what was essentially the area model of multiplication (although the actual term ‘area’ was not used), the representational media being rectangles drawn on 1cm squared paper (e.g. Figure 4-h and Figure 4-i, where students were asked to draw rectangles containing exactly twelve squares). For other tasks, some students used array representations independently, and with others I suggested it. Additionally, the ‘starter task’ set at the beginning of each tuition session involved an array-based enumeration task, this time 3-dimensional and modelled with cubes (see 6.2).

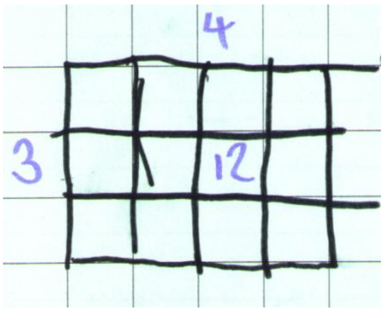


Figure 4-h: 12cm^2 rectangle (Jenny)

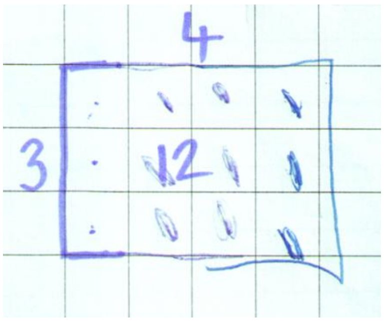


Figure 4-i: 12cm^2 rectangle (Leo)

4.3.3.1 Object Construction grounding metaphor

As the attributes of container representations fit with Lakoff and Núñez's first Grounding Metaphor 'Arithmetic as Object Collection', so some of the attributes of array representations fit with their second: ‘Arithmetic as Object Construction’, i.e. “conceptualizing numbers as wholes made up of parts” where “[t]he parts are other numbers” and “the operations of arithmetic provide the patterns by which the parts fit together to form wholes" (Lakoff and Núñez, 2000, p.p.65). Again, actions within the physical domain are mapped onto the target domain of arithmetic, so 'Objects (consisting of ultimate parts of unit size)' maps to 'Numbers', 'The fitting together of A parts of size B to form a whole of size C' to 'Multiplication', etc. (p.p.66). However, while the formal axioms and entailments of Lakoff and Núñez's metaphor are helpful in articulating the relationships between an array (object) and its component parts, there is no reference to the specific visuospatial structure. Arrays must be considered as *regular* objects, constructed of (near-) identical component parts (e.g. rows and columns, or in 3-dimensional arrays, layers) which are fitted together in a regular structure.

4.3.3.2 Comparing arrays and containers

Container representations consist of closed boundaries within which the individual units may be in any configuration, but arrays are structured and defined by the spatial arrangement of the units, with their relative distances and directions an essential element of the representation type. They integrate numerical and spatial concepts in order to form a visuospatial mathematical object for representing numerical relationships. Unlike container representations, which are topological (i.e. unaffected by proximity or ordering of units, providing they remain within their boundaries), in arrays it is the position of the individual component units that defines the structure of the number, with spacing rather than visible boundaries defining subgroups. Thus any given unit in an array has more complex meaning than one in a container, because it is simultaneously part of a row, a column, and the whole.

Wittmann has said:

> Representations of mathematical objects form a kind of interface between pure and applied mathematics. They can be seen as concretizations of abstract mathematical concepts and at the same time as representations of real objects. Compared with the abstract objects these representations are more concrete than the mathematical objects which they represent, and compared with the real objects which they model they are more abstract. (Wittmann, 2005, p.p.18)

This positioning definition is highly appropriate for the kind of visuospatial representations occurring in this study; however, not all representations are equal in the levels of concretisation/abstraction they provide. On a spectrum of abstraction (in Wittmann's sense of the term), container representations would be positioned further toward the 'real object' end, more intuitive in their closer similarity to the enactive scenarios of sharing and grouping, but embodying fewer of the properties of natural-number multiplication (even considering containers-within-containers). Array structures would be positioned further toward the ‘abstract object’ end – less intuitive, but providing a theoretical link to all the properties of (natural number) multiplication, and expressing them with greater representational economy.

4.3.3.3 Discrete versus continuous

An important visual aspect of rectangular array representations is that they may be formed of ‘discrete’ or ‘continuous’ units. The varieties most commonly seen in classrooms tend to be grids of either widely-separated dots or closely-tiled squares (although Harries and Barmby (2007) have reported the effectiveness of closely-packed circles in supporting KS2 children’s multiplication calculations). In the classroom, arrays of tiled squares are most obviously associated with their use as a central representation for introducing the concept of area, but there is no reason why they should not also be used (as, commonly, are dot arrays) as a tool for demonstrating the commutative law or developing multiplication strategies. For example, Battista et al. (1998) have worked with square grids, identifying various levels of multiplicative sophistication in students’ structuring of rectangular arrays of squares, as have Outhred and Mitchelmore (e.g. 2000; 2004), linking the increased structure observed in young children’s 2D array drawings to the development of multiplicative strategies.

IMAGE REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 4-j: Reproduced from Izsák 2005, p.p.381

IMAGE REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSU

Figure 4-k: Reproduced from Izsák 2005, p.p.377- note ‘3’for height

Regardless of researchers’ choices with regard to discrete or continuous arrays, individual students may display marked representational preferences of their own, as seen in Figure 4-h and Figure 4-i above, where students are working with the ‘continuous’ grids of squared paper, which Jenny reinforces by going over the lines in thick pen, but Leo essentially replaces with a 'discrete' dot array. A study by Izsák (2005) shows similar discrete/continuous array preferences (dots vs. squares), where he presented students with dotted paper rather than squared paper for their rectangle-based tasks. Some of them, as hoped, focused on the square structure (Figure 4-j) while others, unsurprisingly, "found dots more salient than spaces between dots" (2005, p.p.368) and focused on those, to the detriment of their intended progression towards area calculations (Figure 4-k).

Finally, when using dynamic/interactive media, it is possible for array representations to be altered from the discrete to the continuous variety. For example, during tuition I found it visually effective to lay out a discrete array of widely-spaced cubes which could then be brought closer until they formed a continuous rectangle.

4.3.4 Multiplication and division with arrays

4.3.4.1 Arrays in teaching multiplication

Greer's widely-used classification of situations involving multiplication and division of integers includes those he calls the "most important classes": equal groups, multiplicative comparison, Cartesian product, and rectangular area (1992, p.p.276) – in all of which array representations work well. There are many studies focusing specifically on rectangular area, and those that link rectangular arrays to multiplication tend to focus on multiplication with fractions (e.g. Greer, 1992), expanded forms for factors and the distributive property (e.g. Izsák, 2004), or the commutative property, only the last of which is pedagogically within the compass of this study. Barmby et al. propose arrays as the representation to "best convey the most important properties of multiplication" (2009, p.p.224), these being (according to the authors) its binary nature, commutativity and distributivity. In terms of actually employing arrays in multiplication processes, they recommend it for supporting development of the grid method of multiplication, as did Izsák (2004). Of the empirical studies of children working with rectangular arrays, several followed the format of requiring participants to work out how many square tiles of a given size would fit into a rectangle of a given size (Battista et al., 1998; Outhred and Mitchelmore, 2000; 2004). The children's increasingly sophisticated spatial structuring was linked to improved understanding of multiplication and area measurement. In another study, children completed multiplicative tasks using dot arrays on a computer screen (Harries and Barmby, 2007; Barmby et al., 2009), and

Izsák's participants (2004; 2005) completed various rectangular area and equal groups tasks on dotted and plain paper. There are also various examples reported of students having difficulties with rectangular arrays and multiplication, for example, some of Barmby et al's younger (Year 4) participants enumerated their arrays by counting in ones or small steps, and appeared to display lack of understanding of the binary nature of multiplication, while Izsák noted that some of his (similar-aged) students had difficulty in moving on from counting unit squares. However, these are not grounds for discouraging the use of arrays, and none of the above authors do so.

4.3.4.2 Arrays in teaching division

While the above studies, taken together, provide a detailed emerging picture of children representing multiplicative structures, it is noted that all of the tasks are essentially multiplication-based, involving provided or measured rectangle sides, and finding an unknown total (area, number of tiles, etc.) However, multiplicative structures are also involved in division-based tasks and processes, and so array representations could, therefore, also play a key role in conceptualising division. Additionally, while the studies mentioned above (and others) each focus on one of the important representation types (some within a single task type, others with a selection), I have found none that look at the same participants working first with one visuospatial representation type then another (as opposed to one visuospatial representation and one or more symbolic/numeric representations). There are instances (e.g. in the Izsák studies) where children's array representations increase in sophistication, but not where they switch type, e.g. from containers to arrays. In fact, array representations could be used to model many of the same scenario tasks as containers, while 'nudging' students towards what the previously-mentioned authors argue to be superior representations for understanding the properties of multiplicative structures. Within scenario tasks, biscuits may be shared into packets (i.e. columns) instead of into plate-shaped containers, people fitting into vehicles may be lined up in queues (i.e. rows) rather than enclosed in car-shaped containers, and when a student is stuck on a bare division calculation (if the numbers are not too large) rows or columns of units can be drawn while counting up to the given dividend. As will be seen, various examples of these occurred during my fieldwork.

4.3.5 An array-container blend

To facilitate the progression from containers to arrays, I created a specific ‘bridging’ representation incorporating the aspects of both array and container (Figure 4-l). Sfard has emphasised the role that visuospatial representations can play in moving from an operational or process view of a concept (e.g. multiplication tasks) to a structural view (e.g. multiplicative structures as static objects to be examined): “It is the static object-like representation which squeezes the operational information into a compact whole” (Sfard, 1991, p.p.26). The visuospatial object in Figure 4-l was designed to encapsulate the notion of a number (in this case 27) as being constructed from – or capable of being divided into – both three groups of nine and nine groups of three, with the same image also serving as an aid to calculating 27 ÷ 3, 27 ÷ 9, 9 × 3 or 3 × 9.

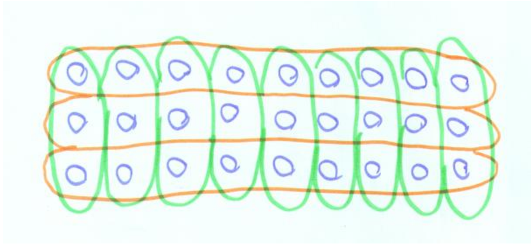


Figure 4-l: 27 as both 3 rows of 9 and 9 columns of 3 (CF)

Giaquinto theorised the following about dot arrays:

> [P]resented with a rectangular array of black dots in columns and rows, if inter-column spacing is not too different from inter-row spacing we can intentionally change from seeing the array as composed of columns to seeing it as composed of rows, and vice versa. In such cases one is exercising a capacity for visual aspect shift. There is some evidence that this can occur in visual imagination too. . . . Visual aspect shift may be a kind of attentional shift; like other kinds of attentional change it can occur without conscious effort. (Giaquinto, 2007, p.p.261)

This encapsulates one of the key attributes of array representations, and its relevance in understanding the commutative property of multiplication. Giaquinto’s term *aspect shifting* is a good description of the way the image in Figure 4-l is designed to work, with the aspect shift between factors aided by focusing on the differently-coloured row and column containers in turn.

However, Giaquinto’s classifications are otherwise problematic, in that he considers the kinds of spatial thinking which occur across the diagrammatic-symbolic spectrum to be classifiable into those that involve visualising motion and those that do not (with aspect shifting defined as one of the most important of the *non-motion* operations). Although in the case of Figure 4-l the image is held static on paper and so lacks physical motion, aspect shifting of this type may also be accomplished by motion (e.g. changing the relative distances between the elements of arrays constructed concretely or on a computer screen). Another example is his classification of 'noticing reflection symmetries' as a non-motion operation, when an obvious way of checking a configuration for symmetry is by 'flipping' it, as is turning a configuration to check for rotational symmetry); if, as stated earlier, the assumption is made that there exist internal (mental) imagic representations in some form, then when one experiences 'seeing' a mental image flip or rotate 'in the mind's eye', that experience includes motion.

So, my array-container blend may be considered a visuospatial manifestation of the 'Arithmetic as Object Collection and/or Construction' metaphors, capable of causing aspect shift. However, whether my students' experienced such a shift when using this representation type, and if so, if it would assist with multiplicative structures presented symbolically with numbers, or verbally within scenarios, remained to be seen.

4.4 Interpreting students' representations

It hardly needs stating that generalisation and the recognition of patterns and similarities play a central role in mathematics, and students' visuospatial representational activity provides clues to their development in this area. One of the motives for using a series of container- and array-based task representations was that students should come to recognise the structural similarity underneath – e.g. plates and buses are both containers of sets of objects (and can themselves function as objects within a larger set); the items within them may also be arranged in array form (particularly effective when considering the seats on a bus). Thus, in building a firm understanding of multiplication and division, we wish them to start to recognise the isomorphisms in tasks with a multiplicative structure, despite differences of scenario, representational modes and media, and superficial aspects of appearance.

To support this study's attempt to understand students' changing understandings, certain theoretical concepts are useful. Note that in a qualitative study of a comparatively small number of cases, there is not the same imperative to definitively categorise representations as there is for statistical analyses of large data sets; however, the concepts underlying quantitative researchers' taxonomies may sometimes also be applied in descriptive mode, and their characteristics seen as points on a spectrum (or multiple spectra) rather than as dichotomous categories.

4.4.1 Depiction and description

A theme present in several different strands of the literature is that of the *arbitrary* nature of aspects of representation and representational strategies. To address this, I have chosen examples from an educational psychology journal, a visual design textbook, and a series of articles aimed at school mathematics educators.

4.4.1.1 Educational psychology

> According to the different sign systems on which they are based, texts and visual displays belong to different classes of representations: descriptive and depictive representations. Texts (as well as mathematical equations . . .) are descriptive representations. A descriptive representation consists of symbols that have an arbitrary structure and that are associated with the content they represent simply by means of a convention. . . . Visual displays, on the contrary, are depictive representations. A depictive representation consists of iconic signs. These signs are associated with the content they represent through common structural features on either a concrete or more abstract level. (Schnotz, 2002, p.p.104)

The distinction between the different sign systems of text and of visual display dates back to Peirce's concepts of iconic and symbolic signs (1998). On attempting to apply these categories to students' inscriptions during tasks, it becomes immediately clear that while there exist some which are made up wholly of conventional symbols (although not necessarily in conventional layout) and others which contain none, a great number combine descriptive (symbolic) and depictive (iconic) elements within the same representation – e.g. Figure 4-m (below). Nevertheless, with data collected from the same students over a period of time, it should be possible to discern trends, if such exist, in the comparative emphasis on the descriptive and depictive in their work, and the way this affects students' performance on tasks.

There is another issue with this two-state classification, however: even if one considers individual elements of a representation rather than the whole gestalt image, there are points of intersection between iconic and symbolic signs – e.g. the short vertical lines in

Figure 4-c, which are at the same time abbreviated forms of stick people (icon), and figure ones (symbol). In fact, icon/symbol intersections such as these may be significant points which allow or provoke connections to be made and shifts in understanding to take place.

4.4.1.2 Visual design

Visual design theorist Ware (2004a; 2008) employed the terms *arbitrary* and *sensory* for distinguishing two aspects of visualisation, with sensory aspects “deriv[ing] their expressive power from being well designed to stimulate the visual sensory system” as opposed to arbitrary or “conventional aspects of visualizations [which] derive their power from how well they are learned” (2004b, p.p.12), and suggests different experimental techniques and interpretive methodologies may be appropriate for the study of these different aspects. In practice (as he allows) most actual visualisations used are hybrids formed of “an intricate interweaving of learned conventions and hard-wired processing” (ibid., p.p.13); however, to some extent the aspects may be teased apart.

Applying this theory to container representations, the image of a set of units held within a containing boundary is a powerful visual stimulus, regardless of what the units look like, the name one ascribes to the container, or the media available with which to create the necessary visual elements. While it may be expected that almost any student will be able to make use of container representations, it cannot be assumed that (low-attaining) students will immediately perceive different container representations as being of the same type and, accordingly, that they will behave in the same way. However, cross-cultural studies of tribes with extremely limited counting vocabulary (e.g. Butterworth and Reeve, 2008; Butterworth et al., 2011) have shown greater use of spatial pattern-matching and visual comparison in the arithmetical strategies of children in these groups, which indicates that it is possible for individuals with limited verbal numeracy to compensate with increased reaction to, and use of, visual stimuli. The evidence discussed earlier in this chapter suggests that while array representations are a powerful visual tool for working with multiplicative structures, the vital attributes (e.g. equal numbers of units across rows and across columns) must be noticed by or deliberately brought to the attention of students, as might general but non-necessary conventions (e.g. equal, ‘square grid’ spacing of rows and columns), and both of these differentiated from irrelevant visual attributes (e.g. colour and shape of units).

4.4.1.3 Mathematics teaching

Arbitrary is also a key term in Hewitt's (1999; 2001a; 2001b) discussions of school mathematics curricula, meaning knowledge which can only be gained by being informed through external means (e.g. another person, a text), and including information such as names, symbols, labels and cultural conventions. In this, it fulfils a similar theoretical role to the visual design 'arbitrary', in that such knowledge cannot be generated by a learner in isolation. Hewitt contrasts *arbitrary* with *necessary*, denoting knowledge which learners might work out for themselves, through perception and/or reasoning, from the knowledge they already have (which is not to say that a given individual learner will do so, merely that it is hypothetically possible). For example, knowledge that particular names (e.g. multiply, divide) have been assigned to certain arithmetical functions is arbitrary, but having defined these functions, the knowledge that (at this level of mathematics) they are inverse functions of each other, and that multiplication is commutative while division is not, are necessary facts. However, the fact that a particular mathematical rule or fact is 'necessary' does not mean that students will come to awareness of it in an appropriate way (through noticing, connecting, reasoning, testing, etc.) – in many cases the necessary is taught as if it is arbitrary, i.e. simply more information to be memorised. As with teacher input on representation (discussed above), my teacher input on strategies, terminology and notation was designed on a principle of minimal 'arbitrary' learning, with as much as possible of the 'necessary' subject matter to be discovered by students.

4.4.2 Function and decoration

The functional aspect of representations was introduced earlier, with regard to research on presented representations. An image or model (whether presented, created or co-created) used in working on an arithmetical task may assist the solver in various ways, such as organising or interpreting the verbal information given, emphasising and reminding of key information or relationships, or re-representing the information in a mode/media in which the student is better able to think. From a longer-term teaching viewpoint, it is worth remembering that a given representation may be assisting a student

- on that individual problem
- on other, subsequent problems of that structure

- in grasping the conceptual basis of the relationships underlying that problem structure
- in linking the concepts and procedures involved to other mathematics encountered
- in other, unpredicted ways

but that this may not be obvious at the time to either teacher/researcher or student. While there is undoubtedly truth in Mulligan's report that "low achievers did not recognise the underlying mathematical similarities between superficially different situations" (2011, p.p.20), I take issue with the later statement that they "replicated, unnecessarily, numerical or spatial features that did not support a coherent model of the mathematical situation" (ibid.), for its assumption that certain marks that an individual chose to make were "unnecessary" – particularly when those marks involve the key multiplicative idea of replication.

4.4.2.1 Mathematical and non-mathematical functionality

For students to recognise different container representations as falling into the same representational type, array representations into another type, (etc.), and to be able to make choices regarding the use of these (or other) representative strategies for unfamiliar tasks, they need to recognise which aspects of the representations are mathematically functional and which are not. By calling a particular part of a representation *mathematically* functional I mean that it directly represents a number or numeric relationship involved in carrying out the task. For example, in Figure 4-f, one could say theoretically that the containing boundaries are mathematically functional, as is the hair on the figures (as it functions to separate the subgroups of boys and girls), whereas in Figure 4-d the units are identical and the subgroups are indicated by another layer of nested containers. In both cases the arms and legs play no role in the calculations, and in most cases the people in the depicted scenarios function just as well when drawn as dots or tally marks as when they are given more human features.

However, it is necessary for the teacher or researcher to take care with how they use terms such a functional and decorative. A representation (or part thereof) which does not have mathematical functionality (as defined above) may fulfil another function. If a child has drawn an image in a particular way, there is some reason why they have done it that particular way rather than another; the representational components which appear to the competent adult as purely decorative (in a given scenario) – wheels, arms, etc. – must by their existence fulfil some function for the student. Bruner stated:

> A representation of an event is selective. In constructing a model of something, we do not include everything about it. The principle of selectivity is usually determined by the ends to which a representation is put – what we are going to do with what has been retained in this ordered way. (Bruner, 1974, p.p.316)

However, as I have observed, when it comes to mathematical tasks, students not only select from the information given, but add new elements to it and alter others to suit themselves. Figure 4-b (above) is from a task scenario in which children share a number of biscuits onto plates: the student has included the biscuits and plates (of which only one reproduced here), but not the children. Figure 4-m is from a scenario about a number of people fitting into taxis: Leo has included detailed taxis and added a road, but used the ‘4’ symbol for the number of people in each, rather than drawing them. Figure 4-n is from a task relating to the number of boxes transported in some vans: while performing the calculation, Vince stated that he was changing the vans into beetles. The fact that beetles are not generally used as delivery vehicles did not prevent him from achieving a correct total. Were these additions and alterations mathematically functional? No. While I made a methodological decision at these particular moments not to stop and ask the students to explain further their representational choices, I may speculate on their function: for Leo, the presence of cars implied the logical necessity for the existence of a road, and for Vince, to draw insects increased his enjoyment of the task.

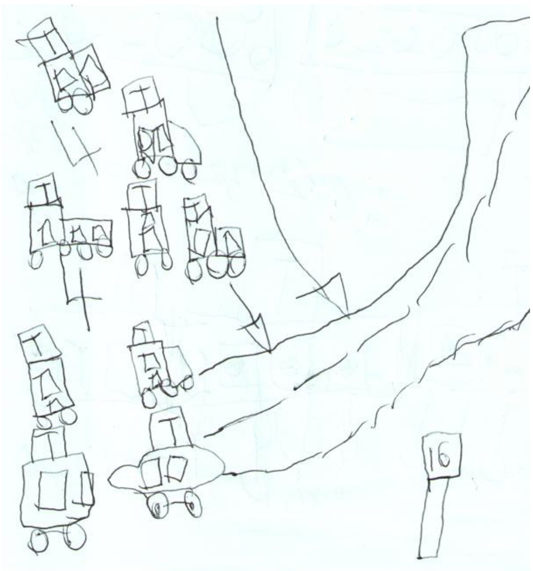


Figure 4-m: Calculating with passengers and taxis (Leo)

Issues have been raised about the potential for decorative images to have a negative effect on students, e.g. by distracting them. Elia et al. conclude that decorative pictures do not enhance understanding of the tasks to which they are attached (2007, p.p.670), and, in this, claim agreement with Carney and Levin. However, while Carney and Levin do cite an earlier study (Levin et al., 1987, in Carney and Levin, 2002, p.p.7) which recorded a negative effect size when decorational images were compared experimentally with their other types of 'picture function', they later state that "Decorational illustrations may help to make the text more attractive" (ibid., p.p.20). Attraction is not to be lightly dismissed when dealing with disengaged students and negative affect surrounding mathematics. To take a professional example, I once taught a boy with autism who loved dogs, and by association was positively disposed toward a particular brand of mathematics worksheets, each of which featured an (entirely irrelevant) small picture of a dog in the corner. As the presence of a dog picture made it more likely that homework would be completed, I drew many dogs during my time teaching him. This anecdote aside, there may be various *non-mathematical* functions for decorative images, particularly when it is the student who creates them. For a student with very poor short-term memory or attention span, a doodle of one car may serve as a useful reminder that the current task scenario involves cars. Then going on to draw a policeman directing the traffic (for instance) would indeed be a distraction from the intended mathematical activity, but if it is a brief, temporary distraction, I argue that the positive effect on a maths-fearing child's mood outweighs the minute or two of calculation time lost.

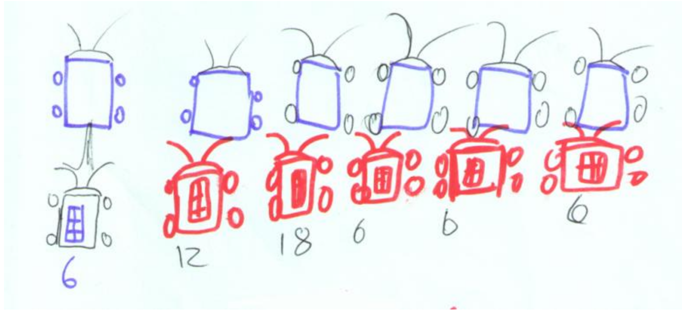


Figure 4-n: Calculating with boxes and beetles (Vince)

4.4.3 Abstract and concrete

A way of considering visuospatial representations which has been particularly popular in educational research, and has already been touched upon, is via the concept of *abstraction*. Originally meaning 'drawing away' (Lat: *trahere*, *ab*[s]), it is generally associated with the development of higher-order cognition. Piaget's stage theory linked concreteness and physical activity in problem solving with children operating at lower levels of thinking, and increasing abstraction with the attainment of higher levels. Similarly, Bruner's three-stage model of cognitive development in representation described a progression from *enactive* (actively manipulating concrete materials) to *iconic* (pictorial representation involving mental images) to *symbolic* (including formal language and mathematical symbols) (e.g. 1974). How do the ideas of abstraction and concreteness fit with those of functionality/decoration, arbitrary/sensory, and depictive/descriptive attributes of visuospatial representations?

4.4.3.1 Levels of abstraction

There are many examples provided, in this chapter and others, of detailed pictorial representations containing various non-mathematically functional elements, drawn in a manner bearing some physical resemblance to the items described in the task scenarios. There are also examples of minimal diagrammatic representations, using marks for units and containers that do not visually resemble items from their task scenarios. Additionally, consider concrete, enactive, physical task representations, such as in Figure 4-o. It becomes clear that there are actually two separate issues – which of the scenario elements are represented, and how they are represented.

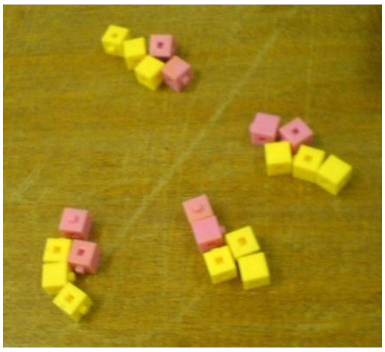


Figure 4-o: Using cubes to represent groups of people (Paula)

Many of my task scenarios involved calculations of numbers of people, grouped in various ways, and I have discussed the ways these person-units may be drawn. In Figure 4-f, there are no decorative elements, only the items and their groupings; however, the items for enumeration (people) are drawn in a pictorial manner. In Figure 4-m, there is an entirely decorative element (the road), the grouping relationships are depicted decoratively (the detailed cars), but the actual items for enumeration are expressed symbolically, in numbers. In Figure 4-d, the people are simplified to iconic 'stick men', and in Figure 4-c, they are not even recognisable as people, but depicted using the universal symbol of the tally mark. The fact that the student in Figure 4-o has used an enactive concrete representation should imply (by Piaget's and Bruner's criteria) a low level of abstraction. However, she appropriates arbitrary markers to represent the person-units – cubes which do not physically resemble people, and which can be re-purposed as whatever items a given task scenario describes – and does not include any other scenario elements. This is minimal, elegant, efficient, and in some ways more abstract than the drawn examples. Of course, in the larger sense of the calculation to be performed, the numbers and operations are being enacted concretely, but the units themselves are at the abstract, symbolic end of the spectrum. Thus, for detailed analysis of these kinds of representational strategy, more than a single spectrum is needed.

4.4.3.2 Alternative views on abstract and concrete

A different way of thinking about the concepts of abstract and concrete is proposed by Roth and Hwang, who redefine concreteness as "that property which measures the degree of our relatedness to the object (the richness of our representations, interactions, connections with the object), how close we are to it, or, if you will, the quality of our relationship with the object" (orig. Wilenski, 1991, in Roth and Hwang, 2006, p.p.335) where this relationship may be to a material object or to an idea. So, if to have a thorough concrete understanding of some mathematical concept is to have "rich representations, interactions, and connections with the (material, ideal) object of activity" (ibid.), it clearly does not imply a less advanced state, in the way that previous theorists' conceptions of the concrete-abstract spectrum do. Or does it? Gray et al. (1999; 2000) have collected the descriptions provided by "low and high achievers" (their terms) of both presented objects or images (e.g. a number of cubes, geometric shapes, "3 ÷ 4") and the children's own mental images in response to verbal cues (e.g. "five", "ball", "fraction"), and concluded there were qualitative differences in the groups' responses, reflecting the degree in which the children were "involved with the abstract qualities of the objects" (2000, p.p.408). "Low achievers" concretised and personalised the items, often with associated episodic memories, whereas "high achievers" filtered out superficial details to concentrate on semantic aspects and general properties. This is not to suggest that those more capable in mathematics cannot or do not invoke and attach rich imagery and episodic memories to mathematical objects – indeed, these authors elsewhere refer to the value of rich *concept images*, which include mental pictures and associations (Tall and Vinner, 1981) – but that they are able to focus on the relevant aspects for a given situation, whereas the less successful are unwilling to reject any information and unable to filter out surface details when required.

4.5 Summary

In practice, then, it is consistent to talk in general terms of a Bruner-type progression in terms of the representations a child has at their disposal at different stages (enactive first, then iconic, then symbolic) when considering broad cognitive development, but not to assume either that the richer, more concrete, interactive representations are completely superseded as the child progresses, or that this would be desirable. In considering the development of individual children, it is highly important to give recognition to the increasing and enriching of their representational experience; this is necessary if they are to recognise isomorphisms and succeed in applying what they have learned to varied and new task types and scenarios. It is, however, also necessary for them to learn to filter out non-functional aspects of images, and make appropriate representational choices.

In analysing the images produced by a student, it will be necessary to engage with them on multiple levels and dimensions. Furthermore, it is vitally important to look not only at which of the task elements are represented and in what form, but exactly how the student interacts with them. Imagine, for example, a scenario task involving calculation of a number of trees. Does the student draw the necessary number of unit trees and count them? If there is grouping involved, is this done before, during or after drawing the unit trees? Does one drawn tree stand only for one tree, or is it re-used multiple times? Are the trees actually used at all in the calculation – perhaps they are a doodle, or just there to remind the student of the task scenario? Having found no prior analytical framework including all the necessary components, I must develop one suitable.

To gain some understanding of a student's understandings through their visuospatial representations, it is important to look for changes over a period of time, but also to allow for variation in the individual factors affecting the particular representational choices made by a given student, on a given task, on a given day. However, co-created representations involve another person, who also makes representational choices regarding sensory/arbitrary, functional/decorative and concrete abstract aspects, and these choices also require analysis. Furthermore, in a personalised tuition setup, unlike with the pre-prepared images on the worksheets used in many of the studies discussed, these representational choices are influenced by the teacher's perception of their student(s). When in 'teacher mode' myself, I was not consciously thinking about these concepts, yet in responding to real-time data from my students, I drew, for example, wheels on some vehicles but not others. Images created under these conditions may thus be functioning as intra- or interpersonal mathematical communications, or as both.

5 METHODOLOGY

5.1 Introduction to methodology

> Research related to education is varied and complex, rarely amenable to precise measurement or given to all-encompassing solutions to its many challenges. (BERA, 2011, p.p.3)

As has become apparent from my review of literature, there is no clearly-defined boundary around the body of relevant research; many different fields of knowledge intersect, and it has been necessary to exercise restraint in what is included. Similarly, there is no single obvious methodological tradition in which this study could follow. I wished to understand better the process of students' thinking, what their numerical concepts are and how they change, the impediments to their learning, and the educational experiences that may allow them to overcome prior limitations. These things are highly individual, complex, and not directly observable. They are not quantifiable without massive simplification, and I argue that it is not desirable here to over-simplify, or, therefore, to quantify. Thus, research methods are required which are suitable for the collection and analysis of fine-grained qualitative data, relating to interior aspects of individuals' behaviour, which may be in a state of change over short time periods.

While my first research objective is straightforwardly descriptive, giving an account of the representational strategies that students are observed using, I do then intend to draw inferences about internal states – why they use certain representational forms as opposed to others. To go further, and arrive at conclusions about the relationship between representational strategies and changes in numerical understanding, I need not only the representations created, but information on the context in which they were created; this requires multimodal data. Furthermore, my intention to deliberately draw out data on students' visuospatial representations via tuition sessions where they are actively encouraged to draw and model has significant implications not only for the details of the chosen tasks, how they are set, and how I interacted with students working on them, but for the nature of the claims I make based on them.

While I have stated there is no prior research paradigm suitable for adoption in its entirety, I have identified a few areas of research which have been particular methodological influences on the design and operationalisation of this research project, discussed in 5.2. I then address methodological issues relating to the specific aspects of carrying out fieldwork with students with SEN, in mainstream schools at which I was not an employee. I provide a rationale for my decisions relating to design of tuition sessions, and lastly a summary of my methods and choices regarding data collection, organisation and analysis.

5.1.1 Ethical research guidelines

The fieldwork for this study was planned and carried out in compliance with the British Educational Research Association's (2004) *Revised ethical guidelines for educational research*, being approved by the Institute of Education ethics committee. The majority of ethical issues pertain to the requirement to "operate within an ethic of respect for any persons involved in the research" (p.p.5). I also drew upon the British Psychological Society's (1992) *Ethical principles for conducting research with human participants*. Between the planning and writing stages, these two organisations brought out updated versions of their documents (BERA, 2011; BPS, 2010), and this thesis will refer to the more recent documents, unless the earlier and later versions are in conflict.

5.1.2 Ethical issues specific to this study

Although my research proposal and plan required ethics committee approval in advance, the reality is that during all stages of planning, fieldwork, data collection/management, analysis and interpretation of findings, it was necessary to constantly self-interrogate regarding continuing adherence to ethical research principles, and in some detail. It was not something which occurred merely as a checklist at the planning, or any other, stage. While some ethical considerations – for example, the appropriateness of the scenarios chosen for tasks – did take place before meeting the students, it was necessary to observe if, once put into practice, the choices had in fact been correct ones. Similarly, the tension of the dual teacher/researcher role is not something which may be considered, allowed for, and then dismissed. Additionally, certain incidents arising during the course of the fieldwork, which could not have been predicted in advance, required ethical consideration. In this kind of research, ethical issues permeate all aspects of the project, and hence, rather than attempt to separate them into an artificial section of their own, I instead address ethical issues in context, as and where they arise throughout the topics covered in this chapter and beyond.

As the fieldwork involved working in schools, including periods where my dual role – particularly as far as the students were concerned – was heavily weighted towards the 'teacher' identity, there are times when concerns and decisions relating to research ethics are difficult to disentangle from the concerns and decisions of educational pedagogy. In some of the examples below, the reader's response may be that what I am discussing is a pedagogical question rather than an ethical one. However, I would argue that the fact that these kinds of considerations are part of teachers' common day-to-day experience does not make them any the less embedded in a professional ethical code which is not altogether different from those of social and psychological researchers, based on respect for the individual child, and concern for their current and future wellbeing. As such, at many points the ethical requirements of researcher and teacher will be in synchrony; if at others they are at odds, this is clearly something that merits consideration.

5.2 Methodological influences

> To understand better how mathematics learning of a highly symbolized type might occur, we worked with a small number of children, observing them in minute detail to determine the steps involved in grasping mathematical ideas. Such an approach is closely akin to the detailed study of the naturalist and clinician. (Bruner, 1974, p.p.426)

A strategy of qualitative enquiry comprises a set of skills, assumptions, and practices that the researcher employs as they move from paradigm to empirical world, and which connects them to specific methods of collecting and analysing empirical materials. Each of these is connected to a complex literature, and has a separate history, exemplary works, and preferred ways for putting the strategy into motion (Denzin and Lincoln, 2003, p.p.36–37). Of the varied methodological practices and paradigms which informed my thinking, research involving case studies, grounded theory, and microgenetic methods have been the most influential, as well as classroom-based practitioner-led research on children's numeracy. Meanwhile, essential to the strategy for this particular enquiry were the professional skills, knowledge and understanding deriving from this researcher's ten years' prior experience as a curious, questioning practitioner.

5.2.1 Case studies

My research aims to make sense of the internal educational experience of individuals by observing and analysing their arithmetical-representational behaviour in a highly detailed manner, seeking out intra-participant connections and patterns. The main benefit of a case study approach is that it allows the researcher to focus in to engage with the subtleties and intricacies of complex situations, and grapple with relationships and social processes in a way that is denied to quantitative approaches with many participants (Denscombe, 2010, p.p.62). However, there have historically been criticisms levelled at case study research, in particular relating to the issues of generalisability and rigour.

To generalise from individual cases, they must be examples of a broader class of things, to which they bear some similarity. I argue that while each individual's collection of experiences and thoughts is unique to them, some of the educational experiences and characteristics of certain secondary school students who struggle with mathematics will bear similarities to the experiences and characteristics of others. I would expect teachers, for example, when reading my accounts of a student's mathematical behaviours, to recognise not an identical student of their own, but certain aspects which are reflected in one or more of their own students; I expect my analyses of the meaning of my participants' mathematical behaviours to be of use in understanding the meaning of similar behaviours expressed by other students.

5.2.2 Grounded theory

In conducting a study exploring a particular set of educational circumstances, aiming to build and develop theory rather than test preconceived hypotheses, I was necessarily influenced by the 'grounded theory' pioneered by (Glaser and Strauss, 1967). In their suggested approach, the majority of hypotheses and concepts not only come from the data, but are systematically worked out in relation to the data during the course of the research, while, by contrast, the *source* of certain ideas, or even "models," can come from sources other than the data (Glaser and Strauss, 1967). While these authors' work is firmly within the realm of sociology, and thus their more detailed descriptions relate to comparing the behaviour of groups of people, similar principles may be applied when comparing the behaviour of individuals. E.g.:

> The constant comparing of many groups draws the sociologist's attention to their many similarities and differences. Considering these leads him to generate abstract categories and their properties, which, since they emerge from the data, will clearly be important to a theory explaining the kind of behavior under observation. (p.p.36)

These principles support the use of an essentially data-first analytical process, where close attention to similarities and differences in the observed mathematical behaviour of my participants would allow me to generate the necessary categories and properties with which to properly analyse their representational strategies and explain their multiplicative thinking.

Later versions of this approach, such as Constructivist Grounded Theory (e.g. (Charmaz, 2003) assume, in particular, the relativism of multiple social realities, and the co-construction of knowledge by researcher and participants; they also tend to place greater consideration on the effect of the researcher’s perspectives, values, privileges, positions, and interactions. This is an ethical stance I have endeavoured to uphold.

5.2.3 Microgenetic methods

Microgenetic methods were developed for the study of the transition processes of cognitive development (Siegler and Crowley, 1991). They have proved appropriate for case studies of individuals with difficulties in mathematics (e.g. Schoenfeld et al., 1993; Fletcher et al., 1998) and been used increasingly in studies of children's arithmetical strategies (e.g. Robinson and Dubé, 2008; 2009a; Voutsina, 2012). The main characteristics, according to Siegler (2000, p.p.30) are:

- Observations span the period of rapidly changing competence.
- Within this period, the density of observations is high relative to the rate of change.
- Observations are analysed intensively, with the goal of inferring the representations and processes that gave rise to them.

Investigators have developed two main strategies for meeting the difficult challenge of observing cognitive growth: to choose a task from the everyday environment, hypothesise experiences that might lead to changes in performance on it, and provide a high concentration of such experiences; or to present a novel task and observe children's changing understanding of it across one or multiple sessions (Siegler and Crowley, 1991, p.p.607). Schoenfeld et al. (1993) describe, metaphorically, a goal of taking "cognitive snapshots" each time a knowledge element or connection in their participant's knowledge structures changed, with the expectation that "running these snapshots in rapid sequence would produce a dynamic picture of growth and change" (p.p.61); however, they also acknowledge that such moments were difficult to find, and that their original model did not do justice to the unstable and nonmonotonic reality of human learning.

A potential pitfall, then, might be going into a microgenetic study with too many, or wrongful, assumptions about the nature of the learning that will (hopefully) take place during the period of observation, in terms of concepts involved, change-inducing interventions, order of events, and occurrences of academic interest. This is where a focus on micro-developmental change may usefully be combined with grounded approaches, i.e. through a stance of limiting, as far as possible, predictions and *a priori* theorising. Of course, a plan to 'simply observe' makes no sense in this context; one cannot observe everything. However, by delineating a particular aspect of focus (visuospatial representations) and the data I could collect in each period (see 5.5), I could gather observations on my aspect of focus in the maximum possible detail, while allowing for the fact that there would likely be key tiny moments of developmental change, some of which would be obvious at the time, but others only on considering the data retrospectively.

5.2.4 Practitioner research, pedagogy and experience

There is a strong tradition in the UK of research into various aspects of early numeracy – particularly counting, addition and subtraction – taking place in the natural environment of the classroom, and carried out by teacher-researchers. Those which focus on children's own representations of number are often quasi-ethnographic in nature, where (usually very young) children are observed in their mark-making (e.g. Atkinson, 1992) or block-play (e.g. Gura, 1993), sometimes questioned about what they were doing, and their representations analysed for 'emergent' mathematics using approaches which draw (as I do) on grounded theory. There are also more structured and quasi-experimental examples, such as Hughes's (1991) classic tasks investigating the relationship between concrete numerosities and children's pictographic, iconic and symbolic mark-making. Key to this body of work is that not only does it focus on children's own, often non-standard, representational strategies, but that it treats children's marks as always meaningful (to them, even if not comprehended by adults), and attempting to see them in terms of "child sense" (Carruthers and Worthington, 2006). This is in the pedagogical tradition of "de-centring" (Donaldson, 1978), i.e. to attempt to shift from an ("egocentric") knowledgeable adult's perspective, and imagine what a scenario, phrase or object might mean to a child. More recently, intervention programmes (such as Every Child Counts (ECC)) have been applying constructivist and connectionist principles to designing 1:1 support for struggling students, initially in KS1, then expanded to KS2-3.

Also relevant is the pedagogical approach of the Realistic Mathematics Education (RME) curriculum of the Netherlands. (Note that 'Realistic' in this case does not necessarily imply a real-life context, as in 5.4.2.1, below.) This derives from Freudenthal's view of mathematics as not subject matter but human activity (1968; 1977; in Van den Heuvel-Panhuizen, 2001, p.p.50), and has been distilled into six basic principles: (1) *Activity principle*, in which students are treated as active participants, developing mathematical tools and insights, rather than receivers of transmitted material; (2) *Reality principle*, in which learning mathematics originates in mathematising reality; (3) *Level principle*, by which aspects of a task scenario become generalised via a modelling process, into levels of increasingly formal, connected knowledge; (4) *Intertwinement principle*, in which mathematics (as a school subject) is not separated into discrete learning strands; (5) *Interaction principle*, where students with a range of abilities reflect together upon their strategies and findings (and tasks are carefully provided to be suitable for students at differing levels of understanding); and (6) *Guidance principle,* described by Van den Heuvel-Panhuizen thus:

> [B]oth the teachers and the curriculum have a crucial role in steering the learning process, but not in a fixed way by demonstrating what the students have to learn. This would be in conflict with the activity principle and would lead to pseudo-understanding. Instead, the students need room to construct mathematical insights and tools by themselves. In order to reach this position the teachers have to provide the students with a learning environment in which this constructing process can emerge. (2001, p.p.55)

5.2.5 Positioning my methodology

Having discussed four research bodies identified as influential – case studies, grounded theory, microgenetic methods, and practitioner research – where does my own project lie in relation to them? Regarding my intervention work with students, while I did not model my practices directly on any of the studies or curricula described above, I find my approaches broadly in line with the RME principles and ECC practices, and my relationship with my participants somewhat in the tradition of the Early Years ‘ethnographers’ – although obviously adapted for an adolescent age group. Regarding my data, firstly, I consider each participant longitudinally as an independent case, but also look for patterns and make connections between the cases; this may be described as *multiple linked case studies*. While I am committed to certain central ‘grounded’ principles regarding theory as emerging from data, I do not claim to be ‘doing’ Grounded Theory, as despite the fracturing of the tradition into many different forms, it is essentially a group process requiring a research team, and although I did discuss my research at all stages with my supervisor, and (during fieldwork) with the teachers at my two schools, this is essentially an individual project. The previous microgenetic studies I read were at their most influential during analysis of my data; however, knowledge of these methods enabled me to plan my fieldwork in a way that would allow me to obtain the kind of data, in the level of detail, that I would need.

An emergent research design does not have to be loose or undisciplined. While acknowledging that it was not possible to predict specific outcomes, or know exactly what would emerge from the research process, my experience allowed me to plan my input to adapt flexibly to individual cases, allow for a range of potential participant responses, predict broad parameters for outcomes, and follow up emerging patterns in the data. Pursuit of complex meanings cannot be simply designed in or caught retrospectively (Denzin and Lincoln, 1994). Rather than planning every detail beforehand, or rigorously applying particular analytical methods after, the discipline required for this kind of study is that it requires continuous attention and an ongoing interpretive role throughout (Stake, 1995).

5.3 Research setting

5.3.1 The teacher-researcher

All research depends on the interpretation of data, and even before formal analysis has begun, this data is not pure and separate from the circumstances of its collection, or from the individual(s) carrying out the research. In quantitative research, a 'value-free' period of data gathering is the ideal; in qualitative case-study work, it is expected that those responsible for interpreting the data will be in the field, "making observations, exercising subjective judgment, analyzing and synthesizing, all the while realizing their own consciousness" (Stake, 1995, p.p.41), and a great deal of research designs have thus placed researchers in the room with teachers and students to document and interpret their behaviours and interactions. However, another layer of methodological complexity is added when one person teaches and researches at the same time. Stake asserts it is "essential to have the interpretive powers of the research team in immediate touch with developing events and ongoing revelations, partly to redirect observations and to pursue emerging issues" (Stake, 1995, p.p.41–2), and a great advantage of being a singular teacher-researcher is the opportunity to immediately seize opportunities to adjust one's plans and attention the better to capture emerging interesting details. For the classic practitioner-researcher undertaking fieldwork in their everyday working environment, there can be serious tension between the business of fulfilling lesson objectives (etc.) as planned, and fulfilling the research objectives, particularly when events develop unexpectedly. In contrast, my tuition sessions and the tasks within them were designed with a high level of flexibility, allowing a space within which unexpected directions could be pursued, should I judge them more promising in terms of collecting relevant data, than the original plan. A crucial factor in this was my sessions' separation from the content of participants' regular Programmes of Study, and the teachers' permission to use the time however I thought best.

This idea of "best", however, itself requires inspection. Is what is "best" from the research angle the same as from the teaching viewpoint? Generally, in fact, yes. For example, if I required knowledge whether a student could solve a certain task, and it became clear that they could not, what then? From a research viewpoint it would appear that the immediate query had been answered, and one could move on to the next. However, to a responsible teacher it would be unethical to ignore the student's need for explanatory feedback on their failed task. In fact, taking the time to give some input in the form of discussion (probes) or instruction (nudges) is beneficial not only to the student's education, but to the working relationship, and, not least, in providing deeper understanding of the problems arising in that type of task.

5.3.1.1 Probes

Asking questions to elucidate participant responses and actions is desirable in terms of both research and teaching. While in individual instances the imperative may come from one, the requests are not detrimental to the other. Presmeg and Balderas-Cañas say:

> The disadvantage of the probing that may take place is that the researchers cannot know to what extent asking about the interviewee's cognition changed that cognition, but it is clear that without the probing, interviewees seldom report the full extent of their use of imagery anyway. Thus the questioning is necessary. (2001, p.p.5)

Here, participants' cognition is already assumed to be in a state of change (micro-and potentially macro- development), the intention being to capture and interpret series of moments from which cognitive changes may be deduced. It has long been observed that children with mathematics difficulties do not readily engage in analysis of their unsuccessful attempts (Allardice and Ginsburg, 1983, p.p.344), and encouragement to self-reflection (i.e. metacognition) is ethically supported, as it has been shown to be beneficial to the improvement of thinking skills (Zohar and Peled, 2008).

5.3.1.2 Nudges

A form of teacher support for students 'stuck' on tasks, I designate *'nudge'* prompts specifically as the smallest 'unit' of intervention that can be given in the circumstances. They draw the student's attention to a single image, model, relationship, representational or strategic component (etc.) that might be helpful to them. In practice, for each task, I prepared a sequence of potential 'nudges' to use; however, these were not to be employed mechanically, but selected, and adapted if appropriate, in response to each individual's circumstances.

This being a study focusing on visuospatial representations, many of the nudges – as might be expected – related to these. As one of my theoretical assumptions was and remains that increasing these students' experience of drawing and modelling in problem-solving is a positive thing for their learning, there is no ethical conflict here. However, it is also a central assumption that students are individuals with different representational preferences and requirements, and that while they should be exposed to various representational forms, and have the opportunity to try them out and discover their own style(s), one should respect their individual learning needs and not coerce them to use representational strategies against their inclination.

5.3.2 Schools

5.3.2.1 Selection of schools

Having decided to conduct research with a sample of the lowest-attaining students in Key Stage 3 mainstream education, I required access to suitable participants. Not currently being employed as a teacher, and my previous teaching posts for several years having been in the special school system, I had no ready-made connections to exploit. The two schools fixed upon were partially a self-selected sample. During the Spring and Summer terms of 2008 I wrote to all mainstream secondary state schools in the three Local Education Authorities closest to my own location (inner London). Of the 25 schools contacted, four responded. Each of these I visited for one day, of which two were then selected as being more promising for maximising the mutual benefit to my research, potential student participants, and the mathematics departments involved. One was a voluntary-aided Roman Catholic boys' school with a roll of 900 students aged 11-16; the other a mixed community school with a roll of 1350 students aged 11-19. During the 2008-9 academic year I was present in one or both schools for usually one full day each per week, over a total of 20 weeks.

5.3.2.2 Initial observations

I began by spending one whole week in each of the schools. These visits were of an ethnographic nature, my intention being to 'get a feel' for the culture and customs of the schools I would be working with. Of course, a week is far too brief a time in which to understand the workings of an organism as complex as a school; however, it was possible to achieve a degree of basic familiarity with the mathematics classes and staff. I observed 47 lessons in total, making sure to spend at least one period in each of the 'bottom sets' in the KS3 age range, and during periods where no such lesson was occurring, chose either a next-to-bottom set in KS3, or a bottom set in KS4. I took only handwritten notes, as I judged that using a camera or any other recording device at such an early stage would have been intrusive to staff and students. I noted briefly the area of mathematics that students were working on, style of the lesson, personnel present, and kinds of activity taking place. I also made a note of any individual students who either (a) appeared to be particularly struggling with the work, or (b) I observed using any kind of nonstandard representations in their calculations.

5.3.2.3 Relationship-building

For each lesson, I asked the teacher if they would prefer me to observe with minimum interaction, or to assume the role of an extra support teacher and interact with students. In the majority of lessons, teachers welcomed having an extra member of staff present, and invited me to "make myself useful". This allowed the students not only to become used to my presence in the department, but to position me as someone who took a direct interest in them (as opposed to, say, an inspector or supervisor observing the teacher), and could perhaps help them with the subject. During and after this observation period I had conversations with the class teachers, support teachers and SEN coordinators about which students might be most appropriate to withdraw from lessons for tuition. I made a shortlist of the students that were described by staff as the "weakest" of their peers, "particularly struggling", or "having the most difficulties" with mathematics, and compared these names with those in my own notes. I did not at this stage wish to know if and which formal SEN diagnoses the students might have, but I did request that staff alert me if any of the students were known to have psychiatric or behavioural issues; it would not be appropriate for me to withdraw students who, for example, had a history of violence.

When not conducting lesson observations, I spent my time in the department offices. In both schools, on my first day, the Heads of Mathematics introduced me at staff briefing, and requested I give a brief summary of my research. Unsurprisingly, some teachers were suspicious of a stranger admitted into their midst and given generous access to the physical and informational spaces of their department. One teacher semi-humorously accused me of visiting under the pretext of observing students, but actually being a spy gathering covert information to be used against teachers “They” wanted to “get rid of”; another was extremely guarded in his speech with me, believing I might be an undercover journalist. Thus, even after the Heads of Mathematics had formally approved my research, gaining the trust and co-operation of other staff was still a serious concern. Some questioned me informally, during breaks, and I was able to tell them more about my teaching background and tuition approaches, which improved their disposition toward me, as did proving my worth as a classroom assistant and an advisor on various individual students' difficulties they brought to my attention. It should be noted, however, that while some of the mathematics teachers were as interested in and supportive of my research as could possibly be hoped, a minority continued to display an attitude I interpret as resentment, and while not necessarily deliberately obstructive, behaved in a way which made my research more difficult (e.g. not providing requested information on their students; refusing to allow a student to leave class in the previously-agreed period).

5.3.2.4 Consent, permissions and feedback

After my initial visits to the schools, and verbal agreements with the Heads of Mathematics to use their departments as research settings, I sent a summary of my research proposal to the headteachers, who gave written permission for all aspects of my proposed fieldwork. Regarding data security and privacy, all raw data collected would be kept under lock and key on non-networked storage devices. All data that I published or presented should have participants' names (and any other identifying information) erased, and pseudonyms substituted.

The Head of Mathematics in one of my schools requested that after I had finished working with students, I give a presentation to the department, summarising what I had been doing and what I had found out. She also asked me to share my analysis of the individual students' difficulties with their class teachers, for the benefit of both. These requests were reasonable: I explained that the formal analysis of my data would not be available for some time, but that I was happy to give an informal presentation, take questions, and write a brief report on each student I had worked with. Although unasked, I volunteered the same in my other school.

5.3.3 Student participants

5.3.3.1 Disclosure and consent

The observation period initially resulted in a longlist of 25 students to withdraw from class for Initial Assessment (IA) interviews. In the interests of giving full disclosure of my aims using language and concepts that students would understand, I told them that I had been a school maths teacher but now worked at a university, and had a particular interest in those kids who found maths difficult. I explained that I was not employed by their school, but writing a ‘book’ about teaching and learning maths, and that if we worked together, I could give them some individualised tuition to help them with some of the maths they had problems with, while observing and discussing their work would help me (and my readers) better to understand how they learned. All students indicated acceptance of this simplified version.

At this point, individual consent becomes an issue, and I was careful to clarify to students that although their teachers had instructed them to go with me, they were free to cease the session at any point and return to their regular lessons, without any penalty. (One student exercised this right.) During the Initial Assessment and the following Tuition phases of research, I asked class teachers their opinion of students’ willingness to leave class and go for tuition with me; in no case did they report unwillingness, and in several cases active keenness, such as repeatedly asking if they were to see me that day. While gratifying, this was something of a concern regarding my relationship with the teachers in question, one of whom appeared to feel some chagrin from this preference! In general, students appeared very positively disposed to my sessions; while individual benefits will be discussed later, I believe it reasonable to state that they enjoyed having the undivided, patient, non-judgemental attention of a teacher figure.

One quite unexpected product of this situation was that a few students who I did not plan to withdraw specifically (usually friends of my selected participants) asked if they could go with me too. On deliberation, I felt that if I was conferring any benefit on those students I was withdrawing (or, at least, if they believed so), it would be unreasonable to refuse a child who directly asked for something that was within my power to grant. With the agreement of their class teachers, I devoted a few extra periods of my time to providing these students one session each. An ethical concern in various prior research has been than withdrawing students from class for special tuition might “lead to ‘labelling’ . . . by the participant (e.g. ‘I am stupid’, ‘I am not normal’)” (BPS, 2010, p.p.14). However, in both of these schools, the sheer number and variety of special educational needs on record (in particular, additional support for the many with home languages other than English), and the resulting highly-active SEN departments, made occasional or repeated withdrawals by staff very unremarkable.

5.3.3.2 Working with students

The students selected for tuition were those who exhibited the greatest difficulties with the IA tasks, and who showed some ability and willingness to discuss their arithmetical strategies and representations. Working with these individuals, particularly when ‘pushing’ them on tasks on which they would normally give up, or asking them to think about their working with an intensity to which they were unused, required constant assessment and pedagogical-ethical choices. Sometimes these were fairly straightforward (e.g. a student in a particularly distractible state of mind, perhaps tired or upset, and disinclined to mental effort), and in other cases more complex: for example, the atypicalities of Leo’s behaviour (contrasted with that of other students), when considered as part of the natural expression of his identity as a person on the autistic spectrum. The question of how much rein to allow his obsessive-compulsive tendencies lead to an example of (minor) conflict between ethical and pedagogical imperatives. In all cases, the engagement of students’ interest in tasks is necessary both for the tuition to be of benefit to them, and for appropriate research data to be gathered.

> Each interview context is one of interaction and relation; the result is . . . a product of this social dynamic. (Fontana and Frey, 2003, p.p.64)

One of the main feminist critiques of traditional research regards the power differential between researcher and subject; where the relationship is also that of teacher-student (not to mention adult-child), such inequality is inevitable. Nevertheless, I took certain small steps with the intention of diminishing the impact of this differential, and fostering a more collaborative, less obviously hierarchical atmosphere. I always sat beside or at right-angles to students, never directly face-on, and used informal speech, for a comparatively natural conversational tone. I also included occasional snippets of personal information (e.g. “I found it really hard learning times tables too”), openly positioning myself as an individual with an educational history of my own, sharing something with them in return for their sharing their thoughts with me.

I have described the circumstances through which participants were selected, and in one of the schools, withdrawn in pairs (see 5.3.4.2). As a result of the selection process and the school timetable, these students were always paired with another member of their mathematics set, but it so happened that none of the four pairs were close friends, or accustomed to sitting together in lessons. At the start of the project it was unknown to what degree students would attempt to collaborate on tasks, and so I neither instructed them to work together or separately. Some individuals were actively supportive of a partner struggling on a task, most behaved independently of the other person, and one pair could be actively unhelpful to each other (and thus required a somewhat more disciplinary style). There was little, if any, of what is generally considered collaboration, but the paired students' activities and conceptualisations may have sometimes been influenced by awareness of what their partners were saying and doing nearby.

5.3.4 During-fieldwork decisions and changes

It goes without saying that the most meticulous planning cannot allow for the unpredictable behaviour of human participants, and in the complex organism of a secondary school, unexpected circumstances arise which necessitate the changing of plans. A sample of these are: student absences and teachers switching lesson plans (in which case I postponed individuals' tuition schedule by a week); one participant leaving the school (early enough in the project to be replaced by a classmate); missing reports, SEN files, etc. (although fortunately my research did not necessitate detailed educational histories of participants); finding my allocated room occupied or unusable (in one case being evacuated by builders, nobody having informed me that the block was scheduled for demolition!); and two teachers attempting to dictate my schedule by promising a session with me to their students without prior agreement (in one case as treat, the other as punishment!) However, as well as these day-to-day issues, there were two that required significant methodological alterations.

5.3.4.1 Age range

At one of my schools, in the middle of the day was a double period with no KS3 lessons taking place. However, there was a girl (Paula) in the bottom set of Year 10 with whom the Head of Mathematics was particularly keen for me to work. Although I had originally specified participants in a certain age range, there seemed no sound reason to exclude a potentially interesting case simply for being one year older. I had also intended to avoid withdrawing students working on GCSE courses, but Paula's class teacher proposed that individual attention addressing her severe numeracy issues would be of greater benefit than her presence in class.

5.3.4.2 Individual and paired tuition

A serious challenge to my methodology came when I was about to start the Individual Assessments, and the Head of Mathematics of one school absolutely refused to let me withdraw students on a 1:1 basis, despite the fact that I was CRB-checked and already had the headteacher's written permission to do this. She was unwilling to discuss this further, but indicated that it was a school-wide policy, and that the SEN department also now only withdrew students in pairs. It appeared that my only options were to find another school in which to carry out fieldwork, or to change my methods immediately. Preferring to avoid potential months of delay, and to work in my two schools concurrently, I decided to embrace this as an opportunity, adapt and work with pairs. In fact, although this had not been part of my original plan, there was the potential for comparisons between participants in the individual and paired conditions, and the possibility of observing students working on tasks with peers.

5.4 Task design

The precise details of the tasks used – scenarios and numbers – were deliberately flexible, allowing response to the differing needs of individual students. However, I worked from a set of basic arithmetical concepts relating to multiplicative structures, a set of task scenarios from which to explore these concepts, and some general principles of representation. These are described below, followed by a summary of the tasks used with students in each stage.

5.4.1 Arithmetical content

As observed in Chapter 3, there is a general assumption that by the time children reach (mainstream) secondary school, they will have mastered addition, subtraction, multiplication and division, and can apply them appropriately. Even if this were generally true, it is clearly not universally so. It may, however, be assumed that all children have encountered these things. One of the functions of the Initial Assessment was to permit the selection of participants who had encountered multiplication and division of natural numbers before (i.e. all of them) but could not reliably or confidently carry out tasks, bare or scenario, involving these operations. The key concepts relating to multiplicative structures that I intended to work on with all students may be summarised thus:

- Numbers may be represented visually by sets of objects (physical, drawn, or imagined), and these sets may then be manipulated in direct correspondence with arithmetical operations.
- Sets of objects (i.e. numbers) may be sorted into equal groups of a given size. (division: quotitive or grouping model)
- Sets of objects (i.e. numbers) may be sorted into a given number of equal groups. (division: partitive or sharing model)
- A set of equal sets of objects may be constructed and combined to give a total number of objects. (multiplication)
- There are standard symbolic notations for the above operations.
- The number of sets and the number in each set may be reversed without affecting the total number of objects. (commutative principle)

I expected my participants to have variable levels of understanding of these concepts, with some aspects partial, hazy, or unstable, and that it would take careful probing and deductive reasoning to determine the state of each individual's capacity for multiplicative thinking.

I had no particular target magnitudes with which they should calculate, in most cases beginning with tasks involving products or dividends I predicted they would find comfortable – usually around 20 – then working upwards (or downwards) as appropriate to the individual's ability on that occasion. The focus being on recognising patterns and understanding principles, I tended to re-use certain number relationships with which the given student was, or was becoming, familiar. There were also no particular calculation formats I pressed students to use. As detailed in Chapters 3-4, the aims were that they should look at a scenario task and determine its arithmetical structure, or look at a bare task and represent it in a way that rendered it soluble; the formats used for enumeration (e.g. repeated addition in columns), while of interest, were not a planned tuition issue (unless participants directly asked for help with a particular calculation type).

I have so far been discussing multiplication and division together rather than separately, and this was also a deliberate tuition strategy. As Anghileri (1995, 1997) has pointed out, much of the language used in talking about numeric structures involving equal sets is shared between multiplication and division, and to use some of the same words and visual imagery is a way of highlighting and or reinforcing their interrelatedness. Even when children do have a firm concept of multiplication, it might be limited purely to a repeated addition model (Fischbein et al., 1985) and division, similarly, to either repeated subtraction (ibid.) or 'sharing', etc. Some students may have had prior instruction that treated the operations separately, and it is sometimes necessary to make the inverse principle explicit (Robinson and Dubé, 2009a; 2009b). There has been debate in the UK and abroad about whether division should be introduced as a single entity, or as two – partition and quotation (e.g. Marton and Neuman, 1996). I chose to focus one tuition session on partitive scenarios and another on quotitive; however, as all main tasks were presented in scenario form, most being represented and solved by students using visuospatial strategies, and I was not teaching set calculation methods, this distinction was not explicit; they were just all tasks which involved some kind of multiplicative structure.

I also deliberately delayed my use of the formal words and symbols associated with the operations. Every single participant, during interview, named division (or indicated the symbol '÷') as something they found difficult, did not understand, or disliked. To immediately plunge struggling students into a topic with which they already have negative associations, without first building up trust, would be unethical. It is also pedagogically counterproductive; as pointed out by various educational authors (e.g. Hewitt, 2001), and as practising teachers know, while Ofsted inspectors may insist that teachers state the specific aims of a lesson at the start, it can actually be beneficial for students to begin by focusing on a task, without necessarily knowing where it will lead. For this reason I only introduced formal terms and symbols at a later stage, in relation to task types in which students had already experienced success. Thus, by changing, in some small way, their relationship with this one deliberately unpopular area, I aimed to increase students' confidence and self-efficacy, and encourage more positive identities in relation to the subject, without 'protecting' them from challenge (Brown et al., 2008).

