Constructing Curvature:

The Iterative Design of a Computer-Based Microworld for Non-Euclidean Geometry

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Abstract

The study charts the iterative development of a computer-based microworld for non-euclidean geometry. Its aim was to explore the possibilities for constructing a suitable context that simultaneously articulated the processes of teaching and learning using computer-based versions of euclidean models for non-euclidean geometry, and the construction of the context.

Using the *microworld* paradigm as the basis for a model of a computer-based learning environment, the study defines a microworld not only in terms of the computational and non-computational tools available to the learner, but also with reference to its pedagogical intentions and cognitive pre-suppositions. The model of the microworld that was created was then used to guide its design and development. The computational element of the microworld employed an object-oriented version of the Lisp-based programming language Logo to implement Turtle Graphics in a non-euclidean context.

The design process for the microworld was iterative. Activities, which brought together software and specific pedagogic approaches to non-euclidean geometry, were trialled and modified in the light of learners' experiences with the microworld. Organised into three developmental cycles, the study describes and analyses each iteration under three interrelated categories: technical refinement of the software and non-computational objects, structuring of the pedagogical framework, and the cognitive development of the learners mediated by their experience of the microworld.

The study concludes with an appreciation of this iterative development process. It proposes a framework for microworld creation based on the principles of design and of learning as the exploration of a knowledge domain.

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Chapter 1

The Aims and Rationale of the Study

1.0 Introduction

One of the more remarkable events in the world of publishing over the past few years has been the success of "A Brief History of Time" by Stephen Hawking (1988). It is a book which many "intelligent non-scientists" possess but few claim to understand. In many ways the fame, or rather infamy, of the book points to the difficulties which many "non-scientific" people experience when faced with trying to understand contemporary mathematics and physics.

Take, for example, General Relativity[1]. It is a central element in our modern understanding of the universe's large-scale structure, but it is very counter-intuitive. Part of its power as a description of gravity lies in its abstract reconceptualisation of the geometry of space and time. This is not the space and time of ordinary experience but a "mathematised" version which enables a connection to be made between the dynamics of physical objects and geometrical quantities such as curvature. From a perceptual point of view, this does not make sense. Our everyday experience of space is described in terms of euclidean geometry. We see and touch curved surfaces and interpret them as such in relation to an assumed background of flat space.

General Relativity, on the other hand, asserts that space is not a flat background to the physical world, nor is it completely explicable in terms of euclidean geometry. Rather, space, together with time, constitute the "fabric" of the material world. By considering space and time as aspects of a single mathematical entity, spacetime, whose curvature is described through local measurements, General Relativity is able to describe the dynamics of objects moving in gravitational fields. On the one hand, therefore, our experience of space and time is visual, qualitative and based upon a

[1] See Appendix A.1.

contrast between "flat" euclidean space and "non-flat" objects *in* that space. On the other hand, General Relativity is non-visual and quantitative, describing geometric objects intrinsically, that is without relating them to external frames of reference.

Historically, the re-interpretation of geometry which provided the mathematical tools for General Relativity began with a variety of attempts to clarify the logical status of Euclid's axioms. These attempts centred on investigations into the logical status of the so-called "Parallel Postulate" or "Euclid's Fifth Axiom" and its relationship to the other axioms of euclidean geometry[1]. A central difficulty for investigators, both in formulating the problem and finding a solution, was the fact that their results ran contrary to common sense and accepted mathematical methods and facts. Until the nineteenth century, euclidean geometry was considered to be *the* example of a deductive science and several mathematicians, including Gauss, seemed reluctant to argue for its removal from pre-eminence (Gray 1989 p. 86). The resolution of the issue, demonstrating that non-euclidean geometries were possible and valid, came about by treating geometry in an abstract and logical way rather than relying on visual intuition.

Riemann exemplifies this process. His inaugural lecture, in 1854, entitled "On the Hypotheses that Form the Foundations of Geometry", outlined a framework for geometry which consisted of two elements: a model for "space" in an abstract general sense, and a function which gives distances and directions between points in that space. He considered space to be a "multiply extended magnitude", capable of various metrical relationships. Those geometric relationships which correspond to the world that we actually live in "can be gathered only by experience" (*Plan* cited in Spivak vol.2 1976).

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[1] Euclid's five axioms can be expressed as:

1. Every line is a collection of points.
2. There exists at least two points.
3. If p and q are distinct points, then there exists one and only one line containing p and q.
4. If L is a line, then there exists a point on L.
5. If L is a line, and p is a point not on L, then there exists one and only one line containing p that is parallel to L.

Definition
Two lines are parallel if they do not have a point in common. (Wilder 1962 p.10)

Although Riemann did not pursue the implications of this notion, others quickly took up his re-definition of geometry to explore the mathematically intriguing possibility that there were many sorts of geometry. His description enabled mathematicians to bring together the results obtained by Gauss, Bolyai and Lobachevsky about non-euclidean geometry, into a coherent framework (Gray 1989 p.141ff).

By conceiving of space as an abstract homogeneous entity, but removing from it any implicit metric relations, Riemann was able to suggest that a number of different types of distance function or metric could be used. This was a profound move, for as Einstein put it...

> "...This (the recognition that General Relativity needed to be expressed in arbitrary coordinate systems) happened in 1908. Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not easy to free oneself from the idea that coordinates must have an immediate metrical meaning."
> (Einstein 1951 reproduced in Smart 1964 p. 285)

Riemann's framework marked a transition from geometry which could be visualised to geometry which could not. In this framework, euclidean geometry was identified only by its metric and as one among many possible geometries.

Notions of curvature were built into this framework later. In 1861, Riemann submitted an essay, in Latin, for a prize at the Paris Academy. This essay contained his expression for curvature. Using the metric function outlined in the 1854 lecture, he was able to define the curvature of a space of arbitrary dimensions. He established an analytic link between the metric function, which determines the local geometry of a space, and the global characteristics of space. "Straight" lines were those lines whose tangent did not change direction as one moved from point to point along them[1]. Since the metric determined how the line changed, it determined which lines were straight.

[1] The mathematical details of this will be given in Chapter 3

The power of Riemann's work came from its capacity to provide a framework which brought together apparently different types of geometry. It achieved this by moving to an abstract and numerical description of geometry, losing on the way geometry's visual and instrumental aspects. A key conceptual change was that curvature could be defined or measured by local means for spaces of arbitrary dimensions, introducing a close relationship between geometry and curvature. One might characterise the shift by saying that lines were not to be thought of as curved in a flat space, but that space "curves" straight lines.

1.1. The Aims of the Study

Visualising abstract mathematical structures such as non-euclidean geometries is difficult, although not impossible. Standard euclidean models for elliptic and hyperbolic geometry, which are described later in this chapter, have been in existence for over a century. As models, they do not preserve all aspects of the geometries that they represent. They are conformal models in that they preserve angle measure between lines representing elliptic and hyperbolic geometry, but do not preserve congruence. Congruence is concerned with the fact that objects which coincide are equal and ensures that measuring rods mark-out the same lengths *irrespective of position*. Our usual sense of distance measure is that it is does not vary with position in space, since a ruler measures a metre wherever it is placed. However, in these models for non-euclidean geometry, this aspect is not preserved: distance measure varies with position in the model.

From a perceptual point of view this represents a difficulty, but, given the successful use of the models by mathematicians for nearly a century, not an insurmountable difficulty. How one comes to recognise and adapt to this change in the perception of distance is an interesting and significant question from both a pedagogical and cognitive point of view. From a practical point of view, there is the issue of how the models should be introduced to learners and how those learners go about developing an understanding of the models. From a theoretical point of view, there are two connected issues. The first concerns identifying those factors, specifically related to the mathematical domain of non-euclidean geometry, which might condition the teaching and learning of the models. The second issue is how these specific factors are related to more general questions of pedagogy and cognitive development. Clearly these theoretical and practical perspectives mutually condition one another and they play a crucial role in exploring how individuals learn to use the models of non-euclidean geometry.

The aim of this study was to explore the possibilities for constructing a suitable context in which to investigate these practical and theoretical issues. The considerations above suggested that any context developed must be able to articulate both the pedagogic and cognitive processes involved, and their mutual interactions. It also raised a methodological question of what was meant by a *suitable* context. *How*, precisely, does a context articulate the issues relevant to pedagogy and cognition?

At the centre of this exploration of context, therefore, lay a dialectical relationship in which the structuring of the context and what might be possible within that context mutually conditioned one another in a dynamic way. To put this metaphorically, the view from a window depends both on where the window is pointed and how it is framed. For a full appreciation of what is seen, therefore, the viewer must take account both of the vista *and* how it is structured by the window. Recasting the aim of the study in terms of this metaphor, it was seeking to provide windows on the process of teaching and learning non-euclidean geometry and to describe how the windows themselves were constructed. It was possible to distinguish between these two poles in the following way. One may describe the articulation of pedagogic and cognitive issues as *local* windows through which to view the processes of teaching and learning non-euclidean geometry. Describing *how* those windows articulated the issues of teaching and learning provided a *global* window on the processes of constructing a suitable context .

Another aspect of interest here was the role that computers could play in teaching and learning non-euclidean geometry. Could the euclidean models of non-euclidean geometry be computerised and what difference did that computerisation make to the teaching and learning?

The overall aim of the study was, therefore, to explore the possibilities for constructing a computer-based context for teaching and learning the euclidean models for non-euclidean geometry. This exploration attempted to provide local and global windows which served the following purposes.

- *Local Windows*

  To provide insights into the processes of teaching and learning non-euclidean geometry mediated by the computer-based euclidean models.

- *Global Window*

  To provide an insight into the process of constructing a suitable computer-based context for teaching and learning non-euclidean geometry which created the possibility for having the local windows of the type above.

1.2 The Rationale for the Study

This section will give an account of the background and rationale for the study. In §1.2.1 the background to the study is described in relation to the interests of the author. Next, §1.2.2 gives an account of the relationship between curvature and its euclidean models, while the processes of visualisation in relation to the models of non-euclidean geometry are considered in §1.2.3. This section concludes with a discussion of the role that computers might play in providing a window on developing a suitable context in which to investigate cognition and pedagogy.

1.2.1 Background to the Study

Originally, the study was intended as an exploration of three interests of the author: differential geometry and General Relativity, Logo's potential as a medium for learning mathematics and, more generally, the processes of teaching and learning mathematics. Initial attempts to bring these areas together in a form that could be investigated were centred on a General Relativity simulator implemented in Logo by Abelson and diSessa (1980) in *Turtle Geometry*. The simulator was programmed by the author and used to explore aspects of General Relativity. The content and style of Abelson and diSessa's exposition was also interesting. They built a Turtle description of spacetime using the idea of curvature as the "excess" created by flattening a curved surface. However, it was reluctantly decided not to use the simulator as the basis of computer-based microworld for pedagogical reasons, which will be described in Chapter 3.

These considerations led to a switch in the focus of the study from working with simulations of General Relativity to a consideration of the ways in which curved objects could be represented in a computational medium. This led, in turn, to an examination of the relationship between three things: non-euclidean geometry, curvature, and the flat representations of curvature, in order to decide whether the representations might be suitable for computerisation. The relationship between curvature and its flat representations raised two issues which formed the basis of the study. The first was how the euclidean models of non-euclidean geometry might be presented in a computational context to learners. The second was how these computer-based representations were understood by those learning to use them. These questions will be examined in more detail in the next two parts of this section.

1.2.2 Representations of Curvature.

Flat two-dimensional models for surfaces with non-zero curvature clearly do not contain the same information as the original surface. By their nature such flat models must lose a spatial dimension and, with it, information about the structure of the surface's geometry. As noted before, Conformal models of non-zero constant-curvature spaces can be obtained either by projecting or otherwise mapping a curved surface onto the euclidean plane. In doing so, they preserve projective properties of the spaces such as angles and the cross-ratios of distances, but not the metric property of distance measure. Two Conformal models were considered for this study. The first, called in the study *Conformal Model A*, is for elliptic geometries (spaces of constant positive curvature). It is also known as the Klein model. The second, called in the study *Conformal Model B*, is for hyperbolic geometries (spaces of constant negative curvature), and is often referred to as the Poincaré Disc model. A qualitative description of the models' characteristics will be given here, but a full mathematical discussion of both will be given in Chapter 3.

1.2.2.(a) Conformal Model A : Elliptic Geometry.

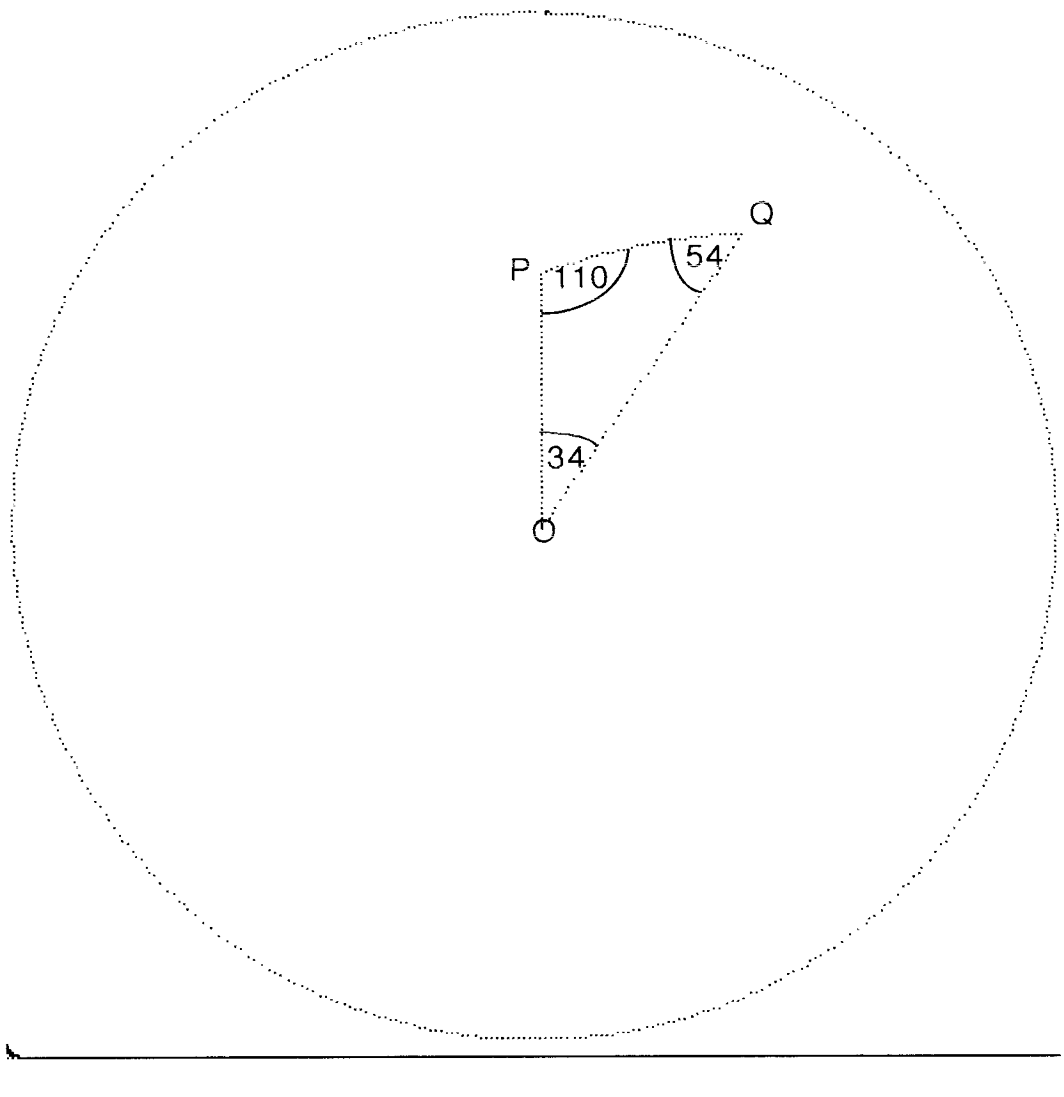

Figure 1.1. Conformal Model A. Triangle OPQ is formed by two portions of the circle's diameter OP and OQ. The arc PQ is part of a circle which intersects the unit circle at opposite ends of the same diameter.

In this model, points of elliptic space are shown by euclidean points inside the unit euclidean-circle whose centre is at the origin. Points at the ends of diameters of the unit circle are identified to give a continuous representation within the unit. "Straight lines" are either diameters of the circle or arcs of euclidean circles that meet the unit circle at the ends of the diameters. The elliptic triangle OPQ, in Figure 1.1, is formed from the diameters through P and Q and the arc PQ which lies on a circle that cuts the unit circle at opposite ends of the same diameter. The measurements shown are euclidean, obtained by measuring the angle made by the tangents to the circle at P and Q. The angle sum is greater than 180 degrees.

Gray (1989 p.215) describes this model as the *Hot-Plate Universe* in which one imagines that the circle is a plate whose temperature increases as one moves out radially from the centre of the circle. Using a metal rule to measure the distance along an arc or a straight line out from the centre, one would be unaware of the ruler's expansion due to the increase in the plate temperature. Only by comparison with a ruler "at room temperature" would the change in length and the increase in "unit step" of measurement be revealed. This variation of distance measure according to position in the model is a key perceptual feature and is also contrary to our usual perception of distance measures.

1.2.3 (b) Conformal Model B: Hyperbolic Geometry

Conformal model B represents the whole of hyperbolic space by the interior of a unit euclidean circle. Hyperbolic points are identified with euclidean points inside the circle. "Straight lines" are either open diameters or open arcs of orthogonal circles. Orthogonal euclidean circles are those in which the radius of the circle is perpendicular to diameters within the unit circle. The lines are open since the points on the boundary are "at infinity" and so cannot be in the interior of the circle. Such boundary points are called "ideal".

In Figure 1.2, the arc of the orthogonal circle whose centre lies outside of the unit circle, cuts the diameter segments at P and Q. The hyperbolic triangle OPQ has angles which add up to less than 180°. The circumference of the circle represents infinity, so that points moving along any orthogonal arc will never reach the circumference.

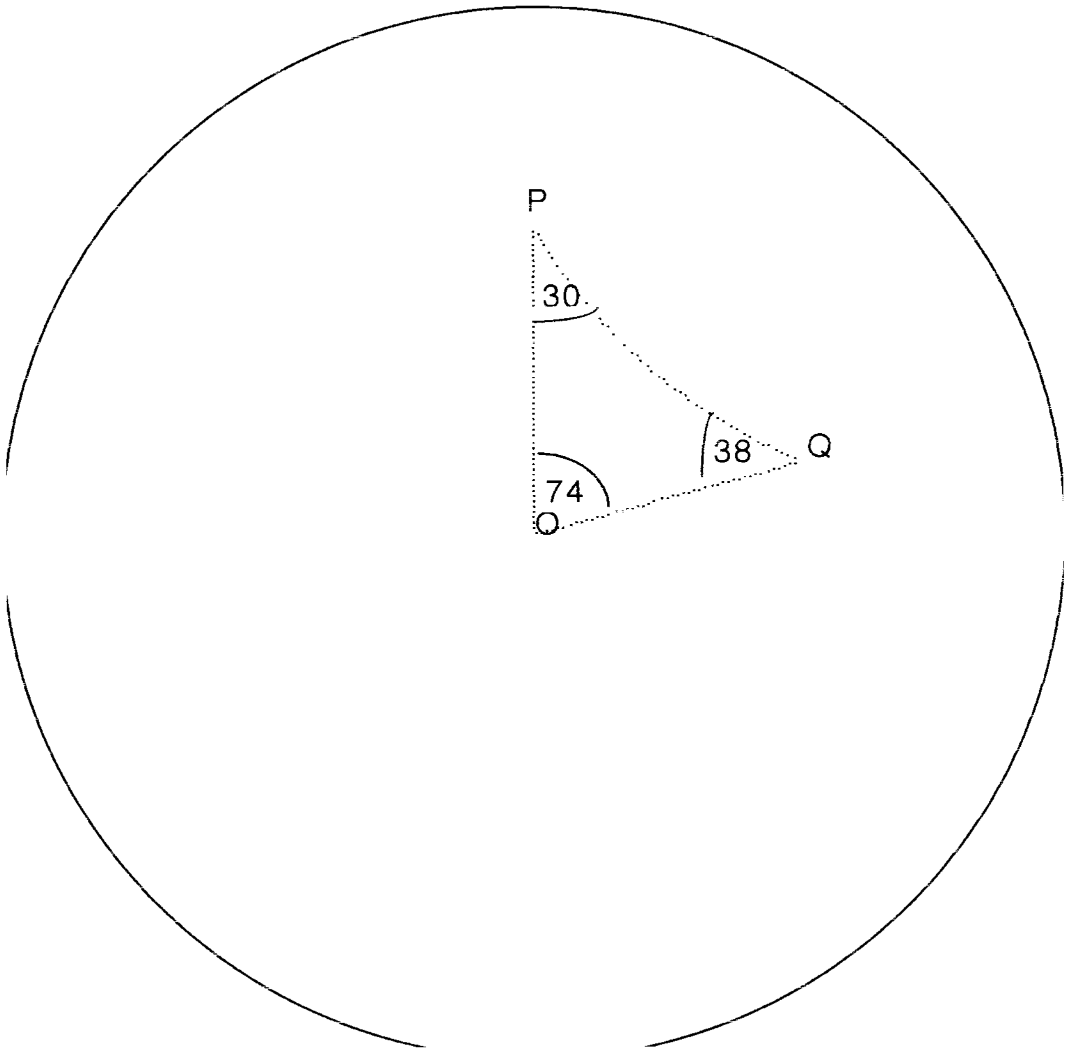

Figure 1.2 Conformal Model B. Triangle OPQ is formed by two portions of the circle's diameter OP and OQ. The arc PQ is part of an orthogonal circle which intersects the unit circle so that their radii are at right angles.

Conformal model B is the *Cool-Plate Universe* (Gray *ibid.*) in which the temperature decreases as one moves radially towards the circumference of the circle. Hence, a metal ruler used to mark out distances along the arcs of circles would contract as it was moved out from the centre and the "unit length" would decrease. However, living in the surface one would be unaware of the variation in length. The change in distance measure would only become apparent if one could compare the rule with that in a "constant temperature" euclidean world.

1.2.3. Visualisation and the Conformal Models

How do we develop an understanding of these Conformal models? As a starting point to discuss some of the issues connected with understanding non-euclidean geometry, this section will consider Reichenbach's model for visualisation of the Conformal models and its implications in relation to this study. Reichenbach (1957) did not advance any empirical evidence to support his model and it is used here to motivate the cognitive issues associated with learning to use the euclidean models of non-euclidean geometry.

In the process of visualisation, Reichenbach argued that two distinct but inseparable functions are at work. The first he calls the "image-producing function" which provides the raw material for visualisation. It comes into action in response to the instruction "imagine a ....". So, "Imagine a cube. Balance it on one face. .... Balance it on one vertex. ....How many vertices are in one layer?" Mason (1988 p.297) says that in response to these instructions people "see" a cube or they have an "awareness" of the cube. Others have a "sort of radar screen with frequent need to refresh the bits of their image which they are not attending to". Image is here taken to mean quite a wide range of things. Reichenbach, in describing the image-producing function, talked about the indistinct nature of the images which are sharpened up by the need to "concentrate much harder" (*ibid.* p39).

The source of this image-producing function is psychological, according to Reichenbach. Limits in our capacity to produce images, however, are the result of infrequent or no contact with, say, infinite planes or objects which are either very large or very small. Our perceptual apparatus gives us only a "slice" of what it is possible to experience. Reichenbach calls the capacity to extend beyond this limitation, the "normative-function", which is logical in nature. The compulsion we see in visual images comes not from our capacity to form images, but from the structure we read into them. As he puts it,

> "We do not read results from (images), but read them into it" (*ibid.* p40).

The images we form are "subject to a directive" that is not based in the process which produces the image. Hence, in the case of imagining the cube, the production of the cube's image is distinct from one's capacity to manipulate the image. Reichenbach believes that this latter ability is based on the process of telling the "eye" what it should "see", using a non-visual model of the logical relations implied by the notion of a cube. A consequence of this view for him is that any structure which is logically consistent can be visualised. Non-euclidean geometries and curved spaces can be represented by euclidean means, provided the interpretation of the euclidean image is guided by the normative function and not by the processes of image production.

As noted in §1.1, euclidean congruence is concerned with the fact that objects which coincide are equal and measuring rods mark out the same lengths irrespective of position. In both the Conformal models, this is not true. In Conformal model B, for example, this failure of euclidean congruence happens in a quite spectacular way since the circumference of a finite euclidean circle represents "infinity". Clearly for the "whole of space" to be contained within a circle suggests that distance measures are not invariant under spatial translation within the unit circle. This failure of congruence makes the models difficult to interpret and suggests that those working with the models *must* be taught how to use them.

Reichenbach asserts that our observation of the variation in the measurement of distance using these models clashes with our euclidean experience of everyday life.

> "We have such a strong visual perception of Euclidean geometry because all our experience of rigid rods constantly teaches Euclidean congruence" (ibid. p.54).

What one sees in terms of congruence in the models clashes with our euclidean intuition. Equal steps along a curve in either of the Conformal models do not give equal distances. Euclidean congruence is replaced by a new definition provided one forgets what one sees and remembers what one knows about the image. The departure from Euclidean congruence is well-defined, however, and relates to a logical structure which can be learnt.

> "euclidean congruence, which we often tacitly presuppose, is based on a definition. This definition too is projected by us into space, not discovered in it" (*ibid*. p.54).

He argues that it is a matter of *habit* that we apply euclidean congruence to "space". Altering the definition of congruence enables different sorts of geometries to be visualised. He implies that by "adjusting" oneself, one is able to visualise geometry in a different way. Representations of non-euclidean geometry and curved space work because we learn to "read into them" logical relationships. One may describe this as a process of "forgetting" what is seen and "remembering" the logical relationships which replace the forgotten euclidean intuitions. The meaning of the geometry derives, Reichenbach claims, from the logical structures of non-euclidean geometry and a reinterpretation of euclidean congruence.

Introducing a different conceptual framework such as non-euclidean geometry implies a "re-learning" of our interpretation of geometry. *How* this re-learning might occur was central concern of the study. It will be considered in greater detail in later chapters. The important point to note here is the possibility, presented by Reichenbach, that the Conformal models have to be explicitly taught and learnt.

1.2.4 Computers and Windows on Thinking

The Conformal models for curved space presented above can be characterised as flat, two-dimensional, euclidean, and static. This static nature of the Conformal models defines their function in non-euclidean geometry. Texts on non-euclidean geometry use these models as visual support for the processes of investigation and proof. Authors, such as Coxeter (1947) and Greenberg (1976), use diagrams to both motivate and illustrate results which they prove logically. Here they are following a style of geometric reasoning as old as Euclid's *Elements* themselves. By contrast, the dynamic nature of the computer image and our ability to manipulate it, provides a different sort of approach to the models. Rather than using the models as a way of supporting counter-intuitive arguments, computers allow users to *explore* the properties of the geometries for themselves. Computerising the models in some way provides a *medium* for thinking with and about non-euclidean geometries rather than just illustrating them.

Take, for example, the distance variation in Conformal Model B. The boundary of the circle can never be reached, but if this could be illustrated in a dynamic and visual way, then it might be possible both to exemplify it and investigate how it is understood by those using such a dynamic representation. Reichenbach's model of visualisation indicates that, at a perceptual level, a geometry is recognised as non-euclidean if it does not give euclidean congruence. If a learner were able to work with the model in such a way that the non-euclidean distance measure was the measure used "on screen", then figures could be constructed which showed immediately the non-euclidean nature of the screen "world". Learners could then form hypotheses to investigate the type of geometry they are working with, and test them using the models.

Underlying this approach is a view of computers which sees them not simply as tools to aid thinking, but central to the structuring of thought itself. Computers fulfil a role similar in many ways to writing in the sense that they provide the possibility for individuals to externalise their thinking in a visual and dynamic way (Pea 1987). In doing so, their "thought" becomes an entity which can be publicly scrutinised, discussed, and evaluated, both by its creator and also by others. This "object" then becomes part of those factors which shape their creator's thinking since it can be manipulated in the computational medium and be integrated into new thought.

This can be illustrated by considering Turtle Geometry in Logo. The importance of the Turtle metaphor at the heart of the microworld lies in it being identified with the self by the learner. Papert describes this process of identification as *syntonic* (1980 p.63). The Turtle becomes a means of externalising a child's understanding of his or her world, since it enables the child to translate his or her own geometric experience of being-a-body into having-a-body (Berger and Luckmann 1966) which must be navigated through space. Related to this is the idea that Turtle Geometry is local and intrinsic geometry (Abelson and diSessa 1980). Procedures which produce shapes in Turtle Geometry describe the shape from a "Turtle's-eye view". In order to create a procedure, the programmer has to "teach the Turtle" by translating his or her visual understanding of the shape to be drawn into a syntonic description. This description can then be translated into Logo commands such as FORWARD and RIGHT, both of which are relative to the Turtle's position and heading. Syntonic descriptions are local, in the sense of being "step-by-step", and intrinsic, precisely because they refer to the Turtle's state and not to any external frame of reference. In this sense, the Turtle plays a role similar to touch. The child is describing in Logo code what it feels like to move over and around a shape, so that the Turtle may be guided appropriately.

From a cognitive point of view, externalising thought by constructing a shape using the Turtle points to two aspects of the way in which understanding develops. On the one hand, there is a sense in which understanding is based in experience, coming about through interaction with objects and people. On the other hand there is the sense in which understanding goes beyond experience and plays a role in shaping the interactions which sustain that experience. Using the Turtle, for example, involves working with the computer in a close way. At a practical level, one is trying to get the Turtle to do what one wants it to do and one is concerned to obtain the correct sequence of commands to achieve a particular result. However, one is also involved in constructing a sequence of actions which can be "objectified" through the Turtle's behaviour precisely because they are the Turtle's actions and not one's own.

The process of objectification enables two things to happen. First, the Turtle can provide the individual with the means to reflect on their own actions and integrate them into their own understanding of the practicalities of a situation. This has both a practical and a theoretical element. What it is possible for the Turtle to do, is determined not only by the practical circumstances (such as the commands available to the Turtle programmer), but also by how the programmer interprets the situation relative to their own understanding and intentions. Secondly, since the actions of the Turtle are public, they provide access to the thought-processes of the person who programmed the Turtle. The computer or floor robot which is being controlled by the programmer can be seen by others and its actions can be discussed and evaluated through a syntonic comparison. Using the Turtle can give a "window" on thinking (Weir 1987).

This had two consequences for this study. The first was that individuals working with computers could create their own understanding of the euclidean models of non-euclidean geometry, perhaps using some version of Turtle Geometry. A consequence of this was that the individuals' processes of understanding could be described, because of the public nature of their interactions with the computer. This provided the possibility of a local window on those processes. Second, the construction of a context to enable this investigation of teaching and learning non-euclidean geometry could be considered as an exercise in model-building. Through the model-building process, with its emphasis on iterative trial and development, it was possible to analyse the way in which a suitable computer-based context for teaching and learning non-euclidean geometry could be constructed. This provided the possibility of a global window on the process of developing a context for teaching and learning non-euclidean geometry using its euclidean models.

1.3 Overview of the Thesis

Chapter 2 describes the theoretical considerations which formed the basis for constructing a suitable context in which to realise the aims of the study. Interpreting the aims of the study in terms of model-building, the chapter begins by discussing, in general terms, the microworld as a paradigm for modelling the learning of mathematics using a computer. The structure of computational-based microworlds is then defined in terms of its computational, pedagogic and cognitive elements. Interpreting the microworld definition as a language-game (Wittgenstein 1953) and as creating a zone of proximal development (Vygotsky 1978), the chapter constructs a framework in which to develop and analyse the microworld. The chapter culminates by describing the outline of a model for the microworld.

Chapters 3 and 4 fill in the details of the models by describing the design of the technical, pedagogical and cognitive elements. Chapter 3 begins by giving a mathematical account of how the euclidean models of non-euclidean geometry are produced by projecting surfaces of constant positive and negative curvature onto a flat plane. Next, it describes the considerations for choice of programming language and how Turtle geometry was implemented using differential geometry. The chapter concludes by giving a set of equations which will form the basis for implementing the euclidean models using Turtle Geometry. Chapter 4 reviews the psychological aspects of curvature and non-euclidean geometry. It describes how teaching non-euclidean geometry may be thought of as inducting learners into a language-game and provides an interpretation of the language-game notion in terms of the interconnection between linguistic signs and actions associated with surfaces.

Next, in chapter 5, the methodology of the microworld's iterative development is described, together with details of data-collection and analysis. Chapters 6, 7 and 8 contain accounts and analysis of the three developmental cycles which took place with the microworld. Each cycle has the same structure, beginning with a description of developments in the technical, pedagogic and cognitive aspects of the microworld, either as the result of a previous phase, as in cycles 2 and 3 (Chapters 7 and 8) or from the design considerations (Chapter 6). Each of the chapters then goes on to describe the activities undertaken in each of the phases together with their rationale. Finally, each chapter concludes with a reflection on the cycle and suggestions for modifications in the light of trials conducted with pairs of microworld users. The study concludes with a discussion of its specific outcomes, limitations and further questions.

Chapter 2

Modelling a Context for Teaching and Learning Non-Euclidean Geometry

2.0 Introduction

The aim of this chapter is to develop a model of a context in which to teach and learn non-euclidean geometry. §2.1 reviews the definition of models and modelling and discusses their cognitive value. In §2.2, the microworld paradigm for computer-based learning environments is used to describe a context for learning non-euclidean geometry. A definition of the microworld is developed in terms of its technical, pedagogic and cognitive elements, which are interpreted in §2.3 - 2.5, to provide a structure that can be implemented practically. §2.6 draws together these considerations to provide a model of the microworld which can meet the aims of the study.

2.1 Models and Modelling

This section considers modelling and models in general. In §2.1.1 a general definition of a model is given and the cognitive value of the modelling process is discussed in §2.1.2.

2.1.1 What is a Model?

Warzel (1989) citing Apostel (1960) describes a "modelling relationship", R, as

> "a structure R(S, P, M, T) which means: 'The subject S takes, in view of the purpose P, the entity M as a model of the prototype T" (p. 121)

The definition is interesting because it identifies two important aspects of models and modelling. The first is that models serve a specific purpose in relation to the needs of individuals. They may serve a variety of functions, but the uses that are made of models are always relative to a problematic situation. Warzel (*ibid.*) identifies five categories of purpose for models. They can be used to increase knowledge, clarify thinking, combine multiple theoretical perspectives, provide novel interpretations of problematic situations and solve problems. The second aspect of the definition is the sense in which a model is an entity used to stand for something else. Models *re-present* situations so that they can serve the purposes outlined. To do this, models use one thing in place of another. A process of idealisation and simplification enables a medium to be chosen which can re-present a problematic situation. (Ogborn 1994).

To put this latter point more formally, one may say that models are representations with specific properties. Kaput (1987) defines a representation as consisting of five elements:

(i) the represented world,

(ii) the representing world,

(iii)what aspects of the representing world are doing the representing,

(iv) what aspects of the represented world are being represented,

(v) the correspondence between the two.

The process of making a representation implies making a connection between two domains so that structural features of one "world", which is understood, may be used to describe another, possibly problematic, "world". Warzel (*ibid.*) citing Stachowiak (1973) provides two properties of this relationship of representation. Firstly, it is a map in the sense that structural aspects of one domain are used to structure the other. Secondly, this map is a "shortening", or partial representation, in the sense that not everything in one domain is mapped onto the other domain.

2.1.2 The Modelling Process

Building models is a process of constructing such representations of systems for specific purposes. Several stages are apparent in this process:

- the selection of a domain, problem or structure;
- the identification of determining factors;

- finding relationships between the factors;
- checking that the behaviour of the model agrees with the domain being modelled, according to some pre-determined measure;
- refining the model as appropriate.

This iterative process, in which the initial attempt to represent a problematic situation is gradually refined, stops when the model is judged to be *sufficient* for the purpose that it was created for. Criteria for sufficiency will vary according to the situation in which the model is to be used and a key part of modelling is identifying when the process can stop. For example, if a model is being used in the planning and construction of a bridge, the criteria for sufficiency will be concerned with ensuring that the bridge does not fall down under a variety of weather and loading conditions. If, on the other hand, a model is used in an attempt to understand an economic system, what constitutes sufficiency may vary between "experts" who build and use it.

From a cognitive point of view, making and using models is an important aspect of learning. Bliss (1994), for example, describes the cognitive effect of modelling as "externalising thought". She draws the distinction between exploratory and expressive modes of model use. The former entails the learner using a model which someone else has created, while the latter describes the process in which the learner creates their own model. The process of building models in "expressive" mode entails the model-maker expressing their own ideas and understanding of a specific system. The process of checking and refining their ideas against the system being modelled leads the learner into an iterative development-cycle of trial, evaluation and modification described above. Using models in "exploratory" mode is also an iterative process. The learner has to internalise the representational connections between the model and the situation being modelled through a process of comparison and questioning. This occurs at the same time as they use the model. The two modes are not completely separate, therefore, and imply one another to some extent. Exploring a model that someone else has created entails relating it to one's own experience and knowledge in an attempt to make it one's own. In this sense, one is representing the model to oneself and operating in expressive mode. On the other hand, part of the process of making a model is that one has to "stand-back" from it to assess its efficacy and this may lead to exploring aspects of the model which were not initially apparent or intended.

These considerations suggest that modelling has both expressive and exploratory aspects. Bliss's view that model-building is concerned with externalising thought may be regarded as an element in a dialectical process of externalisation, objectivation and internalisation (Berger and Luckmann 1966 p.78-80). Models are often generated in the process of articulating a particular view of a given situation and as such represent the *externalisation* of thought through activity. In order to assess whether a model is effective, the model-maker must develop a "critical distance" from the model in the sense of regarding the model as an entity in its own right. Berger and Luckmann describe this process of distancing one's self from one's own products as *objectivation*, which invests the model with a "life of its own". In doing so, the model-maker may begin to explore facets of the model which were not originally apparent and the model, in turn, acts back on the maker to alter the way in which the problematic situation is viewed. The model maker becomes a model user as he or she *internalises* the model's nuances and reinterprets the original problem with the model. Papert (1991) describes the way in which creating a model of something can lead to an understanding of it. He is particularly interested in the plurality in ways of knowing that modelling seems to provide, in the sense that the model which an individual creates reflects his or her perception and interpretation of the situation or thing being modelled. This, in turn, leads to a greater appreciation of the model, its applicability and one's own thought processes.

From the study's point of view, the dialectic of externalisation, objectivation and internalisation was significant in two ways. A central aim of the study was to explore the possibilities for creating a context in which to learn non-euclidean geometry. This context simultaneously had to provide windows on the processes of its own construction and the development of geometric understanding in those who use it. Modelling gave a window on both these aspects. First, model-creation had a metacognitive aspect. The process of externalisation and objectivation implied by the production of the model provided an entity which was examined and reflected on. It supplied an insight into the assumptions and thinking of the model-maker. Building a model of a context for learning non-euclidean geometry, therefore, gave a "global" window on the process by providing an "object to think with". Second, the context created had a specific purpose. It was designed to enable individuals to gain an understanding of and a facility with the standard euclidean models of non-euclidean geometry. A "local window" on this process was provided by charting the way in which learners acquired the euclidean models as they internalised the models' representational aspects and externalised their understanding through the use of the models in solving problems.