5.4.2 Scenario content

With a view to enhancing students' conceptual understanding of arithmetical structures, the scenarios used need to be easily associated with different types of (external) representations (Greer, 1992). Verschaffel and De Corte (1996) suggest a basic list for "rich and flexible variety" including:

- experience-based scripts in which knowledge is organised around real world events or dramatic play,
- manipulatives,
- pictures and diagrams,
- spoken language,
- written symbols.

They add:

> Only after children have had ample and varied experience with describing and exploring additive and multiplicative situations dramatically, physically, pictorially, verbally and symbolically does it make sense to introduce the writing of abstract number sentences involving these operations. (ibid., p.p.116)

As discussed above, my tasks follow these principles.

One of the main weaknesses of literature on arithmetical tasks is the widespread assumption that certain of the more familiar worded task types (e.g. sharing or grouping tasks) are 'standard', 'routine', and 'not challenging' for students. For example, Rosales et al. state:

> Standard problems are those that can be properly modelled and solved by straightforward application of one or more arithmetic operations with the given numbers (for example, "Steve has bought 4 ropes of 2.5 m each. How many ropes of 0.5 m can he cut out of these 4 ropes?" . . .). These problems . . . do not imply a real challenge for students. (2012, p.p.3)

I suggest that this task would, in fact, be a very real challenge for some students in mainstream education, and not only my participants. While it is not wrong for any given piece of research or pedagogy to concentrate on the needs of average-or-above-attaining students, such sweeping statements about the absence of cognitive challenge in tasks ignore and dismiss the day-to-day experiences and struggles of students with difficulties in mathematics. In the example above, the mere use of non-integer numbers would prevent the task from being a routine calculation for some, but even if they were replaced with simpler quantities, it is probable that for students such as mine, the task would still require modelling rather than "straightforward" application of a symbolic operation.

Watson and Mason (2005) propose that rather than categorising textbook questions either as routine (to be answered by mimicking a previously-seen method) or as nonroutine (to be tackled heuristically), it makes more sense to see the range of possible questions as varying on a continuum from routine to nonroutine. I agree, adding that the level of 'routine-ness' of a task also depends on the individual to whom it is set, and when it happens to be set to them. Thus, many of the scenarios I presented to students, being based on sharing, grouping, replication, etc., may look at first glance like highly familiar 'routine' tasks – but, at least to these students, at least initially, they were not. The scenarios, however, were designed to contain familiar items and relationships which would not create additional cognitive challenge for students. As discussed in Chapter 4, biscuits, plates, vehicles and their cargo were the basic scenario elements, to be represented in any and all of the modes on Verschaffel and De Corte's list (above).

Once students were comfortable with a particular task type, it could be 'stretched' so that it became – somewhat counterintuitively – less easily represented. Haylock (1991) describes what he calls a "fuzzy region" of problem-solving, where children can solve a task type with small numbers, but not larger ones – not through lack of computational skill, but from not having explicit enough awareness of the arithmetical structures and relationships of which they are making use. He proposes that on being set a series of tasks with increasing quantities, students experiment with calculators and discover the sequence of keys (and hence the operation) that gives the right answer. While that is not a pedagogical principle to which I subscribe, increasing magnitudes just enough that the comfortable unit-based countable representations begin to be irritatingly time-consuming can indirectly manipulate students out of situation-based strategies into using (and sometimes, in the process, building) knowledge of arithmetical principles (Brissiaud and Sander, 2010).

5.4.2.1 'Real life' mathematics?

In the strictest sense, a 'real-life' mathematics task implies a scenario in which the student might feasibly find themselves, in their current life outside the classroom, which requires of them some kind of calculation or logical reasoning. With a heterogeneous class or set of research participants, there are few, if any, such tasks that could be expected to fulfil those criteria for all students. There are reasons why the sharing of foodstuffs is and has always been such an extremely popular scenario in arithmetical tasks: children demonstrate fair-sharing strategies such as 'dealing' from an early age, and while they may not spontaneously use those same strategies in informal situations as frequently as adults imagine (Davis and Hunting, 1990), they are nevertheless keenly aware of the concept of fairness in sharing, particularly if they are the one who has received an unfairly small share (ibid). The ensuring of a ‘fair share’ is a rare example of an activity involving multiplicative structures, in which children of all ages, genders and sociocultural backgrounds could reasonably be assumed to have actually engaged or observed at some point. Requiring students to dish out imaginary biscuits to imaginary friends is admittedly hackneyed, but this comparative universality (along with its ease of visuospatial representation) is the reason I chose it as my first scenario for division.

As well as mostly using primary-age participants, the majority of ‘word problem’ studies have focused on additive structures, for example, dealing with combination, change and comparison between two sets of objects (e.g. the size of Jane and John’s respective marble collections) – and these have been classified (e.g. (Riley et al., 1983) a great deal more than multiplicative scenarios. In order to discuss specifically multiplicative-structured scenario tasks, I refer to several examples of ‘word problems’ which have been used in the studies cited in previous chapters, and thus illustrate some theoretical and practical issues with the idea of ‘real-life’ arithmetic.

Special case: Buying/selling

Often, scenarios involving shopping, i.e. buying items for cash, and perhaps receiving change, are assumed to have universality for children. If this were ever the case, it is not now. At the supermarket, what many children observe is their accompanying adult collecting a basket of shopping, for which they are unlikely to know in advance the exact total, and depending on economic status (among other things) may or may not have an estimated total. They pay by putting a piece of plastic into a machine and removing it, apparently unchanged; sometimes the situation is further confused by ‘cashback’, where the adult receives both goods and money. Regarding children’s independent purchasing, some do still go to confectioners, but their snacks are rarely sold individually or by weight. Children are also notorious for handing over what they hope is enough money and waiting to be told if they need to give more, then not checking their change.

There are certainly classic studies involving buying/selling scenarios, but they involve careful matching of scenarios (or real situations) and participants, for example, the calculations of children known to work as street market vendors, described in *Street Mathematics and School Mathematics* (Nunes et al., 1993). The majority of children do not have such experience.

Special case: At work

'Real-life' tasks might reasonably be argued to include scenarios that the children themselves would not be likely to have participated in, but which others (i.e. adults doing various jobs) do actually engage in, and of which the children are likely to be aware. This type of task would include scenarios such as a builder working out the number of tiles to cover a given area, or a gardener making calculations where trees must be planted at appropriate distances. Even if children have not seen the actions being carried out, the rules and behaviours involved in well-chosen scenarios are relatively simple to explain.

I chose my scenario of working out the correct number of taxis (buses, etc.) to transport a given number of people as falling into this category. While they have probably not had to make such a calculation themselves, most schoolchildren in the UK have been on a school trip, which involved someone else hiring the correct number of buses. Again, this is a classic scenario used by a many previous researchers, e.g. "19 children are going to the circus. 5 children can ride in each car. How many cars will be needed to get all 19 children to the circus?" (Carpenter et al., 1993)

Realistic scenario, unlikely numbers

Scenarios that fulfil the criteria of being a realistic scenario that a child could reasonably be expected to have experienced, observed, or be able to imagine themselves in, can become unrealistic through the insertion of inappropriate numbers. A fine example of a student's response to numbers they perceive as inappropriate is provided by Goldman (1965; in Dowker 2005, p.p.109):

> Teacher: If you buy a gun for two pence, and caps for a penny, how much do you spend?
> Duggie: Caps is three 'apence. You can't get none for a penny.

Also included is the response of another child, refusing to engage with a scenario they are unable to experience in real life:

Peggy: My Mum says I can’t buy guns.

Of course, if the numbers used in a scenario are unrealistic, there is often a sound reason: the teacher or researcher needs to expand the numbers with which a child calculates, perhaps to nudge them into different arithmetical or representational strategies, but wishes to retain a familiar scenario for presentation of the task. As mentioned above, some children simply accept inappropriate numbers as part and parcel of mathematics classes, but others object. I am unaware of any research on this particular point, but teaching experience and anecdotal evidence suggests that it can be helpful for the adult setting the task to specifically state (in appropriate wording) that they know the specifics of the scenario to be improbable, but request the children accept them temporarily, for the purpose of acquiring the related mathematical concepts and/or procedures. Within the teaching community, an informal term for this is *lampshading* (borrowed from internet-based criticism of fictional media). For example, after Wendy found the ‘Taxis’ scenario particularly helpful for understanding and carrying out quotitive division tasks, I presented her with versions of the scenario involving increasingly large, and increasingly implausible, numbers, lampshading this with comments such as “These are the extra-large 7-seater taxis”, “I don’t think there actually are 9-seater taxis, but for now let’s say they exist”, and later, when calculating with 200-seat aeroplanes, “I’m not sure anyone actually books a set of planes to take hundreds of their friends on holiday; maybe some really rich people do”.

Realistic scenario, unlikely calculation

> John has 3 books, and Sue has 4 times as many. How many books does Sue have?
> Simone has 9 books. This is 3 times as many as Lisa. How many books does Lisa have? (Mulligan and Mitchelmore, 1997, p.p.314)

In some tasks, the items and concepts involved in the scenario are familiar, but the calculation is one that children would be unlikely to ever need to carry out anywhere but a mathematics lesson. In the first example here, children do tend to spontaneously count and compare the numbers of their various possessions with those of their friends, although they are much more likely to use additive than multiplicative comparison. The second example, despite its similarity, is considerably more artificial, and the kind of contrived question only a mathematics teacher or text would ask. However, students generally seem to accept this kind of task as one of the conventions of the mathematics class, as long as examples chosen are not too convoluted or ridiculous, and lend themselves to appropriate and helpful forms of representation.

> You are making hot chocolate. You have _ marshmallows to use up. If you put _ marshmallows in each cup, how many cups do you need? (Kouba, 1989, p.p.150)

This example qualifies as ridiculous: nobody decides how many cups of hot chocolate to make based not on how many people want hot chocolate, but on how many marshmallows they possess, and an arbitrary ruling on how many must go in each cup. Even children experienced at mathematics tasks with internal but no external logic may balk at ones like this, particularly if the arithmetical structure is one that could easily have been set in a more sensible scenario.

Many Cartesian product tasks fall into this category, including 'Holiday Clothes' (6.1). However, there is evidence that children find multiplicative structures based on combination spaces particularly difficult compared to the other types (Anghileri, 1989; Mulligan and Mitchelmore, 1997; Nunes and Bryant, 1996; Verschaffel and De Corte, 1997; Yeo, 2003), and that using a scenario which encourages visuospatial representation can help. It is not easy to come up with Cartesian tasks that resemble 'real life' for children; even *At work* or other quasi-realistic scenarios, with realistic numbers, tend to lead to calculations with too-large numbers or too many dimensions for the child to cope. Thus, for the purpose of addressing this particular kind of mathematical calculation, scenarios are frequently allowed – including by myself – which involve particularly familiar and easily-representable items (e.g. a menu of food items, a suitcase of clothing), but in implausible quantities or item ratios, and with the even more unlikely requirement of calculating all the different possible combinations. My usage of 'Holiday Clothes' benefited from a brief 'lampshading' statement to students initially perplexed by the real-life peculiarity of the task, and, I strongly believe, from being set at the end of the Initial Assessment process.

Ridiculous scenario

> Pretend you are a squirrel. There are _ trees. If you find _ nuts under each tree, how many nuts do you find altogether? (Kouba, 1989, p.p.150)

It is not necessary to address the trees and nuts. If I had chosen to set this task to my research participants (or for that matter, when a class teacher), despite having developed a good and trusting working relationship, I can well imagine some of the more neurotypical looking at me with raised eyebrows, snorts of laughter, or outright scorn. In addition, least one of them (Aspergian Leo; perhaps also his partner Vince) would likely have interpreted the first instruction literally, and gleefully pretended to be a squirrel then and there.

5.4.3 Presenting and representing tasks

5.4.3.1 Verbal presentation

A change from my previous project was the decision to use no pre-printed materials. This complete absence of worksheets was (a) an example of the deliberate informality designed to distance my sessions from students' regular classroom experience, and (b) to reinforce that the sessions were individually tailored just for them. I spoke each question aloud, and as I did so, wrote down the numbers mentioned. On some occasions a single stating of the task was sufficient for students, but on many occasions they required repetition, which I did as many times as necessary. Given that the aim of this exercise was to gain some picture of the abilities and needs in each individual case, not to gather quantitative data to compare, after the initial statement, the support I gave students, while falling within the same overall framework, differed in the details. If they appeared to be having difficulty relating the numbers to their role in a scenario, I wrote down a little more for them, (e.g. in Q4 "6 boxes, 3 vans"; in Q6 "7 groups of girls"), but never whole sentences.

Tasks were posed using the minimum amount of language, but (in line with Chapman (2006), Coquin-Viennot and Moreau (2007), and Rosales et al. (2012)) certain students could be expected to (and did) respond positively to additional scenario information being given, which was not mathematically functional but served a purpose in helping them to picture the scenes (e.g. in Q4, stating that the vans are parked in a car park; in Q6 that the children are being put into teams for a contest). When students were 'stuck', I encouraged them to draw, then, if necessary, drew for them in small 'nudge' increments (e.g. a container, a set of dots), each time giving opportunity for them to take ownership of the representation.

5.4.3.2 Visuospatial (re)presentation

The focus of this study being students' representational strategies, I did not initially use any drawing or modelling in my presentation of tasks. However, working on the assumptions that visuospatial representation can be very helpful to students, but that many lack the metarepresentational competence to create and judge the usefulness of representations for different purposes (DiSessa, 2002), it was necessary to encourage them. I used the ‘nudge’ pedagogical strategy described previously, i.e. providing Pólya-style heuristic problem-solving support in minimum quantities, first asking if there was anything they could think of to help them work on the task, if they could use the same strategies as in previous tasks, then explicitly suggesting drawing/modelling, then beginning a co-created representation if necessary (based on container and/or array forms). If, on the other hand, a student had begun using a representational form which was comprehensible to me, I worked within that as far as possible. Regarding my own drawn contributions, I found that, in the same way that some students required more verbal detail to their scenarios, some preferred more (non-mathematically functional) visual detail to their drawings, and so I adapted my markings to their individual styles.

It is worth noting that while I did sometimes add visual details which were not mathematically functional (e.g. wings on an aeroplane), the various drawn container representations were intended to have some visual similarity to each other, as were the various array forms. Zhang and Norman (1994) state that while both referent and representation are meaningful to the theorist (task-setter), and superficially different tasks may be recognisable as having the same underlying abstract structure, this is not the case for the task-performer, who sees simply different problems to be solved. If this were the case, provoking students to notice regularities and similarities of structure in representations, and thus – perhaps – in arithmetical tasks, could be considered as making them theorise. This would be a significant development. Yeo (2003) has discussed in some detail the particular difficulties of SEN students with using concepts such as (numerical) commutativity in scenarios which are psychologically non-commutative; I suggest recognition of visual similarities in representations could also help counteract this problem.

5.4.3.3 Representational materials

I provided plain light green A4 paper for students to work on, apart from the rectangular area tasks, for which I gave them 1cm squared paper (also green). I provided a set of coloured felt-tips, and allowed them to select their preferred colour(s) from these, or to use their own pen or pencil. I reserved the purple pen for myself, making my own markings easily identifiable. For physical modelling, I provided a bag of coloured $1cm^3$

multilink cubes. I also had short coloured straws which could be easily bundled into tens and hundreds, which I kept back as an additional resource, should I find myself needing to work on the base-10 system with any students moving on to division with larger magnitudes. The use of only simple, cheap, easily available media would ensure that (unlike studies using specialist computer programmes, for example) my tasks and tuition methods could be replicated by any researcher, teacher, or person interested in doing so.

5.4.4 Initial Assessment

Below is a summary of the tasks set in the Initial Assessment stage of fieldwork. (A complete list, each with a rationale for their inclusion, may be found in Appendix A). They derive originally from both my own 1:1 teaching experience and a variety of sources in research and pedagogical literature. Most had already been tested on students of similar age group and level of attainment in the Masters' project which was the direct precursor to this study, after which some small changes were made to the finer details, and some optional extensions to the task sequence added. Note that here, tasks are described with brevity in mind; the wording for students was more akin to natural speech.

I also used these sessions to obtain some information regarding these students' attitudes and feelings towards mathematics. In my previous research I had found it most effective to combine (or alternate) such 'interview' questions with mathematics tasks; children and adolescents can find it difficult or uncomfortable thinking and talking in a self-analytical way, particularly about what they might well view as a problematic subject. After a protracted mathematical effort, I switched to 'chatting' mode, to provide an often-welcome cognitive break, then moving to the next task when the conversational topic came to an end. 'Attitude to mathematics' questions are also listed in Appendix A.

Q1) Shapes

Sketch a rectangle, circle, and triangle.

Q2) Number combinations

Provide pairs of numbers adding, then multiplying, to specified totals.

Q3) Cubes: Visual estimation

Estimate the number of (loose) multilink cubes in a handful, a bag, etc.

Q4) Wheels: Replication-based multiplicative structure

Series of multiplication calculations of the form: *A van has 4 wheels. How many wheels are there on _ vans?*

Extension task of the form: *Each van is carrying _ boxes in the back. Each box contains _ bottles. How many bottles are there in _ vans?*

Q5) Rose bushes: Unconventional arithmetical structure

Two calculations of the form: *A straight path is _ metres long, with rose bushes planted one at each end, and spaced at _ metre intervals. How many bushes are there altogether?*

Q6) Groups: More complex multiplicative structures

Series of calculations based on classes of 20–100 girls and boys being arranged in groups of different sizes, under different criteria.

Q7) Holiday clothes: Cartesian product

Find the total number of possible combinations of x differently-coloured t-shirts and y differently-coloured trousers. (All data from this task are analysed in sub-chapter 6.1.)

5.4.5 Tuition

The theoretical background for these tasks has been discussed in Chapters 3–4. As explained above, I allowed for considerable flexibility in the details of working on tasks with individual students. However, a summary of the basic plan for each of the three tuition sessions follows (with a complete list provided in Appendix B). Students worked at widely differing paces, and so accomplished different numbers of tasks. I created extension tasks, if necessary, based on each student's performance so far.

Each session began with a 'starter' task where I presented the student(s) with a cuboid made of multilink cubes, and they calculated the total number of cubes. (All data from this set of tasks are analysed in sub-chapter 6.2.)

Tuition 1: Biscuits

Partitive division tasks using the scenario ‘Biscuits’, where a number of biscuits are to be shared between a number of children. Each child’s share is calculated.

Extension: finding all the factors of a specified number.

Tuition 2: Rectangles

Tasks based on the rectangular area model of multiplication/division, where students draw rectangles (on squared paper) of specified areas and/or side lengths.

Extension: using formal ‘rectangular’ division notation.

Tuition 3: Taxis

Quotitive division tasks using the scenario ‘Taxis’, where a number of people are to travel in taxis. The number of vehicles needed is calculated.

Extension: use multi-digit dividends and divisors.

Tuition 4: Summary

Recap of work done in previous sessions, reminder of the commutative principle (using the array-container blend), and a selection of tasks from each scenario.

Extension: bare division tasks.

5.4.6 Final session

After I had completed all sessions as described above, I saw students on two more occasions. On the first, directly after the tuition period, I gave them a short semi-structured questionnaire (see Appendix C) to try and gauge something of their feelings about the work we had done, and give them the opportunity for individual feedback. As a result of themes appearing from my preliminary analysis of data, relating to preferences in division strategies and scenarios, I also requested to revisit students near the end of the school year, for one final session (Summarised below; full details in Appendix B).

Students were allowed to choose between tasks which used the same numbers, but were expressed in either a Biscuits, Rectangles or Taxis scenario, or in bare symbolic notation, presented in random order.

Extension: making up a scenario to fit a given bare division task (expressed in symbolic notation).

5.5 Data collection and management

Robson states:

> Case study is a strategy for doing research which involves an empirical investigation of a particular contemporary phenomenon within its real life context using multiple sources of evidence. (2002, p.p.5)

My approach, as discussed above, involved collecting and integrating several forms of multimodal data. The main data collected during sessions with students was:

- all paper marked by students
- audio recordings of complete sessions
- photographs of models constructed by students
- my own notes (made as soon as possible after sessions)

Additional data included:

- preliminary classroom observations (written notes)
- informal conversations with teachers (written notes)
- departmental Programmes of Study
- students' SEN files (where available)

5.5.1 Recording strategies

5.5.1.1 Audio versus video

I considered the advantages and disadvantages of video versus audio-only recording, and opted for the latter. Whilst video might seem the more natural choice for a study of visuospatial representation, it presents various problems. On a practical note, given the spaces in which I had to work, which could be cramped and cluttered, and that I sometimes had to move location at short notice, setting up a video camera on tripod, and adjusting it to focus accurately enough on our workspace to obtain useful data, did not seem the best use of my valuable and limited time with students. Even more importantly, many people are uncomfortable being filmed, and I had serious concerns that students – already being asked by a strange adult to perform tasks in a subject area they found difficult and frustrating – might be inhibited in their work by the looming presence of a camera. Additionally, with the recent heightened concern about child protection issues, some schools have instituted strict policies disallowing or restricting filming of students, and it was possible that individual parental permission might be difficult to obtain.

Audio recording is vastly less intrusive, and could be done through the microphone on my laptop (using *Audacity* software). I informed students I was recording the sessions to help me remember exactly what was done and said, and although in a small number of cases there was some initial shyness, this quickly dissipated. I had trialled the multimodal combination of students' paper-based working with audio recordings in my previous research, and found that by matching up the visual and audio records for each representational entity, sufficiently accurate and detailed accounts of students' strategies for each task could be produced.

5.5.1.2 Transcription: listening versus reading

I have already mentioned matching up visual and verbal data in the context of credibility through triangulation (Bassey, 1999, p.p.76). For this study this was managed by importing recordings into *Audio Notetaker* software, splitting them into individual segments corresponding to each task attempt or example, and presenting each image side by side with the relevant audio recording of its creation and use, and my annotations. Each segment could then be broken down further if required, for example, highlighting the different stages of a representational or arithmetical strategy, resulting in a well-organised multimodal database. I maintain that while transcription of participants' verbal utterances would be the traditional approach here, this was not only unnecessary, but the data loss would be detrimental. To look at each piece of visual data while actually listening to students' words in their own voices (with hesitations, noises, tonal variation, etc.) was a more powerful way for me to engage with the data on a deep level than to type out then read back their stripped-down words in text form. This view is shared by Crighton and Childs, who describe retaining the original "three-dimensional" audio source material past the data collection stage as "honor[ing] the participants' voices" (2005, p.p.40). Thus, I used text transcriptions of participants' (and my own) speech only when quoting passages of particular interest.

5.5.1.3 Photography

I have mentioned that I took photographs, to record numerical structures modelled with cubes. I did not photograph every model, and those that I did were captured in a particular moment of their transient configuration; thus my choices of when to take photographs are pertinent. Despite using my mobile phone to take pictures (being of lower quality but more unobtrusive and naturalistic than a camera), I was still concerned that this intrusion of my researcher-self into the tuition situation could interrupt the flow of the session. At the end of a task, i.e. when a solution had been reached, a brief interruption would be unproblematic; there were also sometimes what felt at the time like natural pauses, where a quick snapshot was possible. I acknowledge the personal and subjective nature of this assertion and my decisions, based though they were on professional experience. With one student (Paula) I made particular use of the cubes, and played a considerably more active role in our co-created representations; when laying out configurations myself, I took the opportunity of a snapshot before explaining or working with my representation. Essentially, on each occasion I had to make a quick judgement on whether the importance of recording the image outweighed the interruption potential at that moment, or whether verbal description would suffice.

5.5.2 Effect of the researcher on data collected

The theoretical underpinnings of my methodological approach to data collection have been addressed. While the term 'transferability', in its strict sense, is not quite appropriate for this kind of case study research, the kind of applicability my data might have to other cases and situations has been discussed. I have described my prolonged engagement with data sources, commitment to persistent observation of emerging issues, and collection of multimodal data for the purposes of triangulation (Bassey, 1999).

In addition to questioning how the observed behaviours of my students might relate to how other students behave (and think, and learn), one might also question how the observed behaviours relate to how those same students might behave (and think, and learn) in other situations. For example, Cox (1999) raises concerns about representations “produced for an audience”, i.e. in response to an investigator’s instruction to ‘draw diagrams’ or ‘show their working’, suggesting representations produced privately may be more valid reflections of externalised cognitive processes. I argue that while there are very likely differences in the external representations produced under different circumstances, any drawing (etc.) produced by a student while working on a task must be considered a valid record of externalised cognitive process – but not necessarily a generalisable one. Cox suggests that

> Private representations are, amongst other things, less fully labelled, sparser and may be only partially externalised whereas those intended for sharing with others will tend to be more richly labelled, better formed and more conventional. (1999, p.p.347)

This study was not designed to allow me passively to observe the external representations that students use when left to their own devices, but to provide an unusual environment in which they were explicitly allowed, approved, and encouraged – including the use of unconventional forms. While I do not deny that my obvious observation of their representational activity will have had an effect on their behaviour, I suggest the absence of time pressure, peer pressure, or judgmental grading of their work (as found in the classroom, or even when doing homework privately) are stronger factors in collecting data leading to a credible analysis of representational strategies.

5.6 Data analysis

5.6.1 Selection of data

In choosing an essentially data-first methodological approach, it was necessary to develop analytical strategies most suited to the data collected. While the main body of fieldwork was taking place, I was concerned with collection of data relevant to the areas of research interest, trusting in the later emergence from it of themes for analysis. Having collected data, questions followed of exactly what kind of findings could be drawn from it. At no point was any data discarded; that which did not initially appear to shed light on the arithmetical-representational thinking of students was tagged and archived for potential retrieval, should it be required at a later stage.

It was immediately obvious that I had a very large quantity of data, and thus a tension between how much of it could be analysed and the level of detail in which it could be analysed. Micro-analytical strategies could not be applied to every minute spent with every student, and thus some potential data must be omitted. I decided to select subsets of data for analysis in three different ways: first in task-defined ‘vertical’ slices (each using all data from all students on a single task or task sequence), then in participant-defined ‘horizontal’ slices (each following an individual student’s progress over time on partitive or quotitive division), then a representation-defined ‘filter’ (including all visuospatial representations fulfilling certain theoretically-derived criteria). While it may be seen that their analysis required some differences in approach, there were overarching analytical principles (as discussed above, derived from a combination of case study, grounded, microgenetic and practitioner approaches) which were applicable to all.

5.6.2 Internal from external

Researchers working in the area of internal representation (e.g. DeWindt-King and Goldin, 2001; Thomas et al., 2002; Mulligan et al., 2005; Mulligan, 2011) have acknowledged the basic difficulty in making inferences about the happenings inside other people’s heads.

> [There are] important limitations to the methodology of inferring internal representational configurations from observed external statements, behaviors, or productions. Internal representation involves ambiguity, and inferences about it entail context-dependent interpretations. . . . [O]ur goal is not (yet) to commit to a definitive, reliable or generalizable coding scheme, but to make as explicit as possible the bases for our inferences, improving task-based interview methodology as we explore individual children’s imagery. (DeWindt-King and Goldin, 2001, p.p.1)

This passage references terminology from traditional research paradigms in a way that appears from my review of literature to have become less of an expectation over the subsequent decade. I am unconvinced it is actually possible to induct or construct a ‘definitive coding scheme’ for children’s internal representations, or indeed, what ‘definitive’ would mean in this context. However, it highlights the need for fully explaining the reasoning behind any inferences that are made, as well as the need to be open to examining and improving one’s own data collection methods.

There is also an ethical component to dissecting people's behaviour and utterances in order to infer and write about their internal thoughts and feelings – this activity must be engaged in within an ethic of respect for the individual. Another of Bassey's (1999) requirements for case studies in educational settings is that raw data are adequately checked with their sources, to ensure fairness of portrayal. One strategy I employed during tuition was to ask students to explain their thinking on a task, and when unsure of their meaning, I paraphrased and asked them if I had understood it correctly. This was fundamentally a teacher practice, undertaken for pedagogical purposes, but also functional in providing additional data for deducing internal processes from external ones.

5.6.3 Use of Initial Assessments and background information

I used the data from the Initial Assessment sessions (my first encounter with students outside their regular classroom) in two ways. Firstly, it acted as a filter for selecting a set for case studies. Secondly, I used the responses to Tasks 1-6 to note my impressions of each student's apparent ability in, understanding of, and relationship with arithmetical tasks, bare and scenario-based – as displayed on that particular occasion. It would be entirely inappropriate to make summative judgment on any of these attributes based on such a short period of time; the assessments were formative in that they provided me with an approximate starting point from which to work. I also noted the representational tendencies and preferences observed, and any affective responses, such as negative comments regarding particular terms, symbols or task types. At this stage I was interested in five broad aspects that, while loosely-defined, could affect individualised aspects of tuition:

- familiarity with additive and multiplicative number combinations, and their usage (Q2)
- visual familiarity with relative number magnitudes (Q3)
- strategies for scenario-based multiplicative-structured tasks (Q4, Q6)
- use of visuospatial representation for tasks (all)
- self-perceived relationship with mathematics (interspersed conversation)

Only after completing all Initial Assessments and reports did I read students' SEN files and educational records. Certain diagnoses which affected my interactions with a student to some degree (e.g. ASD, ADHD, etc.) are mentioned where relevant.

However, the focus of this study is not students' histories but their present conditions as observed during tuition, so I chose to spend minimal time with, for example, reports from past teachers. It should also be noted that the collection of student files made accessible to me was incomplete. However, while the lack of, say, a student's SEN Statement was an inconvenience, it should be remembered that educational records may present, by their nature, biased, incomplete and sometimes inaccurate judgements on students' abilities and difficulties (see Chapter 2).

5.6.4 Writing about students

It will be seen that I sometimes mention students' ages, for descriptive context but not as a factor for analysis. Likewise, potential effects of gender, nationality, ethnicity, socioeconomic status or family are outside the scope of this study. However, I do note that the participant group, like the schools they attended, included a very wide range of ethnic and sociocultural backgrounds. I have chosen to refer to participants with gendered names and pronouns: this decision is in the interest of a naturalistic reading experience, and with the intention of 'bringing them to life' on the page, as opposed to a clinical 'Participant A', a clumsy 's/he' or (perhaps) unfamiliar 'zhie', etc.

On several occasions in this chapter I describe students as 'unable' to complete a certain task: this is a considered choice of word. Although one might argue that the lack of a particular response does not necessarily imply inability to give that response, but could be the individual's choice not to do so, on reviewing my participants' overall input and perceived attitude during my time with them, I saw no indications that they were not willing to take part and keen to demonstrate what they could do.

INTRODUCTION TO ANALYSIS

My analysis is in several sections, spanning the following three chapters, each of which appraises the development of multiplicative structure in students' representational strategies from a different angle. Analyses are presented in the chronological order in which they were carried out, and it may be seen that the experience of analysing each dataset was instrumental in the subsequent work, making my overall analytical strategy a constantly-developing organic whole.

Chapter 6 (Two tasks) relates to the particular tasks 'Holiday Clothes' and 'Cuboid Starters', and treats them as independent mini-studies. In each case, the dataset consists of all work done by all students on the respective tasks. These analyses firstly provided a general idea of the scope of representational and arithmetical strategies students used, and secondly allowed me to start developing an analytical framework for understanding them.

Chapter 7 (Two students) is a pair of linked case studies following in close detail the trajectories of Paula and Wendy, and their progression, over multiple 1:1 sessions, through a series of tasks on (respectively) partitive and quotitive division. For this I used the analytical framework developed in the previous section, and focused particularly on their changing practices, both independent and guided.

Chapter 8 (The key representation types) draws data much more widely, from the complete set of visuospatial representations produced during all tasks, by all students. I filter and organise the collected representations into four interlinked categories, and analyse the different roles of these 'key representation types' in solving multiplication- and division-based tasks, and advancing multiplicative thinking, again with the help of the analytical framework previously developed.

In the Introduction I stated my preliminary research questions for this study. In each of the three following three chapters I include specific questions relating to its dataset. I will then discuss the overall findings in more general terms in Chapter 9.

6 TWO TASKS

This section contains two ‘slices’ of data, each with different qualities, addressed independently. The first slice, ‘Holiday Clothes’ (Finesilver, 2009) comes from Q7 in the Initial Assessment. It was chosen for the immediate richness and variety of representational strategies produced, including words, pictures and dynamic modelling. Being a single task, set before the start of my tuition proper, it is notable that some of the students received no support from me, and their work may be considered independent. The second, ‘Cuboid Starters’, comes from the sequence of 3D array tasks I used to begin sessions. Originally I planned only one of these, as a ‘settling’ task for the first tuition session; however, the variety of responses observed proved sufficiently interesting and rich in qualitative data to extend it, with each variation prompted by student response patterns to the previous version. This task series was unique within the study in that I presented students with a complete prepared representation of a multiplicative structure; the focus, then, is on their interaction with these objects, and the relationship between physical and numerical structures.

To gain the maximum value from these two datasets, it was necessary for me to engage with some additional research literature specifically on Cartesian product and 3D array tasks; this is discussed *in situ*.

As will be seen, during this stage I began to develop my framework of aspects for qualitative analysis of visuospatial representations used in multiplicative-structured tasks, first in general terms, then in a task-specific application.

The (developing) research questions addressed are:

- What representational and arithmetical strategies do the students use?
- What do their strategies indicate about their understanding of multiplicative structures?
- How do their strategies develop over time, and in response to teacher input?

6.1 ‘Holiday clothes’

6.1.1 Additional literature: Cartesian products

A *Cartesian product* in mathematics is the complete set of ordered pairs resulting from the combination of two sets of elements. Tasks based on this type of multiplicative structure are usually described as ‘Cartesian product problems’ (Brown, 1981; Anghileri, 1989; Nunes and Bryant, 1996; Mulligan and Mitchelmore, 1997; Verschaffel and De Corte, 1997; Yeo, 2003), although there are alternate terms, e.g., ‘product of measures’, ‘partnering’ (Vergnaud et al., 1979; Williams and Moore, 1979; both in Dickson et al., 1988).

One way of framing Cartesian products as scenario tasks, used by several of the above authors, is to represent the two sets of elements as top- and bottom-half clothing items, with the product set represented by the possible top/bottom combinations. The total number of combinations may efficiently be calculated by multiplying the number of tops by the number of bottoms; however, if the child does not realise this, it may also be completed through listing and counting strategies. Nunes and Bryant (1996) describe such an investigation of children's use of some provided materials to support their mathematical reasoning. They compared two different conditions for students: the provision of a complete set of miniature cardboard clothing items (six shorts and four t-shirts) or a partial set (two shorts and four t-shirts), the idea being that students in the ‘subset’ group could use their limited materials to “create a model for thinking”. Although examples were cited of some children recognising the solution as a simple multiplication, overall the rate of correct responses was only just over half for ‘complete set’ nine-year-olds, and very low for ‘subset’ nine-year olds, and for eight-year olds in either condition. Cartesian product tasks have been used in various educational research studies into both children’s understanding of multiplicative structures and their methods of solving scenario tasks, and the general conclusion has been that they are more difficult than other types of multiplicative problem involving similar numbers (Hervey, 1966, in Anghileri, 1989; Brown, 1981; Nesher, 1988, in Nunes and Bryant, 1996; Verschaffel and De Corte, 1997; Mulligan and Mitchelmore, 1997; Yeo, 2003).

Nunes and Bryant’s above results appear to provide evidence of the difficulty not only of recognising this problem type as a multiplicative situation, but of completing the task at all – that is, through less efficient listing strategies. However, I propose that the provision of task-specific materials in these studies may have directed students towards a particular representational strategy which was not necessarily the most useful to them, and that it would be illuminating to observe the representational strategies of students who did not receive any model clothes. Assuming that some students will not immediately recognise the multiplicative structure, their success (or otherwise) depends on creating a countable representation of the solution set. Thus this task is rich in terms of opportunities to find out: (a) what kinds of representations students might choose for this simple, closed, but nonstandard problem, and how effective these are in achieving a correct solution; and (b) what may be gleaned from the students' representations about their understanding of multiplicative structures.

6.1.2 Procedure

’Holiday clothes’ was the final question of the Initial Assessment and expected, in line with past published findings, to be one of the most difficult. Although my participants were several years older than those in the studies cited above (who were all primary-age), their functioning in school mathematics generally appeared to be of a kind associated with much younger students (this observation also appearing in several of their educational psychologists’ reports). I used Anghileri's (1989) and Nunes and Bryant's (1996) Cartesian product task as a model, with changes in presentation relating to its intended function: it was chosen not simply to find out if students would recognise it as a case of multiplication, or if they could obtain a correct answer, but because of its potential for the close observing of individual students’ representational strategies. Hence, I gave a verbal explanation of the scenario, wrote down the list of items (six t-shirts and four pairs of trousers), and gave two verbal examples of possible outfits/combinations. As the emphasis was on the students’ own representations, no specific materials were present other than the coloured pens and cubes that I always provided.

Due to the nature of the fieldwork setting, it should be noted that it was not possible to ensure students experienced identical working conditions throughout the task. In particular, time was limited by external factors such as interruptions and alterations to lesson timings. I only gave support in response to direct requests for help, and kept to the minimum ‘nudge’ necessary for a student to continue with the task, prompts being regarding either mode/media (e.g. “How about using cubes?”), organisation of set elements (e.g. “How about putting them in a table?”) or accuracy (e.g. “Are you sure there are none missing?"). Students in the paired tuition condition were seated at a distance such that they could not see details of each other's work, although may have noticed their representational media choices.

6.1.3 Descriptive analysis

The observable representational modes used by students were writing, drawing, modelling, and gesture, and the media employed were pen/paper and cubes. Students' own hands, and anything held in them, may also be considered as media within the gestural mode. Only one Year 9 student, Wendy (subject of case study in 7.2), used no external representations at all in her solution of the problem. However, this does not imply either immediate understanding or ease of solution; she still took ten minutes to arrive at the correct figure, after giving several incorrect responses. From her (scant) verbalisations, it seems clear she was making use of internal visual representation, discussion of which is outside the scope of this chapter. The other twelve students' representational responses are described below.

6.1.3.1 Writing

Jenny worked steadily for around ten minutes without requiring any support. She proceeded in a semi-methodical manner by choosing a colour combination and listing it both ways, e.g., black/blue and blue/black (Figure 6-a). At first, colours were chosen in no discernible order, then clusters appeared, the listing process becoming more systematic as she continued. At later stages she regularly checked her list and looked for missing combinations from colour subsets before presenting a complete list and counting the pairs. As well as occasional duplicates, Jenny made an error in reversing the colours for all her t-shirt/trousers pairs,

Figure 6-a: Writing (Jenny)

when it is not actually possible in the task scenario to do so; however, this is evidence of abstraction – the divorcing of the mathematical aspect of a problem from its original scenario – a potentially positive sign.

Tasha spent around thirty minutes on this task overall, although this contained several lapses of attention. It is surprising that she chose a written strategy (Figure 6-b) as on other occasions she expressed a strong preference for working with cubes. She requested and received support during the task, of which two instances are key. First, she complained about the amount of writing required by her chosen listing strategy, and I suggested using a table. Second, she also became somewhat frustrated as a result of frequently asking if her list was “finished” and me replying in the negative; however, when I provided the prompt of asking if she had all outfits that included the blue t-shirt, she could quickly complete and check the solution set in a systematic manner.

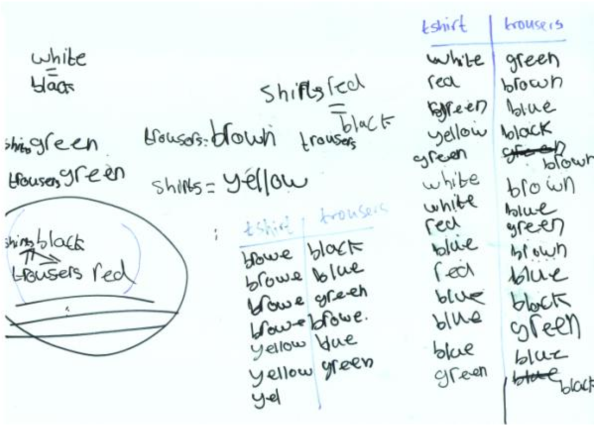


Figure 6-b: Writing with table (Tasha)

Danny generally tended to rely heavily on counting-based strategies, usually involving dot arrays, for multiplicative problems, but used them reliably and with few errors. He spent around five minutes on this task. Danny’s response (Figure 6-c) differs clearly from the two previous written strategies in that it shows an immediate grasp of the problem’s structure, as demonstrated in the orderly working through of groups of combinations. However, two points are of particular interest. First, there is clearly a distinction to be made between (a)

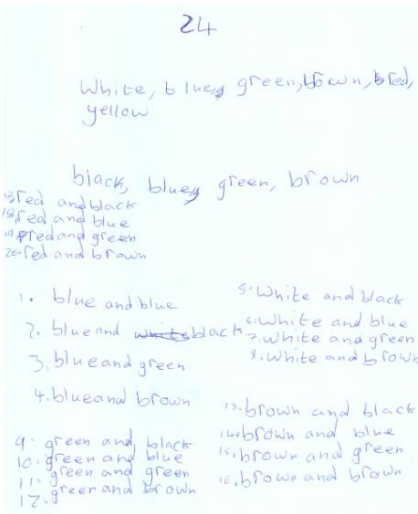


Figure 6-c: Writing (Danny)

comprehending the structure of the complete set of combinations and (b) recognising that the number of combinations may be attained through multiplication. This is perhaps not obvious. Second, Danny's representation is not *complete*: he has carefully and consistently listed 20 of the combinations but omitted the last four – yet still giving a correct answer of 24. I suggest that, given his general reliance on visual counting strategies, it was only at this point in the process that he felt confident enough that he could count the last few (unlisted) combinations without the danger of missing any out.

6.1.3.2 Drawing

Oscar was one of the most capable of my participants, and I presented him with the problem five minutes before the lesson's end, as I considered it possible he might recognise it as solvable by multiplication (and so calculate the total within the remaining time). He did not; thus his work is unfinished, and unfortunately neither was there time for development of strategies. Oscar (Figure 6-d) immediately began to draw one combination after another, without pause. He began with the three matching-colour combinations (blue/blue, etc.) There does not appear to be any pattern (yet) in the other combinations listed.

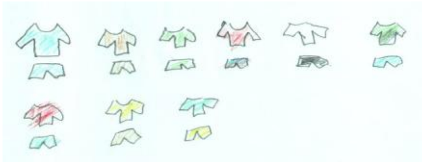


Figure 6-d: Drawing (Oscar)

Kieran had solved prior multiplicative problems using counting-based strategies with tally notation. He spent around 35 minutes on the task (although this included several lapses of attention). Kieran chose to draw (Figure 6-e), barely pausing between the first four combinations. He then made a significant organisational change and reduced the amount of drawing necessary by placing four colours of trousers below each t-shirt. The representation below was presented as Kieran's final solution; it shows 18 outfits, which he submitted as the complete solution set. There was not time to check

Figure 6-e: Drawing (Kieran)

for missing combinations.

George had struggled considerably with prior multiplicative problems, often using counting strategies but with no particular preferred notation. However, he needed only around five minutes on this task, my only contribution being to tell him that several of his early estimated or miscounted answers were incorrect.

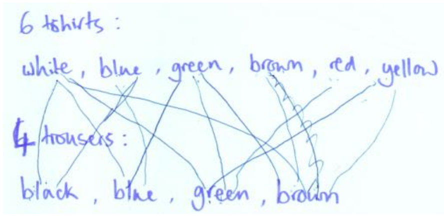


Figure 6-f: Drawing relationships (George)

There is a fundamental difference between George’s use of drawing (Figure 6-f) and the previous two (or any of the responses presented so far), in that he does not represent the individual clothing items of the solution set at all, but draws only the relationships between them. As with Danny, this demonstrates a clear grasp of the problem structure, with the ability to represent the problem scenario in a more abstract way; however, again, it is important to note that this did not lead him to a multiplication calculation. Also, one might reasonably have expected that, on finding such an elegant representation, it would be trivial to work through it in an orderly manner; however, George's difficulties are apparent, with some linking lines missing or repeated. After one of his suggestions of an incorrect total I informed him there were missing links, and although he looked intently at his representation, he was unable to see where. This suggests some kind of difficulty with visual processing.

6.1.3.3 Modelling

Three students made significant use of the multilink cubes in representing the problem scenario (although they also used writing to record the results of their modelling).

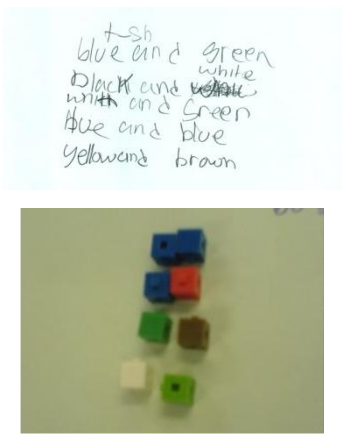


Figure 6-g: Writing and model (Sidney)

Sidney (paired with Oscar, above) unfortunately had only a short time on the task. He began by listing two pairs in written format (Figure 6-g), first just writing unordered pairs of colours, then deciding which were t-shirts and which trousers. However, he was then unable to generate any more. Sidney requested the cubes (without prompt), and took pairs of colours then wrote each down. Note that although he made a green/brown cube pair, he wrote yellow/brown instead, and so although modelling with cubes would seem to be a promising strategy for him on this task, there are concerns raised regarding preservation of information when translating between modelled and written modes.

Harvey had struggled to grasp problems involving multiplicative situations, and was highly error-prone even with counting-based strategies, requiring a great deal of teacher support. However, he was highly motivated and could focus for significant periods. He began by listing four pairs of clothing items (Figure 6-h) before stopping. Harvey's first few answers included both valid and non-valid items. He said he could not think of any more, so I suggested cubes might help (as they had on some prior tasks). He agreed, but required further support. I laid out ten cubes, in two groups corresponding to the six t-shirts and four pairs of trousers, then picked up a black cube, saying "this pair of black trousers, it could go with this one, or this one...". Harvey moved the cube and spoke further combinations. He then picked up and identified another cube as "blue top" but appeared confused as to what to do with it, so I suggested he go through the different trousers that might go with it. After this, he used this system of picking a t-shirt colour and listing it with each of the four trousers colours to complete the set. At first he was very slow to pick out each new combination, and made some recording errors, but became noticeably quicker as he went on. When he reached the end of his list, he immediately said "Finished!" with unusual confidence, which indicates that he was aware enough of the pattern to know that the set of possible combinations was now exhausted.

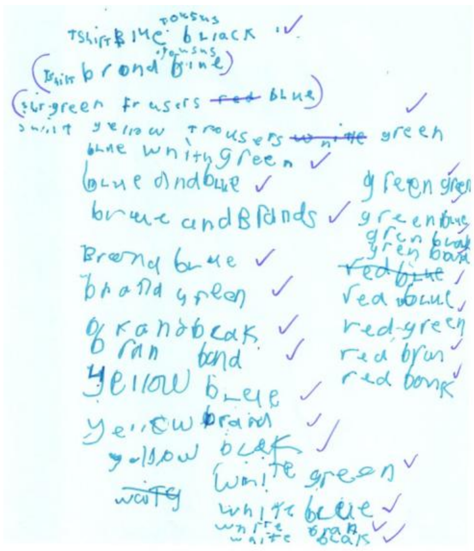


Figure 6-h: Writing accompanying work with cubes (Harvey)

Paula (subject of case study in 7.1) had previously shown exceptionally weak numeracy skills, so I altered the numbers in the task to four t-shirts and three trousers, and was prepared to give her a greater degree of support if required. We spent around 20 minutes on the task. Paula began by suggesting “[she could wear] black, as there’s two blacks”. I suggested she make a note of this, in order to keep track of her outfits, but she could or would not make any move to record it. I offered the prompt to write her combinations in a table; she concurred this was a good idea, but requested help. I drew one (Figure 6-i), and demonstrated how she might write combinations. Her next suggestion was “red”, so I clarified verbally “a red t-shirt, and what colour trousers?”; she specified black. During this exchange, I took a red cube, then a black cube, and stuck the ‘t-shirt’ and ‘trousers’ together vertically. I continued to do this as she spoke then wrote combinations. As the task progressed, Paula did not seem to have recognised that there would be any pattern governing the list of possible combinations, so was unable to systematically check for ‘missing’ combinations. My response was to place all the cube pairs in a visuospatial sequence, arranged by colour. I left gaps in the appropriate places, explained that some combinations were still to find, and first asked “What should go here?”, then made my questioning more explicit, verbally and gesturally, in reference to the visual pattern, i.e., “We have the blue t-shirt with the blue trousers, the blue t-shirt with the green trousers; what else must the blue t-shirt go with?” At this point she identified the remaining combinations.

Figure 6-i: Written table and models (Paula)

6.1.3.4 Gesture

Ellis was the fastest student to complete the task, taking around two minutes. He saw the structure of the problem quickly, but like Danny and George, did not recognise it as being equivalent to calculating 6 × 4. His representation was of the relationships between the two sets of items, but where George drew in the lines connecting them, Ellis simply used a finger to trace them systematically and rhythmically, counting with the fingers of his free hand as he did so.

6.1.3.5 Mixed-mode responses

All the students above who used cubes also used writing to record the combinations they found, but most if not all of their effective thinking about the problem was done through cube configurations. However, my final two students (paired together) made significant changes of direction in representational strategy, struggling to find the way that would be most effective for them. Both required a great deal of teacher support, and unfortunately even with only two students it was not possible to give each of them the constant attention they needed.

Leo had successfully solved several multiplicative problems, his preferred method being repeated addition. He greatly enjoyed drawing, and tended to produce elaborate pictorial representations for scenario tasks. Leo began by choosing to draw combinations (Figure 6-j). However, he ran into difficulties because his special favourite pen was a four-colour ballpoint. He first wanted to change my task to suit the colours of his pen, rejected my offer of individual pens in the appropriate colours, and chose instead to switch his media to cubes. However, during the making of his (again, elaborate) models, he was taken by the idea that they looked like ‘Transformers’, and started to play with them, after which it

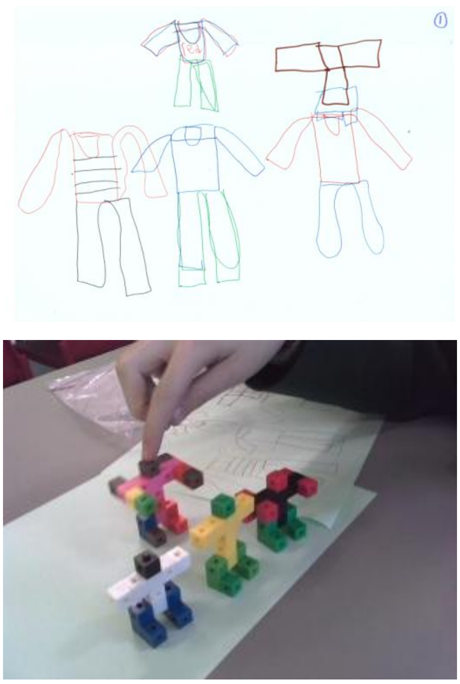


Figure 6-j: Transformers (Leo)

was not possible for me to draw him back to the task. Although Leo's response to the task was not greatly helpful in the understanding of his recognition of multiplicative structures or patterns, it is interesting that in both modes, he began with valid combinations (i.e. from my list of coloured clothing items), that then became increasingly ‘invalid’ (e.g. yellow/green; white/blue (with a black hat); black/red/green; many-coloured) as his focus moved further from the task.

Vince had struggled to grasp prior problems involving multiplicative situations, and was error-prone even with counting-based strategies. He spent around 35 minutes working on the task, requiring a great deal of support. He began with an elaborate drawing, both coloured-in and labelled (Figure 6-k). He then noticed that Leo had already drawn four figures, and appeared to decide that pictorial representation required too much time and effort. He reduced his drawing to symbolic swatches of colour, but still duplicating the information by writing the names of the items in the appropriate colour. He then decided that this representational format still required too much effort and changed again, to just writing down the outfits, but, writing being laborious due to his weak motor skills, requested my help. I suggested then drew for him the table, filling in his outfits so far. He then spoke three more, which I wrote, after which he was willing to take over the writing again. As with Jenny and Tasha's lists, it is possible to see the emergence of systematicity, e.g. his listing together of all the ‘green top’ pairs.

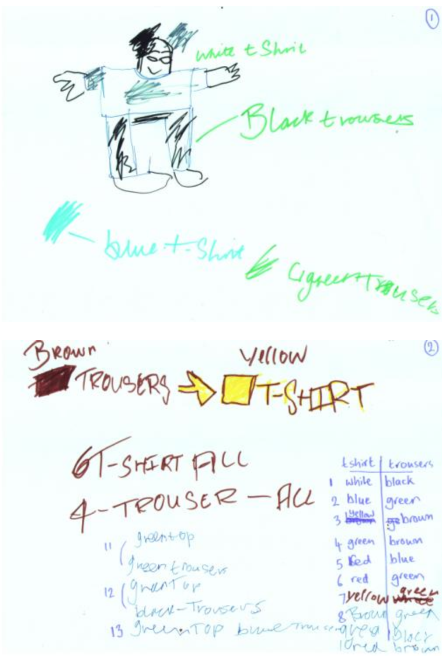


Figure 6-k: Drawing, writing, and table (Vince)

6.1.4 Introducing a framework of analytical aspects

From the various literature on visuospatial representation (Chapter 4) I drew several concepts useful for teasing apart the different aspects of these student- and co-created task representations, for a more structured qualitative analysis of their similarities and differences. Note that in my analyses, *consistency* and *completeness* are both neutral rather than positive terms (as they might be assumed given common usage); for example, *inconsistency* implies an intra-task change of strategy, which may in fact be an improvement.

Mode e.g. modelling, drawing, writing, symbols, gesture ***Media*** e.g. cubes, pen/paper, fingers ***Motion*** e.g. static once created, or involving ongoing movement of elements ***Resemblance*** between the drawing or model and the task scenario described (NB This is related to the concept used by some researchers, ‘abstraction’.) ***Consistency*** i.e. whether a single representational strategy was used from start to finish, or changes occurred ***Completeness*** i.e. whether the external representation had to be ‘finished’ for solution (NB This could be thought of as a special case of inconsistency, with mid-task change from external to internal representation.)

Table 6-a: Initial aspects of visuospatial representation

These allow for succinct summary and comparison. Note that this list is a work in progress, the final product of which is presented in Chapter 9.

Motion: Of the ‘modelling’ students, Harvey's involved motion, in that there were ten cubes, each one representing a single clothing item, which were moved into different configurations, while for the others, each outfit was left in place and new cubes selected for the next. The two schematic representations may be similarly differentiated in that in George’s, the drawn lines provide a static record of all the combinations he had thought of, which could be re-counted and checked, whereas Ellis's finger-movement representation left no permanent trace, so was dependent on accurate and systematic counting.

Resemblance: The 'drawing' students all chose to draw actual clothing items, i.e., of high resemblance, although Vince did then switch to non-pictorial blocks of colour.

Consistency: Six of the students were consistent in choice of mode and media; of those dissatisfied with their initial choice, three switched independently (to modelling with cubes) and one more at my suggestion, while Vince retained the media of pen/paper but altered how he used them.

Completeness: Several students produced (what they believed to be) a complete set of pairs, which they counted in order to obtain a total, and Danny knowingly left his incomplete. Due to external circumstances curtailing some sessions before students had finished, it is not possible to comment fully on this aspect for this task.

6.1.5 Findings

What representational and arithmetical strategies do the students use?

These were described in 6.1.3-4.

What do their strategies indicate about their understanding of multiplicative structures?

Mathematical 'understanding' is notoriously difficult to assess, but tentative assessments may be made of students' understanding of this problem's arithmetical structure based on their representations. In particular, representational inconsistencies during the problem-solving process can indicate the presence of any changes in students' thinking regarding the task.

First, although no student calculated the solution via a multiplication, George and Ellis's schematic representations and Danny's systematic list all indicate immediate perception of the Cartesian structure, i.e., that the solution set would be produced by each of the members of the 'tops' set combining with each of the members of the 'bottoms' set. Their use of counting strategies to enumerate the solution set is comparable with their performance on prior and subsequent straightforward multiplicative tasks when, unable to retrieve multiplication facts reliably, they used grouped counting. In particular, George's understanding of the task's structure is shown to be stronger than his ability to execute the necessary procedure.

Kieran's drawing is unusual for having one clear discontinuity, when he changed from drawing paired outfits to drawing multiple trousers with each t-shirt, which likely reflects a sudden new grasp of the structure of combinations. In contrast, the point at which Danny ceased listing combinations (leaving his representation incomplete) marks not a change in understanding of the structure, but that at which he became confident of counting to a correct total.

It is not possible to make statements regarding Paula’s understanding with much confidence at this stage, as she required such a high level of support. However, the fact that she could (a) suggest some combinations, and (b) respond to being presented with a visual sequence of colour combinations and ‘fill the gaps’ indicates, at least, the ability to combine two separate elements, and some basic pattern recognition.

How do their strategies develop over time, and in response to teacher input?

I have already mentioned several clear intra-task developments, independent and supported. Jenny, Tasha, Harvey and Vince's lists show a more gradual move to systematicity, with the first combinations being chosen in no particular order, then some grouping of combinations (e.g. listing all the blue-trousers combinations together), and eventually using structure and pattern to check that every possible pairing was present.

In order to better study development of students’ representational strategies, more than one task is required, such as in 6.2, where a series of linked tasks is presented.

6.1.6 Discussion

Most students’ representational choices were unsurprising. However, given their high level of visuospatial representational responses (throughout the entire duration of the project) to tasks involving multiplicative structures, and in several cases the great effort it cost some of them to read and write, it is perhaps surprising that writing featured as heavily in students’ recording processes. On the other hand, it is possible that the high status accorded to writing, as a means to communicate one’s working and solutions, has become ingrained to the extent that many students assume that it is preferred, expected, or even the only kind of working which is acceptable to teachers. At this early stage, they were yet to be convinced that my priorities were not those standard in classrooms.

No mode or media stood out as most successful overall in the attainment of a correct solution. It appears that a variety of representational strategies can be effective, and that students (perhaps excepting Paula) were able to choose one which enabled them to engage with and, given time, to complete the task. Encouragingly, this could be taken as demonstrating a basic level of metarepresentational competence. As all students needed to count up the total number of combinations, better organisation of the paired items aided effective and accurate counting, and organisation in turn was dependent on how soon in the problem solving process the Cartesian structure of pairings was perceived.

It would be a great oversimplification to categorise students as either understanding the task's mathematical structure or not, particularly if this understanding were defined as recognition of this scenario's structure as multiplicative. Different levels and different ways of understanding were demonstrated by students at the start, middle and end of the task through the strategies by which they represented the problem, and how they organised the solution sets they produced. However, it is important to remind oneself that the original task as presented did not require students to produce a complete solution set, but to find the number of elements in it. It may be argued that this was the only strategy available to them for finding out the total number, but the fact that during their work on the task several students asked me <u>how many outfits they had to list/draw/construct</u> is of particular note: in focusing on their representations of the solution set they had quite forgotten that enumerating it was the original aim!

Regarding the historic difficulty of Cartesian product problems, Nesher (1988, in Nunes and Bryant, 1996) pointed out that although they are cases of one-to-many correspondence, this is not explicitly indicated in the verbal formulation, i.e., in a 'clothes' task it is up to the problem-solver to figure out the relationship between the numbers of 'tops' and 'bottoms'. (Mulligan and Mitchelmore, 1997) suggest that fundamental to the effective processing of a multiplicative situation is the recognition of equal-sized groups, and it is the fact that these groups are not at all obvious in the Cartesian situation which causes the particular difficulty. However, I suggest another potential issue lies in the 'temporary' nature of the solution set, due to how elements are combined in the scenarios chosen. For example, in the 'clothes' version, the set of all possible outfits is countable, of course, as all combinations may be listed, but within the actual scenario, while each pair may be individually constructed, all outfits cannot exist at the same time.

Despite the difficulty, at least some of the students in the cited Cartesian product studies recognised the mathematical situation as an instance of multiplication. Despite generally successful responses to the task, and in some cases quick or immediate appreciation of the structure of the solution set, not one of my participants did. Why might this have been the case? I suggest a reason may be inadequate and limited mental representations for multiplicative structures. On appraisal of my students' responses to the larger battery of multiplicative tasks, it appears that the majority were aware of multiplication as an operation, with an answer which might be retrieved from memory or found through counting equal groups. Some students were also aware that multiplication is some kind of ‘opposite’ or ‘undoing’ operation to division. However, these understandings are both essentially linear and procedural, and lack the concept of multiplication and division as expressing a static two-dimensional relationship between three numbers (e.g. from the triplet 3, 6 and 18: $3 \times 6 = 18$, $6 \times 3 = 18$, $18 \div 3 = 6$ and $18 \div 6 = 3$). If so, this may have implications for the way teachers support struggling students in their learning of multiplication and division.

6.2 ‘Cuboid starters’

6.2.1 Additional literature: 3D arrays

I introduced the array representation in 1.3.5, and addressed its functionality in depth in 4.3.3- 4.3.4. While clearly a powerful tool for supporting the learning of multiplication (and potentially division), the 2D array provides limited enumeration options (e.g. for a 4×5 rectangle, working out either 4 rows of 5, 5 columns of 4, or simply unit-counting the 20 visible squares). With a 3D array, however, the enumeration options are more complex: for a cuboid with all dimensions >2 units, a one-dimensional strategy of unit-counting the visible cubes will not work, as there are non-visible interior cubes; successful enumeration must then rely on conceptualising the organisational structure of the array as a space-filling object. While the expected final, formal strategy for students would be a three-dimensional multiplication equivalent to the formula for the volume of a cuboid, on the way to this symbolic stage, there are various potential concrete, visuospatial strategies in which the cuboid structure is deconstructed into manageable parts. Two possible deconstructions are to conceptualise the cuboid as (a) a one-dimensional arrangement of 2D horizontal *layers* (i.e. one thick vertical stack of rectangles), or (b) a two-dimensional arrangement of 1D vertical *columns* (i.e. many thin stacks laid out in a horizontal rectangular array).

Battista’s writings on 3D array tasks (Battista and Clements, 1996; 1998; Battista, 1999; 2010) all use the concept of *spatial structuring,* which I adopt.

> Spatial structuring is a fundamental notion in understanding students' strategies for enumerating 3-D cube arrays. We define spatial structuring as the mental act of constructing an organization or form for an object or set of

objects. The process of spatial structuring includes establishing units, establishing relationships between units (such as how they are placed in relation to each other), and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). (Battista and Clements, 1996, p.p.282)

In Piagetian parlance, there are two types of experience required in the production of the mathematical understanding in this context: *physical* and *logicomathematical*. In this case, the physical knowledge available to students would refer to the observable perceptual features of the blocks, in particular the way unit cubes are arranged in rows, columns and layers to make up the object. The logicomathematical knowledge to be gleaned from this would be that successful spatial structuring of the block enables one to calculate how many unit cubes there are, and that neither orientation of the block, count order, nor count grouping affects the total. Note that while a child may pass the traditional Piagetian tests of order-irrelevance when applied to the enumeration of clearly separated discrete objects (e.g. loose cubes, dot patterns), it does not necessarily follow that this knowledge is automatically transferred to this alternative context, where (a) the enumeration is of a continuous mass made up of smaller component objects fixed into position with no gaps, and (b) the counting involved is more complex, involving step-counting or repeated addition, etc.

Figure 6-l: Isometric cuboid drawing

Figure 6-m: Perspective cuboid drawing

Ben-Haim’s work during the 1980s on 3-dimensional arrays involved students interpreting isometric drawings of blocks of cubes (e.g. Figure 6-l), in effect requiring participants to interpret a tiling pattern of identical rhombuses as a solid object – a far from trivial requirement. Thus, his set of proposed error types reflects students’ tendency to interact with the presented 2-dimensional image as a 2-dimensional shape (“1. counting the actual number of faces showing, 2. counting the actual number of faces showing and doubling that number" (Ben-Haim et al., 1985, p.p.397), or to have difficulty picturing the cubes not shown ("3. counting the actual number of cubes showing, and 4. counting the actual number of cubes showing and doubling that number” (ibid.). During the 1990s Battista’s research on 3D arrays also used line drawings, but using perspective rather than isometric projection (e.g. Figure 6-m). His expansion of the set of error categories (Battista and

Clements, 1996) reflects similar difficulties, as do the image-based 3D visualisation tasks of Pitta-Pantazi and Christou (2010). Thus, there is impetus to observe whether, when the problematic requirement of interpreting 2-dimensional representations of 3-dimensional shapes is removed, and participants are simply presented with the solid shape itself, students display similar or different strategies and error patterns.

6.2.2 Procedure

These tasks were presented entirely in the realm of concrete experience, without use of any mathematical language or concepts other than "how many", and while it was not possible to answer the question without referring to the concrete materials present, it was possible to do so without any symbols or formal mathematical language beyond the two-digit natural numbers. This is in contrast to much of the research on concrete representations of arithmetic, in which tasks are presented symbolically, and for the solution of which children may or may not choose (or be instructed to use) concrete materials. Studies of this latter type have led researchers to report children "struggling to attach a concrete model to a written symbolic expression" (Anghileri, 1989), the materials used failing to produce the "expected" or "required positive outcomes" (Maclellan, 1997; Hall, 1998), and children's learning suffering from the "increased processing load" caused by concrete representations (Boulton-Lewis et al., 1997; Boulton-Lewis, 1998). I removed the requirement to mentally translate representations from flat images to solid objects; I dispensed likewise with the requirement to translate the questions from symbolic or scenario-based verbal modes, leaving a task consisting only of a simple question about a fully-present concrete visuospatial representation of a multiplicative structure.

I set a sequence of tasks of this type. On each occasion, students were presented with a cuboid block of multilink cubes, and asked to find the number of unit cubes (i.e. without use of the term 'volume' or any units of measurement). Students were familiar with enumerating multilink cubes, as they had previously counted and estimated loose, unstructured quantities of these same cubes.

The blocks presented to students over the course of four sessions were:

- One 3×4×5 cuboid, with no particular pattern of colours.
- One 3×3×5 cuboid (as above).

- Two 2×3×6 cuboids, one constructed in three differently-coloured 2×6 layers, the other in six differently-coloured 2×3 layers; students were given the choice which of the two to enumerate.
- Two identical 2×2×3 cuboids, both coloured in 2×3 layers; students were asked for the total number of cubes present.

(N.B. Students were informed that the blocks were solid, not hollow.)

I imposed no time constraint on tasks (with the actual time spent varying from 1 to 15 minutes). In paired sessions, there were occasions when one of the students called out a solution while the other was still working; if they appeared not to notice the interruption I did not disturb them, but if they did halt on hearing it, I requested they continue working "to see if they got the same answer or a different one".

6.2.3 Descriptive analysis

Task 1

Initial responses

When presented with the first block, all students used some form of counting-based strategy, and all gave incorrect answers. Battista and Clements (1996) created a set of categories for 'errors of strategy', which, while intended for use with drawn images of 3D arrays, includes descriptors which could apply to counting strategies in this study (e.g. "C2: counts outside cubes on all six faces", p.p.263). However, my students not only made erroneous choices in which cubes to count, but in the counting process itself – for example, skipping or repeating a number. Thus a distinction is necessary between an erroneous strategy, and errors in carrying out a correct strategy.

Two students (Ellis and Wendy) independently made perceptive, effective use of one deconstruction of the array structure, in a strategy which would have been successful had they not made minor counting errors which delivered answers of 59 and 61 rather than 60. With the block on the table, they placed a finger on one of the cubes in the top (4×5) layer and said "1, 2, 3", referring to the touched cube and the two that were vertically beneath it, then moved the finger along one cube, saying "4, 5, 6", continuing to group-count threes for every cube in the top layer of 20. This corresponds to Battista and Clements' strategy B2 "counting of cubes is organized by row or column, but the student counts by ones” (ibid.). It is notable that neither student gave any indication of recognising the cardinal numbers of each count-group as the set of multiples of three, which is consistent with their not noticing when they miscounted.

Ten of the remaining students began by either unitary-, grouped- or step-counting the top layer, then moved onto the other faces of the block, turning it around and attempting to count all the external cubes. Although some students asked for confirmation that the shape was solid as opposed to hollow, their face-based counting strategies nevertheless ignored non-visible interior cubes. Meanwhile, the lack of clear points at which to start and stop counting, and of an obvious ‘route’ around the six faces, also led to some cubes and/or whole faces being counted more than once, while others were missed. Close observation of gestures and comments indicated that four of the students were attempting to avoid double-counting, but the other six gave no sign of noticing that they had double-counted edge cubes or triple-counted vertex cubes. This latter indicates a further confusion: they had replaced a 3-dimensional task with a 2-dimensional one, counting the squares making up the surface area of the block rather than the cubes making up its volume.

One student gave a particularly idiosyncratic initial response to the task. Like the majority, Leo did not make use of the cuboid structure of the block on his initial attempt, and also made enumeration errors. The atypicality was caused, as in Holiday Clothes, by his favourite four-colour pen: he organised his counts by colour, first all the visible red cubes, writing the subtotal in red, then the same for blue, green and black. He ignored all other colours. When I enquired about the pink and brown cubes he said that he “didn’t put them in”. It appears that involving the pen in the task was of higher priority to him than the remaining cubes. (Writing a number down in a non-matching ballpoint colour, or using my proffered felt-tips were unacceptable options.) This example of a participant’s unusual priorities in carrying out a task was reinforced by his subsequent decision to invent his own (incorrect) form of multi-digit addition (Figure 6-n).

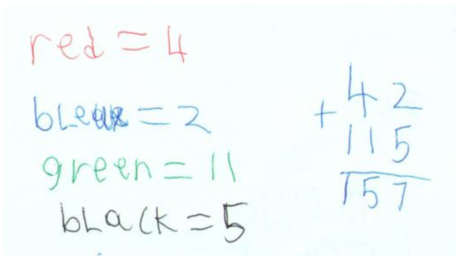


Figure 6-n: Colours and calculation (Leo)

Prompts

If a student was consistently double-counting edge cubes (i.e. they were enumerating the squares making up the surface area) I used two prompts: (a) picking up a single loose cube, and reminding them that these were the items to be counted; (b) pointing to a vertex cube and showing how it might be double- or triple-counted. After this intervention, all students were observed making an effort not to double-count edge and vertex cubes, although there were still errors, and in the following session some reverted to square-counting, and required reminding.

The prompts of particular interest, listed below, were designed to draw students' attention to the layered structure of the cuboid, i.e. that however many cubes were in the top layer, the same number would be found underneath, and again underneath that. My intention was that the students currently focusing on the surface of the shape should notice the replication inherent in the cuboid structure, and use reasoning to develop an appropriate enumeration strategy. The two students who had used the *columns* structure of the cuboid were confident enough in their working strategy that it could reasonably be expected that they would not be confused by discussion of an alternative deconstruction.

The 'layers' prompts were:

- Enquiring how many cubes made up the top layer;
- Enquiring how many were in the layer underneath (and, if necessary, the one beneath that);
- Commenting explicitly that all layers contained the same numbers of cubes;
- Stating the numbers in each layer in the form of an addition (and, if necessary, supporting or performing that calculation).

Of the 13 students, six responded to prompts (a), (b), or (c) by stating the number of cubes in each layer and calculating a total of 60. Three more heard a full demonstration or explanation, and gave verbal indications that they understood. One (Paula) gave no such indication that she understood either the addition procedure or its relevance to finding the total number of cubes. (A complete set of responses to all four tasks may be found in Appendix D.)

The two students who had originally used a columns-based grouped-counting strategy responded positively to my introduction of a layers-based alternative. One, Wendy, appeared to be particularly engaged with the idea of different ways to get to an answer. The dialogue below is quoted in slightly simplified form, with pauses and repetitions removed.

CF: How many are just in the top layer?

Wendy: 20.

CF: Yeah. So if there's 20 just in that top layer, and then there's exactly the same underneath it –

Wendy: Oh! So that would be 40, then 60.

CF: So we could say there's . . . three layers of 20, which is 60. Or, if you happened to have it up a different way *[rotates block]*, how many are in the top layer now?

I work through the process for five layers of twelve.

CF: Or if we happened to have it up that way *[rotates]* to start with –

Wendy: 15.

CF: And how many layers of 15?

Wendy: 4.

CF: So whatever way up it happened to be –

Wendy: Still be 60.

Leo's response to the single prompt (a) indicates how potentially effective a small 'nudge' can be in changing thinking.

CF: Can you tell me how many are in just the top layer?

Leo step-counts 5, 10, 15, 20, and affirms that there are definitely 20 in the top layer.

Leo: Ah!

CF: Does that help at all with getting the total number?

Leo: Well now I think I have a solution to this!

CF: Ok.

Leo: If you were able to split this, if you chop the layers off, it'll be 20 there... underneath that is another 20, and underneath that is <u>another</u> 20.

CF: Spot on.

Leo starts drawing – see Figure 6-o.

Leo: That's 20 there and 20 there. You could just pull it out like a drawer, then pull that out like a drawer. It would be 20, 20, 20.

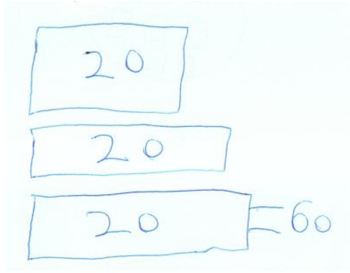


Figure 6-o: 'Drawers' (Leo)

Task 2

The second tuition session, I set all students a 3×3×5 cuboid, with the intention of observing whether they would replicate the successful strategies they had used or observed in the previous session, and if any alternative strategies would appear. Prompts were as before.

- Two students replicated the full layers strategy correctly, and three others (including Leo) began by counting and stating the number in the top layer, but needed one or more prompts to complete the process. Of the two who had used the columns strategy the first time, one re-used it, while the other (Wendy) used horizontal rows instead of vertical columns to organise her counting.
- Four students reverted to the incorrect strategy of counting around the faces, but quickly switched to layers when prompted.
- Paula still gave no discernible sign of understanding the demonstrated process.

Task 3

The teaching aim of this set of tasks had been that students should recognise that the blocks consisted of cubes arrayed in a regular structure, and use the numbers involved in an appropriate strategy to work out the total number, whether it be by multiplication, repeated addition or an appropriate use of counting. (Note that 'appropriate' here does not imply efficient, or even that the student gave a correct answer – only that the steps taken, if implemented without errors, would produce the correct answer.) After the first two tasks, only four students could be said to be able to carry out appropriate strategies without teacher support; one gave no indication of understanding even complete demonstrations, and all others were at a stage of partial understanding and operationalisation.

Hence, for Task 3 I decided to use colour to highlight the physical structure, constructing blocks with each layer a different colour. This is backed up by Battista, who suggests that “using colour [in 3D arrays] . . . might promote the perceptual integration that supports conceptual integration” (Battista, 2010, p.p.196). Rather than force students into one particular colour structure (and thus numerical structure), I gave them the choice of two equal-sized blocks: a 3-colour block in horizontal 2×6 layers or a 6-colour block in vertical 2×3 layers (or pairs of columns). This was intended both to increase the chances of struggling students noticing the array structure, and to provide a basis for discussing the variety of potential enumeration strategies with more confident students.

- This time 11 students used the layers structure in their initial attempts at calculation, and of these, nine independently demonstrated a complete appropriate strategy, while two required prompts.
- Only two students’ initial response was still face-based, and Paula was for the first time able to comprehend and work through the task (with support).

The coloured layers were also indicated to be helpful by student comments, such as “you don’t get confused”, and that if there were “6 cubes there [i.e. in an end layer] then you know there’s six in the rest [of the layers]”. The 3-colour block was chosen by nine students, and the 6-colour by four; implications of their choices are discussed later.

Task 4

With most students now enumerating the cubes in systematic fashion with little or no support, the final task involved an additional structural aspect: I presented students with two identical colour-layered blocks, and asked for the total number from both. For the few students still struggling, this was an opportunity for a second attempt at enumerating colour-structured blocks, while, with a numerical structure of 2(2×3×3) – i.e. a 4-dimensional multiplication – there was an increased number of calculation possibilities for the quicker students.

- Ten of the students produced a correct answer without any arithmetical or strategic support, and the other three succeeded with prompts. All made clear use of the cuboid structure – mostly the coloured layers, but with some spatial words and/or gestures referring to horizontal rows and vertical columns.

Regarding the duplicate blocks:

- Five students used some form of counting (unitary-, grouped- or step-) for the first block, then repeated their actions (continuing the count) for the second.
- Two pushed the two blocks together and treated them as single mass.
- Five worked out there were 18 cubes in the first block and doubled (or added another 18) for the total; one more thought of doing this, but was unconvinced that the two blocks were really the same, and insisted on counting the second before adding.

6.2.4 Task-specific framework of analytical aspects

The set of analytical aspects used in 6.1 is insufficient for this task series. Mode, media and completeness are determined in the task presentation, and resemblance is not applicable as there is no scenario. It is possible to speak of motion in terms of whether the student moved their hand while counting, or rotated the block, and of consistency in their pattern of enumeration; however, more analytical aspects are clearly needed to use this task to diagnose students' understanding of multiplicative structures. Thus, I add three more analytical aspects: *structuring* (i.e. how the physical structure of the blocks is used by students, and the corresponding numerical structures drawn from them for use in the enumeration process); *enumeration* (i.e. the details of how students used the numbers they derived from the physical blocks), and *errors* (what went wrong in their invention, selection and application of enumeration strategies). As before, these terms are familiar from the literature, but expanded and more clearly defined here by my practice. I present an expanded analytical framework specific to this task type, based on these three aspects.

6.2.4.1 Structuring

Battista and Clements (1996) classified their students' strategies into a set of categories and subcategories, on which my system (below) is based. I have adapted their descriptors to apply to actual physical cuboids (as opposed to 2-dimensional images of them), re-ordered them into a loose hierarchy, and expanded the category structure to include certain theoretically-produced strategies not actually observed in this study (e.g. M, C1).

M	**The student conceptualises the set of cubes as a 3D multiplicative structure**	
		Student counts to find the length, width and height of the block, and multiplies the three.
L	**The student conceptualises the set of cubes as forming a stack of 2D layers**	
	1	*Layer multiplication*: Student computes or counts the number of cubes in one horizontal layer, counts the number of layers, and multiplies the two. (It would be equally possible to use A strategies with vertical layers.)
	2	*Layer addition*: Student computes or counts the number of cubes in one layer and uses addition or step-counting (indicating successive layers) to get total.
	3	*Counting subunits of layers*: Student's counting of cubes is organised by layers, but the student unit-counts or step-counts by a number smaller than the number of cubes in a layer
C	**The student conceptualises the set of cubes as forming a 2D array of columns**	
	1	*Column multiplication*: Student counts the number of cubes in one column, counts the number of columns, and multiplies the two. (It would be equally possible to view the block side-on and use B strategies with horizontal rows.)
	2	*Column addition*: Student counts the number of cubes in one column and uses addition or step-counting (indicating successive columns) to get total.
	3	*Counting subunits of columns*: Student's counting of cubes is organized by columns, but the student unit-counts or step-counts by a number smaller than the number of cubes in a column.
F	**The student conceptualises the set of cubes in terms of its faces**	
		Student counts around the faces of the cuboid. They may be counting cubes or counting squares. (It is only possible to obtain a correct answer this way by taking account of cubes appearing on more than one face, and adding on the interior cubes.)
O	**Other**	
		Student uses a conceptualisation other than those described above.
N	**None**	
		Student makes no attempt to enumerate.

Table 6-b: Spatial structurings of a 3D array

Apart from the two C3-strategy students, all initial responses to the task showed little or no awareness of the array structure. Students interacted with the faces only (F strategies), one face at a time, failed to coordinate orthogonal views from different perspectives, and in many cases did not even have a complete faces-based conceptualisation (i.e. surface area). In fact, their strategies did not greatly differ from those they might use when enumerating a loose pile of cubes – but with the considerable inconveniences there was no obvious ‘route’ (or even start or end point) for counting around the six faces, and, being a continuous mass, the counted cubes could not be visuospatially separated from the uncounted ones.

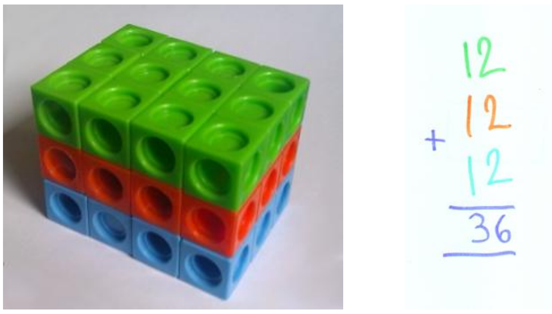


Figure 6-p: 3-colour block, 3-colour sum (CF and Paula)

All students showed increased awareness and use of physical structure following teacher prompts, but the amount of prompting required and strategic change observed varied widely. There was a general move from F towards L strategies, as would be expected given the layers-based prompts and the use of colour in Tasks 3-4. Six students, after being prompted through an L strategy in Task 1, chose it consistently thereafter. Only Paula and one other attempted an F strategy on all four occasions, and both were able to use L strategies (with prompts) by the end of the study. However, there was no clear trend <u>within</u> conceptualisation types, i.e. students did not move sequentially from L3 to L2 to L1 (or the equivalent with C and F). I suggest that strategy choices stem not only from students’ spatial structuring of the arrays, but from an interaction with their ability, confidence, and preferences regarding different forms of enumeration (addressed below). For those still struggling, the use of coloured layers in Tasks 3 and 4 was of significant help in drawing attention to the physical, then numeric, structure. With Paula, it proved particularly effective to link physical and numeric through the use of matched coloured pens for recording (Figure 6-p).

On finding a successful strategy, some students repeated it in precisely the same way for each task, whereas others experimented with strategies based on different structural aspects (layers, columns, rows, and combinations of these). Battista and Clements’ work assumes a clear linear hierarchy of strategies. They consider faces-based strategies to indicate students' "initial conception of . . . an uncoordinated set of faces", whereas layers-based strategies are an indication of their "see[ing] the array as space-filling" and having "completed a global restructuring of the array" (1998, p.p.234). And columns? "Those in transition, whose restructuring was local rather than global, utilized [column-based] strategies . . .They had not yet formed an integrated conception of the whole array" (ibid). My data suggests that the relationships involved are more complex than this.

While faces-based strategies are certainly less sophisticated (except in the unlikely event that a student takes account of all cubes appearing on more than one face, and of the invisible interior cubes), it is unclear why a columns-based spatial structuring should be considered any less advanced than a layers-based one. The former deconstructs a 3D array into a 2D array of 1-dimensional stacks, the latter a 1-dimensional stack of 2D arrays; both are equally valid space-filling conceptions. I suggest that the layers-based structuring wins Battista and Clements' approval not because of structural superiority, but because it encourages an enumeration method with less steps, which for most numerically-capable students is more efficient. However, it is not necessarily more efficient for all students on all tasks (see 9.3).

6.2.4.2 Enumeration

My *enumeration* classification is similar to that used by Anghileri (1997). By using 1 for multiplication, 2 for addition and 3 for counting, the categories may be combined with the spatial structuring categories in the previous section. (A full set of results categorised in this way may be found in Appendix D.)

1	**Multiplication**	
		Student calculates a total without any interim step-counting.
2	**Step-counting/Addition**	
		Student counts in steps formed of the cardinal number of each layer or column, without any interim numbers (i.e. using a number pattern based on addition facts).
3	**Counting**	
	S	Step-counting (within a layer or face)
	R	Rhythmic counting: Student counts each cube individually, but the count sequence is rhythm-driven, with clear emphases on the cardinal number of each (equal) subgroup.
	G	Grouped counting: Student counts each cube individually, but with the count sequence organised into subgroups.
	U	Unitary counting: Student counts each cube individually, without any grouping.

Table 6-c: Enumeration strategies for a 3D array

On encountering Task 1, all students used some form of counting, and overall, counting-based strategies were by far the most popular. Four students could be said definitely to have used multiplication (number in a layer × number of layers) in either Task 3 or 4 (i.e. when the layers were defined by colour), and there were occasions where language used implies some multiplicative understanding (e.g. Jenny in Task 3 referred to “two twelves”, although could not recall an associated number bond). However, between unitary counting and multiplication there may be observed a varied spectrum of ad-hoc grouped, rhythmic, step-counting, and addition, including examples of mixed methods within the same enumeration (e.g. Harvey, Task 1).

The issue of rhythmic counting versus (non-rhythmic) grouped counting is illustrated by two examples.

(1) Ellis began by counting aloud individual cubes organised by column. Then, as his counting increased in rhythmicity, he stopped verbalising the non-cardinal numbers of each subgroup, and represented them kinaesthetically – tap, tap, 3, tap, tap, 6, etc. – a clear progression in use of additive number patterns.

(2) Tasha had group-counted a block in horizontal rows of three, only hovering a finger vaguely above the block rather than tapping or pointing to each individual cube, and had not noticed one of her groups had only 2 in it, giving a subtotal of

17 rather than 18. When she re-counted with exaggerated rhythm, all groups contained three numbers, and the multiples of 3 also received greater emphasis.

However, while rhythm can be very helpful, it may be affected by students' specific weaknesses. For example, Harvey repeatedly tried to count rhythmically but then broke the rhythm when unable to verbalise the next number word quickly enough, causing him both to become frustrated and lose his place in the count sequence.

6.2.4.3 Errors

Under the analytical aspect *error* I propose the four types below, between which are covered all errors observed in this dataset.

Spatial structuring (SS) Student uses an incomplete or incorrect conceptualisation of the array structure, e.g. double-counting edge cubes, not accounting for interior cubes.
Numeric calculation or retrieval (NC) Student makes an error in calculating or retrieving a number fact while multiplying, adding or step-counting, e.g. "three twelves... 12, 24, 38".
Verbal count sequence (VC) Student makes an error in their counting, e.g. "26, 27, 29, 30".
Visuospatial/kinaesthetic (VK) Student makes an error relating to the physical aspect of counting, e.g. desynchronisation of verbal count and gesture, confusion over which units have already been counted, etc.

Table 6-d: Types of error in enumerating 3D arrays

SS: Issues of spatial structuring have been covered in 6.2.4.1. To summarise, while all but two students' first responses to Task 1 involved mis-structuring, there were only 9 subsequent cases of SS error, in particular from Paula and Vince.

NC: On nine occasions, students mis-recalled addition facts and number patterns, or unsuccessfully attempted formal 'vertical' addition notation for the layers; however, the predominant preference of SEN students for counting-based strategies meant that recall of arithmetical facts or procedures was not often required.

VC: Students were all confident in their ability to count individual cubes, but in this case, the confidence may have been misplaced, as pointing to the individual cubes making up a block and counting them was far from error-free. While counting aloud, two students were observed to recite the number sequence incorrectly, Ellis repeating a number and Paula missing out a decade.

VK: By far the most common error type, 22 instances of this were observed. Some were gesture-related, such as the students with weaker fine motor skills moving their finger at a different speed to their verbal count, or taking too large a ‘jump’ and skipping a cube. Other errors were unrelated to motor skills but concerning the motion of the block, such as when students skipped rows, layers or faces, lost track of their start point and rotated the cube too many times, etc. Both spatial structuring and enumeration strategies affected this type of error: start/end point and rotation issues happen when conceptualising the block in terms of faces, but once a columns or layers conceptualisation is achieved, the block can remain immobile throughout enumeration.

Different strategic changes were effective for different errors. For example, Tasha’s move to rhythmic counting helped because her error had been of the VK type; it would not have corrected a VC error. A tendency to different errors may be a factor in explaining why some students preferred columns over layers, and the 6-colour block over the 3-colour block.

Extra-mathematical factors affecting the student’s given answer, while not errors as such, included Leo’s determination to count only colours corresponding to those he could write with his pen, cases of one student overhearing and copying the answer of another (having more confidence in their partner’s abilities than their own), and, in Harvey’s case, once giving an answer based on his “lucky numbers” rather than an enumeration attempt (something he claimed also to do in examinations!) Despite my repeatedly-stated interest in “how you go about working it out” and “the different ways of doing it”, student assumptions that ‘all teachers want is (right) answers’ may have been overpowering.

6.2.4.4 Issues of classification

Although the framework outlined above is useful for identifying individual trajectories and group trends, like many previous classification systems for qualitative data it is not without its issues. These can be to do with identifying the strategy a student is using, e.g. Jenny, who worked silently on tasks and gave mostly correct answers, but did not have the verbal skills to explain coherently how she had obtained them – meaning that the strategy must be inferred from the small amount of gestural data available.

Strategies may also be mixed within the same task, e.g. Harvey, who in Task 3 added the first two layers then unit-counted the third, and in Task 4 multiplied (L1) to enumerate the first block but group-counted the second.

With students who do verbalise their working, there may be inconsistencies between what they report doing and what they are observed doing. On several occasions, students clearly used the language of multiplication (e.g. "it's three twelves"), but, failing to rote-recall the answer to 3×12, instead worked out the total by counting in groups of 12, step-counting in threes, etc. From this evidence, it is likely that some other students, who did not tend to verbalise their thought processes, were aware of the multiplicative structure yet unable to carry out the multiplication operation otherwise than by counting (or preferred the reliability of counting). Another example of mismatch between knowledge of appropriate strategy and ability to carry it out occurred in Task 2, when Harvey double-counted most of the edge cubes; when I pointed this out, he replied (with irritation) that he remembered me showing him why it was wrong the previous week, but nevertheless "couldn't help" doing it that way.

6.2.5 Findings

What representational and arithmetical strategies do the students use?

These were described in 6.2.3-4.

What do their strategies indicate about their understanding of multiplicative structures?

Thinking about the visuospatial patterns within physical structures can reasonably be expected to increase awareness of the numeric structures they embody, perhaps most clearly when a student is pointing with a finger to unit-count the cubes, and the physical motion required to move from one row (or face) to the next causes a pause in count sequence, naturally grouping the counting numbers. Thus even an incorrect faces-based spatial structuring of a 3D array contains enough structure to serve a useful purpose for the weakest students: they may begin to count rhythmically – a stepping-stone to step-counting. However, it cannot be automatically assumed that patterns of three in a given student's counting (e.g. Vince, Task 3) necessarily entail a realisation that three has an integral role in the array structure, if those threes are simply used as a counting short-cut rather than in reasoning about the numerical relationship between the rows/groups of three and the whole/total. Similarly, there are examples of the converse, where students showed awareness of the physical structure but this was not reflected in their enumeration. Wendy and Ellis, for example, verbally demonstrated strong multiple spatial structurings of the arrays, but always counted every cube individually, using grouping and sometimes rhythm, but never making the leap to step-counting.

While it is tempting to assume that students' enumeration of arrays stems directly from their spatial structuring, and initially this might be the case, the relationship is bidirectional. Once a student is familiar with 3D array tasks, enumeration can guide structuring. For example, a student who perceives themself as better able to step-count long sequences of small steps than to add a small sequence of larger numbers may (and quite sensibly so) opt for a C2 strategy, while knowing perfectly well that there is a layers-based alternative (e.g. Oscar, Tasks 3-4). Students with weak numeracy skills can be highly adept at spotting those number patterns with which they feel confident; for example, noticing that there were five units in a row, column or stack could make that the salient grouping of the physical/numeric structure.

If students have access to more than one potential structuring, they can choose the one that best suits their capabilities and preferences. The 'aspect shift' experience of perceiving the same 3D array in more than one way may also be advantageous: an example of this duality was observed in Task 3-4 when a student (Ellis) used <u>vertical</u> layers which he recognised as each being formed of two vertical columns (which he had used successfully in Tasks 1-2). I suggest that the flexibility to switch pragmatically between different structurings is an indicator of sophisticated conceptualisation of 3D multiplicative structures.

How do their strategies develop over time, and in response to teacher input?

On finding their initial solutions were incorrect, one might expect the kind of cognitive conflict which results in reflection and adaptation of strategies; this did not happen. Wrong answers in mathematics lessons were a familiar occurrence for all these students, but their reactions varied. Some immediately started to re-count in the same way as previously, i.e. they were motivated to obtain a correct solution and believed in the efficacy of their strategy, but mistrusted their ability to have carried it out properly. Some accepted the failure of their strategy and simply waited for the next task, with little apparent interest in 'solving the puzzle', their attempt not having immediately produced the desired tick and pleased the teacher. Others were engaged enough to argue with me and insist their answer was correct. However, none independently responded by thinking critically about the strategy they had used and improving it or attempting an alternative. Strategic progression in every case required external teacher input.

I suggest individuals’ willingness (or otherwise) to experiment is linked to their relationship with mathematics and/or school in general; on finding a successful strategy, mathematically insecure students cling to it, while a greater feeling of security allows others to experiment. Cultivating a less formal atmosphere with reduced time pressure but increased discussion (as I did) should encourage the latter attitude. Additionally, different internal motivations make some students prioritise getting a correct answer using a strategy that they know works, while others avoid tedious routine by varying their strategy, and yet others attempt to make their current strategy more efficient.

6.2.6 Discussion

This set of tasks proved extremely rich in information about low-attaining students working with multiplicative structures. The close focus and detailed observation of individuals’ working, as with Holiday Clothes, illuminated the diversity of ways a straightforward-seeming task may be approached by students, the specific difficulties they can face at each stage, and the independent and teacher-prompted strategic changes through which they can surmount these difficulties. (Note that while the approaches observed here represent an important selection of potential enumeration strategies, further research on this subject might profit from a more quantitative approach, for example to rate the relative popularity of different strategies.)

The use of very similar task formats with the same students on four separate occasions allowed tracking of their progression in terms of spatial structuring and enumeration strategies, but also provided cases of individual students varying their strategies in ways which did not constitute linear progression. The richness and detail of the observational data also meant that, when analysed with the framework described above, it could provide diagnostic information about the nature of individual students’ specific arithmetical strengths and weaknesses at a given point, including potentially-significant ‘gaps’ in their understanding. Analysis of these kinds of changes over time, then, can be considered a dynamic qualitative assessment of their conception and manipulation of multiplicative structures.

The literature on this subject suggests that one can expect students with pronounced difficulties in mathematics to tend strongly towards counting-based strategies; however, few authors have acknowledged this degree of variety in counting styles and stages, particularly amongst older children – or the quantity and variety of errors. Presented with a situation where the counting was non-trivial and nonroutine, students had to reconsider this most basic of numerical skills, and how to apply it to the task before them. With those who counted confidently, there was something of a tension between creativity and security. In response to the teacher prompts, in some instances students displayed resistance to change, while others embraced it. Factors not formally evaluated in this study, such as prior relationship with school mathematics, self-perception of numerical and problem-solving ability, and mood, may play a significant role.

Although the layered spatial structure of a cuboid seems obvious to a teacher, and indeed, seemed obvious to some students once given a minimal 'nudge' in that direction, others struggled significantly to conceptualise the array as a coordinated, space-filling structure. The use of minimal, sequential prompts, along with the introduction of colour-defined structure, demonstrated the difference in how much external input and internal effort it took for a student to 'see it'. Furthermore, aside from speed of progression, individual students took their own paths from an essentially 2-dimensional, surface-area, faces-based conceptualisation (or from a position of no spatial structuring at all) to some form of coordinated space-filling structure, with some of these paths through layers-based structuring, some through columns-based structuring, and some through both. While the ability to perceive multiple structurings is unnecessary in the short term (i.e. for solution of this particular task), I assert that in the wider scheme, it is mathematically advantageous and to be encouraged.

7 TWO STUDENTS

This chapter contains two more ‘slices’ of data, each a case study of one student working on a particular task type, diagnosing difficulties and capabilities, and investigating progression over time. This material comes from the Tuition, Summary and Final sessions; by this time, relationships had been built, and the students were familiar with my way of working (e.g. absence of artificial time constraints, encouragement of drawing or modelling, and discussion of strategies). A clear choice for the first case study was the student who stood out as having the most extreme arithmetical difficulties, i.e. Paula (Year 10), and to focus on what is generally thought to be the most basic and intuitive form of division, i.e. sharing (using the Biscuits scenario). For the second study, focusing on the ‘grouping’ division tasks (using the Taxis scenario), I chose Wendy (Year 9) as a contrasting case.

With each, I begin with an introduction to the student, then give a chronological narrative description of all the relevant tasks, including scans of their mark-making, photographs, and excerpts of transcribed dialogue. I then further develop and apply the analytical framework developed in Chapter 6, using it to map trajectories of change in the students’ task strategies. The (still developing) research questions addressed in this chapter are:

- What representational and arithmetical strategies do the students use?
- What do the students’ initial representational-arithmetical strategies say about their particular weaknesses and capabilities?
- How do the students’ individual mathematical functionings change over time, in response to tuition based around tailored, flexible, scenario tasks?

7.1 Paula: Partitive division

7.1.1 Introducing Paula

Paula has been mentioned in previous chapters as having particularly severe numeracy difficulties. For example, although I stated earlier that I expected my participants to be comfortable with additive thinking (but perhaps no further), in Paula’s Initial

Assessment she was unable to demonstrate even single-digit subtraction. A Year 10 student, she turned fifteen during the study, but in certain respects – as the following data shows – her cognition resembled that of a pre-school child. As I have stated in 5.6.3 there were certain difficulties regarding this school's educational records, and although Paula was listed as having an SEN Statement, I was unable to access her file, so have no Statement or Educational Psychologist's report. However, her class teacher was able to inform me she was classified as having Moderate Learning Difficulties (MLD), with a recently-assessed reading age of 8. He also stated that in his ten years' teaching experience (in mainstream schools), her lack of numerical comprehension stood out as exceptional amongst all those he had taught. She relied heavily on unit-counting-based strategies, with all units individually represented in drawn or concrete form. This is not to say she had no experience of symbolic arithmetical calculations; for example, more than once she attempted two-digit additions in the traditional 'columns' layout (unsuccessfully). She was aware of the existence of 'times tables' and sometimes attempted to retrieve number facts from memory. However, her use of number bonds and relationships were infrequent, mostly incorrect, and nowhere near a sound enough foundation for developing pattern-based or derived-fact strategies; thus her counting-based strategies, with which she did enjoy a certain degree of success, are the focus of analysis.

Paula's home language was listed as Portuguese, but she was not considered in need of EAL support, and her conversational English – complete with North London accent – was confident and typical of her peer group. I spent some time sitting with her in class, in addition to my tuition sessions, and during all the time I spent working with, chatting to, or observing her, she exhibited a pleasant, placid disposition. Paula's comments about school led me to believe that in at least some of the more academic subjects she understood very little of what was asked of her, or why it was asked, but nevertheless did not find the lessons unpleasant, and had no objection to participating in the activities as requested. Her favourite subjects were music, drama and art. Paula liked to please the teaching and support staff, with whom she seemed to have an undifferentiated positive relationship – including me, even on my first appearance as a stranger in her classroom.

Paula required the most support on tasks of all my participants, and almost all of the representations used were co-created. This, with the very slow pace of her progress, provides an excellent opportunity to inspect in detail the teacher-student interactions, in particular the effects of teacher prompts on her strategies. Another facet of Paula's work which stands out from the other participants is her initial complete reliance on concrete media. One aim of my tuition, then, was to expand her visuospatial representational repertoire in the direction of drawing (justified in 5.4.3) – which I explained to her as being beneficial for her upcoming GCSE examinations, where cubes would not be available, but paper would). This meant picking scenarios and representation types in which she could comfortably work in concrete form, then replicating the visuospatial configurations in drawn form.

7.1.2 Data included

As with all participants, Session 1 with Paula was spent on Biscuits tasks, i.e. partitive divisions expressed using the scenario of sharing a given number of biscuits between a given number of people. I also set Biscuits tasks in Tuition 2 and the Summary session. In the Final session (where participants were given a choice of different division task types), she chose that type for two tasks. As Paula was reliant on unit-counting, and demonstrated little recall for number relationships either from rote or from previous tasks, I kept dividends and divisors to manageable sizes. The number relationships used were:

Session 1: 15 ÷ 3, 24 ÷ 4, 27 ÷ 3

Session 2: 21 ÷ 3, 21 ÷ 7, 15 ÷ 3, 15 ÷ 5

Session 4: 21 ÷ 3, 27 ÷ 3, 18 ÷ 3

Session 5: 20 ÷ 5, 24 ÷ 4

I have noted previously (6.2.4.3) Paula's tendency to make errors even when unit-counting concrete objects, either in the verbal count sequence or through desynchronisation of finger movement and speech. The focus of this section is not on counting *per se*, but the development of multiplicative structures through a sharing model; hence, while this error type did continue to occur during sessions, it is not discussed further.

Linguistic analysis is also not the focus of this study, and quoted speech is simplified slightly for ease of reading: non-words (um, ah, etc.), stutters, repetitions of speech, my non-task-specific words of encouragement, or the frequent interpolations of "like" between words (a common Londoners' habit), are generally not included in transcription. Indications of cadence and tone of speech are likewise not noted unless judged particularly relevant.

7.1.3 Chronological presentation of task activity

Session 1

15 biscuits shared between 3 people

On my setting the first task, Paula sits silent for some time. I point out that she does not need to work it out in her head, and could draw or use the cubes. She requests cubes, and I prompt her to start with 15 of them, which she counted out.

CF: There's 15 then, and we need to share them out between three people. So that they each have an equal number. Can you do that?

Paula: Like separate them?

Paula pushes the cubes into two roughly-equal groups.

CF: You've separated them into two groups. What I'd like is for them to be separated into three groups of equal size.

Paula pushes the cubes into five groups of three.

CF: You've put them into five groups there. But what I asked is if you could share them between three people.

As Paula appears to understand that sharing is required, but to have confusion over the partitioning, I draw three circles, describing them as "plates".

CF: Can you put the cubes so there's the same number on each of those plates?

Paula: You mean put them on each plate?

CF: I want you to put the cubes on here, so as each plate has got the same number of cubes on.

Paula: Like, in three groups?

Paula pushes three cubes onto each of the three plates, and ignores the remainder.

CF: Ok, that's a start, but can you put these ones on there as well?

Paula picks up each of the remaining six cubes, one at a time, and distributes them between the three circles.

CF: So we took 15, and we shared it into three groups. How many does each person get?

Paula: Does each person get... *[counts]* five?

CF: And they've all got the same, it's evenly shared out, and they each get five.

24 biscuits shared between 4 people

I draw a fourth "plate", tell Paula that there are four people now, and ask her to share out 24 cubes between them.

Paula: So I have to put like four cubes each?

CF: There's four people here, four plates. I would like you to start off with 24 and share them equally between these four plates. So you need 24 to start with.

Paula: So I need 24 cubes?

I help Paula to count out 24 cubes.

CF: Could you share them evenly between these four people, so they each get the same amount?

Paula pushes four cubes onto each of the four plates, and ignores the remainder (Figure 7-a).

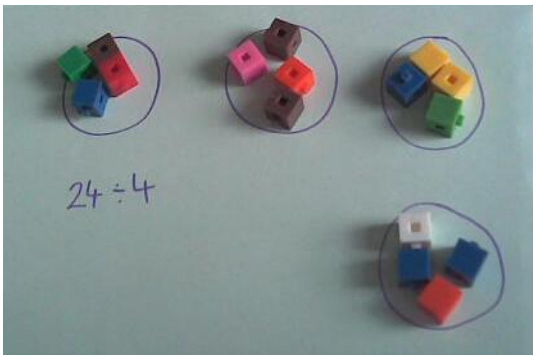


Figure 7-a: 24 biscuits, 4 people (Paula, first attempt)

CF: What about these ones? We can give those out as well.

Paula: Share them?

CF: I don't mind how many are on the plate, as long as everyone has the same amount.

Paula distributes the remaining cubes so there are 7, 6, 5 and 6 in respective circles (Figure 7-b).

Figure 7-b: (as previous, second attempt)

CF: Does everyone have the same amount there?

Paula: I think some of them do.

CF: How many do each of them have?

Paula: That one has six.

CF: Is there a way you can fix it so as everyone has the same?

Paula counts the cubes in each circle, and corrects the shares.

CF: Everyone's got six now. So, we took 24, and we shared it onto four plates. And if we do it equally, everyone has six.

27 biscuits shared between 3 people

I leave the model from the previous task where it is, to see if Paula will adapt it or start afresh. She is silent and motionless for a considerable time, then removes cubes so that there are three in each of the four circles.

I explain that she has given them three each, which is not quite what I was asking. As she does not respond further, I push 27 cubes into a pile, point to three circles, and remind her that all the cubes need to be shared equally. This time she pushes some cubes onto each of three circles, but again, there are not equal numbers.

CF: Do they all have the same?

Paula: I think so.

Paula counts the cubes in each circle, and removes some from the larger groups so they match the smallest group (7).

CF: So they've each got seven on now. Can you do anything with these ones that are left? Can they be shared out too?

Paula puts one extra cube in each of the three circles.

CF:	So you've given them one extra each. And then?

Paula distributes the remaining three cubes, and spontaneously starts to count the cubes in each group.

CF:	So we shared it between three people. How many do they get each?

Paula:	They get three each.

CF:	How many has each of them got?

Paula:	Nine.

Session 2

The following took place one week after Session 1.

21 biscuits shared between 3 people

Paula:	*[long pause]* Can I add it?

I explain that it is not an adding situation, but a sharing one, and recap the task.

Paula:	I don't know. If you give them... *[trails off]*

I remind her of the previous week, when she used cubes and "plates", draw three circles, and give her the bag of cubes. Paula counts out 21 cubes.

CF:	Now put them on these three plates, so that everyone has the same amount.

Paula distributes the cubes between the three circles, taking a few at a time and going back and forth in a non-ordered way between circles, seemingly judging roughly by eye the relative group sizes. On this occasion, it is correct – although she does not seem any more (or less) certain than in previous questions.

CF:	Do you want to check whether they each have the same amount?

Paula:	*[counts]* Yes.

CF:	So they each get how many?

Paula:	Seven each.

21 biscuits shared between 3 people

CF:	How about if I had these same 21, but I wanted to share them between seven people?

Paula:	You want to, like, share them out?

I recap the question, and Paula looks confused, repeating "seven people" to herself and tapping the table.

CF: Would it help to have seven plates? *[draws]*

Paula distributes the cubes, but very unequally. This time there is no obvious divisor/quotient misunderstanding, as she has not constructed any groups of seven.

CF: There's a slight problem with this, which is that these three people are going to be very hungry.

Paula immediately redistributes the cubes correctly.

CF: So they've each got how many?

Paula: Three each.

15 biscuits shared between 3 people

I draw three circles as before, but ask Paula how she might share out 15 if she was in a test, so did not have cubes, but could draw.

Paula: If there's three people. You have to divide it. Divide it like, separate it.

CF: You did it fine with the cubes. How could you do the same thing if you didn't have the cubes available?

Paula: Put it with dots?

I recap again, and Paula draws 13 dots in the first circle, then in the other two (Figure 7-c).

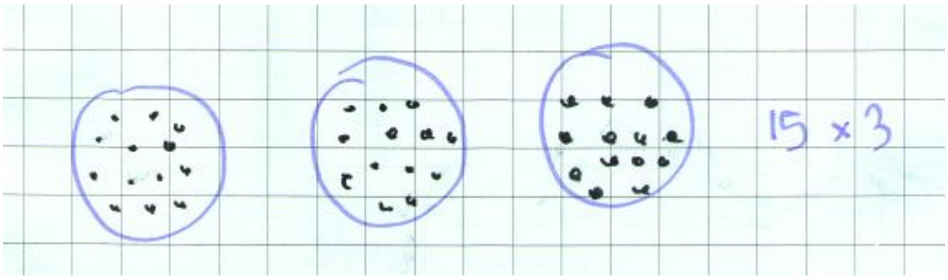


Figure 7-c: 15 biscuits, 3 people (Paula, first attempt)

CF: This would be a perfect way to work it out if I'd said "three people have each got 15". *[I write 15 ×3]* . . . I'm saying if we have 15 altogether, not three lots of 15.

I start a new representation, drawing 15 dots and three circles (see below).

CF: There's our 15 biscuits, and you have to share them out onto the three plates, so as everyone has the same amount. But you've only got 15 altogether. . . . See if you can work out how many they'd each get.

Paula: They'd get – probably six each.

CF: It's going to be something like that. How about if we just share them out one by one until they were all gone? *[I cross out a dot from the row and draw one in the first circle, then the next, etc.]* So – one for you, one for you, one for you. Another one for you. Can you carry on?

Paula continues to deal out the remaining dots. However, she began with the first circle (which already had two dots in) so the final result is 6, 5, 4.

CF: One of them's got too many, and one of them's got not enough.

Paula counts the dots in each circle, crosses out one from the first circle, and adds it to the third. She then crosses out a dot from the second circle, and pauses, looking confused (Figure 7-d).

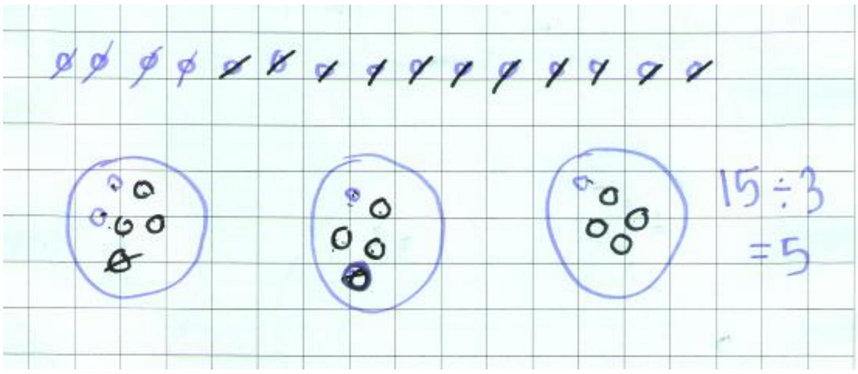


Figure 7-d: (as previous, second attempt)

CF: You didn't need to take that one off as well. Just taking this one and putting it over here fixed it. So now how many have they got each?

Paula: Altogether?

CF: How many has each person got? We know that there's 15 altogether, and we shared it onto three plates. How many does each person get though? *[long pause]* How many on each plate?

Paula: *[long pause]* I think five.

CF: We took 15, and we shared those 15 out into three groups, and each one of them had five in it.

15 biscuits shared between 5 people

CF: Fifteen again, but this time we're going to share it five ways. Fifteen to start with *[draws row of dots]*, and this time we're sharing it into five groups *[draws circles]*. Can you do this one? Try it in the same way.

Paula deals out the dots systematically, crossing each one out and drawing a replacement in the appropriate circle (Figure 7-e).

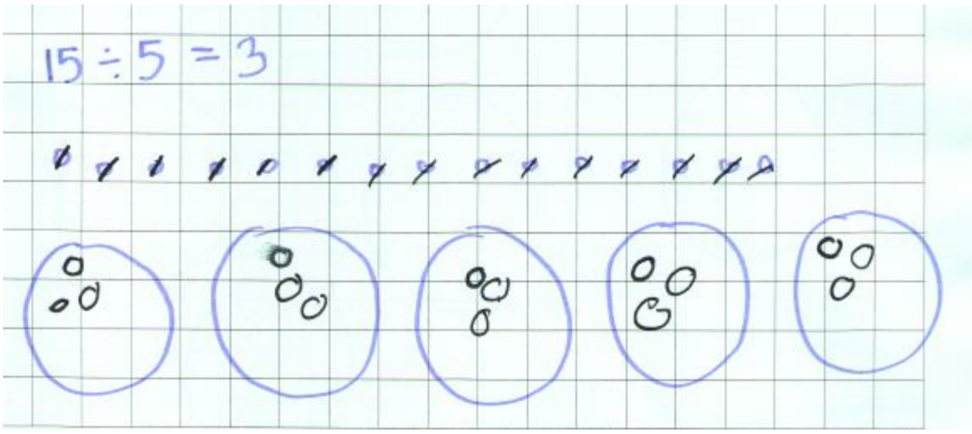


Figure 7-e: 15 biscuits, 5 people (Paula)

CF: So how many do they each get?

Paula: Three.

Session 4

The following took place two weeks after Session 3 (which did not include any partitive division tasks) and three weeks after Session 2 (above).

21 biscuits shared between 3 people

Paula does not respond. I prompt her regarding representational media, and she chooses cubes.

CF: So imagine you've got three people sitting there, and you've got to give them each the same amount.

Paula distributes the cubes non-cyclically into three unequal groups, separated spatially (without containers), looks at them, and moves some between groups. She appears dissatisfied, and reaches for more cubes from the bag.

CF: Wait, you can't have any more. I said 21. 21 is all there is in the packet, and you need to share them evenly. So maybe you need to change your –

Paula continues to move cubes between groups, looking at them, but not visibly counting at any point.

CF: What you've got to do is make sure that they've each got the same number. So at the moment they haven't each got the same number. One of them's got more.

I return the cubes to a single pile, and demonstrate dealing them into groups. After a few rounds, Paula takes over, copies my action and deals the remainder, while I count each round of dealing (e.g. "five each") aloud.

CF: And because you were dealing them out one at a time, and going around giving each person one more each time, then we know that they've each got the same. . . . So how many do they get each?

Paula: Three. No, wait. Seven each.

27 biscuits shared between 3 people

Paula takes 27 cubes, and deals out three, separated spatially, then pauses.

CF: One each...

Paula deals out the remaining cubes. After "two each", she continues and I remain silent. After distributing the cubes, she counts each of the groups (without prompting).

Paula: Nine.

18 biscuits shared between 3 people

CF: If you didn't have the cubes to use, what could you do instead?

Paula: *[long pause]* Draw dots.

I suggest she tries, and recap task. Paula draws a row of 18 dots, marks lines through them, and rings them in groups of 4, 5, 5 and 4 (Figure 7-f, central section in green).

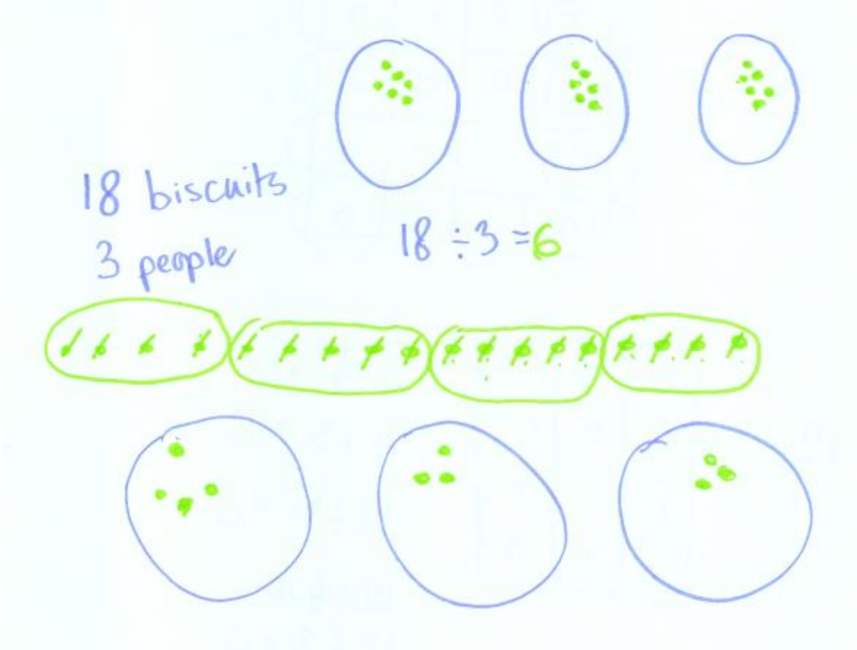


Figure 7-f: 18 biscuits, 3 people (Paula, first, second and third attempts)

CF: This isn't shared between three people.

I remind Paula of how she used the cubes, and draw three circles (Figure 7-f, bottom).

CF: You need to count 18, while sharing them out evenly.

Paula draws 4 dots in first circle.

CF: How will you know how many to put on each plate?

Paula: *[inaudible]* ... by dividing in groups.

CF: Why don't you try doing what you did with the cubes? One each, until they're all gone. So one for the first person, one for the second, one for the third, until you've done all 18.

Paula draws three dots in each of the other two circles.

CF: Now this person's got four, these people have only got three.

Paula looks very confused. I draw three new circles (Figure 7-f, top) and mime the dealing motion. Paula draws dots cyclically in the three circles, while I count aloud to 18. She then counts the dots in each group, and announces "six".

Session 5

The following took place several months after the main fieldwork period.

20 biscuits shared between 5 people

Paula is still and silent for a long time. She then draws four circles, and puts dots in them, without any clear overall system of distribution, the completed representation containing, respectively, 14, 15, 14 and 17 (Figure 7-g).

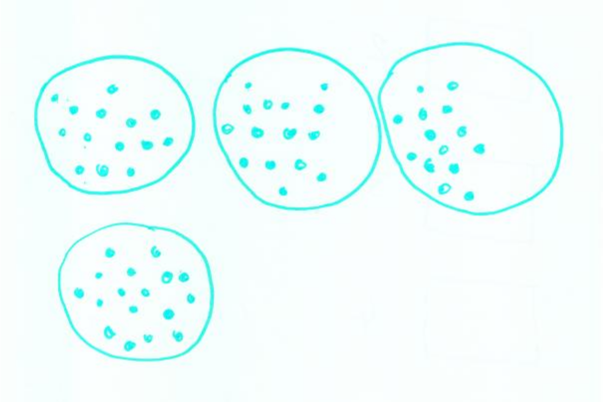


Figure 7-g: 20 biscuits, 4 people (Paula, first attempt)

CF: How are you working it out?

Paula: *[inaudible]* ... separate the biscuits in each plate.

CF: How many biscuits did we start off with?

Paula: 20.

CF: Looks to me like you have more than 20 now. . . . So how many people are we sharing them between?

Paula: Five people.

CF: Why don't you start with the five plates, one for each of the five people.

Paula draws five (new) circles, then waits. I suggest using cubes, and remind her that she will need 20 to start with. After she has counted out 20 into a pile, and re-counted them to check there are the correct number, I recap the task. Paula deals the cubes correctly into the five circles. (No photograph taken.)

CF: So how many does each person have?

Paula: Four.

I remind Paula that the dealing method works with drawing as well as cubes.

24 biscuits shared between 4 people

> *Paula counts out 24 cubes, and begins dealing them onto the five circles from the previous task.*
>
> CF: How many people are we sharing them between?
>
> Paula: Four people.
>
> CF: Right, just four people this time.
>
> *Paula deals the cubes into four groups, steadily until the final deal. However, she pauses when two cubes from completion, and looks confused. She adds the cube to the end group, counts that group and the one next to it, finds them unequal, moves the extra cube to the second group and compares that with the size of the third group (still unequal), moves one again and re-counts (now 6, 6, 6, 5). She looks around and locates one more cube (hidden under her sleeve) to complete the final group.*
>
> CF: How many does each person get?
>
> Paula: Six each.

7.1.4 Developing the framework of analytical aspects

In Chapter 6 I developed qualitative analytical structures suitable for the representations created by (or co-created with) students participating in arithmetical tasks, and the way they interacted with visuospatial representations. I now combine the set of analytical aspects from 6.1 and the general aspects (but not those specific to 3D arrays) from 6.2 into an expanded framework (below).

The process of describing as accurately as possible the changes in the component aspects of Paula’s representational strategies made additional changes and refinements necessary. Previously I included *motion* as a simple categorical variable – either present in a representation or not; however, in analysing Paula’s representational strategies it became necessary to expand the scope of this aspect to give a description of the kinds of movements involved in interacting with a representation. For the purpose of differentiating separate attempts at the same task, I have included the categorical aspect *success*, and two separate types of teacher-student interaction. I have also added the important aspect *unitariness*, which was not applicable in either Holiday Clothes or Cuboid Starters.

Types of representations created: ***Media*** e.g. cubes, pen/paper ***Mode*** e.g. modelling, drawing, words, symbols ***Resemblance*** between the drawing or model and the task scenario described
Relationship between representation and calculation: ***Motion*** e.g. static once created, or involving ongoing movement of elements ***Unitariness*** i.e. whether one representational unit (e.g. cube, tally mark) represents exactly one scenario/numerical unit, or may stand for a group ***Spatial structure*** e.g. grouping of units through separation in space, use of containers, aligning in one or more dimensions ***Consistency*** i.e. whether a single coherent strategy was used from start to finish, or changes occurred ***Completeness*** i.e. whether the external representation had to be 'finished' for solution ***Enumeration*** e.g. unit-counting, step-counting, number fact retrieval ***Errors*** e.g. incorrectly-retrieved number fact, verbal count error ***Success*** i.e. whether the strategy produced a correct solution
Teacher-student interaction: ***Verbal*** e.g. spoken prompts, suggestions for calculation ***Visuospatial*** e.g. participating in modelling or drawing activity

Table 7-a: Aspects of students' representational strategies

A complete catalogue of Paula's representational activity during partitive tasks is provided in Appendix E. Two aspects were immediately inapplicable when looking for change, *unitariness* and *completeness*, as in all tasks, every unit had to be uniquely represented and countable for an answer to be produced. I have also not included *resemblance*, as there was very little variation in this, only simple units (representing biscuits, always) and containers (representing plates, sometimes), and no mathematically non-functional detail at all. (In contrast, some other students drew detailed people – even spontaneously giving them names – amongst whom the biscuits were to be shared). The *consistency* aspect is also inapplicable for the Paula data, as I chose to break down each task into a series of attempts, separated by supporting interjections from me, which tended to be followed by a change of strategy.

7.1.5 Findings

Although Paula presented an unequivocally positive attitude to tuition sessions, always giving her best effort to the tasks, and with little evidence of distraction, her mathematical performance showed considerable variation (in line with the assertion from her class teacher that she had "good and bad days"). Thus, general conclusions must be made with care, particularly regarding longer-term developments.

What representational and arithmetical strategies does the student use?

Paula did not initially have any working representation of her own for use in sharing tasks, but readily adopted those I suggested: modelling, drawing, and mixed-media, mixed-mode representations (concrete units, drawn containers). She was completely dependent on unitary representations. While Paula's first preference was for a dynamic, concrete model of the numerical structure (piles of cubes), this by itself appeared insufficient, and a visual reminder of the required number of groups was also required (in this case provided by the container elements of the representation, i.e. my drawn circles). On my suggestion, she was also willing and able to switch to using drawn units (dots).

Motion was always integral to Paula's representations. Early on, when using cubes, she adjusted her constructed groups by moving cubes back and forth until I appeared satisfied (or, later on, until she herself was satisfied with how it looked). The dealing action then became important, with the regular motion of the hand between the initial quantity and each of the groups (in cyclic order) a key part of successfully using both concrete and drawing strategies.

What do the student's initial representational-arithmetical strategies say about their particular weaknesses and capabilities?

It was clear from the start that Paula was struggling with division as commonly expressed in sharing-based tasks, and initially it could be stated with certainty only that she knew the operation required starting with an initial quantity and separating it into a number of smaller quantities, as this was a consistent response. This may seem trivial; however, is not only a necessary component of the division process, but may be seen as the most fundamental meaning of 'divide'.

Paula’s errors came in three forms. On eight occasions she broke the rule that groups must be equal, and on seven that the initial number of units must be preserved (i.e. no cubes left over, and no increase through taking extra cubes or drawing extra dots). Generally either one or the other of these errors occurred (although there were times both occurred within the same representation), indicating a tension in satisfying both demands at once. The third form of error was to share the units into an incorrect number of groups. These errors seemed fairly evenly distributed throughout the sessions, with no immediately apparent trend.

Paula’s tendency to produce unequal groups implies that she did not see it as important for them to contain an equal number, and/or that she did not have a reliable method for distributing them fairly. The latter is indicated, as when I reminded her that units should be shared equally, she counted each individual group and took action to even them up (by moving the excess in larger groups). Regarding the former, group sizes only varied by one or two cubes, i.e. similar enough in size that one could not immediately judge their (in)equality by eye alone; it is possible that groups that looked of a similar size were judged equal enough.

Paula’s retention (or non-preservation) of units implies that she did not see it as important that all of the initial quantity should be distributed, that she believed that including them in the groups already created would conflict with another requirement of the task (e.g. equal group size), and/or that she had simply forgotten about them. The second interpretation seems likely, as she distributed the remainder when asked, and then re-counted the group sizes.

These observations together suggest that Paula had a three-part conception of division, corresponding to three independent requirements: separation of the initial quantity into groups, that the groups are of equal size, and that all of the initial quantity have been distributed. The priority relationship between the second two requirements was unclear. For most students using a unit-based concrete model, these three stages would be subsumed into one through ‘dealing’ units cyclically into groups until all of the initial quantity is gone. It is notable that Paula did not initially do this, instead using an unsystematic and disordered distribution process. It is inconceivable that a Year 10 student in mainstream education has never encountered dealing; however, Paula appeared unconvinced of its utility in these situations, and not to have connected the physical dealing action with the numerical structure or arithmetical operation. I further suggest the possibility that the dividends, while well within her counting range, were large enough that, unless prompted to deal with them as individual units, her tendency may have been to treat a pile of, say, 20 cubes as a continuous rather than a discrete quantity, and thus to perform a ‘rough’ division with visual approximation rather than exact enumeration.

Lastly, there are the occasions when Paula initially confused the number of groups with the number in each group. At a higher level of arithmetical functioning, of course, this would not make a difference to the answer in a division calculation; however, when working at this fundamental, concrete level, the distinction is important. As my verbal prompts did not contain any new information, but essentially repeated the task set (or components of it), using the same language (share, equal, etc.) with minor variations in sentence structure, I suggest that the issue is not that Paula did not know what was meant by the terms, but that she initially picked up on key words from the task (e.g. ‘share’, ‘groups’ and ‘three’) without registering the relationship between them. If this were the case, the resulting action could well be to count out a group of three, or three groups of three, or continue counting groups of three until either all the cubes were gone or the teacher indicated the solution was satisfactory.

Paula’s extreme focus on individual countable units, taken with the instances of her sharing into an incorrect number of groups, indicate the possibility that she may have difficulty with the very idea of groups being countable objects, i.e. with shifting her focus from unit-level to group-level. This interpretation is consistent with both the fact that my drawn containers were helpful to her (through visually reinforcing groups-as-units), and the fact that, despite this, she was somewhat disinclined to draw them independently.

How do the student’s individual mathematical functionings change over time, in response to tuition based around tailored, flexible, scenario tasks?

Because my level of support and involvement was so high, it is more helpful to consider the overall content of my teacher prompts and their effect on Paula, rather than individual instances. As well as emphasising the concepts underlying 'fair sharing', I introduced a practical method for accomplishing this: dealing. I also introduced the possibility of an alternative mode of representation (drawing) while maintaining the necessary elements of iconic representation of all units in visually separated groups, and the systematic dealing motion. These proved influential on Paula's ongoing task strategy choices.

In each session Paula required teacher support on the first task, but was then able to complete subsequent tasks using the same strategy with reduced or no support, more quickly and with greater efficiency of movement. I provided both verbal and visuospatial forms of support. My verbal ‘nudges’ each related to a specific rule that had been broken (e.g. unequal sharing) or to a single aspect of the task that had been misunderstood (e.g. number of groups). In each case, Paula immediately corrected her error (although sometimes making another while doing so). My visuospatial interactions consisted of drawing containers and demonstration or miming of dealing; in each case, Paula was able to take over and use the representation.

Intra-session changes in representational strategy only occurred in response to my suggestions, which I made either to extend her when successful (i.e. from cubes to dots) or to remind her of a trusted format when struggling (i.e. cubes/circles). While within individual sessions she transferred from using concrete to drawn representations, in each subsequent session this temporary confidence had been lost somewhat, and it was necessary to return to concrete methods. However, it is possible that on further repeated exposure to the two formats in this way, the connection between them might be strengthened, and the drawn forms regained more quickly.

While there is no clear trajectory across all four sessions in terms of media/mode preferences or resemblance of representation to task scenario, the data shows an increase in systematicity of distribution. I describe Paula’s movements as ‘non-ordered’ when there was no apparent sequence in the distribution of units; this was often accompanied by either pushing cubes from one group to another or drawing additional dots. I describe some task attempts as ‘partially systematic’, where there was some order visible, such as picking up groups of three cubes at a time, or ‘dealing’ the last few cubes in a remainder group. After I had explicitly demonstrated the dealing process, Paula began increasingly to use this method. While it is true that she required reminding of it in each of the subsequent sessions, she began to carry out the action with surer and more efficient movements. Based on the way she enumerated dealt units, it may also be inferred that the repeated success of dealing contributed to strengthening her belief that this method would ensure fair shares – which in turn might encourage her to employ it again.