These considerations suggest that the central aim of the study was to be addressed by *building a model* of a context for non-euclidean geometry. This model would provide windows on both the process of construction and the way in which individuals came to understand the euclidean models.

2.2 Computers, Modelling and Learning

Having outlined the way in which the notions of model building and model use were significant for the study, the role of computers in the modelling process will be considered. Computers have a significant role to play in modelling for two reasons. First, there is a sense in which building a model using a computer *embodies* the model, giving it a reality which is independent of the maker. A computer representation of a model provides both the "critical distance" needed to assess it and also allows the model to be used in a dynamic and interactive manner. Second, this tends to support the interpretation of the model as both an "entity" and a "tool", separate from its creator, but which can be used by its creator to deal with problematic situations. Given that computers can play an important role in the modelling and in interpreting the aims of the study in terms of modelling a suitable context for learning non-euclidean geometry, the next stage is to consider the specific role that computers might play. This section will first consider, in general, possible modes of computer use in learning and then go on to consider the microworld paradigm which was used in the study.

2.2.1 Computer-based Learning Environments

Darby (1992), in reviewing the ways in which computers are used in the UK Higher Education sector, identifies three approaches. In various ways, these approaches describe how the control of the learning process is distributed between computer and "the learner". The first approach that he describes, taken from MacDonald *et al.* (1979), employs the nature of the interaction between students and computers to produce five styles of use. Starting with students having a passive role in their use of computers, the classification goes on to describe an increasing degree of active participation by them. This culminates in "constructive understanding", where the student is in control of the learning and uses the computer as tool for processing information. A second typography, taken from Allison and Hammond (1990), classifies computer use in terms of the three styles of Computer Assisted Learning (CAL): programmed learning, intelligent tutorial systems, and learner support environments. In programmed learning, the computer decides the learning path and mode of understanding, assuming that the user is an "errorful expert" rather than a "naive learner". Such programs are usually constructed on the basis of either a linear development or a conditional branching mechanism. By contrast, the learner-support environment provides the learner with a number of tools, related to a specific area, which have been chosen to optimise the opportunities for learning. Control of the process and the strategy for learning are left to the individual. Intelligent tutor systems form a "half-way house" between the two. (Elsom-Cook 1990). These systems attempt to match the learner's current state of knowledge and understanding against a model of the required state and provide appropriate support for the learner. The third approach, again from MacDonald *et al.* (*ibid.*), is based on describing types of software: instructional, revelatory, conjectural and emancipatory. Again the gradation is from the situation in which the software instructs the learner, through to providing tools which support the learner in some way but leave decisions about the mode and pace of learning to the learner.

Sewell and Rothery (1987), in reviewing the use of computers in primary and secondary schools, draw a similar distinction between those pieces of software and modes of computer use in which the computer instructs the child and those that facilitate child learning with the computer. The former they characterise as being:

> (a) highly structured learning environments in which the program controls the path or paths of learning—flexibility of learner input is restricted;
> (b) detailed analysis of the task;
> (c) use of successive approximations to the desired end point, usually defined as a 'behavioural objective';
> (d) an emphasis on extrinsic re-enforcement which may be divorced from the nature of the task. (p.380-381)

The second type of computer use they describe as having:

> (a) A high degree of learner control over the learning paths — the computer merely provides an environment in which important ideas are thought to be embedded;
> (b) An emphasis on process rather than product; learning is believed to arise organically from the structure of the interaction;
> (c) (A) freedom of interaction (which) is believed to be intrinsically motivating and no external re-enforcement is needed. (p. 381).

As an example of this type of software they cite the computer-based microworld as "an environment in which the learner is free to carry out procedures and operations which in some way embody important ideas" (*ibid.* p. 382). This way of using computers will now be considered in more detail.

2.2.2 What are Microworlds?

In *Mindstorms* (1980), Papert described microworlds as "computer-based interactive learning environments where prerequisites are built into the system and where learners can become active constructing architects of their own learning" (p.122). Two elements are important here. First, a microworld is a learning environment. Papert refers to it being computer-based and interactive. However, any environment which uses a restricted and well-defined collection of tools to aid understanding is a microworld. A set of wooden blocks or a construction kit both provide the opportunity for learning by exploring the range of possibilities and constraints they offer (Papert 1987). "Prerequisites" are assumptions and constraints imposed by particular "task domains" or "problem spaces" (Lawler 1984). Second, they provide children with the means to actively engage in building their own meanings and to control the process of their learning. Starting with a set of tools, microworlds encourage children to build ever greater understanding of specific mathematical domains through use of those tools. As Hoyles (1993) puts it, in a review of the genesis of the microworld concept in relation to mathematics,

> "At the core of a microworld was a knowledge domain to be investigated by interaction with the software........they (microworlds) aimed to facilitate the building of conceptual and strategic foundations - from simple entry points to deep ideas." (p.3)

A related theme in explicating the meaning of microworlds is the relationship between the pupil, computer and domain of knowledge. An important aspect of microworlds is their pedagogic function and Hoyles and Noss (1987) make this a basic consideration in microworld construction.

> "A microworld cannot be defined in isolation from the learner, the teacher, or the setting; activity in the microworld will be shaped by the past experiences and intuitions of the learner, and by the aims and expectations of the teacher." (p.587)

Microworld design should take into account the context and intentions of the users. Computer-based tools are important, but as part of a wider collection of activities, structured for the purpose of teaching and learning. Hoyles, Noss and Sutherland's experience in developing a microworld for ratio and proportion led them to extend the definition of a microworld to take account of the pedagogical context in which it is used.

> "A microworld consists of software designed to be adaptable to pupils' initial conceptions together with carefully sequenced set of activities on and off the computer, organised in pairs, groups or whole classes each with specified learning objectives."
> (Hoyles, Noss and Sutherland 1991 Vol.3 p.3)

Noss and Hoyles (1992), in reviewing a decade of research on Logo and its microworld, further emphasise the point that microworlds are more than a collection of software tools. They caution against the assumption that children's use of software tools will lead, necessarily, to an appreciation of the ideas which underpin the tools. They point to three ways in which children may not become aware of the epistemological base of the microworld: unreflective use of tools, avoidance of using certain tools, and an avoidance of mathematical analysis. Their observations centre on the way in which children do not reflect or apparently discriminate as they use the tools of Logo microworlds. Noting that microworld tools are built according to pedagogical criteria, they indicate the importance of teacher intervention to provoke the process of reflection on the underlying structure of the microworld by the child.

This implies that any definition of a microworld must take account of both the "creator's" intention and the likely behaviour of the learner. Bearing these considerations in mind, and drawing on Hoyles and Noss (1987), a working definition of a microworld can be given. A microworld is a computer-based learning environment which consists of three aspects: the technical, the pedagogical, and the cognitive. These three categories seek to take account of the computer-based tools, the learner, the teacher, the setting, and to facilitate the analysis of the interactions which take place between them. Each of the elements will be described in more detail.

The *technical element* of the microworld is defined to be "a set of tools" designed to provide the learner with access to a specific mathematical domain in ways which suits the learner. (Hoyles and Noss 1987). Designing the tools is a process that starts with an initial assumption about how the learner will interact with them. The tools are then modified in light of the experience of individuals using them. (diSessa 1986a). For Hoyles and Noss (*ibid.*), this element is concerned with the programming language which the learner will employ. Procedures, constructed by the authors of the microworld, are given to the learner and the learner then adapts them to suit his/her own mathematical investigations. However, tools may also consist of non-computational aspects which are demanded by the knowledge domain and the activities devised to introduce that domain. Hence, the technical element consists of any set of tools, computational or otherwise, needed by the activities associated with the microworld.

The function of the *pedagogic element* of the microworld "is to structure the investigation and exploration of the concepts embodied in the technical component" (Hoyles and Noss 1987). What is implied here by "structure the investigation" is a careful mixture of activities and informal interventions by the "creator". These direct the learner in specific ways, provoking reflection, suggesting new lines of enquiry and consolidation. Such "guided discovery learning" (Hoyles, Noss and Sutherland 1991 p.3) seeks to provide learners with a clear understanding of the "tools" and to "encourage them to reflect on their own thinking and upon the embedded mathematics" (*ibid.*). From a pedagogical point of view, however, this creates what Hoyles and Noss call a "critical tension" (1992) between allowing learners to go their own way and the pedagogical agenda of the microworld design. Both timely intervention by "a teacher" and an understanding of how learners are likely to react are needed to ensure that the pedagogical objectives of the microworld are met.

The *cognitive element* of the microworld refers to those factors which condition the learner's response to the activities of the microworld. Several factors are important, such as the assumptions of the learners about mathematics and computers, their ability to work with others, and their willingness to enter into apparently open-ended activities. Hoyles and Noss (1987) indicate that the way in which problems are presented effect the approaches adopted for solution and they highlight the need for sensitivity towards what learners of the microworld will understand by the activities given and accepted as "problems". Hoyles and Noss refer to the "social setting in which the programming activity takes place" as having an effect on the way in which the users of a microworld interpret what is required of them and how they react to it. Clearly, the perceptions, assumptions and knowledge of the learner are vital for effective learning with the microworld and must play a major factor in designing computational tools and activities, as subsequent chapters indicate.

Microworlds can be defined, therefore, in terms of three elements.

- A *technical* element, consisting of computational and non-computational components required by the knowledge domain;

- A *pedagogical* element, which describes the aims of the microworld, the type of activity which learners engage in, and the pedagogical strategy adopted;

- A *cognitive* element, which consists of the learners' responses to the experience of working with the technical element within the pedagogical structure.

The remainder of this chapter will be concerned with interpreting the microworld's definition so as to develop a model for it. Each element of the microworld's definition will be examined theoretically to establish a suitable framework in which to interpret it. This framework will then be used to create a practical structure for each element . The chapter concludes with the presentation of a model for the microworld, constructed from the structures developed for each of the microworld's elements, and used in its design and development.

2.3 The Technical Element of the Microworld

This section will consider the technical element of the microworld. §2.3.1, discusses the general nature of tools, both computational and non-computational. The second part, §2.3.2, presents the rationale for using Turtle Geometry to implement the computational aspects of the technical element and concludes by outlining the structure of the element used in the model.

2.3.1 Tools and their Uses

A "craft" paradigm is useful for understanding the meaning and significance of tools. Working with some form of raw material, craft practitioners shape it into objects and artefacts using implements and techniques suited to the material. Selecting a tool that is appropriate to a particular purpose is part of the knowledge that practitioners have. Such knowledge has been acquired through experience and participation in a community of practitioners, sharing common values and modes of communication.

At an individual level, tools have a significance in so far as they serve a purpose: not just any old purpose, but *my* purpose. Objects become tools in so far as they can be used to achieve *my* ends. For example, a hammer is a tool if I wish to bang in a nail but if I have to plane a piece of wood, it is an object. Heidegger (1962) refers to this distinction as being *ready-to-hand* and *present-at-hand*. What makes the difference is my need of the thing in order to act for a specific end. If one uses a hammer as a tool, however, one is not aware of it as a hammer, only that it serves the purpose to which it has been put. The hammer, as an entity, is hidden in the process of using it and is only disclosed as an entity when it can no longer serve the purpose to which it has been put.

Our sense that tools are ready-to-hand is essentially practical, obtained through active involvement with tools and the world around us. This practical engagement with things and people is fundamental to our experience of being human. We do not come to know through detached contemplation, but through *praxis*, understood as an active and necessary practical engagement with the world. For Heidegger, this "thrownness" into the world, over which we have no control but which we are necessarily oriented towards, characterises our "Being-in-the-World". A dialectical relationship exists between the "world", as the backdrop to all action, and "things", which lie in the foreground and are the central focus of attention. The "World" and "things" mutually condition one another in the sense that all activity with things pre-supposes a knowledge of the world and knowledge of the world is disclosed though acting with things. (Heidegger 1962 p.102-107).

Computational tools are built in a similar way to "ordinary" tools. They presuppose a background of shared knowledge possessed by both creators and users against which the function and significance of the tool is understood (Kammersgaard 1990). However, it is also clear that the more specialised the tool, the more specific is the type of user. There is a close connection between "tools" and knowledge domains, in the sense that they mutually condition one another. Tools are an integral part of the practices which make up specific domains of expertise. Effective use of tools requires knowledge of the domain in which they are used and, conversely, competence in the use of tools helps to develop knowledge of a domain. Dubinsky and Tall (1991) argue that computer-based mathematical tools can only be used effectively by those initiated into the type of mathematical discourse that generated them. The processes that underpin the execution of the commands in Logo or a Computer Algebra System are not visible to the user. For the knowledgeable user, this does not matter. In fact, it is often the case that the algorithms used by the computer are quite different to those employed by the user. Here, what counts is the functional equivalence of the computer tool with other methods: it gives the correct answer. Such "opaqueness" is significant since it encourages "blindness" to the underlying process by the user. (Love 1993). For the experienced user, this is not significant. For the learner, it is of crucial importance, since it is precisely the mathematics implicit in the "tool" which is to be learnt.

2.3.2 Structure of the Technical Element : Turtle Geometry

Microworlds usually contain one or more "computational tools" which are available to the user. The tools are designed in such a way that they provide a learner with the means to explore a particular domain of knowledge for themselves. This section will consider the sort of tools needed for a microworld concerned with non-euclidean geometry. Three issues will be examined in light of the previous considerations. The first concerns how the Conformal models for non-euclidean geometry were to be implemented in practice. The second was the relationship between the Conformal models and the curved surfaces that they represented. The third issue was concerned with the implications of these two aspects for the make-up of the technical element of the microworld.

Perhaps the most familiar example of a microworld is the Logo implementation of Turtle Geometry in which the child learns to navigate a small floor robot or an object on a computer screen using four simple commands: FORWARD, BACK, LEFT and RIGHT. The child can then "teach the Turtle" new commands using the programming structures of Logo, so that he or she learns to program through exploration of the Turtle's world. The child is given a simple entry point to powerful geometric ideas through the Turtle metaphor and can develop their understanding through active engagement with it and reflection about it.

The relevance of the microworld idea as implemented by Turtle Geometry for this study lay in two aspects. First, Turtle Geometry provided an example of a *geometric* learning environment and the aim of the study was to develop a context for learning non-euclidean geometry. In particular, the study was concerned to implement the standard Conformal models of non-euclidean geometry, outlined in §1.2. The Turtle Geometry approach suggested the possibility of a simple entry point to the domain of non-euclidean geometry, *if* the Conformal models could be implemented using the Turtle metaphor.

The second aspect of Turtle Geometry which was interesting from the study's point of view was that it provided the learner with a simple entry point to the knowledge domain but did not prescribe any learning path, in the sense of the first category of software in §2.2.1. If no learning path was prescribed for those using the microworld, one may conjecture that the path that the learners actually took reflected, in some way, the development in their understanding. This was important since an aim of the study was to provide a window on the learner's thinking as they engage with the Conformal models of non-euclidean geometry. By observing how the learners interacted with the software and the type of support that they required in terms of non-computational components, it would be possible to chart the learner's development. The type of open-ended activity which might provide such a window implied that the pedagogical strategy should be one of guided-discovery, mentioned in the §2.2.2. This approach attempted to balance the desire to allow the learner to explore the knowledge domain for themselves against the pedagogical objectives of teaching them about non-euclidean geometry. The discussion of Turtle Geometry also implies that this approach might be a suitable way to implement the technical element of the microworld, so that it provides a "local" window on the cognitive development of those using the microworld.

From a software design point of view, this raised two questions. The first concerned the feasibility of implementing the Conformal models using Turtle Geometry and the second was concerned with the sorts of "tools" that the learner should be given. As regards the first question, it turned out that it was possible to use Turtle Geometry as a medium for implementing the Conformal models and this is discussed in Chapter 3. The second point meant reviewing whether the usual commands of Turtle Geometry were appropriate for the purposes of the study and whether other tools, specifically related to the Conformal models, were needed. This issue of the type of tools which might be needed is also discussed in detail throughout the development of the microworld.

Mathematically, the Conformal models are obtained by projecting surfaces of constant positive curvature, such as a sphere, and constant negative curvature, such as a hyperboloid, onto a flat plane[1]. In terms of the technical element of the microworld, this raised the possibility that its non-computational component might consist of such surfaces, although their specific function would have to be defined as the interpretation and structure of the other elements were developed.

[1]The details are given in Chapter 3.

2.4 The Pedagogic Element of the Microworld

This section, again in two parts, examines the pedagogic element of the microworld with a view to elaborating a structure for it. It begins, in §2.4.1, from the assumption that learning is a collaborative process and interprets the role of tools under such an assumption. This leads, in §2.4.2, to a specific structure for the pedagogical element.

2.4.1 Tools and Learning

The principle which will be examined in this section is the contention that learning is a collaborative enterprise embedded in, and structured by, human social interactions. Connected with this, the role of tools in learning will be explored from a psychological point of view.

The starting point in examining this principle is described by Berger and Luckmann (1966) thus.

> "As soon as one observes the phenomena that are specifically human, one enters the realm of the social. Man's (*sic* .) specific humanity and his sociality are inextricably intertwined. *Homo sapiens* is always, and in the same measure *homo socius*." (p.69).

This has important consequences for the development of "the human child". As Bruner (1985) puts it, it is not the case that "a lone child struggles single-handedly to strike some equilibrium between assimilating the world to himself (*sic*) or himself to the world" (p.25). The child is neither alone nor struggling. He or she is inducted into a "form of life" which structures the child's experience and provides the basis for its development. For Vygotsky (1978), the process of

> " the child's cultural development appears twice: first on the social level and later, on the individual level; first *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to all voluntary attention, to logical memory, and to the formation of concepts. All the higher mental functions originate as actual relations between people." (p. 57)

Here, the child internalises the social relationships in which it is embedded and these relationships structure all aspects of the child's mental development: they form the subject until the subject can form itself.

Vygotsky argued that there was a dialectical opposition between the external, social world and the internal, personal world of the subject. Emerging from an undifferentiated social experience, human subjectivity develops through language, where language was understood as a mediating tool which used signs. These signs are reversible in the sense that they acted back on the user and can be used to control behaviour. Consciousness developed, not independently of the world, but from the interaction of beings with the world. In using signs, the subject can be conscious of its own experience as an object, but also understand itself as subject. However, this structuring of consciousness by social relationships is not a deterministic process in the sense of a passive "imprinting". Rather it is considered to be a process which is,

> "a continual movement backward and forward from thought to word and from word to thought. In that process the relation between thought and word undergoes changes which themselves may be regarded as development in the functional sense. Thought is not merely expressed in words; it comes into existence through them." (1962 p. 125).

This dialectical process of interactions between word and thought in which each conditions the other, forms the basis for a complex network of interconnections between thoughts and things.

These considerations have important consequences for learning. They suggest that "the human child" learns through interaction with other humans and this interaction sets the pace for the child's development. Vygotsky, in trying to define the correct object of psychological study in this context, described the "zone of proximal development" (1978 p.86) both as a basic methodological concept and tool. This referred to the

> "distance between the actual development level (of the child) as determined by independent problem solving and the level of potential development as determined through problem solving

> under adult guidance or in collaboration with more capable peers." (*ibid.*)

Two points are significant. First, development was seen as a collaborative process between the child and those others, more competent than the child, including peers working in computational settings (Healy 1989), in non-computational settings (Forman and Cazden 1985, Forman 1989), text books and posters, as well as teachers. Second, development of the child occurs *through* learning rather than the other way around.

From a pedagogic point of view, Vygotsky explored the relationship between what he describes as "spontaneous" and "non-spontaneous" or "scientific" concepts (1962 p.84f). Disagreeing with Piaget, he saw a dialectical relationship between the ideas that a child develops on his or her own and those that are acquired through formal instruction. His investigations

> "warrant the assumption that from the very beginning the child's scientific and his spontaneous concepts..........*develop in reverse direction*. Starting far apart, they move to meet each other." (*ibid.* p.108)

It is important to note the structure of the process he outlined. Essentially, this is a dialectical process in which children's intuitions are formed through formal instruction to produce a mature understanding of abstract ideas. However, the pupil is not seen as a passive recipient of pre-packaged knowledge. Rather, the learner is encouraged through activity to use the formal framework of a specific knowledge domain as a tool to shape their intuitions. It is the role of the "collaborative other" to create the activities and structure their interventions and instructions in such a way that the learner may "benefit from it".

2.4.2 Towards a Pedagogic framework for the Microworld

Microworlds are fundamentally pedagogic devices which enable learners to be inducted into specific domains and provide them with the tools and activities to explore the knowledge domains for themselves. This interpretation suggested that the process of induction and development implied by the notion of a microworld as a pedagogic tool may be characterised by Vygotsky's concept of the "zone of proximal development" (ZPD) referred to above.

Wood *et al.* (1976) use the metaphor of "scaffolding" for ZPD to describe the connection between the child and the "competent other" in practical learning situations. Greenfield (1984) elaborates the metaphor to draw out five major characteristics. Scaffolding provides: support; acts as a tool; extends activity; opens up new possibilities and enables specific task execution. Further, scaffolding may come from a variety of sources: teacher, other pupils, books and computers. Bruner (1985) refers to "props, processes and procedures" which facilitate development through "transactional learning" (p.25). He outlines the role of each of the "three P's" for both learner and teacher in facilitating the development of the learner. Hoyles (1991) argues that the resources of the microworld should be rich enough to enable the user to develop their own micro-structure. She wishes to move away from the notion of scaffolding as surrogate tutor towards a more flexible and creative structure created partly by the "originator" and partly by the user. To use the scaffolding metaphor, the originator of the microworld provides the main support over the area to be learned, while the learner uses the microworld's resources to cover particular portions in more detail or extend beyond the given in specific directions. Greenfield (*loc.crit.*) describes as "fading" of the scaffolding the process by which the learner internalises the structures and content of the microworld's epistemological base and can work with confidence and competence on self-generated tasks.

If the scaffolding metaphor is to form the basis of a pedagogical strategy for microworlds, then it is necessary to consider what the role of the "competent other" that provides the scaffolding, is going to be in the process. On the one hand, there is the issue of how the pedagogical process is to be structured in terms of what sequence of activities and forms of support are to be given. On the other hand, there is the question of who controls the pedagogical process. To use the scaffolding metaphor; what is to be used to create the scaffolding structure and who decides how the learner is to navigate it?

The first question of what was to form the scaffolding structure in relation to the microworld's design was answered partly by considerations of the technical element of the microworld given in §2.3.2. Since the Conformal models were to be computer-based and those models were obtained by projection, then it followed that the activities must be structured around the relationship between the curved surfaces and their projections. The second question of who controls the learning in terms of content and direction was more problematic. As §2.2.2 indicates, the desire to provide learners with autonomy and, at the same time, to realise the intentions of the microworld led to "guided-discovery learning" being the preferred pedagogical strategy. This strategy pointed the learner in a certain direction though the use of investigations and activities, but left open the possibility that learners may adopt their own strategy for dealing with the activity and, perhaps, reformulate the investigations to suit their own interests. The resulting tension between the intentions and interests of the teacher and those of the learner which may surface during this process highlights what Brousseau and Otte call the "didactical contract" in which: "the teacher is obliged to teach, and the pupil is obliged to learn". (1991 p.18).

Coupled with this issue of informality in the process of teaching was the question of teacher intervention. Given that the learner's activities with microworlds were governed in their overall direction by the objectives of those who constructed it, an important issue was that of the role of the teacher in the actual process of learning with a microworld. As §2.2.1 shows, at one end of the spectrum there was the view that the learner should be left with a minimum of support from the teacher as they explored the knowledge domain of the microworld. The other end of the spectrum was characterised by the need for some form of explicit structuring of the learner's introduction to the microworld. Hoyles and Sutherland (1989), reviewing the issue of teacher intervention in a Logo-based environment, suggested three sorts of intervention in pupil's learning with computer-based microworlds. Interventions, they found, were characterised by those that aimed at keeping control with the pupil; those based on the introduction of teacher-directed tasks, and those interventions which introduced formal teaching episodes. (p.143) These reflect a spectrum of approaches which tried to preserve autonomy for the pupil and balance it against the need, on some occasions, for more formal instruction by the teacher. This provided a way of resolving the potential conflict between the role of formal instruction, felt to be important by Vygotsky, and the autonomy of the learner implicit in the microworld concept.

These considerations suggested a three-stage pedagogical framework for the microworld which was structured by the notion of ZPD, interpreted as "scaffolding". The learner was inducted into the microworld through activities which are intended to establish what she/he already knew about the epistemological base of the microworld. There then followed a period in which the learner developed an understanding of non-euclidean geometry using the microworld which was "scaffolded" by activities and investigations. The culmination of the process was fluency with the microworld characterised by confidence and competence with its technical element. The role of the teacher varied as the learner negotiated the activities of the microworld, with the degree of formality and overt control changing according to the needs and achievements of the learner.

2.5 The Cognitive Element of the Microworld

This section will discuss an interpretation of the cognitive element based on the assumption that meaning construction is a social process built on an understanding of words as tools. §2.5.1 discusses this assumption in relation to Wittgenstein's notion of language-games. In §2.5.2, the idea of language-games is interpreted so as to provide a means of analysing a learner's cognitive response to their work with microworld.

2.5.1 Tools and Meaning Construction

Meanings which individuals construct are conditioned by the particularities of the social context in which they find themselves and are characterised by specific ways of speaking and acting[1]. Drawing on an ethnomethodological approach, Suchmann (1987) describes this type of behaviour more generally as "situated action". By this she means "the view that every course of action depends in essential ways upon its material and social circumstances." (p.50). She describes the inability of humans to "stand back" completely from their circumstances to deliberate, form plans and act. Rather they are locked into a set of circumstances which determine what is possible. Lave (1988) has argued that mathematical understanding is closely related to the social and material circumstances in which it is used. Noss and Hoyles (1992) describe as "situated abstractions" those images, "rules of thumb" and partial generalisations by "which people make mathematical sense of everyday activities". They point out the need to examine the constitutive function of "context" in teaching and learning mathematics.[2] Ackermann (1991) describes the way in which we are embedded in social activity, but must also emerge from it to make sense of it. Echoing §2.1.2, she sees this "sense-making" process as linked to modelling and dialogue .

> How could people learn from their experience as long as they are totally immersed in it? There comes a time when one needs to translate the experience into a description or model. Once built, the model gains a life of its own and can be addressed as if it were "not me". From then on, a new cycle can begin, because the dialogue gets started (between me and my artefact), the stage is set for new and deeper connectedness and understanding." (p.274)

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[1]This position may be contrasted with the view that meaning construction is a process which takes place as the result of individuals trying to make sense of their own experiences. Noddings (1990) characterises this Constructivist position as a combination of epistemology, psychology and pedagogy which builds on the notion that all knowledge is constructed by the knowing mind. Such constructions takes place as a consequence of the knower continually adapting, cognitively, to new situations rather as an organism adapts to its environment to achieve equilibrium with it. The social dimension of experience is regarded as extrinsic and secondary to this process of cognitive re-organisation, providing a background to the process, but about which the individual must remain agnostic. Proponents of the constructivist position have tried to reconcile its Piagetian roots with Vygostkian notions of the social dimension of cognition (Steffe and Tsur 1994). However, the results seem to highlight the incommensurability of the two views, rather than create a new synthesis (Lerman 1994 , Ernest 1994).

[2]For a more recent exposition of this notion see Hoyles and Noss (1995).

Wittgenstein (1953) draws on the tool metaphor to describe the relationship between human activity and linguistic meaning.

> "Think of tools in a toolbox; there is a hammer, pliers, a saw, a screwdriver, a rule, a glue-pot, nails and screws.-The functions of words are as diverse as the functions of these objects."
> (1953 11. p.6)

Tools exist to do jobs and they can be used in a variety of ways according to the type of job required. Although not every tool can be used for every job, every job may require tools to be used several times and in particular sequences. Similarly, words perform diverse functions and the sense that they convey can be found only by paying attention to how the words are being used. He developed two technical concepts to explore the implications of the insight that meaning was defined by the uses made of language: the "language-game" and "grammar". The first describes the connection between language and human activity, and the second delineates the rules by which specific words perform their functions. The central focus of the language-game concept is that word usage is tied to particular ways of acting. He draws on the tool metaphor to explain that the meaning of words is not be found by considering some underlying logical structure for language which can be uncovered by analysis, but instead can be discovered in the actual practices of human life.

> "The word 'language-game' is here meant to bring into prominence the fact that the speaking of language is part of an activity, or a form of life." (1953 23 p.11)

Language has its roots in human interaction and these characteristic "forms of life" provide the reference point for exploring meaning. Characteristically, Wittgenstein employs the notion in a variety of ways and Specht (1969 p.42ff) identifies three main uses. "Language-game" can refer to a primitive or simplified form of language which might be used in learning. It also represents a partial system or function of language which may include specific linguistic acts, combinations structured for some purpose, or complete partial systems. Finally it is used by Wittgenstein to refer to the totality of language and its uses.

What each of these has in common is that language and games are both rule-governed activities. Rules may be arbitrary and conventional, but they are definite and regulate social activity. There are similarities here with Heidegger's account of how tools disclose the world[1]. Using the notion of word-as-tool simultaneously provides a means of understanding how words function in particular ways and discloses the background of human activity in the words that are being used. To participate in a language-game means to participate in a "form of life" which gives the various language acts their sense. Here the dialectic of "tool" and "World disclosure" is given a structure, so that its specific moments can be identified. The notion of language-game provides a way of moving from the word-as-tool in the "foreground" to the human forms of life in the "background".

The reverse process of how the language-games as forms of life determine specific meanings can be found in the Wittgensteinian notion of "grammar". As he put it, "Grammar tells us what kind of object anything is" (1953. 373 p.116). The rules of word usage are not derived from the structure of things, rather the rules *constitute* the objects in and through language since the world comes segmented and organised via particular language games. Linguistic signs appear at the same time as the objects which they signify within the language-game. Rules express the unity of sign and object and how the sign may or may not be used. Hence the rules provide the essence of an object, not as descriptions, but as protocols for actions which define the object as object. There are major similarities with Walkerdine (1989) who describes these meaningful unities of word and action as *discursive practices*, in which "the object world cannot be known outside the relations of signification in which the objects are inscribed" (p.119). We are inducted into these discourses though learning a combination of activity and language, which positions both the objects of the discourse *as* objects and us *as* "subjects".

[1] For the similarities between Heidegger and Wittgenstein see Mandel (1978)

One difference between the language-game notion and that of discursive practice is the issue of the extent to which subjectivity is determined by the social practices in which individuals participate. Walkerdine's deconstructive programme dissolves the subject in discursive practice, arguing for the removal of the notion of the unitary self and replacing it with the idea of the self as the site of multiple practices. Wittgenstein, on the other hand, in his discussion of private language, holds out the possibility that ontological questions are meaningful, although not in Cartesian or Behaviourist terms. There is room for the construction of new language-games which develop from social interaction. (1953 PI 23.) Winograd and Flores (1986), drawing on Heidegger's notion of disclosure through "breakdowns" in understanding, identify those areas which give rise to new language-games in which discontinuity in cognition serving a significant role in learning. Breakdowns are for them,

> " the interrupted moments of our habitual, standard, comfortable 'being-in-the-world'. Breakdowns serve an extremely important cognitive function, revealing to us the nature of our practices and equipment, making them 'present-at-hand' to us, perhaps for the first time." (*ibid.* p.77 -78)

Breakdowns create spaces in which new interpretations can be constructed, both about things and about ourselves. This implies that not only genuine new interpretations of the problematic situation emerge from breakdown, but also a new sense of self is created in the process.

In this context, "breakdown" represented the possibility of "breaking through" to a new domain. It became the condition of the possibility of transition from the "old" to the "new", since it makes the person explicitly aware of their presuppositions by calling them into question. Further, it produces "the learner" in the space created by the breakdown in their intuition. The person did not try to make sense of the microworld, but , one would like to say, the microworld made sense of the person.

2.5.2 Exploring a Language-game for Curvature

Two important practical issues for the cognitive element of the microworld followed from these considerations. The first concerned the use of the language-game notion in developing a method for analysing the meanings of geometric terms associated with curvature. The second related to the notion of breakdown in cognition as the condition of the possibility for developing an understanding of new knowledge domains. In particular, how such breakdowns might be used to introduce the Conformal models for non-euclidean geometry.

Having outlined language-games and the related idea of grammar in §2.5.1, the next step was to connect them to the curvature, which was the domain of the microworld. Language-games were concerned with the way in which the meaning of linguistic terms were embedded in social relationships and the rules (grammar), which govern usage, had a constitutive function in relation to the objects of linguistic practice. The intention was to apply these ideas concerning the construction and maintenance of meaning within specific discourses to obtain a structure which could be used in two ways. The first was to analyse the meaning of terms used to describe curved surfaces and their respective geometries as a way of developing a pedagogic approach for the microworld. The second use of such a structure was as a way of analysing the meanings created by those using the microworld.

A suitable structure was found to consist in the combination of linguistic sign and actions on surfaces. This followed from observing that language about curved objects was connected with everyday experiences of seeing and touching surfaces, as well as the more formalised speech and activity associated with geometry. This combination of linguistic sign in relation to specific action on a surface was useful in identifying ways of speaking and acting with curved surfaces. Pedagogically, the sign-action-surface combination was useful for *identifying* the meaning of terms in relation to euclidean and non-euclidean geometry as a precondition for teaching these meanings to the microworld's participants. It served to de-lineate the various language-games associated with spatial and geometry intuition. Cognitively, the sign-action-surface description was used to analyse how those engaged with the microworld developed their understanding of the linguistic terms in the context of the microworld.

The second issue to be considered was the role that "breakdown" would play in the development of an understanding of the Conformal models. If it was accepted that such breakdowns play a key role, as §2.5.1 indicates, then it followed that they should be a part of the learner's initial experience with the microworld. The pedagogical strategy which was suggested by this notion was that learners should be *deliberately* confronted with a situation which challenged their usual understanding of geometry. In the case of the microworld, this implied that the learner's euclidean intuitions should be called into question through contact with the microworld, creating the space for new geometric intuitions to be developed. Using the sign-action-surface structure, the pedagogical element of the microworld could then used to "re-structure" the learners' intuitions, gradually leading them to fluency with the language-game of the computer-based Conformal models.

2.6 Modelling the Microworld

The purpose of this section is to draw together the reflections of §2.3 - §2.5 by developing a suitable model for the microworld. §2.6.1 will outline the rationale for a model of the microworld in light of the study's aims. The subsequent sections, §2.6.2 and §2.6.3, will use the structures developed in the previous sections for each of the microworld's elements to elaborate a model.

2.6.1 Rationale for the Model

As §2.1.1 indicates, models can be used for a variety of purposes. This section will describe the reasons for constructing a model of the microworld, before going on to outline its structure. The central aim of the study was to explore the feasibility of creating a suitable computer-based context in which individuals could learn non-euclidean geometry. Building a model of the microworld served three purposes in relation to the study. First, it provided a framework for clarifying thinking about the design and development of the microworld by generating an entity "to think with". Second, from a practical point of view, a model enabled the organisation of activities by providing a structure which guided the setting up and analysis of individual's work with the microworld. An intention of the study was to chart the development of the microworld's construction. From this point of view, a detailed model of the microworld enabled the description and analysis of this process by providing a framework in which to compare the elements and describe changes to their structures. Related to this was the final reason for having a model, which was to provide a window on the processes of teaching and learning with the microworld.

Two criteria for the model were identified from the aims. First, a model of the microworld must provide the means to describe the microworld's development, as a whole, *over time*. This was related to the need for a global window on the process of designing and developing the microworld. Second, a model must provide a local window on the way in which the individuals using the microworld developed their understanding of non-euclidean geometry. The first aspect referred to the temporal development of the microworld and was described as *diachronic*[1]. The second aspect, providing local windows on cognitive development and pedagogic process as required by the aims, was described as *synchronic*. These synchronic windows gave a "snapshot" of what was happening in the cognitive and pedagogic elements at any given time, using the sign-action-surface structure developed in §2.5.2.

Having outlined the criteria which the model had to fulfil, the next step was to develop an outline of its structure. A central purpose of the microworld was to provide a context in which participants in the microworld could use a computational version of the Conformal models to investigate non-euclidean geometry. This suggested that a suitable starting point to elaborate this model of the microworld was to consider the changes in geometric domains necessary to enable the learner to use the conformal

[1]For an elaboration of the distinction between diachronic and synchronic axis, see Walkerdine 1989 p.184

models. This was the "end point", so to speak, of the microworld. The starting point for the user of the microworld, however, was something entirely different. Those using the microworld would have the usual understanding of geometry based on Euclid and rooted in their everyday intuitions of space. In order to connect the microworld users' euclidean intuitions of geometry with the Conformal models, two other stages had to be passed through. The first was the introduction of the learner to non-euclidean geometries and, second, to their flat projection which produced the Conformal models, implemented in the computer software. The overall structure of the microworld was, therefore, one which started with the introduction of the learner to surfaces of non-zero curvature with their non-euclidean geometries, proceeded to the projection of these surfaces, and culminated in the fluent use of the microworld's computer-based Conformal models by its participants. These three distinct phases provided the basic structure for the development of the microworld.

2.6.2 The Diachronic View of the Microworld

The diachronic view of the microworld was concerned with its temporal development, based on the microworld's definition in terms of technical, pedagogical, and cognitive elements. As §2.6.1 indicates, the structure of the microworld's technical element consisted of a movement from objects with non-zero curvature, such as spheres, to projections of spheres on paper, finishing with computer-based tools for investigating the Conformal models. At a pedagogic level, the structure described in §2.4.2 implied a period of induction in which the microworld's participants had their euclidean intuitions challenged as they were introduced to non-euclidean geometries. A process of instruction followed in which the microworld scaffolded the participant's understanding in a variety of ways, leading to a "fading", in which the participants worked entirely with the computer images. Cognitively, these corresponded to three phases. First, a breakdown in understanding. Second, a period of re-structuring, in which the participants learned about non-euclidean geometries and their representations. Third, "fluency" in the new language-game of computer-based Conformal models of surfaces with non-zero curvature. However, this did not constitute a "model" of the learner, since part of the aim of the study was to chart what happened to the participants as they used the microworld. Rather, the three-phase structure was regarded as a set of working assumptions, implied partly by the pedagogic structure and partly by the considerations in §2.5.

Figure 2.1 shows the full diachronic view consisting of the structures outlined above.

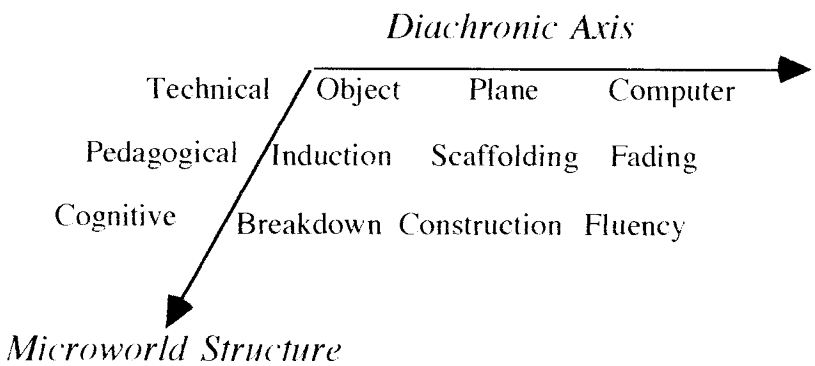

Figure 2.1 The Diachronic View of the Microworld. This describes the temporal development of the microworld's elements, starting with the use of physical objects, moving to plane projects and ending with computer models. The pedagogic and cognitive structures corresponding to these phases are also shown.

The horizontal axis, in Figure 2.1, is the diachronic axis, indicating how the microworld develops over time. The other axis represents the microworld's technical, pedagogical and cognitive elements, and the detail of Figure 2.1 shows the structure of each of the microworld's elements .

The diachronic view had two uses. When it was used to show the development of the microworld as a pedagogic unit, in the sense of having a starting and a finishing point, it was referred to as the *structure* of the microworld over the time that it was used. The second use of the diachronic view was in charting the changes in the microworld as it moved through various stages of development. Used in this way, the diachronic view enabled comparisons to be made between stages of its development so that changes to the structure of the microworld's elements could be documented and analysed.

2.6.3 The Synchronic View

The synchronic view was used to obtain "snapshots" of what was happening to the various elements of the microworld at any given time and hence reflected their static structure at given moments. The view was obtained by using the sign-action-surface structure to analyse either the pedagogical intentions at a given stage or the cognitive responses of those using the microworld.

The link between the elements of the microworld (technical, pedagogic, cognitive) and the sign-action-surface structure was obtained by identifying the "surface" component of the sign-action-surface model with the technical element of the microworld. Figure 2.2 illustrates this.

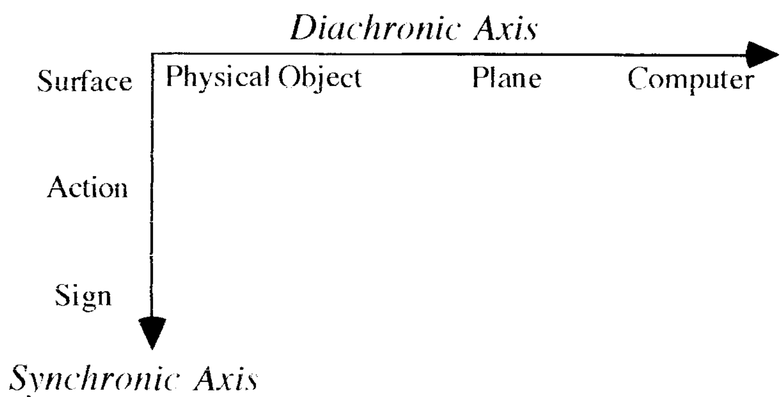

Figure 2.2 Synchronic Structure of the Microworld. The diachronic axis indicates the stages of progression from physical objects with non-zero curvature, through projections of the surface onto the euclidean plane to computer-based Conformal models. The synchronic axis shows the structure carried through the geometric domains.

The synchronic view was used in two ways. First, it was used to explore the meanings of geometric terms needed in the pedagogical element of the microworld. This was preliminary to establishing what should be taught via the microworld, and how. Second, it served as a tool for analysing the participant's appropriation of the microworld by focusing on how they acted on its technical element and what they said about it.

2.6.4 A Model for the Microworld

The diachronic and synchronic view can be combined into a three-dimensional model that displays the interaction between the two views. Figure 2.3 shows how the identification of the technical element of the diachronic view with the surface element of the synchronic axis enabled the sign-action-surface structure to be applied to both the pedagogical and cognitive elements of the microworld. The structure was used in analysing the meanings which are created by the pedagogical element of the microworld and also in exploring the meanings created as the learners engaged with the activities of the microworld.

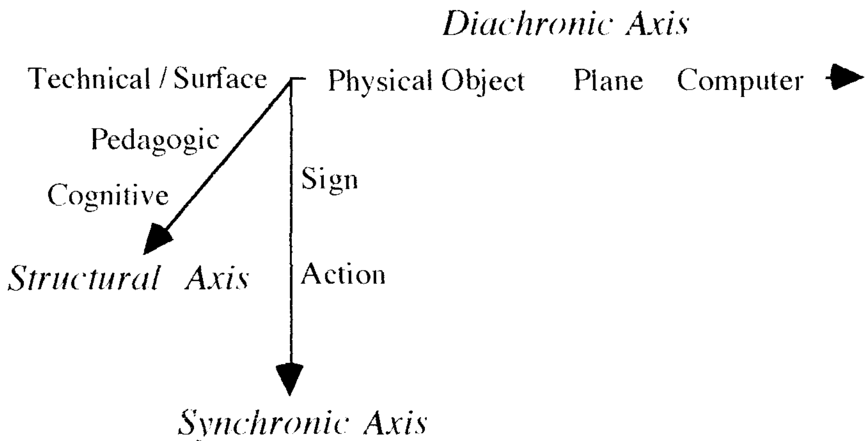

Figure 2.3 The Full Model of the Microworld.

This "three-dimensional" model enabled the construction of both the local and global windows required by the aims. The global window was given by the changes to the model and its structure over the developmental cycles. The local windows on the cognitive development of the participants and the pedagogical intentions of each phase were given by the synchronic views on each of those elements in the microworld.

2.7 Conclusion

In this chapter, a model has been developed with the aim of providing a context for teaching and learning non-euclidean geometry. The model used the microworld as paradigm for computer-based environments. The internal structure of the model, defined in terms of its technical, pedagogical and cognitive elements, was viewed in two ways. The diachronic view, divided into three phases, showed the structure of the microworld's elements as a complete unit and provided the means to describe how the elements themselves changed over the period of the microworld's development. The synchronic view was used to describe the meaning of the microworld's pedagogical and cognitive elements during each of the microworld's phases.

The next two chapters will examine the structure of the microworld's technical, pedagogic and technical elements in order to elaborate the structure provided by the model. This will involve considering, in detail, those factors which were relevant to the specific knowledge domain of the microworld. In Chapter 3, the implementation of the Conformal models using Turtle Geometry will be given, together with an account of the computational tools provided for the learner. Chapter 4 will examine in detail the perceptual and cognitive aspects of curvature together with their relationship to geometry and Logo. This leads to the elaboration of a language-game associated with non-euclidean geometry using the sign-action-surface structure of the synchronic view. This will inform the construction of a pedagogical strategy for the microworld. The cognitive implications of the synchronic view will also be reviewed

Chapter 3

The Design of the Microworld (1): The Technical Element

3.0 Introduction

The aim of this chapter is to provide details on the background, rationale and implementation of the computer-based components of the microworld's technical element. §3.1 describes the background to the design of the computer-based component of the microworld through a review of microworlds connected with curvature and non-euclidean geometry. §3.2 provides a rationale for the use of Object Logo as the medium in which to implement the Conformal models. The mathematics associated with implementing the Conformal models using Turtle Geometry are given in §3.3. and §3.4.

3.1 Background to the Design: Other Computer-Based Microworlds

This section describes other computer-based microworlds, related to the themes of non-euclidean geometry, curvature and Relativity, that were considered as a preliminary to designing the computational component of this microworld. Starting with the General Relativity simulator in Abelson and diSessa (1980), the section covers the ReLlab microworld designed for Special Relativity, the *Gravitas* microworld for Newtonian gravity, and a Logo microworld for non-euclidean geometry based on the Conformal models.

3.1.1 General Relativity Simulator: Abelson and diSessa

Logo representations of curved surfaces are among the many topics covered by *Turtle Geometry* (Abelson and diSessa 1980). Several chapters are devoted to curvature and related issues, such as Turtle representations of spheres (Chapters 5 and 7), piecewise flat surfaces (Chapter 8), and curved spacetime and General Relativity (Chapter 9). Several lines of enquiry were followed in the initial stages of the study using ideas from the book. Particularly interesting was the "wedge" representation of curved surfaces used by Abelson and diSessa to explore the idea of curved spacetime.

Here symmetric curved surfaces are divided into identical wedges and laid on a flat plane. The Turtle moves across these wedges in a straight line until it comes to a gap. A "demon" then moves the Turtle across the gap, making the necessary correction to the Turtle's motion that represents the difference between walking on flat and curved surfaces. Abelson and diSessa model a curved world, therefore, as "a flat one plus a very dependable 'force' " (p.347), which for them is the essence of General Relativity. Computer representations of curved surfaces can be constructed to reproduce this "flat walk + demon jump" and they go on to give details of a General Relativity simulator built on the idea.

Building the "GR Simulator" was interesting and absorbing since it provided insights into the conceptual framework of General Relativity and how it might be expressed without the use of abstract differential geometry. However, appreciating the value of the GR simulator came as a result of having some knowledge of the mathematics and this was the main difficulty with the simulator from the study's point of view.

The principle aim of Abelson and diSessa, and of this study, was to introduce novices to General Relativity through the use of the "demon jump" as a metaphor for the effects of curvature. Like all metaphors, this had both strengths and weaknesses. Part of the difficulty which many experience with General Relativity is its abstract and counter-intuitive nature. Some kind of metaphor was needed to enable the naive user of the simulator to grasp that there was a difference between their normal experience and that of spacetime. The power of the metaphor was in its explanation of curvature as a "dynamic deficiency" when compared to the euclidean case. The demon both moved *and* corrected the Turtle's motion. The weakness of the metaphor, from a perceptual point of view, was that it did not appear to directly connect with an individual's intuitions of curved objects and, consequently, might have seemed contrived to the learner. It became apparent to the author of this study while the simulator was being developed that, precisely because the ideas in GR were so abstract and counter-intuitive, some kind of progression from known and familiar objects to the unknown and unfamiliar had to form the basis of the approach to building the microworld. There needed to be something other than just the computational aspect for individuals to work with so that their own spatial intuitions could be developed to take account of curved surfaces. This progression did not seem likely using the "demon jump" metaphor, since it was unclear how it could be made to relate to an individual's understanding in a natural way.

A second issue concerned the ways in which the simulator might have been introduced in practice. The simulator required extensive experience of Logo programming to implement it, as well as a willingness to engage in modelling counter-intuitive entities such as curved spacetime. The approach, adopted by Abelson and diSessa, was an invitation to active exploration through programming. An obvious methodology for the study would have been to chart the progress of individuals as they constructed the GR simulator for themselves. However, this pre-supposed knowledge of Logo programming and individuals would have to spend time *learning* Logo before they could begin learning *with* Logo.

The experience of building the GR simulator was valuable and interesting. The difficulty for this study's author was how to turn personal enthusiasm for the simulator into a viable set of activities which gave access to other learners' understanding of curvature. In particular, the need for extensive Logo experience and an apparent need to *teach* users the conceptual base were thought to be drawbacks. Certainly the requirement to be able to programme in Logo before dealing with the content of the simulator was thought to shift unnecessarily the focus of attention away from the aims of the study. It was decided, therefore, to look for another approach.

3.1.2 RelLab

RelLab is "A Computer Tool for Experimenting with Relativity". Developed at BBN by Paul Horowitz and Wally Feurzig, this microworld is intended to provide a "computer-based tool for creating thought experiments involving both Galilean (low-speed) and special (high-speed) relativity" (Horowitz and Feurzig 1992 p.1). It provides a variety of screen objects, such as cars, rockets, and plants with which to create "events" that can be viewed from different reference frames.

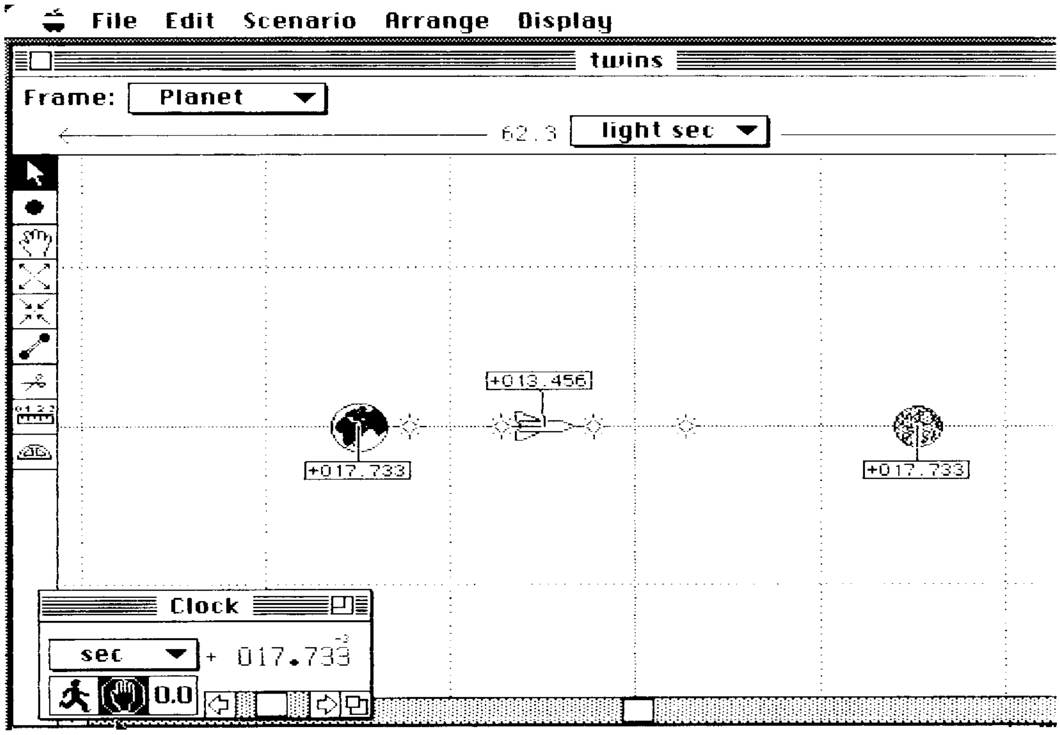

Figure 3.1 RelLab Screen Shot of the "Twins Paradox" in Relativity. A description is given below.

The screen shot in Figure 3.1 illustrates the type of situation that can be set up using RelLab. It shows the "Twins Paradox" in Relativity. A rocket is accelerated away from a planet which sends light pulses to synchronise the clocks. Times are shown for the planet and the rocket. As the rocket completes a round trip, its clock runs slower than the synchronised times on the planets. The situation can be viewed from either of the two planets or the rockets.

RelLab software is accompanied by example files, curriculum support materials and a reference manual. They are part of a programme to develop software tools to support advanced ideas in physics.

This microworld was interesting because it provided a computer-based medium with which to explore ideas of Special Relativity. Its user-interface was particularly interesting because it provided easy access to those aspects of the software that were needed regularly, and placed less-frequently used elements on menus. This approach was thought to be valuable in the design of the interface for this microworld. By contrast with the previous microworld, the visual nature of the software and the ease of access to it provided a more effective means of connecting intuition with the physics. By concentrating on the phenomenology of Special Relativity, RelLab provided an experiential dimension which was lacking in the GR simulator.

Although it was decided not to use RelLab because it was designed specifically for Special Relativity, the microworld gave a valuable insight into the qualities of effective software. RelLab was dynamic, interactive, visual and easily assessible, with "tools" that were derived from its epistemological base. Individuals could "play" with pre-defined situations or set up their own situation and investigate it. What characterised this microworld was a sense of immediacy and contact with a medium which could facilitate thinking about Special Relativity. It was a good example of a "direct manipulation" interface. (Hutchins, Hollan, Norman 1986).

3.1.3. *Gravitas* : A Microworld for Newtonian Gravity

Gravitas is a "Discovery Learning Environment" (Sellman 1994) for exploring the effect of gravity on systems of Newtonian objects. Written in Object Logo and consisting of a basic object called *MassOb* , it provides the user with both a graphical and a linguistic interface to set-up gravitating systems and observe how the systems evolve over time. Each *MassOb* is gravitationally affected by any other *MassOb* defined in the system, so that the development over time of complex arrangements of planets, rockets and orbiting satellites can be set up and observed.

Sellman takes an uncompromising constructivist view in designing the software, mirroring Logo's syntonic commands with four commands: `boost`, `boost.back`, `boost.left`, `boost.right`. Drawing on arguments by Papert (1980), he proposes that *MassOb* is a kind of "transitional object, partner to the Turtle" which enables the gap to be bridged between formal and informal understanding of gravitating systems. He also compares the graphical and linguistic interfaces with Turtle Geometry, pointing out that the "natural sequence" is from graphical to linguistic representations. Although the two interfaces are functionally identical in *Gravitas*, he notes differences in approach to each by the user.

The software was used with two school students (15-18) and five adults. The sessions with the school students, who had relatively strong backgrounds in school science, were video-taped. Sellman reports a number of episodes, including writing programmes in *Gravitas*, extensions to *Gravitas* and "surprises". In the latter category he compares the sort of surprises found by Abelson and diSessa (1980), using simple recursive procedures in Logo with similar types of procedures in *Gravitas*.

Sellman uses *Gravitas* to exemplify a class of computer-based learning environments which he calls "objectworlds".

> "A computer-based objectworld is the combination of a simulated object (or objects) and an interactive programming language. The object should be continuously visible and its attributes should derive from, or be a representation of, some fundamental concept. The language should contain a set of commands that allow the inspection and manipulation of the object's attributes, and must support data types corresponding to those attributes. At least some of these commands should act on the objects in ways that we would expect learners to grasp with little difficulty."
>
> (Sellman 1994 p.48)

He identifies Turtle Geometry, Dynaturtle (Papert 1980) and Boxer (diSessa and Abelson 1986) as three such worlds, since they all contain the combination of object and programming interface, based on some "fundamental" concept.

This microworld was also interesting because of its use of Object Logo as the medium both for programming the software and for exploration of the microworld by the learner. Object Logo is ideally suited to the sort of system which involves the interaction of entities such as *Gravitas*. Another interesting aspect was Sellman's use of the software with adults and school students. He reports several episodes and analysed them, pointing to the differences in the way in which they used the graphical and linguistic interfaces. His methodology was similar to that adopted in this study.

However, *Gravitas* was not used for the microworld because it was concerned with classical Newtonian gravitation rather than General Relativity and non-euclidean geometry.

3.1.4 Computer-based Versions of the Conformal Models

Two examples of computer-based versions of the Conformal models were examined. The first, by Hemmings (1985), consisted of a booklet about hyperbolic geometry and software. The program allowed the user to produce tessellations of the type shown in Figure 6.7, together with a "zoom-in" and "zoom-out" facility. The images are static representations, which can be constructed to the user's specifications.

A second piece of software which uses the Conformal models is *Non-Euclidean Logo* by Sims-Coomber (1993). The aim of her work was to produce a non-euclidean Logo interpreter as a research project in computer graphics. It was implemented in the high-level programming language C on a Sun Workstation and was developed to include a subset of Turtle Graphics commands which described the Turtle's motion both in the Conformal models and on an arbitrary parameterised surface.

The implementation of the two-dimensional Conformal models was based on the use of circles and arcs of circles to produce the Turtle tracks, described in §1.2. Two sets of non-standard Logo commands were introduced. The first, called SETGEOM and GEOM, referred to the choice of geometry. These selected either of the elliptic or hyperbolic models. Related to this were the commands SETCOORDS and COORDS which enabled a variety of coordinate systems to be introduced (ibid. p.50). The second type of new command related to features of the non-euclidean geometry. Two sorts of lines; *h-line* and *e-line*, one for each model, were implemented to show the large-scale behaviour of the Turtle in each model. These, coupled with FD INFINITY and BK INFINITY in the case of hyperbolic geometry, enabled the user to explore the different sorts of "straight" lines in each geometry.

In the second part of her study, Sims-Coomber (1993) implemented a version of Logo to operate on arbitrary surfaces specified by intrinsic coordinates. SURFACE LOGO, as she called it, made extensive use of differential geometry. Again, a subset of Turtle Geometry commands relevant to the topic were implemented, introducing new keywords such as SURFACE. This enabled a coordinate definition of the surface to be entered which the Turtle would then "walk over". Extensive use was made of computer graphics routines to display the resulting surface and Turtle path. However, the software has not been used by naive users and there are no plans to do so (Martin 1994)[1].

Martin and Sims-Coomber (1991) was important for the development of the microworld since it led to considering a computer version of the Conformal models as the basis of the technical element. However, the similarity between the implementation of this microworld and that of Sims-Coomber was purely phenomenological. Although both studies implemented the same models, they did so in different ways, using different mathematics and computer languages and for different purposes, as the following sections will show.

3.1.5 Reflections on the Microworlds Reviewed

Although the microworlds described above were not used for the study, they each contributed to the formation of the design for the computational component of the microworld's technical element. Abelson and diSessa's GR simulator pointed to the importance of having some computational metaphor which could connect the learner's intuitions and knowledge with the epistemological base of the microworld. RELlab suggested that effective software should have a "user-interface" that provided easy assess to screen objects and computational tools. *Gravitas* indicated the possibilities of Object Logo as a medium for implementing the software and the *Non-Euclidean Logo* of Sims-Coomber showed a practical implementation of the Conformal models.

---

[1] Private Communication.

3.2 Rationale for the Microworld's Computational Element

Non-Euclidean Logo, described in the previous section, was particularly important for the development of the technical element of the microworld, since it showed that it was possible to make a Turtle geometric version of the Conformal models. There were two differences between *Non-Euclidean Logo* and that version created in this study. First, the high-level language used to create the software was different, as indeed was the power of the machine needed to use it. *Non-Euclidean Logo* was created using C on a Sun workstation, while the microworld of this study used a variant of Logo on an Apple computer. Second, the mathematics used to implement the Conformal models was differential geometry, while *Non-Euclidean Logo* used the circular geometry of the Conformal models and a variety of coordinate systems. This section will describe the rationale for using Object Logo, an object-oriented version of Logo as the programming medium. It will also describe the structure of the software developed for the microworld. The mathematical base of the microworld will be described in §3.3.

3.2.1 What is Object Logo?

Object Logo is a dialect of Logo designed to give an object-oriented programming environment which enables *entities* to be created with their own states and behaviour (Dreshler 1987). Each entity forms a *class* which can interact with other classes by passing messages between instances of the classes. Object Logo is particularly useful for building systems of related, interacting objects, since the objects can be given their own variables and functions interactively and incrementally.

An important aspect of the object-oriented approach is the notion of *inheritance*. Classes can be created which share some or all of the characteristics of other already existing objects. This may take the form of a subclass or "child" of a given type which shares all the properties of the "parent" class but modified in some way. Alternatively the inheritance may be a "hybrid" in the sense of an object sharing some or all of the characteristics of a number of different entities. Inheritance enables objects to be created with the same procedure names but which do different things. In Object Logo there are a number of objects which come already defined: *Turtle and Turtle Window* for Turtle geometry, *Listener* for entering commands, *Window* for creating text windows and *Menu* for making menus. There are several others but these are the ones relevant to the study. More details can be found in the Object Logo manual (Paradigm 1990).

3.2.2 Rationale and Structure of the Software for the Microworld

The object-oriented approach was useful in the context of the study for three reasons. First, the paradigm offered the possibility of creating Turtles which "lived" in their own geometric world and responded in their own way to "ordinary" Logo commands. For example, a Turtle could be created so that it moved either according to the Conformal or euclidean models for geometry when FORWARD was typed. This capacity to "shadow" commands such as FORWARD was significant. Second, the subset of commands to be shadowed were relatively small, since the non-euclidean models were conformal. Hence, FORWARD and BACK, both of which effected only distance, were shadowed. RIGHT and LEFT, on the other hand, retained their usual meaning with the usual euclidean measures for angles. Third, users could type Logo commands in the usual way but obtain screen behaviour which is based on the models described. This suggested a means to "breakdown" the euclidean intuitions of the individuals and, perhaps, give access to changes in their understanding.

As a result of these considerations, it was decided that the program needed three main elements: a screen with its Turtle, a visible means of selecting Turtles, and a means of entering Logo commands. Each will be described.

3.2.3 Turtle Object

Object Logo has a generic object called Turtle, which behaves in the usual way for Turtle geometry. The microworld created a variation of this which inherited all the usual properties (e.g. POS HEADING), but shadowed the commands FORWARD and BACK. The Turtle also had its own set of variables which enabled the selection of different kinds of geometric behaviour. This was achieved through the mechanism of inheritance already described. One Turtle was created which could provide three types of geometry: elliptic, hyperbolic and euclidean. However, the Turtle's behaviour would not be identified to the user as hyperbolic, elliptic or euclidean. Rather, the aim was for the user to work out the properties of each Turtle from its screen behaviour.

3.2.4 Selecting Turtles

It was decided to make the means of selecting Turtles a "button pad" which involves "pointing and clicking" the mouse. Such a pad would remain on the screen at all times to encourage the user to try one of the three different Turtles.

3.2.5 Listener

This was another generic object which enabled text to be entered and printed output to be read. The system could not operate without it! It also had the advantage that it could be saved to a file and so act as a "dribble" file, which is useful during data collection.

3.2.6 User Interface

Putting these considerations together, it was decided that the user interface would consist of three parts:

- "Surface" : a window in which the Turtle draws.
- "Turtles" : a button-pad which is used to select the Turtles
- "Listener" : a window in which text commands are entered and printed

3.3 Finding theTurtle's Path Using Circles

Having outlined, in general terms, the structure of the software, the next stage is to describe how the Turtle was to be moved on screen. Two approaches were tried and this section is a description of the method developed, based on the circular geometry of the Conformal models. Both Conformal models are based on arcs of circles. The basic problem in moving the Turtle was to find the equation of the circle or euclidean straight line on which it lay, given its position and heading. This equation could then be used to plot the Turtle's path in the model. In this section, the equations will be obtained for both models, assuming that the Turtle had screen position (p, q) and heading $m = \tan \alpha$, where α is the heading of the Turtle relative to the positive x-axis[1].

The equation of the unit circle is $x^2 + y^2 = 1$ and the general equation of a circle with centre (a, b) and radius r is $(x - a)^2 + (y - b)^2 = r^2$...................(1)

Turtle lies on (1) at point (p, q) with $\left.\dfrac{dy}{dx}\right|_{x=p} = m.$

Hence from (1), $(p - a)^2 + (q - b)^2 = r^2$. ....2)

Differentiating (1) gives $\dfrac{-(p-a)}{(q-b)} = m$................(3)

3.3.1 Conformal Model A

In Conformal model A, the straight lines consist of either straight euclidean lines from the centre, or the arcs of circles which intersect the unit circle at opposite ends of the same diameter. Figure 3.2 illustrates the situation.

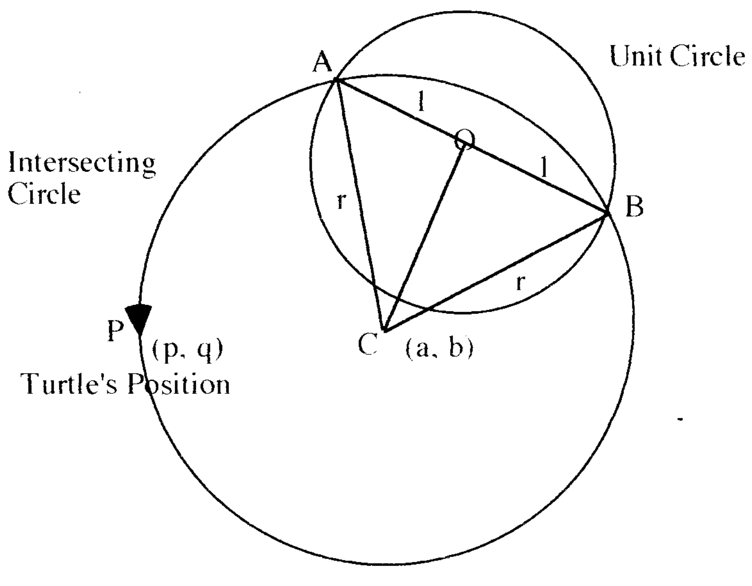

Figure 3.2 Mathematics for Conformal Model A. The intersecting circle has centre C with coordinate (a, b) and radius r. By Pythagoras, $r^2 = a^2 + b^2 + 1$

[1]For the purposes of the derivation, the heading will be measured anti-clockwise from the positive x-axis and modified to the Turtle's screen heading by simple calculation. The derivation of the equation also assumes that the Turtle is moving on the plane containing a unit disc and then the unit disc will be scaled to fit the screen

C at point (a,b) is the centre of the circle, radius r, which intersects the unit circle at A and B. These points lie on a diameter of the unit circle, centre O. This gives,

$$OC^2 = a^2 + b^2 \text{ and, in triangle ACO, } r^2 = a^2 + b^2 + 1.$$

The equation of the intersecting circle is therefore: $(x - a)^2 + (y - b)^2 = a^2 + b^2 + 1$.

Solving for (2) and (3) gives:
$$b = \frac{p^2 - q^2 + 2mpq - 1}{2p + 2mq}$$

$$a = m(q - b) + p$$

These give the coordinates of the centre of the circle on which the Turtle moves in terms of an initial position (p, q) and heading α.

3.3.2 Conformal Model B

For Conformal model B, straight lines are either euclidean straight lines from the centre, or arcs of circles which lie within and are orthogonal to the unit circle.

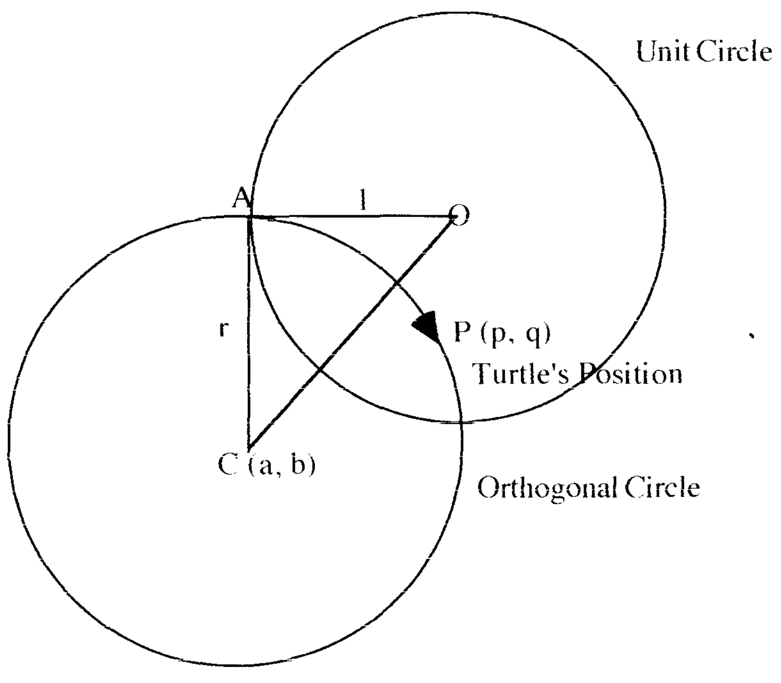

Figure 3.3 Mathematics of Conformal Model B. The centre of the circle orthogonal to the Unit circle is C, coordinate (a, b) and $r^2 = a^2 + b^2 - 1$

As Figure 3.3 shows, O is the centre of the unit circle and C is the centre of the orthogonal circle, with coordinate (a,b). A is the point of intersection between the circles, so that AC is perpendicular to OA.

In triangle OAC, $OC^2 = a^2 + b^2$ and $r^2 = a^2 + b^2 - 1$.

The equation of the orthogonal circle is $(x - a)^2 + (y - b)^2 = a^2 + b^2 - 1$

Solving for (2) and (3) gives: $b = \dfrac{p^2 - q^2 + 2mpq + 1}{2p + 2mq}$

$$a = m(q - b) + p$$

Again, given the Turtle's position (p, q) and m = tan α , these give the coordinates of the centre of the circle on which it will move.

3.3.3 Moving the Turtle

The next issue to be considered was how the Turtle could move on the circles whose equations have just been found. Two problems have to be resolved: orientation and step size for the Turtle. Orientation refers to whether the Turtle should move in a clockwise or anti-clockwise sense around a given circle, for a particular position and heading. Step size refers to the screen step around the circle corresponding to one Turtle step. These points can be illustrated by considering a procedure for drawing a circle.

```
to circle :step :turn
fd :step
rt :turn
circle :step :turn
end
```

The orientation is concerned with the sign of **:turn**, given a position and heading. Since the coordinates of the circles for the Turtle motion in both models are calculated using m = tan α related to the Turtle's heading, a heading of $\alpha°$ and $(180 - \alpha)°$ would give the same value of m. However, there is a clear difference between the subsequent motion of the Turtle with a heading of $\alpha°$ and that of $(180 - \alpha)°$! The two headings will produce opposite senses of motion around the circle. It is necessary to develop an algorithm for deciding whether the Turtle would move in a clockwise or anti-clockwise sense given a particular quadrant of the screen coordinates in each model.

How far the Turtle should move on the screen for each "Turtle step" also has to be considered. The curvature of the circle is given by the ratio of **:turn** to **:step** in the procedure above and it is constant for a given circle. Both models are position-sensitive in that the size of step taken on the screen varies according to the Turtle's position. In Conformal model A, if the Turtle is moving out from the centre of the unit disc, its steps are increasing in size on the screen and when it is moving towards the unit disc from outside, its step size is decreasing while, for the Conformal model B, the reverse happens. Values for both **:step** and **:turn** had to be calculated so that:

- **:step** had the correct value according to screen position.
- **:turn** was adjusted to maintain the correct curvature at each step.

Although this was investigated, the whole approach was considered to be unsatisfactory for two reasons. First, an algorithm had to be developed which decided on how the Turtle should move according to the quadrant that it was in and its heading. This involved analysing a large number of possibilities for both Conformal model A and B, and proved to be both complex and time-consuming. Second, the author of this study felt there was a certain lack of elegance in the approach! This led to another strategy being considered using differential geometry.

3.4 Finding theTurtle's Path Using Differential Geometry

The second approach to finding the equations for Turtle motion was based on differential geometry. This uses a model for geometry which consists of the pair (**M**, **g**), where **M** is a differential manifold and **g** is a function which maps vectors in **M** to the real numbers. As a differential manifold, **M** has two properties. The first is that it can be covered in "coordinate patches" which give a unique representation of each point in **M.** Second, the patches are created from functions that can be differentiated as many times as one wishes.

§3.4.1 introduces the idea of a vector as the tangent to a curve in a surface and the metric as providing the geometry for the vectors. §3.4.2 describes the relationship between the metric and generalised straight lines, called geodesics. §3.4.3 shows how the projection of curved surfaces induces a metric in R^2 which gives the geometry of the conformal models. This metric is used in §3.4.4 to find the equations of motion for the Turtle which are solved numerically in §3.4.5.

3.4.1 Vectors and Metric in Two-Dimensions

A *curve* in a two-dimensional surface is a map C: (a, b) —> S, where S is two-dimensional surface and (a, b) is an open interval of R^1. s is a *parameter* of the curve if a< s < b and C(s) is its image in S .

Let p be a point on the curve and let $p = C(s_0)$ in S.

A *vector* in S can be obtained by considering the tangent to the curve at p, which has unit length and is given by $\mathbf{t} = \left.\frac{dC}{ds}\right|_{s=s_0}$. Figure 3.4 illustrates this situation.

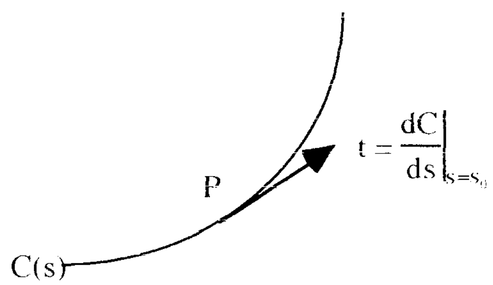

Figure 3.4. Vector at P in C(s). A vector is defined by the tangent to the curve C(s) at p

The set of all such tangent vectors to curves through p is the *tangent vector space* to S at p, denoted S_p.

A *metric* is a map **g**: S_p —> R which acts on pairs of vectors in S_p. If **u** and **v** are in S_p then the image of **g** acting **u** and **v** is a real number with **g(u,v)** = **g(v,u)**. For any three vectors **u, v, w** in S_p, **g(u , v + w)** = **g(u, v)** + **g(u, w)** and **g(u + w, v)** = **g(u, v)** + **g(w, v)**. Similarly, if λ is a scalar quantity, then **g(** λ**u, v)** = λ**g(u, v)**.