Earlier in the tuition process, where Paula had distributed cubes or dots in a non-ordered way, she often looked at her representation and made adjustments to it via visual approximation, which nevertheless in all but one case still resulted in unequal groups. After a verbal prompt regarding fair sharing, she counted each group (with finger-pointing) to check for exact equality – a significant improvement. However, in later tasks, after performing the dealing process, she looked carefully at the groups without prompting and pronounced them equal (or not, in which case she took action to correct them), sometimes spontaneously giving the number in each group. Although there was no outward indication of counting (such as the finger-pointing), the fact that the shares in her completed distributions were exactly equal, and her ability to give a precise answer, indicates that something different was taking place: a visual count (i.e. of discrete units) rather than a visual approximation (i.e. assessing equality by the amount of visual space taken up). The change to this form of counting is an additional indicator of increased confidence in the dealing process.

7.1.6 Discussion

The data from this arithmetically exceptionally weak student cannot be expected to show the same markers of comprehension and progress that would be found in a student of more typical ability. Nevertheless, a microgenetic level of observation both brings to light some of the difficulties in conceptualising and carrying out division-based tasks which may go unrecognised in a mainstream secondary mathematics class, and demonstrates the possibility of improvement even in such cases.

Analysis of such slow, painstaking mathematical activity is helpful in deconstructing partitive division, into the separation of a quantity into a given number of parts, where those parts are equal, and the original quantity is preserved. It also highlights the interplay, and potential for conflict, between those requirements, or the overriding of them by particular understandings and connections (e.g. “three” powerfully signifying a group of three units, as opposed to three groups). Without the cognitive capability to consider more than one ‘rule’ at a time, what would be a simple one-stage calculation becomes a complex multi-stage process.

The nature and degree of Paula's individual difficulties make progress not only extremely slow and effortful, but uneven and non-linear; nevertheless, I assert that my analysis of her representational and arithmetical strategies over this task sequence does constitute evidence of positive change, however small it might seem when measured against the rate of progress of typically-attaining teenagers. However, does this change constitute development in arithmetical/multiplicative thinking? Yes. It is true that children may carry out action sequences without understanding their significance, e.g. dealing units without realising the process ensures conformance with the rules of equal groups and preservation of original quantity. However, Paula's increasing use of dealing indicates that her belief in its reliability was increasing, while changes in her enumeration methods indicate a strengthening understanding of the link between the repeated distribution action and sharing into exactly-equal groups.

Regarding representational modes, the ease with which Paula switched from concrete to graphic representations of numeric relationships – thanks to carefully-designed scenario tasks emphasising similarity of visuospatial form and motion – is also significant. However, although her visuospatial representations allowed change, her narrative remained firmly embedded in the narrative scenario of people and plates (etc.) While I have argued against a direct linear interpretation of Bruner's (and others') Enactive – Iconic – Symbolic stage model, this student was increasingly capable of moving from a highly enactive strategy to a more iconic one, and demonstrated increasing independence in the target mode. While meaningful symbolic thinking about multiplicative structures was still a long way off for Paula (and may never be achieved), there was undeniable movement in that direction, for which her efforts deserve appreciation.

7.2 Wendy: Quotitive division

7.2.1 Introducing Wendy

Wendy was thirteen at the start of the study, in the lowest set in Year 9, with a history of poor performance in school Mathematics, and the avowed particular distaste for division common to all my participants. Her counting and addition were comparatively reliable in outcome, but she still made heavy use of unit-counting-based strategies, tending to represent all units individually in her drawings. Within the context of this study she was one of the stronger students: whereas with Paula I had to adapt my tasks to make them easier, with Wendy the opposite was the case – I had to create increasingly difficult extension tasks to adapt to her intra-session speed and progress. As she seemed to comprehend well my comments about number relationships (although not necessarily remembering them), and was willing to tell me when she did not understand something I had said, I took the opportunity to engage her in more detailed discussions about how and why the arithmetical strategies worked, which she appeared to enjoy. Wendy was very positively disposed to my tuition in general, and consistently gave her best effort. In comparison to almost all my other participants, her ability to maintain concentration during sessions was excellent, meaning no 'wasted' time, and plentiful task data generated.

I was able to access Wendy's Statement of SEN (dating from two years previously), which identifies her as having dyslexia, leading to difficulties with numeracy, memory, organisation and sequencing skills. These descriptions seem reasonable to apply at the time I was working with her. However, the Statement also contains the judgement that she "lacks the ability to retain and process academic subjects that require logical thinking, analysis, sequencing, rationalising and accuracy"; this is an example of the kind of dismissive labelling of students with SEN as predetermined future, as well as past, academic failures, which I criticised in Chapter 2. Her Wechsler Individual Achievement Test (WIAT-II) scores at age 11 were in the 2nd percentile for numerical operations, and the 0.5th percentile for mathematical reasoning. Like the quote above, these scores do not square well with the mathematical behaviour I describe below, and while there may well have been a certain degree of catching-up relative to peer group in the intervening time, Wendy more than once independently expressed extreme negativity toward “tests”, and I suggest she may perform worse in more formal assessment environments.

Wendy's Statement also mentions "social immaturity", a lack of self-confidence, and low self-esteem. While these may have been true in the past, Wendy appeared now quite self-possessed, socially and emotionally mature for a 14-year-old. She was well aware of the importance placed by the education system on the memorisation of 'times tables', and of her own long-term failure to do so ("even with constant review and revision", according to records), which had left her, unsurprisingly, with a low opinion of her own mathematical ability; however, she did not radiate general low self-esteem. From my observations of Wendy in a classroom environment, she seemed generally well-behaved, with unproblematic relationships with teachers and assistants. However, she showed an active preference for my withdrawing her for 1:1 work, to the extent of asking if she could have more frequent or longer sessions. While an analysis of teaching/learning relationships is not the focus of this analysis, the particularly positive nature of the educational environment I was able to provide for this individual must be considered relevant to the case study.

7.2.2 Data included

Of Wendy's data, I focus on the quotitive division tasks which were introduced in Tuition session 3 using the ‘Taxis’ scenario, and set in similar and extended ways in Session 4. In Session 5, she chose one Taxis task, but I also include data from extension questions, and, for comparison, two where she picked the Biscuits option. As Wendy was barely ever able to recall number relationships either from long-term memory or from previous tasks, in the extension tasks with three-digit dividends and two-digit divisors I selected the divisors for their ease of repeated addition or step-counting, but sometimes included non-multiple dividends. The number relationships used were:

Session 3: 16 ÷ 4, 20 ÷ 4, 28 ÷ 4, 32 ÷ 4, 35 ÷ 7, 45 ÷ 7, 45 ÷ 25, 96 ÷ 25, 391 ÷ 50, 612 ÷ 200

Session 4: 30 ÷ 5, 38 ÷ 5, 343 ÷ 50, 147 ÷ 21, (100 ÷ 20, 650 ÷ 50)

Session 5: 36 ÷ 4, (30 ÷ 5), (105 ÷ 21), 180 ÷ 20, 240 ÷ 20, 300 ÷ 25

7.2.3 Chronological presentation of task activity

Session 3

16 people in 4-seater taxis

> *On my setting the first question, Wendy immediately drew a row of four dots, then repeated the action three times below.*
>
> Wendy: Fit four in there.
>
> CF: Sorry?
>
> Wendy: Four.

20 people in 4-seater taxis

> CF: What if it was 20 people instead?

Wendy uses the same process to draw five rows of four.

Wendy: Still four.

I explain, pointing at the rows in Wendy's drawing.

CF: So here's four people. If they get in the first cab, and the next four people get in the second one – *[long pause]* These four people get in the third one -

Wendy: Oh, they have another one there.

CF: We'd need another one, yeah. . . . So that's the first taxi, that's the second one, that's the third one. How many do they need altogether?"

Wendy: Four. No.

CF: Fourth one. How many taxis is that?

Wendy: Five. . . . I'm still not sure.

I explain again, first reminding Wendy that each taxi can fit exactly four people in it, and as I relate the scenario of groups of people getting into taxis, I ring each of her rows (Figure 7-h).

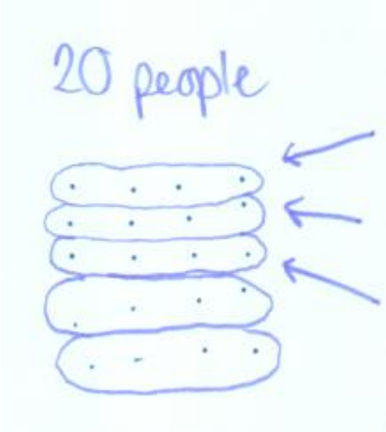


Figure 7-h: 20 people, 4-seater taxis (Wendy and CF)

Wendy: Oh!

CF: Does that make sense?

Wendy: Kind of.

I explain that this is a slightly different way of using dot patterns to the Biscuits tasks: with those, we were putting biscuits onto plates, and knew how many plates we had; now we are putting people into cars, but don't know how many cars we'll need. I refer to her successful use of dot arrays previously, and reassure her that they can still be used here.

28 people in 4-seater taxis

I suggest counting on from 20, and draw another row of dots below the previous array, while Wendy counts along.

CF: Now that's filled up another taxi. *[I ring the dots]* So for 24 people we need six. But that's not 28 yet.

I draw another row, counting aloud.

Wendy: We need seven taxis.

32 people in 4-seater taxis

CF: What if there were 32 people altogether?

Wendy quickly draws a new dot array (without rings), then counts the rows aloud while pointing to each.

Wendy: Eight taxis.

35 people in 7-seater taxis

CF: . . . How many of those should we book, if we put people in in sevens?

I extend the scenario by introducing extra-large 7-seater taxis. Wendy draws two rows of four dots, and I interrupt.

CF: Before you go any further, seven -

Wendy: Oh yeah *[laughs]*

CF: - people in each now.

Wendy adds an extra three dots to the first row, an extra four to the second (incorrectly), and then three further rows of seven. She seems dissatisfied, starts re-counting her dots, notices the extra dot and crosses it out, then re-counts the whole array.

Wendy: 35? *[quietly]*

CF: Sorry?

Wendy: 35, and then you need *[counts rows]* five.

I mention that we can check the answer by working out five sevens is 35, because we had 35 split up into five groups of seven.

45 people in 7-seater taxis

Wendy starts drawing rows of seven. She counts the first row aloud, then counts 1, 2, 3, 4, 5, 6, 7 under her breath for subsequent rows. She stops after five rows, and counts the total number of dots. She then adds three more rows, now keeping a running total.

Wendy: Oh-oh, I've gone further. I went on to 56.

Wendy goes back and re-counts from the start, but stops at around 35, looking confused and complaining she's "got them muddled". I step-count the groups in sevens, pointing at each, up to 35, then unit-count the rest. Wendy crosses out the remaining 11 dots. I draw attention to the fact that we have a row that's only half full.

Wendy: They can get the small taxi. *[NB continuing to relate to scenario rather than abstract]*

CF: . . . So we've filled up six whole taxis *[points]*, but we've got some people remaining behind *[points]*. How many people are remaining behind?

Wendy: Three.

. . .

I recap the task verbally, in terms of the scenario, at each piece of information stopping to write the number or symbol (45 ÷ 7 = 6 r3).

CF: Or if we just got them another one the same as the others, we could just order seven.

Wendy: But then it would be more expensive.

I draw attention to the fact that her general strategy (dot arrays) still works even when the numbers don't work out into an exact number of rows.

45 people in 25-seater coaches

I suggest a slightly altered scenario, with the same people going on a trip in 25-seater coaches.

CF: Now, you could put them in rows of 25. That would work. But the numbers are getting a bit big, so it's quite a long way of doing it. *[W laughs]* Is there any way that you could work out how many coaches we'd need?

Wendy: Not sure.

CF: . . . Ok, here's one coach, and we're going to put the first 25 people on that. *[I draw 'coach' container and write '25' in it – see Figure 7-i]* That hasn't taken all of them, has it? So let's get another one. *[draws another]* We could fit 25 on that one as well. How many people is that altogether, that we can transport?

Wendy writes 25 + 25 as a column addition, but misremembers the procedure, getting 41. When I point out her error, she recalls where to write the 1 and 0, but does not know how and why this works. I promise to explain it later.

CF: Ok, so we've got 50 seats altogether there. Is that enough to carry 45 people?

Wendy: No. Think you might need... Oh, yes. *[laughs]*

CF: Do you know how many empty spaces there'd be? 50 seats, only 45 people.

Wendy: Five.

CF: Can you tell me how you got five?

Wendy: From 42, I counted on to 50.

96 people in 25-seater coaches

Wendy: Can I still use that one? *[pointing at drawing from previous task]*

CF: Sure, if it helps.

Wendy draws two more coaches, with 25 written in them (Figure 7-i). I suggest checking by seeing how many seats there are altogether with two more coaches. Wendy correctly carries out an addition of four twenty-fives. I point out that the division hasn't worked out exactly, and there would be spaces in the final coach. Wendy counts on from 96 to 100 and writes '4 spaces'. I then write the calculation symbolically, and give the alternative answer of three (full) coaches and 21 people remaining.

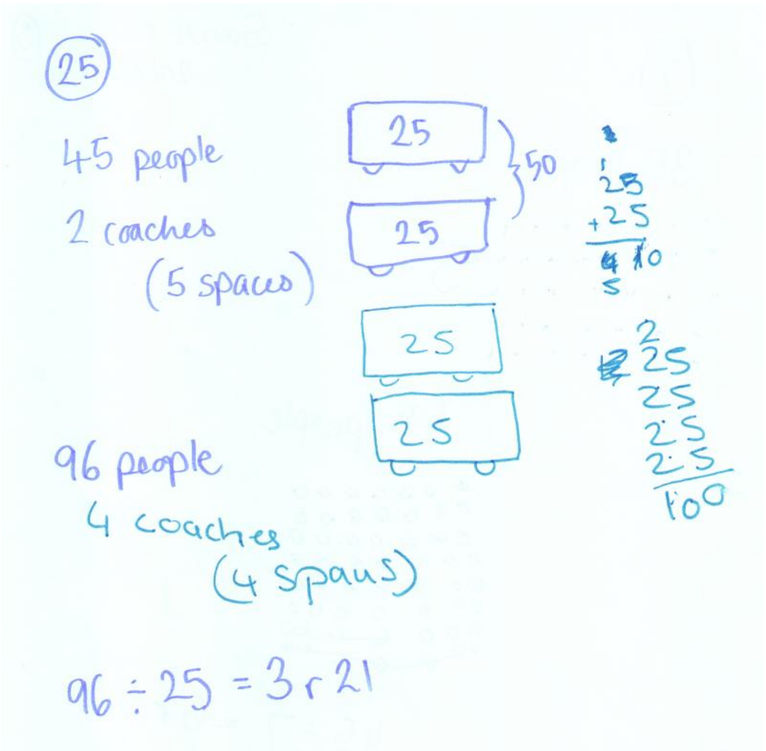


Figure 7-i: 45 people, 25-seater coaches (CF); 96 people (CF and Wendy)

391 people in 50-seater coaches

I increase the magnitudes again, staying within the scenario (coach tours are becoming increasingly popular, so the company gets bigger coaches), but also including some slightly more abstract language.

CF: So how many fifties do we need to make 391?

Wendy abandons the coach drawings this time, in favour of just numbers. She writes two fifties ("That's 100")*, then thinks silently.*

Wendy: Four fifties... I think?

I suggest that it is difficult to keep count in her head, and that she might write something down to make a note of how many fifties she's got. She writes four more fifties in a column.

Wendy: Think I might need a couple more.

She writes another two fifties, and stops. I suggest checking the number of seats. She attempts to do the addition in her head.

Wendy: I had it but I lost it.

I suggest grouping them, and bracket two fifties together to make 100 (Figure 7-jFigure 7-j).

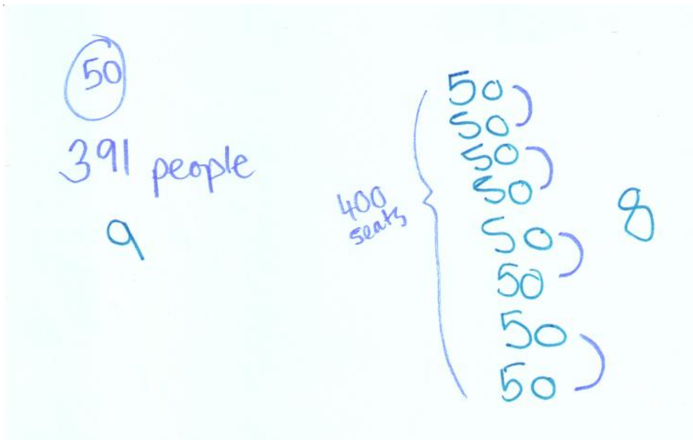


Figure 7-j: 391 people, 50-seater coaches (Wendy and CF)

Wendy gives 400 as the total number of seats, 8 coaches. She states she has "no idea" how many empty spaces, but after I write '400 seats', she counts up from 391 to 400 as before, counting the numbers aloud while keeping track on her fingers, then writing 9.

612 people in 200-seater aeroplanes

I further extend the scenario (the holiday company organising coach tours also organises flights, and they use 200-seater aeroplanes).

CF: So we've got 612, grouping them in two-hundreds.

Wendy: Can I draw some planes?

CF: Yeah, whatever helps.

Wendy draws one aeroplane ("That'd be 200"), *then a second* ("Another 200") *and stops (Figure 7-k).*

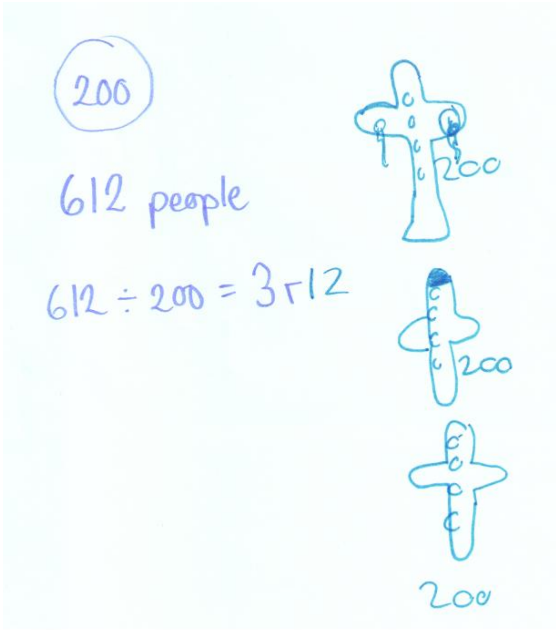


Figure 7-k: 612 people in 200-seater planes (Wendy)

CF: Is that enough? *[Wendy seems unsure]* Have we got 612 seats yet? 200 and 200?

Wendy: Nn-nn. *[draws another]*

CF: Have we got enough seats yet?

Wendy: Oh yeah, that's enough.

CF: How many seats have we got so far? *[pause]* 200 and 200 and 200.

Wendy: 200, 400, 600 seats

CF: Is that enough?

Wendy: No, one more plane. Smaller plane.

I recap that we have 612 people, 3 full planes (while writing symbols), and point out that some people are left over. Wendy unit-counts aloud from 600 to 612, with fingers, and writes '12'.

Session 4

The following took place one week after Session 3.

30 (then 38) people in 5-seater taxis

Wendy answered these quickly and easily. Her drawings (e.g. Figure 7-l) differ from previous unit-based representations in that the units are not in a rectangular array, although they are grouped in a repeating pattern.

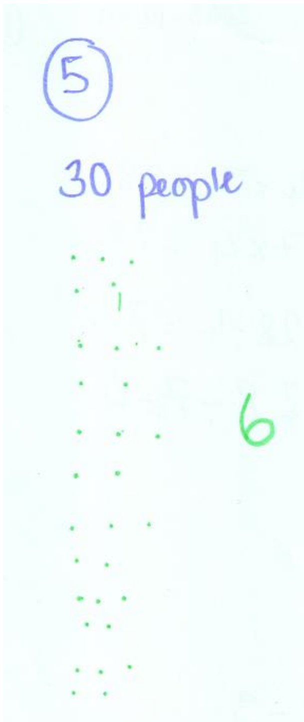


Figure 7-l: 30 then 38 people, 5-seater taxis (Wendy)

343 people in 50-seater coaches

Wendy draws four coaches (Figure 7-m).

Wendy: I think we might need to add on one more.

I ask how many people we can take with the existing coaches.

Wendy: That's 50, 100, 150, 120... *[trails off]*

CF: 150 and another 50? *[no response]* How many are on these two? *[bracketing first two coaches]*

Wendy: Hundred.

CF: And how many on these two? *[brackets]*

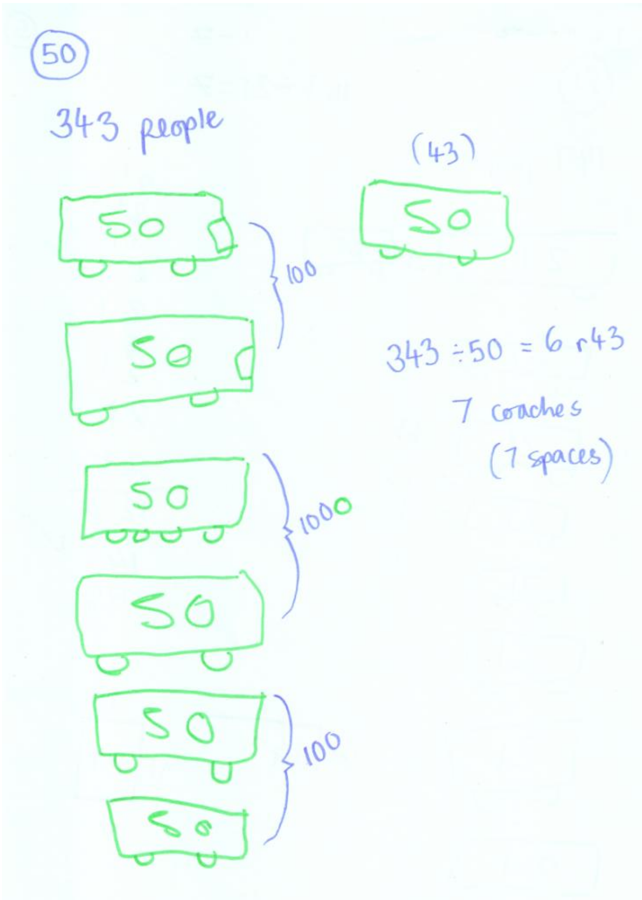


Figure 7-m: 343 people, 50-seater coaches (Wendy and CF)

Wendy: Hundred.

CF: So, altogether you've got? So far?

Wendy: One... something. *[tentatively adds an extra 0 to my 100]*

CF: 100 plus another 100? . . . 100 people here, another 100 people here, how many hundreds do we have?

Wendy: Oh, 200!

Wendy draws another two coaches.

CF: Now how many have we got?

Wendy: 300. Add on one more. *[draws]*

Wendy tells me how many coaches altogether and how many are full, but struggles with the number of people left over, guessing 3 then 4.

CF: We've got 343 there. 300 of them go...

Wendy: Oh, 43!

147 people in 21-seater coaches

Wendy draws four coaches, then pauses (Figure 7-n).

CF: Do you think that'll be enough or do you think we'll need more?

Wendy: I think might need some more. *[draws two more]*

CF: Ok. If we have too many, can always cross them out.

Wendy: Add them all together?

Wendy counts the coaches, and writes six twenty-ones in a column, and adds them to get 126.

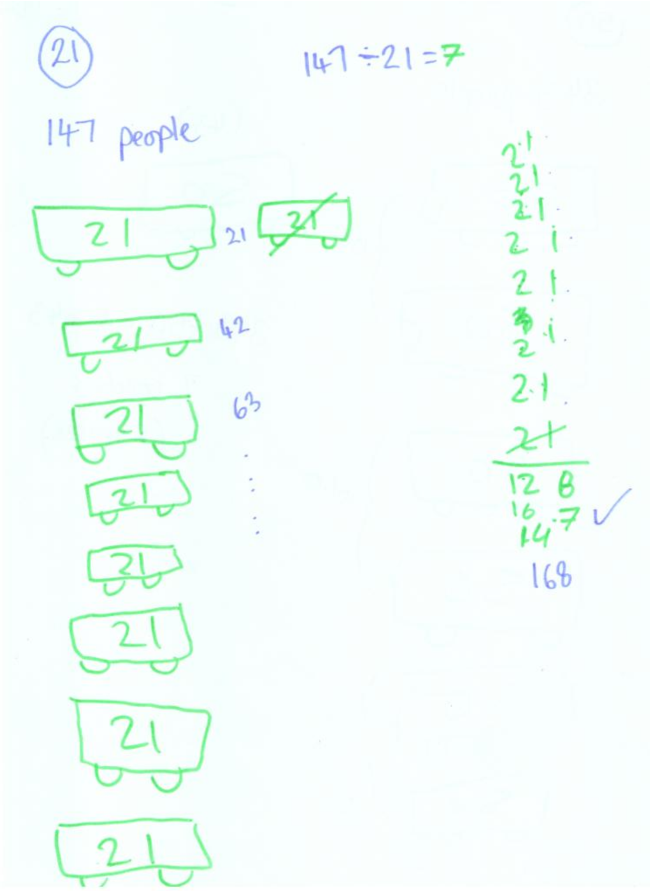


Figure 7-n: 147 people, 21-seater coaches (Wendy and CF)

Wendy: So I need some more.

Wendy draws three more coaches. She adds two more twenty-ones to the addition column, and re-adds it, changing the 6 to an 8, and writing the 16 (from the tens column) below.

Wendy: 16? *[confusedly]* . . . Going to need some more.

CF: Do we? What number have you got there?

Wendy: Oh, one six eight. What? No, I think we've got enough.

CF: We've certainly got enough . . . to transport those 147 people. But do we need all of these?

Wendy: No. Get rid of that one. *[crosses out coach]*

CF: If we remove one of those twenty-ones, how many people can we take?

Wendy crosses out a 21 from the column, and re-adds to get 147. She then goes back and counts her coaches (8). I point out that while her strategy worked, and her addition gave the correct answer, it actually doesn't match her drawing. I mention the possibility of keeping a running total (writing beside coaches).

100 ÷ 20

At the end of the session, I decide to set Wendy some 'bare' division tasks, written symbolically with no scenario. Those with small numbers she worked out using dot arrays. However, when I set this one, Wendy was not keen to draw 100 dots, but unsure how else to do it.

CF: What if I said it was 100 people trying to fit onto 20-seater coaches? Could you do it this way – see how many twenties we need to make up 100?

Wendy counts in tens under her breath, while putting out ten fingers.

Wendy: 10.

CF: You were counting in tens there, which would be fine if they were 10-seater coaches. But they're twenties.

Wendy laughs, re-counts and gets five.

650 ÷ 50

I include the task that followed, even though it was set only in bare numbers, as the strategy was influenced by the preceding scenario.

CF: What if it was – massive number *[writes 650]* – dividing it into fifties?

Wendy writes 650, with 50 and a line beneath (Figure 7-o). Her gestures with the pen indicate she is trying to perform a columns subtraction.

Wendy: Would it be six?

I show her the earlier task 343 ÷ 50 as a reminder of the building-up strategy.

CF: Can you work out how many fifties we need to make 650? *[pause]* Two fifties are?

Wendy: 100. Another hundred – 200.

CF: How many fifties is that?

Wendy: Four fifties.

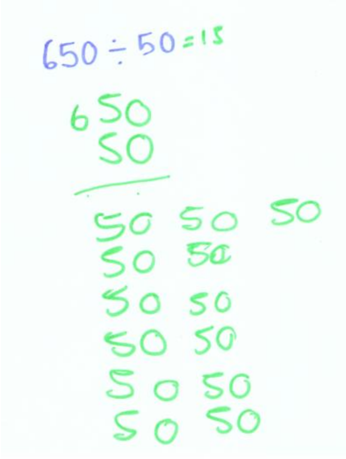


Figure 7-o: 650 ÷ 50 (Wendy)

Wendy spontaneously decides to start writing fifties, to keep a track. After confirming 4 of them are 200, she continues writing pairs of fifties.

Wendy: 300, 400, 500, 600. And I think you might want one more 50. *[counts, with noticeable confidence]* 13.

Session 5

The following took place several months after the main fieldwork period.

36 people in 4-seater taxis (and 30 biscuits shared between 5 people)

Wendy uses an array of tally marks rather than dots (which she had done previously), ringing each row as she goes along (Figure 7-p).

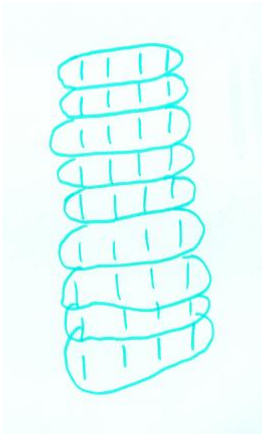


Figure 7-p: 36 people, 4-seater taxis (Wendy)

Figure 7-q: 30 biscuits, 5 people (Wendy)

This may be compared with her representation of the previous (Biscuits) task, where she had constructed a similar tally array, but ringed the columns at the end (Figure 7-q).

105 biscuits shared between 21 people

For these numbers, Wendy had the option of a Coaches scenario, where she might have built up the total from adding twenty-ones; however, she picked the Biscuits option, i.e. partitive division, and carried out a lengthy unit-based strategy (Figure 7-r).

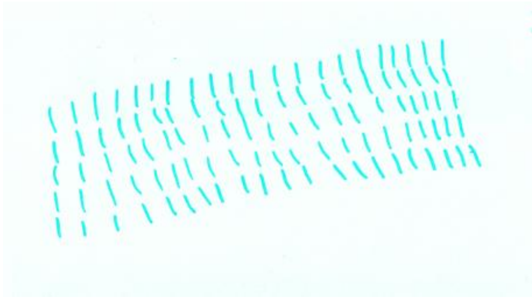


Figure 7-r: 105 biscuits, 21 people (Wendy)

180 ÷ 20, 240 ÷ 20

I set this calculation as a bare division, written in standard symbols.

Wendy: Something tells me it's going to have two zeros on the end of it. I think.

I remind Wendy of the dangers of using half-remembered 'tricks', and the need to think about how and why strategies work.

CF: What kind of problem could we turn this into, to make it easier to think about? If you have something you can picture...

Wendy: I don't know.

CF: I'm going to think of that as 180 tourists being put onto 20-seater coaches. So 180 altogether, being divided into twenties. . . . Does that help at all? *[pause]* No?

I draw two containers, each with 20 in, while talking in the scenario terms of people and coaches.

CF: How many people is that altogether so far?

Wendy: 40.

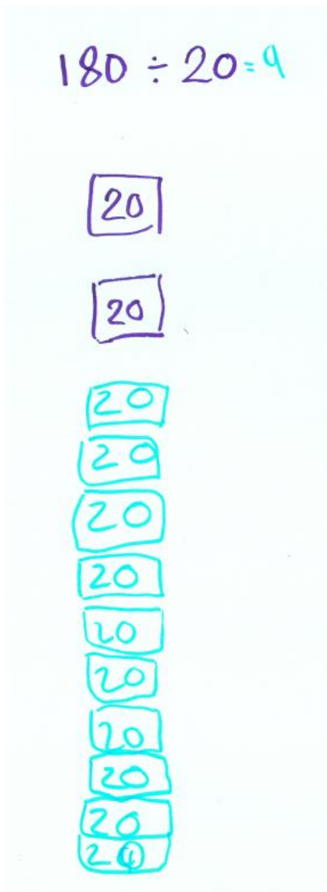


Figure 7-s: 180 people, 20-seater coaches (CF and Wendy)

Wendy continues the representation, continuing to step-count in twenties as she draws/writes, and answers 9 (see Figure 7-s – also used for following task).

CF: Did it help, making it into something real?

W: Yes.

CF: What if it was – *[writes “240 ÷ 20”]*

Wendy extends her previous representation, adding three more twenties, and answering 12.

300 ÷ 25

Again, this was set initially as a bare task, but Wendy and I made occasional references to people and coaches during the working.

Wendy began by drawing containers with twenty-fives in, each time carrying out a separate written addition (Figure 7-t). (In one place she writes 5 for 2, but it does not affect the calculation.) Unusually, she seemed to find this laborious, and it was accompanied by some sighing. After her third addition (25 + 25 + 25 + 25 = 100), I comment, intending to draw her attention to this significant number relationship.

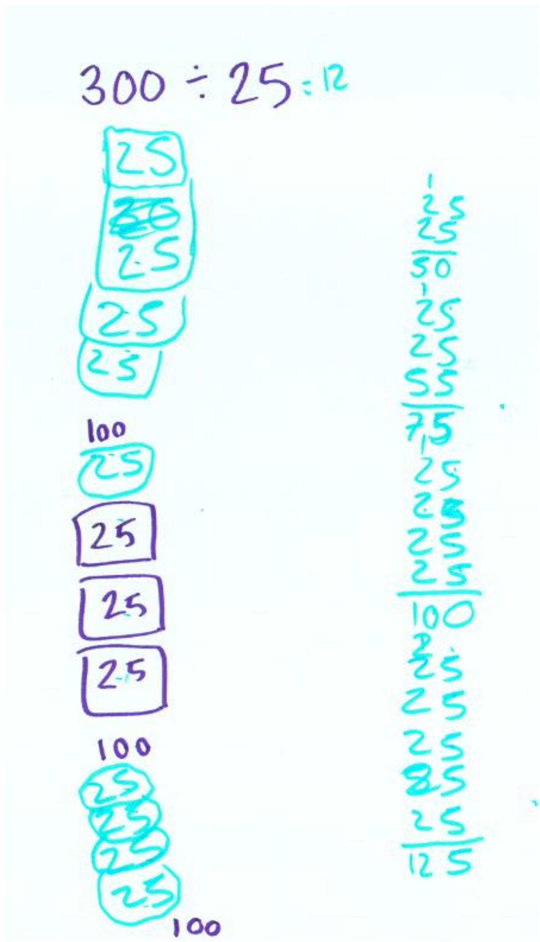


Figure 7-t: 300 people, 25-seater coaches (Wendy and CF)

CF: Four coach-loads of people so far, that’s 100 of them *[writes 100]* . . . How many more coach-loads of people would you need for 200?

Wendy: Three? One?

CF: That’s 100 people sitting there, on four coaches *[pointing]*. Altogether, we need enough for 300 people.

Wendy pores over her additions, thinks, says she is confused, and decides to continue with her previous strategy, drawing another 25-container and carrying out another (longer) addition.

CF: Ok, so that would be the next coach-load of people. . . . Before you go on any further - you could do it like that, and that would work . . . You know how to do it. If that was 100 people *[pointing to first four coaches]*, I'm just going to put in the next three coaches for you *[completes second set of four]*. How many people are there altogether? *[pointing to second four]* It is a calculation you've already done.

Wendy: Oh! Yeah! One hundred!

CF: Right, so that's 100 people *[writes 100] . . .*

Wendy: That'd be 200. We need another four.

Wendy draws four more 25-containers, I write 100, and she counts the containers, answering 12.

CF: You saw it then, didn't you?

W: Yeah.

7.2.4 Applying the framework of analytical aspects

A complete catalogue of Wendy's representational activity during quotitive tasks is provided in Appendix E, using the same framework as for Paula (7.1.4). When looking for change, the aspects *media* and *motion* are inapplicable, as Wendy always used static drawn graphics. *Completeness* is also redundant here, as whether using individual units or containers with figures in, the complete quantity was always represented. As with Paula, the *consistency* aspect is unnecessary here, as on occasions where one strategy was tried, then another, I have broken down the task into separate attempts.

In terms of teacher input I include those comments regarding representation of the arithmetical structure (but not general encouragement, restatements, etc.), and when I interacted with our visuospatial representations (e.g. modelling counting out the people into their vehicles). While the use of numbers which were not exactly divisible into factors was relevant from a teaching point of view, and the discussion around 'empty seats' and 'people remaining' further highlights Wendy's attachment to imagined real-life scenarios, they are only tangentially relevant to the main analytical issue, of representing multiplicative structures and understanding the meaning of division.

Therefore, under *enumeration* I consider only her methods for enumerating the sets of equal groups, rather than noting every instance of her unit-counting a ‘remainder’.

7.2.5 Findings

Wendy’s performance was comparatively consistent throughout my time with her. While she was always focused and engaged on tasks, it is to be expected that there would be times when, for example, she was tired, which could easily be the cause of those few occasions when she became confused on a task type with which she had previously had no difficulties.

What representational and arithmetical strategies does the student use?

Wendy showed a strong continuous preference for working with pen and paper over concrete materials, and occasionally attempted to do without even these, using only fingers or internal representation for the numbers. She employed two main representational-arithmetical strategies, one unitary (all units individually represented, arranged in spatially-structured patterns, enumerated by unit- or group-counting) and one non-unitary (a written number representing each group, organised spatially in one dimension, enumerated by step-counting or addition). The first, which she introduced independently, was used for smaller quantities and the second, which I introduced and she readily adopted, for larger – although there is likely an overlap of magnitudes where either might be chosen.

What do the student’s initial representational-arithmetical strategies say about their particular weaknesses and capabilities?

Wendy wished to obtain answers she was confident were correct, with what she considered a reasonable expenditure of time and effort, and the strategies she used reflect this. In general her representations were highly mathematically functional, drawing and/or writing the minimum content which allowed her to complete the task successfully. I suggest this as evidence of logical and meta-strategic ability, and a very sensible attempt to make the most of her limited computational skills and minimal recall of number facts and relationships.

Wendy produced 2D array representations (or similar patterns) quickly, confidently and accurately. While this initial heavy reliance on countable unitary representations was a weakness, her use of this representation type also highlighted important and useful strengths in creating, recognising and utilising structured visual patterns. Observation of her interactions with dot arrays demonstrated informal, non-declarative, awareness of multiplicative structures, enhanced by the rhythmic nature of her counting. I also consider Wendy's immediate willingness to experiment with representational form and style (such as the presence or absence of different kinds of units and containers), and different forms of enumeration (e.g. step-counting, addition) – depending on her confidence with the particular numbers involved – to be itself a valuable component of metarepresentational competence.

Wendy's performance on the Taxis task 45 ÷ 7 in Session 3 was particularly telling on two counts. Firstly, she started constructing the array without keeping a running count of the total number of people/dots, and had to go back and re-count before continuing as normal. This is the only time she did this while using a unitary representation; however, it demonstrates a (temporary) lack of concern for the running total, and a tendency to estimate the number of groups required. These issues surfaced again in later tasks (using a different representation type). Secondly, on keeping a running total, she nevertheless overran – perhaps due to my picking a non-multiple dividend for the first time, and thus the target number not being at the end of a row.

Of particular note is Wendy's continued strong preference for scenario tasks, even when they were obviously unrealistic (5.4.2.1). While her sensible comments about the comparative expense of different sizes of taxi and aeroplane demonstrate a real-world awareness of money and pricing that is often considered an important part of being 'numerate', this tendency to cling to the scenario aspect of the calculations could also indicate an unwillingness to engage in abstraction, and demonstrates a profound need for something extra-mathematical to grasp and cling on to. Numerical relationships, by themselves meaningless, were given meaning when they became numbers of people (etc.). This finding would be no surprise to an Early Years teacher, who has heard many exasperated children ask "but three <u>what</u>?" (for example), but this strength of attachment to imagining numbers <u>of things</u> may be surprising in a Year 9 student. However, note that Wendy's need for extra-mathematical detail in the narrative for arithmetical tasks does not imply an equivalent need for non-mathematically functional detail in visuospatial representations.

There are a couple more points of note on the subject of Wendy's general numeracy. She occasionally made verbal slips, meaning one number but saying another: on working out the empty spaces (in 45 ÷ 25) she stated she had counted from 42, but had clearly counted from 45; then, later (in 391 ÷ 50), she substitutes 389 for 399 without noticing, and without it affecting her answer. In carrying out addition in columns, her misplacing of 1 and 0 was not simple digit reversal (common with dyslexia) but indicative of insufficient understanding of the workings of the place value system. This was confirmed by her needing to unit-count from 600 to 612, rather than immediately seeing the difference.

How do the student's individual mathematical functionings change over time, in response to tuition based around tailored, flexible, scenario tasks?

During the first session, Wendy adopted the representational forms I suggested, but also tried different variations on them. In subsequent sessions, she appeared to have forgotten about the option of using written numbers in containers (vehicles) for larger quantities, but recalled and used it confidently once reminded. As each session progressed, she experimented with representational variations, e.g. using containers to group units, dots versus tally marks, using fingers to keep track of the number of groups.

The main intra-session progression was Wendy's tackling of increasingly large dividends. At the end of each session, I briefly recapped what we had been working on, and pointed out the calculations she had completed – to her apparent surprise and pleasure. While division tasks involving two- and three-digit numbers – particularly if written in formal symbolic notation – would very likely have paralysed her if presented at the start of a session, repeated successes and gradual increases maintained the momentum of her confidence, allowing her to forget for a time her dislike, fear, and negative beliefs about her ability to work with larger quantities.

Wendy started the tuition sessions secure in her dot array strategy for partitive division. Although it took some time for her to see that it could be equally useful for quotitive division (and multiplication), once grasped, this was clearly added to her strategic toolkit for future sessions. Although not addressed directly here, the triple function of this visuospatial representation has implications for deepening her understanding of the relationship between multiplication and division, and the commutative principle.

During the tuition period, Wendy learned to use a building-up, additive strategy to construct larger quantities from larger equal-sized groups, which no longer required individual units, but combined number symbols with pictorial/iconic containers (coaches, etc.) linking to the scenario narrative. Whilst she did require a reminder of this in the final session, overall she progressed to using the strategy with increasing confidence. The fact that she chose to continue using drawings side by side with the numerical calculation indicates they must have been serving some continuing purpose for her. Unlike some other students, Wendy did not doodle, and showed almost no inclination for decorational drawing, so I suggest that in these longer, multi-stage tasks, the drawn element, while minimal, served to remind her of the scenario, and thus of what it was she was actually trying to work out. However, as she generally showed a tendency to minimalism in her representational strategies, it may be expected that given more time, she would cease to feel a need to duplicate the numbers (once in container drawings, once in addition calculations), and indeed, in the tasks involving 'easier' numbers such as multiples of 20 and 50, there is evidence of this beginning to happen.

With both her preferred representation types, Wendy initially showed a tendency to use the arithmetical strategy of estimating the number of groups (usually a very low estimate), counting, adjusting (by adding or removing groups), re-counting, and repeating until the desired total was achieved. By the end of the tuition period she was independently using the more efficient strategy of keeping a running total as she added each group.

Lastly, there is the issue of Wendy's attachment to the ‘real-life’, human details of the task scenarios. While she was able to carry out some 'bare' tasks with smaller quantities, using unitary representational strategies, this was not yet the case for tasks with larger quantities expressed in symbols. However, the building-up strategy was new to her, so it is not surprising that learning to adapt it in this way might take more time, and she was at least starting to reduce the amount of scenario detail, while I started to introduce more formal language. Encouragingly, Wendy also appeared to appreciate the tactic of taking a bare task and making it comprehensible through recasting it as a familiar scenario – although as this discussion occurred at the end of the tuition process, there was not the opportunity to see her try this independently.

7.2.6 Discussion

Analysis of Paula’s work helped deconstruct partitive division on a fundamental level; what it means to ‘share’ a number. While Wendy worked at a considerably more advanced level and with greater speed, similarly detailed analysis of her work shows up the conceptual fault lines and fissures in her prior learning; for example, that she had been taught to ‘perform division’ without explanation or exploration of the arithmetical structure underlying “how many x go into y?”. This instigated the first key instance of change following teacher input, via my appropriation of the array representation she already used for partitive division, for use with quotitive division and multiplication. This multi-functionality of the same visual form can thus create a powerful link between what were previously thought of as quite separate kinds of calculation.

While some adjustment may be made for progress since assessment, there is still a huge discrepancy between the formal judgements that had been made on Wendy’s abilities and potential, and the clever maximisation of cognitive resources captured in my observations. Forced to avoid received strategies that rely on memorisation, she analysed tasks to work out alternative strategies relying instead on sequence, pattern, and a realistic self-assessment of the level of arithmetic she could reliably manage (counting and addition). She showed great interest in the numerical structures underlying arithmetical processes, and in suggestions for their use in solving different types of task. Moreover, there was evidence these strategies stood a chance of being remembered better than those which had previously been ineffectively rote-learned. These are not the behaviours of a child who "lacks the ability to retain and process academic subjects that require logical thinking, analysis, sequencing, rationalising and accuracy". However, it took my focused individual investigation and a flexible mixture of tuition, discussion and assessment to gain a fair picture of the nature of Wendy’s mathematical thinking (and this in only one area of the syllabus). If one accepted the statements in her Statement at face value, to expend this time and effort on her would seem irrational. This is one of the factors leading to students with difficulties in mathematics being limited from reaching their potential.

Alongside the issue of understanding the multiplicative structure within quotitive division, there is the actual process of carrying it out. In this case guidance was necessary for Wendy’s realisation, through experience, of the advantage of keeping a running total when constructing a dividend. Before this, it had seemed perfectly reasonable to her to repeatedly estimate then adjust the number of groups; in this sense the role of running totals is a leap in procedural systematicity equivalent to the introduction of dealing for Paula. The third key input from me, which instigated massive strategic change for Wendy, was the introduction of a non-unitary representational form that replaced the unit marks with symbols (while keeping container elements to maintain some visual continuity). This prompted arithmetical change (from counting) towards repeated-addition enumeration.

Is this evidence that Wendy was not just encountering better methods, but actually moving towards more symbolic thinking in arithmetic? Yes. Her mathematical journey has been and will be a slow and complicated one, with end point unknown, but this short section of it shows unambiguous substantive change. Initially Wendy showed a strong attachment to representing individual units (i.e. a simple iconic form). However, our co-created array-container blend turned out to be a significant bridge between quantities engaged with as iconic units in spatial arrangements, and as numeric symbols. Enclosing equal groups of dots in rings visually transformed the groups so a particular aspect of the structure was made salient; the ‘building blocks’ making up the total were now a set of contained equal groups rather than individual units; visually, these containers became the new ‘units’. This was vital for the significant cognitive leap of replacing a container with a number of dots inside by a container with a number symbol inside.

In fact, Wendy’s stacks of number-containers were already becoming less pictorial, and the next stage – were I to have had further tuition sessions with this student – would have been to encourage the removal of the containers altogether, leaving a vertical stack of numbers. I would have then drawn her attention to their looking very similar to the common form of addition in columns. Now, although Wendy had clearly come into contact with common arithmetical forms and methods, she had not yet independently made the connection between these and her own developing symbolic system. Given time, space and encouragement, her learning trajectory implies this might happen next.

8 THE KEY REPRESENTATION TYPES

8.1 Numbers as containers, numbers as arrays

In the previous two sections I focused on the ‘slices’ of data generated from two selected tasks and two selected students. I now widen the analysis to encompass my entire collection of visuospatial representations, from all students and all tasks, and look for indicators of the emergence and development of multiplicative structure. For this, a different organisational approach to the data is required.

In 4.3 I theorised *Numbers as Containers* and *Numbers as Arrays* as two fundamental ways students can come to understand and work with multiplicative structures, and thus when students required support on tasks, my prompts and demonstrations were based on these types of representation. Additionally, I designed a particular form of *array-container blend* (4.3.5), which I introduced to all students, and which some adapted for use in subsequent tasks. Some students independently introduced *number containers* (as seen in 7.2) – drawn container forms combined with number symbols – and these proved effective enough for me subsequently to incorporate them into my teacher support on tasks involving larger numbers.

These key representation types were theoretically generated, and although all had certainly been observed in students’ independent work, their fitness as a set of clearly-delineated analytical categories required testing. I went through my set of visuospatial representations (including both independent and co-created representations, but excluding the few that I created during explanation or demonstration which contain no student markings), and attempted to assign them to the group Unit containers, Unit arrays, Array-container blends, Number containers, or Other. By refining my criteria for inclusion (see individual sections below), it proved possible to filter the great majority of data into one of the four main types, with representations fitting none of the criteria a minority, and those fitting the criteria for two categories (e.g. Figure 8-ff) very rare. Complete sets of each of the four representation types are reproduced in Appendix F.

It is important to restate that I did not demand students adopted these representation types, or discourage their alternative representations. Recall that students were always given encouragement and generous time to work independently on tasks, in their own preferred ways. It was a methodological requirement for me to keep my support to the minimum that allowed students to move forwards on a task, although in some cases that minimum level of support was necessarily a significant involvement.

'Other' representational strategies employed by students, not involving either containers or arrays, included simple, unitary representations (for multiplication-based tasks) with no visible grouping, units arranged either one-dimensionally (i.e. in an unbroken line) or with no spatial organisation at all. There were also examples (for division-based tasks) in which the cubes/dots/etc. were separated by a gap but no container elements; there are similarities in the way that groups of units with and without delineated container markings are used by students, but as Paula made considerable use of both of these, non-contained groups have been already been discussed at some length in 7.1. Similarly, a few non-array spatial patterns were produced, e.g. Figure 7-l. Meanwhile, certain students used only their fingers for visuospatial representation of tasks (involving smaller numbers), and the most arithmetically ambitious of my students attempted purely symbolic calculations (with varied success in both recall and execution of procedures). The remaining unclassified representations, including an instance of non-unitary use of tally marks (by Kieran) and several of Leo and Vince's highly-decorative creations, may be found under 'Miscellaneous' in Appendix F.

Research questions addressed in this section are:

- What types of representational strategy do the students use, and how do they use them?
- What relationships can be found between representation type, scenario, calculation and multiplicative understanding?

To answer these, I first take each of the four key representation types in turn, and give a descriptive analysis of each dataset produced by the filtering process, comparing, contrasting, and highlighting both individual representations and sequences of particular interest. I then draw out patterns which shed light on the relationships between the representation types observed and the scenarios within which tasks were set, the calculations used, and my diagnoses of students' multiplicative understanding.

To interpret the roles played by the different representations within tasks, and as part of longer patterns of student progress, I use the set of analytical aspects from the framework developed in Chapters 6-7 (see 7.1.4: Table 7-a for full definitions). Rather than address each individually, they have been grouped for ease of discussion thus:

(1) Visual elements: *media*, *mode*, *resemblance*, *unitariness, consistency*
(2) Spatial relationships: *motion, spatial structuring*
(3) *Enumeration*
(4) *Successes* and *errors*

Observations regarding teacher involvement (verbal and/or visuospatial prompts) are threaded throughout, where relevant. The aspect *consistency* could also theoretically be applied to motion, spatial structuring and enumeration, but in this dataset related particularly to the visual elements, so is grouped with them. The aspect *completeness* is omitted, as all examples of these four representation types included the full number of visual marks for each total quantity.

I include examples from all my students except Paula (from whom there have already been many, in 7.1.3) and Oscar, the one student who never used any of these representation types (although I introduced them in demonstrations and explanations). With the exception of the Initial Assessment (where he drew rose bushes and clothing items), he used only fingers and written symbols – although not necessarily conventionally.

8.2 Types of representational strategy used

8.2.1 Unit containers

Criteria: Groups of two or more units enclosed by visible boundaries. Includes representations where units are aligned in rows and/or columns, but these do not represent divisor/quotient or multiplier/multiplicand.

Overall, this was the most common type of representation – eleven of the cohort (all apart from Oscar and Ellis) chose to draw unit containers at some point while working on a task. Seven of them introduced containers independently during the Initial Assessment, followed by two more in early tuition tasks. Paula and Harvey both made considerable use of containers, but only after my direct suggestion and co-creation of examples. Some students choose to use container representations much more frequently than others. I have previously stated that Wendy showed a strong preference for array-based forms over containers; so did several others. However, Harvey, Kieran, Vince, and, of course, Paula used unit containers heavily throughout.

This was the representation type involving the least teacher involvement (with the exception of Paula). I only suggested it explicitly when students were completely stuck on a task, and of those, few required verbal support throughout the process. If visuospatial support was also required, I drew a set of circles (‘plates’) for Biscuits, and one or more rectangles (sometimes with an indication of wheels) for Taxis, followed by, if necessary, distributing the first few units.

Some students, once comfortable with and reliably using unit containers, began independently to experiment with other forms; in other cases I actively prompted change. Depending on the individual, this involved either nudges towards aligning the units (i.e. array-container blend), or replacing them with symbols (i.e. number containers). The benefits of these two paths of progression are discussed in 8.2.3 and 8.2.4.

8.2.1.1 Visual elements

Container representations were for the most part completely drawn, but Leo and Vince also sometimes made use of mixed-mode, mixed-media representations with cubes or other physical units placed in drawn containers, and I have previously described Paula’s partial transition from concrete to drawn units.

Levels of resemblance varied widely. While it might be argued that a large circle does indeed visually resemble a plate, and the small circles within it biscuits, circular containers and small circle/dot-shaped units were also a common feature of container representations in other scenarios and bare tasks. Tally units (i.e. multiple figure ones) were also popular. For many students, while using visuospatial representation made tasks possible, all that was necessary was for the number of groups and number within each group to be countable – it was not important that the drawing resemble the scenario in anything more than structure. However, for others, mathematical functionality was not enough, and they chose to add additional details and variations, reflecting narrative elements which helped connect the calculation to the scenario.

As an example of the former, in Tasha’s two representations (Figure 8-a), it is not possible to determine from the drawing that the first is one person’s share of a quantity of biscuits, while the second is a group of people travelling in a five-seater taxi. Although using one of the most basic representational types, with individually countable units, Tasha’s re-use of the same visuospatial form for the two different types of division (set in two different scenarios) is relevant to the development of her understanding of the relationship between them. Of course, the process of creating the two (complete) representations was not the same; in the partitive case all containers were drawn first and the units ‘dealt’ into them, whereas in the quotitive case each was drawn and ‘filled’ in sequence. Nevertheless, these student-owned identical visuospatial structures provided an excellent starting point for discussion of multiplicative relationships and their notation.

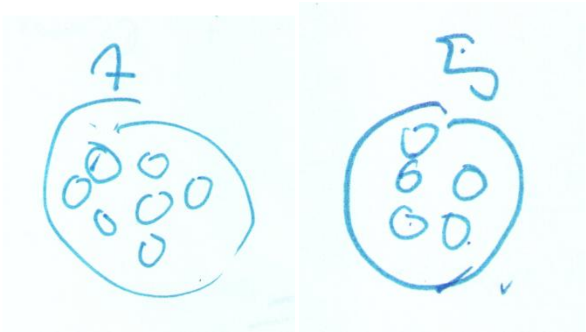


Figure 8-a: A share of biscuits, people in a taxi (Tasha, partial images)

In comparison, Vince’s Biscuits representations look very like Tasha’s, but most of his vehicles include non-mathematically functional details (Figure 8-b, Figure 8-c, Figure 8-d). In (d), he decided that my drawing (rectangular containers and units) was too minimal, and elaborated on it himself.

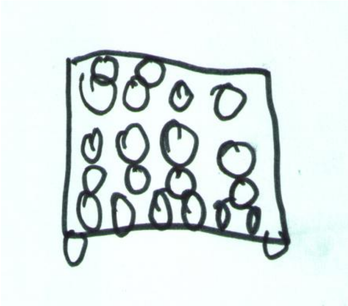


Figure 8-b: Coach (Vince)

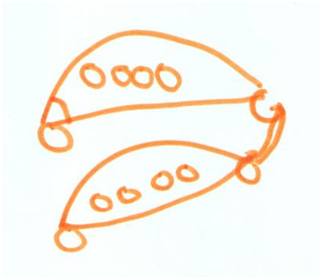


Figure 8-c: Taxis (Vince)

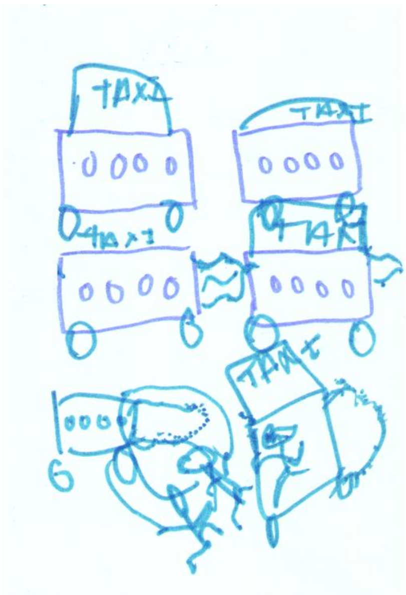


Figure 8-d: Taxis (CF and Vince)

I have argued earlier that time spent on detail-heavy drawings should not be considered wasted, as the decorative elements may fulfil a valuable function in enabling the student to process and work on tasks. However, multiplicative structures involving repetition, and repeated drawing of the same visual elements may become tiresome: this is actually a pedagogical advantage in that the students' drawings become more minimal (when, for instance, they can no longer be bothered to draw wheels, or bodies for their stick people), and they realise they no longer need that much detail. Figure 8-e shows George's only container representation (also his only instance of pictorial detail) – a style which he immediately abandoned as unnecessary, stripping it down and transitioning towards an array format.

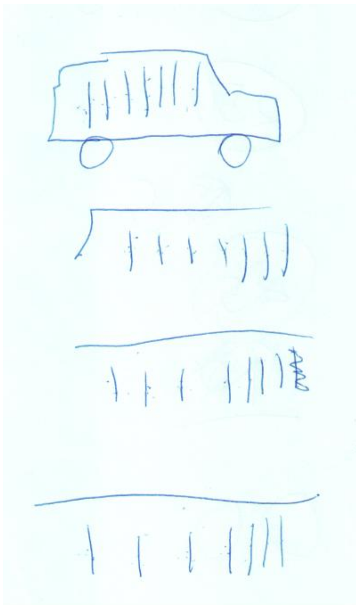


Figure 8-e: Taxis (George)

Likewise, my vehicle drawing (Figure 8-f) enabled Wendy to make sense of the three-dimensional multiplicative structure (bottles in boxes in vans), but she then extracted the units and their spatial relationships, discarding containers and pictorial details.

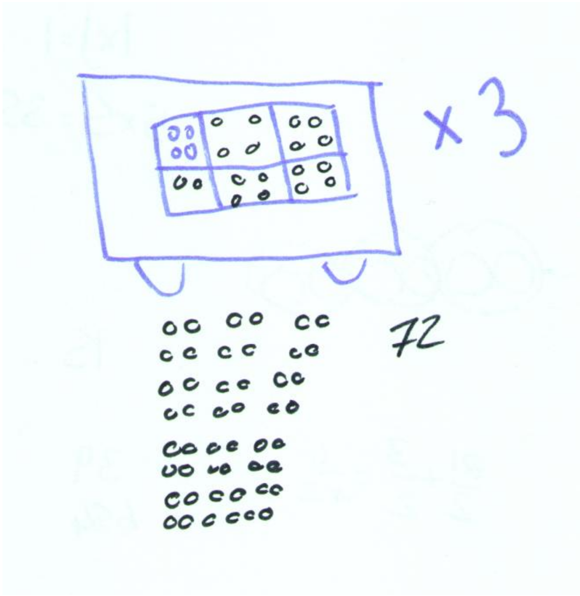


Figure 8-f: Discarding container elements (Wendy)

In some cases students alleviated the temporary boredom of repetitive drawing through superficial changes, for example of colour or unit shape; this is not useless from a teaching/learning point of view, as it reinforces the fact that the superficial appearance of representational elements is irrelevant to calculation. However, it can become problematic when the level of extraneous detail is detrimental to mathematical functionality (e.g. in Figure 8-g).

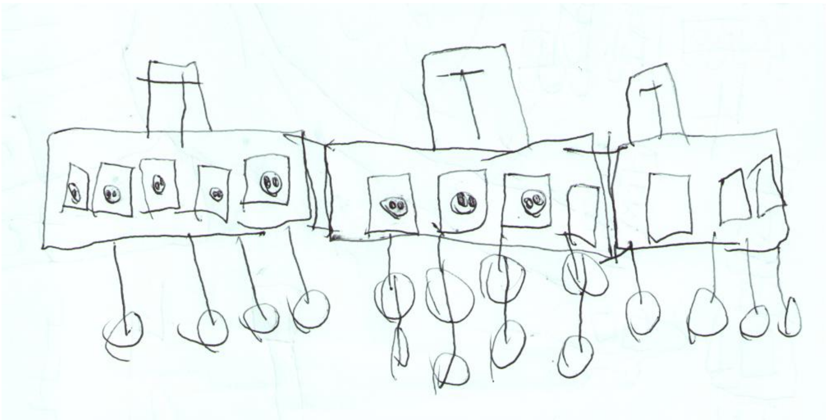


Figure 8-g: Incorporating decorative elements (Leo)

This same balance may be struck with students giving tasks a strong narrative element. I have discussed the importance of real-life considerations for Wendy; Leo also chose to bring in additional human elements to the scenarios – with mixed success. In Figure 8-h he familiarised a sharing scenario by labelling the plates with the names of two friends and himself: this enabled him to understand my request and attempt the task, but it also enabled him to decide to give himself two extra imaginary biscuits (as his tuition was the period before lunch, and he was hungry).

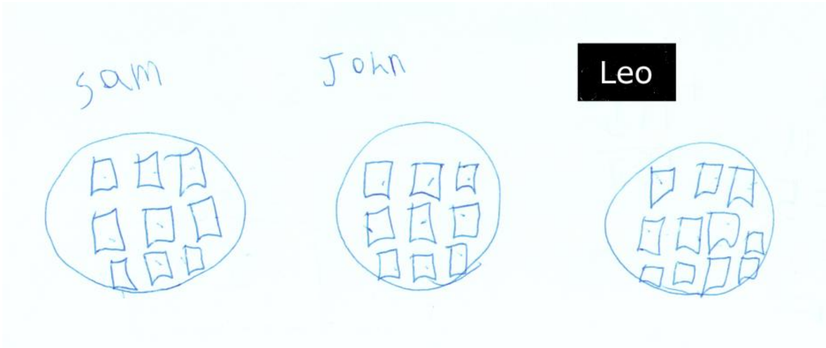


Figure 8-h: Extra biscuits (Leo)

8.2.1.2 Spatial relationships

The only container representations to contain actual motion were those with concrete elements, which could be moved from group to group (for students using an estimate-adjust strategy), or dealt out. However, drawn partitive division representations could also be said to contain a particular, organised, kind of motion if they were created in a structured, rhythmic way – by the drawn equivalent of dealing. Making this motion explicit was very important to Paula's progress, and while it was most clearly demonstrated with her, it is reasonable to presume it at least somewhat important to students whose difficulties are not quite as severe.

If dealing may be said to have a rhythm that irregular forms of sharing do not, then may something similar be said for container representations and quotitive division? Indeed. In Tasha's Taxis drawing above, there was rhythm of movement corresponding to the spatial structure: *[ring]* 1, 2, 3, 4, 5 *[ring]* 6, 7, 8, 9, 10 *[ring]* 11 . . . etc. In contrast, this was not present in Kieran's Taxis representations (Figure 8-i). In each case, he began by marking out the total number of people in rows of ten, then going through and ringing them in groups of the required size; in two of the examples he allowed the rings to extend over more than one row, in others he crossed out the remainder at the end of each row and added them to the next one. While his strategy produced correct answers, the ongoing adjustments broke any rhythm he might have built up in ringing fours, and so did not reinforce the multiples of four as a verbal sequence.

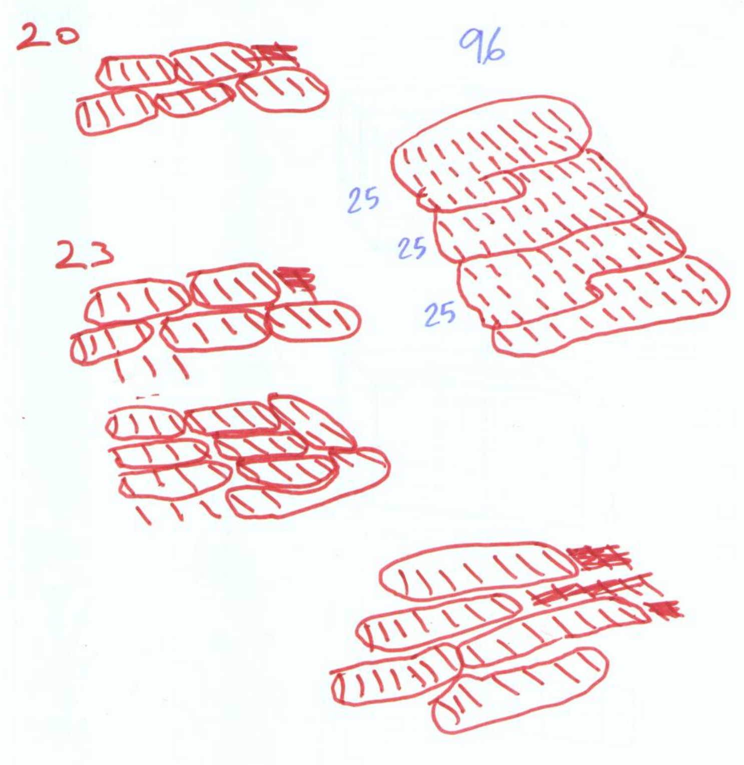


Figure 8-i: Selection of containers (Kieran)

The images above display the favourite representational strategy of this particular student, which he used with reasonable efficiency and accuracy for quotitive scenario and bare division tasks, and attempted to use (with considerably less efficiency) for partitive scenarios too. Note that Kieran's representations have some visual similarity to arrays (and array-container blends), so it was in these directions that I focused his individual tuition.

8.2.1.3 Enumeration

The students using container representations were reliant (at least on those occasions, if not always) on counting individual units, and all examples except one are unitary. The counting varied in that it was sometimes verbalised aloud, sometimes not, sometimes unstructured and sometimes grouped. There was increasing rhythmicity of verbal count during longer tasks for some students (those tending to be more rhythmic in their physical movements), but not others. Thus counting out or re-counting units grouped in containers can certainly encourage grouped counting, but if there are irregularities in the spacing and layout, it may still lack the regularity of rhythm which aids the move to step-counting.

The one non-unitary containers representation was produced by Wendy in Tuition 4. After an (unplanned) discussion on the base-10 system and the underlying reasons for the ‘tricks’ she had been taught regarding zeros, she used containers to divide large numbers by 2, with each mark representing 10 (Figure 8-j). In the first calculation (purple), I ‘dealt’ tally marks between the two circles, while she step-counted aloud in tens; the second (blue) she carried out by herself.

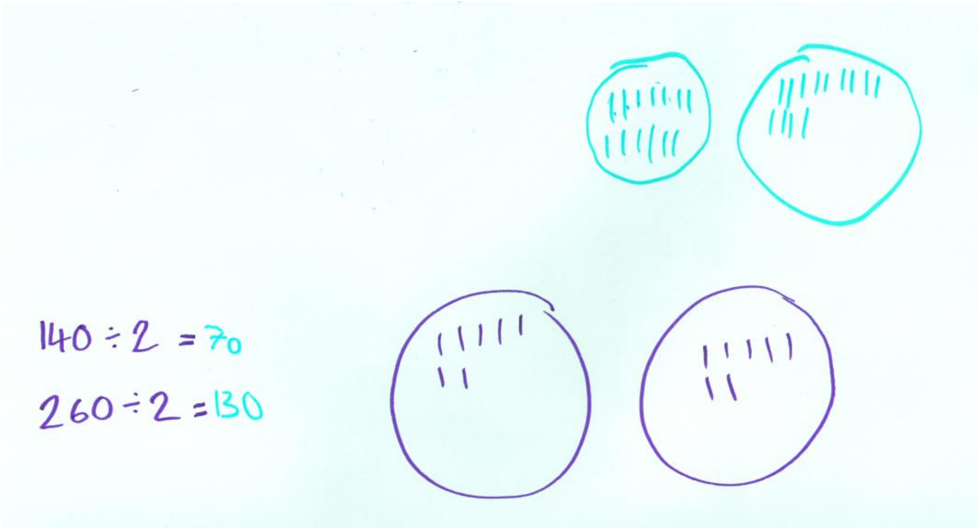


Figure 8-j: Division with one mark representing 10 (CF, Wendy)

8.2.1.4 Successes and errors

All students were able to use container representations successfully, and even the students with the severest difficulties began to use them independently. While the arithmetically more capable students progressed to more structured and/or symbolic representations, containers remained a backup option to which they could all return when needed. This is not to say that container use was error-free. I have analysed in detail Paula’s struggle with the concept of equal shares; I now look briefly at the error patterns of the three other students who obtained incorrect answers from unit containers.

Leo has already been mentioned on several occasions due to the idiosyncrasies of his mathematical behaviour. The evidence suggests he was well aware of the rules underlying division (e.g. equal groups) but ignored them when some extra-mathematical factor took over, for example, imagining a plate of appealing food (Figure 8-h), producing elaborate drawings of multiple vehicles (e.g. Figure 8-g), or simply the artistic rather than the mathematical function of arrays of dots and attractively looping coloured rings.

Harvey also understood the objectives in both partitive and quotitive scenarios, and could use container representations independently, but was sometimes let down by what appear to be sensorimotor difficulties. His markings of the paper are characterised by uneven spacing, overlapping elements, and general lack of clarity (e.g. Figure 8-k). However, on examining his representations carefully, the majority are in fact correct, and were clearly intelligible to him, if not necessarily to others.

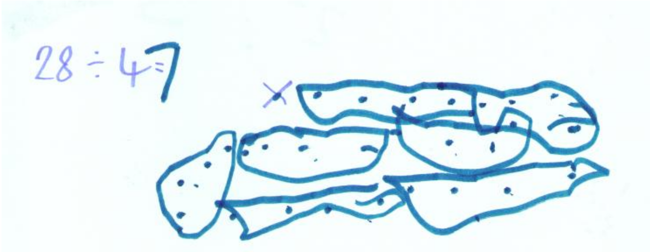


Figure 8-k: 28 ÷ 4 (Harvey)

Vince appears to combine the difficulties of Paula, Leo and Harvey, albeit each to a lesser extent. Early on, he distributed grossly unequal piles of cubes; he could get caught up in decorative detail; and confusing irregularities of drawing stemmed from his weak pen control.

While these factors affected the students' effective use of this representation type, they are factors that would also affect their use of any pen-and paper strategies, and thus do not constitute an argument against its use, for as long as it remained their best way of engaging with multiplicative structures.

8.2.2 Unit arrays

Criteria: Groups of two or more units aligned in rows and columns, where number of units in the rows/columns represents divisor/quotient or multiplier/multiplicand.

Both plain unit arrays and those with added rings (i.e. array-container blends, 8.2.3) were used frequently, with around half as many examples of each as container representations; however, while the majority of array-container blends involved considerable teacher input, the majority of array representations were produced independently. Nine of the cohort chose to draw their own unit arrays at some point while working on a task, and this format was particularly popular with three (including Wendy). Jenny also began with a strong preference for array-based forms, but once she had seen an array-container blend, switched almost exclusively to that type. Of the others, some used arrays alongside a mixture of other forms, while some only tried them briefly after a co-created example or demonstration from me. Apart from demonstrations, I did not intervene in students' choice to use, or not use, array representations. The few occasions I suggested or demonstrated them took place when students were completely 'stuck', and I judged it worthwhile experimenting with an alternative to container representations.

8.2.2.1 Visual elements

All examples of student-produced arrays were drawn, and none constructed with cubes or any other materials. This may be somewhat surprising, as it is easy and visually effective to produce cube arrays – as indeed I did when demonstrating (e.g. Figure 8-p, below). However, in general it was the weakest students who made greatest use of concrete media, and this group also tended to prefer container representations over the more spatially structured array form.

In container representations, much of the scope for resemblance-increasing detail derived from the taxis, coaches and aeroplanes in quotitive division scenarios, and to a lesser extent the people inside them; in array representations of these scenarios, however, there were no containers to decorate, and none of the students made their units resemble people. I had observed, though, that the addition of resemblance-increasing details, even when not part of the actual calculation, could 'anchor' students to the task scenario. Thus, when Harvey was unable to make a start on sharing 21 biscuits between three people (in Session 4), I prompted him by drawing three stick people and labelling them with his initials, mine, and his co-tutee's (Figure 8-l, identification redacted), and carrying out the first round of the dealing process. I was interested to see whether he would accept this combination of pictorial detail and plain unit array, be able to continue it, stop at the appropriate point, and enumerate correctly (in particular, avoiding confusing the stick people with units, and counting them too). He did: this transitional representational strategy was successful, and enabled to him to complete that task, and then the following one (Figure 8-m) in the same way but unaided. Note that in the second of these images, Harvey's units are inconsistent: he changes from tally marks to dots, then back again. The dots are not well aligned, and there are also extraneous dots made during the re-count, but none of these things prevented him from reaching a correct answer with this strategy.

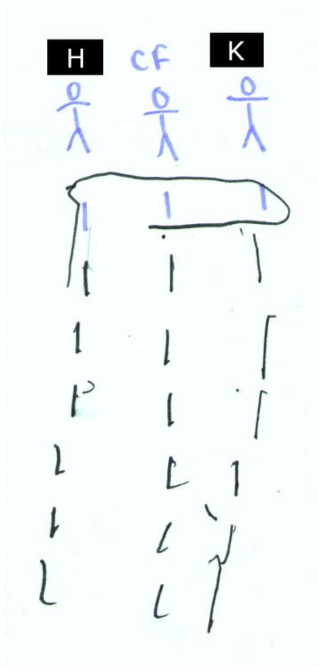


Figure 8-l: 21 biscuits, 3 people (CF and Harvey)

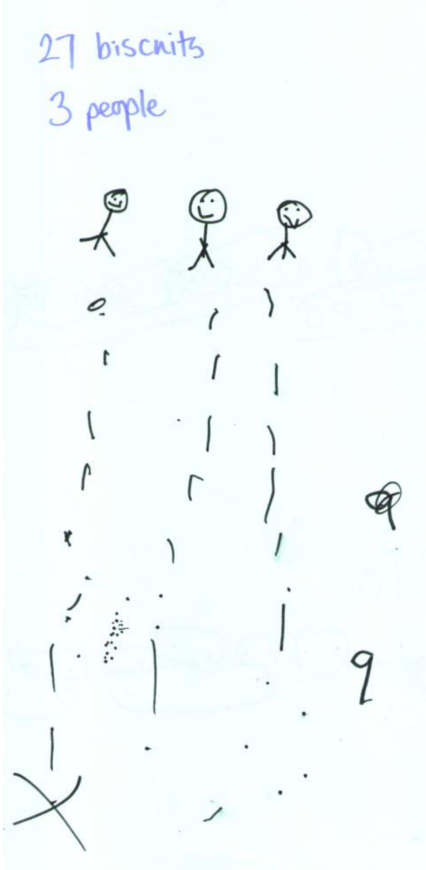


Figure 8-m: 27 biscuits, 3 people (Harvey)

8.2.2.2 Spatial relationships

I described earlier the rhythmic motion involved in the creation of container arrays by a drawn analogue of 'dealing'; this aspect was even more pronounced in array representations, due to the comparatively even, regular spacing of units. Visually, the separation of the total quantity into groups is not as intuitive as when those groups are enclosed in visible containing boundaries, but once the aligned rows or columns of an array are perceived as subgroups, it is easier to visually compare them for equality.

Moreover, with a shift of perspective between vertical and horizontal structure, it may be seen that both rows and columns are formed of a set of equal groups, which underlies the commutative principle. This was seen more readily by some students than others; compare the responses of two students with an array-preference on being asked to work out 28 biscuits shared between four people followed by "and if those same 28 were shared between seven people?" (Figure 8-n and Figure 8-o). Both students quickly and efficiently worked out the first question (one using rows of four, the other columns); however, Wendy did not need to do further calculation for the second (effectively using the commutative principle as a derived fact strategy), while George repeated his previous process, and did not see the connection between the two until it was pointed out.

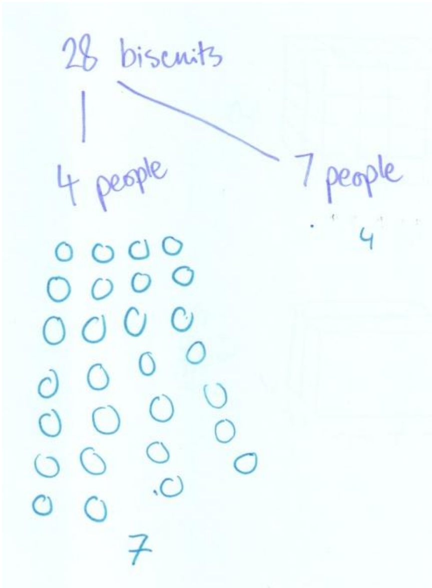


Figure 8-n: 28 biscuits, 4 then 7 people (Wendy)

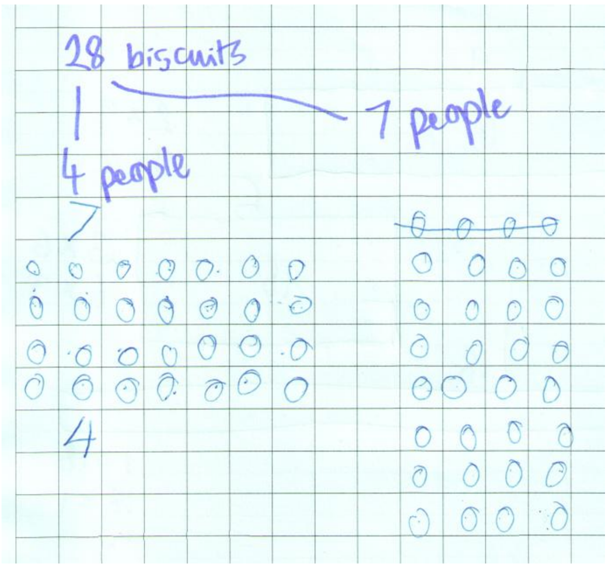


Figure 8-o: 28 biscuits, 4 then 7 people (George)

Understanding of the commutative principle had been one of the key concepts in my tuition plan. While the above activity was sufficient for some (including Wendy and George) to grasp the principle, others such as Paula, who were barely grasping the fundamentals of division, required considerably more concentrated demonstration and interaction to begin to comprehend the commutative nature of multiplicative structures. The array-container blend (see 8.2.3) was designed for this purpose, but I also used cube arrays (e.g. Figure

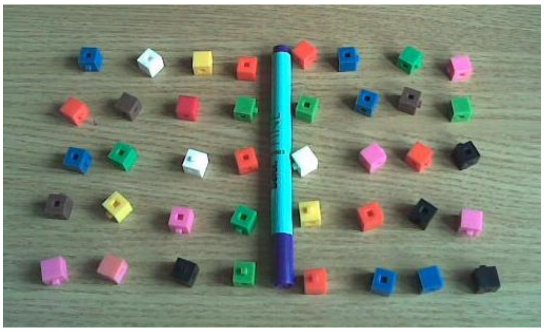


Figure 8-p: Loose cube array (CF)

8-p), moving the rows and columns back and forth (with pen or ruler) to emphasise the two possible groupings which combined to create the spatial – and arithmetical – structure (in this case, eight columns of five = five rows of eight = 40).

8.2.2.3 Enumeration

All unit array representations were unitary, as the students who used them still needed to count out all the units to reach an answer. However, while basic unit-counting was common, the regularity of the structures encouraged rhythmic counting more strongly than container representations.

I have mentioned that some students were particularly attached to using unit arrays, and while it is an extremely useful representation type, it makes for an unwieldy strategy when the numbers increase. When students seemed 'stuck' at the counting stage (e.g. Wendy constructing a 104 ÷ 21 array, Figure 8-q), I intervened to nudge them toward a non-unit-based form, i.e. number containers.

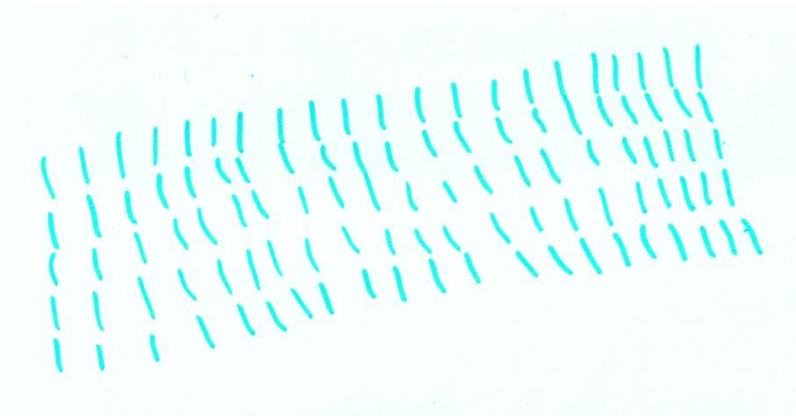


Figure 8-q: 104 ÷ 21 as unit array (Wendy)

8.2.2.4 Successes and errors

It is not useful to compare the actual numbers of times students used array representations, etc., as due to the nature of the tuition, individuals worked at very different paces and were not given identical sets of tasks. However, the array representations created by Wendy, George, Danny, Kieran and Jenny give an overall impression of confident, efficient, reliable usage that those created by Sidney, Harvey and Vince do not. It is worth looking for any descriptors the former group have in common, and differences from the latter group. (I exclude Paula because of the unusually high level of support in her one example.)

Firstly, it is not the case that the more successful students' arrays contain no errors: they sometimes begin dealing with the wrong divisor (i.e. incorrect row/column length), overrun past the dividend, or lose count, necessitating a re-count and correction. These errors, however, were easily spotted by students and fairly easily self-corrected. It is also not the case that their arrays are the most perfectly rectangular in shape; rows and columns sometimes bend around or spread out (e.g. Figure 8-n, above). In most cases, though, there is enough regularity and clarity of pattern that the numerical structure is easy to 'read', and a row/column with too many or few units would be easy to spot. Initial clarity is sometimes obscured during the answer-checking process, when a student uses single dots with the pen for units, and touches the paper again with the pen while re-counting, e.g. Figure 8-m).

Sidney, Harvey and Vince's arrays took longer to produce – which is unsurprising, as these students' fine motor skills were weaker – and, to an observer, are more difficult to read (although, as seen in Figure 8-m, clearly the students themselves found them adequate for calculation purposes). In cases such as Vince's array (Figure 8-r), the combination of cramped spacing, non-parallel lines and inconsistent colouring makes it much harder to see whether the rows contain equal numbers of units or not.

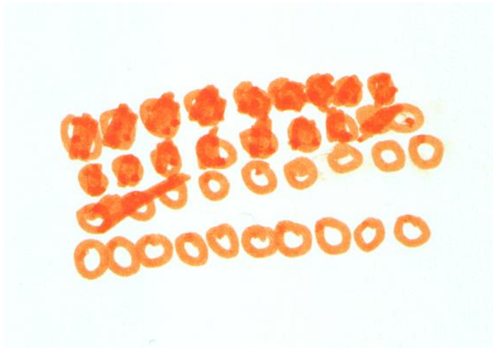


Figure 8-r: Irregular unit array (Vince)

Some of the students' more irregular array images may bring to mind Piagetian flower/vase number conservation tasks, in which preschool-age children famously confuse a row of items spanning a greater distance with a row containing a greater number of items. My examples suggest that this misunderstanding was not a concern for any of my students using arrays. They may have made occasional enumeration errors (verbal number sequence, desynchronisation of verbal count, etc.), but they knew that each row/column ought to contain the same number of units (except in cases with remainders, when the last row/column might be incomplete).

8.2.3 Array-container blends

Criteria: Unit array representation with additional containing rings, where number of units in each row/column/container represents divisor/quotient or multiplier/multiplicand.

Although my fieldwork data includes many array-container blends, examples produced entirely independently by students during tasks are considerably rarer. Co-created examples include those where a student was using a unit array for a task, and I had instigated ringing of rows or columns to clarify the relevant grouping. There are also examples where I introduced this representation type to students working primarily with simple containers, as a visuospatial nudge toward greater structuring of representations (e.g. Harvey, in Figure 8-s). Notice that in this example, I only drew the first container, but Harvey chose to continue the ringing process, even though the task could have been completed with the rest of the representation drawn as unit array alone. Taking the additional time and effort to superimpose rings onto an array was clearly worthwhile for certain students in certain tasks, and cases of this are discussed below.

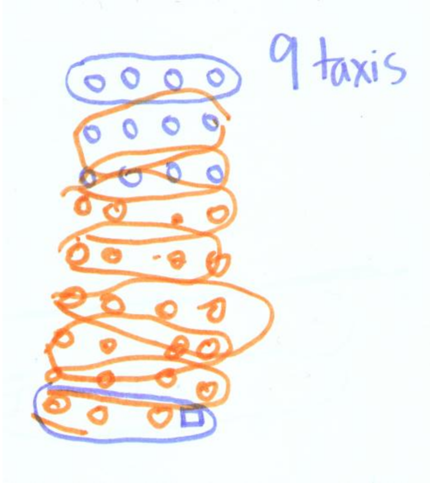


Figure 8-s: 35 people, 4-seater taxis (CF and Harvey)

I also chose to introduce a particular form of array-container blend, in which sets of <u>both</u> rows and columns were ringed, specifically to address the commutative principle and relationship between multiplication and division. Having used my judgement (as teacher) on appropriate timing for this discussion, it occurred at different times and in differing levels of detail depending on the individual student(s) involved. Additionally, in Tuition 4 one of the tasks I set to all students involved my drawing a unit array, and requesting they ring subgroups of the main two factors (in different colours), before completing a set of symbolic statements. I stated earlier that none of Oscar's working appears in this chapter; in this particular task he traced my presented array with his finger, but declined to mark it, before completing the symbolic statements correctly.

8.2.3.1 Visual elements

All array-container blend representations were drawn, and contained the complete complement of units, although not necessarily a complete complement of containing rings. The particular point of interest in this representation type is that although every unit is visibly present, the visual emphasis on ringed subgroups serves to change focus, drawing attention away from the units and towards the groups. Thus, it encourages the possibility of seeing containers (enclosing aligned sets) as the new ‘units’. A greater proportion of examples of this representation type (when compared to those above and below) were used for bare arithmetical tasks, but even when used for scenario-based tasks, no resemblance-increasing details at all were seen. Students were generally, but not always, consistent in their use of containers within the same array.

I provide four examples from Jenny, the most prolific user of this representation type (Figure 8-t, Figure 8-u, Figure 8-v, Figure 8-w). These were independently created and are chronologically presented. They display a variety of styles regarding units and layout, and show that she sometimes chose to carry out a visual pattern consistently throughout the representation, but not always. Looking at the representations chronologically, it is clearly not the case that she began by drawing consistent patterns and realised this was unnecessary in order to obtain an answer. Efficiency, then – in terms of minimum expenditure of time and effort spent on a task – was not her main concern. (It is also notable that she chose to continue working with this form despite experiencing confusion and making errors with it; this is further discussed below). Did she, and the other students who chose to draw array-container blends when simple arrays (or containers) would have sufficed for the given tasks, merely enjoy drawing the patterns – in the same way that others enjoyed drawing detailed people and vehicles? I suggest that there is no ‘merely’ about it, and that this ‘unnecessary’ visual pattern-creation was part of students’ personal exploration of multiplicative structures, driven by natural curiosity about numbers, and enabled by a flexible, non-time-constrained learning environment, where experimentation was encouraged and mistakes allowed without judgment.

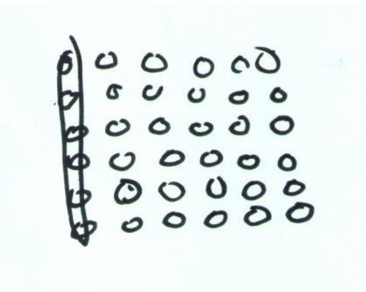


Figure 8-t: Array with single vertical container (Jenny)

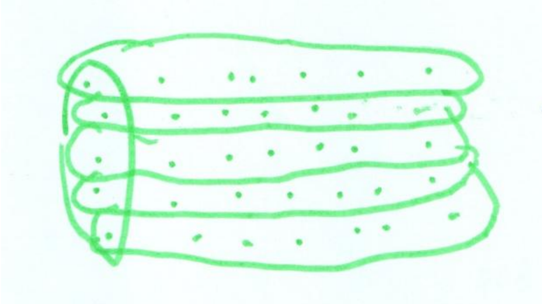


Figure 8-u: Array with single vertical and complete horizontal containers (Jenny)

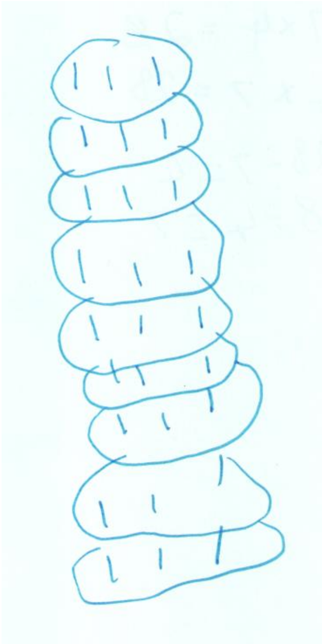


Figure 8-v: Array with complete horizontal containers (Jenny)

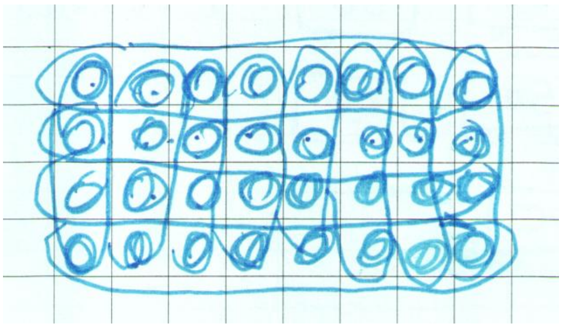


Figure 8-w: Array with complete vertical and horizontal containers (Jenny)

8.2.3.2 Spatial relationships

As with drawn container and array representations, while the completed image was immobile, the drawing (or re-counting) action had rhythm – which this combination representation type, with its additional level(s) of structuring, was intended to enhance. I have discussed the potential effects of this emphasis on replicative spatial structure on counting, and on developing understanding of multiplicative structures. However, these depend on students being able to look at a rectangular unit array and perceive it as a set of equal rows and/or columns. If the units are grouped in an irregular manner (i.e. the containing rings do not follow the rows/columns), divisions may still be carried out and correct answers obtained, but the power of the image is diminished. This is the case in Figure 8-x and Figure 8-y, from the two-colour array-ringing activity in Tuition 4.

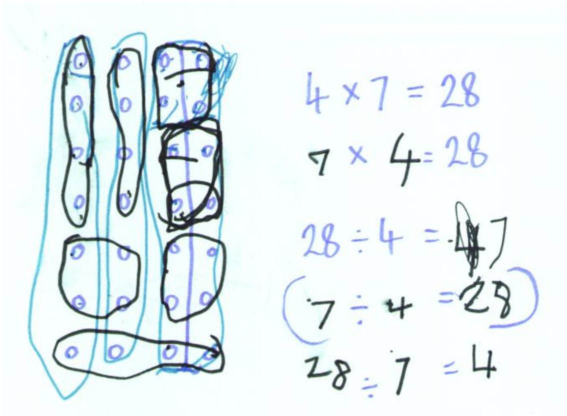


Figure 8-x: Irregular and unsystematic array-container blend (Harvey)

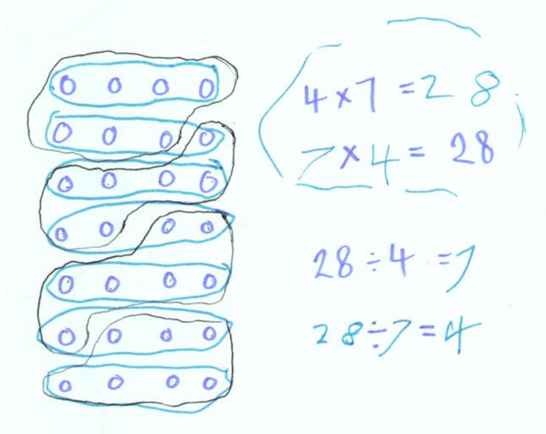


Figure 8-y: Irregular but systematic array-container blend (Ellis)

Students were asked to ring groups of four and then seven; given that the majority ringed rows and columns, it is reasonable to question why these two did not (plus Tasha, whose pattern is similar to Figure 8-y). In general it is unwise to make assumptions that students who do not complete a given task in the expected, or obvious, way, <u>cannot</u> have spotted the ‘expected’ pattern: they may well have spotted it but decided on a more interesting alternative. However, on this particular occasion, given the contextual data on these individuals, I interpret these representations as implying lack of perception of the two-dimensional nature of the spatial structure. I have already mentioned Harvey’s weak fine motor skills, and suggest sensorimotor neurocognitive issues as a factor in perception of visuospatial pattern. Ellis’s case, I suggest, rather reflects a lack of interest in visuospatial representation; throughout the tuition he strongly prioritised ‘getting answers’ as quickly as possible, preferably by rote-based methods, disdained the use of drawing or concrete strategies (other than fingers – and these used surreptitiously), and was very difficult to involve in post-task or general discussion of arithmetical relationships and processes. His pen control is neat and confident, and the grouping process proceeds in orderly, conventional horizontal motion, left to right, moving down the rows – my spatial arrangement of units entirely irrelevant to him.

In contrast, for Wendy, a similar rectangular array presented her with a springboard from which she decided to explore all the factors of 30. Figure 8-z shows her visuospatial representation of 2-dimensional arithmetical structure taken to an extreme.

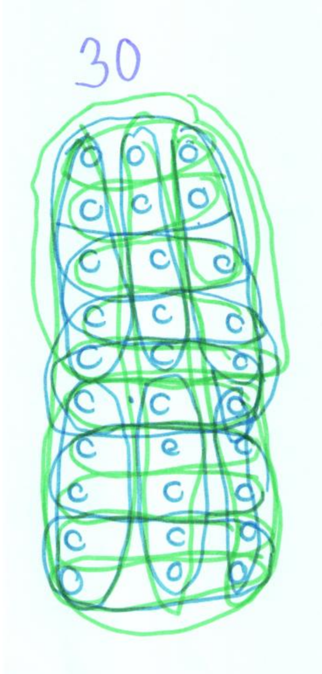


Figure 8-z: Extended array-container blend (Wendy)

8.2.3.3 Enumeration

The regular, replicating pattern of clearly-delineated identical groups seen in array-container blends supports the concept of multiplication as replication of a set, and division as either sharing or grouping into equal sets. Reusing the completed pattern also highlights the place of numbers as part of a static multiplicative relationship, rather than only as parts of a process.

Does this change of focus from the individual unit to the group correspond to any observable change in enumeration? I have already linked increasingly-structured representations (containers, then arrays) to increasingly-structured counting: the verbal count sequence is grouped, and becomes more rhythmic. Following this logic, it would be expected that emphasis on groups rather than units might facilitate the move from unitary to step-counting. Unfortunately there is not enough relevant data in this study to indicate unequivocally that this took place; it will require specifically-designed tasks. Although very few examples of this type include students verbalising their counting, in most cases the audio recording picked up the noise of their pen, and combining this with the scanned images gives a clear impression of the rhythmic element. This was firmer and more regular for the arithmetically stronger and more independently-working students (in this case, Danny and Wendy), while more erratic for the more struggling students (Harvey, Paula).

8.2.3.4 Successes and errors

Firstly, note that on no occasions did students place unequal numbers of units in rows, columns, or containers (excepting remainders, of course). Where students themselves chose to use an array-container blend as part of a multiplication or division calculation, the outcome was, for most, an independent correct answer. Where I instigated this representation type, and/or more support was needed, the process took longer, but still eventually reached a successful conclusion. However, occasionally the issue of confusing number-in-a-group and number-of-groups still appeared. This is not surprising, as not only was this representation type very low on resemblance, and thus lacking visual reminders of the relevant scenario (if there was one), but its use was skewed strongly towards the later sessions, by which time I was setting tasks from a wider variety of scenarios, and increasingly, bare tasks. I had also gently increased my use of formal arithmetical language (e.g. 'divide'), and was nudging the students in the direction of abstract arithmetical structures and relationships.

As a student who struggled with this representation type, but persisted with it, Jenny's responses are particularly interesting. She chose to use it for one Biscuits, seven Taxis, and three bare tasks, on each occasion first drawing a basic unit array, then doing the ringing as a second, separate, stage. Her use of rows and columns was inconsistent: in most (but not all) cases she constructed her arrays in rows of the appropriate divisor, but then on several (but not all) occasions then used rings to group the array in columns. This created confusion, particularly within the Taxis scenario, and caused her sometimes to give the original divisor as her answer. This may be seen in Figure 8-aa, her first attempt at '20 people in four-seater taxis'.

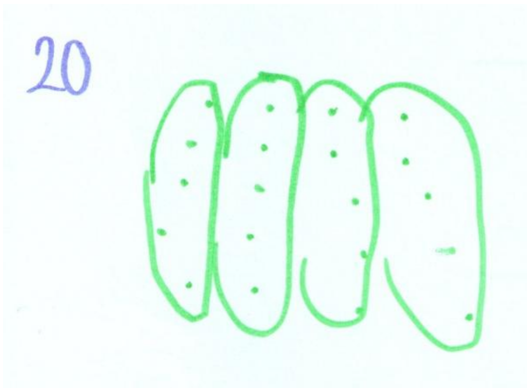


Figure 8-aa: 20 people, 4-seater taxis (Jenny, first attempt)

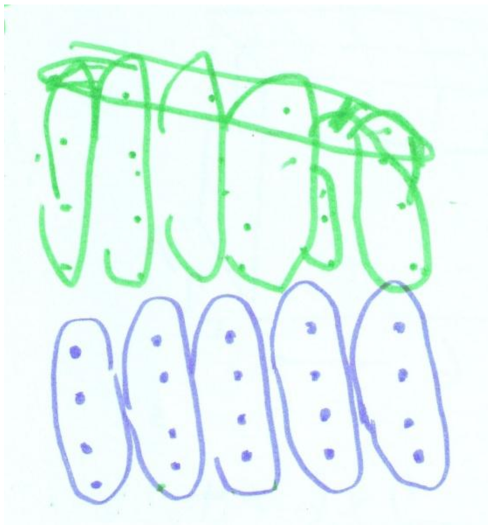


Figure 8-bb: (as previous, second attempt)

I pointed out that Jenny's drawing appeared to show five passengers per taxi, whereas I had specified four. In her second attempt (Figure 8-bb), she switched to constructing the array in columns of four (rather than rows), and almost made the same error in reverse, before self-correcting. This confusion did not arise when students ringed each row/column immediately after constructing it, so I demonstrated this strategy underneath Jenny's representation, accompanied by verbal scenario-based narrative ("Four people *[dots]* go in the first taxi *[ring]…*", etc.). I expected Jenny either to adopt this more reliable version, or to reject the array-container blend as ineffective for her; however, she strongly persisted in drawing full arrays followed by containers, sometimes correctly on first attempt, sometimes self-correcting, and sometimes requiring comment from me before re-grouping. This ongoing use of what seemed to be a confusing representational strategy was one of the ethical-pedagogical tensions that occasionally occurred for me as teacher-researcher, but I decided it was methodologically appropriate to maintain a student-led approach to tasks, and allow Jenny to continue in her chosen way. This representation type was clearly important to her, and I suggest the difficulty associated with it is an indicator of her difficulty with comprehending multiplicative structures and carrying out division. Furthermore, her ongoing struggles with it indicate an encouraging willingness – desire, even – to work through that confusion.

8.2.4 Number containers

Criteria: Container representation with numerals (rather than unit marks) representing the number in each group written inside, or close by, each container.

Of the four representation types which are the focus of analysis, this was the one least frequently used, by some margin. Usage is skewed towards the later sessions, with nine of the students using it at some point, but some individuals clearly having a stronger preference than others. Five introduced number containers independently during tasks, without input from me. I demonstrated the strategy to three more (including Wendy, who subsequently used it independently, as discussed in 7.2). I address certain of Leo's representations as a special case (see 8.2.4.3), as while he introduced number containers independently, he later produced several representations which share superficial visual characteristics with this type (containers with number symbols) but do not fit the strict criteria.

8.2.4.1 Visual elements

All examples of number containers were, of course, drawn. They are also, by their nature, non-unitary – in that no longer is there a one-to-one correspondence between quantities specified in the task and units individually represented; now one mark stands for >1 unit. At the numeracy levels in which this study deals, this stage must be considered a significant cognitive leap, and one which not all participants securely achieved. While most of the students independently carried out at least one division-based task using numerals (although rarely in standard 'received' notation), three did not. Although Harvey and Vince, and to some extent Paula, seemed able to work symbolically for addition (provided the quantities were not too large), when it came to multiplicative structures, their use of symbolic representation was erratic, unreliable, including a great degree of guesswork, and always with some form of unitary visuospatial representation required. Although Harvey did use number containers (with significant teacher support), all three of these students remained at a level where I believe they would benefit from further work with unit containers before progressing to number containers (and then, eventually, fully symbolic notation).

In general, the number containers have low resemblance to their scenarios. While a rectangle might be interpreted as a simplified taxi or coach, only Wendy's had any wheels, and many containers representing vehicles were circular. However, some examples from Sidney are curious in that he adds non-mathematically functional detail from the scenario to his containers, but does so in writing rather than pictorially (Figure 8-cc).

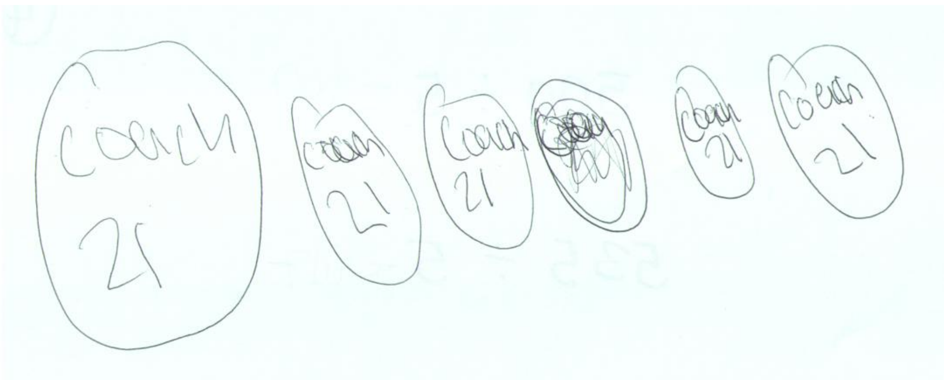


Figure 8-cc: 21-seater coaches (Sidney)

As I observed when discussing the Wendy data, from an enumerative point of view, students using number containers might as well just be using plain columns of numbers – so the containers clearly fulfil some other important function. In most of the examples of this type, students were still speaking almost exclusively in scenario-based language; when I set bare tasks (towards the end, written formally), we frequently ended up then creating a scenario in order to solve them. These two things together indicate students part-way to thinking symbolically and abstractly about multiplicative structures. It is reasonable to expect that as confidence is gained, the containers begin to disappear, along with the scenario narrative – and indeed, this appeared to be happening for some. I did wonder if some students might tire of drawing containers halfway through a representation, and switch to numbers alone, but overall, the number container representations were internally consistent, and any such changes happened between tasks rather than during.

Number containers were also used by students in a completely different way; one not related to the enumeration part of the task at all. I gave one example of this back in Chapter 6 (Figure 6-o), where Leo gave his 'drawers' explanation for a cuboid's volume. Similarly, throughout the first tuition session, Tasha independently chose to use number containers as a form of notation to record results that she had already worked out by other means. For example, she worked out the factors of 30 in movable concrete form, grouping and regrouping cubes, but then recorded her results as in Figure 8-dd and Figure 8-ee. An example of a student happily using symbols for quantities, but as yet uncomfortable with symbols representing operations, Tasha had created a clear and comparatively efficient non-unit-based way of

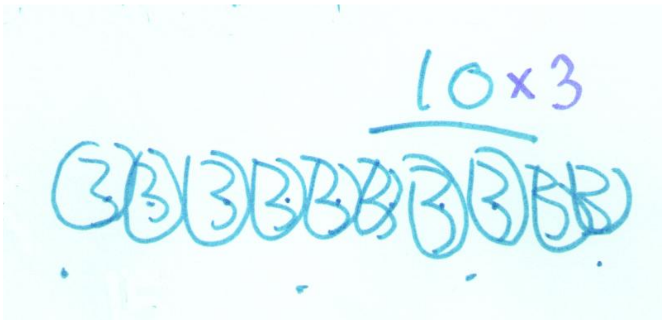


Figure 8-dd: 30 as 10 sets of 3 (Tasha)

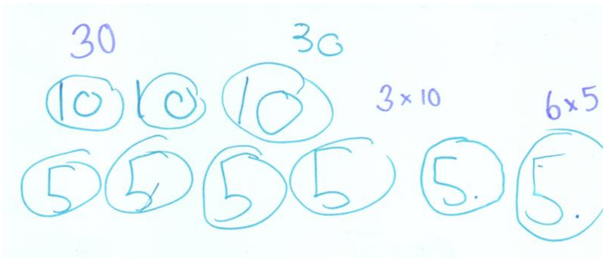


Figure 8-ee: Further factors of 30 (Tasha)

translating concrete actions onto paper. While unexpected, this is perfectly consistent with (and provided preparation for) my suggested later use of number containers in carrying out Taxis tasks. It is highly possible that this after-recording may have also been instrumental in helping her make sense of the numerical structures as static relationships rather than grouping processes.

8.2.4.2 Spatial relationships

With the absence of directly-represented units, there is no dealing, no back-and-forth motion along rows or columns, and no rhythmic ringing on reaching the cardinal number of each subgroup. However, as I have suggested, if the container becomes the ‘new unit’, it is still possible to compare layouts. Looking at the various examples seen so far (and in Appendix F), the containers are not strewn around randomly; they are vertically or horizontally aligned. This both highlights their repetitive nature, and the visual similarity to formally-notated multi-part additions.

I have mentioned previously that some students were much more inclined to array structures than others. Tasha was one of those not keen on arrays, but through replacing unit containers with number containers could still progress in stages to more symbolic (and less unitary) thinking. Figure 8-ff: shows a transitional representation, including a complete set of the units which she had, until that point, been counting, but also including a numeral for each group.

Figure 8-ff: Containers with both units and numbers (Tasha)

8.2.4.3 Enumeration

In general, students used number containers not only later in the data collection period, but later in individual tuition sessions, in particular when managing larger quantities. This is to be expected: one of my objectives in setting tasks involving larger numbers was to see how students comfortably using unit-based representations of multiplicative structures would react when set tasks for which their previous representations were possible yet annoyingly long-winded. With divisors 20, 21, 25, 50 and 200, unit-counting (unstructured, grouped or rhythmic) was much less likely, and I only observed one instance of a student doing it (George, 96 people in 25-seater coaches). In many cases, there were written or verbal indications that the students were using repeated addition of the divisor to build up to the required total, drawing each container then performing the corresponding calculation. In some cases it is not possible to tell whether a student was step-counting (e.g. in twenties) or repeatedly adding, but the significant point here is that units were not directly involved.

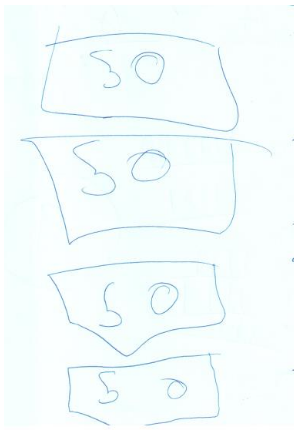


Figure 8-gg: 648 people, 50-seater coaches (George, first attempt)

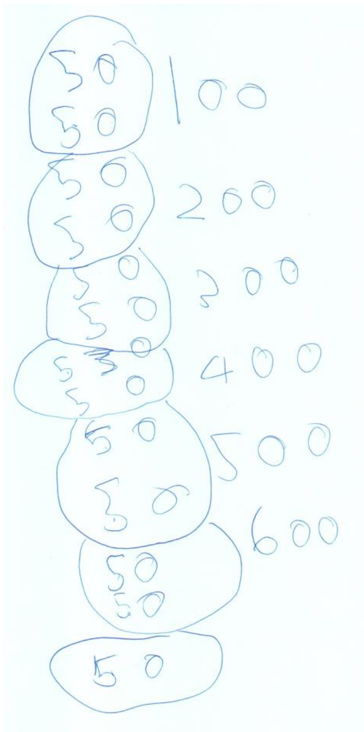


Figure 8-hh: (as previous, second attempt)

In fact, sometimes the more able students spotted ways to speed up the enumeration process. For example, in Figure 8-gg and Figure 8-hh, George starts the task ‘648 people in 50-seater coaches’ using number containers to represent coaches, but then stops, realising he can pair the fifties and count in hundreds. This was encouraging in terms of his growing ease and flexibility with multiplicative structures, although in this particular case it caused him some brief confusion over what the answer actually was (being no longer the number of containers drawn).

8.2.4.4 Successes and errors

Some students were able to produce independent number container representations leading to correct answers; often, though, some form of teacher support was required. This frequently occurred at the very start of a task: suggesting drawing the scenario, or turning a bare task into a vehicles-based scenario; then, if the student was unable to proceed, beginning the representation myself and allowing them to take over. The other type of support required was regarding enumeration: helping keep track of the running total, or assisting with summing after a number of containers had been drawn. Note that this representation type was effectively used (albeit with support) to tackle not only significantly larger quantities, but bare division tasks, written in informal symbolic notation – precisely the type of task which, at the start of the study, many of the cohort had reported disliking, hating, and fearing. Although praise for achievement and effort was a normal part of my practice, after completing this kind of task I made a particular point of drawing students' attention to the calculations they had just successfully performed – and observed many reactions of pleasure, pride and frank surprise.

Many of the errors occurring in unit-based representations did not occur for this type, and those that did were generally enumeration errors in carrying out an addition of running totals, or the adjustment calculations of those students who estimated the number of containers (of which Wendy's examples have been already discussed).

While not exactly 'errors' as such, on several occasions Leo produced visual representations which look somewhat like number containers, in that they include a mixture of container shapes and numerals (full collection in Appendix F), but were not successful in giving a correct answer. In each of these he began by drawing vehicles and writing a number in or alongside each, as he went along. This might be the number of people in (or wheels on) that particular vehicle, but could alternatively be the running total (e.g. Figure 8-ii). I have argued before for the importance of scenario-specific pictorial elements to this student's working, but there was also a tendency for these elements to overtake the mathematically functional elements in importance, resulting in desynchronisation of imagery and numbers, or complete loss of focus on the arithmetical aspect of the task.

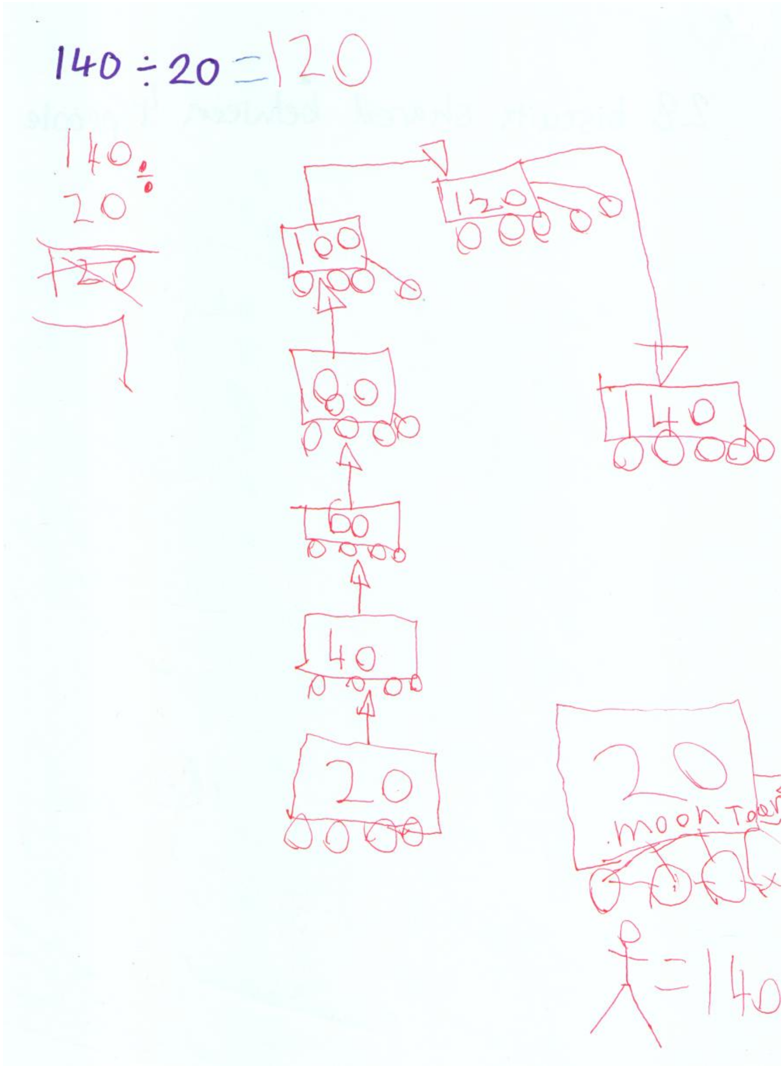


Figure 8-ii: 140 people, 20-seater coaches (Leo)

8.3 Representational relationships

8.3.1 Representation type and scenario

Container representations worked very well for both partitive and quotitive scenarios, provided the numbers involved were not too large. They were invaluable to the students with little or no concept of division or multiplicative structures, allowing them to engage with tasks for which their usual strategies appeared to be guessing, attempting some mangled partially-remembered sequence of symbol manipulations, or giving up. Container representations allowed these students instead to create manipulable simulacra of realistic scenarios, with as much visual resemblance and accompanying narrative description as required by that student at that time.

Array representations likewise worked well for both partitive and quotitive scenarios (with small to moderate quantities, and for somewhat abler students). However, the lower degree of resemblance meant that some students lost their connection to the task scenario; in these cases it proved helpful to add either non-mathematically functional decorative detail (e.g. Figure 8-l, Figure 8-m) or containing rings (e.g. Figure 8-aa, Figure 8-bb). This is, in fact, how most array-container blends used in scenario tasks came about. Number containers featured most strongly in Taxis (etc.) tasks – and, conversely, when struggling to get to grips with a bare task, students frequently ended up creating a vehicles-based scenario to give the calculation meaning.

This need to work with *imaginable* (even if not technically 'realistic') scenarios was very strong for the majority of my students, and working with equal-groups relationships was made possible not just by thinking (for example) about a number of items being shared, but by picturing the actual items, the people between whom they were being shared, and why. Even relatively capable Danny, when working with triple-digit figures, was able to grasp a task better when giving it additional, personal, context, by imagining all the boys in his school climbing onto a row of coaches, to go on a trip. However, as with drawn non-mathematically functional detail, the need for verbal scenario-specific detail decreased in general when students were (or became) more confident working within a particular scenario. Any such decrease in either drawn or verbal scenario detail (or increase in abstract or symbolic working, if preferred) was not a smooth progression, and did not occur simultaneously across different scenarios.

With the exception of number containers, which were linked strongly to the transport scenarios, the choice of representation type appeared to be more a function of students' individual tendencies in terms of spatial organisation (i.e. their liking or otherwise for array forms) and the numbers involved, than the type of scenario within which tasks were set. However, this does not detract from the importance of there being <u>some</u> kind of scenario – the familiarity of which supports and scaffolds cognitive development in the direction of more symbolic notation.

8.3.2 Representation type and calculation

For students at the most basic stage of calculating with multiplicative structures, who were trying to make up equal groups of units without some form of structured sharing or grouping strategy (e.g. Paula, pre-dealing), unit containers provided by far the clearest way to see the groups.

For those students who had acquired a systematic sharing or grouping procedure (e.g. dealing), but still needed to count out all units, there was a general split between those using primarily container or array forms, which I have noted as reflecting individual preference. To this I add a second factor, regarding the level of rhythmicity of their creation and counting of unitary representations. I theorised in 3.3.1 the importance of making a distinction between arrhythmic and rhythmic grouped counting, as a precursor to step-counting and dual counts, and it is in this area that the distinction has shown itself most relevant. The lack of visible, unambiguous boundaries in array-based forms (compared to containers) means that regularity of spacing is of much greater importance in defining subgroups. Regularity of spacing of representational elements is linked to regularity of motion, and so to regular rhythmic counting (as opposed to irregularly-timed grouped counts).

Number containers are quite different to the others in terms of calculation: although it is possible to use them to record groups, while counting the units verbally, on fingers, etc. (as George did) they come into their own when students are able to carry out repeated additions up to a given point. Note that while addition was the only arithmetical operation that my entire cohort stated themselves entirely comfortable with, completing a set of repeated additions while remembering to keep track of the running total in some way is considerably more challenging.

The affective issue of students becoming bored or irritated by the amount of drawing they had set themselves (in their initial choice of representative strategy) seemed to be an internal factor in some cases (although not all) for ‘stripping-down’ and removing non-mathematically functional detail (i.e. the impetus for change came from student, not teacher). It was also a factor in them switching the representation type used, in particular replacing the units with numerals – and so changed the kind of calculation required (from counting-based to addition-based enumeration). Again, the impetus for this came sometimes from the students themselves (for example, Danny transitioning independently to number containers in Session 3, based on the increasing quantities involved – coaches full of people rather than taxis), while others required one or more teacher's 'nudges' in that direction.

Figure 8-jj: Use of '=' sign (Danny)

Lastly, I have described number containers as being a significant step on the path to full symbolic notation. However, my students' use of this representation type, and other written calculations, indicate that there is a period of their development – perhaps quite a long one – where they are quite comfortable with the idea of symbols representing quantities, but not so comfortable with symbols representing operations. It is within this stage that number containers are particularly useful for bridging the gap between pictorial/iconic and symbolic working, and I have discussed individual examples of students doing this (in particular Wendy and Tasha). As examples of the beginning of a transition to traditional addition in columns, complete with operation symbols, see Figure 8-jj and Figure 8-kk.

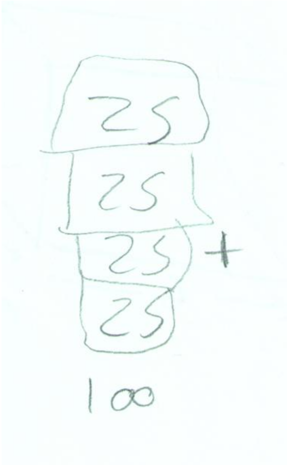


Figure 8-kk: Use of '+' sign (Danny)

8.3.3 Representation type and multiplicative understanding

All students in this study were given the opportunity to experiment with the various representation types described above, and those they created, and the way they changed, provided an outward expression of changes in students' multiplicative thinking. I have mentioned multiple examples of students producing visual representations of decreasing resemblance – i.e. including scenario-specific pictorial details at the start, but then realising that not all of these were necessary for completion of the task. The ability to work with more minimal representations is a positive step in itself, in terms of efficiency; however, more importantly, this change results in representations of different scenarios looking more alike, enhancing structural similarities – and each time this happens, it is a small but significant step on the path to both abstract understanding of multiplicative structures, and symbolic thinking.

However, the aspect which particularly differentiates multiplicative structures from additive ones – their two-dimensionality (or more) – is particularly relevant to representation types. Those individuals effectively using containers could clearly conceptualise division as process, of sharing or grouping. However, they did not necessarily perceive multiplicative structures as a static relationship between quantities: this understanding grew with use of array forms, and particularly exposure to the array-container blend.

To work successfully with arrays, students had to be aware of the visuospatial structure inherent in these objects, in effect, ‘seeing’ the invisible gridlines (through what is sometimes termed perceptual organisation). For some, this structuring, or spatial relationship, of units was highly visible from the start, whereas others seemed able to focus only on the individual units. The cases of Ellis and Kieran show that students may have a strong sense of the horizontal rows in a multiplicative structure, but not the vertical columns – processing the marks as if they were a passage of text. This allows for the carrying-out of multiplication and division-based tasks through repeated addition, and strengthens the concept of equal-groups relationships, but it does not address at all the two-dimensional or commutative nature of multiplicative structures. For this it is necessary, then, for students to perceive <u>both</u> row- and column-based groupings. This was the rationale behind my inclusion, for all students, of a full co-created array-container blend, and the direct stating of the numerical relationships within it.

The ability to switch perspective between horizontal and vertical patterns is undoubtedly a welcome sign of cognitive flexibility. However, when arrays are used in arithmetical tasks, row/column confusion can occur if the spatial organisation is insufficiently strongly linked to the task being carried out (as with Jenny). Again, using array-container blends, rather than simple arrays, proved helpful for clarifying task-relevant groupings. However, not all completed array-container blends are equal: students were observed constructing them in two different ways – drawing an array first, then the rings; or ringing each row/column after drawing it. While the first was useful in teacher-led explorations of the commutative principle (or factors in general), the second was unambiguously better for carrying out quotitive division tasks.

8.4 Discussion

In 5.4.1 I listed the main arithmetical concepts relating to multiplicative structures, which formed the basis of the pedagogical content in this study. As I have described in this chapter and the preceding ones, students at the start of the study were at various points regarding these concepts. I had never expected it to be a clear-cut case of either knowing these principles or not knowing them, but the reality was even more complex. Individuals had varying levels of confidence in the validity of these concepts (particularly commutativity), and might or might not remember about them on a given occasion, when needed. Moreover, believing one or more of the above concepts to be true did not necessarily imply a student being able to use that knowledge to choose and carry out an appropriate strategy for a task. Conceptual understanding was partial and unreliable; the fact that some of the students were able to recall some multiplication facts, or manipulate numbers and symbols in certain learned ways, merely served to mask underlying weaknesses.

Students made considerable use of container- and array-based representation types throughout the study; overwhelmingly so compared to other representational forms. Some students appeared to have a general inclination toward the use of array forms (i.e. where units were laid out regularly throughout the representation, and boundaries between groups invisible), while others did not. In no way am I suggesting a two-state categorisation of students as array-based or container-based thinkers; however, observation of their general inclinations in this respect could be useful in choosing the best form of support to offer a struggling individual. It is also important to note the persistence of container forms in students' changing representational strategies: students who preferred array forms used containing rings for rows/columns to emphasise particular groupings, and students starting to replace grouped units with numerals frequently retained the containers around them, at least for a while. In all, this suggests a powerful visuospatial/perceptual phenomenon relating to equal-groups number structures and relationships.

There were multiple instances of unit containers, unit arrays, and number containers produced independently by students prior to my intervention in their representational strategies. I introduced these representation types, along with the array-container blend, to those students who had not demonstrated them, at an appropriate point in their tuition, using professional judgement. This enabled students to complete tasks for which they did not previously have a reliable strategy. When intervening in students' representational strategies, via verbal and/or visuospatial interaction, I worked on the principle of the 'nudge', i.e. attempting on each occasion to give the smallest possible amount of support to allow the student to continue, and only one kind of support at a time. This included actions such as counting aloud, drawing a group of units, ringing a row of their array, etc. – in each case stepping back once the student could take over the process. It is clear that a minimal quantity of teacher support, if chosen and timed appropriately, can be significant in making the difference between being unable and able to attempt a task.

As well as the small but specific instances of intervention noted above, students benefited from general suggestions, such as "Is there something you could draw that would help?" or equivalent. This was more frequently so at the beginning of the study, but was still the case surprisingly often towards the end. Even students who had previously experienced considerable success on tasks by drawing sometimes appeared to forget that this was an option until reminded, particularly when there had been a longer gap between sessions. This is paralleled by students' verbalising and imagining of the scenarios involved, and particularly the strategy of taking a bare arithmetical task and creating an appropriate grouping or sharing scenario to make it manageable. The students who found this strategy very helpful also did not necessarily recall it when needed, but were responsive to quite minimal reminders (e.g. "What if it were people getting on buses?"). It is not surprising that my cohort of students did not tend to make immediate, lasting changes in their representational strategies, but required repeated experience before fully adopting them as part of their arithmetical toolkit, and reminders of their work in previous sessions. It is not unreasonable – unfortunately – to assume that they did not have the opportunity or encouragement to use visuospatial, narrative, or other nonstandard representational strategies in their mathematical activity in classwork or homework.

Whilst the majority of students responded positively to my interventions and suggestions, and particularly in the moments where they suddenly 'saw' a numerical relationship or structure, I believe it was particularly effective when they made discoveries for themselves, using their preferred representation types and styles. For this reason, it is beneficial to remind a student to draw (or model), but not instruct precisely what they should draw, or in what way. This applies also to the changes and developments in students' representations; for example, it is better for them to discover for themselves that certain elements of their representations are mathematically functional and others unnecessary than to be told so. Likewise, the progression from more pictorial, scenario-resembling representations to more stripped-down, iconic representations, to the incorporation of formal symbols and layouts, is slow, nonmonotonic, and made up of many small stages of change, and it is beneficial for students not to be hurried through this, but to take their own time and make changes when they feel ready. In the right circumstances (such as this flexible, dynamic 1:1 or small-group work) the tedium induced by carrying out a task with a time-consuming repetitive strategy may prompt independent strategic adaptation.

I have devoted the majority of discussion to students' use of the various representation types during tasks, and less to the more teacher-led use of them in discussion and demonstrations. Obviously, all of the representation types were used to a great extent for enumeration of quantities, and for the visuospatial organisation of these quantities so the correct set of objects (units or groups) could be counted or added. However, the representations created were not immediately rendered useless once an 'answer' had been achieved. Students completed visuospatial patterns when an incomplete pattern would have been sufficient for an answer; they sometimes added further organisational (or, for that matter, decorative) detail <u>after</u> giving an answer. Occasionally they even created a whole new representation to record their work, or to help them explain to me an exciting discovery they had just made about numerical relationships. The fact that these representational activities were important to the students <u>in their own right</u> (i.e. not just as a means to the short-term end of a correct answer, or to please the teacher) suggests they are an important part of the learning process. The completion or repetition of visuospatial representations, even if the teacher cannot immediately see the point of it, should therefore not be rushed, or glossed over.

It seems that using any of the representation types involving containers – even without any pictorial detail – worked as a kind of visuospatial shorthand, which preserved or created the link between the imagined scenario and the numerical relationships involved. This power was seen in some of Jenny's work, when she had grouped an array incorrectly, and simply referring to a ringed row/column as a "taxi" was enough for her to see the error and correct her representation and answer. Also evidence for the power of this simple representational element is the fact that sometimes, merely the idea of a scenario involving containers (as opposed to what the drawn rings, in fact, actually contained) was enough to be helpful. Consider some of Harvey's drawings: boundaries overlap or dots go astray, yet despite these inaccuracies of execution, planning and carrying out this kind of drawing process was enough for him to obtain a correct answer. Likewise, the arraying and grouping of units in more than one dimension was powerful in triggering the concept of multiplication/division as an arithmetical structure or relationship rather than as a process with an end point (The Answer) – and this did not depend on the straightness of the lines of units or the squareness of the grid, but did require *regularity* of pattern. There is cause for speculation about the role of horizontal versus vertical 'reading' of arrays; while this study can offer little on this topic, future research could be valuable, perhaps making use of eye-tracking software.

Use of these representation types, in the ways described above, provides a series of links between fully concrete, enacted simulations of multiplicative-structured scenarios, and the use of symbolic notation required by the mathematics curriculum. It is not a prescriptive teaching programme for students with difficulties, as it is clear that their patterns of strength, weakness, and the representations which work best for them, are complex, interrelating, and individual. There is no single ideal path through from, for example, dealing out a pile of cubes to a set of actual present people, and carrying out a fully symbolic division calculation. The important point from a teaching/learning perspective is that for any student, at no stage is the leap too wide from one representational strategy to the next, so the connection between them is unclear. Bricks are bricks, sums are sums, and there are many possible stepping-stones in the path leading from one to the other.

9 FINDINGS

In this chapter I begin with findings on three overarching themes which emerged from the analyses in Chapters 6-8: the conceptualisation of division by students with difficulties in mathematics; the role of time when working on tasks; and the question of ‘efficiency’ in representational strategies. For each, I consider the pedagogical implications. I then revisit the theoretical underpinnings and methodological decisions from the planning stages of this thesis, and reassess them in light of the data collected during fieldwork and its analysis. I also discuss certain products and outcomes of my work in terms of their relationship to past (and potential future) research in the field.