3.4.2 Geodesics and The Metric

"Geodesic" is the name given to the generalisation of the idea of a straight line. It is a curve with a tangent vector that does not change direction as one moves from point to point along it. It defines "straightness" relative to properties of the curve, rather than to how the curve is located in space. There is, therefore, a very close connection between the metric and geodesics in the sense that a curve is a geodesic relative to some metric. To show this connection in general for a two-dimensional surface, it is necessary to establish the equations which determine the geodesics of a surface and relate them to the metric

The first stage is to introduce a pair of basis vectors for S_p at some point p. This is done by considering two curves in S, C_1 and C_2, parameterised by arc length s_1 and s_2 which both pass through p and are shown in Figure 3.5

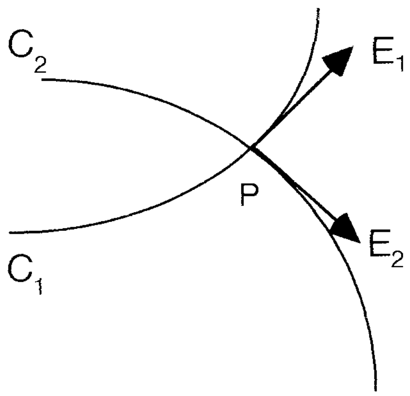

Figure 3.5 Basis Vectors in a surface S. Two curves C_1 and C_2 pass through a point p. Vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ are the tangents to the curves at p

The vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ are the tangents to the curves at a point P on C_1 and C_2 so that

$$\mathbf{E}_1 = \frac{dC_1}{ds_1} \text{ and } \mathbf{E}_2 = \frac{dC_2}{ds_2}.$$

Choosing $\mathbf{E}_1$ and $\mathbf{E}_2$ as orthogonal unit vectors and introducing coordinates x_1 and x_2 with $\mathbf{E}_1$ and $\mathbf{E}_2$, they can be written as :

$$\mathbf{E}_1 = \frac{\partial C_1}{\partial x_1}\frac{dx_1}{ds_1} = \frac{\partial C_1}{\partial x_1} \text{ and } \frac{dx_1}{ds_1} = 1. \text{ Similarly, } \mathbf{E}_2 = \frac{\partial C_2}{\partial x_2}$$

$\mathbf{E}_1$ and $\mathbf{E}_2$ form a *basis* at P. Any other vector **V**, tangent to a curve through P can be expressed in terms of them: $\mathbf{V} = v_1\,\mathbf{E}_1 + v_2\,\mathbf{E}_2$, where $v_1 = \mathbf{V} \bullet \mathbf{E}_1$, $v_2 = \mathbf{V} \bullet \mathbf{E}_2$ and $\bullet$ is a metric on S_p. This is shown in Figure 3.6

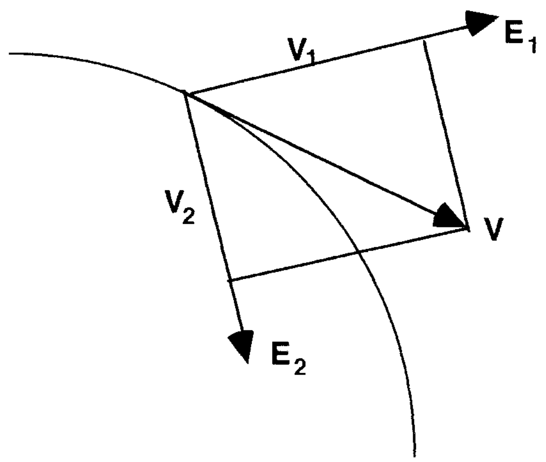

Figure 3.6. Components of a vector **V** using the basis vectors $\mathbf{E}_1$ and $\mathbf{E}_2$. The components of **V** are $v_1 = \mathbf{V} \bullet \mathbf{E}_1$ and $v_2 = \mathbf{V} \bullet \mathbf{E}_2$ so that $\mathbf{V} = v_1\,\mathbf{E}_1 + v_2\,\mathbf{E}_2$

The *geodesics* at P of **V** can be found by considering the derivative of **V**, relative to the arc length of the curve for which it is tangent, denoted by $\dot{\mathbf{V}}$. If $\dot{\mathbf{V}}$ has no component in the direction of **V**, then the curve to which **V** is tangent is a geodesic. This follows from the fact that $\dot{\mathbf{V}}$ measures the change in **V** along the curve and if $\dot{\mathbf{V}}$ has a component in the direction of **V**, then **V** must be changing in some way. Hence, for geodesics, $\dot{\mathbf{V}} \bullet \mathbf{V} = 0$. Putting this in kinematical terms, **V** represents the "speed" of the curve and $\dot{\mathbf{V}}$ represents the "acceleration". The condition that $\dot{\mathbf{V}} \bullet \mathbf{V} = 0$ implies that for straight lines, the acceleration of the curve is orthogonal to its speed at all points on the curve. This condition will be used to deduce the general equations for a geodesic.

Since $\mathbf{V} = v_1\,\mathbf{E}_1 + v_2\,\mathbf{E}_2$, its derivative, $\dot{\mathbf{V}}$, is $\dot{\mathbf{V}} = \dot{v}_1\,\mathbf{E}_1 + v_1\,\dot{\mathbf{E}}_1 + \dot{v}_2\,\mathbf{E}_2 + v_2\,\dot{\mathbf{E}}_2$.

The components of $\dot{\mathbf{V}}$, relative to the basis $\{\mathbf{E}_1, \mathbf{E}_2\}$, are given by:

$$\dot{\mathbf{V}} \bullet \mathbf{E}_1 = V_1 \text{ and } \dot{\mathbf{V}} \bullet \mathbf{E}_2 = V_2$$

Hence

$$V_1 = \dot{v}_1 (\mathbf{E}_1 \bullet \mathbf{E}_1) + v_1 (\dot{\mathbf{E}}_1 \bullet \mathbf{E}_1) + \dot{v}_2 (\mathbf{E}_2 \bullet \mathbf{E}_1) + v_2 (\dot{\mathbf{E}}_2 \bullet \mathbf{E}_1) \quad ..(1)$$

$$V_2 = \dot{v}_1 (\mathbf{E}_1 \bullet \mathbf{E}_2) + v_1 (\dot{\mathbf{E}}_1 \bullet \mathbf{E}_2) + \dot{v}_2 (\mathbf{E}_2 \bullet \mathbf{E}_2) + v_2 (\dot{\mathbf{E}}_2 \bullet \mathbf{E}_2) \quad ..(2)$$

Two sets of notation will be introduced. The first denotes the components of the metric by taking the dot product of the basis vectors: $(\mathbf{E}_i \bullet \mathbf{E}_j) = g_{ij}$ with $(\mathbf{E}_1 \bullet \mathbf{E}_2) = 0$ since the vectors are orthogonal. The second piece of notation represents the components of the derivative with respect to the coordinates of the basis vectors:

$$\Gamma_{ij,k} = \frac{\partial \mathbf{E}_i}{\partial x_j} \bullet \mathbf{E}_k$$

Using this notation, the components of the basis vector's derivative, from (1) and (2) above, can be written as :

$$(\dot{\mathbf{E}}_i \bullet \mathbf{E}_k) = \frac{\partial \mathbf{E}_i}{\partial x_1}\frac{dx_1}{ds} \bullet \mathbf{E}_k + \frac{\partial \mathbf{E}_i}{\partial x_2}\frac{dx_2}{ds} \bullet \mathbf{E}_k$$

$$(\dot{\mathbf{E}}_i \bullet \mathbf{E}_k) = \Gamma_{i1,k}\frac{dx_1}{ds} + \Gamma_{i2,k}\frac{dx_2}{ds}$$

where s is the arc length for the curve defining **V.**

The components of $\dot{\mathbf{V}}$ can be written as follows[1].

$$V_1 = g_{11}\dot{v}_1 + v_1\left(\Gamma_{11,1}\frac{dx_1}{ds} + \Gamma_{12,1}\frac{dx_2}{ds}\right) + v_2\left(\Gamma_{21,1}\frac{dx_1}{ds} + \Gamma_{22,1}\frac{dx_2}{ds}\right)$$

which can be written as : $V_1 = g_{11}\frac{d^2x_1}{ds^2} + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij,1}\frac{dx_i}{ds}\frac{dx_j}{ds}$ ....(3)

Similarly,

$$V_2 = g_{22}\dot{v}_2 + v_1\left(\Gamma_{11,2}\frac{dx_1}{ds} + \Gamma_{12,2}\frac{dx_2}{ds}\right) + v_2\left(\Gamma_{21,2}\frac{dx_1}{ds} + \Gamma_{22,2}\frac{dx_2}{ds}\right)$$

[1]The details of this are in the Appendix A.2.

which can be written as : $V_2 = g_{22}\frac{d^2x_2}{ds^2} + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij,2}\frac{dx_i}{ds}\frac{dx_j}{ds}$ ....(4)

These equations describe how the components of **V** change with s.

The next step is to link (3) and (4) with the metric. Consider the derivative of $(\mathbf{E_i} \bullet \mathbf{E_j})$ with respect to coordinates.

$$\frac{\partial}{\partial x_k}(\mathbf{E}_i \bullet \mathbf{E}_j) = \frac{\partial \mathbf{E}_i}{\partial x_k} \bullet \mathbf{E}_j + \frac{\partial \mathbf{E}_j}{\partial x_k} \bullet \mathbf{E}_i$$

This can be written as:

$$\frac{\partial g_{ij}}{\partial x_k} = \Gamma_{ik,j} + \Gamma_{jk,i}$$

Permuting the indices gives,

$$\frac{\partial g_{ki}}{\partial x_j} = \Gamma_{ij,k} + \Gamma_{kj,i} \text{ and } \frac{\partial g_{jk}}{\partial x_i} = \Gamma_{ki,j} + \Gamma_{ji,k}$$

Combining these three expressions gives:

$$\Gamma_{ij,k} = \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right\} \quad ....(5)$$

The significance of expression (5) is that it can be used to relate the equations for geodesics, in the form of the components V_1 and V_2, to derivatives of the metric. For geodesics, $\dot{\mathbf{V}} \bullet \mathbf{V} = 0$ which implies that $\dot{\mathbf{V}} \bullet (v_1 \mathbf{E}_1 + v_2 \mathbf{E}_2) = 0$. So $\dot{\mathbf{V}} \bullet \mathbf{E}_1 = 0$ and $\dot{\mathbf{V}} \bullet \mathbf{E}_2 = 0$, since v_1 and v_2 are non-zero by hypothesis. Using (3), (4) and (5) gives a set of equations which link the metric with the equations of the geodesics. The next stage is to find the metric which describes the Conformal models to put in the system of equations.

3.4.3 Metric Induced in R^2 by Projection[1]

Projections of curved surfaces induce a metric on the x-y plane in the following way. Pick a point on the surface and a unit tangent vector to a geodesic of the surface through the point. Next, project the vector onto the flat plane and show that its image is itself a vector, scaled in length by the projection. This section will describe the process for a sphere and a hyperboloid, starting with the general case for projecting a three-d surface.

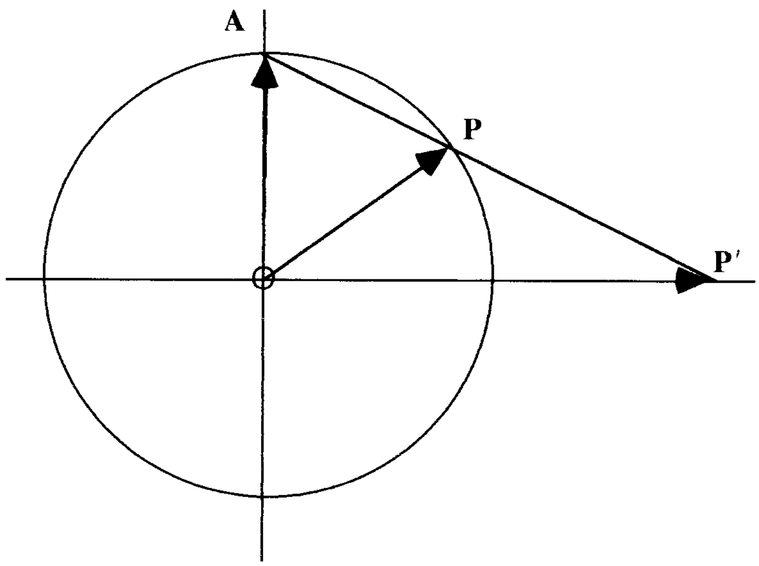

Figure 3.7 Projection of a 3-D surface. A is the point of projection, P is the point on the object being projected, and P' is its image in the x-y plane.

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[1] An alternative approach suggested by Dr. J. Gray (private communication 1995) based on Gray (1989), is to introduce longditudinal and "co-lateral" coordinates θ and φ on the sphere and positive branch of the hyperboloid. These coordinates are then projected stereographically from the North Pole (0, 0, 1) onto the equitorial plane.

For any point P on the sphere, for example, with coordinate $(\mathrm{Cos}\theta\ \mathrm{Sin}\phi, \mathrm{Sin}\theta\ \mathrm{Sin}\phi, \mathrm{Cos}\phi)$ corresponding to (X, Y, Z), its image in the Z = 0 plane is $\left(\frac{X}{1-Z}, \frac{Y}{1-Z}\right) = \left(\frac{\mathrm{Cos}\theta \mathrm{Sin}\phi}{1-\mathrm{Cos}\phi}, \frac{\mathrm{Sin}\theta \mathrm{Sin}\phi}{1-\mathrm{Cos}\phi}\right) = re^{i\theta}$,

where $r = \frac{\mathrm{Sin}\phi}{1-\mathrm{Cos}\phi}$. This induces the metric $ds^2 = \frac{4dz\,d\bar{z}}{(1+z\bar{z})^2}$, where $z = re^{i\theta}$.

Gray proposes that the length of the Turtle's path on a given heading can be calculated as follows. Rotate and translate to the origin the complex number which describes the path, and then calculate its modulus and argument before using the inverse transformations to locate the Turtle at the "end" of its walk. This gives the Turtle's new position, but it is necessary to calculate the intermediate positions to move it in a "step-by-step" manner. This method has the virtue of its relative simplicity. There are two difficulties, however. First, the approach is similar to §3.3 in that it is global and extrinsic, with the difficulties mentioned in §3.3.3. The second is that in the hyperbolic case, stereographic projection was not used in the study to induce Conformal B. Using such a projection produces a reversal of sense in the model since one is effectively looking through the inside of the hyperboloid rather than from outside and underneath, as is the case in this study. However, it is a very interesting approach which yields the induced metric quickly. It also highlights the link between the Conformal models and complex functions which is used in Chapter 7.

In three-dimensions, A is the projection point with coordinate (0, 0, a), P is the point being projected on a unit sphere with coordinate (X, Y, Z). P′ is the image of P on the x-y plane with coordinate (x, y). AP is a straight line so that in vector form

$$\mathbf{OP'} = \mathbf{OA} + \lambda\,\mathbf{AP} \Rightarrow \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} + \lambda \begin{pmatrix} X \\ Y \\ Z - a \end{pmatrix}$$

This gives: $x = \lambda X$, $y = \lambda Y$, $\lambda = \dfrac{a}{a - Z}$.

3.4.3 (a) Projection of the Sphere

For the unit sphere $X^2 + Y^2 + Z^2 = 1$ and the projection point is (0, 0, 1). Let any point P on the sphere with coordinate (X, Y, Z) be projected to a point P′ on the x-y plane with coordinate (x, y) so that:

$$X = \frac{2x}{x^2 + y^2 + 1}, \qquad Y = \frac{2y}{x^2 + y^2 + 1} \qquad \text{and} \quad Z = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \quad \text{.......(1)}$$

Geodesics on the sphere are obtained by plane sections through the origin[1], that is planes with equation aX + bY+ cZ = 0. Re-arranging this as Z = kX + mY and using equations (1) gives :

$$\frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} = k\,\frac{2x}{x^2 + y^2 + 1} + m\,\frac{2y}{x^2 + y^2 + 1}$$

This simplifies to $(x - k)^2 + (y - m)^2 = 1 + k^2 + m^2$., with $k = -a / c$, $m = -b / c$ and $c \neq 0$. This is a circle with centre (k, m) and radius^2 $= 1 + k^2 + m^2$ and gives the "straight lines" of Conformal Model A. Any line on the sphere passing through the point of projection must lie in a plane parallel to the Z axis and so cannot lie in the X-Y plane. This means that k X + m Y = 0 and so the equation of the projection is kx + my = 0, which is a straight line throughout the origin of the x-y plane.

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[1]The details of this are in Appendix A.3.

Let C(s) be a geodesic of the sphere, parameterised by arc length s. Let **V** be a unit tangent vector at P (X, Y, Z) on C(s), $\mathbf{V} = \left.\frac{dC}{ds}\right|_{p=C(s)}$. The coordinates of P's image under projection are (x, y). Using the usual metric on the unit sphere in component form,

$$1 = |\mathbf{V}|^2 = V \bullet V = \left(\frac{dX}{ds}\right)^2 + \left(\frac{dY}{ds}\right)^2 + \left(\frac{dZ}{ds}\right)^2$$

Calculating the total derivatives of **V**'s components from (1) gives[1] :

$$|\mathbf{V}|^2 = \mathbf{V} \bullet \mathbf{V} = \frac{4}{(1 + x^2 + y^2)^2}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) = \Omega^2(\mathbf{v} \bullet \mathbf{v})$$

where $\mathbf{v} \bullet \mathbf{v}$ is the usual dot product on R^2 and (x, y) are the coordinates of the image of (X, Y, Z) on the sphere. This indicates that the projection induces a conformal metric on the flat plane in which the vectors are scaled uniformly but the angle between them is preserved (Thorpe 1979).

3.4.3 (b) Projection of the Two-Sheet Hyperboloid

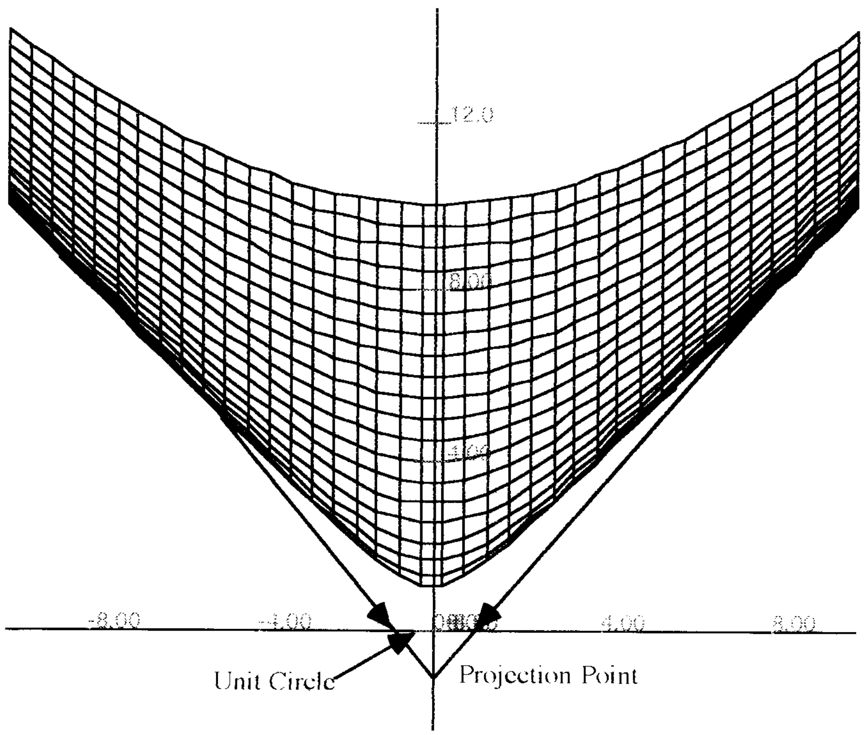

Figure 3.8 Projection of a Hyperboloid. The two-sheet hyperboloid with equation $z^2 - x^2 - y^2 = 1$ is projected onto the x-y plane from the point (0, 0, -1) and intersects the x-y plane within the unit disc or circle.

[1] Checked using CAS MAPLE V

The hyperboloid sheet $Z^2 - X^2 - Y^2 = 1$ has two branches Projecting the positive sheet which passes through (0, 0, 1) using the point (0, 0, -1) is shown in Figure 3.8. Given a point (X, Y, Z) on the hyperbolic sheet and a corresponding point (x, y) in the X-Y plane, let (0, 0, -1) be the projection point. The relationship between the coordinates is given by:

$$X = \frac{2x}{1-(x^2+y^2)}, \qquad Y = \frac{2y}{1-(x^2+y^2)} \qquad \text{and} \quad Z = \frac{x^2+y^2+1}{1-(x^2+y^2)} \text{.......(2)}$$

Again using the fact that geodesics of the solid are obtained by plane sections through the origin[1] with equation aX+ bY+ cZ = 0. Re-arranging this as Z = k X + m Y, where k = -a/c, m = - b/c, $c \neq 0$ and substituting in equations (2) gives:

$$\frac{x^2+y^2+1}{1-(x^2+y^2)} = k\frac{2x}{1-(x^2+y^2)} + m\frac{2y}{1-(x^2+y^2)}$$

Multiplying through by the denominator and simplifying gives :

$$(x-k)^2 + (y-m)^2 = k^2 + m^2 - 1.$$

This is a circle centre (k, m) and squared radius equal to $k^2 + m^2 - 1$ and orthogonal to the unit circle. Hence this is Conformal Model B of the geodesics on the hyperboloid.

Let C(s) be a hyperbola parameterised by arc length s, with $\mathbf{V} = \left.\frac{dC}{ds}\right|_{p=C(s)}$,a unit tangent vector at P (X, Y, Z) on C(s). The coordinates of P's image under projection are (x, y). Imposing the hyperbolic metric on the hyperboloid, in component form,

$$1 = |\mathbf{V}|^2 = V \bullet V = \left(\frac{dZ}{ds}\right)^2 - \left(\frac{dX}{ds}\right)^2 - \left(\frac{dY}{ds}\right)^2$$

Calculating the total derivatives of **V**'s components from (2) gives :

$$|\mathbf{V}|^2 = \mathbf{V} \bullet \mathbf{V} = \frac{4}{(1-x^2-y^2)^2}\left(\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2\right) = \Omega^2(\mathbf{v} \bullet \mathbf{v})$$

[1]The details are in Appendix A.3.2.

where **v** • **v** is the usual dot product on R^2 and (x, y) are the coordinates of the image of (X, Y, Z) on the hyperboloid.

Putting together the metrics corresponding to the projection of the sphere and the hyperboloid, they can be expressed by a transformed euclidean metric:

$$ds^2 = \frac{4\ ds_e^2}{(1 + k(x^2 + y^2))^2}$$

where $ds_e^2 = dx^2 + dy^2$ is the usual dot product on two-dimensional euclidean space. This is very useful since it enables one to switch between metrics by changing one variable, k, whose sign corresponds to the value of the curvature of the spaces which produced the metric.

- If k = 1, then the metric is that induced by projection of the sphere.
- If k = -1, then the induced metric is that obtained by projecting the hyperboloid.
- If k = 0 then $ds^2 = 4\ ds_e^2$, which is euclidean.

3.4.4 Finding Equations of Turtle Motion

Having found the metric induced by the projection of the surfaces, it is now possible to combine these equations derived in §3.4.2 to obtain equations which govern the motion of the Turtle.

Recalling §3.4.2, the components of the derivative of any vector, tangent to a curve, relative to the coordinates basis are:

$$V_1 = g_{11}\frac{d^2x_1}{ds^2} + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij1}\frac{dx_i}{ds}\frac{dx_j}{ds}$$

$$V_2 = g_{22}\frac{d^2x_2}{ds^2} + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij2}\frac{dx_i}{ds}\frac{dx_j}{ds}.$$

For geodesics, $V_1 = V_2 = 0$. Computing the value of

$$\Gamma_{ij,k} = \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right\}$$

using the components of the induced metric:

$$g_{11} = g_{22} = \frac{4}{(1 + k(x^2 + y^2))^2} \text{ and } g_{12} = g_{21} = 0$$

This gives a pair of second-order differential equations, which relates how the coordinates of the Turtle's position, (x_1, x_2), vary with arc length s.[1]

$$\frac{d^2x_1}{ds^2} = \frac{2k}{1 + k(x_1^2 + x_2^2)}\left[x_1\left(\frac{dx_1}{ds}\right)^2 - x_1\left(\frac{dx_2}{ds}\right)^2 + 2x_2\frac{dx_1}{ds}\frac{dx_2}{ds}\right]$$

$$\frac{d^2x_2}{ds^2} = \frac{2k}{1 + k(x_1^2 + x_2^2)}\left[x_2\left(\frac{dx_2}{ds}\right)^2 - x_2\left(\frac{dx_1}{ds}\right)^2 + 2x_1\frac{dx_1}{ds}\frac{dx_2}{ds}\right]$$

3.4.5 Solving the System

The method adopted to solve these was to turn the equations into a pair of first-order differential equations by substitution and then use a numerical method of solution. The equations, together with the information given by the Turtle's position and heading, could then be considered as an Initial-Value Problem (Matthews 1992). Taking the pair of equations above and making the following substitution :

$$U = \frac{dx_1}{ds} \text{ and } V = \frac{dx_2}{ds}.$$

the system becomes:

[1]The details are in Appendix A.4.

$$\frac{dU}{ds} = \frac{2k}{1 + k(x_1^2 + x_2^2)}\left[x_1 U^2 - x_1 V^2 + 2x_2 UV\right]$$

$$\frac{dV}{ds} = \frac{2k}{1 + k(x_1^2 + x_2^2)}\left[x_2 V^2 - x_2 U^2 + 2x_2 UV\right]$$

The initial conditions were: $x_1(0)$ = x-component of Turtle's position

$x_2(0)$ = y-component of Turtle's position

$U(0)$ = x-component of Turtle's heading

$V(0)$ = y-component of Turtle's heading.

The Turtle parameters were all scaled to fit a unit circle, since the equations were derived on that assumption. To solve the equations, a simple numerical algorithm such as Euler's method was used, based on the assumption that the first derivative of a function may be considered as a linear approximation to the function over small regions of its domain (Matthews 1992). The size of the region is critical for the accuracy of the method. Suppose, in general that $\frac{dx}{ds} = f(s,x)$, with initial conditions $X_0 = x$ and $S_0 = s$. Then the Euler method gives a "step-by-step" solution to the equation of the form:

$$X_{n+1} = X_n + \Delta s \left.\frac{dx}{ds}\right|_{s=s_n} \quad \text{.............(1)}$$

where X_n is the value of X after n applications of the rule and Δs is the size of increase in the independent variable s for each use of the rule.

For the equations described above, the Euler system is as follows.

$$\left.\frac{dU}{ds}\right|_{s=s_n} = \frac{2k}{1 + k(x_{1,n}^2 + x_{2,n}^2)}\left[x_{1,n} U_n^2 - x_{1,n} V_n^2 + 2x_{2,n} U_n V_n\right]$$

$$\left.\frac{dV}{ds}\right|_{s=s_n} = \frac{2k}{1 + k(x_{1,n}^2 + x_{2,n}^2)}\left[x_{2,n} V_n^2 - x_{2,n} U_n^2 + 2x_{1,n} U_n V_n\right]$$

$$U_{n+1} = U_n + \Delta s \left.\frac{dU}{ds}\right|_{s=s_n} \qquad V_{n+1} = V_n + \Delta s \left.\frac{dV}{ds}\right|_{s=s_n}$$

$$x_{1,n+1} = x_{1,n} + \Delta s U_n \qquad x_{2,n+1} = x_{2,n} + \Delta s V_n$$

where $x_{1,n}$ represents the value of x_1 after n applications of the system and $x_{2,n}$ represents the value of x_2 after n applications of the system. The solution involves calculating ($x_{1,n+1}$, $x_{2,n+1}$) on the basis of knowing the Turtle position ($x_{1,n}$, $x_{2,n}$) , its heading (U_n ,V_n) and k . The Turtle is moved from screen coordinate ($x_{1,n}$, $x_{2,n}$) to the new coordinate of ($x_{1,n+1}$, $x_{2,n+1}$) during one Turtle step s .

The Euler algorithm is controlled by the parameter Δs, since its size controls the accuracy of the solution. Reducing the size of Δs increases the accuracy but requires more steps of computation. Since the geodesic equations govern the Turtle's movements, the value of Δs has a direct effect on the speed at which the Turtle moves. The equations of motion contain second-order differentials and so the algorithm has to be used twice, which introduces the possibility of greater inaccuracies. A "trade-off" is required between speed and accuracy, which could only be produced through experimentation.

From the point of view of the study, this step-by-step numerical approach was interesting for two reasons. First, differential geometry is concerned with the *local* and *intrinsic* descriptions of surfaces using the metric. As §1.2.3 points out, Turtle Geometry also uses a local and intrinsic description of shapes to create geometric objects. It was clear that the differential geometric solution to the problem was eminently suited to Turtle Geometry for two reasons. First, the Conformal models are two dimensional, as is Turtle Geometry, and so any description of the former in local and intrinsic terms could be used by the latter. Second, the Euler technique gave local values that could be interpreted as moving the Turtle forward in a "step-by-step" manner, with each step defined with reference to the Turtle's current position and heading. By contrast, the approach to Turtle motion using the geometry of circles, shown in §3.2, would have entailed finding the equation of the circle on which the Turtle should move and then using an algorithm to develop a local description for Turtle motion along the curve. The second reason for adopting differential geometry was that it produced a *working solution* which was elegant, powerful and implemented over a short period of time. The full power of this approach will be apparent in Chapter 6 when it is implemented in Object Logo.

3.5 Conclusion

This completes the discussion of the technical element of the diachronic view of the microworld. It has shown how the Conformal models are the result of projecting the sphere and hyperboloid onto the flat plane. This indicates that the physical objects mentioned may form part of a non-computational component for the technical element. The chapter has set out the rationale for the choice of programming language and given a mathematical justification for the choice of a differential geometric approach to finding and solving the equations of Turtle motion. The actual implementation of these equations and the changes that were made will be described in Chapters 6, 7 and 8.

Chapter 4

Designing the Microworld (2):
The Pedagogic and Cognitive Elements

4.0 Introduction

The aim of this chapter is to examine the pedagogic and cognitive elements of the microworld from both the diachronic and synchronic views so that detailed structures can be developed for them. The chapter begins with a review, in §4.1 and §4.2, of how space and curvature are perceived from a visual and tactile point of view. This is then discussed in relation to Turtle Geometry in §4.3. In §4.4, the sign-action-surface combination is applied to aspects of euclidean and non-euclidean geometry to identify a specific "language-game" associated with the term "straight line". This forms the basis for developing a pedagogic strategy in §4.5 and identifying the strategy's cognitive assumptions in §4.6.

4.1 Perception of Space and Curvature

Psychological understandings of space range from the physical to the abstract. Liben (1988) points to the potentially confusing way that psychologists refer to "space" in accounting for such things as spatial awareness. It may be referring to "individuals' knowledge about particular places" in some contexts, while in others, it refers to "individuals' understanding of space in the *abstract*" (p.169).

A further sense of the term "space" is that of a *domain*; be it of representation or psychological activity. Here a structural connotation of "space" is used to identify a delimited region which has specific characteristics and modes of operation. To some extent it reflects the Kantian notion of space as the *condition of the possibility* of discourse about physical objects. Space is not, therefore, a thing but a category of experience with a characteristic logic. Unlike the Kantian sense, "space" is not absolute; there are many possible sorts of space, providing different structurings of experience. Emphasising the structural aspects indicates two things. First, the term is used in a metaphorical sense, in order that sense can be made of psychological experience: "space" implies structure, and structure implies "order" and "meaning". Second, the spatial metaphor is powerful because it draws on a fundamental experience of being human; our spatiality.

At a perceptual level, there would also appear to be a close connection between curvature, objects, and space. Our perception of curvature starts to develop very early. Fantz and Miranda (1975), for example, report that new-born children can distinguish curved from straight lines. Interacting with the world around us through sight and touch, we develop a sense of "curviness" and "straightness". Contact with and sight of curved things provides a variety of perceptual information which is coordinated in the later stages of development with linguistic signs.

Sight and touch provide different sorts of information. Visually, objects are judged to be curved through contrast with surroundings. For example, if one is looking at a ball, the circular outline of the object defines it as an object through contrast with a non-circular background. Touching a ball gives different information. Variations in pressure as one moves one's fingers across the surface leads to the conclusion that the ball is not "straight". Similarly, holding a ball in one's hands so that it is partially enveloped gives a sense of the surface bending. This information is local, since touch deals with portions of the ball's surface and it is extrinsic to the surface: one's hand pushes against the surface.

Lakoff (1988) describes this type of involvement with the environment as "the basic level". It is characterised by gestalt perception, mental imagery and motor movements, which together place a pre-conceptual structure on experience. However, Lakoff is at pains to point out that this level is neither primitive nor atomistic, rather it is an initial but intermediate form of dealing with the world. Gestalt perception and motor interactions enable different kinds of things to be identified as different. Coupled with mental imagery, these structures form the basis for metaphorical projections onto abstract domains. An example of one such "image schema" is what Lakoff calls the

"Container Schema". He argues that we experience ourselves as containers through the process of ingretion and excretion: an "in" and an "out". This experience forms the basic logic of the container metaphor which uses the distinction between interior, exterior, and boundary, as a way of structuring other sorts of experience.

The container schema plays an important role in developing a sense of space. As Flavell (1963) puts it, echoing Piaget

> "What the child needs eventually to establish - and does not at first possess - is a picture of space as a kind of all-enveloping container made up of a network of sites and subspaces. Within the container are objects, the things contained, which move from site to site, now occupying or filling a given site, now leaving it unoccupied." (p.335)

Space is, initially, a collection of relationships between significant places, which provides both a map of the physical environment and also defines it. Space *surrounds* them and they are located *in* it. Liben (1988 p.170), following Ittleson (1973), elaborates the distinction. She describes the space which surrounds as "large-scale" and that which is contained as "small-scale". Hence our sense of both being-a-body and having-a-body (Berger and Luckmann 1966) are related to an awareness that we are embedded *in* space. Contrasts between the two scales of space, structured by the container schema, provide the underpinning for subsequent categorisation of things, such as "objects". Later, according to Piaget (1956), this "topological"[1] network takes on metrical properties, which progress from a "projective" to a euclidean space as our perceptual apparatus matures. Piaget's developmental framework views this process as diachronic. Mandler (1988), on the other hand, argues that the "topological" and metric properties of space are synchronic, with one or other of them dominating at different times. Whichever is adopted, the important point for this study is the dominance of the euclidean sense of space as the "container" in which we "live and move and have our

[1] Whether Piaget meant this in the mathematical sense of the term is an open question. He was trying to capture the sense of a non-metric ordering of objects developed through sense which corresponds to mathematical ideas of interior, exterior, and boundary of sets. However, the idea of "place" and "significant sites" suggest that the representations which we develop contain more that just transformations of actions. Adam (1973),working in the other direction in seeking a perceptual basis for mathematical concepts, so-called "naive topology", points to the relationship between touch and connectedness of a surface. He thinks that this may be more fundamental than the point set analogies of Piaget.

being". This provides the background against which our physical conception of curvature develops.

Curvature, as a property of the boundary in the container schema, is related therefore to the development of our ideas of space. The "large-scale" / "small-scale" space distinction leads to an understanding of curvature as the result obtained by embedding objects *in* space. Curvature of a surface is defined, therefore, with reference to a higher dimension, which is "not-curved". This implies both that curvature is understood by making a contrast and that an infinite regress in the definition occurs unless there is an accepted understanding of "flatness" in some space. Our intuitive interpretation of large-scale space is flat and euclidean. So what might be called our "primary" experience of curvature is the result of embedding objects in three-dimensional euclidean space. Curvature is defined extrinsically by this process.

Both sight and touch are involved in the development of ideas about curvature. Each contributes something different to our understanding of curved and straight objects. The next section will consider the role of the visual in mathematics and its function in developing intuitions about curvature.

4.2 Visualisation: Seeing and Knowing

Since Plato, *seeing* has been a powerful metaphor for *knowing*[1]. **θεωρια** (theoria) is knowledge obtained by contemplation and is related to the word for god, **θεος**. Knowing is, therefore, a kind of "godlike seeing". A number of factors contribute to this identification. "Seeing" has a sense of immediacy and certainty which leaves one feeling sure that what is seen is true. "Seeing a situation" suggests taking "everything in at once". A global view is given from which details can be discerned. Seeing, a synchronic and, apparently, all-encompassing act of perception, therefore, possesses many of the qualities ascribed to knowing: certainty, truth and universality.

[1] For example, "The Sun", "The Line" and "The Cave" In Republic VII, 507b - 511c; 514a - 518d

Visualisation as "the ability to represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowitz 1990 p.75), plays an important part in our understanding and knowledge. Bishop (1983) distinguishes between the ability to interpret figural information (IFI) and the ability for visual processing (VP). The former is concerned with being able to understand visual representations in all their various forms and is something which can be taught. The latter "involves the visualisation and translation of abstract relationships ...into visual terms" (*ibid.* p.185) and implies both an internal and an external sense of visualisation. External visualisation involves the "embodiment" of some abstract relationship so that it can be understood. Mathematics uses many such forms of external visualisation, such as inscriptions, which stand for something else (numerals, symbols, equations), and images, such as the number line, which both illustrate the structure of the Real numbers and have a function in problem solving. Internal visualisation is the formation of "pictures in the mind's eye". It is "imagery abstracted from phenomena that we have actually witnessed" (Miller 1991 p.37) which is used to illuminate ideas and solve problems.

4.2.1 Visualisation in Mathematics

Mathematical concepts have both structure and function. Real numbers, for example, have an internal structure which may be defined independently of any use, while their uses are many and varied. Similarly, one may distinguish the *derivative* as a mathematical construct from the *operation* of differentiation. Learning mathematics must involve one in coming to terms with both the structure and function of its ideas. There is a close relationship between the two elements: *using* mathematical concepts leads to improved *understanding* of them, and this in turn leads to finding a wider range of uses for the concepts. (Sfard 1991)

Visualisation, both internal and external, plays an important part in the process of learning and doing mathematics. Krutetskii (1976) identifies three styles of mathematical thinking: analytic, harmonic and visual. The analytic thinker has a "predominance of a very well developed verbal-logical component over a weak visual-pictorial one" (p.317). Geometric thinkers on the other hand, have "a well developed visual-pictorial component" (p.321) in their mathematical reasoning. Harmonic thinkers have a mix of the two previous forms which dominate. Van Hiele's hierarchical model (1958) of geometric thinking begins with a level relating to visualisation and recognition of shapes. As the levels change, the model predicts that the visual element will give way to analytic, deductive and rigorous treatments of geometry. Non-euclidean geometry is the highest level, requiring no visual stimuli but rather logical thinking and rigour. Subsequent modifications (1987) still maintain the visual-logico division.

What is interesting about Van Hiele's model is its hierarchical nature, which places non-visual geometric reasoning at the top level. This suggests an implicit positive valuation of the logico-analytic approach to mathematical thinking, while the visual-intuitive approach is at a lower level. Van Hiele's model has been investigated from a number of viewpoints which are reviewed by Clements and Battista (1991). They ask several critical questions about the research findings obtained so far, relating to the capacity of researchers to identify the levels in practice. Eisenberg and Dreyfus (1991), reviewing findings about older students' uses of visualisation, note a distinct reluctance among them for visual thinking in advanced mathematics. They speculate that students prefer a more "mechanical" or "algorithmic" approach to learning mathematics because visualisation makes higher cognitive demands than analytic methods. However, they did find that among professional mathematicians there was a greater variety in approaches to solving problems.

Another sense of internal visualisation is highlighted by Vinner, who draws a distinction between the formal definition of a mathematical concept and its "image".

> A concept name when seen or heard is a stimulus to our memory. ......Usually, it is not the concept definition, even in the case where the concept does have a definition. It is what we call a "concept image".......The concept image is something non-verbal associated in our mind with the concept name. It can be a visual representation..in the case where the concept has a visual representation; it could also be a collection of impressions and

> experiences, the visual representation ......can be translated into verbal form. (Vinner 1991 p. 68)

From a cognitive point of view, these images play an important role in providing an interpretation of abstract ideas. The power of visualisation resides in the capacity of individuals to "call up" visual referents that enable them to make sense of mathematical concepts. However, these can also be a barrier to understanding if they are inappropriate. The very aspects which make visual images useful, such as immediacy and self-evidence, can also obstruct development in understanding if the images cannot be extended or adapted to new situations. Concept images, as outlined above, are more like "cognitive hooks" used to aid understanding. Eisenberg and Dreyfus (1991), on the other hand, are interested in visual reasoning where images, both external and internal, are used to develop mathematical arguments and solve problems.

Expectations about visual images play an important role in understanding and interpreting mathematical concepts. Graphs, for example, are important because they give information about the large-scale behaviour of analytic functions. They are coded visual artefacts which one must learn to "read" and manipulate if they are to be effective. Stevenson and Noss (1991) report the results of a small-scale study in which 17 and 18 year-old students who are specialising in mathematics at pre-university level, sketch Cartesian graphs. The strategies which the students adopted seem to have depended on two things. The first was whether they recognised the function which they were asked to sketch. If they did, the students proceeded in a "top-down" manner, being able to identify the global shape of the graph and deduce details from the global structure. If they did not recognise the function from its equation, they proceeded in a "bottom-up" manner, filling in local details as they developed the global structure of the graph. The second important aspect was that the students had expectations of the graph based on previous experience of sketching and exposure to a large variety of Cartesian graphs. These expectations guided how the students interpreted the shape of the graph, particularly in the case where they did not recognise the equation they were given to sketch. Experience of working with visual images which have mathematical meaning gave them expectations about the images and these expectations guided their understanding and use of the images.