9.1 Conceptualising division

Numerical division is often spoken of in literature and classroom as a single concept; the inverse operation of multiplication. While the use of actual ‘division tables’ (as separate from multiplication tables) is currently unfashionable, it is still common for teachers and quantitative researchers to expect students to carry out division-based tasks through the recall and inversion of memorised multiplication facts. Of course, there are many students who do become proficient at this strategy, and have been able – independently or via good teaching – to form connections between this manipulation of facts and symbols, the underlying multiplicative structures, and the relationships between or actions upon quantities of different sizes. There are many more who, as discussed above, dutifully perform maths-like actions with little connection to the meaning of what they are doing, and may well be rewarded for this behaviour to the extent that it becomes habitual and normalised.

When first introducing the idea of division to young children, or doing remedial work with those students who are not performing division as expected, the dual models of partitive and quotitive division are likely to be invoked, perhaps with some kind of scenario narrative and/or visuospatial representation. By the time these students reach secondary school, they will certainly have had the experience of working on sharing-based scenario tasks, and this may be enough for some more students to grasp the necessary underlying concepts and make the required connections. However, for others it is still not enough, and those underlying concepts must be separated out and made explicit, with the pace slowed enough for each click of the conceptual and procedural cogs to be distinct.

9.1.1 Division as a componential concept

On a fundamental level, the act of division may be considered as:

- the separation of a quantity into a number of parts,
- where those parts are exactly equal,
- and the original quantity is preserved.

In a division task, two pieces of mathematically functional information are given:

- the original quantity,
- and either the part size or the number of parts.

It may seem that these aspects of division are too obvious to be stated as rules; this is not so. There are many examples of my students carrying out sharings and groupings which resulted in unequal sets, and while some of these resulted from enumeration errors (e.g. missing number in verbal count sequence, desynchronisation of verbal count and pointing finger) or short-term memory problems (e.g. dividing original quantity into wrong number of shares or group size), the evidence suggests that others resulted from conceptual misunderstandings – such as when presenting me with groups of obviously different sizes, or increasing the total number of units.

The majority of data behind this assertion comes from Paula, in particular the sequence of 'sharing' activities explored in 7.1, although the finding is also supported by interactions with other students. I have demonstrated that low-attaining students may not actually be aware of all these rules, or may be aware of them but consider them desirable rather than necessary. I include the word 'exactly' in my definition above, because learners may also be aware of the 'equal groups' rule, but not consider <u>numerical</u> equality necessary, considering approximate visual equality to be adequate. This is effectively treating a discrete quantity as a continuous one – something which mathematically competent people frequently do, and deem perfectly appropriate when dealing with larger discrete quantities (e.g. a bowl of pasta). A person with severely limited numeracy may well consider it appropriate to treat much smaller quantities as continuous. (The range of discrete quantities which are treated as continuous of course varies depending on context and individuals – I knew a pair of young brothers who counted their peas at mealtimes, to check for any sign of favouritism in their servings!) There may also be a difficulty with the cognitive demands of addressing more than one of the rules at a time (see 9.1.2).

The conceptual basis for division-related processes, such as dealing, may likewise not be assumed. Although it appears obvious to most, it is quite possible for a learner to mimic the motions without necessarily realising that the dealing process absolutely ensures conformance with the rules of equal groups and preserving the original quantity. When working with students whose division requires enactive, concrete dealing, accompanied by a narrative, it should also not be forgotten that ‘sharing’ has a cultural component. For example, what is their experience of mealtimes? With school dinners, a regulation ladleful is given to successive students until it runs out (quotitive model) whereas at home, successive quantities may be dealt to the set of people around the table (partitive model). Or perhaps the norm is for people to help themselves from the pot at will?

9.1.2 Division as a multiplicative structure

Analysis of the Paula data illuminated partitive division on a fundamental level: what it means to ‘share’ a discrete quantity. While Wendy worked at a more advanced level and progressed at a greater speed, analysis of her work highlighted the fact that students can go through school, being ‘taught division’ without really comprehending the two-dimensional numerical structure behind “how many *x* go into *y*?”, with its innate inverse and commutative attributes. Again, there are principles which may need to be made explicit and experiential for students with numeracy difficulties:

Discrete quantities (i.e. numbers)

- can be made up of a set of smaller equal-sized quantities,
- where there are different possibilities for the size of those quantities,
- and picking a different group size will result in a different number of groups (and vice versa).
  (NB prime numbers and division by 1 not considered here.)

In 4.3.1.1 I discussed Lakoff and Núñez's Grounding Metaphors: while *Object Collection* was particularly relevant for understanding the issues of partitive division, *Object Construction* may be more relevant for quotitive division.

In Wendy's case, the first key development in structural understanding was my appropriation of the array representation she already used for partitive division, to be used also for quotitive division, multiplication, and finding factors. This use of the same visual form can thus forge connections between what a learner has previously thought of as quite separate kinds of calculation. Along with the issue of understanding the multiplicative structure within quotitive division, there is the actual process of carrying it out. Possible concrete modellings of quotitive division tasks may be easily imagined by the reader, or the way in which an equivalent process for dealing might work.

It is important to note, however, that difficulties conceptualising division may be present long after the concrete stage, and in students used to working in symbols and with larger numbers. For example, guidance was necessary for Wendy's slow realisation, through experience, of the advantage of keeping a running total when constructing a target number from equal groups; before this, it had seemed perfectly reasonable to her to repeatedly estimate and then adjust the number of groups. Here, the role of running totals in constructing a final total (through repeated addition) is a leap in systematicity equivalent to that of the dealing process in creating equal shares.

9.1.3 Implications for learning

For students with a weak or partial (or entirely absent) understanding of division, the above rules need to be accepted as absolute. Regardless of a student's current method for creating groups (by eye and adjustment, or via a more organised strategy), they must first know they are aiming for exact equality of groups, and must account for the whole original quantity (and no more). Any sharing/grouping that they are asked to do (i.e. into a specific requested number of groups or group size) must comply with this. At the start of tuition, Paula found it a considerable challenge to manage the apparently-conflicting triple requirements of forming a specific number of groups of units, and making the number in each group the same, while keeping the total number of units constant. Even in contexts where the third constraint is managed through the representational media used (i.e. counting out the total number of cubes allowed beforehand), a student being able to give attention to only one rule at a time effectively turns a single-stage calculation into a multi-stage process. There may be times when an action which complies with one rule (e.g. moving some cubes to equalise groups) breaks another (e.g. creating the wrong number of groups; having cubes left over), and the student perceives themself as at an impasse: this is a valuable learning experience, as the experience of tension between requirements powerfully reinforces the quantity relationships involved.

The dealing method is very important in development of early division, in that it provides an automatic way to solve partitive division tasks by creating a specific number of exactly-equal groups. However, it is a meaningless activity without emphasis on the fundamental concepts of division, and the rules they entail for what the 'answer' should look like. Also, as shown by my analysis of the Paula data, the connection between the dealing process and the rules may not be immediately clear, and a learner may require some convincing that this method really will always produce equal groups. A teacher saying 'this is so' is not enough, but it is not valueless, if a connection is explicitly drawn with the arithmetical structure. However, there needs to be enough personal experience of the dealing process for belief to strengthen – and 'enough' depends on the individual. It helps if the learner has previously been given the time and opportunity to explore 'inefficient' sharing/grouping strategies (as described above), and for students whose numerical difficulties are less severe, it will take place much more quickly. It should also be remembered that actually carrying out a dealing pattern successfully – at least, in the early stages of its use – may also require greater cognitive work for some students than is generally assumed. Learners with specific limitations in hand-eye co-ordination, visual processing, and/or short-term memory may have difficulty carrying out the pattern of movements without losing their place and repeating or missing stages.

The experience of sharing/grouping the same quantity into a variety of different possible equal-groups arrangements is also important at an early stage – as soon as the learner is able to perceive and create equal groups of units. The early sharing/grouping behaviour of my weaker students indicated the possibility that on successfully producing an equal-groups arrangement (e.g. twenty as four groups of five), they were confused either by being told that this was not the number of groups (or equivalent) specified in the task, or by my request for an alternative grouping. The experience of grouping the same total quantity in different ways (and explicitly discussing this) could be quite a profound learning moment; it is not by chance that this was one of the ideas of which students themselves chose to take control, extend and explore (e.g. Wendy's multiple-factor array-container blend (Figure 8-z), Tasha adding the layers of a cuboid three times, in its three different orientations). While multiplicative structures may be grasped abstractly by some learners, for my students and those like them, the abstract must be made experiential.

9.1.4 Implications for teaching

Historically, a common theme in research on cognitive development has been debate over whether certain accomplishments arise in a concepts-first or procedures-first manner – probably most famously in the long-running debate between Gelman and Baroody (and various others) on how children learn to count discrete quantities. Arguments over the precise chronology of learners acquiring the ability to divide discrete quantities are, however, not particularly relevant to the teaching of very low-attaining, struggling, older individuals – not least because, by definition, they are those whose development has been atypical. In Paula's case, the division concepts were somewhat present, and the dealing procedure was not; it is also theoretically possible – although not seen among my participants – for the inverse to be the case. However, in both cases there is a benefit to starting by focusing on the conceptual knowledge required for a successful division. Depending on the particular nature of an individual's weaknesses (and strengths), when given the opportunity, they may discover the dealing strategy for themselves, or may need explicit teaching of it. Once taught it, they may grasp it immediately or require practical experience. Through observation and verbal interaction, the teacher (or other provider of learning support) may determine whether it is the conceptual or procedural aspect which requires a 'nudge' at a given point in the learning process; the microevolution of knowledge involves both.

I mentioned in 9.1.2 that learners may need the 'evidence' of a number of successful divisions to believe that a strategy (such as dealing) is truly reliable. Thus, it is also important to watch for any errors in carrying out the strategy, and clarify the nature of discrepancies (verbal count error, cube hidden by sleeve, etc.), as incorrect answers may seem to the learner to provide evidence that the strategy is unreliable.

I have demonstrated the importance of imaginable scenarios and narratives in developing multiplicative thinking. In addition to creating initial appropriate task scenarios from which to explore equal-grouping patterns, continuing to reference the real-life aspects while working, or after an answer has been given, can be beneficial. This might take the form of appealing to 'fairness' in sharing, or reinforcing the physicality of scenarios (e.g. 'full up', 'empty seats', etc.)

Lastly, there is the question of which representational media to use, and when and how to change. For students with particularly weak, incomplete, or no conceptual understanding of division, the obvious (and correct) choice is to begin with concrete units. Concrete unit containers such as bowls may also be helpful, particularly if visual impairment is present. However, there are advantages to using drawn containers as I did, as it provides a stepping-stone between concrete and drawn representations of numbers (which are themselves a stepping-stone on the path to symbolic representation). Saundry and Nicol (2006) described students manipulating their pictures on the page as though they were physical objects; I have had the opportunity to compare how my students interacted with both concrete and drawn representations. I conclude that for each stepping-stone to be as secure as possible, perceptual links between representational forms are helpful – visual similarity, spatial layout, movement sequence, or rhythmic element – and the way the fundamental concepts (such as equal groups) manifest within the representation type(s) should be emphasised.

9.2 Tasks and time

In my discussions of learning and teaching implications thus far, I have frequently emphasised learners being given the chance to experiment with arithmetical tasks and explore numerical structures and relationships, to make mistakes, to use inefficient strategies which (may) naturally develop into more efficient ones, to talk about what they are doing, and progress at their own pace. This, of course, takes time – a valuable commodity that almost always feels in short supply in schools. In this section I address the issue of time spent on tasks in relation to mathematics lessons and (individual) curriculum planning, and then in 9.3, unpick the idea of within-task arithmetical 'efficiency'.

During this study it was both an ethical and a methodological decision to be as flexible as possible regarding the amount of time students spent on tasks; a congruence of teacher and researcher imperatives. (Note, though, that my own 'teacher imperatives' in this situation were likely to differ from those of a class teacher.) School-based mathematical activity generally involves time pressure, either explicit (e.g. a teacher's stated expectation of a certain number of tasks being completed during the lesson) or implicit (e.g. comparison of one's own quantity of work with that of peers), and this pressure can cause stress and associated negative consequences for struggling students. Removing this time pressure was intended to contribute to a positive, low-stress environment for my participants; I also suggested that it would allow the kind of constructive experimentation with unconventional representations, comparison of strategies, and discussion of concepts to take place which would be beneficial for students' learning, while also allowing the collection of fine data appropriate for qualitative analysis of representational strategies (5.3.1).

In the discussion and conclusion sections of each of the Analysis chapters are comments on aspects of students' mathematical behaviour which would not have come to light without the freedom of unconstrained task time; it has not been a popular methodological choice for past research into arithmetical strategies or visuospatial representation, but it is one that I hope to see increasingly in the future. However, one of the emergent themes of my analysis has been the powerful influence of unconstrained task time on the nature and path of students' learning itself.

9.2.1 Relevance of unconstrained task time

In this study I have endeavoured to gain better understanding of the multiplicative thinking of students with a history of low attainment. Observing these students charting their own courses (as much as possible), at their own paces, through seemingly-simple division tasks has made visible their individual patterns of capability and limitation, developing organisational structures, knowledge of number relationships, and gaps in that knowledge. It has also highlighted representational needs and desires that students are unlikely to express in the classroom, particularly when completing a set of traditional exercises, but perhaps also in less formal group activities. Students working in such a constrained condition will, consciously or unconsciously, be pressured to use what appear to be the most time/space-efficient strategies to produce a stream of answers; to engage in the maths-like behaviours they believe appropriate. They will be unlikely to take exploratory detours from tasks, looking for alternative examples or counterexamples, comparing different representational or arithmetical strategies, extending patterns, generalising, and yes, even theorising – and observation of these are at the heart of recognising the presence of mathematical thinking in an individual. Of course, my observation of these behaviours in students with significant numeracy difficulties does not mean that such instances of genuine mathematical thinking are guaranteed to be present or happen in the same way for all students with significant numeracy difficulties, if only given enough time. However, it means that they might do – and this possibility is currently not the default assumption of curricula, textbooks, or many teachers.

I have also endeavoured to increase my students' multiplicative thinking; that is, to address this particular subject area in a way which encourages conceptual, connected understanding, and prioritises awareness of mathematical structures and relationships over recall of facts and reproduction of procedures (in line with Dowker's principles for DFS, 3.3.2). Allowing them to take their time and set the pace functioned in a similar way to allowing them freedom to choose their representational strategies; in fact, these are linked, as the lack of time constraint allowed them to choose more time-consuming representations, should they wish. I believe this kind of choice to be not only important for individuals' thinking to progress at their own appropriate pace, but also empowering for the mathematically disadvantaged. Mathematicians of high ability are described as "knowing to fool around with examples" (Watson, 2001, p.p.464); this should not be their preserve alone, as the benefit can apply to those at all points on the spectra of mathematical abilities.

9.2.2 Implications for learning

This study has shown that whether carrying out a single division task or considering the concept of division more generally as a static, examinable multiplicative structure (as suggested in Sfard, 1991), progress is not a direct function of time, and it is not possible to predict after how many minutes (or how many similar tasks) an 'a-ha' moment, such as those I have described, will arrive. This unpredictability may be assumed to apply more widely than the narrow subject area on which I have focused.

Working on the constructivist assumption that, in general, it is better for a learner to build an understanding of arithmetical concepts and procedures than to receive them in transmission form, it follows that they may need more exploration time than one might expect. But how much is 'expected'? This will vary depending on the individual, the teacher, and many other variables; however, to err on the side of generosity of time is better than to interrupt or rush struggling students. It is also worth noting that even in my paired tuition condition, in the presence of just one other student of comparable ability, and despite my efforts to release them from time constraints, there was a small but ineradicable competitive element: one student indicating (in quite neutral manner) that they had finished a task put pressure on the other to complete more quickly. This time-based peer-pressure can be expected to increase in groups of >2, and is not confined to work on closed tasks; while mixed-ability investigational work has been shown as advantageous for some low-attaining students (e.g. by Boaler, 1997), there is a strong possibility of the quicker students' chains of reasoning and exemplification leaving behind those who struggle most with mathematical thinking.

Due to the nature of class mathematics teaching, the slower and/or struggling learners will frequently feel time pressure, as discussed above. This may result in tasks being left incomplete, or, frequently, with a cognitively un-traversable gap between some intermediary stage of working and The Answer (provided by teacher or peers in a briefly satisfying but educationally pointless act of 'closure', which further reinforces the importance of product (final answers) over process (strategies or methods of working), and the performance of maths-like behaviour over mathematical thinking.) The effects of rushing learners through a larger number of tasks rather than allowing them to take their time over fewer, are firstly that they will have less time to digest their work and form connections, and, secondly, that they will not learn to value this kind of thinking, instead prioritising the (short-term) quantity of answers produced over long-term structural understanding. I do not suggest that the students are self-aware of these prioritisations and values on a metacognitive level, but I do suggest these perceptions of relative value are present and, for perceived-as-low-attaining students, increasing.

There are two main counter-arguments to this stance: (1) that completing a large quantity of tasks of a similar type aids procedural competence; and (2) that completing more tasks provides more data, which means there is more chance for learners to engage in pattern recognition (universally agreed as central to mathematical thinking). I do not dispute the likely truth of (1) for most learners – at least, in terms of short-term success. (Long-term retention is quite another matter, addressed ably elsewhere, e.g. by Hewitt, 1996; 2001.) However, as I have made clear from the start of this study, I do not believe that rote reproduction of mathematical procedures should be the primary goal for students at this or any level of learning mathematics. (2) is also true, but my concern is regarding normal(ised) classroom practice: that in an educational context where a page full of sums (or equivalent) is clearly prized, taking ‘time out’ from visibly-productive ‘work’ to think more deeply about it will not be prioritised. (This attitude is exemplified by the countless textbook and worksheet exercises which consist of half a page of sums (or equivalent), followed by an optimistic but widely-ignored “What do you notice?” below or beside.)

9.2.3 Implications for teaching

Although the majority of mathematics teachers with whom I engaged during my fieldwork grew to respect the work I was doing, at the start of the project I was looked upon with some suspicion, and the students I was offered as participants were ones whose attainment was so low, and progress so slow, that missing a few lessons “wouldn’t make any difference” – in effect, a judgment that even if I were entirely incompetent, I could not harm them mathematically. This, of course, fitted perfectly well with my methodology. Although I have just stated it in a rather negative manner, the positive aspect of this perception might be interpreted as an acknowledgement that the standard mainstream classroom had not been working (well) for a particular student, and it would be a reasonable use of their time to try something different, rather than just continue with more of the same.

However, I do not believe this was the case with the SEN support staff who were generally present in the classes from which I drew my participants. Classroom assistants are not the focus of this study, but the observations and interactions I happened to make and have with them do make up a part of my broader ethnographic data, and among this population there was a general tendency running counter to any long-term concerns of *economy of learning* (Gattegno, 1970), and which I sum up as an overriding ‘fear of wasted time’. That is to say, there were cases where they exhibited the same valuing of quantity over quality; of maths-like replication of patterns of meaningless symbols over genuine mathematical thinking. This was particularly the case with Paula’s main support teacher, which is particularly unfortunate, as of all the students, she was the one least likely to gain any long-term benefit from this approach. At the end of a lesson, to be able to present (for example) a page of multiplications carried out by performing a series of actions with a multiplication grid is tangible evidence of Work Done, in a way that reporting on half an hour spent exploring multiplicative structures with cubes is not. It is also a way of ensuring praise from the teacher – for both student and LSA. This is problematic.

I have talked a great deal of the importance of unconstrained, or at least, generous, time being allowed for tasks, and for stepping back and letting students make their own discoveries. However, in all my analyses of tuition sessions, I have also described moments where I chose to intervene. Each of these interventions was a judgement call that I, as teacher, had to make regarding the individual concerned, the point they were at, and their affect at that moment. There is no metric for ensuring that the teacher gives the right support at the right time. Several principles, though, apply. The non-interference stance I applied does not involve taking one's attention away from the student and their work and leaving them to get on; it is watchful, approving, and supportive. This watchfulness will give some idea as to whether a student who has been silent and inactive for some time is thinking intently (perhaps engaged in internal representation) or has become disengaged, with wandering thoughts. It may be necessary to ask a student if they are 'stuck' or need help, rather than assuming it is the case. Where support is needed, the teacher should not cover multiple steps at once, but provide the smallest incremental 'nudge' which will allow the student's thinking process to continue. Meanwhile, taking the time, at appropriate moments, to invite the student to articulate their working, can give access to tiny steps forward in their thinking (or 'microprogressions') that might not otherwise be seen.

9.3 Efficiency in representational strategies

9.3.1 What is meant by 'efficiency'?

'Efficiency' in general usage, can be described as the extent to which time, effort and/or other costs are well used for an intended task or purpose, with minimum wastage. As has been seen in the previous section, for the specific case of learning arithmetic, the concept of 'wasted' time and/or effort is both important and contentious. Along with various interrogators of classroom practice previously cited, I believe that an approach based on "repeat and repeat, review and review, correct and correct" (Gattegno, 1970) is not economical in the long term, and can result in significant time wasted in activity which has only short-term gains. It is clear, however, that many teachers, support staff, and others involved in education (including parents and, perhaps, Ministers for Education) are more likely to consider mathematical activities (such as those in this study) that do not produce pages of easily-recognisable 'work' to be time wasted.

In the previous section I discussed the issue of how lesson time is allotted, and how the encouragement of mathematical thinking relates to considerations of long-term 'economy' in learning, valuing of quality of tasks over quantity, and making potentially significant allowances for individual learners' different paces. I mentioned that one of the outcomes of unconstrained task time was greater freedom of representational choice, as the learner is not limited to trying to produce answers by using up minimum time (and space), but in alternative ways which may end up conferring more long-term benefit in terms of mathematical thinking and understanding. While I have adopted Gattegno's (and others') term 'economy' regarding macro-level lesson- or curriculum-planning choices, I use *efficiency* on a micro-level, for comparing the various strategies that might be used for a given task type on a particular occasion.

The National Curriculum mentions efficiency: Key Process 2.4(c) states that "Pupils should be able to consider the elegance and efficiency of alternative solutions", and Attainment Target 2 contains the descriptor "They use efficient written methods of addition and subtraction and of short multiplication and division" (QCA, 2007). The National Numeracy Strategy, likewise, includes "calculate accurately and efficiently" (DfEE, 1999, p.p.4) within its definition of numeracy, and later states that some methods of calculation (e.g. "standard written methods") are more efficient and reliable than others, and that "you can guide pupils towards choosing and using the methods which are the most efficient" (ibid., p.p.7). Both documents assume there could be no ambiguity over what is meant by this. In many educational contexts, it might indeed be straightforward; for example, when teaching more numerically capable students (with a solid conceptual understanding of multiplicative structures and place value, and a reliable stock of either retrievable multiplication facts or derived fact strategies) compact symbolic notation options for multidigit division. It is also somewhat straightforward when, say, comparing students' differing arithmetical strategies for a given task, although there may be acknowledgement that the strategy which obtains one individual the correct solution in the shortest time may not do so for someone else. The idea that students may be attempting the same task from a basis of varied and varying patterns of individual differences, strengths and weaknesses, and that this may lead to a different 'most efficient' is nowhere acknowledged. The further the individuals in question are from expected norms of mathematical behaviour and attainment, the more problematic this is.

Efficiency is often assumed to align with certain arithmetical and representational strategies, particularly the retrieval of memorised multiplication (or division) facts and the use of traditional forms of received symbolic notation. Fact retrieval is certainly quicker than working things out, and symbolic notation takes up little physical effort or page space; however, these gains are irrelevant if the 'facts' and procedures are unreliably deployed. In the case of students like many of my participants, where there were significant memory issues, attempting retrieval involved a great deal of guessing. I describe below an attempt by Jenny at symbolic division (9.4.2.3), and the reasons this wildly unsuccessful strategy might have seemed attractive to her. I have also discussed the effects of time pressure (9.2.2): the need students feel to produce answers quickly, in large quantities, and with only standard symbols used – and the way that teachers and classroom environments reinforce this. For students such as those described in this thesis, this is a false efficiency. If <u>on a particular occasion</u>, a unitary drawing-based strategy (e.g. dot arrays) is fully understood and reliably deployed by a student to solve a task, then that may well be a more efficient strategy for them to reach a correct solution than floundering hopefully through half-memorised facts and procedures.

Therefore, actual efficiency of strategies is context-dependent rather than absolute; it depends on the individual, and the knowledge, understanding, and representational toolkit they can access at any given time. That said, it is possible to compare relative efficiency of the various representation types used in this study.

9.3.2 Efficiency of representation types

When considering nonstandard visuospatial representation of arithmetical tasks (i.e. partial or no use of standard mathematical symbols), an *efficient* representational strategy:

- includes all the elements which enable the student to solve the task (correctly) more quickly and/or with less effort than they would be able to do without those elements;
- does not include any elements which do not help the student to solve the task more quickly and/or with less effort.

This means that all kinds of imagery (4.2.1) may be part of an efficient representation for a given individual at a given time. Note that this includes decorative imagery, which while not mathematically functional, may still serve a valid task-related purpose that accelerates solution – for example, to serve as a reminder of some relevant aspects of the task scenario, and so ‘anchor’ students with a tendency to wander.

I suggest that it is normal for people working on mathematical tasks at all levels to want to minimise their expenditure of time and effort; i.e. to use the most efficient representational strategies available to them. However, this does not mean that they <u>do</u> actually work in the most efficient way, as this tendency may be modified by their beliefs about desirable and undesirable strategies. Some students, unaware of teachers’ implicitly-expressed hierarchy of strategies and expectations of progress, happily use (e.g.) counting strategies that they perceive to be most efficient (for them, on that task, at that time), and in many cases they are correct to do so. Other students, for whom number fact retrieval and compact symbolic notation have repeatedly proved unreliable, nevertheless continue to attempt those methods because (a) it is quick and easy to produce <u>an</u> answer, if not necessarily a right one, (b) they see peers using them, (c) as discussed above, this behaviour is rewarded and reinforced by adults, (d) they believe that nonstandard alternative representations would unacceptable (or at least discouraged), or (e) they do not have the metarepresentational competence to create their own reliable representations, and guessing at answers or half-remembered procedures is their only option. Once my students were away from their peers, with time pressure removed, discouraged from engaging in guessing and other maths-like behaviours, given a variety of visuospatial representational options, and encouraged to use whichever forms and elements they felt best suited their needs, they began to make better representational choices. They increasingly prioritised obtaining an answer <u>in which they had confidence, via a strategy they understood</u>, over producing a quick answer, regardless of accuracy.

Furthermore, analysis of the students’ representations over a period of time and multiple tasks showed ongoing changes in their representational strategies, which demonstrate a level of judgement which is not generally assumed to be present in those categorised as low-attaining. By this I mean that they effectively judged which representational elements were helpful to their thinking, discarding those that were not, or no longer, necessary. Those using unit-based representations decided when to drop organisational or decorative elements, and when it was appropriate to introduce number symbols. To use DiSessa’s term (again), they exhibited metarepresentational competence. While there were occasions when certain individuals worked inefficiently because their engagement had shifted from the original task to that of producing an attractive picture or pattern, in general the evidence supports the idea of students with arithmetical difficulties being able to make strategic representational choices that allow them to work on tasks in a way which is the most efficient for them at that point.

9.3.3 Transitioning between strategies

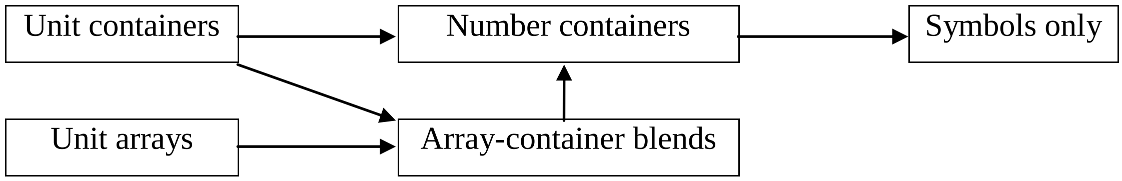
In Chapters 6-8 I provided many examples of students making small changes in the representational elements they included, which made whichever representation type they were using at that point slightly (or considerably) more efficient. However, I have also shown that the same collection/arrangement of marks on a page can be used in different ways, and so students' interactions with representations must also be considered – in particular, their style of enumeration, and any changes therein.

Students' enumeration in tasks with multiplicative structures develops (independently or through appropriate support) through some or all of the following stages:

- Counting: unitary → grouped → rhythmic → step
- Repeated addition (or, theoretically, subtraction)
- Multiplication or division using directly retrieved or derived facts

The main development, though, is a gradual change of focus from units to groups. Whether the tasks are multiplication-based (e.g. Holiday Clothes, Cuboid Starters) or division-based (e.g. Biscuits, Taxis), there is a total quantity which is made up of, or can be separated into, equal groups, and the most basic enumeration strategies involve counting without awareness of the repeating structure, while the more advanced ones make use of it to enumerate more efficiently. This move from units to groups as primary focus is mirrored in the representational strategies used.

I have described the way a student can move from manipulating concrete or drawn units (into equal groups) one at a time, to making use of visuospatial repeating structures, to manipulating component groups as though they were units, to, eventually, replacing groups of units with number symbols. In Chapter 8 I described how students used four key representation types, the fundamental containers and arrays, the combination array-container blend, and number containers (an intermediate stage on the path to fully symbolic notation). There is not a single linear path through these representation types (and on to standard symbols), but there are various possibilities.



Note that while continuous models such as number lines have not featured in this study, it would not be difficult, if so desired, to create bridging representations with which to connect, say, number containers (perhaps, within-scenario, as rows or stacks of buses) and a *double number line* (Küchemann et al., 2011).

While I described students discarding no-longer-needed container elements as an efficient representational choice, it is also important to note that the replacing of the units within containers (or array-container blends) with numbers was not only important for its inclusion of standard symbols, but a major step in this change of focus from units to groups. It is going from using one mark to stand for one thing (unitary representation), to using one mark to stand for many. Thus, the use of more minimal representations is not, taken by itself, directly linked to progress in understanding, or greater efficiency of representational strategies.

9.3.4 Implications for learning

As I have stated previously, the key to moving from the most basic forms of arithmetical representation (scenario-embedded/concrete/enactive/unitary) to the desired end-point of maximally-efficient and versatile symbolic notation is a chain of small and well-connected steps. There should never be too great a cognitive leap between one and the next. This chain metaphor (or others I have used, such as stepping-stones) should not be taken to imply there is only one path of representational progression: there are many possible paths. Neither does it imply that low-attaining students are not capable of making sudden cognitive leaps, and skipping several possible interim stages at once: of course this happens sometimes, more often for some than others, depending on the individual nature of their difficulties and strengths. However, as progress cannot be expected to be steady, linear or monotonic, a cognitive leap made on one occasion does not mean that the connection made is secure and always available for future tasks.

In discussing issues of improving economy and efficiency in young (typical) children's arithmetic, Hewitt (1996) uses the analogy of learning to walk. For students such as mine, an apt analogy is with the mobility of a person with significant physical illness or injury. A pair of crutches may be necessary, then maybe just one crutch, then a cane, before walking freely. On one day, the person may be keen to be rid of the supports, and to see if it is possible to manage without; on another, they may feel weaker, more vulnerable, or lacking confidence for a particular trip, and need those previously-discarded supports. It may be appropriate to set out on a journey walking normally, but with a fold-up cane tucked away, ready to be deployed if movement becomes difficult. So it is in arithmetic, with the range of visuospatial representational strategies which act as optional supports for thinking and problem-solving: likewise, a student may on one day manage without the cognitive support of, say, container representations, but on another, be very glad to have it as a backup strategy.

It is worth noting that a more minimal drawing (i.e. with less use of representational supports) does not automatically imply a task being completed more quickly. With less in the way of external markings, more may have to be remembered, or worked out in the head, and a new strategy may be slower or more effortful, at first, than a well-practised draw-and-count, for example. Like Yeo (2003), I have observed students reliant on counting-based strategies who have become very quick, and very attached to them; like Karsenty et al. (2007) I have also observed students attempting to discard visuospatial representational elements too soon, in a rush to conform to culturally-approved symbolic notation. Meaningful progress in metarepresentational competence requires a balance between these two states. In an ideal learning environment where time pressure is removed, a variety of representational options have been experienced, and understanding all stages of working is prioritised, students' general tendency to make choices that avoid unnecessary effort should theoretically lead them to use the most efficient strategies (by my definition) in their arsenal. However, for this to happen, low-attaining students will need particular kinds of teacher support.

9.3.5 Implications for teaching

I have discussed the reasons students might be attached either to poorly-understood symbolic notation, or to unitary counting-based models, for multiplication and division tasks. I have argued that in progressing securely, there need to be many small steps with incremental changes (incorporating and discarding representational elements), and some more radical changes (switching between representation types). What can a teacher do to help release students' mark-making, and allow them to make good representational choices? Firstly, they need to foster an environment where multiple representational modes and media are available and acceptable, and representational exploration and experimentation are encouraged. Recall also that changing mode may be easier than expected, if examples are well chosen (9.3.3). For example, even students as weak as Paula can switch from concrete to graphic mode – provided there is high visuospatial similarity between the two representations used, in terms of their layout on the workspace. This is particularly the case if there is also similarity of motion (e.g. in the back-and-forth movements between the remaining original stock of units and the distributed groups, when dealing).

Secondly, on having tried more than one way of representing a task type, many students will require prompting to stop and compare the advantages and disadvantages of the different strategies. It is also worth stressing explicitly to them that a nonstandard strategy which is well-understood, and that the student is confident of using successfully, is better than a half-remembered, half-guessed standard one. Likewise, it must be reinforced that if drawing or modelling in a certain way enables the student to solve the task (correctly) more quickly and/or with less effort than without doing so, then that strategy is acceptable. If their individual representational preferences and strategic choices are respected, this will additionally benefit their self-esteem and independence as students of mathematics.

Lastly, it is necessary somehow to disincentivise maths-like behaviours, such as shuffling symbols around hopefully, guessing, and producing as many ‘answers’ as possible. Mathematics is not a disconnected mass of arbitrary rules and right/wrong judgements, but this may be the main prior experience of low-attaining students (and others). It is, nevertheless, possible to build structured, connected understanding of arithmetic, and while it may seem to take a long time, that time is not wasted.

9.4 Additional research outcomes

9.4.1 Theory

In Chapters 2-4 I addressed three bodies of research literature: difficulties in mathematics, arithmetical strategies, and visuospatial representations. These were all instrumental in developing an appropriate specialised methodology for data collection, organisation and analysis; in this I include the many ethical and pedagogical considerations guiding the day-to-day interactions with my young participants. They allowed me to form several theoretical assumptions regarding learners' difficulties, strategies and representations – clarifying, strengthening and adding to the beliefs drawn from my teaching experiences and my previous research. These assumptions I now revisit, with commentary on the extent to which the findings within this thesis align with the relevant literature.

9.4.1.1 Difficulties in mathematics

I asserted that this country's current SEN framework is compatible with an *interactionist* model of causation of special educational needs, which assumes that environmental and child factors interact over time to result in the difficulties that give rise to special educational needs (Wedell, 2008, in Warnock, Norwich, and Terzi, 2010, p.p.70). Moreover, when applied in a non-discriminatory and ethical way (Warnock et al., 2010), it is not incompatible with a *capability approach* philosophy, where student's specific learning difficulties are seen as *limitations in particular functionings* resulting from the interaction of the personal characteristics of the child with the schooling environment (where the latter is not appropriately designed and – perhaps because of this – the individual is not receptive). Using this philosophical basis, I designed a programme of study with the flexibility to adapt dynamically, in 'real time', to individual students' changing specific limitations, and assert that my students benefited from this philosophy, as in fact would all students, were this possible.

Studying the history of Special Educational Needs also informed my philosophy and practices. Although aware of the long-term prior difficulties with mathematics that were the case for most of my students, and the global nature of the learning difficulties attributed to some by educational psychologists, I chose the deliberate stance of assuming *specificity* of difficulties (i.e. that their abilities were componential; finding one arithmetical concept or process problematic does not imply that all related concepts and processes will necessarily be problematic), and *non-permanence* of difficulties (i.e. an openness to the possibility of significant change in capability). I also followed a policy of minimal labelling of students. Would either the tuition or research have been improved by greater use of categorisation of students into more general 'types' of learning difficulty? I do not believe so.

Finally, for the purposes of this study I asserted that understanding the issues of a small number of students who find mathematics extremely difficult would help in understanding the issues of the larger number of students who find mathematics moderately difficult. This is in line with a general assumption of *normality* of difficulties – providing one is considering an individual's current limitations to be specific and componential. This is not to argue that some students – Paula springs to mind – do not in fact struggle with all aspects of the subject; however, their struggles with different aspects may still differ in severity and kind. Moreover, the same representational strategies which allow a student with extreme difficulties to slowly come to understand a particular mathematical idea ('layers' in 3D arrays, as one example) may provide a swift moment of clarity for a student with moderate to mild difficulties.

9.4.1.2 Arithmetical strategies

In 3.5 I discussed the evidence for persistence of multiple arithmetical strategies in both young children and adults (Dowker, 1992; et al. 1996; 2005), with the possibility of different strategies dominant in different contexts (Siegler and Shipley, 1995; Shrager and Siegler, 1998), different routes of strategy preference (Gray 1991; Brissiaud and Sander, 2010), and new effective strategies being used (with increasing frequency) alongside old ones rather than simply replacing them (Fletcher et al., 1998). While direct testing of these theories was not one of my research objectives, I did consider, prior to fieldwork, how well they might fit my particular participants, these generally not being of a comparative age or 'ability' to those in the studies cited. As has been seen, the data from my low-attaining secondary-age participants is in line with these previous findings. My students did indeed display multiple arithmetical strategies, choosing different strategies depending on the scenario in which the task was presented, the magnitudes involved, the representational media they perceived as available to them, and in response to both substantial and minimal teacher prompts. New strategies certainly did not immediately replace old ones; even after clearly visible 'a-ha' moments of clarity and comprehension, pleasure and pride, progress was not linear or unidirectional. Students required time and multiple examples to be convinced of the utility and applicability of new strategies, and in the absence of reinforcement outside of my sessions, sometimes forgot about them until reminded.

I also remarked upon Dowker's proposed 'U-shaped curve' of strategy variability (novices and experts exhibiting a wide variety of strategies compared to those at intermediate level). My data does not directly contradict or support this hypothesis. Due to the severity of difficulties my participants experienced with multiplicative thinking, for significant portions of the study some were in the position of having no viable strategy for the set tasks until supported, or had only the most simple of concrete or drawn unit-counting-based strategies; others, (currently) more able, displayed a greater degree of strategic capability and choice. On the other hand, those of my students performing best in their respective mainstream classrooms (particularly Oscar and Ellis) showed little strategic variability or inclination to experiment, relying more heavily on direct retrieval and standard algorithms. This implies a possible relationship between level of ability (or maybe, level of confidence in one's ability), but not a simple U function – perhaps some kind of wave form? After all, it would not be unexpected if struggling students clinging by the fingertips part-way up the cliff of received methods (even if unable to move) were less inclined to look around them than those at the bottom, with no grasp at all. I also suggest that the peer-pressure which can affect strategy choices (particularly in mainstream classrooms) may not affect everyone to an equal degree, perhaps acting more strongly on Dowker's 'intermediates' than novices or experts. This would be an interesting area for further research – perhaps comparing students' strategy choices in 1:1 situations to when surrounded/observed by peers.

Regarding the use of counting, I cited sturdy evidence for children with arithmetical difficulties having much greater reliance than their typically-attaining peers on counting-based strategies (as compared with direct retrieval or derived fact strategies) – and my participants indeed made heavy use of counting (of various types). When successful in their use of counting strategies for multiplication- and division-based arithmetical tasks, there was a tendency to then become over-attached to unit-counting, this inhibiting further development, as predicted by Yeo (2003). However, while Yeo's conclusion that in these cases a 'reasoning habit' must be deliberately fostered is not incorrect, it carries an implication that where counting is being used, the student is not reasoning. This may be broadly the case with additive-structured tasks, but the appropriate organization/representation of multiplicative structures for counting can be not just a 'link in the conceptual chain' (Gray, 1991) but require significant reasoning about numbers, as can the introduction of structured counting.

I looked at various analytical frameworks that have been applied to children's single- and double-digit natural number arithmetic, and identified weaknesses. In the case of both counting development and higher-level strategies, there is a tendency in frameworks to mix strictly enumerative elements (e.g. grouped counting) with representational elements (e.g. counting of concrete objects). It is my view that while there is a strong and complex relationship between representation and enumeration, they occupy different analytical dimensions (as do other factors, such as interpersonal interactions). One arithmetical strategy can be represented in a variety of different ways; one visuospatial representation can derive from or be used for a variety of different calculations, and any proposed path of arithmetical development should not mix elements of both as if they were interchangeable. Furthermore, the role of rhythm in enumeration is under-theorised and would benefit from greater emphasis.

9.4.1.3 Visuospatial representation

In 4.2 I addressed research on the use of different types of representation in arithmetical tasks: that external representation can reduce the amount of cognitive effort required, compared to holding information in short-term memory; that the way the information is represented can make problem-solving easier or more difficult; and that graphical elements in a visuospatial representation may constrain the kinds of inferences that can be made about the underlying represented world (Scaife and Rogers, 1996). While the evidence for these assertions is strong (and corroborated by my own data), I found the frameworks proposed for the analysis of visuospatial representations used in arithmetical tasks unsatisfactory. Many, particularly from the more traditionally experimental studies, are only applicable to pre-prepared graphics presented to students, and do not even allow for the way different students might interact with them; they also tend to divide representations (presented or created) into categories which mask the breadth and variety of students' representational activity, and – as mentioned above – mix representational with enumerative elements. For this reason, it was necessary to develop my own framework.

Drawing on the theoretical work of Lakoff and Núñez (2000), I considered how two fundamental 'grounding metaphors' for arithmetic – *Object Collection* and *Object Construction* – might apply when providing materials and prompts in division-based tasks. I theorised the particular importance of *container* and *array* representations as two archetypal ways of organising units into the necessary equal-groups structures (independent of the mode, media, scenario resemblance, etc. with which they were represented). While arrays are more advanced in their spatial organisation and affordances, there is clearly something profoundly graspable – even for the most numerically limited – in the topological nature of containers; even if one cannot process the relative alignments or distances between units, enclosing sets in containers delineates them from all outside. Prior to fieldwork, I also designed an *array-container blend*, which functioned (as planned) as both a bridging image between the two types, and a tool for effecting *aspect shift* as a visual manifestation of the commutative principle. While there have been studies in which children have been observed using representations which fall under the two main types – generally containers for very young children, and dot, square or cube arrays for older ones – they have not looked at participants who are free to choose their representation type, and how their choices regarding representation type changed over time. I endeavoured to do this, and to address not just whether these representation types helped, but how they helped (see 9.4.3.2).

9.4.2 Practice

In Chapter 5 I developed a methodology based on both research literature and professional experience. I now appraise the effects of my key methodological and pedagogical choices, and make some observations regarding participant experience.

9.4.2.1 Tuition programme

I cited Pólya's work on problem-solving as an inspiration, particularly the instruction to 'draw a figure' when first trying to understand a new problem, and made "Is there something you could draw that might help?" (or equivalent) my first metastrategic 'nudge'. However, both prior research and my own experience indicate that secondary mathematics classroom environments may create strong disincentives to use informal drawing-based strategies, and that even if students do not feel this negative pressure, they may lack the basic metarepresentational competence to create any kind of usable visuospatial representation when unguided. Withdrawing from the class for individual or paired tuition certainly appeared to increase all students' openness to nonstandard and informal representational strategies (although to varying degrees), based on observation and their own verbal statements; however, a change of scene cannot undo the habits and limitations of years of standard mainstream instruction and practices. In a more flexible environment the question arises of how much, and what form, of guidance to give to a struggling individual.

The different schools of thought on this range from a taught curriculum including principles for choosing appropriate representations for the information given and task requirements (e.g. Cox) to very open-ended, child-directed investigative work (e.g. DiSessa). While inclining towards DiSessa, I had concerns that his approaches would not work as well with my low-attaining adolescents as with his mid- to high-attaining primary schoolchildren. These concerns turned out to be warranted, as there were many occasions when my students were completely ‘stuck’. This is where my practices of minimal ‘nudges’ and constant close observation are effective, as they allow each task to be started in an open manner, and at each point for the student to explore independent representational choices, while at those points where they cannot continue independently, receiving teacher support tailored to their own previous work. Another key requirement for this kind of independent exploratory work is the absence of constant time pressure (9.2).

In 3.4.3 I questioned an assumption frequently found in the literature: that children dislike ‘word problems’. I suggested that the supposed difficulty actually stems from either comprehension of the text itself, or the nature and presentation of the extra-mathematical content involved, both of which making it difficult for some students (e.g. those who are not strong readers, or natural automatic visualisers) to turn the information provided into a working model of the scenario. In fact, I have presented evidence of students displaying a very strong preference for scenario over bare tasks, and when working on them, made even more use than I had expected of elements which were not directly mathematically functional, often adding their own narrative and/or pictorial details.

What, then, makes a good scenario task, suitable for this kind of tuition? Firstly, it is introduced verbally and conversationally, with opportunity for the student(s) to interrupt and ask for clarification or add their own personalising details. It is flexible, with a single scenario allowing opportunity for different sub-questions, support and extensions as appropriate to the individual’s changing needs. Lastly, it must be genuinely imaginable – by which I do not necessarily mean ‘realistic’ (5.4.2.1), as students are happy to overlook many unlikely scenario elements if acknowledged and internally consistent. (For example, I recall from my teaching days one of my bottom sets becoming particularly engaged with multidigit division through an invented scenario forever after referred to as ‘Alien Space Biscuits’.)

9.4.2.2 Data collection

For the most part, the combination of the marks made on paper by students (and me) with their contemporaneous dialogues or monologues provided a vivid record of our working on each task. Not only that, but the sounds of pens scratching, cubes being tapped or pushed, etc., increased my awareness of the importance of learners’ developing rhythmicity in working with repetitive structures. Of course, rhythm does not go unmentioned in the discourse on counting development, but, as mentioned above, it is an under-researched aspect, our understanding of which would benefit from paying close attention to small changes in the timing and regularity of student’s counting as it progresses (in addition to the visuospatial regularity of their representations). For this, the incorporation of techniques from research in music education could be helpful.

Regarding this project, I still assert that the benefits of video recording would not have outweighed the negative consequences, and my phone camera ‘snaps’ were an appropriate pragmatic option. However, in an appropriate environment where video equipment could be set up easily, the participants would not be discomfited by its presence, and concrete modelling was the main representational mode being used, it could provide extremely useful data for the further investigation of students interacting with representations of multiplicative structures.

While all the tasks I set students follow firmly in the tradition of practical research in mathematics education, I have made certain refinements which I believe beneficial for their use in research, without being detrimental to their learning potential for students. As a general principle, allowing students to create their own representations provided a more informative window on their thinking than forcing them to interpret a (strange) adult’s drawings or diagrams, and if they need support, the sequence of actions in a co-created representation can be studied. Meanwhile, removing the requirement to read passages of text avoids the possibility of outcome being affected by reading comprehension difficulties. Observation of students’ spatial structuring in the Cuboid Starters tasks was much clarified by providing actual cuboids, rather than drawings of cuboids.

9.4.2.3 Participants' expectations and attitudes

CF: What are you up to in maths at the moment?

Jenny: Angles.

CF: How's that going?

Jenny: All right.

CF: Good!

Jenny: I don't really get it though.

CF: Is there anything in particular you're finding hard?

Jenny: Er, angles.

Later on in the same session as the above quote, I set Jenny a division task in a vehicles-based scenario. Her first attempt (Figure 9-a) produced the figure of 310 coaches, which she presented to me as her answer. Why do this? She did not appear to have any confidence in her work, rather just the knowledge that she had done something that looked maths-like, and a vague hope that I might pronounce the end product correct. But if 310 coaches were by some chance right, what then? What would be the benefit to her or to me of an answer without an understanding of the process that led to it?

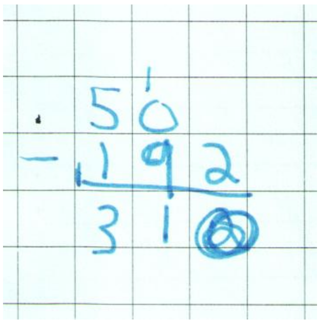


Figure 9-a:192 people grouped into 50-seater coaches (Jenny)

These two brief episodes, neither particularly unusual, exemplify a serious problem in the attitudes of my students (and undoubtedly others) to school mathematics, quite apart from the previously-discussed issues of subject-specific fear and loathing. This is that they do not necessarily expect to understand what they are doing in mathematics lessons, or consider it of any importance to do so. They may consider their educational experience to be satisfactory even while aware they 'don't get' the subject matter. They may believe it an entirely appropriate response to write down some number and operation symbols, perform a mixture of half-remembered calculation steps, then present the product of these actions to their teacher, without any personal consideration of the likelihood of its being correct. This is not a new finding, but it is a concerning one. How has this situation arisen, and given that it has, what may be done about it?

My participants were generally cooperative, and chose to engage in mathematical behaviours which prior experience had taught them would be pleasing to teachers; this attitude of performing mathematics as a series of increasingly-challenging hoop-jumping exercises – or worse, as a lottery in which one picked a number for each answer and was told if it was good or bad – must have been well-reinforced over the years to embed itself so thoroughly. While not entirely unexpected, the degree to which my participants exhibited this kind of behaviour was striking, and the degree of difficulty I had in convincing them that guesses were unwelcome, and the rightness or wrongness of a given answer was of less relevance to me than the way they had obtained it, was remarkable. Nevertheless, my repeated avowals of sincere interest in students' thinking during tasks did have an effect; they were increasingly willing to take their time working through points of confusion rather than jumping to guesses, and to share their thought processes. This reinforcement was an important part of fostering the beginnings of a 'reasoning habit' which ought to be a central part of all learners' mathematical experience.

9.4.3 Analysis

It was clear from a very early stage in the research that with multiplicative thinking, there are many potential stages for students in between 'not getting' and 'getting it'. Concepts and structures are componential, and may develop piecemeal. With very close observation, changes in systematicity may be seen (e.g. emerging pattern in the listing of combinations, increasing use of rows or columns, more regular dealing motion) before the individual is aware of the complete structure.

With this necessary close observation, my fieldwork had generated a large quantity of complex multimodal qualitative data – enough for multiple theses – necessitating decisions regarding selection of certain parts for analysis. Meanwhile, my examination of literature had not provided an appropriate analytical framework, requiring me to develop my own by combining, adapting and extending ideas drawn from both literature and practice. Thus, there was an ongoing relationship between theory and data.

9.4.3.1 Data selection and usage

The Two Tasks were not chosen at random from the many tasks available; they both produced small, self-contained data sets which could be treated independently from the main tuition material; they were both also new adaptations of tasks familiar from the literature. Holiday Clothes provided a first glimpse of the huge variety of representational strategies used, information on individual students' representational preferences and capabilities, and a starting set of analytical aspects through which to accurately describe, compare and contrast visual and interactive attributes of student- and co-created representations. The Cuboid Starters series additionally allowed comparison of the same individual's responses over time; it also prompted development in my thinking about spatial representation of multiplicative structures. Both tasks would be suitable for use by teachers and researchers, either in diagnosing individual capabilities and limitations, or for comparison of a cohort.

All thirteen participants were interesting, but those chosen for the Two Students case studies were especially suitable for focus because of the particularly pronounced nature of their arithmetical-representational capabilities (or lack thereof) – in Paula's case, her extremely limited conceptualisation of division, high level of support required, and painfully slow progress; in Wendy's, the highly effective use of pragmatic alternatives to memory-dependent practices, responsiveness to appropriate support, and fast-developing metarepresentational competence. Even with only two cases, I could still not possibly include all their respective data while examining it in such detail, so limited each subchapter to one form of division only. An obvious alternative approach would have been to compare the two students on the same type of division and task scenarios; however, I believed there were greater benefits for theory-building from examining both partitive and quotitive models. They also contribute to the small but growing literature genre of microgenetic case studies.

By Chapter 8, I had had refined my qualitative frameworks enough to enable the definition of four key representation types, including within them the great majority of my collected visual data. These are discussed below.

9.4.3.2 General framework of analytical aspects

This project required an analytical framework capable of separating a visuospatial representational strategy into component strands, in order to compare and contrast them independently, and to organise the qualitative observations made. This enables one to take representations that are superficially similar but structurally dissimilar (or vice versa) and ask questions such as: ‘What, precisely, is the same and what different?’ or ‘What has changed?’. I present the following set of named analytical aspects, not as exhaustive, but as of proven use in qualitative research on student-created visuospatial representations of arithmetic. I suggest it may also be of use to those providing 1:1 support to struggling students, both in initial diagnosis and tracking progress which is much slower, and involving much smaller steps, than typical.

Visual aspects	**Spatial aspects**	**Numerical aspects**	**Interactive aspects**
Mode	Unitariness	Enumeration	Consistency
Media	Motion	Success	Errors
Resemblance	Spatial structuring	Completeness	Verbal prompts
			Visuospatial prompts

Table 9-a: Aspects for the analysis of visuospatial representational strategies

Note that I consider consistency ‘interactive’, as a given representational strategy may be inconsistent in terms of its resemblance (e.g. decreasing decoration), spatial structuring (e.g. changing alignment, use of containers), enumeration (e.g. unit- to step-counting), etc. Likewise, errors made may be of the enumeration or spatial structuring variety, or other kinds (6.2.4.3).

Some aspects are clearly categorical (mode, media) or even just yes/no (unitariness, success, completeness), whereas others stand for spectra of variation (e.g. resemblance). I first noted motion as simply present or absent in a representation, before deciding it was necessary to describe the kind of motion involved. In this manner, aspects can be expanded depending on the needs of the particular analysis being undertaken.

9.4.3.3 Expanded analytical aspects

I have used the set of aspects as a framework for qualitative, descriptive analysis; however, it would also be possible to adapt it for larger-scale research using quantitative methods, by taking one or more aspects and expanding each into a categorical variable. Examples of task-specific expansion into categories (some hierarchical) have been seen in 6.2.4 – for spatial structuring, enumeration and errors in using presented 3D arrays.

Similarly, the four Key Representation Types may be considered specific kinds of spatial structuring (three unitary, one non-unitary) used in student- and co-created drawn representational strategies, and these could be incorporated into future research. Regarding pedagogy, I have stated that one of the aims of this analysis was to ascertain how the representations functioned: while the complexity, variability and individuality of this 'how' was explored in 8.3, it is possible to give a simplified overview:

Representation type	**Students' current stage**	**Subject content**	**How it helped**
Unit containers	Partial or no concept of division	Partitive and quotitive division; relationship with multiplication	Model for sharing and grouping units; seeing repetitive structure
Unit arrays	Partial or procedural concept of division	As above, plus: commutativity of multiplication; multiplicative structures as static relationships	As above, but seeing 2-dimensional repetitive structure
Array-container blends	Using either unit containers or arrays to divide	As above, plus: factorising numbers	As above, plus: seeing equal groups as 'units' in a larger structure
Number containers	Ready to transition from unitary to symbolic representation	Quotitive division; multiplication; recording work	Introducing symbolic notation via familiar imagery

Table 9-b: Use of key representation types

10 CONCLUSIONS

10.1 Summary

I began this study with a clear idea of certain characteristics: the participants I wished to study, and the settings in which to find them (the most low-attaining students in secondary mainstream education), the subject material they would work on (division), and the aspect of their mathematical activity on which I would primarily collect data (their visuospatial representations). I had also decided on a dual teacher-researcher role for myself, in which I could make use of my professional experience and respond flexibly to participants in real time. Less clear at the start were the exact research questions I could ask and answer, and the main thrust of argument did not coalesce until the analysis stage. I stated in the Introduction that I was fascinated to find out more about these students' mathematical experiences, keen to share my findings with the research community, inspired to try to find some way to help them, and impelled to share pedagogical implications with the teaching community. I believe I have done so.

My interactions with students were designed via a methodology at the intersection of ethical pedagogical and research practices. From a teaching viewpoint, by checking the integrity of the conceptual foundations of division for each individual, then fixing or filling in some of the weak, incomplete, incorrect or missing links, I supported students' development in multiplicative thinking and progress towards a more solid, understanding-based use of those mathematical symbols with which they were all familiar, but far from comfortable. From a research viewpoint, this illuminated many of the difficulties low-attaining students may have in the move from additive to multiplicative thinking. It also provided information on the various ways visual representations can function in both the short term (for solving tasks) and longer term (forming and linking arithmetical concepts and processes). Although this study has comprised the analysis of several distinct subsets of data, they are linked by one overarching theme, which is the emerging and developing of multiplicative structure in students' representational strategies.

In each of the analysis chapters I posed slightly different, dataset-specific, research questions. The overall questions addressed by this thesis are:

- **What representational strategies do low-attaining students use for multiplicative-structured tasks?**
- **What relationships can be found between their representational strategies and their multiplicative thinking?**
- **Under what circumstances do these students produce work in which progression of arithmetical understanding can be seen?**

The first is essentially descriptive: I have observed, catalogued and described the representational strategies my students used over various different types of task, before, during and after teacher intervention. The second is more complex, and has involved looking at students' changing usage of different representational types and styles within the context of their individual intra- and inter-session trajectories. The findings from these analyses have enabled me to draw certain conclusions regarding the teaching and learning of students labelled low-attaining by the current education system. I address the three questions in subsequent sections, after first considering the limitations of my research.

10.2 Limitations

Although I have included specific limitations of methodology and analysis as and where they occurred in previous chapters, there are some general ones which require addressing.

As stated early on, I am myself an active participant in this study, and it is reasonable to question the effects of both aspects of my teacher-researcher role on the data gathered and conclusions formed. How would the study have differed if carried out by someone else? Individual students respond differently to different teachers, and so interpersonal relationships with individual students would have been different; however, it is reasonable to suppose that at least some participants would have responded positively to another researcher following my methodology, been willing to experiment representationally and share their thoughts. It having been necessary to develop my own analytical framework, it thus does not come pre-tested by previous research. I certainly do not suggest that my list of thirteen qualitative analytical aspects is the only possible framework for analysing this kind of data, and neither are my four key representation types exhaustive, but I have shown them valid and useful systems. Similarly, as a single-author study, the findings are reliant on one person's interpretation; however, while acknowledging subjectivity, I have endeavoured to clearly substantiate all claims made, and shared my findings in order to receive feedback at various stages (e.g. as conference papers).

My data was gathered from thirteen specific individuals, during 2008-2009: how would the study have differed if carried out with other participants, and how does this affect the possibility of generalisation to other learners? Although I have discussed extensively the mathematical behaviours of particular students, and findings based on such, all learners have different patterns of capabilities, limitations and preferences, and the range of characteristics found within the individuals of my cohort could be expected to be found likewise within other individuals, at the time the fieldwork took place, at time of writing, and in the future. In this sense, generalising is possible, although must be done with particular care not to over-identify patterns of mathematical behaviour, preferences and tendencies with SEN diagnoses. For example, some of Leo's characteristics reminded me of autistic students I have previously taught, and some of Wendy's my past dyslexic students. This is not a basis from which to start making general recommendations for students with dyslexia or on the autistic spectrum, or assuming intra-diagnosis homogeneity.

Later in this chapter I comment on implications regarding school-based SEN support. Support teaching was not a focus of this study, but my recommendations do stem partly from informal observations in my two fieldwork schools (as well as prior professional experience). I do not suggest that characteristics observed in the two schools in which I researched, or those in which I have taught and visited, are universal; however, the schools were not particularly atypical, so it is reasonable to suppose the SEN support observed to be similar to that of at least some of the support in at least some other schools. It should also be remembered that my participants were at secondary school, and so had also experienced many years of primary mathematics teaching (on which I can only conjecture), and, in the case of those with early SEN Statements, had received additional support there too (the content of which may only be gauged by omission).

This study focuses on individual and paired tuition only: how relevant is it to school mathematics in general? I am aware that the implications for teaching discussed in Chapter 9 are most applicable to a 1:1 context, and will become increasingly difficult to engineer with larger group sizes. Nevertheless, a considerable amount of 1:1 work takes place in mainstream schools, with many students with SEN allocated a set number of hours of individual support, either via withdrawal or in-class, and in some cases, further supplemented by outsourced or private tuition. Additionally, while close individual attention will obviously be harder to achieve with larger groups, many of the principles I recommend (see 10.5) do have wider application to the mathematics classroom and curriculum.

10.3 Representational strategies used

As has been seen in Chapters 6-8, students' independent representational strategies varied a great deal throughout the period of study, and across all analytical aspects. As they became used to the encouraging, non-judgmental environment, where their nonstandard arithmetical-representational strategies were valued, they allowed greater rein to their individual representational preferences, both in their independent choices and in their responses to teacher prompts. Although classroom observations have not been a focus for analysis, I observed enough of the students in mathematics lessons to be confident that the individual and paired tuition sessions drew from them more varied mathematical behaviours and more informative discussion of strategies (both providing a window into their multiplicative thinking) than they would have either opportunity or inclination to express in class.

The students, as a group, made much greater use of counting-based strategies than one might expect of 11-15-year-olds (even those in 'bottom sets'), coupled with mostly unitary visuospatial representations which provided complete sets of external units to count. This is an indicator that their multiplicative thinking was essentially not symbolic; however, while I have used terms such as enactive and iconic, these should not be treated as discrete homogeneous stages, as there are clearly very many micro-stages along the way. In fact, it is an oversimplification to use even a single spectrum of development, as various aspects of visuospatial representation (e.g. resemblance, spatial structure) can change independently and asynchronously.

Although the students used many task-solving strategies which might appear inefficient to the casual or untrained observer, I have argued that in fact many could make very good choices based on realistic self-assessment of the limited knowledge and skills at their disposal, and that these choices can and should be considered *efficient* for those students at that time, and evidence of a level of metarepresentational competence. However, some found it very difficult to let go of the maths-like behaviours to which they had become accustomed, and persisted in unsuccessfully performing manipulations of configurations of symbols they did not fully understand, and stating number ‘facts’ for which they did not have reliable recall. These behaviours may take up little time or space, but they should not be called ‘efficient’, as they are unsuccessful in terms of either short-term task solution or longer-term arithmetical development.

10.4 Representational strategies and multiplicative thinking

Investigation of this complex bidirectional relationship required teasing apart the multiply interlinked dimensions of arithmetical and representational strategy, then looking at how they were connected, in particular the relationship between changes in one and the other. From the tangled mass of nonlinear, nonmonotonic changes in representational and arithmetical strategies observed over the tuition period, one particular theme emerged, which was the emergence and development of structure; an anti-entropic tendency from a state of disorder to increasing order. In terms of the visuospatial structure in external representations, and its corresponding arithmetical structure, this could mean something akin to the progressions in Table 10-a (examples taken from a single student, Kieran).

Although presented as such here for the purpose of summary, these, again, must not be considered discrete or homogeneous stages. One verbal count sequence can be more or less rhythmic than another, and a count can become increasingly rhythmic as it progresses; similarly, one unitary visuospatial representation can be more or less spatially structured than another, and a single representation can become increasingly spatially structured during its creation. As seen in Chapter 8, small changes to individual representational elements constitute microprogressions in strategy. An incremental increase in the structuring of a unitary visuospatial representation can cause increased structure in counting, but counting with greater emphasis on component groups can also cause a student to create representations with increased visuospatial structure. Impetus for these changes may come from student or teacher. In particular, it appears beneficial for students who struggle with multiplicative thinking to become familiar with multiplicative structures and relationships through varied unitary representations, increasingly structured in as many incremental stages as is appropriate for the individual, before attempting to work with only non-unitary symbols.