4.2.2 Visualising in 2-D and 3-D

Representations of curvature have developed over the centuries for a number of practical purposes, such as navigation and trade. The sixteenth-century and seventeenth-century map makers used both two and three-dimensional models of the earth to plot new discoveries of land, and to develop means to explore and exploit them (Holt and Majoram 1973). Together with the invention of accurate chronometers, the introduction of world maps opened the way for the expansion of Western Europe. Among the earliest form of flat representation of the earth's surface is that which bears Mercator's name. This uses stereographic projection of the sphere onto a plane and then maps the polar coordinates produced into rectangular coordinates. Several other types of projection produce maps which preserve some aspects of the sphere at the expense of others (*ibid.* see also Selkirk (1982)). What exactly is lost depends on the purpose to which the flat representation is put. For example, Mercator's projection is conformal (Kreyzig 1959) which is important for navigational purposes. However, if one wants a comparison of land mass sizes, it is not very useful.

Flat representations of curved objects raise several difficulties from a perceptual point of view. Precisely because they describe objects which are not flat, plane representations imply an "entropy", a loss of information (Janvier 1987). In the case of the spheres, for example, their two-dimensional representations lose a spatial dimension. The issue is also significant for computer representations of three-dimensional objects on flat, two-dimensional screens. Hershkowitz *et. al.* (1990) have reviewed the question of two-dimensional representations of three-dimensional euclidean space in both computational and non-computational environments. They report that

> "factors of culture, experience, and familiarity with the conventions of transforming 3D shapes to their 2D representations and vice versa have considerable effects on the drawing and interpretation of 3D shapes." (p.78)

Culture, experience and conventions are interlinked so that we learn to "read" 2D images of 3D objects. This would tend to support Reichenbach's contention that we can learn to use euclidean images of non-euclidean space in the sense that the process is a constructive rather than a "faculty" of the mind. However, as Hershkowitz *et.al.* note, the research evidence approaches the issue in a number of different ways. The extent to which one accepts the contention that it is possible to teach individuals to use euclidean images for interpreting non-euclidean geometry, depends upon the extent to which one considers teaching and learning to be a matter of "culture, experience and convention".

4.3 Touch, Logo and Curvature

Having discussed the visual aspect of perceiving curvature, this section will consider the role of touch. Ordinarily, curvature is a property of lines and surfaces which is usually understood in a *comparative* and *global* sense. Clearly, curved lines are not straight, but the basis for this judgement relies on an implicit comparison of two lines over portions of their lengths. It is the need to make this comparison over portions of lines (and surfaces) rather than just at points on the line which makes the usual understanding of curvature global (Janvier 1978). Specific measures of curvature derived from these comparisons are *extrinsic* to the curve being measured. Text book measures of curvature, given in terms of either the radius of curvature or the rate of change of the direction of the tangent, are obtained by using a frame of reference outside of the object being measured. These global and extrinsic qualities of the definition of curvature are closely related to the *visual* nature of curvature.

The sort of information given about curvature by touch is different. Variation in the pressure exerted by objects on the hands gives a *local* understanding of curved surfaces. As one moves one's hand over an object, information is obtained in a "step-by-step" manner, exploring first one direction and then another. Clearly this information is *non-visual* since it relies on touch and not sight. Touch data can also be said to give a sense of the *intrinsic* properties of the surface in the following way.

Although fingertips, for example, may be thought of as extrinsic to the surface (they lie *on* the surface) the point of contact of the finger with the surface can also be thought of as being *in* the surface. Hence variations in pressure on the fingertips correspond to variations *in* the surface and so touch provides an intrinsic measure of the surface.

A key feature of the Turtle metaphor is the identification of the screen Turtle with one's own body. Papert (1980 p. 63) describes this process of learning as *syntonic*. Programming the Turtle to draw a circle is identified with "walking" the circle, using stepping and turning. Since the input to Turtle is textual, children must translate what they see and feel into commands for the Turtle. Logo's power, from this point of view, is the ease with which children adopt the Turtle metaphor, enabling them to represent textually, and hence symbolically, what they see. An illustration of this can be seen in a typical LOGO procedure for drawing a circle.

```
to circle
repeat forever [fd 1 rt 1]
end
```

It is interesting to note what this procedure does not mention. No allusion is made to either radius, centre, or an external frame of reference. However, the procedure does describe the Turtle's motion in a *local, intrinsic* manner. It tells the Turtle how to move in a step-by-step way, giving its new direction of movement relative to its current heading and not an external reference system. Curvature is measured, therefore, by the bracket [**fd 1 rt 1**] which gives the rate of change of the Turtle's direction. Procedures, such as those which produce circles in Logo, are textual and numerical, and thus non-visual. The Turtle approach to the geometry of the circle corresponds closely to the sort of information obtained by touch: local, intrinsic and non-visual. The Turtle cannot "see" what is being created, it follows the commands given. Programming the Turtle to draw a circle requires a translation of the visual, global synchronic perception of the shape into non-visual, diachronic text string which contains local and intrinsic information about the shape.

Curvature may be thought of in two ways, therefore, corresponding to the different kinds of information given by sight and touch. Sight provides a global and extrinsic definition of curvature derived from the properties of visualisation referred to in §2.4.2. Touch gives local and intrinsic information about curvature, since hands can only cover finite areas of objects but are very sensitive to variation in texture and contour. Contrasting these two ways of thinking about curvature, psychologically, suggests that the extrinsic and global definition is perhaps dominant. It is not difficult to see why this might be the case. The definition relies, implicitly, on visualisation and contrast to provide it with a plausible basis. Characteristic of vision is the ability to see things both in relation to others and as objects *per se* at the same time. Curvature shows itself both through contrast with non-curved objects and as a property of an object, in the process of being seen. Given that early explorations by children of their immediate environment rely heavily on sight and touch (although not exclusively!), it is not difficult to appreciate that the extrinsic, global definition is more immediately accessible than the Logo definition. However, the connection between Logo and touch, suggested by the similarities in the type of information they provide about a surface, becomes more important when surfaces are difficult to visualise. This will be explored in more detail in the next section.

4.4 Geometry and Intuition

Historically, the origin of geometry was the science of physical space. Meaning, literally, earth (γεο) , to measure (μετρειν), geometry arose chiefly as the study of land measurement to cope with abstract calculations posed by practical problems of building and commerce (Gray 1989). It was transformed into a logical structure consisting of axioms, theorems, and proof by Euclid's *Elements* around 300 BC. The development of non-euclidean geometries in the nineteenth century led to the separation of the logical and practical aspects of geometry. (Hershkowitz 1990).

From a perceptual point of view, this separation implied by non-euclidean geometry has important consequences. The support which euclidean geometry receives from everyday experience is no longer useful as a mode of understanding. The geometry of spheres, for example, introduces a new set of perceptual referents for terms such as "line" and its properties differ in significant ways from those of a flat surface. The key role which everyday experience plays in finding and proving properties of euclidean geometry is also no longer valid. Given this break with everyday experience of geometry which the introduction of non-euclidean geometry implies, how is geometric intuition to be developed? To explore this in more detail the connection between geometric practices and language will be discussed in relation to the differences between euclidean and non-euclidean geometry.

4.4.1 Euclidean Geometry

As indicated in the introduction, part of euclidean geometry's success lies in its apparent correctness, from a perceptual and practical point of view. Terms such as "line" and "point" which appear in Euclid's axioms are understood in an intuitive and common-sense manner, reducing the need, methodologically, to provide a formal justification for them. This "self-evidence" of the terms in the axioms played an important role in the development of the geometry. The logical requirement that all deductive sciences have indemonstrable first principles (Aristotle *Metaphysics*) was clearly satisfied by euclidean geometry so that it became *the* example of deductive science. However, there was another element in the geometry which played a significant role but was not logically significant. Euclidean geometry is *synthetic*. It is an instrumental geometry (Bkouche 1989) in which rulers and compass play both a practical and a theoretical role in the process of establishing its results. It was only with Hilbert's treatment of Euclid that the constructional assumptions implicit in the process of proving results became apparent (Stillwell 1992).

The importance of this constructional element can be exemplified by considering two terms: "straight" and "parallel". Lines are implicitly understood to be straight in euclidean geometry. This is not defined by the axioms but is understood in two ways. Perceptually, there is the contrast between "straightness" and "curviness" alluded to in the previous section, and from which we develop a clear sense of what a straight line is. Operationally, we have the experience of using a straight edge to draw a straight line. A prototype (Lakoff 1988) is developed by the motor-sensory action of "ruling", combined with the linguistic sign "straight" and the Gestalt perception of the resulting mark on a surface. The ruler and pencil are pieces of equipment which reproduce what is meant by "straight" and, as a result, re-inforce its meaning.

Parallel lines are more complex. Although they are defined to be straight lines which never meet, perceptually pairs of such lines always meet when viewed over a large distance. Consider, for example, a straight railway track which stretches "as far as the eye can see". Local observation of the tracks show that the rails are the same distance apart, and riding on a train would not be possible if they were not. However, our perception of the track is that the two rails do meet in a "vanishing point" at the horizon. There is a contrast between the local perception of the rails as being a fixed, constant distance apart and a global perception in which the lines do meet. Constructing parallel lines plays an important role in establishing and maintaining the geometric concept. The process of drawing a straight line, finding a point, and constructing a line parallel to the original through the point as far as the edge of the paper, is essentially a local procedure. One imagines that both lines extend "to infinity" and what is seen on the paper is a "window" on infinite entities. One *remembers* what one knows about parallel lines and *forgets* one's perceptual experience that straight lines do meet. Again, the combination of geometric construction, local perception, and linguistic usage, provide a basis for the understanding and use of the concept.

Self-evidence of the terms "straight" and "parallel" in relation to euclidean geometry, therefore, is founded on the dynamic interaction of the sensory-motor, perceptual, and linguistic factors, which together define and validate the concepts. Euclidean geometry, with its processes of proof and construction, its language, and ways of structuring the environment, may be regarded as a collection of language-games. Understanding euclidean geometry, therefore, entails being able to act and communicate in specific ways with geometric apparatus.

4.4.2 Non-euclidean Geometry

Non-euclidean geometries are not those which we normally experience. Transferring an understanding of geometric terms such as "straight line" from a euclidean to a non-euclidean context, implies retaining the linguistic sign "straight line" but making it refer to an alternative set of images, actions, and experiences. Pimm (1987), in discussing such phrases as "spherical triangle", points to the way in which metaphors are used to provide signification of terms, such as "triangle", across domains of geometry. The meaning of the term is supported by embedding the metaphor in some geometric practice, such as a method of construction. As the previous parts of this section indicate, our perception of curvature is based on both sight and touch and so any context for the metaphorical identification of terms across geometric domains requires attention to both the visual and the tactile elements. These will be considered in turn, starting with the visual element in the metaphorical identification of the term "straight line" in non-euclidean geometry.

Visual referents for terms such as "straight line" in non-euclidean geometry can be produced in several ways. Two will be considered which have analogies with euclidean practices of geometry and are based on the euclidean idea that lines can either be *produced* by the intersection of two planes or that straight lines are *generators* of the plane through rotation. In both of these approaches, there is an operational connection between those things termed "straight line" and surfaces. Similar processes can be applied to non-euclidean geometry. In the case of elliptic geometry, the term "straight" is taken to refer to "great circles" on the surface of a sphere. These are obtained either by the intersection of the sphere with a euclidean plane which passes through the origin of the sphere or by rotating a circle about a fixed axis. The significant point here is that the straight lines of elliptic geometry are so called by means of drawing some analogy with euclidean geometry.

For hyperbolic geometry the case is more complex still, since there are several surfaces which can be used, but they are all incomplete since the hyperbolic plane cannot be completely embedded in euclidean space (Coxeter 1969 p.377). For example, the hyperboloid used in §3.4.3(b) to obtain Conformal model B is composed of two branches, but only one of the branches was used for the projective model. In the case of the hyperboloid, the "straight lines" of the surface are either produced by the intersection of the surface with a euclidean plane through its origin or the hyperboloid is generated by rotating a hyperbola about a fixed axis.

The situation is equally complicated in relation to the Conformal models obtained by projecting the sphere and the hyperboloid onto the euclidean plane. In both models, the term "straight" refers to either euclidean straight lines or arcs of circles. The situation is summarised in Table 4.1 which shows the signification of the term "straight line" in each geometry, together with the actions and surfaces used to produce them.

Geometry	Action of Construction	Surface
Euclidean	• Use of ruler and pencil. • Intersection of two planes. • Line generator of the plane.	Plane.
Elliptic	• Intersection of surface with a plane section through the origin of the sphere. • Circular generator of the sphere.	"Great Circles" on the sphere.
Hyperbolic	• Intersection of section of the positive sheet of a hyperboloid with a plane section through the origin. •Hyperbolic generator of the hyperboloid.	"Great Hyperbolas" on the hyperboloid.
Conformal Model A	Euclidean straight lines and circles which cut the unit circle at opposite ends of the same diameter.	Euclidean Plane.
Conformal Model B	Euclidean straight lines and arcs of circles which cut the unit circle orthogonally.	Euclidean plane

Table 4.1 Significations of the term "straight line" in various geometries produced by the interrelationship between objects and actions.

Table 4.1 highlights the discontinuity of the geometric domains and poses the question of how they can be related so as to provide continuity in development for the learner. It also provides a clue as to how the continuity can be constructed. Although the table shows differences, it does so by bringing together the elements which provide signification of "straight line": a linguistic sign embedded in actions of construction on specific surfaces. The term has meaning because of the way in which it is associated with actions, images, and surfaces in different geometrical contexts.

Table 4.2 illustrates how this construction process might occur in practice for the term "straight line". Moving between the phases of the microworld, corresponding to a change of geometric domain, the term "straight line" is re-assigned by a combination of actions and linguistic practices on specific surfaces. This enables the transfer of meaning to be made across geometric domains.

Technical /Surface	Physical Objects	Plane	Computer
Sight	Great circles or hyperbolas from plane section or as surface generator	Euclidean straight lines and arcs of circles as projections	Euclidean straight lines and arcs of circles
Touch	Tracing great circles across surface	Tracing lines and following projections on a flat surface	Use of the Turtle with FD command

Table 4.2 Signification of "Straight line" across the three phases. The pedagogical construction of meaning across the geometric domains associated with each phase is related to both touch and sight.

In the first phase, for example, the microworld's participants would be introduced to curved surfaces, such as a sphere. The term "straight line" would refer to "great circles" on the sphere supported by a combination of sight and touch. During the second phase, the mathematical connection established between a curved surface and its flat image would provide the basis for the physical and semantic transfer between the two domains. The projected images of those lines identified as "straight" on the curved surface would be used to transfer the meaning of the term "straight line" through a combination of sight and touch. Finally, in the third phase, the touch aspect of this transfer could be identified with the type of local and intrinsic information needed to move the Turtle and so provide a computer screen designation of "straight line"

Tactile support for these metaphorical extensions of the term "straight line" may be found by considering the identification between the motion of the Turtle and touching a surface, made in §4.3. The power of this resides in reflecting on the mathematical solution to the problem of defining a straight line. The mathematical definition of a straight line or "geodesic" is as follows. Given any two points in a surface, the straight line is the path between them which has the shortest distance. Abelson and diSessa (1980) reformulate this idea of the shortest distance in terms of Turtle steps: "A (turtle) line is an equal-stride turtle walk" (p.204). They wish to capture the local and intrinsic sense of Turtle motion which is measured by "steps" and "turns" relative to the Turtle's current heading. Straight lines are those in which the Turtle takes equi-distant strides without turning and hence picks out the geodesics "naturally". Curved lines are those which involve the Turtle either in turning after each step, or those taking unequal strides on its "right or left legs".

As Abelson and diSessa point out, this applies to any surface that the Turtle may be "walking over". It provides a way of describing the mathematical sense of a straight line in any geometric context. If the connection between the Turtle's motion and touching a surface is valid, then one has a syntonic understanding in terms of sight and touch which provides a context for the metaphorical transfer of the term "straight line" between geometries.

Two notions were suggested by these considerations for the pedagogical element of the microworld. First, the analysis of the term "straight line" using the surface-action-sign combination provided the basis for a pedagogical strategy which created meaning across geometric domains. As Table 4.1 suggests, by approaching the introduction of the various non-euclidean geometries through the metaphorical application of terms such as "straight line", embedded in some method of construction, it was possible to build new geometric meanings based on the euclidean intuitions of the user. Second, Turtle Geometry had a potentially important part to play in the process of meaning creation, by the identification of the motion of the Turtle on the screen with the motion of one's finger across a curved surface. Both of these considerations were used in the design of the pedagogic strategy.

4.5 Towards A Pedagogic Framework

Recalling the model of the microworld described in §2.6., the diachronic view of the microworld was concerned with its temporal structure as a pedagogical unit. The pedagogical element was described in terms of a linear structure. This began with the induction of the learner into the microworld, followed by a period of scaffolding, leading to "fading" of overt pedagogical structures and support as the learner developed confidence and competence with the computational element. The aim of this section is to describe the elaboration of a structure for the microworld's pedagogical element based on the considerations of §4.4.

The pedagogical element was based around the idea of teaching a "language-game" about non-euclidean geometry. As §4.4.2 indicates, this was devised using the combination of linguistic sign and an action of construction on a surface, as a means of supporting meaning-creation across geometric domains. Table 4.3 summarises this process in connection with the corresponding components of the microworld's technical element.

	Physical Objects	Plane	Computer
Technical Element	Solids with non-zero curvature and their geometry. e.g. sphere.	Flat projections of the solids on paper to produce Conformal models.	Dynamic and interactive versions of the Conformal models on computer.
Pedagogic Element	*Induction* Introducing non-euclidean geometry by: • using objects such as spheres. • challenging euclidean intuition.	*Scaffolding* Activities to direct and support the progression from objects to projections of surfaces.	*Fading* Independent use of the computer-based models by the participants without support.

Table 4.3 Pedagogical Structure of the Diachronic View. The pedagogical development of the microworld begins with induction into non-euclidean geometry, is followed by scaffolding of projections, and ends with independent use of the software by the participants.

Each phase of the pedagogic element is described in more detail in §4.5.1- §4.5.3.

4.5.1 Induction into the Microworld

Two issues guided this phase of induction. The first issue was the need to establish what was already known about non-euclidean geometry and how individuals expressed their intuitions about curvature. This established a reference point which could then be used to chart the development of an individual's understanding as a result of participating in the microworld. The second issue was the need to challenge the participants' euclidean intuitions in order to prepare them for their work with the computer-based Conformal models. In particular, the participants were confronted with a version of Logo which did not agree with their euclidean assumptions. The intention was to call into question the connection between commands such as FORWARD in Logo and the intuitive euclidean support which the commands ordinarily received. This prepared the ground for the microworld participants to be made aware of a non-euclidean interpretation of the Turtle's actions.

These considerations suggested three objectives to this first phase of the microworld, working with physical objects such as spheres.

- Establish a "base-line" for the participants, understanding of non-euclidean geometry.
- Explore, with the participants, specific aspects of geometry on the surface of curved objects.
- Challenge the participants' euclidean intuitions about Logo.

These objectives entailed a combination of both computer-based and non-computer-based activities concerned with investigating "every-day" intuitions of curvature and knowledge of non-euclidean geometry. Non-computer activities included getting the participants in the microworld to describe curved objects to one another, discuss the idea of a "straight line" on a curved surface, and check geometric facts such as the angle-sum of triangles on spheres. Participants in the microworld would also be challenged to make sense of the computer-based Conformal models without any formal support. They were invited to try out each of the Turtles using ordinary Logo commands, to make comparisons, and to conjecture about the nature of the images. The intention was to get the participants to question their euclidean perceptions.

4.5.2 Scaffolding

In this phase, the participants were led to consider flat projections of curved surfaces as a preparation for using the screen images. Two factors had to be considered. The first was the introduction of the participants to the idea of projection as a way of producing flat representations of curved surfaces. The second was to make the connection between the geometry of curved surfaces and the corresponding properties of their projections. Two objectives were identified corresponding to these factors.

- Introduce the idea of a stereographic projection.
- Check with the participants which geometric facts found in the first phase were preserved during projection.

Through the use of a combination of objects and diagrams, the participants of the microworld were introduced to the stereographic projection of a sphere. They were invited to explore what happened to "straight" lines under the projection and introduced to Conformal model A. They checked that the facts about angle sums, found in the first phase, held using model A. However, an immediate problem at this stage in designing the microworld was a lack of a suitable surface to project which could give Conformal model B. How this was managed will be described in Chapter 6.

4.5.3 Fading

The focus of attention would switch to the computer in this phase. The ultimate intention of the pedagogic strategy was for the participants to reach a stage where they could use the computerised Conformal models confidently and in an independent manner. Consequently, the participants had to be introduced to the computer-based versions of the models in a manner which enabled them to connect the screen images with the projections of the previous phase. The second step was to encourage the participants in using the software to check which geometric properties of the sphere were preserved under projection. The final aspect of the phase was for the participants to explore the Conformal models themselves, by posing their own questions and investigating their solutions. Three objectives governed this phase.

- To relate the computer-based Conformal models to the projections of curved surfaces obtained in the second phase.
- To work with Logo in checking which geometric "facts" about curved surfaces were preserved under the projections.
- To investigate the properties of both the Conformal models using the software.

4.6 Cognitive Element of the Microworld

Any pedagogic strategy must make assumptions about learners if it is to be made practicable. In this section, the "expected" cognitive development implied by the technical and pedagogic elements of the microworld will be outlined. The cognitive element described here does not constitute a model of cognitive development, but rather represents a collection of working assumptions and expectations about the microworld's participants, derived from the pedagogical structure outlined. A central aim of the study was to chart the actual cognitive development of the participants as they worked with the microworld. The structure outlined in this section represented a starting point, not a hypothesis to be tested.

The intention was to teach the participants a collection of language-games connected with non-euclidean geometry. From a cognitive point of view, this implied the need to create the possibility for new interpretations of the participants' intuitions by provoking a breakdown in their euclidean assumptions. This was followed by a period of encouraging them to construct new understandings of geometry through using the microworld. Consequently, corresponding to the structure of surfaces usage (Physical Surfaces —> Plane —> Computer) in the technical element and their associated pedagogical function (Induction —> Scaffolding —> Fading), there was an expected cognitive development (Breakdown —> Construction —> Fluency). Table 4.4 summarises this outline of the expected cognitive development, derived from the diachronic view of the microworld in §2.6.

	Physical Objects	Plane	Computer
Technical	Solids with non-zero curvature and their geometry. e.g. sphere.	Flat projections of the solids on paper to produce Conformal models. (e.g. stereographic projection).	Dynamic and interactive versions of the Conformal models on computer.
Cognitive	*Breakdown* in euclidean intuitions by introduction of objects and "confusing" screen images.	*Construction* of understanding through activities with objects and static images.	*Fluency*. Confident and independent use of the computer versions of the conformal models.

Table 4.4 Cognitive Development of the Diachronic View. The cognitive development of participants in the microworld begins with breakdown of euclidean intuitions , followed by a re-structuring of intuitions, and ending with independent use of the software by the participants.

Each phase of the development will be described in detail.

4.6.1 Breakdown

As the title suggests, the intention of this phase was to challenge the participants understanding of geometry in two ways. The first was to establish what was known by the user about non-euclidean geometry as a reference point for further development. The second was to call into question the euclidean intuitions of the participants in terms of their knowledge of both euclidean and Turtle geometry. As §1.2.4 pointed out, Turtle geometry, although local and intrinsic, has a strong euclidean support. A command such as FORWARD, for example, has a syntonic aspect, but it also uses euclidean intuition because of the command's visible effect in moving the Turtle in a euclidean straight line on the screen or over the floor. The connection between Turtle geometry and euclidean geometry had to be called into question, therefore, if the non-euclidean version of Turtle Geometry was to be understood. Participants in the microworld were encouraged to place less trust in their euclidean-based understanding and work instead on the local and intrinsic view which placed less reliance on sight and more on touch.

4.6.2 Construction

In this phase the intention was for the participants to build up their understanding of the Conformal models as projections of curved surfaces. They were encouraged to check the transfer of geometric properties from curved surfaces to flat representations, such as the preservation of angle and the angle sum of triangles. In this phase they were "taught" the connection between the Conformal models and their flat projections as a pre-cursor to using the computational element of the microworld.

4.6.3 Fluency

As the title of this phase suggests, the intention was for the participants to use the computerised versions of the Conformal models in a confident and independent way. This entailed making the connection between the flat projections and the computer versions; i.e. getting the participants to check geometric facts and set their own investigations using the computer. The extent to which the participants were able to work in this manner and the type of geometric facts which they discovered were of interest in this phase, since these two factors were very important in relation to the aims of the study.

4.7 Conclusion.

This chapter has provided a rationale for the detailed structure of the microworld's pedagogic element, together with the cognitive development implied by it. Based on the premise that the pedagogic element was concerned with teaching language-games about non-euclidean geometry, §4.5 and §4.6 have provided specific objectives for each of the phases of the microworld's diachronic structure. In the next four Chapters, the development of the microworld will be described using the detailed model established in Chapters 2, 3 and 4.

Chapter 5

The Methodology for the Microworld's Development

5.0 Introduction

In the previous three chapters, the diachronic view of the microworld has been described in detail. In this chapter, the methodology for developing the pedagogic and technical elements will be considered together with a framework for analysing the cognitive developments in the participants. In §5.1, a rationale is given for the iterative design methodology which is developed in detail in §5.2. The synchronic view of the microworld is used to develop an analytical framework for the microworld in §5.3. The chapter concludes with an overview of the developmental process in preparation for the detailed description of each cycle in Chapters 6, 7 and 8.

5.1 Rationale for the Developmental Methodology

The study's aim has been interpreted in terms of producing a computer-based microworld in which participants could develop an understanding of non-euclidean geometries, which, at the same time, provided windows on the participant's cognitive development and the microworld's production. This section is concerned with the methodology for translating the microworld's design into practice. The method chosen to achieve this was based on diSessa's notion of iterative microworld design (1986b, 1989), which relates closely to the process of modelling described in §2.1.

Two factors are important in the microworld's developmental process. The first factor may be described as "documented redundancy", in which a large number of activities were tried and their outcomes carefully observed and documented. The intention is to build up experience for the designer of what works and what does not within the area of investigation, and to "calibrate" a range of possible cognitive responses from the participants. Activities, therefore, fail, but from these failures emerge a set of viable opportunities for learning. The second element in this process of iterative development is a framework of pedagogical aims to guide its overall direction. From this point of view, the pedagogic framework described in the previous chapter played an important role, both in guiding the development of activities and in assessing their outcomes. It set the direction and provided the structure for the microworld and the sequence of activities for the participants.

5.2 Methodology for Developing the Microworld

The picture which emerges from this approach is that of a microworld design process that is dynamic. Successful activities are refined and repeated, and unsuccessful activities serve to delimit the boundaries of investigation ever more clearly. The participants' cognitive responses are crucial in this regard, since they serve both as a possible source of activities and as a guide to what the participants' intuitive understanding of the domain might be. This latter aspect is important for the aim of the study, since it was precisely the process by which participants developed their understanding of the microworld's epistemological base that was being examined.

On the basis of these considerations, the methodology for developing the microworld was as follows. It was conceived as an iterative process in which sets of activities were devised, tried out, refined, and tried again together with fresh insights gained from reflecting on the cognitive responses of the participants. The coherence and direction of the whole process was maintained through the use of the pedagogic framework of induction, scaffolding, and fading. This framework provided continuity and a way of organising the process of trial and development, so that the designer was not continually having to begin again with each new iteration.

The developmental process consisted of *phases* and *cycles*. *Phases* referred to the internal structure of the diachronic view provided by the temporal progression from curved surfaces to computer images, which constituted the technical component of the microworld. Phase 1 was concerned with activities associated with physical surfaces; Phase 2 was concerned with those activities that dealt with the flat projections of the physical surfaces; Phase 3 related to computer-based activities. *Cycles* referred to the iterative development of activities which covered one or more phases and were related to the overall development of the microworld.

Each cycle of the developmental process was concerned with the same set of issues relating to the technical, pedagogical and cognitive elements of the microworld. These are summarised in Table 5.1.

	Developments	Activities	Review
Technical	What changes are to be made to the: • software • non-computational elements ?	How are the: •software •non-computational elements to be used?	How effective are the changes to the: • software • non-computational elements?
Pedagogic	What changes are to be made?	What is to be achieved?	What effect did the changes have?
Cognitive	• What is understood about the cognitive processes so far? • What changes are there in expectations of cognitive development?	What is expected?	•What is understood? •How is it understood? •What assumptions are being made by the participants?

Table 5.1 Structure of each Developmental Cycle. Each iteration of the microworld's development had the same structure.
(a) Developments from previous cycle.
(b) Activities devised for the cycle.
(c) Review of the results.

The first column of Table 5.1 is concerned with a systematic description of developments in each of the three elements of the microworld's definition. These came either from previous iterations or, in the case of the first cycle, initial attempts to implement the technical and pedagogical aspects of the microworld. The next column contains a description of the issues relevant to activities of the cycle which, together with their rationales, formed the main focus of work with the participants. Finally, in the third column, the results of the cycle are analysed to indicate how the technical and pedagogic elements supported cognitive change in the participants.

The questions which formed the basis of the table reflected the need to identify clearly what was intended by each aspect of the cycle and to assess what happened. The technical and pedagogic issues were concerned mainly with analysing and implementing any changes to the structure and function of the respective microworld elements as a result of using the microworld. The three questions concerned with the cognitive outcomes of each cycle relate to the description and analysis of the participants' responses to the activities, using the synchonic view of the microworld. This will now be discussed in detail.

5.3 The Synchronic View: Analysing the Cognitive Element of a Cycle

The model outlined in §2.6 indicated that the synchronic view of the microworld was used in two ways. The first use was to identify the geometric meanings which were created in the pedagogic element of the microworld. The second use was to analyse the meanings created by the participants through observing what they said and how they said it, using the sign-action-surface combination identified in §2.5.2.

The synchronic view provided an analytic framework in which to assess the development of both the microworld and the participants' understanding of non-euclidean geometry. The structures of the microworld's technical element were combined with the synchronic view through the identification of the "technical" element of the first with the "surface" component of the second, so that the "surfaces" consist of physical object, flat plane, and the computer screen. Since each phase of the microworld's diachronic structure corresponded to the use of a particular surface, the meanings created by the participants of the microworld were identified, described, and analysed by considering the actions and language used with each surface.

Using the categories of the sign-action-surface combination, it was possible to chart how the participants' understanding developed, both within and across each phase, in answer to the three questions given in Table 5.1

- What was understood by the participants from the activity in which they were engaged?
- How was that understanding developed?
- What assumptions, intuitions, and prior knowledge, did the participants bring to bear?

The first question was explored by attending to what the participants said and did during the activities. The second issue was concerned with the genesis of that understanding. This entailed attending to the participants' sequences of action with the technical elements of the microworld and their linguistic behaviour. With these it was possible to document and analyse their developmental processes using the sign-action-surface structure. Finally, any assumptions made on the part of the participants about what they are seeing and doing was inferred from their actions and through discussion.

5.4 Overview of the Developmental Process

This section will give details of the timetable for the three developmental cycles of the microworld design and their relation to the phases of the microworld. Each cycle will be described briefly, giving its duration and date, together with a short account of its participants. Full details are given in subsequent Chapters.

5.4.1 Timetable of the Developmental Cycles

Cycle 1 took place between March 1992 and March 1993. The intention during this first cycle was to try out activities in all three phases. It was exploratory in nature, with the intention of finding out what was feasible in terms of activities. A prototype of the software was prepared to investigate two aspects of its operation. The first was the efficiency and accuracy of the algorithms developed to implement the equations of Turtle motion found in §3.4.5. The second aspect of the investigation was to explore the "user interface". In particular, the screen layout of the three components described in §3.3 (The Listener, Graphics area, and Turtle selection "pad") was tested for its ease of use and visibility. These will be described in detail in Chapter 6.

Cycle 2 took place between April 1993 - March 1994. During this cycle, the intention was to develop those activities from Cycle 1 which were found to be successful in terms of the aims of the study. Other activities were added and modifications to the software were made as a result of issues raised by the first cycle. However, there were still problems with the software's speed and accuracy. Cycle 2 covered Phase 1, Phase 2 and some of Phase 3. A detailed account of the Cycle is given in Chapter 7.

Cycle 3 covers the period from April 1994 to March 1995. Again, activities from Cycle 2 were used and others introduced. Problems to do with the speed and accuracy of the software were resolved and the main focus of the Cycle was on Phase 3 activities connected with hyperbolic geometry and its Conformal model. For that reason, the cycle was concerned with Phase 2 and Phase 3. The Cycle is described in Chapter 8.

The relationship between the phases and cycles are summarised in Figure 5.1.

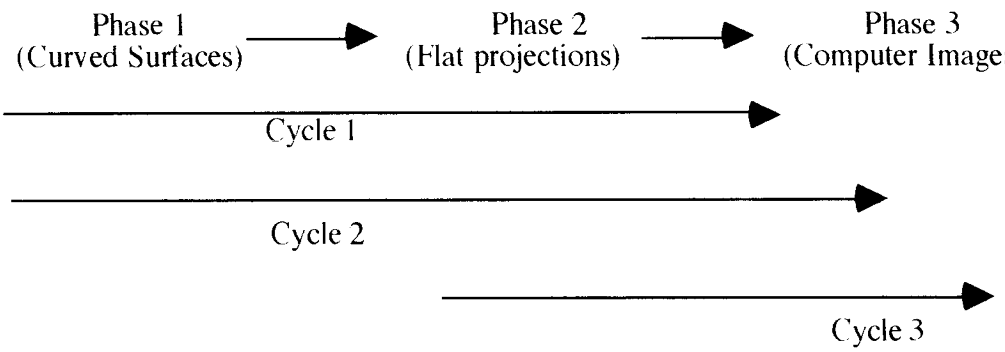

Figure 5.1 Relationship between Phases and Cycles. Cycle 1 was concerned with phases 1, 2 and some of phase 3. Cycle 2 focused on all three phases. Cycle 3 dealt only with phases 2 and 3

5.4.2 Participants in The Cycles

This section gives details of those who participated in the three developmental cycles. Each set of participants is described and the amount of time that they spent on activities with the microworld is reported.

5.4.2(a) Participants in Cycle 1

Two volunteer sets of pairs were used. They were mixed sex and came from the same course; a one-year pre-service course for intending secondary mathematics teachers. All the volunteers were mathematics graduates.

Pair A: *Bill & Christine*

Both Bill and Christine, in the age range 23-35, had some experience of Logo as a result of their course. They spent about five hours working on activities, but did not do any work with Object LOGO due to constraints on their time.

Pair B: *Erica & Michael*

Erica and Michael, in the age range 35-45, had LOGO experience from their course and both had knowledge of non-euclidean geometry, although this only emerged during the activities. They spent about five hours on the activities, including some work with Object Logo. Again, time constraints meant that they were unable to do more.

Timetable of the Participants

Table 5.2 gives the approximate time spent on each phase by the participant pairs. Pair A's work on the second phase was split into two sessions on the same day, but all Pairs B's sessions were continuous.

	Phase 1	Phase 2	Phase 3
Pair A	2 hours	3 hours	0 hours
Pair B	1.5 hours	2 hours	1.5 hour

Table 5.2 Timetable for the Participants in Cycle 1

5.4.2(b) Participants in Cycle 2

It was decided to continue with pairs, as they seemed to produce discussion. However, the first cycle had used mixed-sex pairs and this had not been successful since one person (usually male) tended to dominate, so it was decided to use same-sex pairs of volunteers in this cycle.

Pair C: *Sarah and Anne*

Sarah and Anne were two females in age range 25 - 30. Both were following the same four-year honours course leading to qualification as teachers in the Primary age range. As part of their degree, Sarah and Anne had to study an academic subject and they had chosen to specialise in Mathematics. They had some LOGO experience prior to using the microworld, obtained as part of both their academic and curriculum studies.

Pair D: *Tim and Steve*

Tim and Steve were two males in age range 30 - 45. Both were following the same two-year shortened degree course in Mathematics and were planning to teach mathematics in the secondary age range. Steve had some experience of non-euclidean geometry as part of his qualifying studies for the course. Like Pair C, Tim and Steve had some Logo experience prior to using the microworld, obtained as part of their studies.

Pair E: *Paul and Sean*

Paul and Sean were two males in age range 25 - 40. They were following the same degree course as Pair D and, like them, had some Logo experience prior to using the microworld, obtained as part of their studies. Paul had been a professional photographer prior to starting the course and Sean had a degree in Physics.

Timetable

Table 5.3 gives details of the time spent by the pairs on each phase. SG and HG refer to Spherical Geometry and Hyperbolic Geometry respectively.

	Pair C	Pair D	Pair E
Phase 1	1.5 hours	1.5 hours	1.5 hours
Phase 2 (SG)	1 hour	3.5 hours	3.5 hours
Phase 2 (HG)	1 hour	2.5 hours	2.5 hours
Phase 3		1 hour	1 hour

Table 5.3 Involvement of the Participants with the Microworld during Cycle 2

Due to pressure of other commitments, Pair C were not able to complete all three phases and there was a gap of several months between the first session in December 1993 and the other two in the following April 1994. Pairs D and E, on the other hand, were able to do all three phases on a regular basis in a continuous block from January to March 1994.

5.4.2 (c)Participants in Cycle 3

Two pairs of participants were used in this cycle, both of whom had been involved in Cycle 2. Pair D (Tom and Steve) spent one session of about one hour 45 minutes on the activities; Pair E (Paul and Sean) spent about 2 hours on the work in two sessions, one of 30 minutes and one of 1 hour and 30 minutes. Unfortunately, this was all the participants could spare due to the pressure of their commitments.

It was decided to use the same pairs for two reasons. The first was that since the objectives of the cycle were to introduce Turtle C and develop Phase 3 activities, it was desirable that the participants were familiar with at least some aspects of the microworld. This meant that it was not necessary to repeat the entire process of the microworld with them and the limited time available could be used effectively. The second reason was that it was exactly a year since their first encounter with the microworld and although their recollection of it would be vague, it would be sufficient for them to make comparisons with the software.

5.4.3 Data Collection.

Data was collected in three ways during each of the cycles. All sessions with the participants were taped, using a video camera and microphone. This provided a record of what they said, did, and produced on screen, during the activities. Edited transcripts of these sessions will be included in the following chapters. The participants' key strokes were saved using a dribble file obtained from the "Listener" window of the software. Second, any written material or drawings generated by the participants during their microworld sessions were retained. Third, extensive notes were kept about all aspects of the microworld's development over the period of the study.

5.5 Structure of Each Developmental Cycle

The three cycles of the microworld's development are described in Chapters 6, 7 and 8. Each chapter has the same structure, consisting of four sections:

- *Developmental*: concerned with changes to the technical and pedagogic elements in light of any previous cycle.
- *Activities*: concerned with describing the activities and their rationale used during the cycle.
- *Review:* concerned with analysing the performance of the technical element, the effectiveness of the pedagogic strategy, and describing the participants' responses to the activities.
- *Reflection:* concerned with highlighting those technical, pedagogic, and cognitive issues that are to be carried forward to the next cycle.

This reflects the iterative process which formed the basis of the microworld's development. Activities, which brought together the technical elements of the microworld with approaches to teaching non-euclidean geometry, were trialled with the microworld's participants. Their responses to the activities were analysed using the synchronic structure of sign-action-surface and these were used to generate new activities for the next cycle. The same structure for the microworld was maintained using its model, so that a stable framework was established within which the developments could be analysed.

Chapter 6

The First Developmental Cycle: Exploring the Terrain

6.0 Introduction

The first cycle of the microworld's development was intended to explore possibilities for activities and try out the technical and pedagogic elements. This chapter describes the process of preparation, the activities and their outcomes. It begins in §6.1 with a description of the preparations made for the activities of the cycle. These include the construction of the software which was to form the computer-based phase of the microworld's technical element and a re-iteration of the pedagogic objectives outlined in §3.6. Next in §6.2, the activities and their rationales are described. After a short account of the participants and the time they spent with the microworld in §6.3, the chapter goes on to describe and analyse the technical, pedagogic and cognitive outcomes of the cycle in §6.4. The conclusion, in §6.5, contains three issues which were carried forward to the next cycle.

6.1 Preparations for the Cycle

This section will describe the developments made prior to the implementation of the first set of activities with the microworld's participants. It consists of two parts which deal with the construction of the software and the pedagogical strategy respectively.

Designing the software was a major priority since, without it, the microworld would lose its rationale! Accordingly, the first part of this section describes the process of creating the three components which formed the computer-based part of the technical element of the microworld. It also provides a rationale for the choice of differential geometry and an object-oriented approach to implementing the software. These two aspects came together in the creation of a screen Turtle for Logo which could move according to either of the Conformal models or the normal euclidean model by changing the value of a single variable. The second component of the computer-based element of the microworld which will be described is the mechanism for selecting the geometry by which the Turtle would move. This consisted of a "screen pad" which could be used to select a particular Turtle and its geometry, using the mouse.

The pedagogical aspect of this section takes the form of a summary of the objectives, described in §3.6, which guided the construction of the activities for this cycle.

6.1.1 Technical Issues for Cycle 1

The structure of the software was determined by the need to create the three components described in §3.3.3 - §3.3.5: a Turtle and its graphics area; a "button pad" to select Turtles that moved according to one of the Conformal models or the euclidean model, and the Listener window to communicate with Object Logo. A description of the first two objects will be given in this section, since the function of the Listener window has been described in §3.3.4. The only variation of the Listener system object required by the software was to change its shape and position on the screen, therefore it does not warrant a separate description here

6.1.1 (a) Creating Objects

Creating objects in Logo followed the same procedure. First, the object to be created is identified either as an instance of a built-in system object (Turtle, Turtlewindow, Listener, Menu, Window) or as a "something". In the case of this software, all the objects created were variants of the systems objects through the use of the primitive command *kindof*. Next, a new class was created by asking the object *to exist* and this had two functions. The first was that the created object inherited all the properties of the system object through the *kindof* command. Second, a new object could be given properties of its own, including variables, procedures, or modifications of shape or colour. In the case of the Windows, for example, this meant that a window could be created with a specific shape, size, and position. Turtles, on the other hand, could be created with their own variables and procedures. The third step was to "shadow", if necessary, any of the usual

Logo commands. It was at this stage that the procedures to shadow FORWARD and BACK, needed to make them move according to the Conformal or euclidean models, were created for the new Turtle object. Finally, an instance of the new object class was made using the primitive *oneof* command. The process is summarised in Figure 6.1

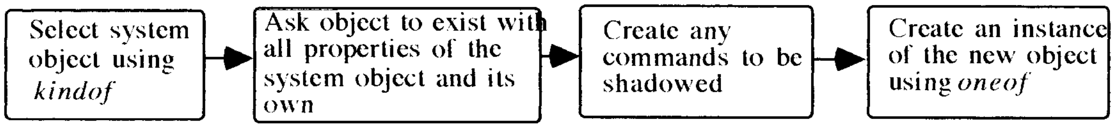

Figure 6.1 Process for creating objects in Object Logo. (a) select system object using *kindof*; (b) make the new class of objects exist; (c) give the new object class its own procedures; (d) create an instance of the new class.

6.1.1 (b) The Turtle Object

The four stage process for creating objects was used to create the Turtle Object which formed a central part of the software. By way of illustration, and because of its significance in the software, the creation of the Turtle object will be described in detail.

Stage 1: Select system object make "t kindof turtle

The object class being created was to be a variant of the Logo Turtle and so the first step was to select the system object Turtle as the basis of the new class. This was achieved using the *kindof* command

Stage 2 : Create new class of objects

```
ask :t [to exist]
usual.exist
have "k
have "x1
have "x2
have "dx1
have "dx2
havemake "scale 250
havemake "step 0.01
end
```

Each new object must be told to exist and so the second stage of object creation is to use the **exist** procedure to establish the new class of objects in Logo. This serves two purposes. The first is to give **:t**, the new Turtle, all the properties of the system object, Turtle, via the command *usual.exist*. The second function is served by the *have* command which introduces variables that **:t** alone can possess. In this case, three types of variables were created for the Turtle which were not available to any other object and they will be described in more detail later in the section. Communication with the new class **:t** is achieved by use of the command *ask*. This is the mechanism in Object Logo for communicating with objects by passing them messages .

As §3.4 and §3.5 indicate, the Conformal models and the equations of Turtle motion are based on the assumption that the Turtle will move in or around a unit circle. All calculations concerned with the Turtle's motion were done at the level of the unit circle and then scaled-up to fit the screen coordinates. The first set of variables assigned to the Turtle, therefore, were those containing its position and heading within the unit circle of each model. They were labelled (**:x1**, **:x2**) for position and (**:dx1**, **:dx2**) for heading. The second type of variable was that concerned with selecting the geometry of the Turtle and, in this case, it was called **:k** . As §3.5.3 indicates, the geometry induced on R^2 by the various forms of projection can be described using the metric:

$$ds^2 = \frac{4\ ds_e^2}{(1 + k(x_1^2 + x_2^2))^2}$$

- If k = 1, then the metric is that induced by projection of the sphere.
- If k = -1, then the induced metric is that obtained by projecting the hyperboloid.
- If k = 0 then $ds^2 = 4\ ds_e^2$, which is the usual euclidean distance measure.

The connection between this metric and the equations of the Turtle's motion, shown in §3.5.4, also indicates the importance of the variable :**k** in determining the Turtle's behaviour. It is at this stage that the power of the object-oriented paradigm and the differential geometric approach can be seen. By changing the value of a single variable, **:k**, which is owned only by the newly-created Turtle, one of three different geometries can be selected for the same Turtle. The third type of variable controlled the screen behaviour of the Turtle. These were the screen scaling, **:scale**, and **:step** for the step size used by the Euler algorithm to solve the equations of motion. The *havemake* command was used, which both created the variables and assigned specific values to them .

Stage 3 : Shadowing Commands.

Two commands, FORWARD and BACK, were to be shadowed. This meant that when the participants of the microworld typed either of the commands, the Turtle would move according to the geometry selected and not in its usual euclidean manner. Again, the capacity to implement this shadowing of commands in Object Logo made it very easy to create the type of behaviour which was required by the aims of the microworld. Initial contact with the Turtle by the microworld's participants, for example, could create the type of confusion required by the pedagogic strategy in the following way. As the participants instructed the Turtle to go forward with the usual Logo command of FORWARD or FD, they would find that the Turtle moved in unexpected ways. The resulting confusion could then be used to start the pedagogic process. Later, as the participants came to understand each of the Conformal models, they could use the shadowed commands to develop their own intuitions about non-euclidean geometry.

Implementing the Conformal models, which preserved angles but altered distances, required three procedures. These were called; **fd, direct** and **solve**. **fd** actually shadowed the FORWARD command, while the other two were subprocedures used by it. Their structures are described below, starting with **fd**.

fd worked like FORWARD in that when it was called, it was followed by a number which represented the number of steps to be taken by the Turtle. The procedure was divided into two parts. The first established the position of the Turtle within the Unit Circle for each Conformal model and gave values to the position variables (x1, x2) and the heading variables (dx1, dx2). The second part of the procedure used a REPEAT control structure to do four things. First, it called the procedure **direct** to calculate and return values for the coordinates of the Turtle's position after one step. The procedure's second function was then to translate the new coordinates into a length and direction for the Turtle. Third, **fd** moved the Turtle to the new position and, finally, saved the new coordinates for the next iteration of the REPEAT command. The process is summarised in Figure 6.2.

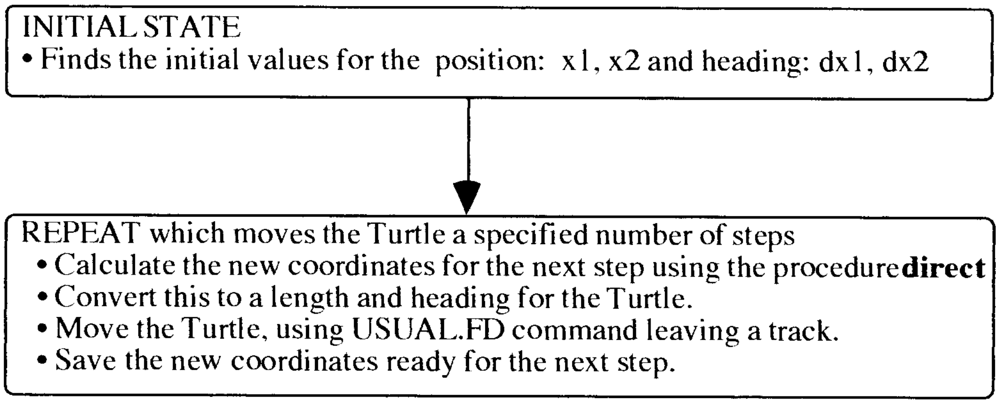

Figure 6.2 Structure of the **fd** or FORWARD procedure.

The procedure **direct,** referred to in the flow diagram, contains the equations of Turtle motion found in §3.5.6 and actually calculates the new values for x1, x2, dx1 and dx2. It consisted of two parts. The first part of the procedure used the equations of Turtle motion to calculate the second derivatives of the Turtle's coordinates in the Unit Circle, using the current values position and heading, (x1, x2) and (dx1, dx2) respectively. These were then passed to the procedure, **solve,** which implemented Euler's algorithm. Using this algorithm twice enabled the new position and heading to be calculated. Figure 6.3 summarises the process.

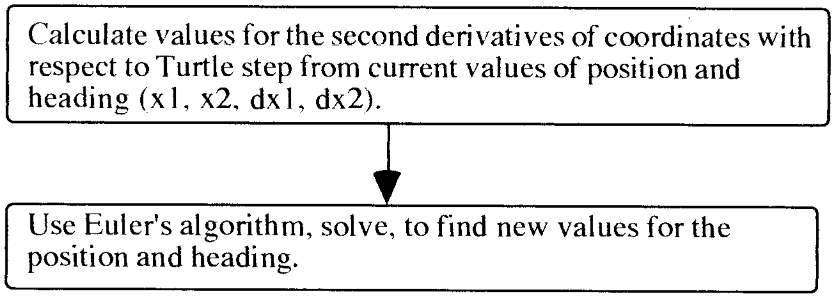

Figure 6.3 Structure of the procedure called **direct.**

Recalling §3.5.6, Euler's algorithm uses the first derivative as a linear approximation of a function to give an estimate for the value of function over a small region of its domain. It is a relatively simple procedure with three inputs which provides a new numerical estimate of the function. The procedure, **solve**, which implemented it, is shown below.

```
ask :t [ to solve :initial :rate :step]
op (:initial + :rate * :step )
end
```

The variable **:initial** is the current value of the function being approximated, **:rate** is the value of the function's first derivative at its current position and **:step** is the increase in the domain of the function. OP in **solve** stands for OUTPUT and is a standard Logo command which returns a numerical value to the procedure which called it. The procedure **direction** uses **solve** twice to find the new values for x1, x2, dx1 and dx2 from equations of Turtle motion.

Stage 4: Creating an instance make "t1 (oneof :t)

The *oneof* command here makes an instance, **:t1**, of the new Turtle class, **:t**, which can be used on its own Turtlewindow. Communication with **:t1** was then provided by the *ask* mechanism, as used in Stage 2. A typical message might be **ask :t1 [fd 20]** and this would move **:t1** twenty units forward, using its own version of the FORWARD command, in what ever model had been selected for it by setting the value of **:k**.

6.1.1 (c) Selecting Turtles

The selection of Turtles was made as simple as possible and it was decided to create a "button pad" so that a Turtle could be selected by pointing and clicking on a button with the mouse as shown in Figure 6.4

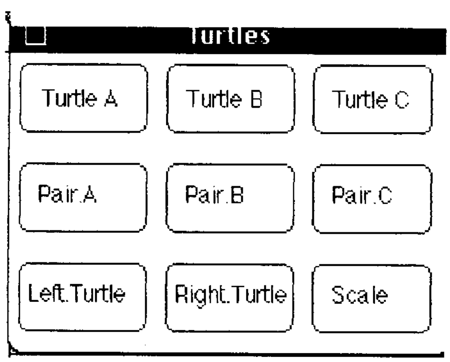

Figure 6.4 Turtle Selection Button Pad.

The top row of the pad contains the buttons to select one of the three possible Turtles and their respective geometries. The second row enables pairs of Turtles to be selected and the third row allows the position and heading of the left and right hand Turtles in each pair to be changed. The "Scale" button allows the scale of the screen to be varied and it also contained an option for changing the step size used in the algorithm for solving the equations. This enabled the user to experiment with the speed and accuracy of the Turtle's motion by varying the value of **:step** as shown in the procedure for Euler's method in the previous section.

The pad was created from the system object Turtlewindow. Each button could be selected by pointing and clicking the mouse through the use of a built-in Logo command, WINCLICK. Several screen layouts were tried so that the shape and size of the Turtle's drawing area and the button pad could be established. The software was tested by drawing polygons in each of the geometries and observing the large scale-behaviour of the Turtle.

6.1.2 Pedagogical Issues

The purpose of this section is to recap on the pedagogical objectives described in §4.5 as a precursor to describing how they were implemented in the activities of this cycle. The pedagogical element is divided into three phases according to the type of surface which was being used. Phase 1 focused on physical objects, such as spheres, and corresponded to the process of inducting the participants into non-euclidean geometry. Phase 2 used flat representations of curved surfaces and aimed at facilitating the progression from working objects to working with their two-dimensional models. The final phase was concerned with using the computerised version of the flat representations of curved surfaces. The objectives for each phase will be summarised from §3.6 as preparation for the design of the activities.

	Phase 1 Physical Objects	Phase 2 Plane	Phase 3 Computer
Technical	Solids with non-zero curvature and their geometry. e.g. sphere	Flat projections of the solids on paper to produce Conformal models. e.g. stereographic projection	Dynamic and interactive versions of the Conformal models on computer
Pedagogic	*Induction* into non-euclidean geometry using objects such as spheres and introduce a challenge to euclidean intuition	*Scaffolding* to aid the progression from objects to projections of surfaces	*Fading*. Activities which develop independent use of the computer-based models by participants
Objectives	•Establish a "base-line" in terms of the participants' understanding of non-euclidean geometry. •Explore with the participants specific aspects of geometry on the surface of curved objects. •Challenge the user's euclidean intuitions about Logo.	•Introduce the idea of a stereographic projection. •Check which geometric facts found in the first phase were preserved during projection.	•Relate the Logo screen images to the Conformal models obtained in the second phase. •Work with Logo to check geometric "facts" found with the projections. • Investigate the properties of both the Conformal models using the software.

Table 6.1 Pedagogical Objectives for Cycle 1.

6.2 Activities for the Cycle

The aim of the first cycle was to bring together the software, activities, and a pedagogic sequence, to explore what was possible. It was planned to work with pairs of participants, since this had proved effective for collection of qualitative data from small groups during previous studies (Stevenson 1990). By encouraging the pairs to talk with one another, it was hoped to gain access to their reasoning and understanding of the microworld.

A large number of activities were planned in an attempt to find those which might be useful. A preliminary outline for the pedagogic sequence was drawn up from the diachronic view, which involved the pairs in a number of computational and non-computational activities. These activities were to begin by exploring the language used about curvature and were to be followed by some "ordinary" Logo experience. Here the participants were to be asked to explore ideas of curvature using Logo. The general process was then to move gradually from working with objects to working with screen images. The intention was both to familiarise the pairs with the geometry of surfaces having non-zero curvature and to introduce them to the images of the geometries when they were projected on to a flat surface. After the Logo activities, therefore, they would work with spheres in establishing some facts about geometry on its surface and how this related to flat projections. Hyperbolic geometry would then be investigated using a print by M.C. Esher, shown in §6.2.3(b) below, based on Conformal model B. Finally the pairs were to be introduced to the software and asked to investigate it. The intention was for them to relate the non-computational actives to the Logo images.

Five sets of activities were tried during this cycle and the approach to them was both exploratory and experimental in an attempt to "discover the terrain". They will be described with their rationales and protocols.

6.2.1 Absent Friends

The aim of this activity was to investigate how curvature is perceived in everyday objects and is communicated to others. Based on an activity of the same name reported in Hershkowitz (1990 p.76), *Absent Friends* was devised to examine the vocabulary associated with curvature. This was thought to be useful in establishing what a common sense view of curvature might be, as a kind of reference point for subsequent discussions. It was also thought to be useful for gauging the development of individuals' intuitions as they used the other aspects of the microworld.

The protocol of the activity was as follows. Working in pairs, the participants were to be asked to sit back-to-back. One of the pair was to be given an object and asked to describe it to their partner. The partner had to make a drawing of the object from the description given. Details could be queried and questions asked, but the drawer could not look at the object being described. The activity was called "absent friend" as it might be a way in which a person would describe something to someone else over the telephone. It forced the person describing the object to translate their visual perception into a linguistic format.

A variety of shapes and objects were to be used from random 2-D doodling to everyday 3-D objects such as a port bottle, a hemisphere, and a coffee pot. The objects were to be selected either because they were "regular", such as the hemisphere, or because they were not.

6.2.2 Logo Activities

The aim of the activity was to explore the local and intrinsic definition of curvature which Turtle Graphics makes possible. Two procedures, derived from Chapter 1 of Abelson and diSessa (1980), were to be used to create circles and spirals.

```
to c :step :turn
fd :step
rt :turn
c :step :turn
end
```

```
to spi :step :turn
fd :step
rt :turn
spi :step+ 0.1 :turn
end
```

Procedure **c** draws a circle recursively by moving forward a small distance **:step** and then turning through an angle **:turn**. Procedure **spi** draws spirals by recursively increasing the length of **:step** at each step, while keeping **:turn** constant. Different values for **:step** and **:turn** created circles and spirals that had varying degrees of curvature. For example, **c 1 1**, created a circle by turning 1° after a step of 1. **c 0.5 0.5** produced an identical circle since in both cases "turn per step" = 1. However, **c 1 2** and **c 2 1** produced circles with different curvatures, since the "turn per step" = 2 for **c 1 2** and "turn per step" = 0.5 in the case of **c 2 1**.

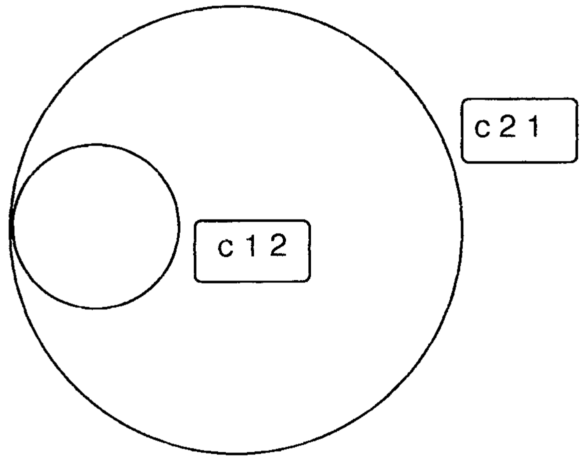

Figure 6.5 Circles procedure **c**. **c 1 2** is the circle obtained by using the procedure **c** with a step of 1 and a turn of 2° after each step. **c 2 1** has a step of 2 and a turn of 1°.

The **spi** procedure produces spirals by modifying the length of each **:step** each time the procedure was called for a fixed value of **:turn**. Hence **spi 1 10** gives a spiral in which the Turtle turns 10° at each step, and the step increases by 0.1 each time.

The intention was for the participant pairs to reach a local and intrinsic definition of curvature in terms of Turtle "turn per step" by experimenting with the values in the given procedures and also by modifying the procedures as they thought appropriate.

6.2.3 Non-euclidean Geometry

The overall strategy of the microworld was to move the participant pairs from working with objects to working with projections of the objects, ending, finally, with computer images of the projections. At the same time, it was intended that the developments in the participants' understanding of non-euclidean geometries would also be charted. For this to be achieved, some kind of reference point had to be established for the participants' understanding of both spherical and hyperbolic geometry and two sets of activities were developed to facilitate this so that comparisons could be made. In particular, three aspects of each geometry were to be investigated: what "straight line" meant in each geometry; what the angle sum of a triangle was in each and what "parallel" meant in spherical and hyperbolic geometry.

There was, however, one practical difficulty for hyperbolic geometry. There was no "surface" that could be given to the participants in this case. During this first cycle there was only the model, Conformal model B, available in the form of the Escher Prints and grid, shown in §6.2.3(b) below. The work with "objects", required by the overall methodology, was restricted to spheres. This was thought at the time to be useful in the following way. Since Conformal model A for the sphere could be shown directly as the result of stereographic projection, contrasting it with the "model" provided by Conformal model B would be interesting for two reasons. First, there was the issue raised by Reichenbach's assertion that by "adjusting oneself" it was possible to make sense of non-euclidean structures. Confronting the participants of the microworld directly with Conformal Model B could give some insight into how they understood hyperbolic geometry purely through a model. It also could provide a reference point against which to assess their progress, or otherwise, in developing their understanding of hyperbolic geometry. Secondly, by contrasting this with the approach based on stereographic projection for spheres, it might be possible to examine whether a less "direct" introduction to the Conformal models was effective in developing understanding. The protocol for the activity had to be split, therefore, to take account of the different approaches to introducing the geometries. In both cases there was first to be a discussion of the meaning of "straight line" in each geometry.

6.2.3 (a) Spherical Geometry

For spherical geometry, the participants were to be given spheres, elastic bands, and string, and asked to apply their definition of a straight line to the surface using them or any other means they thought appropriate. Having decided on what constituted a straight line, they were to be asked to form triangles on the sphere using straight lines and find their angle sum. The activity was to finish with a discussion of what "parallel" meant in these circumstances.

The next part of the activity for spherical geometry was to consist of an introduction to stereographic projection and what happens to "straight lines" on the sphere when the lines are projected onto the plane. Figure 6.6 shows the diagram which was to be used in the activity to illustrate Stereographic Projection. Point q, on the sphere, is the projection point. Each point on the surface of the sphere is projected onto the plane through the "equator" by drawing a line from q through the point onto the plane. "Great circles" which do not pass through q are projected as circles on the equatorial plane. Those "great circles" which do pass through q become straight lines on the plane. A relatively straight-forward process of projection enables lines marked out on the sphere to be traced on the plane.

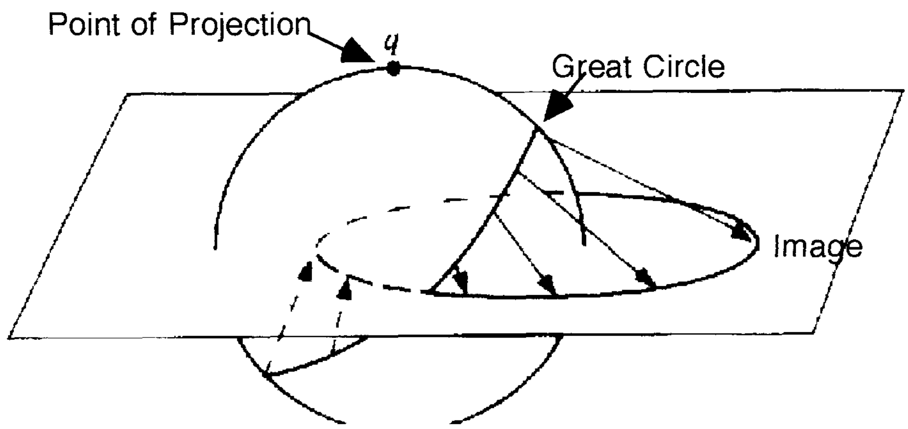

Figure 6.6 Steregraphic Projection. of the Sphere. Point q, is the projection point. "Great circles" which do not pass through q are projected as circles on the equatorial plane. "Great circles" which do pass through q become straight lines on the plane.

The stereographic projection was to be used to introduce Conformal Model A and to relate the geometric properties on the sphere to those of the model. In particular, two aspects of the projection were important. The first was the fact that congruence was not preserved by the projection and the second was that angle measures were preserved. The cognitive aspects of this were to be investigated by asking the participants first to check the facts of the geometry they had found on the sphere surface against the projected images of lines and triangles in the model. Second, they were to be asked to describe the model in their own words.

6.2.3 (b)Hyperbolic Geometry

Since the approach for this geometry was to be different, the participants were presented with an Escher print to illustrate the Conformal model B for hyperbolic geometry.

IMAGE REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 6.7 Circle Limit I by M.C. Esher. This is a tessellation using Conformal model B

IMAGE REDACTED DUE TO THIRD PARTY RIGHTS OR OTHER LEGAL ISSUES

Figure 6.8 Tessellation Grid for the Escher Print

The pairs were to be asked to look at Figure 6.7, while the researcher told them about its counter-intuitive aspects. Their reaction would then be discussed. Following this, the grid in Figure 6.8, which could have been used to form the basis of the print's tessellation, would be introduced. Both are taken from Hemmings (1985). The pairs were asked to explore the geometry of the grid in light of what they had been told. In particular, their attention was directed to the angle sum of triangles and the properties of parallel lines.

6.2.4 Computer-Based Activities

The desired outcome for the microworld was for the participants to use the objects and printed images as scaffolding to enable themselves to work entirely with the screen images of the Conformal models. Hence, the activities described in §6.2.3 were designed to familiarise the participants with the Conformal Models so that they could connect the behaviour of the screen Turtle with the models they had met in the activities. They were to be asked to use the software in checking and extending their geometric findings, and encouraged to make and investigate their own conjectures about each of the non-euclidean geometries.

6.2.4 (a) Screen Layout

The computer screen was designed so that the participants had access to three Turtles and could select any one of them using the mouse. A sample of the screen is shown in Figure 6.9. Three windows were available to the learner. The "Surface" window was where each of the three Turtles drew. The "Turtle" window allowed the learner to select either of the three Turtles. The "Listener" was where commands were entered and any non-graphic output was shown. Turtle A behaved according to Conformal model A, Turtle C used Conformal model B and Turtle B was the "usual" screen Turtle.

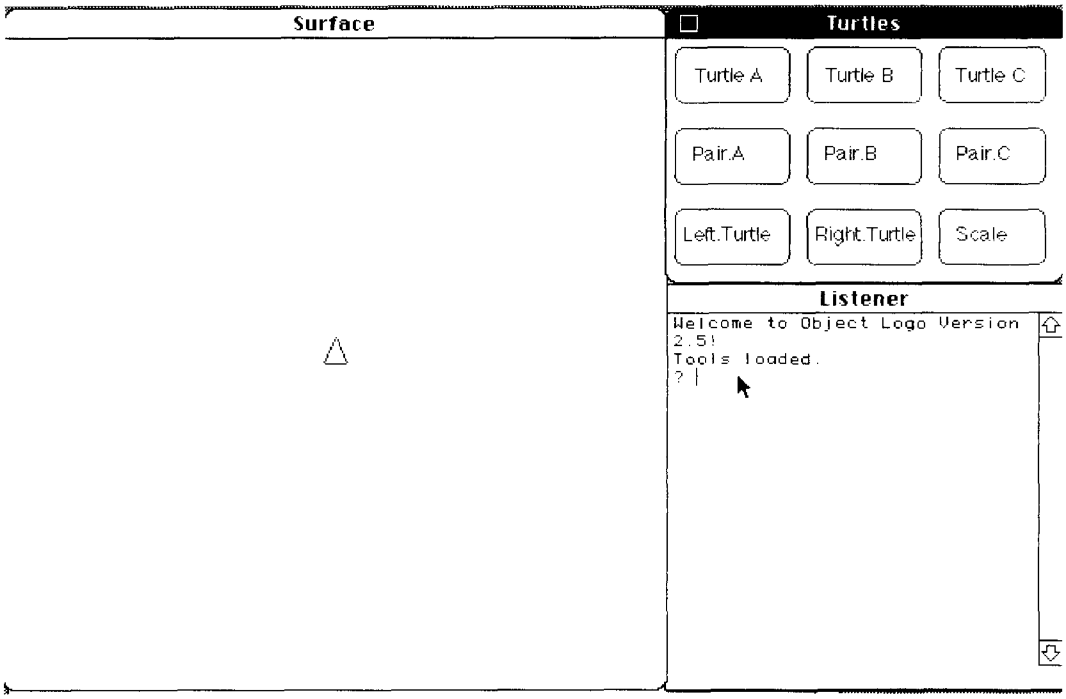

Figure 6.9 Layout of the Computer Screen.
(i) "Surface" window is for Turtles Graphics.
(ii) "Turtle" window allows the user to selectTurtles.
(iii) "Listener" allows commands to be entered.

6.2.4 (b) Breakdown of Euclidean Intuitions

Building on the idea of "breakdown" described in §4.6.1, the participants were first to be asked to investigate what each Turtle did, without being given any support. The first aim was to challenge their euclidean intuitions through the "odd" actions of the Turtles. The second intention was to help the participants connect the behaviour of the Turtles with the Conformal models. Hence the procedure outlined in §4.6.1 could be developed, in the sense that the participants were initially confused by the Turtle behaviour. Resolving the confusion and guiding the participants to an understanding of the images was the chief pedagogic aim of the computer-based work in the microworld.

The process of resolving confusion and guiding the participants to work with the computer images was to be achieved through guided discovery. Using the scaffolding of stereographic projected images and the model of hyperbolic geometry, the participants were to be encouraged to make the connection between the behaviour of the screen Turtle and the Conformal models. Through a combination of discussion, and activities, which reproduced on the computer the geometric properties found in the previous activities using the sphere and grid of Conformal model B, the participants were to be guided to an understanding of the software.

6.3 Reviewing the Cycle

The central concern in assessing the outcomes of the activities was to decide how they contributed to the development of a coherent framework for the microworld. Being exploratory in nature, the first cycle contained initial attempts at finding a combination of activities (computational and non-computational) which would meet the two aims of the study. Recalling the aims of the study, two aspects were important. First, the microworld was to provide a context in which participants could develop their understanding of non-euclidean geometry. Second, the microworld activities were to facilitate access to the processes by which the participants came to their understanding of its epistemological base. Bearing these two concerns in mind, specific issues for the cycle will be discussed using the three categories of the microworld's diachronic view: technical, pedagogic and cognitive.

6.3.1 Technical Issues of the Cycle

Two interconnected issues emerged in relation to the software: speed and accuracy. The Euler algorithm, adopted to solve the geodesic equations needed to move the Turtle, was relatively simple (cf. §3.4.5). It relied on one parameter, called **:step**, to control its accuracy. Reducing the value of **:step** produced a more accurate motion for the Turtles, but it also gave a slower screen plot. The two Conformal models implemented in the software were position sensitive. This showed itself in two ways. Close to the centre of the screen, both Turtles A and C (Conformal models A and B) behaved in a reasonable manner in the sense that closed shapes could be drawn. Reversing the Turtle, so that it re-traced its steps, also returned it more or less to where it started. However, further out from the centre of the screen both Turtle A and C behaved in unpredictable ways and these will be described.

6.3.1(a) Turtle A

Turtle A had inaccuracies when it was used in the large scale. For example, if Turtle A was moved from the centre by say 50 steps, turned thorough 90° and made to go forward, it should have returned to its original position, completing a circle. This corresponds to a circuit of the sphere projected onto the plane. However, because of software inaccuracies, the circle did not close with Turtle A, as shown in Figure 6.10

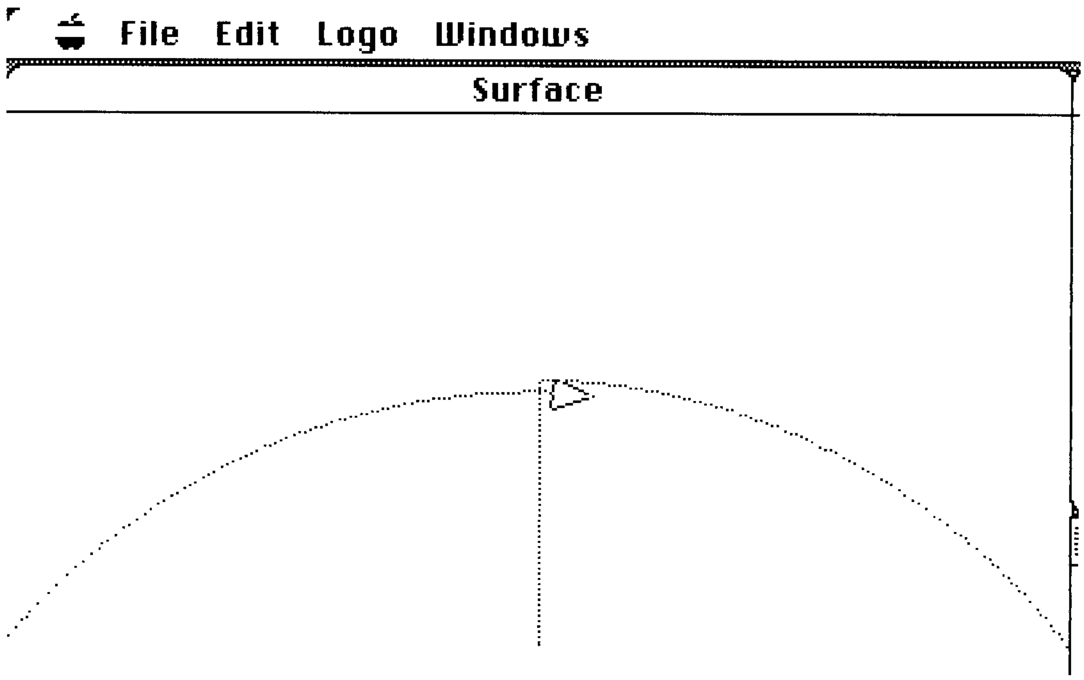

Figure 6.10 Errors in Turtle A's large scale behaviour. Turtle A should "close" the circle, but cumulative inaccuracies gave a spiral.

This could be modified by changing the **:step** parameter mentioned above, but at the expense of speed. Turtle A moved very slowly.

6.3.1 (b) Turtle C

Turtle C was erratic near the "boundary" of its model. This showed itself in two ways. First, if Turtle C was moved when it was very close to its boundary, it tended to rotate around the point at which it had settled and did not move forward at all. Second, if Turtle C was at the centre of the screen and was moved towards the Boundary, it tended to slow and "hover" as it got close to the edge of the model. This was expected behaviour. However, if the Turtle was returned to the centre from its position near the Boundary, it overshot and produced quite unpredictable behaviour.

Both these large-scale errors came to light during the first cycle, because the large-scale behaviour was not tested when the software was first written. Since it was intended to concentrate on the geometric properties of angle sum and parallel lines, the large-scale behaviour was not thoroughly tested. It was only when Pair B started to investigate the Turtles in an unstructured way that the problems came to light. They chose to examine the large-scale behaviour of the Turtles, rather than draw closed shapes. They seemed intrigued by the Turtle's erratic behaviour and investigated it. However, this posed important issues for the computer-based element of the microworld. The Turtle's motion was counter-intuitive, but it had to be "correctly" counter-intuitive for the user to come to understand what was happening. A compromise had to be found so that the Turtle's motion was fast enough to be interesting and accurate enough to be useful. Some experimentation with both the algorithm and the value of **:step** was necessary.

6.3.2 Pedagogical Issues for Cycle

The implications of the cycle for the pedagogic element fall into two categories. The first is a review of the basic strategy of guided discovery learning. The second is a consideration of the activities tried during the cycle and the selection of those which were to be used in the second cycle.

6.3.2 (a) Reviewing the Pedagogic Strategy

As §2.4.2 indicates, the pedagogical approach planned for the microworld was that of "guided discovery", with the researcher "teaching" through structuring the environment in an appropriate way. Pairs would be given activities which directed their attention in certain directions and the researcher would give informal "encouragement" to re-enforce particular lines of enquiry. This would be coupled with periods of reflection to consolidate the pair's understanding of the topics being investigated. However, it became clear during the first cycle that this was inadequate. Some of the difficulties are illustrated in the following extract from video transcripts. It involved Pair A while they were working with the Escher Print and Grid shown in §6.2.3(b).

After being introduced to the Escher print and using the grid as a possible template for the patterns, Pair A were asked to regard themselves as "living in the surface", so that the circumference of the circle was infinitely far away. Further, they were told that the apparently curved lines on the grid were in fact straight and the same length in the surface. B is Bill, C is Christine and R is the Researcher.

Extract A

B: So you want comments?!!

R: Yes... What are you feeling at the moment?

C: I think you're lying!

R: I'm lying! Ok. Why do you think I'm lying?

C: Cos I can't imagine all those being the same size and two-d.

R: Why can't you imagine it?

C: If I knew that I'd be able to imagine it!

R: Let's explore that a little bit. When you say "imagine", what do you mean?

C: I suppose I can't think of a situation where that's two-d and all the same size and, because you're so used to looking at things in three-d, and things get smaller, like a sphere as you go towards the edge or something..its hard to get into your head that that's not three-d. Therefore I can't imagine it.

Bill and Christine carry on in some confusion, trying to make sense of the grid. They were unable to reconcile the finite model in three-dimensions with the notion of the infinite plane, if one is "in the surface".

Extract B

C : If you went out a certain distance, would you get some kind of shape? If you make a cut-off point? .. could you make a cut-off point?

R: Explain that to me.

C: I don't really know ...I'm trying to get information out of you.