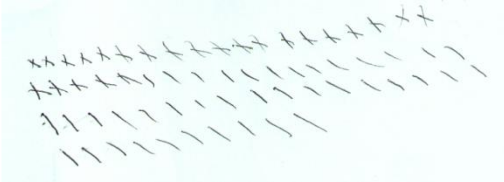
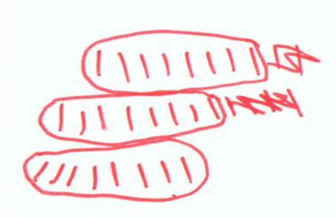
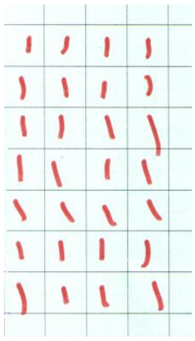
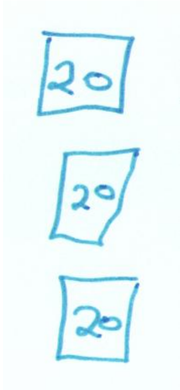
Enumeration	Spatial structuring	Example
(unstructured) unit-counting	unordered units	
grouped unit-counting	grouped units	
rhythmic unit-counting	spatially ordered units	
step-counting or repeated addition	spatially arranged symbols	

Table 10-a: Possible progressions of enumeration and structuring

In scenario tasks, the level of resemblance tends to reduce over time. As has been seen, sometimes decorative elements are mathematically functional and so can enhance arithmetical structure; however, too many or too detailed non-mathematically functional decorative elements can also obscure the underlying structure. Sometimes it is unclear to the observer how the representational elements chosen by a student contribute directly to task solution, but it is likely that they have not been chosen randomly, and fulfil some particular function (e.g. pictorial elements as an anchor to task scenario, multiple ringing patterns as an exploration or affirmation of multiplicative structure). It was clearly important, for example, to Vince to be allowed to decorate his vehicles (4.2.1.1) – so why not allow him to do so? Unless one of those rare occasions when the student's drawing is actively dissociating them from the task (as with some of Leo's), there is no good reason to prevent them. Recall, an *efficient* representational strategy is one that includes all the elements which enable the student to solve the task (correctly) more quickly and/or with less effort than they would currently be able to do without those elements (9.3.2).

Representations with low resemblance and simple spatial structuring (e.g. dot arrays) can be ambiguous in their meaning, and this ambiguity may be either a negative or positive trait, depending on context. It may be evidence of a student's confusion over how a particular representation type represents the task scenario (e.g. Jenny, 8.2.3.4); on the other hand, it may demonstrate versatility, and the ability to re-use the same form (i.e. the visuospatial manifestation of a single multiplicative structure) for different purposes (e.g. multiplication, partitive and quotitive division). This is something for the teacher to judge in each case.

Finally, I briefly mention finger counting, which I have noted certain students using, but which has not been a focus of analysis. Anghileri (1995) described how what is often thought of as a single 'finger counting stage' actually comprises a progression from unstructured unit-counting, grouped unit-counting (and, I would add, rhythmic counting) to step-counting. Although fingers are an obvious media for students at the counting stages, they are not actually very helpful in the early stages of developing multiplicative thinking, as (assuming the total >10) the complete representation is never seen at once, and so the internal structure is not visible; drawn units are better (or concrete, if available and the student is willing to use them). Finger counting does have its uses for those students who can reliably count rhythmically (e.g. for keeping track of number of replications when building up to a total), but even then, the equal-groups multiplicative structure is kinaesthetic/temporal rather than static/spatial, so cannot be examined as a static object embodying permanent numerical relationships.

10.5 Circumstances for observable progression

Constructivist-connectionist educational theory carries the implication of allowing students as much representational autonomy as possible, but in practice, this is often ignored when working with low-attaining students, who are assumed to be incapable of comparing and choosing arithmetical and representational strategies, and are instead trained to replicate a set of supposedly-efficient methods. On the basis of my interactions with the participants in this study (and my previous teaching career) I believe these students are capable of greater metarepresentational competence and independent strategising than they are generally given credit for – but require the right kinds of attention and scaffolding to draw out and harness their creativity in helpful ways. A balance must be struck which provides them with the necessary prompts while maximising their independence, and this may be done by making each prompt as minimal as possible, and not delivering multiple suggestions or pieces of information at the same time. Appropriate prompts may be intentionally vague (e.g. 'can you draw it?'), or more specific (e.g. referring to the strategy used for a previous task); they may be verbal or visuospatial (e.g. drawing containers to group a student's units); they may nudge the student towards a within-type change (e.g. changing the alignment of units) or a major type change (e.g. unitary to symbolic). Also, even when a student has successfully chosen appropriate representational strategies in the past, they may still need reminders on future occasions.

It has become clear that for this kind of learning, there are many fine judgements to be made on the part of the teacher (or other supporting adult). I have stated that there is no one path of development, so when a student is completely stuck, which strategic change to suggest? When are narrative and/or decorative elements beneficial to thinking about a task, and when do they become a distraction, or lead off on a tangent? How does one discern between a student building confidence through repetition, and one becoming 'stuck' in a successful but inefficient strategy, needing encouragement to modify it or try something different? There are no universal answers to these, as every student has their own individual thought processes, but the fact that there are not universal answers does not mean those in a teaching role should not be asking themselves these questions, and often. I have shown that close observation of how a student represents tasks, and how they respond to prompts, can yield a great deal of diagnostic information about how they think about numbers and arithmetical relationships; what they understand, and where the gaps lie. By observation, I do not mean only of the marks students make: one thing which I did not realise at the start of this project, but which became clear during analysis, is the importance of task narrative to students, and of the relationship between visual and verbal to the researcher (or observant teacher). Listening to what a student says <u>while</u> watching what they do, is key to understanding their current state of multiplicative thinking, and thus to deciding how best to prompt and support their next step. Nobody can expect to make the 'right' decision every single time (supposing there to be such a thing), but in a learning environment which values time spent on exploration, and multiple strategies, the only 'wrong' decision is one which stifles creativity and limits opportunities for mathematical thinking.

I assert that, in general, the allocation of 1:1 SEN support already in place for these students is not being used as effectively as it might. This is based not only on my observations of classroom-based support, but the results of the great many hours of individual support that students had already received. It is shocking that in all this time, nobody had taken the time or trouble to work through the dealing process with Paula, the place value system with Wendy, or the various other fundamental numeric concepts poorly understood by others of my participant group – but would contentedly spend time slogging through whatever exercises or worksheets were on the menu in class that day, repeating and repeating the performance of various maths-like behaviours with meaningless (or only partially-meaningful) configurations of symbols. This is at once acting at too high and too low a level, mathematically: too high in that these students were not ready to work fully symbolically, and too low in that they were believed incapable of rational thought or conceptual understanding. Moreover, the students themselves came to believe that they were incapable of conceptual understanding, and/or that this was not important for the learning of mathematics. These things are wrong. The progress I have described in previous chapters was based on under five hours per student, which suggests that their prior tuition was not fit for purpose. Firstly, unfit tuition prioritises the performance of maths-like behaviour over genuine mathematical thinking, and secondly, it does not properly consider the individual. It is a waste of valuable 1:1 time to use the same teaching methods as when teaching a whole class, or inflexibly to follow the same programmes for disparate individuals. In order to find out what really helps a particular student progress, it is necessary to study their strengths and weaknesses, understandings and gaps in understanding, and the representations of number relationships that work best for them. All of these, of course, will change over time, so a single assessment will not do; it must be a continuous, formative process.

Is it possible, or reasonable, to expect adults with less experience and training to work with these kinds of students in the way I did? To some extent. There are several aspects:

having the necessary subject knowledge (including awareness of multiplicity of arithmetical procedures, their underlying structures, and the value of derived and heuristic strategies), the observational acuity to make deductions and inferences about students' mathematical thinking (enabling them to work from an individual's current level, rather than the level at which the National Curriculum expects them to be), the willingness to work flexibly and dynamically for the real benefit of the students (as opposed to taking the easy route of proceeding through a standardised plan), the patience to take as much time as needed on a concept rather than rushing to the next, and the belief that all students can improve. Some school support staff have all these qualities, as do some private tutors, parents, and others involved in tutoring low-attaining students. Unfortunately, many do not. This study has focused on students, not staff, so I am unable to comment on the training currently provided and received, but these principles need to be a part of it. Of particular concern is the use of mathematics graduates or undergraduates for this work: an enthusiasm for, and personal capability in the subject, while helpful, are not remotely enough.

In recent years there has been something of a growth in intensive 'catch-up' numeracy programmes, first aimed at primary-age children, but now beginning to extend to the secondary age group. Where these are research-based and involve significant specialist teacher training, there is some evidence for their effectiveness, although reports are mixed and – due to their recent nature – as yet, few. The intensive intervention *Numbers Count* appears to be effective at least in the short term, if not necessarily economically supported (Torgerson et al., 2011), while *Numeracy Recovery* and *Catch-up Numeracy* have their support (Dowker and Sigley, 2010). This is an encouraging sign for those students whose numeracy difficulties do require specialist support, and perhaps if more receive this at primary level, there will be fewer like my students, requiring help understanding basic numerical relationships at secondary level.

In the mainstream classroom, while some secondary-age students would benefit from using concrete media, it is understandable that they may be unwilling, so drawing – being relatively discreet – should be encouraged. If students think of visuospatial representations as totally separate from formal symbolic ways of working, and do not initially appear to value them, this does not mean that those strategies lack value, or those individuals are 'not visualisers'. It may mean that the connections between 'bricks' and 'sums' have been insufficient or otherwise not been made meaningful for them by their experiences so far, perhaps because the cognitive leaps have been too wide. Flexible scenario tasks, particularly with an engaging narrative, are particularly helpful in forming connections between representations, as well as highlighting arithmetical structures, and should not be avoided out of misplaced fear that students dislike 'word problems'. (If using a particularly unrealistic scenario, students appreciate an acknowledgement of this; it confirms the teacher has a particular educational reason for choosing it, and demonstrates respect for their intelligence – something which, unfortunately, some may not experience often enough.)

There is a great difference between the lasting satisfaction of working through a task, understanding all the steps, and knowing it is right, and the brief, quickly-dissipating buzz of submitting a semi-guessed answer, waiting tensely, and then having the teacher pronounce it correct. The latter can be tempting to teachers as a way of keeping low-attaining students' attention, but this creation of a lottery-like atmosphere runs counter to the view of mathematics as meaningful, connected knowledge which they deserve as much as their faster-progressing peers. This is particularly the case with those number relationships which are still taught to a large extent through repetition and (attempted) committal to memory of verbal strings – the 'times tables'. My students were not able to do division tasks because of problems at a conceptual level, and no amount of rote knowledge of facts or symbolic procedures would cure the problem without improving understanding of the underlying numerical/arithmetical structures. To reward those occasionally-correct disconnected retrieved facts, then, could actually be counterproductive. In particular, students get particular satisfaction from working through tasks they have posed themselves or worked through in their own way (for example, Wendy's single multi-ringed array-container blend or Jenny's series of them (8.2.3), although the value of these is not always immediately obvious to the observer.

It is a problem when staff, schools, and curricula permit an environment which stifles low-attaining students' creative, imaginative, narrative and visuospatial tendencies in mathematics when these could be supporting learning. I have reported multiple occasions of my students being both pleased and surprised at how helpful it could be to employ nonstandard visuospatial representational strategies, and pleased and surprised to find themselves more arithmetically capable than they had believed. With larger groups, while it is impossible to observe all students so closely, generosity of time constraints is a possibility, as is working to create an environment which does not over-reward the performance of maths-like behaviour, but emphasises explorative mathematical thinking at all levels. In the words of Zoltan Dienes (in Sriraman 2008, p.p.3):

> Children do not need to reach a certain developmental stage to experience the joy, or the thrill of thinking mathematically and experiencing the process of doing mathematics.

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APPENDIX A: INITIAL ASSESSMENT TASKS

A complete list follows of tasks set in the Initial Assessment stage of fieldwork, each with a brief rationale for their inclusion.

Q1 Shapes

(a) Draw a rectangle.
(b) Draw a circle with a square inside it.
(c) Draw a big triangle that will fit around the rectangle [OR, if there is not enough space left, a triangle around all the other shapes].

The first aim of this task was for the students to 'claim the paper'. It is common for students to dislike showing their working, especially if it involves nonstandard representations, and I conjectured that they might be more willing to mark the paper during their calculations if the paper already something on it, particularly when the something in question was drawings (as opposed to just numbers) made by their own hand. The second aim was to provide some informal data about students' ease and comfort with drawing as an action, as it was to play a considerable role in tuition. I made personal observations on aspects such as the ease with which they manipulated a pen, the care they took to make the drawing accurate (e.g. how straight and parallel the sides of the rectangle actually were), their ability to follow verbal instructions relating to space and shape (e.g. to place the square inside the circle rather than vice versa), and their ability to make spatial judgements (e.g. to judge the dimensions and orientation necessary to place a triangle around one or more of the other shapes).

Q2 Number combinations

(a) Two numbers that add up to 10?
(b) Two numbers that add up to 24?
(c) Two numbers that add up to 75?

(d) Two numbers that multiply to make 10?
(e) Two numbers that multiply to make 24?
(f) Two numbers that multiply to make 75?

This task was chosen to put students at their ease by asking something that observational evidence suggested they would find comparatively easy, and then increasing magnitudes to see if it remained easy. Also, having more than one possible answer for each enabled accepting one correct answer and then asking for another (from the second student in the paired condition, or from the same student if appropriate). Regarding the numbers, 10 was chosen because it is extremely common in practice of number bonds, 'chunking' mental arithmetic strategies, etc.; 24 is above 20, i.e. the commonly-practised range for basic addition bonds, doubling strategies, etc. and also has many factors; 75 is outside the range of 'times tables', odd (so not divisible by 2 without recourse to non-integers), but a multiple of 5 (one of the most well-known number patterns).

Q3 Cubes: Visual estimation

I take a handful of multilink cubes.
(a) Estimate how many I have in my hand.

I allow them to count the cubes.
(b) Are there are twice as many in the bag, or more than that? Estimate how many cubes are in the bag.

I tell them the actual number in the bag - around 120.
(c) Estimate how many cubes it would take to fill up my rucksack.

The aim of this task was to gain some idea of students' sense of magnitude of numbers, with both direct and proportional estimation. For the handful it is simple direct visual estimation; for the bag, students could either reason proportionally using the first quantity, or just make another direct visual estimate. The last question, about the rucksack, is obviously difficult (and not one I would expect answered with great accuracy); its inclusion was intended to give an idea of students' concept of higher orders of magnitude.

Q4 Wheels: Replication-based multiplicative structure

(a) A van has 4 wheels. How many wheels there would be altogether on 3 vans?
(b) How many wheels there would be altogether on 6 vans?
(c) Each van is carrying 6 boxes in the back. How many boxes altogether are carried by 3 vans?
(d) How many boxes altogether are carried by 6 vans?

Extension

(e) Each box contains x bottles. What is the total number of bottles in y vans? [Choose numbers as appropriate to student.]

This set of tasks was the first to directly test arithmetic involving multiplicative structures. The scenario was chosen for its use of familiar objects, minimal level of verbal explanation, and ease of visual representation (drawing, particularly). It could be solved by recognition of the multiplicative structure followed by direct multiplication, repeated addition, grouped counting, etc. or by externally representing all items followed by unit counting. Increasing magnitudes were expected to result in changes in strategy in some cases. Magnitudes could be adjusted if necessary, depending on speed, accuracy and potential stress levels observed in student responses.

Q5 Rose bushes: Unconventional arithmetical structure

(a) A man has a path in his garden; a straight path which is 5 metres long. He plants rose bushes along it, one at each end, and all along the path with 1 metre gaps in between them. How many bushes is that altogether?

(b) Another path is 12m, and the rose bushes are planted with 2m gaps. How many bushes is that altogether?

When using this scenario in the precursor to this research project (Finesilver, 2006), it proved one of the most difficult for students, both in initially comprehending the task, and then when realising a single operation involving the two numbers would not suffice. While acknowledging its likely difficulty in this situation also, it was retained as I believed it important to include a task which did not conform to one of the expected arithmetical operations, and which participants frequently solve through their own or co-created imagery (Booth and Thomas, 1999; Elia and Philippou, 2004). The magnitudes and number relationships were chosen to be deliberately smaller and simpler than other tasks, with the intention that the numbers would not be an additional cause of stress.

Q6 Groups: More complex multiplicative structures

(a) A teacher has 20 kids in her class. She puts them into groups of 5. Each of those groups has 2 girls in it. How many boys and how many girls are in the class?

(b) There are now 100 [or 50] kids, and the groups are not mixed, but either all

girls or all boys. There are 7 [or 3] groups of girls. How many girls and how many boys are there altogether?
(c) There are 60 kids, and exactly twice as many boys as girls. How many girls and how many boys altogether?

This set of tasks, like 'Wheels', was designed to directly test arithmetic involving multiplicative structures, but with more complex grouping patterns instead of straightforward equal-groups replications. Different components of the tasks might be recognised as division and multiplication, and solved as such, but again, the scenario was easy to draw, and soluble by counting-based strategies. If students appeared to be struggling significantly with either the larger magnitudes, or proportional reasoning, (c) could be adapted or omitted.

Q7 Holiday clothes: Cartesian product

A boy/girl has 6 [or 4] t-shirts and 4 [or 3] pairs of trousers. The t-shirts are white, blue, green, brown, red and yellow. The trousers are black, blue, green and brown. Examples of different outfits: all in blue; blue trousers with white t-shirt. How many different possible outfits can be made?

This task was the only completely new addition to the Initial Assessment toolkit, and is discussed in detail in 6.1. I originally included it as an 'extension' item, to be used if (thanks to a student working particularly speedily, or any logistical issues) there was spare time between finishing the assessment and the end of the allocated time period. However, the data collected proved so interesting that for those students who did not have the opportunity to work on it in the first session, I found another occasion for them to do so.

Interspersed conversation

How do you get on with maths in general?
- Has it always been like that, or has it changed?

Do you usually find the work easy, hard, or in between?
- Has it always been like that, or has it changed?

Is there anything in maths you particularly like/dislike?

Does your teacher ever use drawings [like this] to help you?

Do the teaching assistants . . . ?

Do you ever draw [like this] to help work out an answer?

Follow-up questions as appropriate.

APPENDIX B: TUITION TASKS

A complete list follows of tasks set to all students in the Tuition stage of fieldwork. In some cases they took up the whole session; in others, they provided a springboard for further individually-tailored tasks based around the same scenarios and models.

Tuition 1: Partitive division with containers ‘Biscuits’

Starter: Cuboid 1

I present a cuboid made of 3 × 4 × 5 cubes.
(a) Estimate the number of cubes.
(b) Try to work out the exact number of cubes.

Division as sharing

Scenario: the student has baked a number of biscuits, and wants to share them among a group of friends.
(a) There are 15 biscuits to share between 3 students. How many do they each get?
(b) ... 24 to share between 4 ...
(c) ... 24 to share between 6 ...

Students may comment on the repetition in the numbers. If they do not, I prompt by asking whether they notice anything about the last two questions and answers.

(d) ... 27 to share between 3 ...

Finding all the factors

Scenario: The biscuits are to be put in packets for selling.
If there are 30 biscuits, what are all the different ways they could be put into equal-sized packets?

Tuition 2: Multiplicative structures with 2D arrays ‘Rectangles’

Starter: Cuboid 2

I present a cuboid made of 3 × 3 × 5 cubes. How many cubes altogether?

Rectangular area representation

On 1cm squared paper:

(a) Draw a rectangle which has exactly 12 squares inside it.
(b) ... 18 squares ...

I draw the left side of a rectangle and the start of the two adjoining sides, then write the total number of squares 'inside'. Complete the rectangle so it has the right number of squares in it. (Repeat with different numbers, chosen as appropriate.)

On plain paper:

I draw a rectangle, give the side lengths as numbers, and ask how many squares would fit in it.

I draw a rectangle, give the left side and total, and ask for the length of the upper side.

Extension: Formal division notation

I relate rectangular area representation to formal division notation, and set questions as appropriate.

Tuition 3: Quotitive division with containers 'Taxis'

Starter: Cuboid 3

I draw a cuboid diagram (either 5 × 3 × 2 or 4 × 3 × 2) and ask student/s to calculate how many cubes it would take to make it.
- If they find this too difficult, I suggest they try to construct it from cubes.
- If they find this quite easy, I draw another (6 × 4 × 2) without marking in the individual cube edges.

Division as grouping

Scenario: A large group of people are on holiday, and are arranging a trip.
(a) There are 16 people, and they need to travel somewhere by taxi. Each taxi will carry 4 people. How many taxis do they need?
(b) ... 20 people ...

(c) ... 23 people ...

(d) ... 35 people ...

This is a lot of taxis! The taxi firm also does extra big cars ('people carriers') for large groups. These can each hold 7 passengers.

(e) If the 35 people travelled in people carriers instead, how many do they need?

(f) ... 45 people ...

The holiday company decides to arrange tours for bigger groups of people. They use coaches with 25 seats in them.

(g) If the 45 people travelled in coaches instead, how many do they need?

(h) ... 96 people ...

Continue on same theme with either larger coaches (50 seats) or aeroplanes (200 seats).

Tuition 4: Summary

In this session I recapped and set tasks of the types covered in the three previous sessions. While this was planned in advance, I waited to fix the details of the tasks until after the previous sessions had taken place.

Starter: Cuboid 4

I present two 2 × 3 × 6 cuboids, one coloured in 3 (2 × 6) horizontal layers, one in 6 (2 × 3) vertical layers. I tell students they are both the same size, invite them to choose one, and ask them to work out the total number of cubes in it.

The reasoning behind this task (and Cuboid 5) is discussed in detail in 6.2.

Recaps: Baking and Transport

I remind students of the scenario tasks they worked on, showing them my collection of their work, and drawing their attention to any representations they had found particularly helpful.

Recap: Rectangular array representation

I draw an array of 4 × 7 small circles.

Pick a colour and ring the dots in groups of 7

Pick another colour and ring the dots in groups of 4.

I write 4 × 7 = 28.

If we know this, we also know that... __ × __ = 28?

And what is 28 ÷ 4?

28 ÷ 7?

Baking tasks

21 biscuits are shared between 3 people.

(a) How many do they each get?

(b) ... 27 shared between 3 ...

Rectangles tasks

Draw two rectangles, each with exactly 15 squares in it.

Transport tasks

A taxi firm has taxis which can fit 5 passengers in each.

(a) If there are 30 people who want to travel, how many taxis do they need?

(b) If there were 38 people, how many taxis?

(c) How many of the taxis are full?

(d) How many empty spaces are there?

Bare division tasks

If support is required, give prompts of potential visuospatial representational strategies (rectangular areas, dot arrays, container arrays, modelling with cubes).

Easy:	*Medium:*	*Hard:*
10 ÷ 2	*20 ÷ 2*	*30 ÷ 2*
15 ÷ 5	*30 ÷ 5*	*45 ÷ 5*
12 ÷ 4	*24 ÷ 4*	*36 ÷ 4*
40 ÷ 20	*100 ÷ 20*	*180 ÷ 20*
		650 ÷ 50

Final session

Starter: Cuboids 5

I present two identical 2 × 3 × 3 cuboids (2 × 2 × 3 for Paula) and ask them to work out total number of cubes in both.

Division preferences

I present students with three sets of pre-handwritten tasks written down on slips of the usual green paper. Each set consists of one calculation expressed in four different formats: Baking, Transport, Rectangle and (formal) symbolic. I read out the questions (in random order) and ask them to choose which one of the questions they would prefer to do, then to work out the answer to that one.

30 ÷ 5 (Paula: 20 ÷ 5)
36 ÷ 4 (Paula: 24 ÷ 4)
105 ÷ 21 (Paula: 60 ÷ 12)

After students complete the tasks, I ask them if the answers they worked out would be helpful in answering the other questions (that they didn't choose).

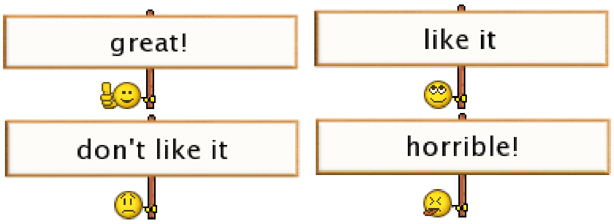
Extension: Creating division scenarios

I set the student a division task expressed only symbolically, and ask them to make up a scenario to go with it.

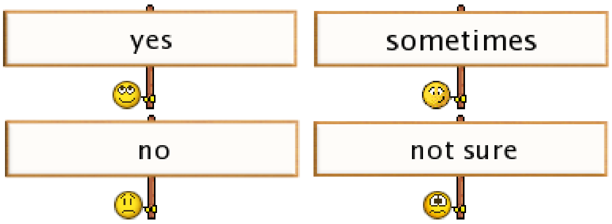
Appendix C: Student Questionnaire

Name: Age (years):

How do you feel about maths in general? [*pick an answer or write or draw your own*]



Did you enjoy the maths tuition sessions? *[pick an answer or write or draw your own]*



What did you like? What did you dislike?

Did the tuition help you understand any of these things better?

+ addition, plus

− subtraction, take away

× multiplication, times

÷ division, sharing, grouping

?? word problems

✎ solving problems by drawing or modelling

Was the tuition helpful in any other ways?

What would make Ms Finesilver's tuition sessions better?

Is there anything else you want to say?

APPENDIX D: RESPONSES TO 'CUBOID STARTERS'

Complete set of the students' response to the Cuboid Starters tasks, classified by spatial structurings (pre-prompt), enumeration strategies, and errors.

White	Complete correct strategy
Green	Correct structuring but enumeration error (NC or VK)
Yellow	SS error (counting cubes on faces)
Orange	SS error (counting squares on faces)

	Task 1: mixed-colour cuboid, 3x4x5	**Task 2:** mixed-colour cuboid, 3x3x5	**Task 3:** choice of striped cuboids, 2x3x6	**Task 4:** twin striped cuboids, each 2x2x3
Ellis	C3g VC error	C3r -	C3r * -	L3g -
Wendy	C3g VK error	L3g VK error	L3g * -	L3g -
Jenny	F3u (cubes) SS, VK errors	L2 -	L3g -	L2 (2nd block L3g) -
Tasha	F3g (cubes) SS, VK errors	L2 NC error	L3r -	L3r VK error
Sidney	F3s (cubes) SS, VK errors	L3u VK error	L1 VK error	L2 (2nd block L3s) NC error
Leo	O (colours) SS, NC errors	L2 NC error	L2 -	L1 * -
Kieran	F3u (squares) SS, VK errors	L3 -	L3 -	L2 -
Danny	F3s (squares) SS error	F3s (cubes) SS error	L1 * -	L2 (NC error on 2nd)
George	F3g (squares) SS, VK errors	*No data* *	L3g -	L3g (NC error on 2nd)
Oscar	F3g (squares) SS, VK errors	F3u (cubes) SS, VK errors	C2 * NC error	C2 * -
Harvey	F3g (squares) SS, VK error	F3s (cubes) SS, VK, NC errors	F3g (cubes) SS, VK errors	L1 (2nd block L3) (VK error on 2nd)
Vince	F3g (squares) SS, VK errors	F3g (cubes) SS, VK errors	F3g (cubes) SS, VK errors	F3u (cubes) SS, VK errors
Paula	F3u (squares) SS, VC, VK errors	N	F3u (squares) SS, NC errors	F3u (squares) SS, VK, NC errors
Total Layers	0	6	8	10
Total Columns	2	1	2	1
Total Faces	10	4	3	2

* On Task 3, Wendy and Danny did not use the expected layers (as delineated by colour)

* On Tasks 3, Ellis refers verbally to layers but counts in columns. On Task 4, used vertical layers, subdivided into columns.

* On Tasks 3-4, Oscar used columns, but may have been mentally grouping these into vertical layers. Uses multiplicative language.

* On Task 2, George’s data was lost due to a technical malfunction.

APPENDIX E: SUMMARY TABLES FOR CASE STUDIES

Paula's partitive division representations

		Media	Mode	Motion	Spatial structure	Enumeration	Errors	Success	Verbal	Visuospatial
1	15 ÷ 3	Cubes	Model	Non-ordered distribution	2 groups	Visual approximation	No. of groups	No	-	-
		Cubes/pen/ paper	Model-drawing	Partially systematic	3 containers of 3, remainder 6	-	Incomplete	No	-	Draw containers
					3 containers of 5	-	-	Yes	Share all	Pointing
	24 ÷ 4	Cubes/pen/ paper	Model-drawing	Partially systematic	4 containers of 4, remainder 8	-	Incomplete	No	-	-
				Non-ordered distribution	4 unequal containers	Visual approximation	Unequal	No	Share all	-
				Non-ordered distribution	4 containers of 6	Counts all groups	-	Yes	Share fairly	-
	27 ÷ 3	Cubes/pen/ paper	Model-drawing	Non-ordered distribution	3 unequal containers	Visual approximation	Unequal	No	-	-
				Partially systematic	3 containers of 7, remainder 6	Counts all groups	Incomplete	No	Share fairly	-
				Partially systematic	3 containers of 9	Counts all groups	-	Yes	Share all	-
2	21 ÷ 3	Cubes/pen/ paper	Model-drawing	Non-ordered distribution	3 containers of 7	Visual approximation	(correct by chance)	Yes	-	Draw containers
						Counts all groups	-		Check fair	-
	21 ÷ 7	Cubes/pen/ paper	Model-drawing	Non-ordered distribution	7 unequal containers	Visual approximation	Unequal	No	-	Draw containers

				Partially systematic	7 containers of 3	Visual count	-	Yes	Share fairly	-
	15 ÷ 3	Pen/paper	Drawing	Partially systematic	3 too-large containers	Visual approximation	Excess	No	-	Draw containers
				Dealing	3 unequal containers	Visual approximation	Unequal	No	Maintain quantity, share fairly	Demo 'dealing' dots
				Dealing	3 containers of 5	Visual count	-	Yes	Share fairly	Pointing
	15 ÷ 5	Pen/paper	Drawing	Dealing	5 containers of 3	Visual count	-	Yes	-	Draw containers
4	21 ÷ 3	Cubes	Model	Non-ordered distribution	3 unequal groups	Visual approximation	Unequal, excess	No	Maintain quantity, share fairly	-
				Dealing	3 groups of 7	Visual count	-	Yes	-	Demo dealing cubes
	27 ÷ 3	Cubes	Model	Dealing	3 containers of 9	Counts all groups	-	Yes	Counting deals	-
	18 ÷ 3	Pen/paper	Drawing	Non-ordered distribution	4 unequal containers	Visual approximation	Unequal, no. of groups	No	-	-
				Partially systematic	3 unequal containers	Visual approximation	Unequal, incomplete	No	Maintain quantity, share fairly	Draw containers
				Dealing	3 containers of 6	Counts all groups	-	Yes	Counting units	Draw containers, mime dealing
5	20 ÷ 5	Pen/paper	Drawing	Non-ordered	4 too-large unequal containers	Visual approximation	Unequal, excess, no. of groups	No	-	-
		Cubes/pen/ paper	Model-drawing	Dealing	5 containers of 4	Visual count	-	Yes	-	-
	24 ÷ 4	Cubes/pen/ paper	Model-drawing	Dealing	4 containers of 6	Visual count	-	Yes	No. of groups	-

Wendy's quotitive division representations

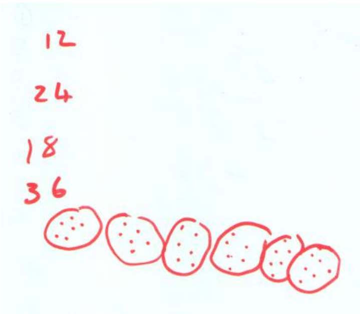
		Mode	Resemblance	Unitariness	Spatial structure	Enumeration	Errors	Success	Verbal	Visuospatial
3	16 ÷ 4	Drawing	Dots	All units	Array 4 rows of 4	Visual count	-	Yes	-	-
	20 ÷ 4	Drawing	Dots	All units	Array 5 rows of 4	Visual count	No. in group	No	Group size	-
		Drawing	Dots/containers	All units	Array 5 ringed rows of 4	Counts units, groups	-	Yes	Counting (in scenario)	Ring rows
	28 ÷ 4	Drawing	Dots/containers	All units	Array 7 ringed rows of 4	Counts units, groups	-	Yes	Counting (in scenario)	Ring rows
	32 ÷ 4	Drawing	Dots	All units	Array 8 rows of 4	Counts units, groups	-	Yes	-	-
	35 ÷ 7	Drawing	Dots	All units	Array 5 rows of 7	Counts units, groups	No. in group, drawing/counting	Yes	-	-
	45 ÷ 7	Drawing	Dots	All units	Array 6+ rows of 7	Counts units, groups	Total count	Yes	Counting (in scenario)	-
	45 ÷ 25	Drawing/numbers	Coaches (+ wheels)	Groups	Column of containers	Adds groups	Arithmetical	Yes	Addition method	Draw containers with numbers
	96 ÷ 25	Drawing/numbers	Coaches (+ wheels)	Groups	Column of containers	Adds groups	-	Yes	-	-
	391 ÷ 50	Numbers	-	-	Column of numbers	Adds groups	-	Yes	-	Bracket numbers
	612 ÷ 200	Drawing/numbers	Planes (+ details)	Groups	Column of containers	Step-counts groups	-	Yes	-	-
4	30 ÷ 5	Drawing	Dots	All units	6 groups of 5	Counts units, groups	-	Yes	-	-

	38 ÷ 5	Drawing	Dots	All units	7+ groups of 4	Counts units, groups	-	Yes	-	-
	343 ÷ 50	Drawing/numbers	Coaches (+ wheels)	Groups	Column of containers	Step-counts groups	-	Yes	-	-
	147 ÷ 21	Drawing/numbers	Coaches (+ wheels)	Groups	Column of containers	Adds groups (several iterations)	Representation inconsistency	Yes	-	-
	100 ÷ 20	Fingers only	-	-	-	Step-counts groups	Group size	No	Scenario	-
		Fingers only	-	-	-	Step-counts groups	Arithmetical	Yes	Group size	-
	650 ÷ 50	Numbers	-	-	-	Numerical calculation	Wrong operation	No	-	-
		Numbers	-	-	-	Adds, step-counts paired groups	-	Yes	Scenario, pairs of 50	-
5	36 ÷ 4	Drawing	Marks/containers	All units	Array 9 ringed rows of 4	Counts units, groups	-	Yes	-	-
	105 ÷ 21	Drawing	Marks	All units	Array 5 rows of 21	Counts units, groups	-	Yes	-	-
	180 ÷ 20	Drawing/numbers	Containers	Groups	Column of containers	Step-counts groups	-	Yes	Scenario	Draw containers with numbers
	240 ÷ 20	Drawing/numbers	Containers	Groups	Column of containers	Step-counts groups	-	Yes	-	-
	300 ÷ 25	Drawing/numbers	Containers	Groups	Column of containers	Adds groups (several iterations)	-	Yes	Sets of 4 × 25	Continue drawing

APPENDIX F: REPRESENTATION TYPES

Unit containers

Initial Assessment



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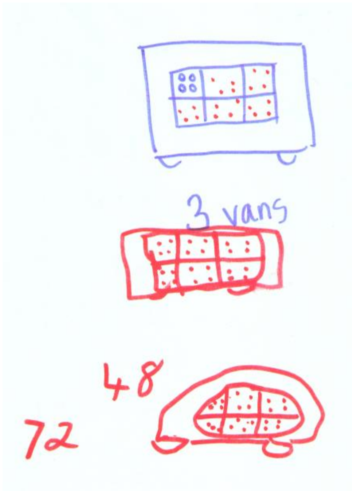

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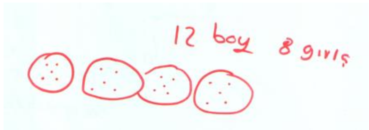

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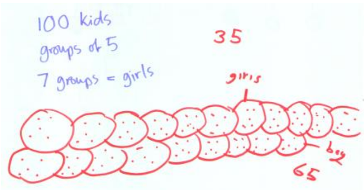

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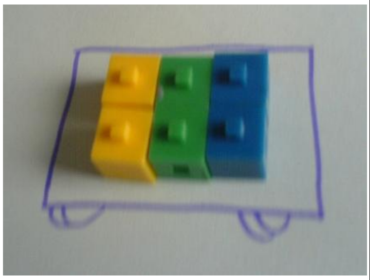
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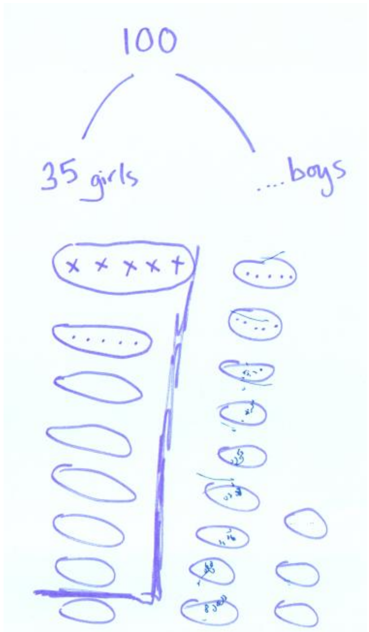
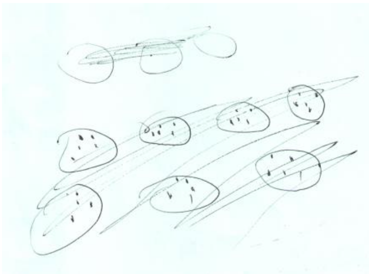
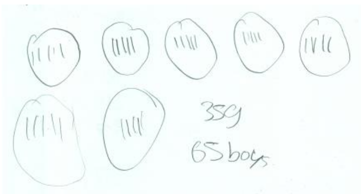
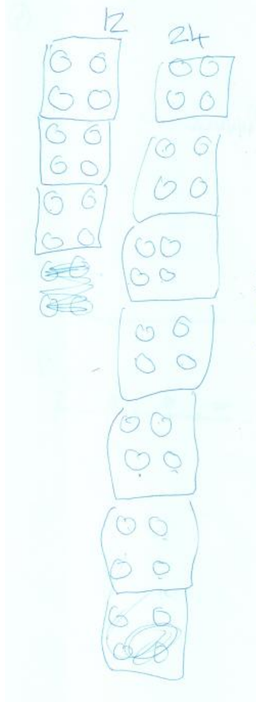
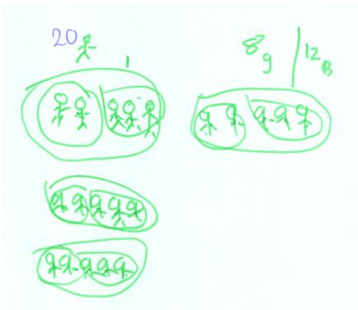
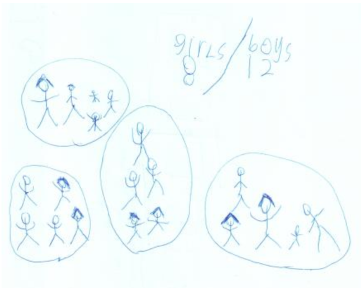
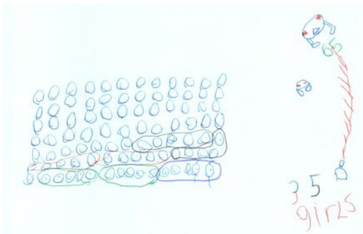
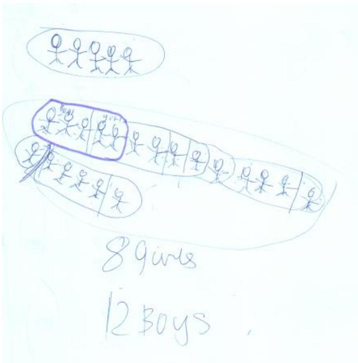
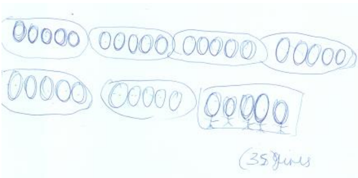
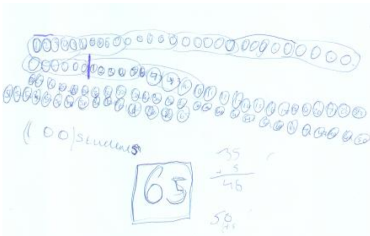
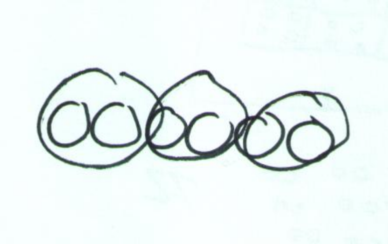
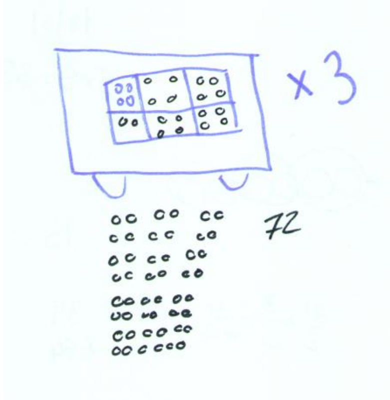
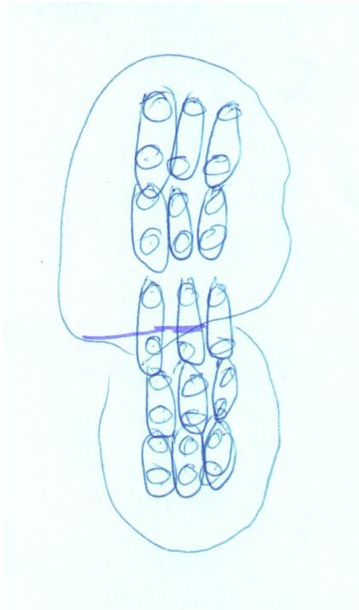
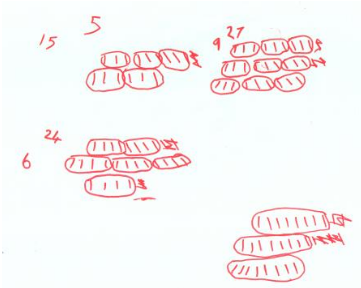
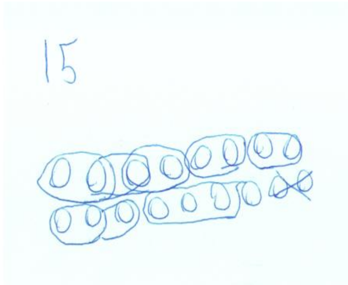
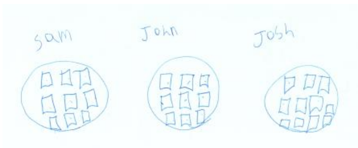


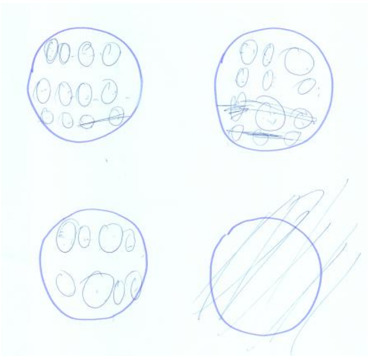
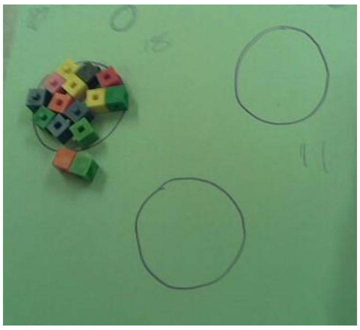
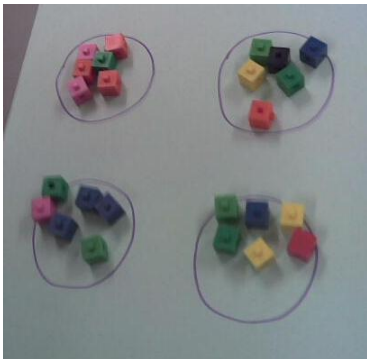
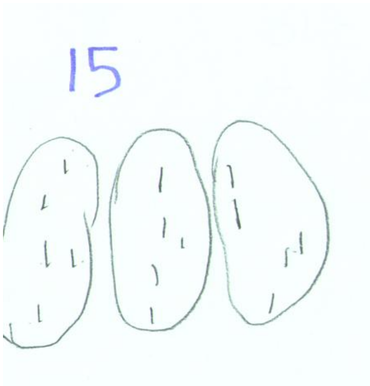
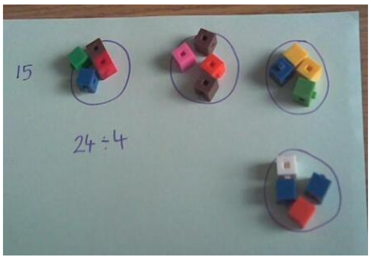
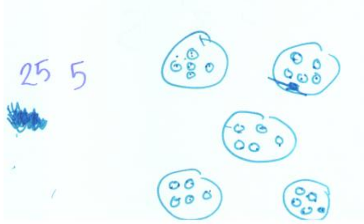
IA Harvey C1

 IA Kieran C1	-  IA Sidney C1	 IA Tasha C1
 IA Tasha C2	 IA Leo C1	 IA Leo C2
 IA Vince C1	 IA Vince C2	 IA Vince C3

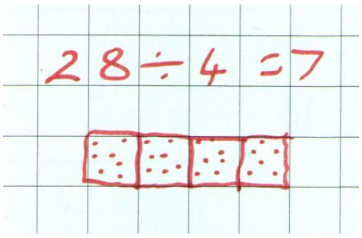
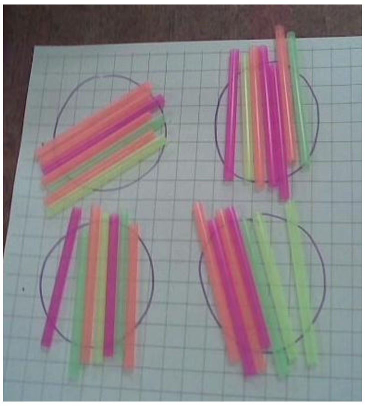
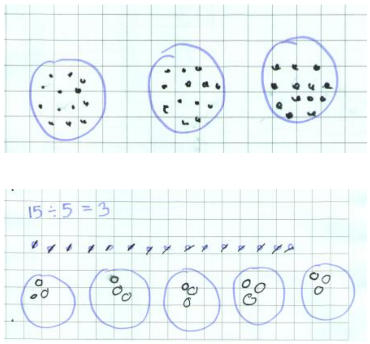
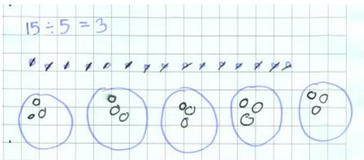
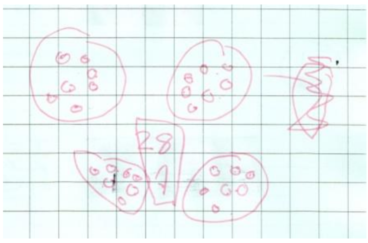
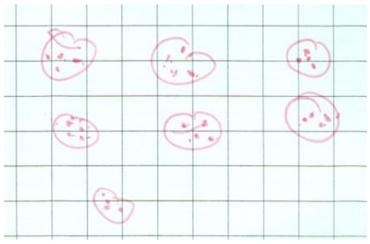
 IA Wendy C1	 IA Wendy C2

Tuition 1 (Biscuits)

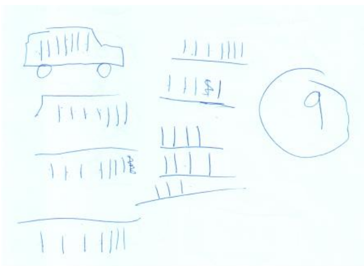
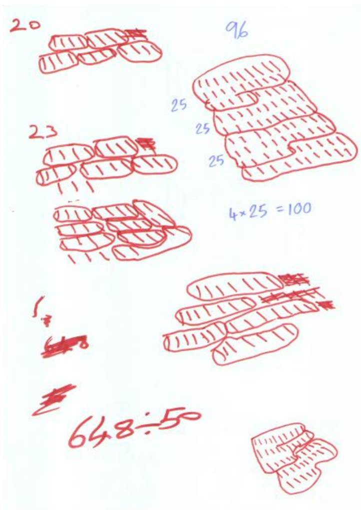
 T1 Danny C1	 T1 Kieran C1-4	 T1 Leo C1
 T1 Leo C2	  T1 Leo C3	 T1 Leo C4

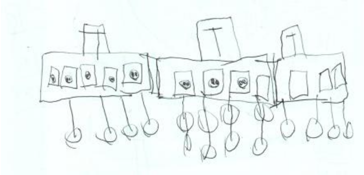
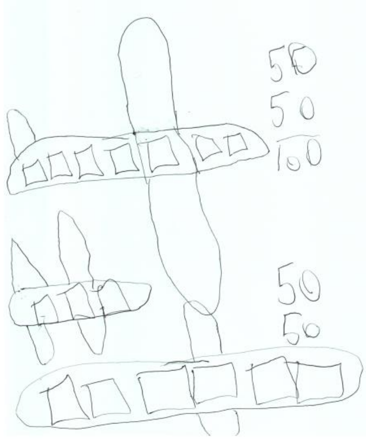
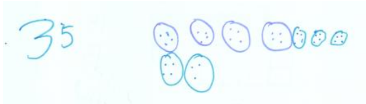
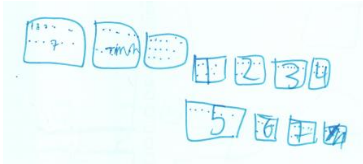
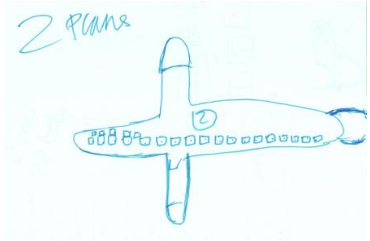
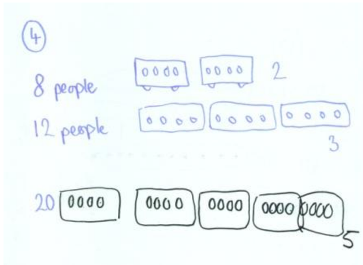
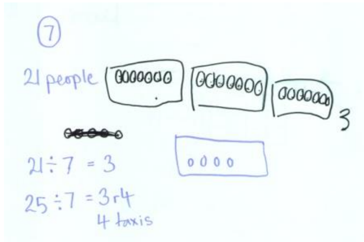
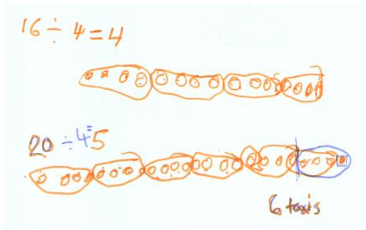
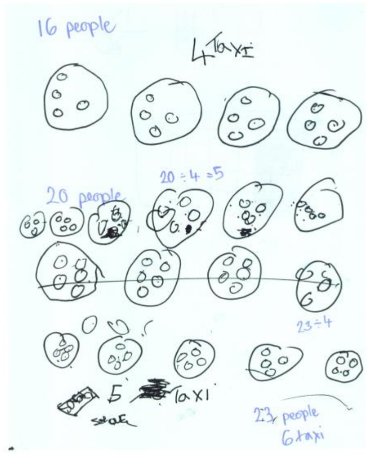
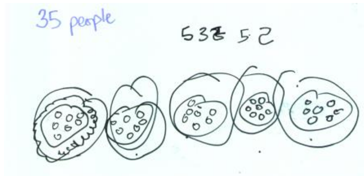
 T1 Vince C1	  T1 Vince C2	 T1 Vince C3
 T1 Jenny C1	  T1 Paula C1	 T1 Tasha C1

Tuition 2 (inc. Biscuits recap)

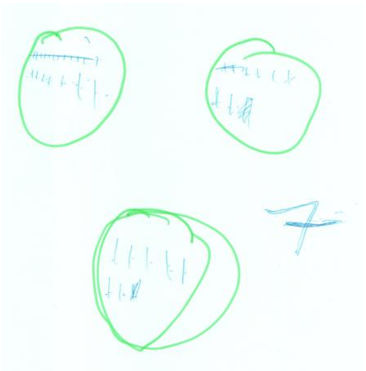
 T2 Kieran C1	 T2 Vince C1	 T2 Paula C1
 T2 Paula C2	 T2 Tasha C1	 T2 Tasha C2

Tuition 3 (inc. Biscuits recap, Transport)

 T3 George C1	 T3 Kieran C1-6	 T3 Leo C1

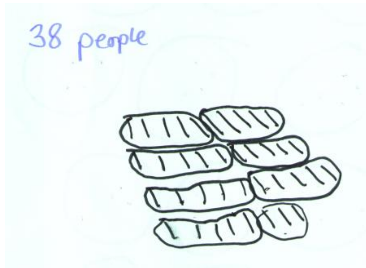
 T3 Leo C1	 T3 Leo C4	 T3 Vince C1
 T3 Vince C2	 T3 Vince C3	 T3 Vince C4
 T3 Sidney C1	 T3 Paula C1	 T3 Paula C2
 T3 Harvey C1-2	 T3 Tasha C1-2	 T3 Tasha C3

Tuition 4 (Summary)

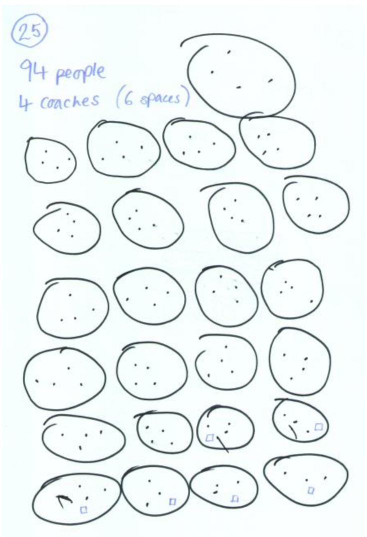


T1 Vince C1

T4 Kieran C1



T4 Kieran C2



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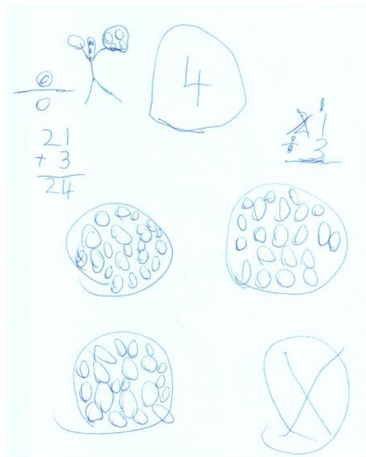
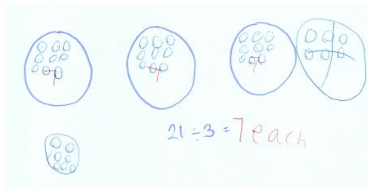


T4 Leo C1



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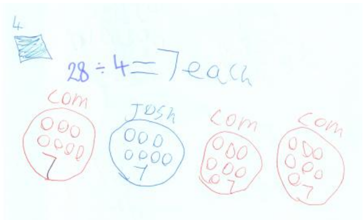

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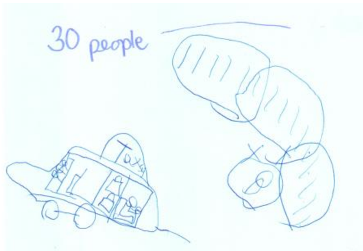

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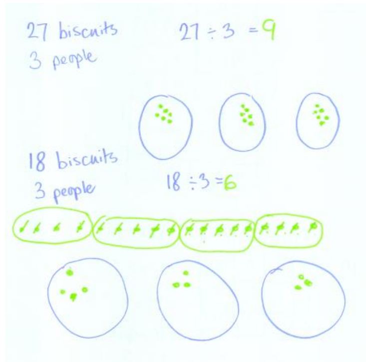
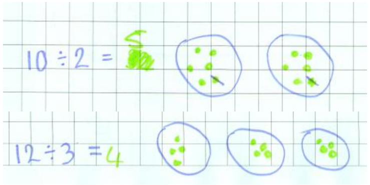
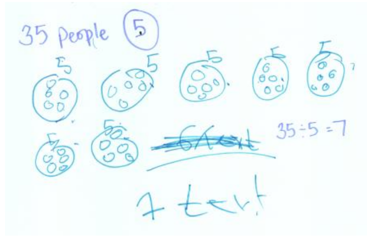


T4 Paula C1-2

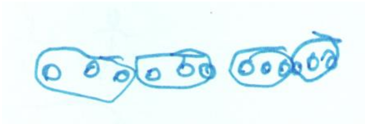


T4 Paula C3-4

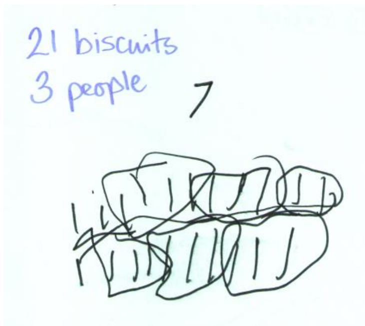
T4 Tasha C1



T4 Tasha C2



T4 Harvey C1



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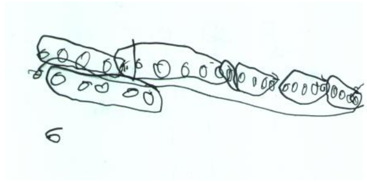
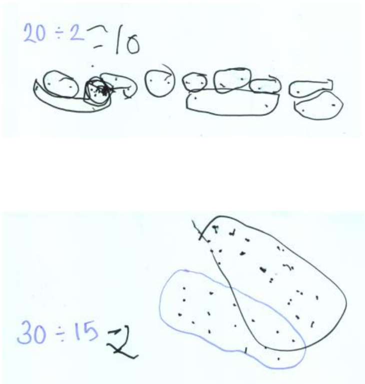
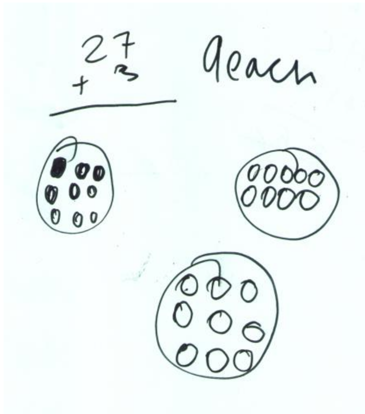


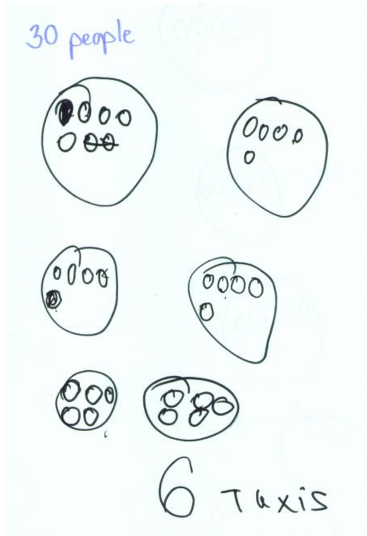
T4 Harvey C3



T4 Harvey C4-5



T4 Vince C1



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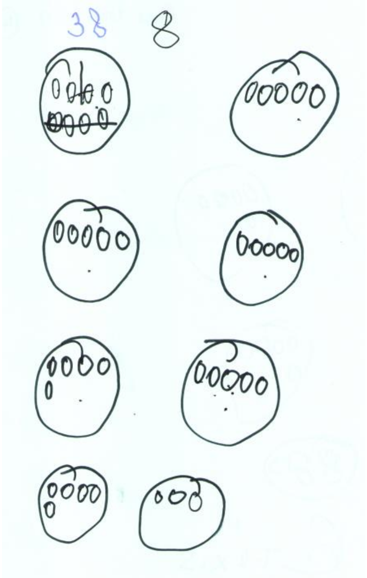
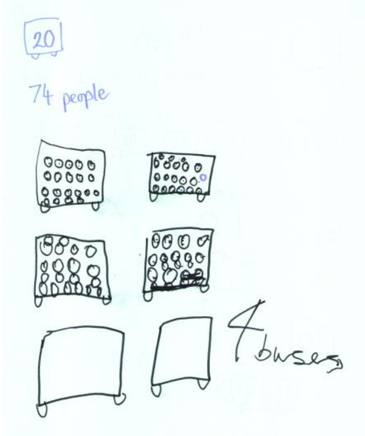
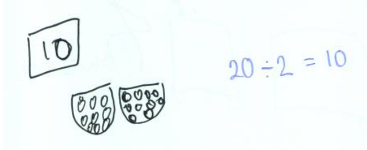


T4 Vince C3

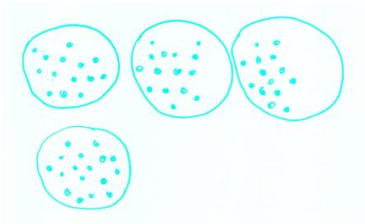


T4 Vince C4



T4 Vince C5

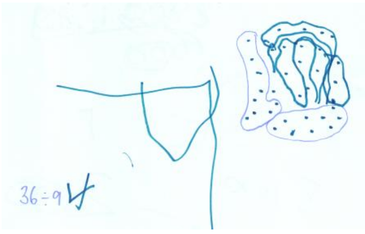
T4 Wendy C1



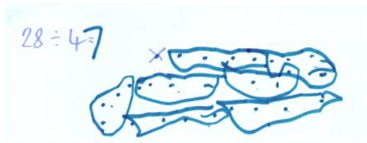
T4 Paula C1

T4 Paula C2

T4x Harvey C1

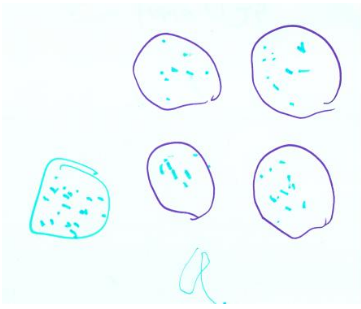
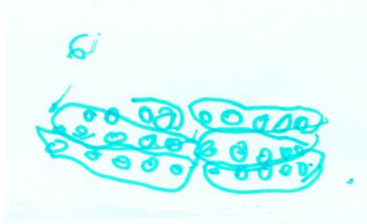
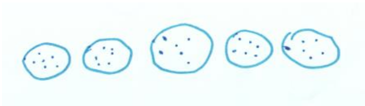
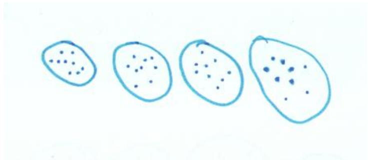
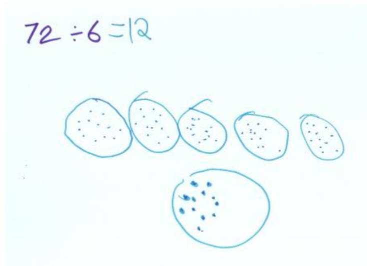
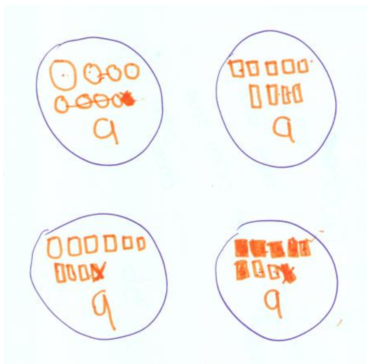
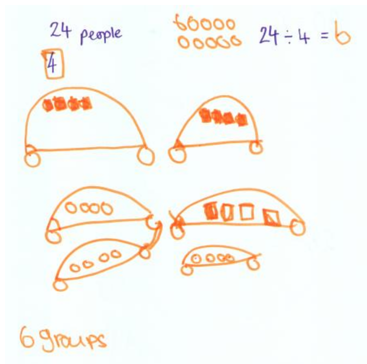


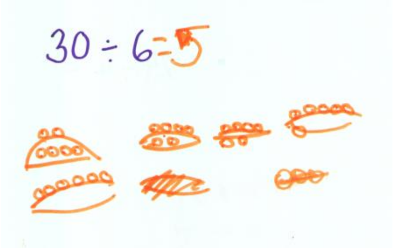
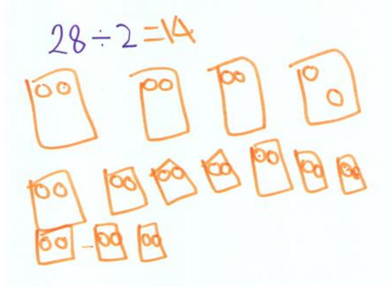
T4x Harvey C2



T4x Harvey C3

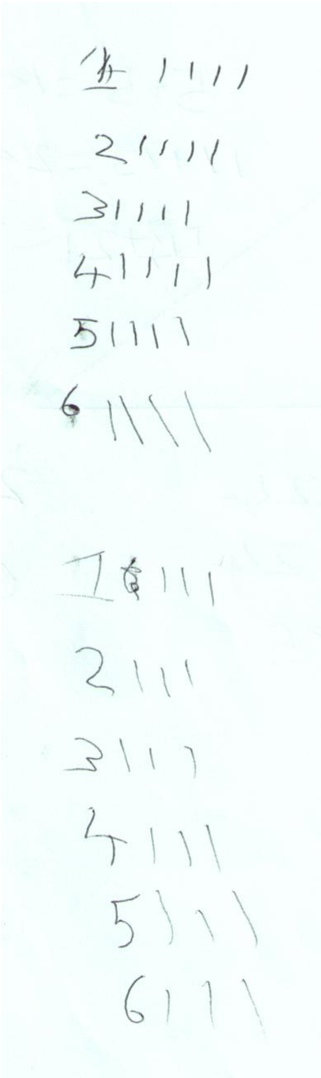
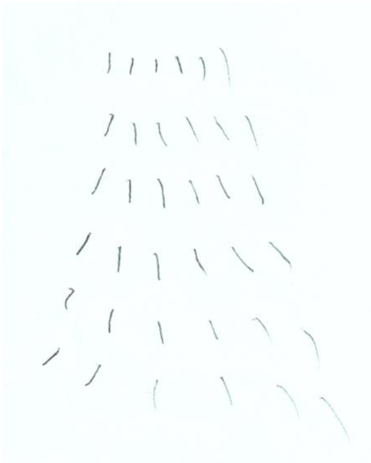
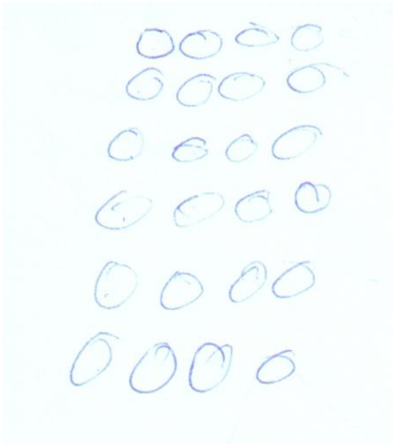
Tuition 5 (Division preferences)

 DP Harvey C1	 DP Harvey C2	 DP Harvey C3
 DP Kieran C1	 DP Kieran C2	 DP Kieran C3
 DP Harvey C1	 DP Vince C1	 DP Vince C2

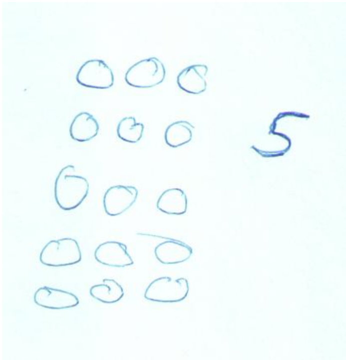
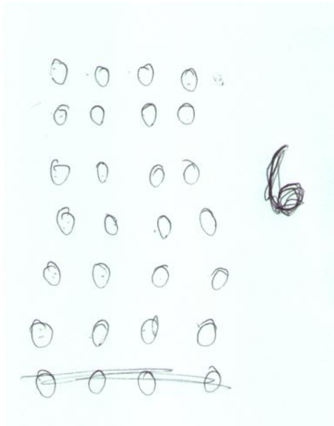
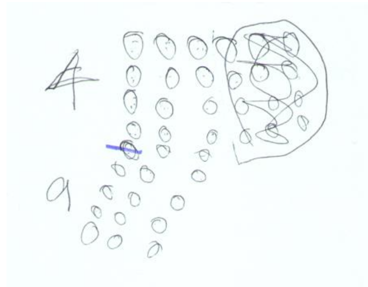
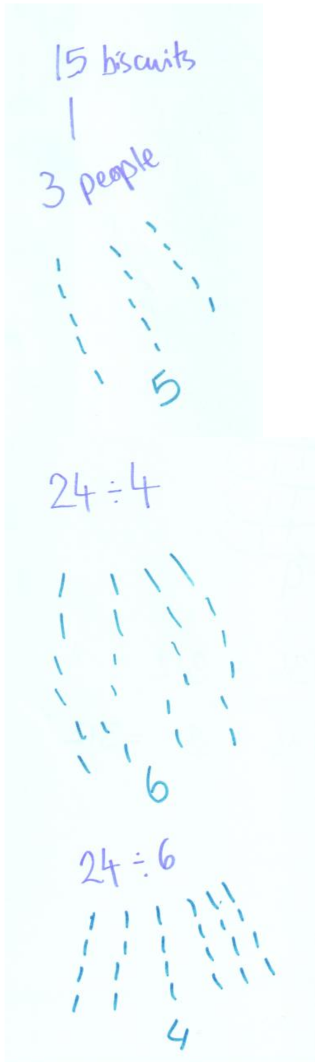
 DP Vince C3	 DP Vince C4

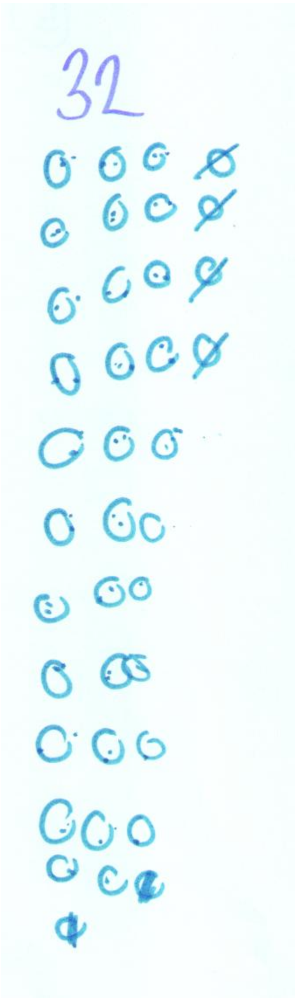
Unit arrays

Initial Assessment

 IA Kieran A1-2	 IA Kieran A3	 IA Danny A1

Tuition 1 (Biscuits)

 T1 Danny A1	 T1 George A1	 T1 George A2
 T1 George A3	 T1 Kieran A1-2	 T1 Wendy A1-3



T1 Wendy A4

Tuition 2 (inc. Biscuits recap)

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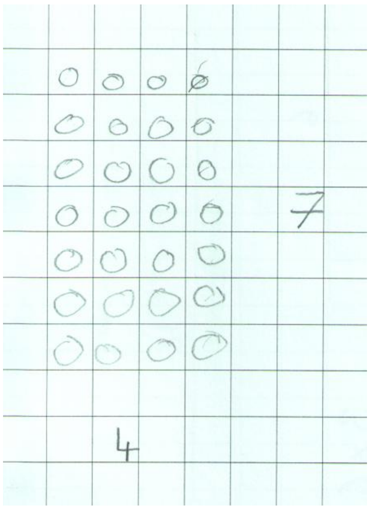
<td>

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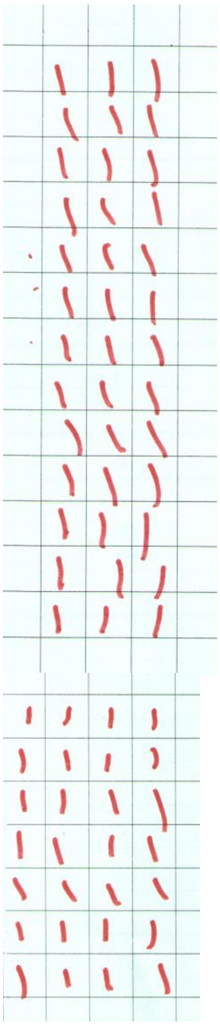
<td>
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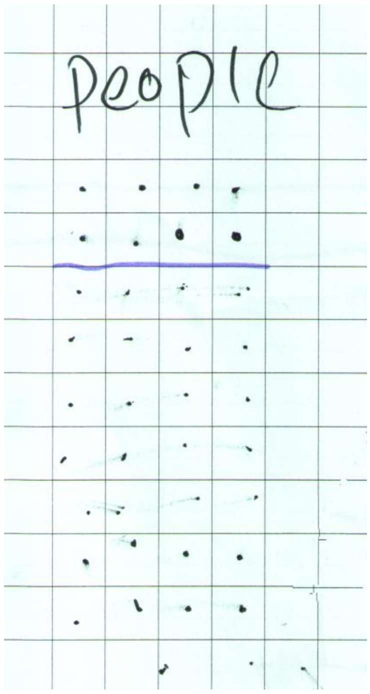
<td>

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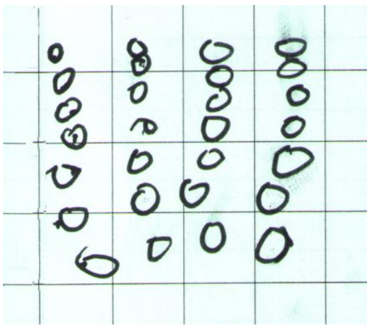
<td>
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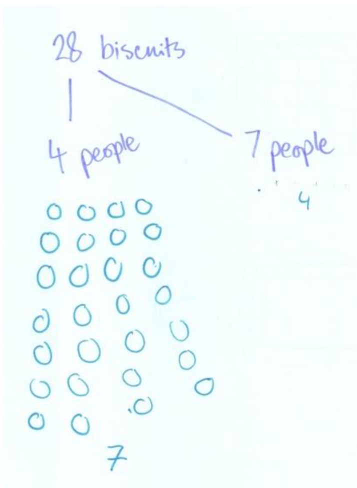
<td>

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Tuition 3 (inc. Biscuits recap, Transport)

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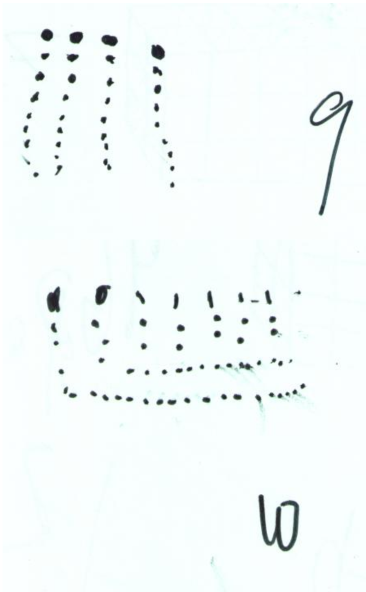
<td>
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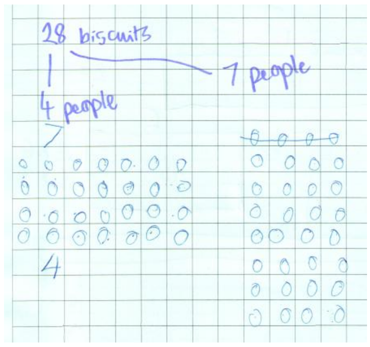
<td>
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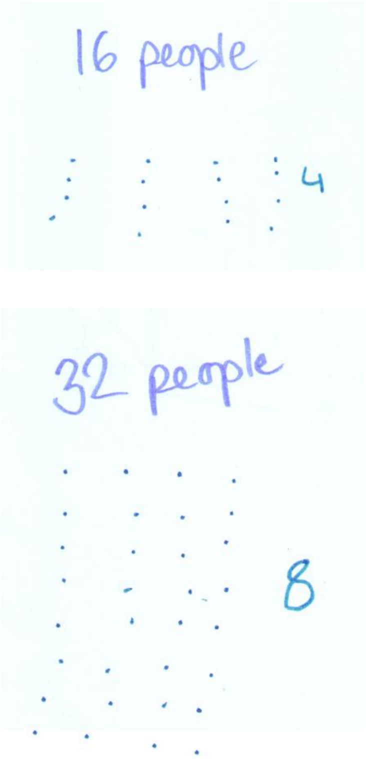
<td>
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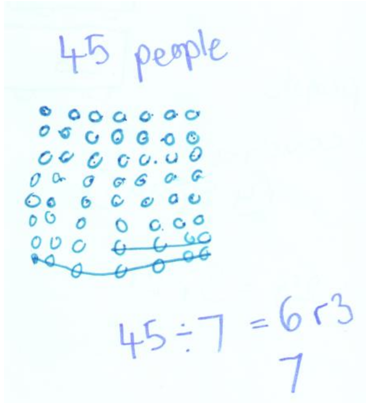
<td>
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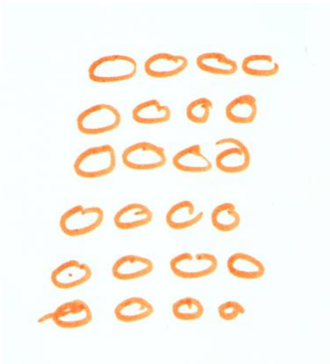

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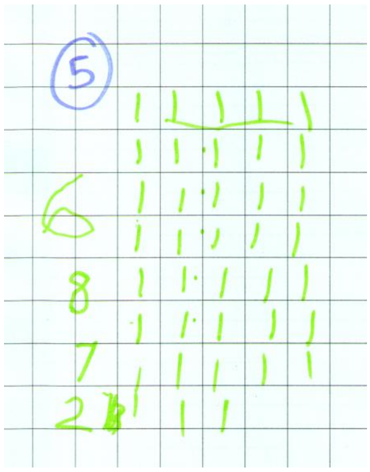
T4 George A1



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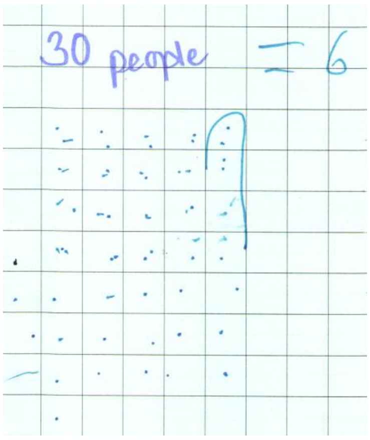

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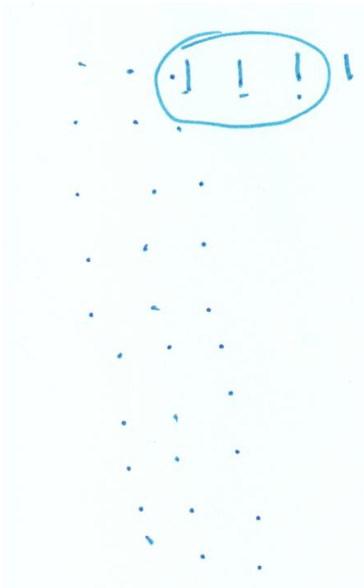

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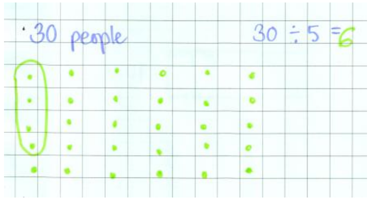

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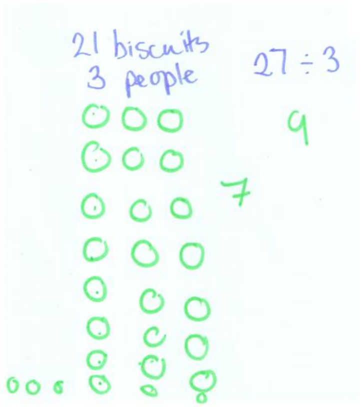
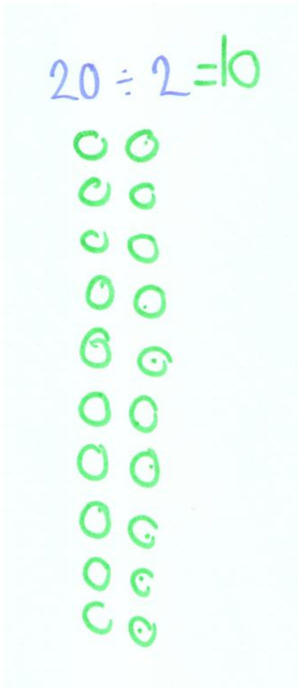
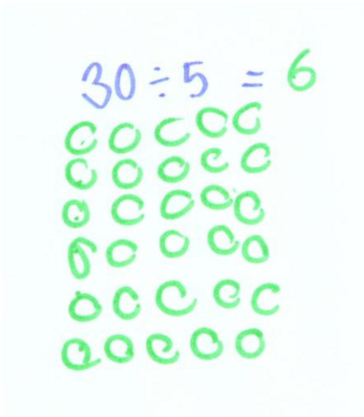
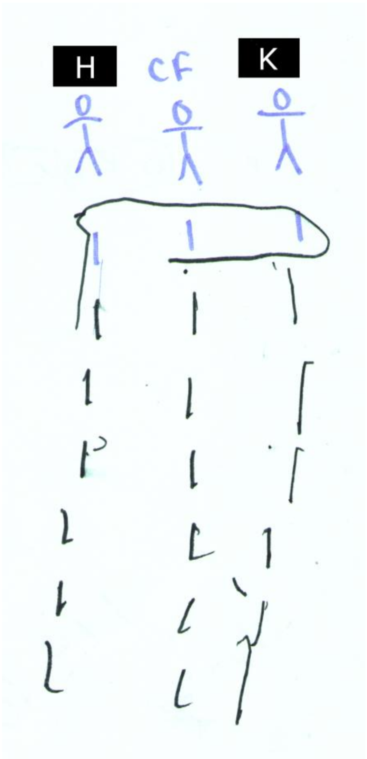
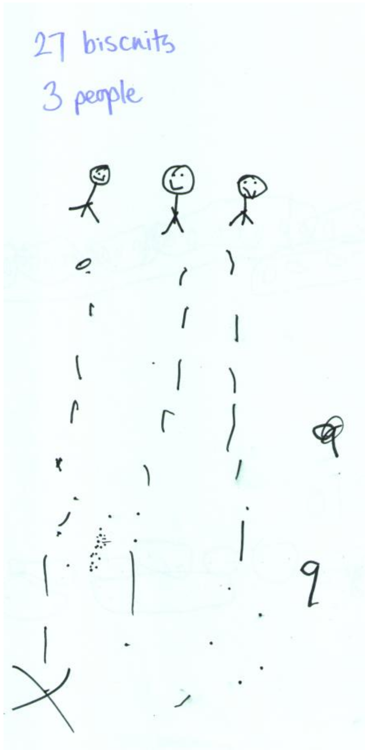
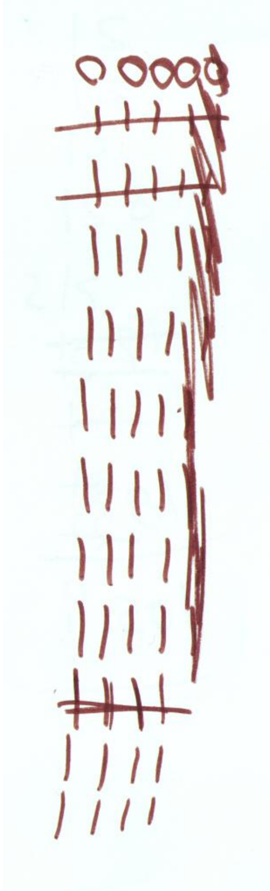
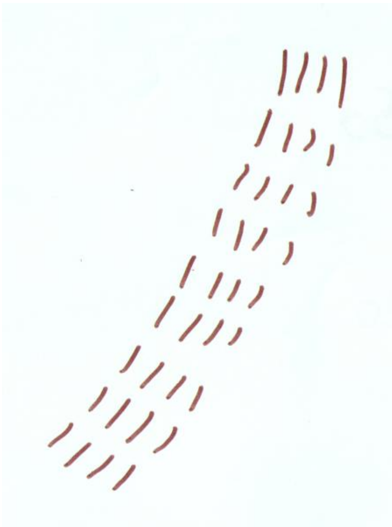
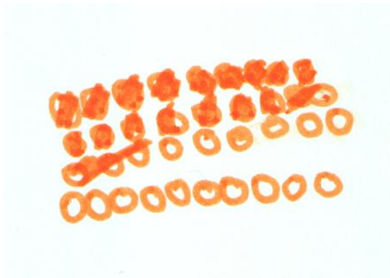
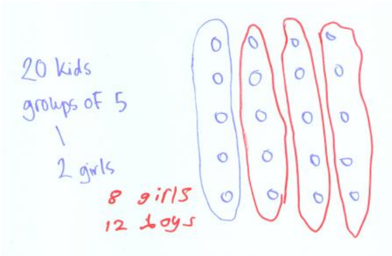
 T4 Wendy A1-2	 T4 Wendy A3	 T4 Wendy A4
 T4 Harvey A1	 T4 Harvey A2	 T4x Wendy A1

Tuition 5 (Division preferences)

 DP Harvey A1	 DP Sidney A1	 DP George A1
 DP George A2	 DP Vince A1	

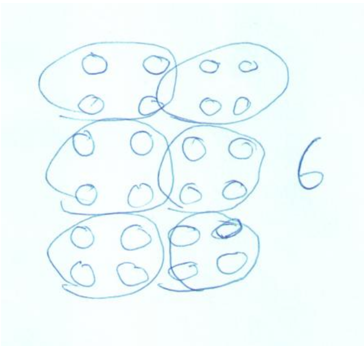
Array-container blends

Initial Assessment



IA Ellis AC1

Tuition 1 (Biscuits)



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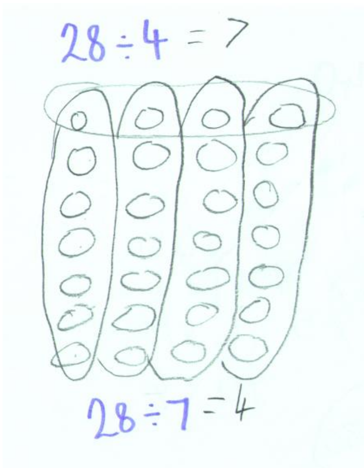

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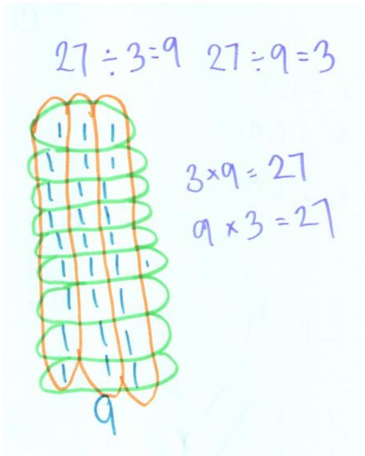

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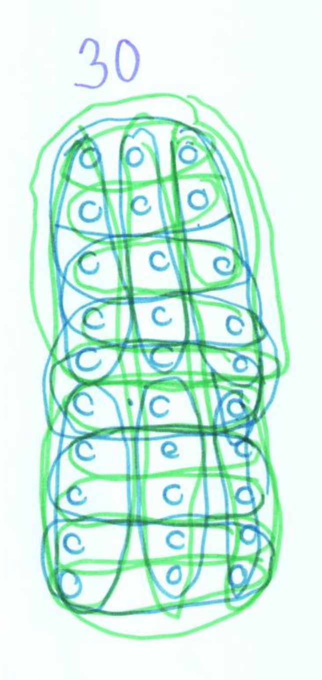
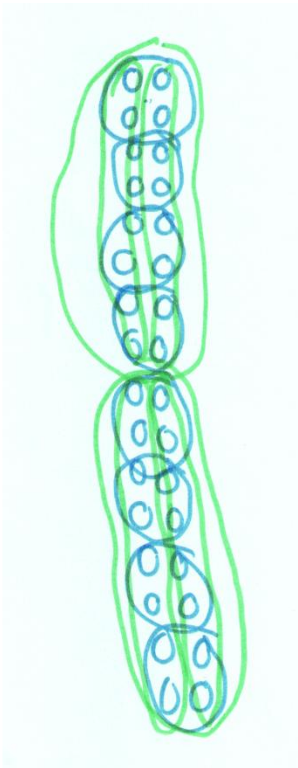
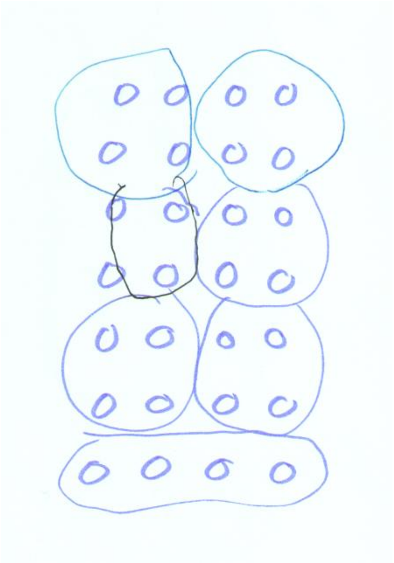
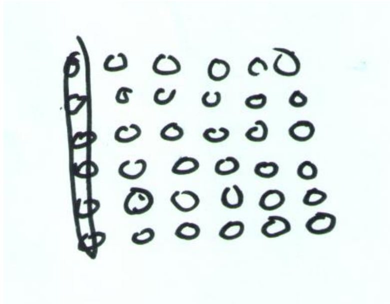
 T1 Wendy AC2	 T1 Wendy AC3

Tuition 2 (inc. Biscuits recap)

 T2 Leo (blue) Vince (black) AC1	 T2 Jenny AC1

Tuition 3 (inc. Biscuits recap, Transport)

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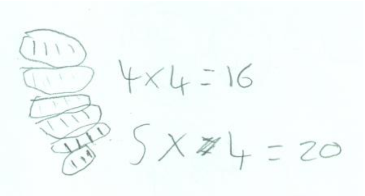
<td>
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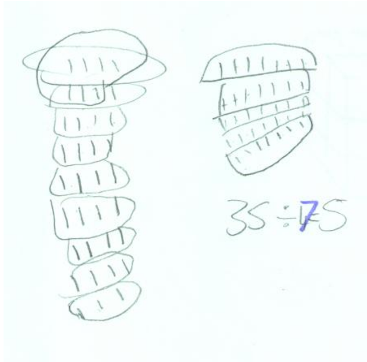
<td>
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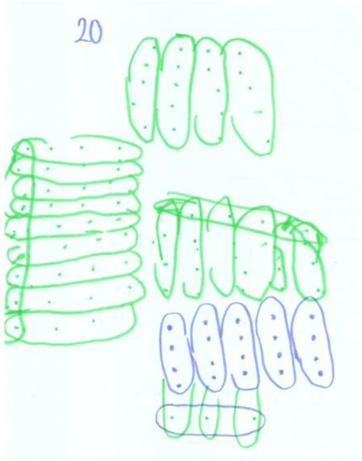
<td>
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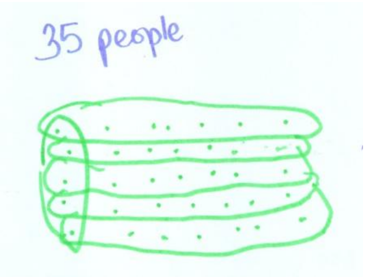
<td>
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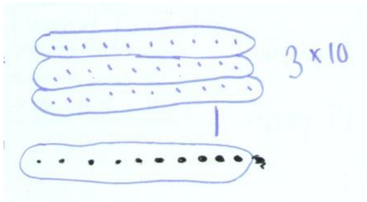
<td>
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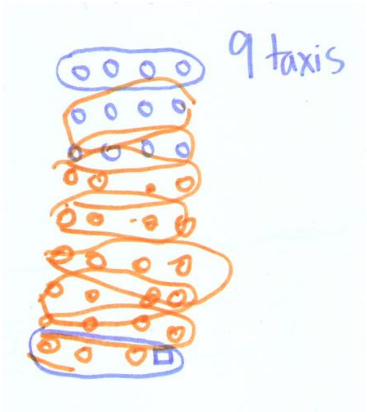
<td>
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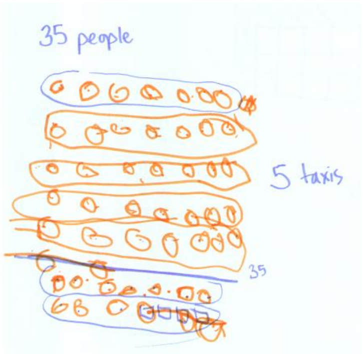
<td>
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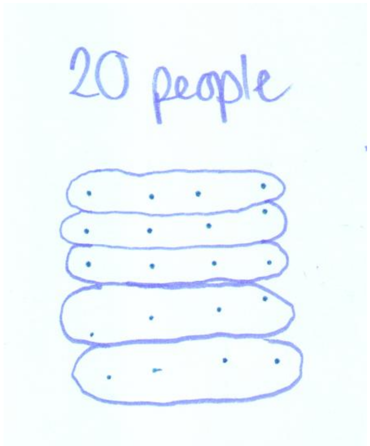
<td>
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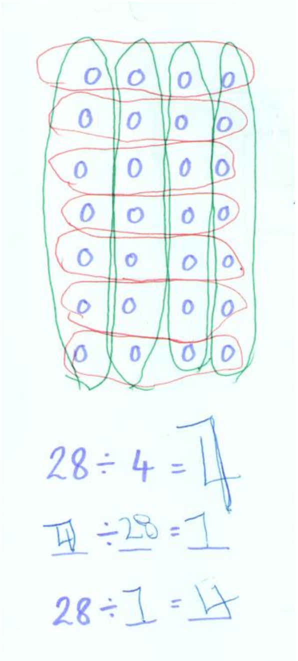
Tuition 4 (Summary)



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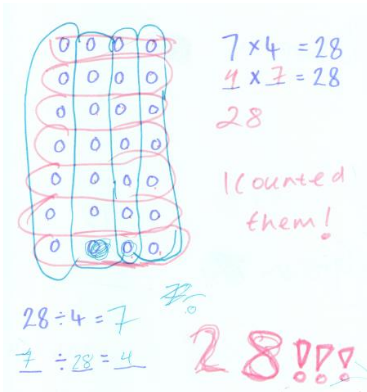

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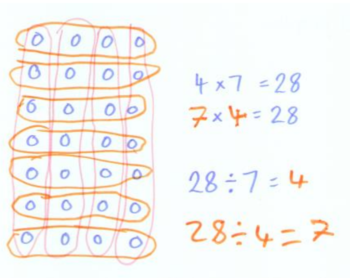

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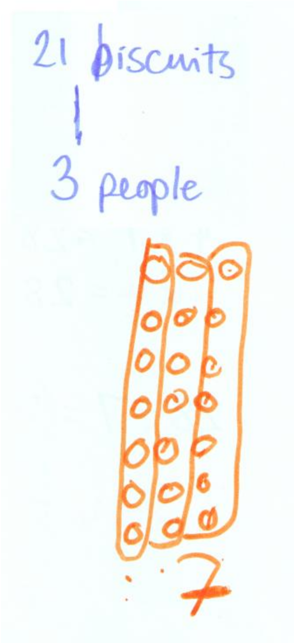

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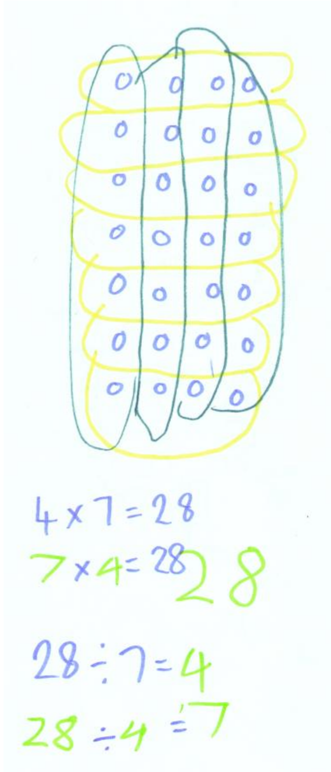

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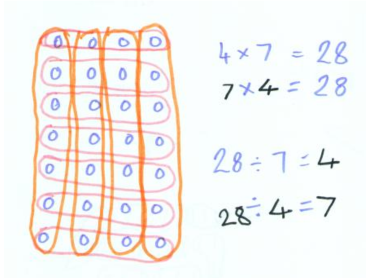

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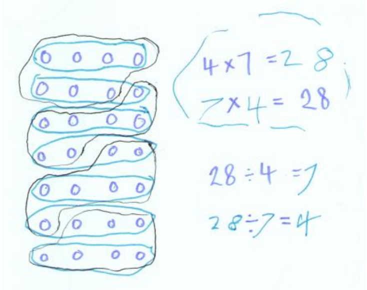

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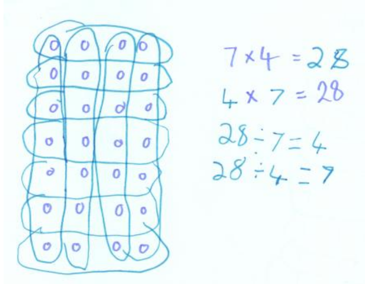

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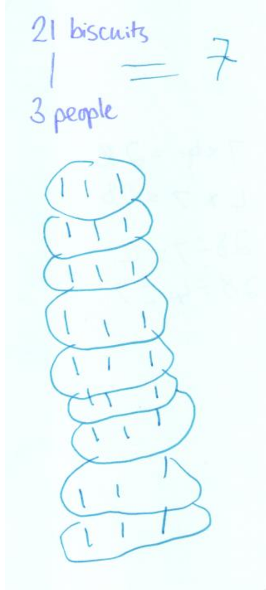

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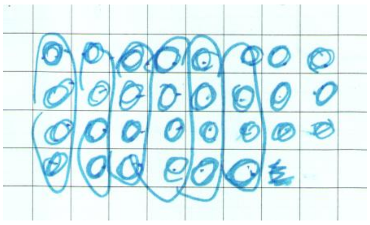
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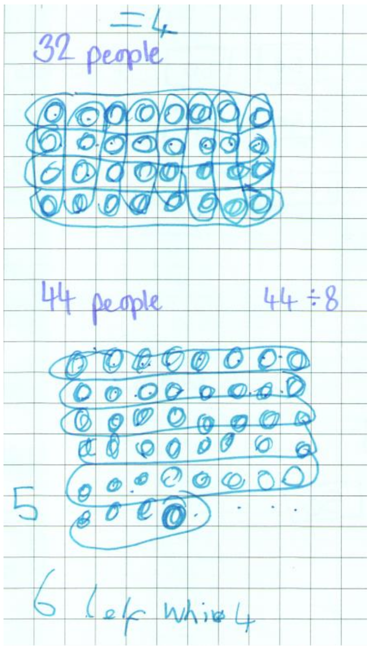

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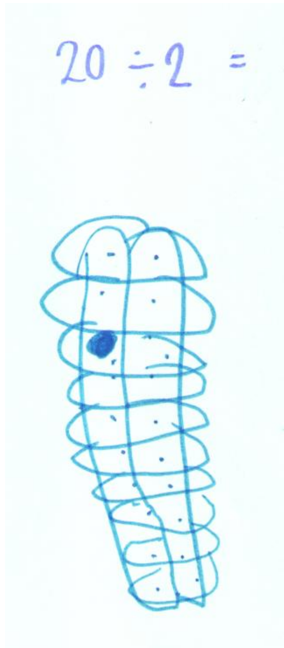

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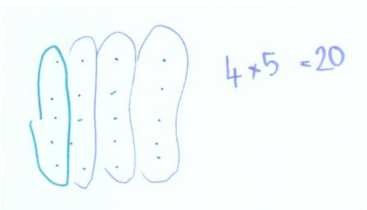

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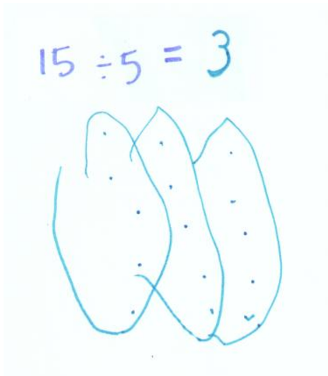

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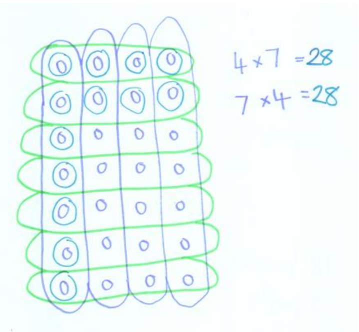

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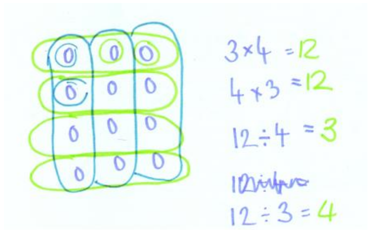

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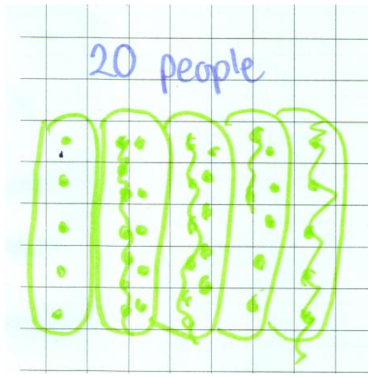

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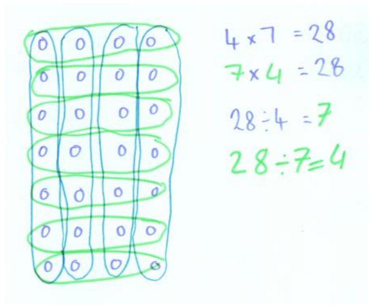

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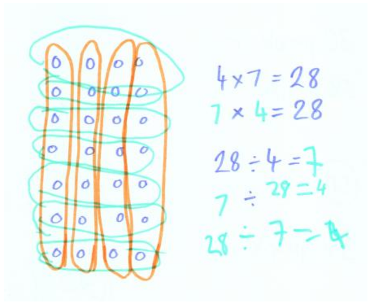

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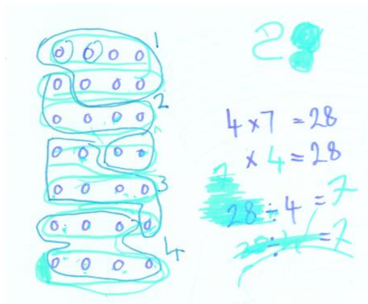

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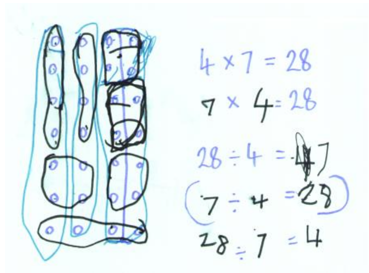

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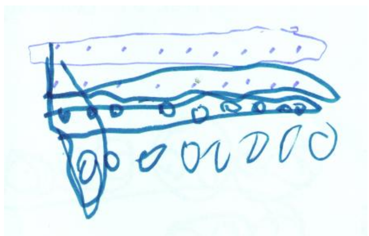

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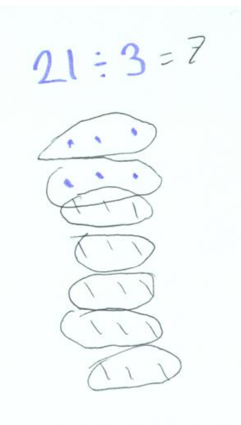
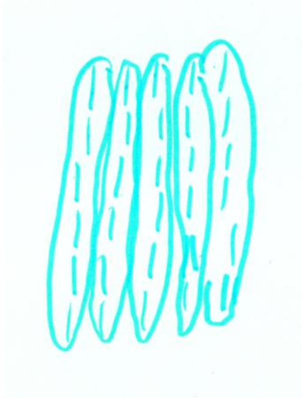
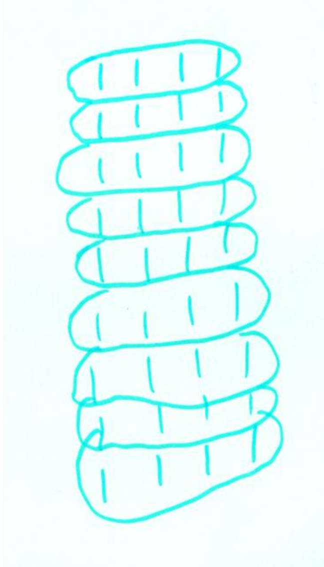
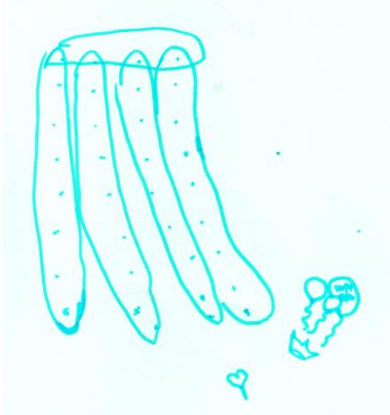


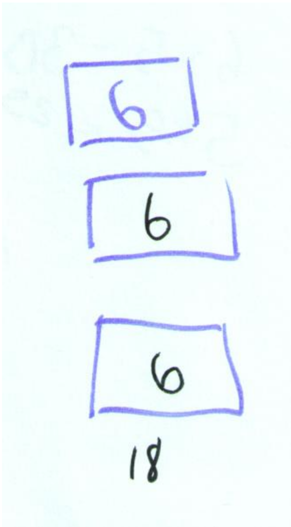
T4 Wendy AC2

Tuition 5 (Division preferences)

 DP Wendy AC1	 DP Wendy AC2	 DP Jenny AC1

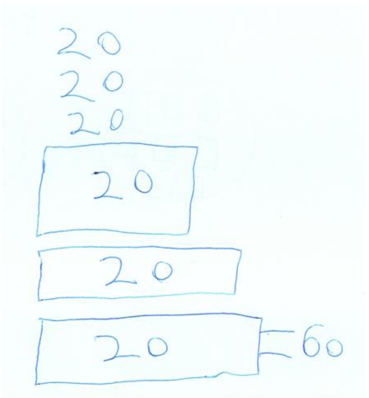
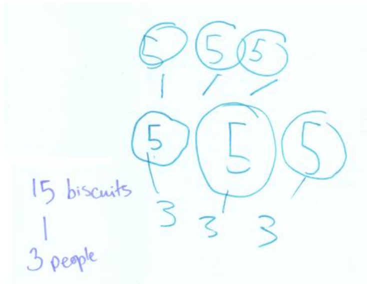
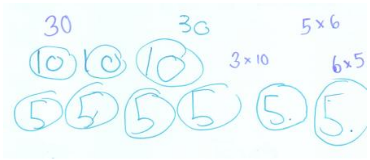
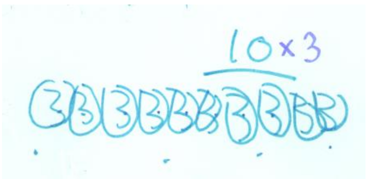
Number containers

Initial Assessment



IA Wendy NC1

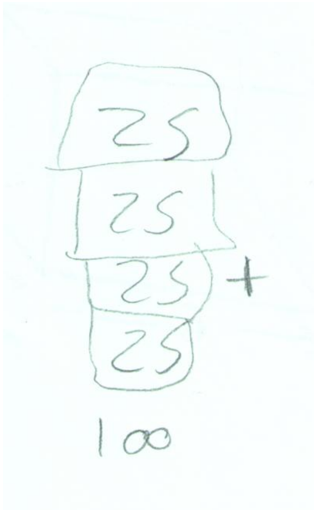
Tuition 1 (Biscuits)

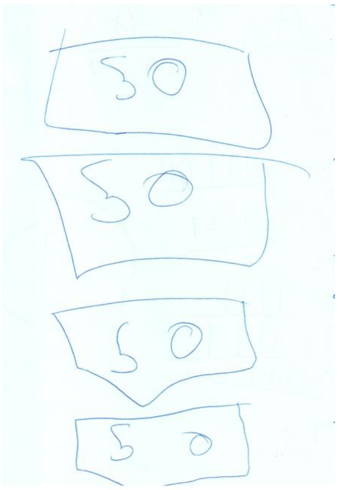
 T1 Leo NC1	 T1 Tasha NC1	 T1 Tasha NC2-3
 T1 Tasha NC4		

Tuition 2 (inc. Biscuits recap)

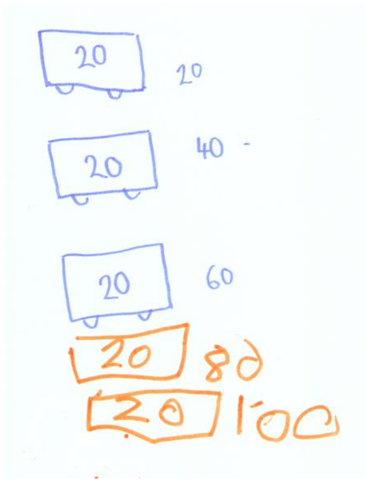
None used this session.

Tuition 3 (inc. Biscuits recap, Transport)

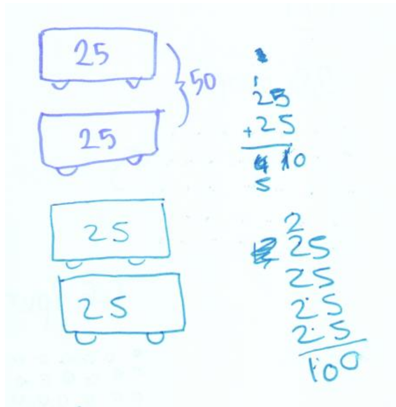
 T3 Danny NC1	 T3 Danny NC2	 T3 George NC1



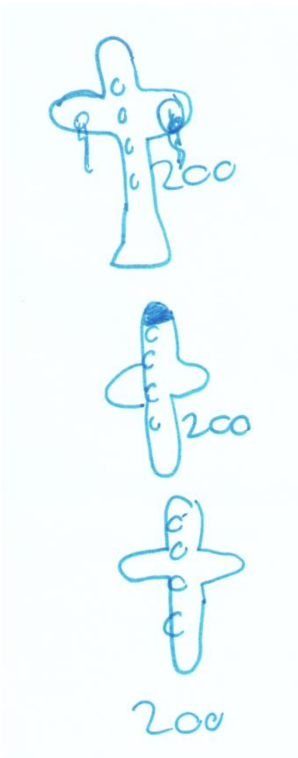
T3 George NC2



T3 Harvey NC1

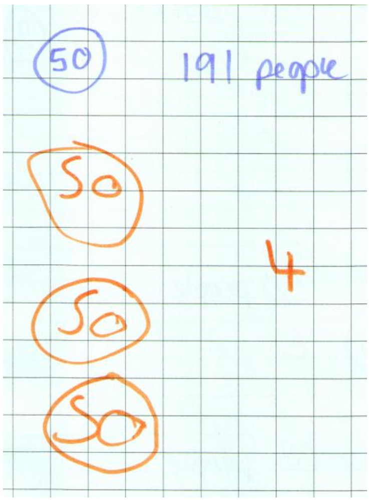
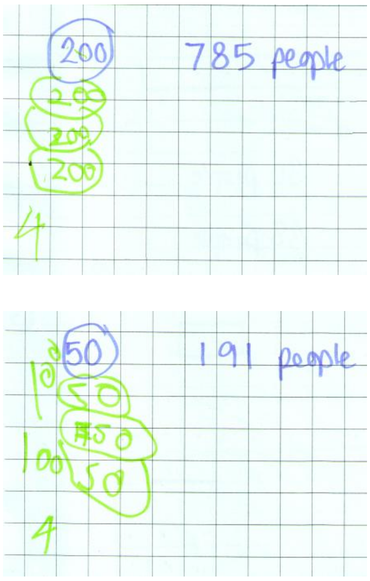
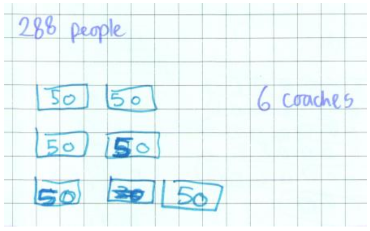
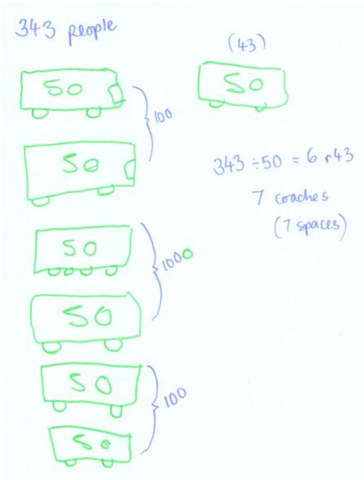
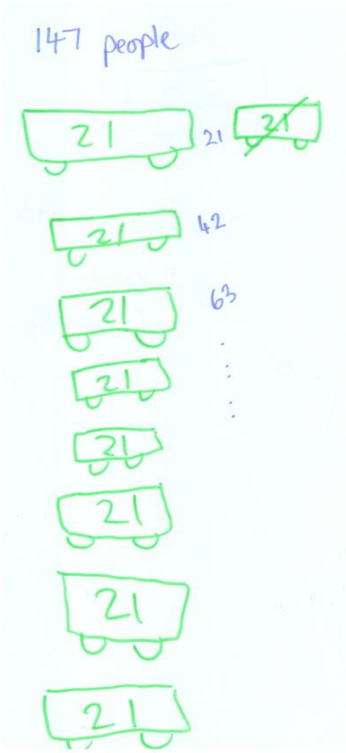
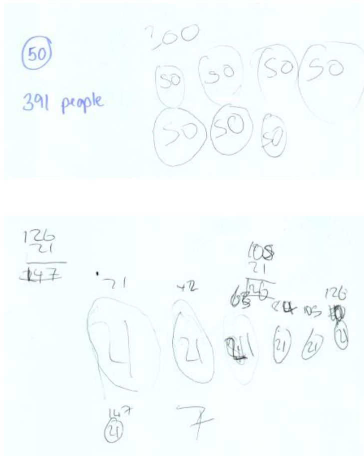


T3 Wendy NC1

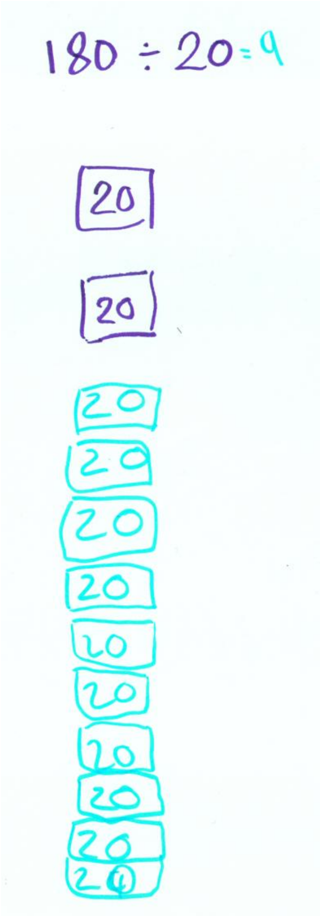
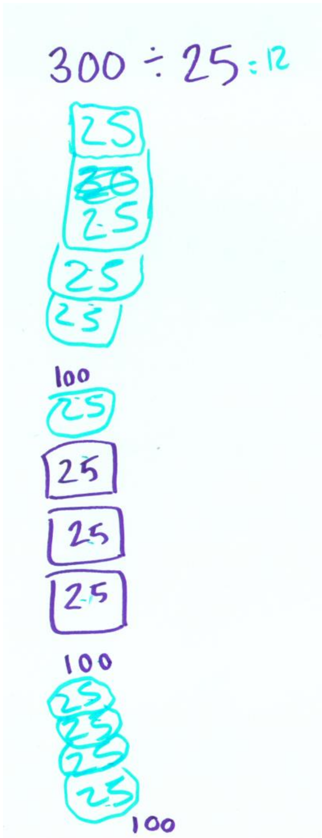
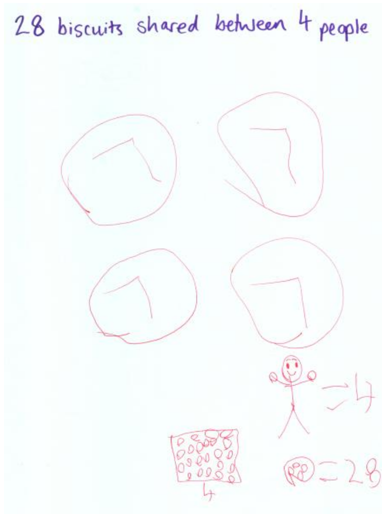


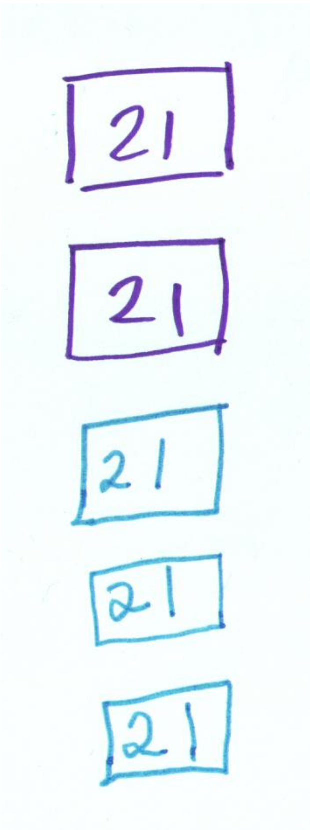
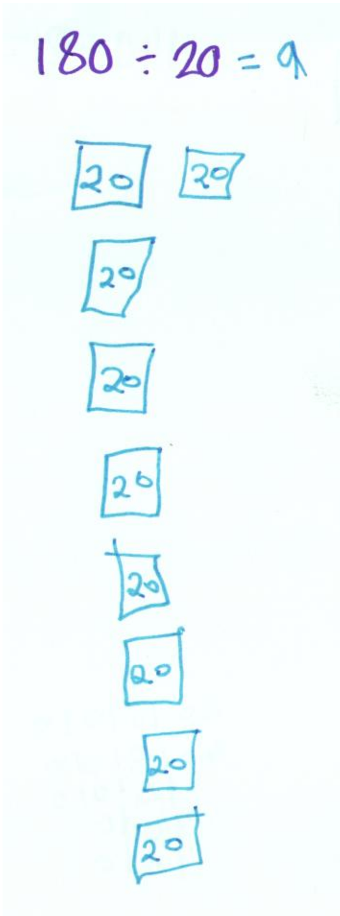
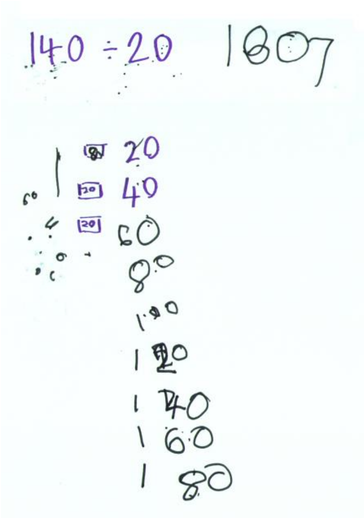
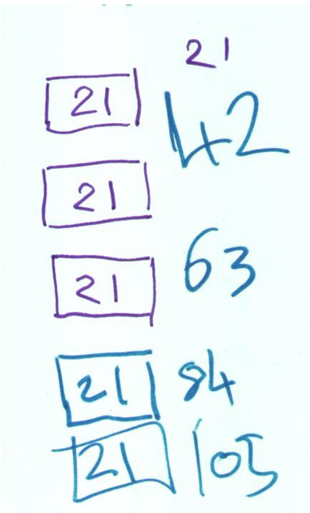
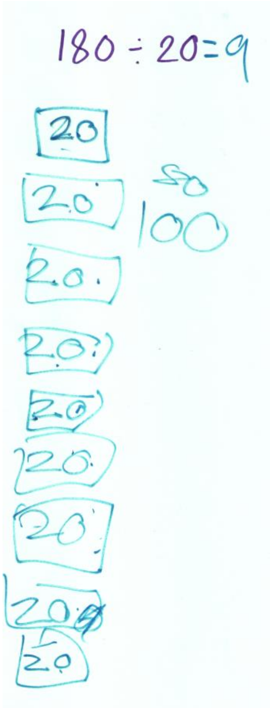
T3 Wendy NC2

Tuition 4 (Summary)

 T4 Danny NC1	 T4 George NC1-2	 T4 Jenny NC1
 T4 Wendy NC1	 T4 Wendy NC2	 T4 Sidney NC1-2

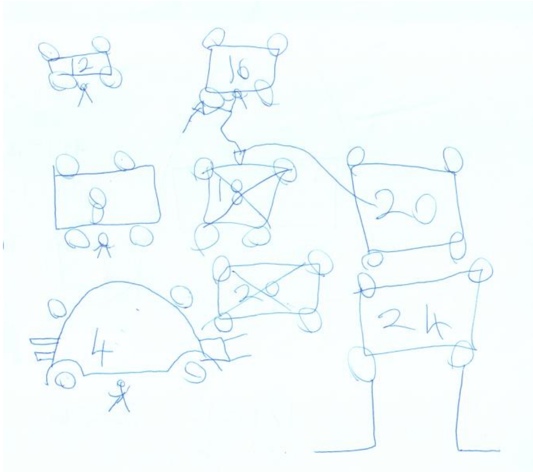
Tuition 5 (Division preferences)

		
DP Wendy NC1	DP Wendy NC2	DP Leo NC1

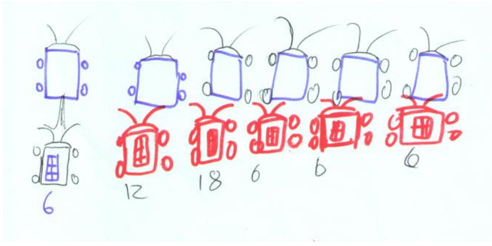
 DP Sidney NC1	 DP Kieran NC1	 DP Kieran NC2
 DP Harvey NC1	 DP Tasha NC1	 DP Tasha NC2

Miscellaneous visuospatial representations

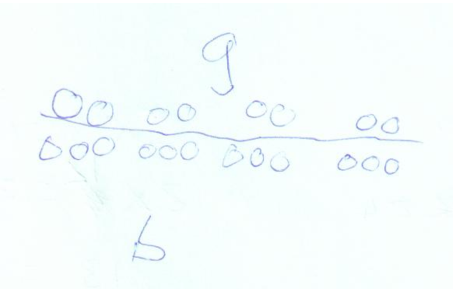
Initial Assessment



IA Leo M1



IA Vince M1



IA Danny M1

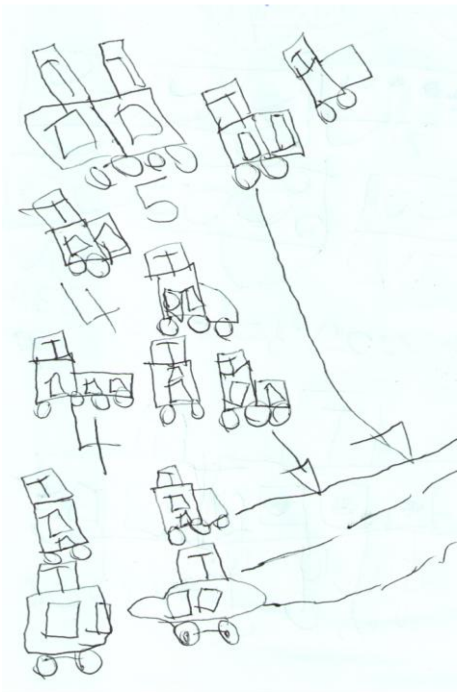
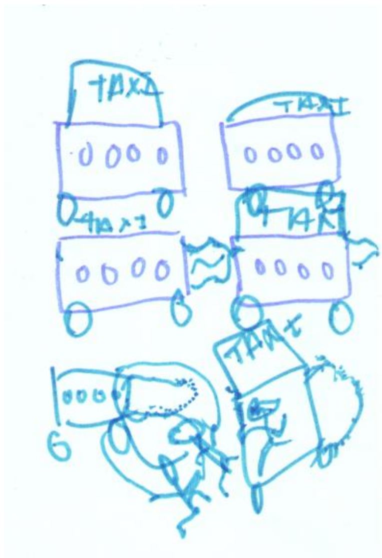
Tuition 1 (Biscuits)

None used this session.

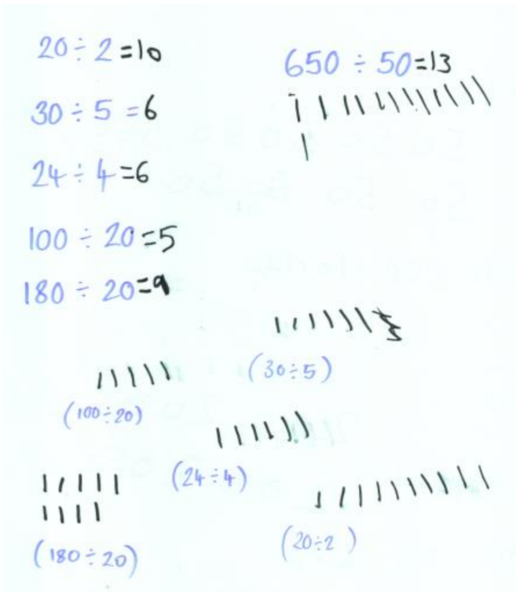
Tuition 2 (inc. Biscuits recap)

None used this session.

Tuition 3 (inc. Biscuits recap, Transport)

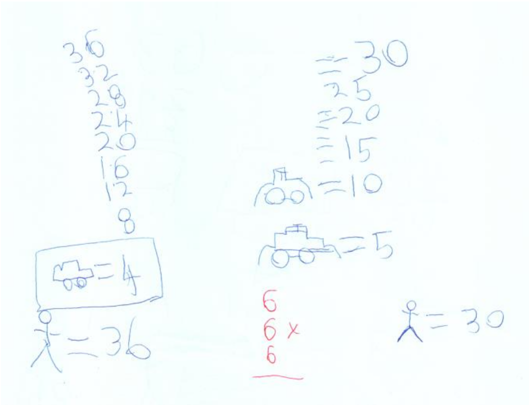
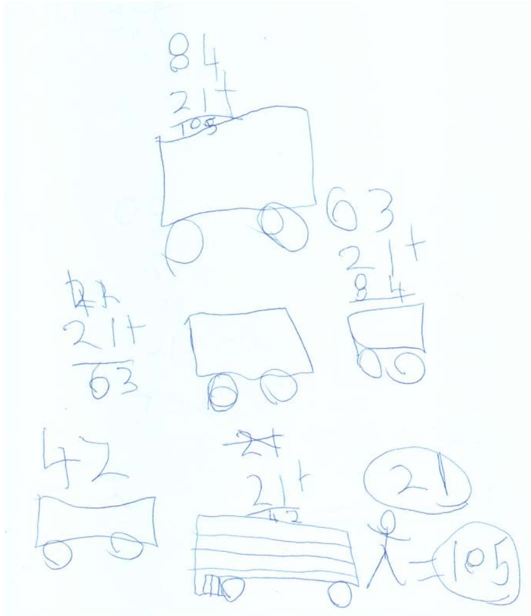
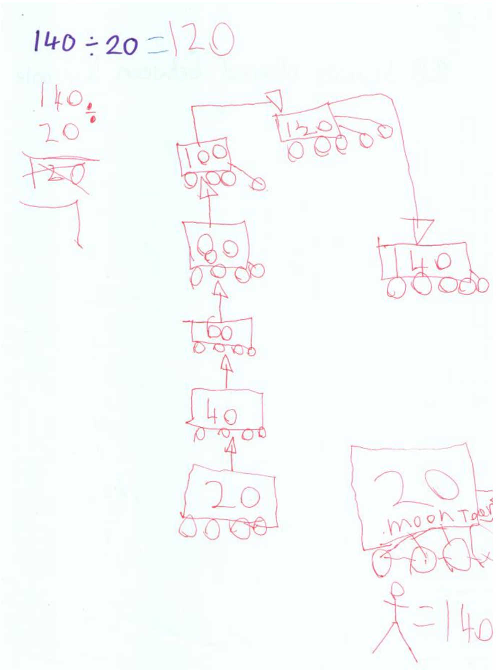
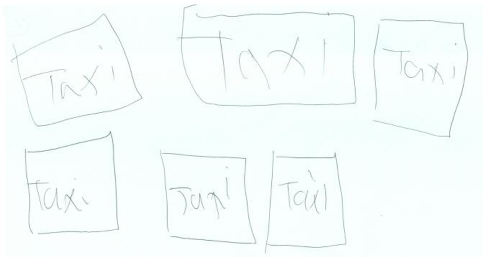
 T3 Leo M1	 T3 Vince M1

Tuition 4 (Summary)



T4 Kieran M1

Tuition 5 (Division preferences)

 DP Leo M1	 DP Leo M2
 DP Leo M3	 DP Sidney M1