Christine and Bill's confusion about the grid seemed to be at two levels. First, there was the perceptual level: the grid presented a visual puzzle. They were asked to operate in two-d when in fact they were observing the grid in three-d and they tried to interpret it in terms of spheres. This was a reasonable conjecture given that in the activity prior to this they had been using spheres and stereographic projection. The second level of confusion was the pedagogic setting of the problem. Bill and Christine were given a puzzling image and were being asked to make sense of it by the researcher. They assumed and, perhaps, were led to assume, that there was an answer because of the context in which they were being asked questions. They perhaps felt that there was a "correct answer" and were anxious to obtain it, but felt unable to provide a satisfactory account. Hence Christine's remark about the researcher "lying". (*Extract A* line 3).

More important was Christine's comment that she was "trying to get information out of you (researcher)" (*Extract B* line 5). Her strategy seemed to be as follows. There is an answer, but it is in the head of the researcher, and it is my job to get that information either by asking questions or by wearing down the researcher. This highlighted an important tension in the guided-discovery approach as used during this cycle. Awareness on the part of the participants that they were engaged in a pedagogical situation led them to form assumptions about their role and what the outcomes might be for them. Since they were not able to reconcile their perceptions of the grid with the statements being made about it, they made many conjectures in the hope of gaining a "clue" as to the answer from the researcher. The activity became one of "guess what's in the researcher's head" rather than "can we make sense of the image?".

What the extract did point to was the need to re-evaluate the pedagogic strategy for the microworld. The counter-intuitive nature of the grid and, consequently, the screen image, meant that a more formal instruction was needed to ensure that the user could make sense of what the grid meant and what the Turtle might do. Hence, guided-discovery learning would play a part, but it would be later in the process rather than sooner.

A guided-discovery approach failed for another reason: the activities were not well-defined. The approach relies upon the activities orienting the participants in certain directions, while leaving them a degree of freedom to follow their own lines of enquiry. There is a tension between direction and exploration which can be only be preserved by careful attention to the type of activities undertaken. It became apparent during this first cycle that the activities which had been devised for work on projections and models, both of which were crucial to the development of the microworld structure, lacked a clear focus. They required so much informal input by the researcher that the pairs were being taught rather than being guided.

This was illustrated most clearly in relation to the grid described in §6.2.3(b) above. Since this was a model rather than a projection, participants were told several things about the grid which conflicted with its appearance. They were asked to imagine themselves lying in the surface and told that the circumference of the circle which forms the boundary of the grid was "at infinity". "Straight" lines which formed the grid were both "straight" in the euclidean sense and also arcs of circles. Clearly this was a confusing set of images to contend with and the researcher was necessarily engaged in much informal "question and answer" with the participants, with the difficulties alluded to in the extract above. If the pedagogical strategy was to be guided discovery, then the activity with the grid, for example, had to be re-thought. An alternative was to allow other sorts of pedagogical approaches mixed with guided discovery as was deemed appropriate. Introducing stereographic projection and the Conformal model for hyperbolic geometry required a lot of explanation to ensure that the participants "got the point" sufficiently well to work with them in later activities. This implied that a more didactic strategy was needed at some times and a less formal approach at others. What the balance was to be could only be determined by further experimentation and careful attention to the activities used.

6.3.2 (b) Reviewing the Activities

These considerations suggested a reduction in the number of activities. A sharper focus and clear objectives in the activities were also needed to enable the participants to come to terms with the microworld and to use it effectively. Time was needed between the sessions with the microworld for the participants to reflect on what they had been doing. Time was also needed at the start of each session, devoted to re-capping what had been understood from the previous sessions.

The desired operational outcome for participants in the microworld was that they should be able to use the software tools to investigate non-euclidean geometry for themselves. To this end, the activities were judged according to how they facilitated both fluency in the use of geometric tools and the "openness to scrutiny" needed by the aims of the study. Three sets of activities were discarded, since they were considered interesting but unenlightening from the study's point of view. They were the three "Reference point" activities: "Absent Friends", Logo, and the initial geometry activities.

"Absent Friends" did provide a set of "ice-breaker" activities for both sets of participants, enabling them to become familiar with the circumstances, (including being recorded on video tape), and with one another. However, the episodes did not provide material which was relevant to the development of the microworld's methodology, since they did not illuminate any of the issues connected with learning non-euclidean geometry. It was decided not to use the activity in the next cycle.

Similar considerations led to the exclusion of the Logo activities in the next cycle. Their purpose was to establish a local and intrinsic definition of curvature in terms of "turn per Turtle step". Such activities require a balance between investigation and "leading" the participant to the desired result. In these activities, neither pair were able to make the connection for themselves and eventually they had to be told what they were looking for. Two consequences followed from this. The first was an awareness on the researcher's part that such "leading" was a difficult and subtle process, particularly when there was a very specific result to be obtained. In these circumstances, "guided discovery" was not appropriate. Second, on a more general level, the study's basic pedagogic strategy of guided discovery had to be re-assessed carefully.

Finally, the exercises in establishing "reference points" for the pairs' knowledge of non-euclidean geometry were assessed in light of the overall aim of the study. It was decided not to use activities in the next cycle for two reasons. First, all those participating in the first cycle had some knowledge of non-euclidean geometry, so that the activities were thought to be trivial. They were performed by the participants, therefore, for the sake of the researcher rather than as an exploration of the geometry. A consequence of this was that the participant pairs were trying to work out "what was in the researcher's head" rather than focus on the mathematics. Again, the more specific the point about the geometries, the more they tried to elicit a response from the researcher. Second, the spherical geometry activities were designed around a specific method for constructing straight lines using elastic bands, which for one pair (A) met with disapproval. Activities for hyperbolic geometry did not meet with much success, since the pairs had considerable difficulty in understanding the model and another approach had to be considered.

6.3.3 Cognitive Issues for the Cycle

The cognitive issues of the cycle were identified using the categories of the synchronic view described in §5.3. Three issues were considered to be important for the study in trying to assess the cognitive changes in the participants. The first was concerned with establishing what the participants actually understood as a result of engaging with the microworld. The second issue related to the way in which the participants had come to their understanding of non-euclidean geometry. Together, these two areas were crucial for the aims of the study, which sought to establish both the content and processes of cognition in the context of the microworld.

However, a third aspect which plays an important role in the development of cognition is the knowledge and experience of geometry which the participants bring to the microworld. They are not "blank sheets" on which the microworld may write, so to speak. Their prior knowledge and experience can affect their understanding in two ways. First, their general experience, both of geometry and of making sense of visual and tactile information, form a background to their encounter with the microworld. Second, they may have specific knowledge of some types of non-euclidean geometry and this must be accounted for in the assessment of cognitive outcomes.

The three issues described above will form the basis for this account of the cognitive outcomes. The first part of the section will present and analyse an extract from the transcript of Pair B's video to show the role of prior knowledge in the development of understanding. The second part will describe an episode in which one of Pair B comes to understand an aspect of non-euclidean geometry and analyses how this has come about using the surface-action-sign structure of the synchronic view, described in §5.3. The extracts of transcripts from the video will make reference not only to what is said by the participants, but also to their actions with the various elements of the microworld.

6.3.3 (a)The Role of Prior Experience in Understanding Conformal Model B

This extract illustrates the importance of knowledge that the participants brought to the activity and which helped them to make sense of the novel image they were presented with.

Erica and Michael were introduced to the Escher print shown in §6.2.3(b) above. They were asked to imagine that they were "in the surface, so that the circumference of the euclidean circle was infinitely far away and hence the "fish" shapes were all the same size. Michael experienced similar difficulties to Pair A, described in §6.4.2 (Extract A), above. He was initially unable to reconcile his three-dimensional perception of the image with the notion of living in the two-dimensional world. His reaction was interesting because he used his knowledge of three-dimensional solids to explain the image. Michael was discussing with R how the Escher print could not be the result of tessellating a sphere.

Extract C

M: It (*tessellation of the sphere)* wasn't that I had in mind......it was shapes that are all the same size but when viewed from a....a certain point off the surface (*points to the middle of the Print with a pen and then raises the pen to a point above the centre*) .....just trying to think, if you could .....(*points to the centre of the print again*). I was saying if you had a thing like a hyperboloid.

R: Urhmm.

M: With a circular cross-section and you were looking from a point at the centre of the hyperboloid.....At the join of the asymptotes, sorry,........ the asymptotic cone...then when you looked at the surface, if this thing was drawn on the surface (*indicating the "fish" pattern of the print*) the horizon would be infinitely far away wouldn't it ?(*indicating the circumference of the print*) which would be the rim here (*moving pen around the circumference of the print*) If you could imagine that this is a curved ermm (*forms hands into a cone and moves them down towards the print*) this was the nearest point (*indicating the apex of the cone in relation to his eye*) and it curves away from you if you are looking down from above ....and this thing (*indicating the print*) is asymptotic to a cone, vertex at your line of sight...err your point of view.

E: What you're saying is that this (*indicating the centre of the print with a pen*) is the top of the cone...

M: No that's the point ......that's the perigee on the surface.

R: Would you like some paper?

M: Looking at the ......(*draws two lines at an angle*) If you imagine a cross-section of a cone, like that, where the cone is wrapped around and you've got your hyperboloid like this (*draws in an hyperbola between the lines*), you are looking at it from above and so you'd be looking at the rim. When you look towards the edge...this point is this point here (*M points first to the centre of the print and then to the turning point of the hyperbola on the diagram he has drawn*)...your pattern is drawn on that surface, so as you look towards the horizon you're getting closer and closer to the asymptote but never actually touching it.

Michael proposed that the two-dimensional Escher print could be accounted for in terms of a three-dimensional surface: a hyperboloid. Figure 6.11 shows the arrangement he described:

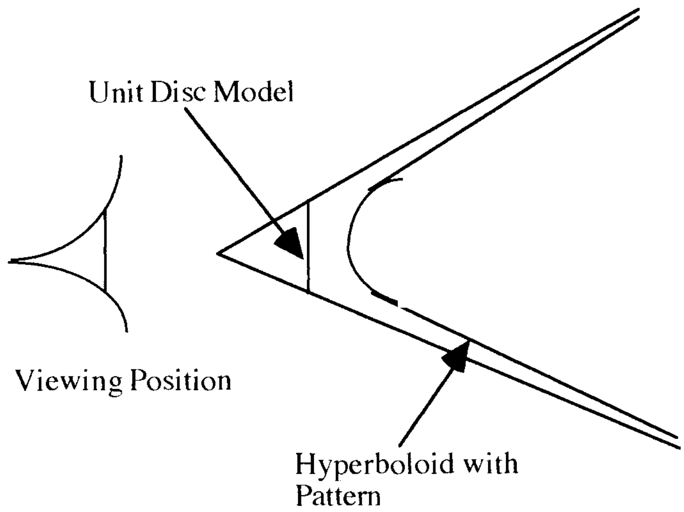

Figure 6.11 Michael's Model of Conformal model B as a Projection.

The pattern shown in the Escher print tessellated the hyperboloid. Viewing the three-dimensional surface so that one's eye is close to the apex of the cone, which forms the asymptotic boundary of the solid, produces the two-dimensional print.

This extract was interesting for two reasons. First, Michael was essentially correct in his description. As §3.4.3(b) shows, Conformal Model B, which forms the basis of the Escher print, is the result of a projection from a hyperboloid.

The second point of interest was the way in which Michael used his own understanding of three-dimensional surfaces to account for the two-dimensional image. There were no "clues" given by the researcher. Pair B were presented with a two-dimensional image and asked to interpret it. Michael tried initially to relate it to a sphere, which seemed reasonable given that the activity prior to that with the Escher Print had been concerned with stereographic projection. However, the juxtaposition of the activity with one concerned with projections of surfaces may have influenced

Michael in that the previous activity opened the possibility that the Print was related to a three-dimensional surface.

What is clear is the way he gradually developed an understanding of what the surface might be. Just prior to this extract, he noted that the "crowding" at the edges of the Escher print might be the result of viewing an asymptotic surface and that the surface must be convex and "gradually sloping away". Putting together these perceptions, he able was able to integrate them into a coherent structure by referring it to a surface that he understood, as lines 1-5 indicate. He interpreted the Escher print as the view obtained by looking at a surface which was sloping away from the eye and suggested a possible surface, the hyperboloid (*Extract C* line 5). From there, he describes the situation (depicted in Figure 6.11 above) by a combination of diagrams (lines 22-30), gestures (shown in italics at lines 10-14) and mathematical language such as "cones" (lines 22-23) and "asymptotes" (line 28).

The identification of the Escher print as the projection of a known mathematical surface by Michael was a real achievement. It did not appear to be something that Michael knew already, but he was able to use both his geometric knowledge and ability to visualise three-dimensional objects in drawing the necessary conclusion.

6.3.3 (b) Developing an Understanding of Conformal Model B

The second extract from Erica and Michael's work with the Escher print relates to the way in which Michael came to understand the logic of the model's geometry. In particular, it illustrates how he resolved the conflict between what he saw and what he had been told about the print.

Michael had been discussing the Escher print with Erica. She had had experience of the model before, as part of her degree, and so she was quite happy with the apparent visual contradictions of the print. As the extract begins, she has been trying to explain the fact that every point in the model is meant to be infinitely distant from the circumference of the circle which bounds the print, and thus the "centre" of the print is an illusion.

Extract D

M: Maybe I'm trying to think of it in euclidean terms and shouldn't be.

R: What suggests to you that it isn't a euclidean object?

M: Ermm

E: Because if it was euclidean it would be definite.

R: What do you mean Erica?

E: Well, you have definite limitations..you know if it was in two-dimensions and you drew a circle that would be a circle!

R: Right. So something is going on here (*indicating the circumference of the print*)?

E: Yes

R: You are clearly confused by this Michael. What I'm interested to know is what's created the confusion.

M: Partly definitions I think.

R: What definitions?

M: Well you said that this point (*pointing with a pen to a point about a quarter of the way into the Escher print from the circumference*) you chose any point here. You said that an observer would see the same pattern.....If you were taking this as a plane shape ..you're representing this in a plane, but it's not meant to be a plane shape.

R: Why not?

M: You define this as being infinitely far away (*pointing to the boundary of the print*), from where?

R: From any other point.

M: Yes. so that ..so you would need something other than euclidean to define it.

R: Something other than euclidean....?

M: Geometry to define it.

R: What would this new geometry look like?

M: Sounds more like projective geometry.

Michael then goes on to describe his understanding of projective geometry . After this the researcher summarises what has been agreed

R: Ok, you said that it (*the print*) was non-euclidean and I said that this, the edge, was infinitely far away. What would you have to change for that to be the case? What would you have to modify?

M: You'd definitely modify your definition of radius, I suppose, errm.

R: So that's apparently a circle, but isn't. That's apparently the centre, but isn't. What would be happening as you moved further out? What's happening to the shapes?

E: They keep repeating themselves.

R: Are they the same size or do they get smaller or bigger?

M: They certainly appear to get smaller.

R: They appear to get smaller.

E: They are the same size here (*near the boundary*) as they are here (*pointing to the centre of the print*).

M: Yes.

R: So what are you having to do mentally to make sense of what I'm telling you and what you are seeing?

E: Just position yourself in another place and imagine that it looks like that (*again pointing to the centre of the print*).

R: When in fact it doesn't.

E: Not on paper it doesn't, but if you were there it would!

R: Right. So what would happen to your sense of distance as you move along here (*moves finger from centre of the print towards the circumference*)? What would you have to do to reconcile what you are told with what you see?

M: You'd need some gradually.......yes (*pause*). You'd need to say that if you were going to talk about distance that .....your perception of distance reduces as you ....I can define that (*to himself*).......(*pause*). Yes you are defining .....You could define the size of things as being inversely proportional to the radius.

The extract is interesting for two reasons. First, Michael comes to a conclusion which, in general terms, is correct. There was an inverse relationship between the size of the patterns which make up the Escher print and their distance from its centre. Michael did attempt briefly to define it precisely, but was not able to. However, he did draw the basic conclusion about the Conformal model which indicates the "position sensitive" nature of distance measures. His definition, which relates size to position, draws a crucial distinction between the distance measured and position in this model, and hence enables the model to work.

The second point of interest is Michael's progression to this relation. In the extract, he begins by noticing that he could not reconcile what he was seeing with euclidean geometry (line 12) and so he began to look for possible variations. He first tried to draw on his experience of geometry other than euclidean, which was projective geometry (line 27), but this was not adequate to explain what he saw. The starting point of his new understanding is at line 35, in which he recognised that the problem was something to do with the radius of the circle. The researcher asked about Michael's sense of distance as one moved along the radius (line 51) and Michael took up the distinction between radial measure of the euclidean and distance measure in the model. In lines 55-57 he notes that one's perception of distance reduces as one moves along the radius and this leads to a mathematical relationship between position in the circle and distance: distance is inversely related to position in relation to the centre of the circle.

The overall process began with confusion, which Michael tried to resolve by reference to his own experience of projective geometry. He moves on to an understanding of the print by making the distinction between distance and position, culminating in a mathematical relationship. This seems to exemplify the assertion of Reichenbach that in order "to adjust ourselves" to the model, we must lose euclidean congruence. Indeed, the model only works because of the introduction of a new form of distance measure, which varies according to position. It is also interesting to note the way in which Michael formulated his solution. He developed a mathematical relationship which accounted *logically* for what he saw and this enabled him to reconcile his perceptual difficulties.

6.4 Reflections on Cycle 1

These reflections are divided into two parts. The first, in §6.4.1, concern the cognitive issues of the cycle. The second, in §6.4.2, deal with those technical and pedagogic issues which were carried forward to the next cycle.

6.4.1 Cognitive Issues

The episodes described in §6.3.3(a) show the different roles that visual intuition can play in understanding images such as the Escher print. First, there was Michael's intuition about interpreting the Escher print as a projection of a curved surface. He was able to bring together several local features of an apparently ambiguous visual image and integrate them into a coherent projective structure, which related the image to a surface familiar to him. Such an inductive recognition strategy, which draws together aspects of an image into a global structure that can be used, in turn, to interpret those aspects as "local", relies on a number of processes. These include internal visualisation, as suggested in §4.2.1, where the individual draws on his or her own experience of imagery to aid their understanding. There is the individual's mathematical experience, projective geometry in Michael's case, which, although it is not the direct explanation for the image, conditions the type of interpretative structures which can be called upon to make sense of the image. Perhaps, also, there is the awareness that there is an answer, communicated by the pedagogical context of the work, which motivates the need to find a way of interpreting the Escher print.

The second way in which visual intuitions can play a part can be seen from the second episode described in §6.3.3(b). Paradoxically, here the intuitions initially hindered Michael in his efforts to understand the construction of the Escher print. In common with most people, his everyday experience was euclidean and this led to difficulties with the interpretation of the print. Michael was only able to reconcile his euclidean perception with his understanding of the print's structure by noting the variation of distance measure with position. This recognition was built on an understanding at logical and non-perceptual levels that the distance measure could vary with position, since it was clearly at odds with his normal experience and the extract charts this progress.

Related to this move to abstract relationships to resolve perceptual difficulties was the role of discussion in the process. In the first episode, Michael came to his recognition of the Escher print as a projection without much discussion. He "saw" the solution and communicated it to others without much prior dialogue. In the second episode, Michael gradually built up his understanding through observation of the Esher print, and question-and-answer with his partner and the researcher. The dialogue seemed to form a basis on which he could develop a counter-intuitive interpretation of the print and formulate the mathematical relationship between position and distance measure.

6.4.2 Technical and Pedagogical Issues for the Next Cycle.

The major technical concern was the speed and accuracy of the software, which proved to be very unsatisfactory in the first cycle. This needed to be investigated in great detail as, without the software, the study lost a major part of its rationale. The second technical issue was the lack of a suitable object to introduce the Escher prints.

From a pedagogical point of view, several activities were removed and the aims and objectives of the remainder needed to be defined more carefully. There was also a need to revise the pedagogic framework of the microworld. In particular, the pedagogic structure needed to be examined in order to define it more precisely and find appropriate teaching styles. There was also the question of how to introduce Conformal model B, since the Escher print and grid, on their own, were not satisfactory.

Chapter 7

The Second Developmental Cycle: Making a Map

7.0 Introduction

The second cycle of the microworld's development was intended to build on the outcomes of the first cycle in two ways. First, it was to take those activities which had been effective in the first cycle and develop them further. Second, it would try to improve and refine the technical, pedagogical, and cognitive elements of the microworld in light of the first cycle. In §7.1, there is an account of the technical and pedagogical developments that resulted from the first cycle, including changes to the software, and a revision of the pedagogic strategy. In the second section, §7.2, a new set of activities are proposed which take account of the results of the first cycle and the changes to the pedagogic strategy. §7.3 gives an account of the technical, pedagogic, and cognitive outcomes of the cycle. The conclusion, §7.4, reflects on the cycle and summarises the issues which were carried forward to the next cycle.

7.1 Developments from the Previous Cycle

This section describes the changes which were made to the technical and pedagogic components of the microworld as a result of the first cycle. The technical changes were concerned mainly with improvements to the software in an attempt to improve the speed and accuracy of the Turtle's motion. Central to these issues was the need to increase the efficiency of the algorithm which governed the equations of Turtle motion described in §3.4.5 and to improve the execution of the program by Object Logo. These twin aspects of improvement to the algorithm and "run-time" speed will be considered in §7.1.1. In the second part of this section, the changes to pedagogical strategy which were highlighted in the first cycle will be described. The central concern about this aspect of the microworld centred on the need to increase the variety of pedagogical approaches, so that other strategies besides guided-discovery could be identified and introduced into the activities of the second cycle.

7.1.1 Technical Developments in the Cycle

As indicated in the introduction to this section, the technical developments fell into two categories, concerned with changes to the algorithm for solving the equations of motion, on one hand, and increasing the speed of execution, on the other. Four specific topics will be described. They are the use of complex numbers in the equations of motion, changes to the initial conditions of the algorithm for solving the equations of motion, the increased use of the object paradigm in constructing the software, and the compilation of the code.

7.1.1 (a) The Use of Complex Numbers in the Equation of Turtle Motion

The central concern about the software which was carried forward from the first developmental cycle was that of the relationship between speed and accuracy of the Turtle's motion. The connection between the two follows from the use of Euler's algorithm for solving the equations of motion, consisting of a pair of second-order differential equations. As §3.4.5 shows, given a first-order differential equation such as $\frac{dx}{ds} = f(s,x)$, with initial conditions $X_0 = x$ and $S_0 = s$, the Euler method gives a "step-by-step" solution to the equation. The key factor in its efficiency is the size of the region in the solution's domain (Δs) over which the approximation is used: the smaller the values of Δs the greater the accuracy but the more iterations of the algorithm must be used.

From the point of view of this software, both the speed and accuracy depended on the value of Δs. If Δs was small, the Turtle produced very accurate motion, but slowly. If the size of Δs was increased, the Turtle moved in a quick but inaccurate manner. The entire approach, therefore, relied upon one parameter for its accuracy and efficiency, and a balance had to be found between them so that the software was accurate enough to be useful and fast enough to be interesting. Two approaches were investigated, with the first concentrating on simplifying the mathematics, so that the equations were less cumbersome, and the second focusing on the implementation of the algorithm and its initial conditions.

The first approach, revising the mathematics, led to a simplification based on writing the equation of motion in complex form. Recalling the equations of motion for the Turtle in §3.4.4

$$\frac{d^2x_1}{ds^2} = \frac{2k}{1 + k((x_1)^2 + (x_2)^2)}\left[x_1\left(\frac{dx_1}{ds}\right)^2 - x_1\left(\frac{dx_2}{ds}\right)^2 + 2x_2\frac{dx_1}{ds}\frac{dx_2}{ds}\right]$$

$$\ldots..(1)$$

$$\frac{d^2x_2}{ds^2} = \frac{2k}{1 + k((x_1)^2 + (x_2)^2)}\left[x_2\left(\frac{dx_2}{ds}\right)^2 - x_2\left(\frac{dx_1}{ds}\right)^2 + 2x_1\frac{dx_1}{ds}\frac{dx_2}{ds}\right]$$

It was noticed that the equations had a familiar structure and this became more apparent when expressed in complex form.

Let $z(s) = x_1(s) + i\,x_2(s)$, so that its conjugate is $\bar{z}(s) = x_1(s) - i\,x_2(s)$, where $x_1(s)$ and $x_2(s)$ are the coordinates relative to the Turtle Step s, then

$$\frac{dz}{ds} = \frac{dx_1}{ds} + i\frac{dx_2}{ds} \quad \text{and} \quad \frac{d^2z}{ds^2} = \frac{d^2x_1}{ds^2} + i\frac{d^2x_2}{ds^2}$$

The complex representation of the equations of motion was seen by considering:

$$|z|^2 = (x_1^2 + x_2^2) \quad \text{and} \quad \bar{z}\left(\frac{dz}{ds}\right)^2 = (x_1 - ix_2)\left(\frac{dx_1}{ds} + i\frac{dx_2}{ds}\right)^2$$

Expanding the second equation gave:

$$\bar{z}\left(\frac{dz}{ds}\right)^2 = (x_1 - ix_2)\left(\left(\frac{dx_1}{ds}\right)^2 - \left(\frac{dx_2}{ds}\right)^2 + 2i\frac{dx_1}{ds}\frac{dx_2}{ds}\right)$$

This had real part $\quad x_1\left(\frac{dx_1}{ds}\right)^2 - x_1\left(\frac{dx_2}{ds}\right)^2 + 2x_2\frac{dx_1}{ds}\frac{dx_2}{ds} \quad$ and

imaginary part $\quad x_2\left(\frac{dx_2}{ds}\right)^2 - x_2\left(\frac{dx_1}{ds}\right)^2 + 2x_1\frac{dx_1}{ds}\frac{dx_2}{ds}$

It followed that the original pair of equations (1) above could be written as a single equation:

$$\frac{d^2z}{ds^2} = \frac{2k\bar{z}}{(1 + k|z|^2)}\left(\frac{dz}{ds}\right)^2$$

This had two advantages. The first was that Object LOGO was able to work directly with complex numbers and so there was no need to implement a special code for complex arithmetic. The second reason was that it made the code for the section of the program which dealt with calculating the Turtle's motion more compact. A comparison from the two sections illustrates this.

The first extract of the code below shows the procedure **direct,** which calculated the new coordinates for position (x1, x2) and heading (dx1, dx2) from the turtle's current position and heading. The procedure makes use of the **make** command in order that the various formulae can be seen. The **solve** procedure implements the Euler algorithm.

```
ask :t [to direct]
local [den ndx1 ndx2 ]
make "den (2 * :k / (1 + :k * (:x1^2 + :x2^2)))
make "d2x1 :den * (:x1 * :dx1^2 - :x1 * :dx2^2 + 2 * :dx1 * :dx2 * :x2)
make "d2x2 :den * (2 * :dx1 * :dx2 * :x1 + :x2 * :dx2^2 - :x2 * :dx1^2)
make "ndx1 solve :dx1 :d2x1 :step
make "x1 solve :x1 :dx1 :step
make "ndx2 solve :dx2 :d2x2 :step
make "x2 solve :x2 :dx2 :step
make "dx1 :ndx1 / :len
make "dx2 :ndx2 / :len
op (se :x1 :x2)
end

ask :t [to solve :initial :rate :step]
op (:initial + :rate * :step)
end
```

By comparison, the complex version of the equations of Turtle motion in **direct** is more compact, in which the procedures **d2z** and **cong** return the value of the second derivative in complex form and the complex conjugate of any complex number. They are shown over the page.

```
ask :t [to direct]
local [ndz]
make "z solve :z  (solve :dz  d2z :step) :step
make "ndz solve :dz (d2z ) :step
make "dz :ndz / (abs :ndz)
end

ask :t [to d2z ]
op (2 * :k * (cong :z) * (:dz) ^ 2)/(1 + :k * (abs :z)^2 )
end


ask :t [to cong :z]
op  complex realpart :z minus imagpart :z
end
```

These provided a code which was both shorter and easier to read and it was felt that it might provide an improvement to the speed of the software's execution.

7.1.1(b) Initial Conditions of the Algorithm for Solving Equations of Motion

As §6.3.1 indicates, there were inaccuracies in the way that Turtles A and C behaved over the large scale and both produced erratic results. In the case of Turtle A, this meant that circular paths did not close (cf. §6.4.1(a) and Figure 6.10) as they should have. For Turtle C, the inaccuracies occurred as the Turtle moved towards the boundary of the unit circle (§6.4.1(b)).

To try and counteract these errors, a "unit speed" condition was imposed after each step of the algorithm for solving the geodesic equations. Hence, if the Turtle had position (x_1, x_2) with heading $\left(\frac{dx_1}{ds},\frac{dx_2}{ds}\right)$, then the heading was divided by its modulus to give a unit complex number in that same direction. This ensured that the algorithm had the same initial conditions at the start of each new step, in the sense that it began with a unit vector in the new direction of motion for the Turtle. As a result, the screen Turtle's motion was more uniform and accurate for motions near the centre of the screen.

Several tests were made to check for accuracy. These involved moving the Turtle and then reversing it to see at what point it would finish. With the unit speed condition, the construction of geometric shapes such as polygons was accurate in the sense that if the Turtle travelled 70 units along a particular path, reversing it returned it to its original position. Without this condition the Turtle's return journey was unpredictable and it certainly did not return to its starting position. However, it neither solved the problem of the Turtle's behaviour near the boundary of Conformal model A, nor the difficulties for long journeys in the Conformal model B, since the unit speed condition still did not give reliable results.

An alternative approach was to separate explicitly the Turtle's local behaviour from its global behaviour. It was possible in Object Logo to have access to the *Quick Draw* graphics facilities of the Apple Macintosh computer and these allowed geometric shapes to be created and manipulated very easily. The projected images of spherical and hyperbolic geometry were either straight lines or arcs of circles, whose equation could be determined from the Turtle's position and heading (see §3.3). It was possible, therefore, to create geometric images using *Quick Draw* , which described the large-scale behaviour of the Turtle without getting the Turtle to trace the path out completely. To implement this, a procedure called **path** was introduced, which sensed the type of geometry the Turtle was in and drew its large-scale path very rapidly from the Turtle's current position and heading. The advantage of this arrangement was that the Turtle could be used with the unit speed condition to give accurate descriptions of local geometry (angles in a triangle, methods of tessellation etc.) and its large-scale behaviour could then observed quickly and precisely by typing **path**. Figure 7.1 shows the sort of behaviour produced by the use of **path** in this case for Conformal model A. The Turtle is shown with a certain position and heading and the circle which passes through the Turtle's position indicates the path that it will travel on if it continued its motion.

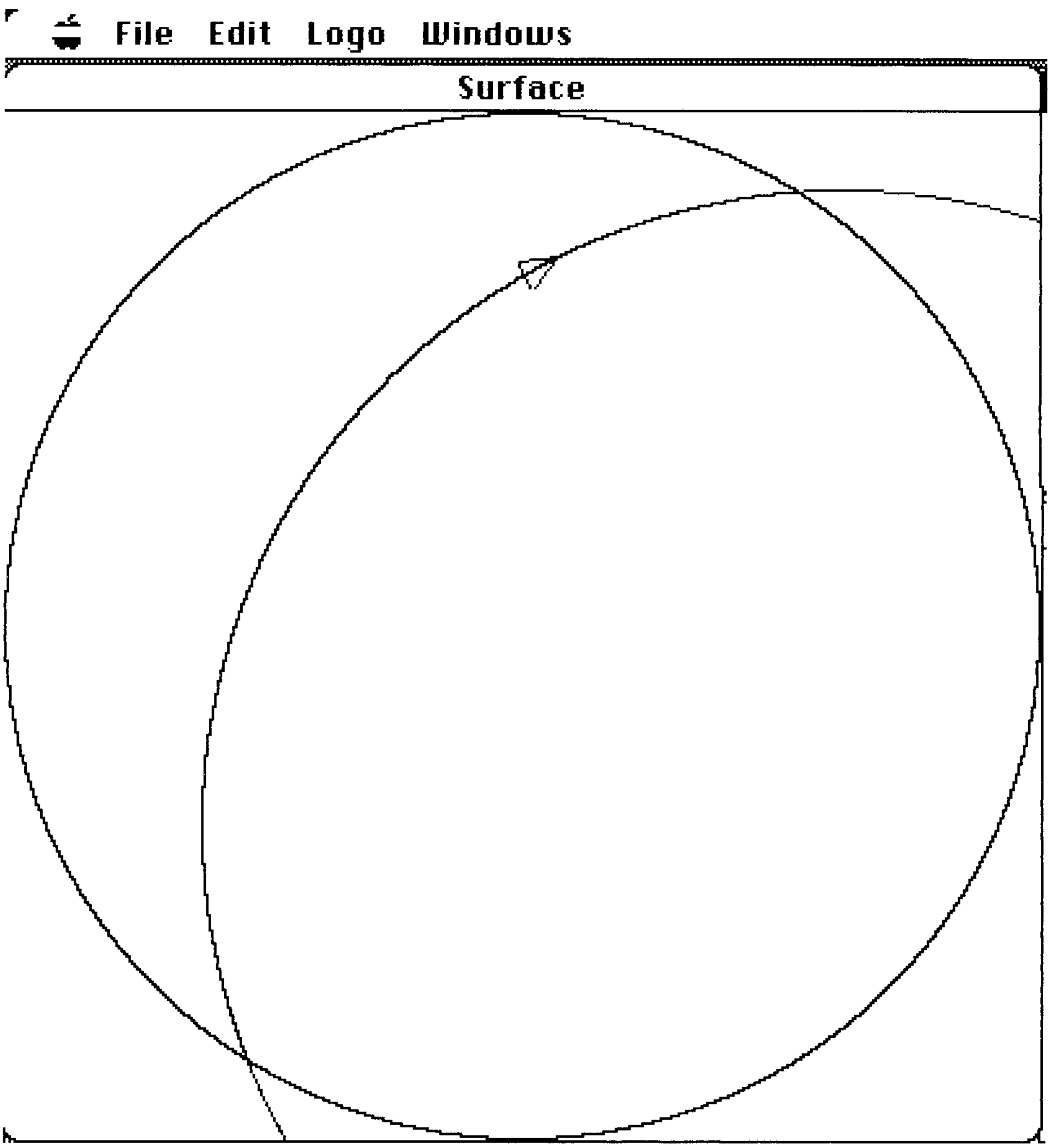

Figure 7.1 Use of **path** Procedure for Conformal model A. Given a heading and position, the procedure draws the large-scale behaviour of the Turtle.

7.1.1 (c) Execution of the Program

Three other changes were made to the software with the intention of speeding up the execution of the code. The first was to experiment with the value of "step" in the Euler algorithm to find a value which balanced speed and accuracy. A second change was to re-write the software making greater use of the object-oriented paradigm, so that the complex variables, which carry information about the Turtle's state, could be declared as belonging to the Turtle object. Previously they had been held as a list because of the way in which the programme operated and this slowed down the execution of the code. The third aspect to be considered was that of compiling the code.

Object Logo allowed the creation of a compiled version of the software so that it could be turned into an application and this had the advantage of speeding up the "run-time" execution of the software. As a result of these modifications, the software was more accurate and operated at a speed which was acceptable. However, speed and accuracy remained an issue throughout the cycle and were constantly reviewed.

7.1.2 Revising the Pedagogical Strategy

A major pedagogical outcome of the first cycle was to identify the difficulties associated with the strategy of guided discovery. As §6.3.2 showed, there were tensions introduced by the strategy, partly due to the difficulties that the participants experienced in using the software, and partly due to the lack of clarity in the design of the activities. On a more positive note, the overall structure for the pedagogic strand, described by the diachronic view, seemed to be effective and this was developed further. These two issues will be discussed, starting with the pedagogic strategy.

In reviewing the pedagogical interactions around the microworld, it became apparent that a number of techniques were used implicitly by the researcher in an attempt to guide the participants in discovering for themselves various aspects of geometry, both with and without the aid of the computer. Drawing on the styles of pedagogic interventions described in §2.4.2 and reflecting on the experience of Cycle 1, four specific roles were identified as being adopted by the researcher at various times: the didact, the expert, the counsellor, and the guide.

The Didact: The researcher led sessions of formal instruction in a didactic manner. The participants were invited to ask questions for clarification and to answer questions as part of the exposition of the models and projections. This role became necessary because the participants had difficulty in interpreting both the screen images and the Escher print together with its grid (§6.2.3). They also required more explanations of the stereographic projection than the researcher had anticipated. These three factors contributed to a situation in which the desire on the part of the researcher— to enable the participants to discover the geometry for themselves— was in a constant tension with the participants' need for support and explanation. It was decided, therefore, that there would be sections of the microworld in which the participants would be taught about stereographic projection, the Escher model, and the screen images, in a formal way. This would then be followed by the participants being invited to ask questions of clarification and be asked questions to check their understanding.

The Expert: The researcher answered questions about the software or the task in hand to aid clarity and to provide information, but did not volunteer anything else or provide hints. This role occurred quite naturally as the participants were using the software, since they would ask for support concerning some aspect of what they were doing and could be given an answer to their query. It acknowledged the fact that the researcher had constructed the situation and was guiding it.

The Counsellor: The researcher helped the participants to reflect on their activities, in a non-directive way, through the use of open-ended questions and by "reflecting-back" what they said. This informal process arose from the participants' attempts to understand what they were doing. As the name suggests, this role was designed not to guide or instruct, but to aid the participants' cognitive development and the researcher found himself doing it on a number of occasions.

The Guide: The researcher helped the participants to direct their attention in particular directions and draw out aspects that were significant. This was done through informal interventions and questions, but, unlike the "counsellor" role, there was a conscious attempt to direct the process, albeit informally, in specific directions thought by the researcher to be significant.

Having identified the roles which might be used as part of a revised strategy, it was then necessary to review the pedagogical objectives in light of the decision to eliminate some of the activities tried during the first cycle. An important factor in this review was the recognition that the three-phase structure for the pedagogical element outlined in the diachronic view, was appropriate for the development of the microworld. Together the review of roles, activities, and structure, suggested three changes to the pedagogic objectives.

The first was to remove the need to establish a reference point of knowledge concerning non-euclidean geometry. During the activities of the first cycle, two points became clear as regards the participants' use of knowledge about non-euclidean geometry. First, if the participants understood that an activity was specifically about non-euclidean geometry, they tended to do the activity because they had been asked to. There was a sense of them "going through the motions", which was related to their interpretation of the didactic contract in the situation. Second, if they did possess some knowledge, they did not tend to relate it to either the work with projected images or the computer activities. As a pedagogical and cognitive requirement, it seemed superfluous and tended to interfere with the development of the microworld.

A second change to the objectives was concerned with the related activity of checking geometric facts associated with different geometries and following their development through the microworld. Again, the requirement for the participants to verify the angle sum of a triangle in each geometry seemed, from a formal point of view, correct. It provided a thread of fact running through the pedagogic activity which could be used to assess cognitive developments in the participants. However, emphasising this aspect tended to produce a tension with the desire to let the participants explore the microworld for themselves and, again, make the participants more aware that there was a pedagogical agenda.

A final consideration was to re-structure Phase 2 of the microworld. In this phase, both stereographic projection and the Conformal models for hyperbolic and elliptic geometries were introduced. In light of the review of pedagogical strategy, it was decided to make the introduction of the projection and Conformal model B more formal and didactic. The reason for this was that the participants had difficulty in understanding both stereographic projection and even more difficulty with the Escher print and grid. As *Extract A* in §6.3.2(a) indicates, this confusion created a tension with the pedagogic strategy of guided-discovery. As a result, it was decided to teach the participants about the models and develop their understanding in a more formal way. Table 7.1 summarises the changes, together with a re-iteration of the microworld's technical and pedagogical structure.

	Phase 1 Physical Objects	Phase 2 Plane	Phase 3 Computer
Technical	Solids with non-zero curvature and their geometry. e.g. sphere.	Flat projections of the solids on paper to produce Conformal models. e.g. stereographic projection.	Dynamic and interactive versions of the Conformal models on computer.
Pedagogic	*Induction* into non-euclidean geometry using objects such as spheres and to challenge euclidean intuition.	*Scaffolding* to aid the progression from objects to projections of surfaces.	*Fading*. Activities which develop independent use of the computer-based models by participants.
Objectives	•Explore with the participants specific aspects of geometry on the surface of curved objects. •Challenge the user's euclidean intuitions about Logo. • Act as guide, expert, and counsellor.	•Introduce the idea of a stereographic projection •Introduce the Escher Print and Grid as a model of hyperbolic geometry • Act as a didact	•Relate the Logo screen images to the Conformal models obtained in the second phase. • Investigate the properties of the Conformal models using the software. • Act as counsellor, guide, and expert.

Table 7.1 Revised Pedagogical Objectives of the Microworld for Cycle 2.

The table shows the new objectives in light of Cycle One's review. The pedagogical structure of induction, scaffolding, and fading was retained since it was found to be appropriate, but the objectives were revised. Also included were the pedagogic styles which were to be adopted during the various phases of the microworld.

During Phase 1, the intention was to challenge the euclidean intuitions of the participants by giving them the computer-based Turtles to investigate, without any explanation or support. The central concern was to establish what reactions the participants had to the screen images. This open-ended strategy required an approach which combined the need to observe what the participants said and did with the option of directing their activity, if they raised interesting issues. A combination of guide, counsellor and expert was envisaged, therefore, which would facilitate a flexible and informal pedagogic style for the phase. Phase 2, however, required a different approach, as the results of the first cycle indicated. It was decided that a more formal approach was needed to introduce the stereographic projection and Conformal models. This implied that the principal pedagogical role was to be that of the didact. In the final phase, however, the emphasis was on encouraging and facilitating the participant's fluency in the use of the computer-based versions of the models. This suggested an informal and flexible pedagogical strategy which provided "starter" activities, but allowed the participants to develop their own investigations. It was intended that by this phase the participants would understand the Conformal models sufficiently to enable a return to the guided-discovery strategy. Consequently, the pedagogical roles envisaged for this phase were those of guide, counsellor and expert corresponding to the desire to adopt a flexible and informal approach. As with the first phase, this allowed the researcher to pay attention to what the participants said and did, as required by the aims of the study, while also providing the possibility for exploring issues of interest to the participants.

7.2 Activities for Cycle 2.

The activities retained from the previous cycle were those which fitted with the structure of the microworld that had also been developed in Cycle 1. These included the Phase 2 activities associated with stereographic projection and the Escher print and the idea of investigating the angle sum of triangles. However, there were several new activities needed for Phases 1 and 3, since most of the activities planned for the phases in the first cycle had been rejected. The most pressing need was for Phase 1 activities which introduced the Turtles and aimed at challenging the participants' euclidean intuitions. Activities for Phase 3 were also needed, but they were not so urgent as it was planned to try and develop them from the participants' work. The primary interest in this cycle, therefore, was the development of Phase 1 and 2 activities and these will be described, together with their rationales.

7.2.1 Phase 1: Induction into the Microworld

As the title of this phase suggests, its function was to introduce the participants to the microworld. It was intended to challenge the euclidean intuitions of the participants concerning Logo, as a preparation for working with the computerised versions of the Conformal models. A further requirement was that this be done in an open-ended way, so that the reaction of the participants to the new version of Logo could be observed, documented, and analysed. A "starter" activity was needed which introduced the Turtles to the participants and was open-ended to enable the participants' investigation of each Turtle.

The activity selected was to use a short procedure which the participants tried with each Turtle in turn. The procedure used is shown below.

```
to t
repeat 3 [fd 60 rt 120]
end
```

With a "normal" turtle, this drew an equilateral triangle, but, with the Conformal models, it produced different effects. To enhance the participants' potential confusion on using the procedure with the Turtles, it was decided to ask them to describe what they thought would happen with the procedure before they used it with the Conformal models (Turtles A and C). All participants would have some knowledge of Turtle Graphics and so were expected to be able to "walk through" the procedure.

7.2.2 Phase 2: Scaffolding of the Process

In this second phase, the participants were introduced to the Conformal models through stereographic projection and the Escher print. As the pedagogical objectives indicate, this was to be done in a formal and didactic manner by the researcher. This would enable the participants to gain an understanding of the models before they moved to the computer. It was clear from the previous cycle that such formal support was needed, making use of physical and pictorial illustrations. Accordingly, the activities of this phase were based around objects, such as a sphere, and diagrams. Two sets of introductory activities were planned, one for stereographic projection and the other for the Escher grid.

The first part of the work on stereographic projection and Conformal model A was to be with a sphere. This was to provide a focus for some preliminary discussion on the geometry of the sphere and to establish, informally, what may be known by the participants about it. In particular, the notion of what was meant by the term "straight line" in relation to the sphere was to be discussed and the Turtle metaphor was to be introduced as an appropriate way of thinking about a local and intrinsic definition of the term. This was to be followed by a description of the projection using the diagram found in Figure 6.6 of the previous chapter, together with a discussion about how straight lines, as identified on the sphere, were projected.

In the second part of the work, the Escher print and grid (to be found in Figures 6.7 and 6.8) were to be introduced as a model of hyperbolic geometry. Although the outcomes of the previous cycle indicated that this was not the most appropriate way to proceed, it was all that was available at the start of this cycle. However, quite late in the cycle it was found that the Conformal model B could also be obtained by projection from a three-dimensional object, the two-sheet Hyperboloid and this was used with one pair of participants. However, at the beginning of the Cycle, the planned protocol was to introduce the participants to the Escher print and ask for their comments. This was followed by the grid, which was to form the basis of discussion on its properties, such as the angle-sum of hyperbolic triangles and the meaning of "straight" in this geometry.

7.2.3 Phase 3: Fading of Scaffolding

The final phase of the process was for the participants to work with the computer-based versions of the models. As Table 7.1 indicates, two objectives guided this phase.

The first was for the participants to make the connection between the diagrams of the second phase and the behaviour of the Turtle on the screen. This was achieved through discussion and, where necessary, demonstrating to the participants how the models of curved surfaces found in Phase 2 were related to the screen behaviour of the Turtle. The second objective was to encourage the participants to develop fluency with the computer-based models and investigate them for themselves. Here a combination of "starter" activities and developments of their own intuitions were used. The final Phase's activities were to be built, therefore, on the participants' understanding of the models, as far as was possible. The intention was for Phase 3 to be led by the participants' work and so specific activities were not developed during the initial part of the cycle, other than indicating general areas which might be considered. These included investigating "straight lines", angle-sums, parallel lines, and tessellations.

7.3 Reviewing the Cycle

This section describes the technical, pedagogical and cognitive issues arising out of this second developmental cycle. Again, the priority was to establish which aspects of the cycle contributed to meeting the aims of the study. The focus was, therefore, on finding activities and ways of proceeding that enabled the development of a suitable context in which the participants could develop an understanding of non-euclidean geometry and which also provided the researcher with insights as to how the participants came to their understanding.

7.3.1 Technical Issues Arising from the Cycle

From a technical point of view, two issues were important. The first related to the continuing difficulties with the speed and accuracy of the software. A second development was the introduction of a surface which could be used to obtain Conformal model B by projection; the two-sheet hyperboloid.

7.3.1(a) Speed and Accuracy of the Software

Issues of speed and accuracy were still significant. The central problem was the position sensitive nature of the non-euclidean models and, in particular, Turtle C

(Conformal model B). In this model, a finite euclidean circle represented infinity and a Turtle moving inside the circle should, in theory, never reach it. However, the unit speed condition imposed to give an accurate screen image ensured that the Turtle reached the circle in a finite time! Removing the unit speed condition meant that the Turtle behaved in the expected way, "hovering" near the circle boundary but never crossing it. However, accuracy suffered without the unit speed condition, and the Turtle did not behave correctly when asked to return from the boundary.

A possible compromise, described in §7.1.1(b), was to separate the local and global behaviour of the Turtle by introducing a command called **path**. This drew the large-scale behaviour of the Turtle without the need for it to actually move the whole way along the path. The result was that the difficulties associated with the large-scale motion could be overcome whilst retaining the accuracy which the unit speed condition gave near the centre of the screen. The **path** command was tried with Pair A towards the end of the cycle and, although the introduction of the "fix" gave encouraging results, it was not completely satisfactory. The participants could use the **path** command once they had experimented with the software and knew when to apply it to the Turtle's motion. Its use, therefore, required some knowledge of how the Turtle moved and so could not be used, unaided, in exploring the geometry of the Turtle.

A second approach was to improve on the relatively simple Euler algorithm for solving the pair of second-order differential equations governing the Turtle's motion. The appeal of Euler was its simplicity, since it could be implemented quickly for the equations. An alternative was the Heun algorithm, which was tried and produced some benefits in terms of speed, although its accuracy was not really improved. Two further options considered were using the Runge-Kutta method (Matthews 1992), which was a more accurate numerical algorithm, and finding an analytic solution to the complex equations. Both were actively investigated during the cycle. The latter was discarded after some investigation because the equations could not, apparently, be solved! This left the Runge-Kutta method which seemed to offer possibilities, and investigations of how to use this technique were actively pursued.

7.3.1(b) Projection of the Hyperboloid Surface

The second issue, which was partly technical and partly pedagogic, concerned the introduction of Conformal model B for hyperbolic geometry. Initially, it was decided to use an Escher print and "instruct" the user in its characteristics. This was unsatisfactory for the reasons outlined in previous sections, since the participants found it difficult to understand. It also did not fit with the framework outlined by the diachronic view of the microworld which described the microworld's development in terms of a progression from curved objects to flat images, finishing with computer images. Some other approach was considered necessary and the "answer" came in the form of a projection of a two-sheet hyperboloid.

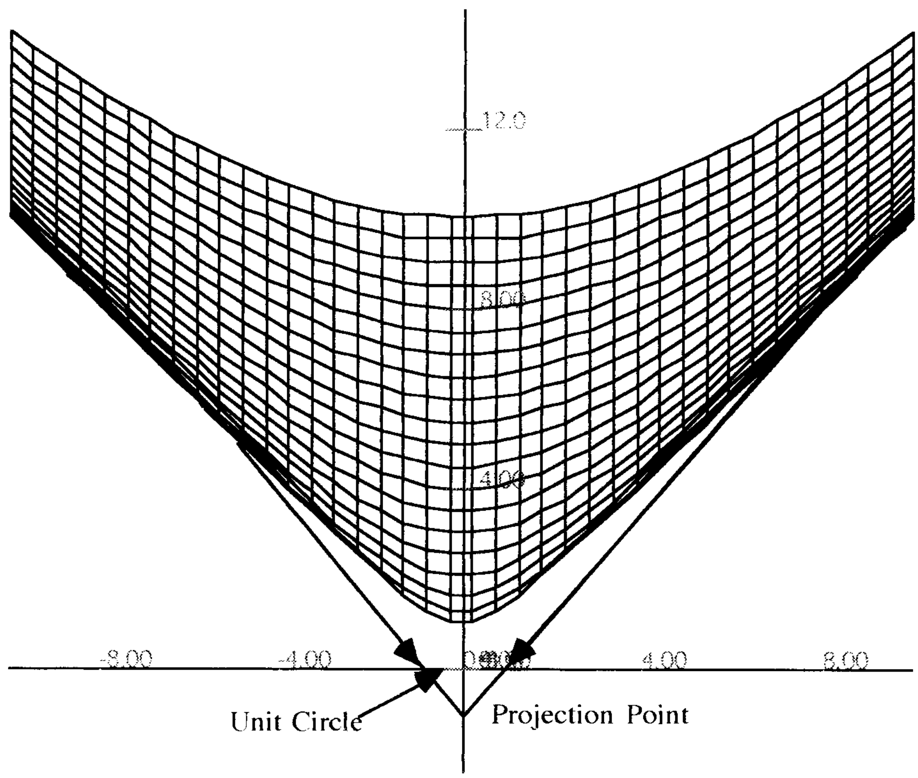

Figure 7.2 Projection of the Positive Branch of the Two-Sheet Hyperboloid $Z^2 - X^2 - Y^2 = 1$ to produce Conformal Model B. See §3.4.3(b) for details.

Although hyperbolic space cannot be completely represented in euclidean space, parts of it can be. Using the positive branch of the surface $Z^2 - X^2 - Y^2 = 1$, and projecting it through the point (0, 0, -1) into the x-y plane, produces the Conformal model B. Figure 7.2 shows this process of projection. "Straight lines" on the surface are produced by plane sections of the hyperboloid. The images of all projected lines lie within the unit circle on the x-y plane, producing the Conformal model B.

This was valuable for two reasons. As §3.4.3 indicates, the mathematics of the projection and the resulting metric which is induced in R^2 was analogous to that of the stereographic projection of the sphere. In a similar manner, the production of the model by projection fitted with the spherical case since it was the result of a projection which started with an object. This enabled a consistent method of presentation of the models for the pedagogical structure and fitted with the technical structure of the microworld provided by the diachronic view.

7.3.2 Pedagogical Issues Arising from the Cycle

One of the major contributions of the cycle was the emergence of a coherent pedagogical framework which the diachronic view summarises. The three-phase approach of induction, scaffolding, and fading was developed and refined. This section will concentrate on those points related specifically to the process of refining the phases. Two issues concerned with the pedagogical framework which will be considered are the need for an "object" for hyperbolic geometry in Phase 1 and a further review of the pedagogic strategy.

7.3.2(a) Projections, Objects and Conformal Model B

Working with the overall methodology of objects —> projections —> images, it became clear that activities given to the participants during the section on hyperbolic geometry were unsatisfactory. The fundamental problem was that the model for hyperbolic geometry, Conformal model B, was precisely that: a model. It was not obtained by projection, but had to be learned. Initially this was thought to be an advantage, since a contrast could be made between projections and models and how they were assimilated by the users. In fact, it was a disadvantage and this became apparent during work with the pairs.

As the transcribed extracts of video which follow in §7.4 indicate, there was a need for objects and diagrams both "to think with" and to "teach with". In the work with elliptic geometry, the "props" consisted of a sphere and diagrams of the stereographic projection for spherical geometry. However, the Escher print plus grid for hyperbolic geometry were all that were available. There was no "object", corresponding to the sphere, which could begin the process. All three pairs in this cycle made successful use of the sphere during their work on spherical geometry. However, the activities with the Escher print were less successful both methodologically and practically.

As noted in the previous section, the researcher became aware of the possibilities offered by the projection of a hyperboloid surface towards the end of the cycle in time to use with Pair A. Unfortunately, a solid object was not available and so three-dimensional plots, shown in Figure 7.2, were produced using a Computer Algebra System[1] and given to Pair C. They found the plots useful in understanding the model and it was decided to obtain a solid object for the next cycle, since there was not enough time to explore the implications in depth during this cycle.

7.3.2 (b) Review of the Pedagogic Strategy

An important issue in the first cycle was the effect that the participants' awareness of their being in a pedagogical setting had on their response to it. From a methodological point of view it was important, because one of the study's aims was to examine the processes by which the participants came to an understanding of non-euclidean geometry. Their awareness of a pedagogical agenda could, potentially, affect their behaviour.

This was an issue in the second cycle as the following extract indicates. It concerns Pair E (Paul and Sean) during their first session with the microworld; that of "confusion". After about 55 minutes of working with the three Turtles, Sean (S)

[1] MAPLE V by Waterloo Maple Software

commented about the title of one Turtle window, called "Surface". R is the researcher who is working at a desk nearby, and P is Paul.

Extract A

S: Is "surface" your name for the Turtle Screen R?

R: Yeah

S: The Turtle is going over a curved surface....What would that do to the path of that, if the surface it was going over wasn't flat, but was curved?.......If it's flat it is going to go in a straight line.

P: Yeah?

S: If it's curved, you'd probably see it .....is that what it is? is the surface different for each turtle?...or it could be another red herring?!!

One of the pair, Sean, noticed an aspect of the screen which he thought of as a "clue" to working out what was going on. The extract highlights a complex of issues to do with the context of the microworld. The microworld is essentially pedagogic: there is something to be taught and something to be learned. On the other hand, there is a commitment in the methodology of the microworld to "guided discovery". The aim of the study was to establish in what ways the microworld may scaffold *leading* the user to an understanding of the screen images produced by the Turtle *through* structured investigation.

In this episode, Sean understood that there was a pedagogical agenda and "something" must be learnt. He interpreted the screen structure as a source of clues to this. After the session, he wrote some notes which he gave to the researcher unprompted. Here is an extract.

> Noticed name of the turtle screen "surface".
> -could be a clue. Work on that.
> (KICK MYSELF- OBVIOUS CLUE-SURFACE)...
> ....
> Note - knew getting closer when talking about "surfaces"
> with cues from R.

1st non-verbal clues - body language when hearing comments.
Then verbal- "So what do you think that Sean? "

The researcher, who had set up the situation and knew that there was something to be learned, wanted the participants to "discover" this for methodological reasons. However, if the participants understood the microworld as a pedagogical tool, then he/she must consciously enter into a didactical contract of discovery, with the consequence that what the researcher *said* was not necessarily what he *meant* or that he was not being honest in his responses to the participants.

From this study's point of view, the significance of the episode was that it raised awareness of the role of the didactical contract in the operation and interpretation of the microworld, both by the researcher and the participants. What effect the awareness of the pedagogic nature of the microworld had on how participants both interact with it and understand it remained to be seen. However, it was a factor which had to be both recognised and accounted for. The extract also suggested that being conscious of the pedagogic nature of the microworld does not preclude the participants being drawn into it in an active way, and responding in an unconscious way to its demands. Doing "interesting" problems may offset the fact that they are someone else's problems which may have answers.

7.3.3 Cognitive Issues Arising from the Cycle

The cognitive outcomes of the cycle were again identified using the categories of the synchronic view and centred on the three questions suggested by the aims of the study:

- What is understood by the participants?
- How did they come to that understanding?
- What assumptions and intuitions did they bring to bear on the situation in so far as they can be identified?

This section will present extracts from transcripts of the videos which highlight these areas. In particular, three episodes will be presented which both summarise participants' reactions to aspects of the microworld and provide insights into developing its next phase. The first extract reports the participants' responses during the first Phase which aimed at challenging their euclidean intuitions about Logo and prepared the way for work with objects. The second extract is taken from Phase 2 and shows the way in which one pair of participants came to understand the properties of stereographic projection by finding errors in the Turtle's behaviour. A third extract reports a conversation between one pair of participants and the researcher about the need for "objects to think with", while trying to make sense of the Turtles' behaviours.

7.3.3(a) Phase 1 : Confusion

During this Phase, each pair of participants were asked to use a short, pre-written procedure which drew equilateral triangles when used with a "euclidean Turtle". They were asked to compare the Turtle's screen movements in two Conformal models and the euclidean. Each pair reacted in a different manner when faced with the Turtle's behaviour. Their approaches were experimental and descriptive, but their differences in strategy will be summarised. An extract from Pair E will be presented in more detail to illustrate the scaffolding they created to help them understand what they were seeing.

Pair C

Their approach was to work initially with the given procedure that drew equilateral triangles. Modifying the procedure to draw right angles, they guessed at the conformal nature of the screen images. Through experimentation they were able to produce a "spherical square" for Turtle A, but ran out of time while using Turtle C. Making comparisons between the Turtles, they noted:

Extract B

S: So for C, the bigger the line, we need to make the angle bigger.

A: Yes.

S: But with A it would be to make the angle smaller, because the curve is going in the opposite direction.

By this they meant that the further away from the centre Turtle C moved, the larger was the angle of turn needed to move the Turtle to a pre-assigned position because of the orientation of its circular arcs. On the other hand, Turtle A needed a shallower angle as it moved out, since the circular arcs curved in the opposite direction. This kind of empirical generalisation was quite common as they tried to make sense of the Turtles' behaviours.

Pair D

Exploring Turtle C, this pair worked methodically, using the given procedure to create nested triangles as a test tool. They were able to make comparisons between the triangles and investigated several conjectures on the basis of their findings. However, these conjectures were not true, but the Pair worked in a systematic fashion when they tested them. In summarising their results, they noted that each Turtle moved on the arcs of circles and these arcs were opposite in curvature.

Pair E

Pair E made repeated use of the given procedure (see §7.2.1) with each Turtle in an attempt to produce a closed shape. Detailed examinations were made by the pair of the various images produced. By comparing the paths of the Turtles, they were able to exclude random behaviour and identify common characteristics. They noticed the circular nature of the paths and identified some characteristic features of Turtles A and C.

Pair E came the closest to understanding what might be happening on the screen. Sean, one of the pair, noticed that the Turtle window on the screen was entitled "Surface" and speculated that the images were related to the Turtle moving across a surface (see *Extract A*). Later, while discussing whether the sequence of commands in a procedure to draw a square affected the way in which the turtle drew the shape, they made two conjectures based on this idea of the Turtle moving on a surface. The following extract reproduces these conjectures and shows how Pair E created their own scaffolding in an attempt to understand what was happening on the screen. The extract begins with Paul (P) and Sean (S) using the commands **repeat 4 [fd 30 rt 90]** in a short procedure using Turtle A. Figure 7.3 shows a screen shot of their shape.

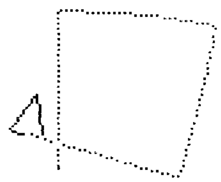

Figure 7.3 Paul and Sean's Square. It used **repeat 4 [fd 30 rt 90]** with Turtle A

Extract C (Part 1)

S: We're on turtle A, yeah?

P: That is now.

S: That wasn't the same pattern as we were getting for the triangular moves: straight curved curved..this one's doing straight, curve, curve, straight

P: Hum.

S: Its position, ...it's not to do with the sequence of moves, it's for the place it's in, isn't it?....I wonder if it *is* a surface?! I'm trying to think what kind of surface could cause it to do it.

P: So you're saying that we should now get a straight ....

S: If its on position ..I think it will curve slightly to the... right.

P: Slightly?...because it is not quite vertical?

S: Yeah.

Paul directs Sean to get the turtle vertical. R, the researcher, intervenes, introducing the command **seth,** which points the Turtle vertically upwards on the screen.

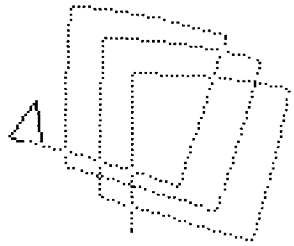

Figure 7.4 Sean and Paul's modification of the shape.

Sean describes a gravitational analogy in which there is a valley "pulling" the Turtle, like a marble inside a bowl.

Extract C (Part 2)

S: It's slightly away from it (*pointing to the previous turtle track*) and its curving inwards.

P: It's a pattern.

S: It might be going over a surface, yeah.

P: So why isn't....?

S: If there's like a valley in the middle, you can walk straight down, but if this slopes to the side as it tries to go up, it's pulled back.

P: Eha (non-committal)

S: (*pointing down the screen, tracing out an arc*) ..is it opposite this way?..as it's trying to go this way, it's pulled towards the valley...pass !!.....because it looks like it's being pulled. If it's going this way, (*making a left hand downward arc*) it's coming down. If it's going that way (*moving finger upwards from right to left*) it's straight because it can run through. Now is that what we are looking at?

P: I'm trying to picture what you are saying.

R: So what do you think you are looking at?

P: We think it's a locus of something moving..going up a hill and away.

S: It's whether it's a valley that's being pulled ...as its trying to go up it's being pulled back towards the gravitational well, for want of a better word.

Sean suggested two related ways of understanding the Turtle's behaviour. He imagined the Turtle moving from one side of a valley to the other. The curved lines produced as the Turtle moved vertically, either towards the top or the bottom of screen, were the result of its motion along the side of the valley when viewed from above. The lines near to the horizontal were considered to be straight and so represented the motion across the valley "floor". Figure 7.5 illustrates Sean's conjecture.

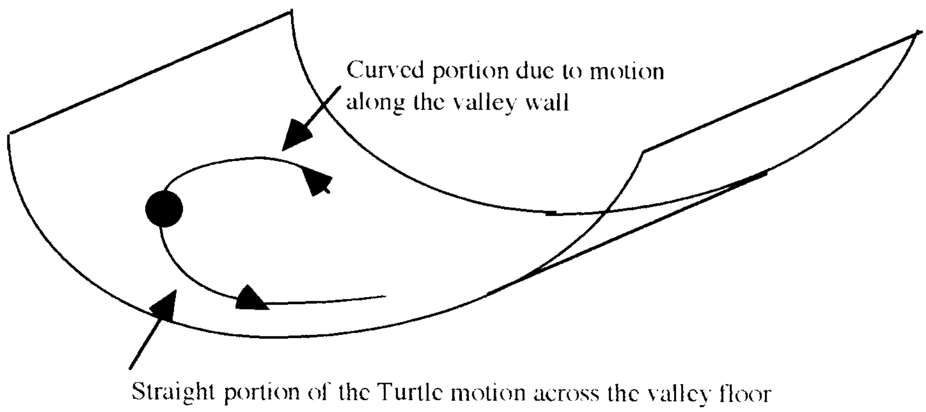

Figure 7.5 Sean's Gravitational Model. Sean uses this model to interpret the Turtle's screen behaviour.

Paul and Sean go on to test this idea by considering lines produced by large values for FD. Circles which do not close are produced, giving a series of lines which Sean interprets as a marble moving inside a bowl. However, what is interesting is the way in which Sean tries to make sense of the Turtle's behaviour, linking it to the notion that it is moving on a surface. The "gravitational" model is not followed up, but starts a train of thought for Paul who suggests that the Turtle could be moving on the outside of a sphere.

Extract C (Part 3)

S: So the Turtle's walking over a sphere?

P: Yes.

R: So what you are seeing on the screen?

P: We're seeing.....

S: Sort of flattened out.

R: How would it be flattened out?

S: Good question.....

R: Do you have in mind those footballs...black and white?

P: Yes ...if you were to flatten out the surface of one of those, .. you would have regular shaped polygons.

R: With curved edges?

P: Yes.

R: Show how would that relate to this? (*pointing to the screen*)

P: I don't know which way round we are doing it but ... it could be showing the path that the turtle is taking on a flat surface, which is then converted to a spherical surface.....or it could be the path of a spherical turtle flattened out.

R: Could you decide between them do you think?

P: Well...

R: Do you agree with this Sean?

S: Yes.

P: Is this the difference between one turtle and another...?

Sean suggested trying out these options. First, they folded paper over a sphere, drawing a straight line on the paper and then flattening out the paper. Next they drew a straight line on the paper, then folded it over the sphere and compared the two results.

This episode indicates how participants, in this case Paul and Sean, created their own scaffolding to settle a question which they have set themselves. They used paper and a small sphere to decide whether the screen image was the result of "unwrapping" the surface of a sphere and flattening it out, or wrapping a flat surface on a sphere. Paul and Sean wrapped paper around a ball, drew a line and then unfolded the paper. They compared this with drawing a line on a piece of paper first and then folding it over the sphere. There is no conclusion, but it is interesting to note how they build on Paul's intuition, that Turtle A is related to the sphere since the shape of the path (lines 5-10) reminded him of a type of football which can be produced by tessellating "curved" polygons.

A second point here is the use of intuitions and prior experience by both of the pairs in trying to make sense of what they were seeing. Sean made use of a gravitational model to explain his conjecture that the Turtle was moving on a curved surface in *Extract A* and was being viewed from above. It is interesting to speculate about Sean's choice of metaphor, since he had a background in physics. Paul then tried "to picture" Sean's conjecture, but was not able to. This may be because he was unfamiliar with the type of image to which Sean refers (i.e. gravitational wells in *Extract C, Part 2*). This idea is common in physics, but may not be familiar to those without a such a background. Paul, in turn, goes on to refer to a more familiar object, footballs, to explain the behaviour. He compared the shape on the screen to the type of "curved" pentagons and hexagons which can be joined together to make a sphere.

7.3.3(b) Phase 2: Constructing Understanding

This section describes an episode from the second phase in which Pair D begin to develop an understanding of the microworld's epistemological base. It will focus on how the elements of the microworld provided scaffolding to aid understanding for the participants during this phase of guidance and consolidation, albeit accidentally!

Tim, one member of Pair D, was able to reason that the screen images produced by Turtle A were incorrect, using the sphere and his recently acquired knowledge of stereographic projection. The second session with the Pair, from which *Extract D* is taken, started with Tim and Steve returning to their discussion about the screen they had produced in the first session. This contained a collection of lines produced by positioning the turtle at various points, turning 90°, and letting it go forward for a large number of steps. The Turtle path did not close and they interpreted this as the Turtle "spiralling" inwards, although in fact the Turtle's behaviour was produced by inaccuracies in the software. Figure 7.6 illustrates their initial screen.

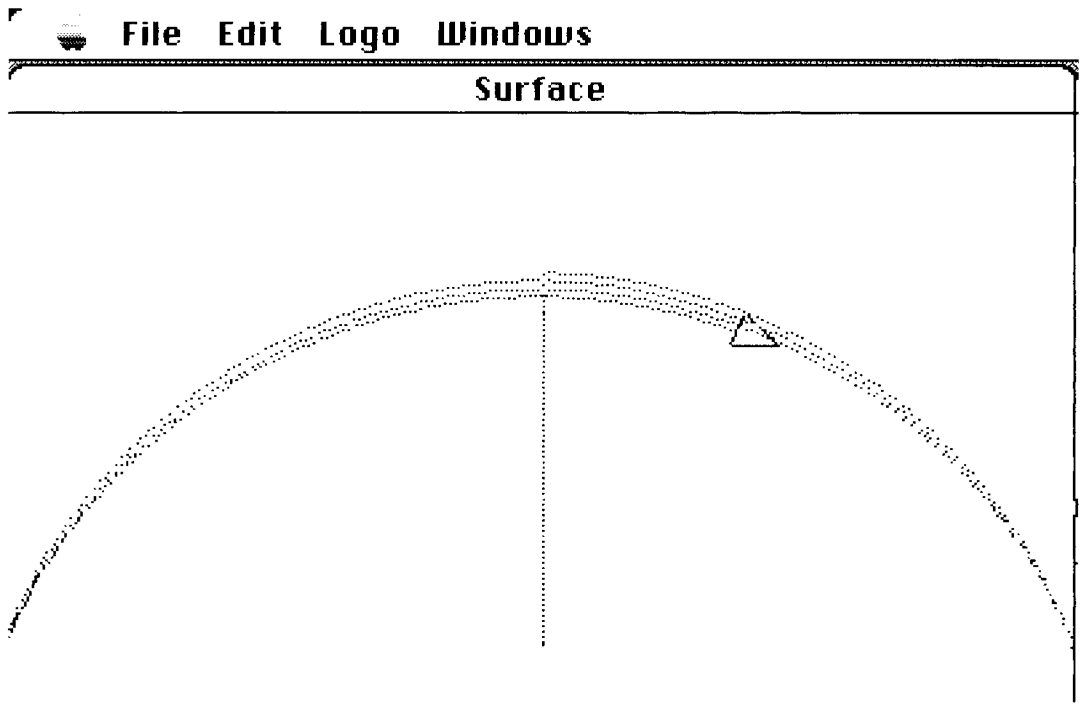

Figure 7.6 Tim and Steve's Initial Screen. Turtle A's path did not close and they interpreted this as the turtle "spiralling" inwards. In fact the Turtle's behaviour was produced by inaccuracies in the software

After being introduced to the stereographic projection for the sphere, Pair D spent some time checking their understanding of it, with Steve leading Tim through the properties of the projection. This in turn led the researcher, R, to start a discussion about what straight lines were on the surface of the sphere. Using the metaphor from Turtle Geometry in which straight lines are produced by "equal turtle strides", Tim and Steve distinguished between lines of longitude and latitude. The former were straight while the latter led to unequal steps and so were not straight lines. Tim returned to the question about what they had seen on the computer. Tim was concerned that he could not make sense of the screen images in light of his understanding of the stereographic projection.

Extract D

T: I don't think there's anything different (to lines of longitude). (*T is pointing to the line formed by the join of the two halves of the sphere*). Why does it ...why do they not come back and meet up (*pointing to the screen image*)? That line would be ..(*pointing to the line on the sphere*) ......I'm thinking that that line would be a circle (*pointing to the screen image*).

R: Yeah

T: But if you come to a point (*pointing to the sphere*) forgetting where you've moved on, if you've just come to a point here (*pointing to a place on the sphere but not on the join*) and go in a straight line in any direction ..why do you not..? (*moves finger across the sphere and round to indicate a complete circuit*)

R: Right.

T: Just meet up (*pointing to the sphere*).

R: Right. So thinking about that, that's what you think should happen. Thinking of walking a straight line (*pointing to the sphere*), that even if you were projecting onto this flat plane through the middle, which is what we are saying this is, (*indicates flat surface parallel to the equator of the sphere*) ...that this should still meet up.

T: Yeah, that's right.....say you're anywhere, you're there (*pointing to a point on the top of the sphere*) and you walk in a straight line (*makes circular motion in the air*) with a piece of string.....on the edge of that (sphere).

R: Yes.

T: Surely you'd come back (*indicates the same point on the sphere*).

S: Yeah..it's like you'd said if you're starting there (*pointing to the sphere*) and kept going, you'd always come back to the same point, 'cos that's the starting point.

R: You're assuming its going in a straight line?

S: Yeah.

T: You move to there (*pointing to a point on the side of the sphere*) but then ...it's like (*rotating the sphere so that a line of longitude is under the original point*) if you move it underneath yer, it's still moving on a circle. So if you go in a straight line, why don't you come back to there?....(Pause) I think you should.

R: Right, so what do you think about that picture (*on the screen*) do you think its correct?

S: No.

T: Don't think it is. It should all be circles.

Three things are significant. The first is that the problem about the shape of the Turtle path arose out of a discrepancy between what Tim and Steve had produced on the screen *before* they knew about the projection and their understanding of the projection *after* both being told about it and having worked on it themselves. Tim and Steve were *remembering* what they had been told and had understood for themselves. This led them to question what they saw on the computer screen (lines 1-7).

Second, implicit in Tim's noticing that there was a discrepancy between the motion of the Turtle on the screen and the motion of his finger on the sphere was an identification between the two: the finger on the sphere was like the Turtle on the screen. Lying behind this identification seemed to be two structures, one mathematical and the other metaphorical. The first was an acceptance of the mathematical connection between the sphere and the screen through the process of projection (lines 14-20), which had been established previously. This was the basis of a two-way association between the screen and the sphere: points on the sphere could be mapped onto the screen and back again. The second structure was that of the metaphorical association between the screen Turtle and Tim's finger, which can be seen in a number of places beginning with his statement of the problem (lines 1-5) and its restatement (lines 30 -35), together with his reference to "walking a straight line" (line 15) on the sphere. Tim's circular motion with his finger on the sphere, which he made in several places (lines 8-11 and 18-20), was supposed to match the motion of the Turtle on the screen but did not. This is interesting, since it illustrates the way in which Tim has appropriated the transfer of meaning across the geometric domains suggested by the synchronic view of the microworld. Pedagogically, the researcher sought to build the connection between the screen and the sphere using a combination of mathematics, vision, touch, and linguistic signs. Here, Tim has used those elements to reason that the computer images of the projection could not be correct.

A third point of interest was the way in which Tim made use of the sphere. He noticed an equivalence between moving his finger across the surface of the sphere (lines 10- 11) and keeping his finger still and rotating the sphere (lines 26 - 30). Both produced straight lines on the surface of the sphere for him. He also used the longitudinal markings on the sphere to emphasise his point that the circular path on the sphere must be mapped to a circular path on the screen. This suggested that a combination of visual and tactile information was being used to develop his understanding of the sphere's geometry and hence the meaning of terms such as "straight" in the context.

7.3.3(c) The Need for Objects To Think With

During their final session, Paul and Sean, Pair E, were asked to give their reaction to the sequence of activities that they had been involved in. They both point to the need for objects to "think with".

Extract D

R: I'd like to discuss what your reaction has been to the whole thing.

P: It's very hard to get get your head round what it's doing.

R: What the computer is doing?

P: Every week.......I've got to concentrate hard..to think, when it goes there, that's what I'm seeing.

R: So what are you doing there Paul..you're having to....?

P: I have to look at that each week (*pointing to the sphere and diagrams*).

R: Right.

P: With the projections on the first one...Turtle A and on the other one, I just can't comprehend.

R: Why is that do you think?

Pause

R: Last week Sean, you mentioned about, ermm, not having something to think with.

S: Yes, like Paul is saying, he has to go back to the sphere...I don't need the sphere now. I can actually use that in my head. Effectively I've created the sphere in my mind and think about it that way. But with C there is nothing, as we said last week, there's nothing even close to that, nothing you can use.

Paul has a background in photography and relates Turtle C to the effect produced in a "fish-eye" lens. Sean later sums up his feelings about Turtle C.

S: It's just attempting to find some way of rationalising it because....it's almost like it doesn't happen. A sphere is something you normally use and see, but there's nothing like negative curvature............(*later*) there's no real object we can look at.

R: To what extent do you think the software helped or hindered in coming to terms with it, given that you were thinking about what's moving on the sphere?

S: I don't think I would have got a picture of A without having the sphere there to work with.

R: Right.

S: The software was right ...we did get some way towards it.

P: Yeah.

S: But we had to move to a physical model...very early on we moved to the ball wrapping the thing (*paper*) around the ball, we had to do that. I mean in some ways we had decided that we were working with a surface which wasn't flat. But to prove it to ourselves we could not just use the machine (*pointing to the computer*) to prove it, we had to have something physical.

R: And so when you had the sphere, how did that help you?

S: We'd already made moves towards using it and then we started using the sphere in the second week it was like finalising it, we could see things on the sphere and used ......pointers on it and started making predictions in that way. It's almost like reversing the process. The first week we moved from the machine to the sphere and when we found what the physical thing was, we started doing it the other way around. We were looking at things in the physical world and using the virtual world in the computer to see if we were right.

R: But you found that that was not possible with C , you couldn't go to something.

S: ...we had no feeling for C...with A you've got a feeling for it . With C ....I'd been very reticent.......I was fighting like crazy to come up with this model which isn't there.

P: Yeah.

S: I like doing this physically before I use the machine.......with C there was no feeling.

Two issues are raised by this account. The first is the importance of having some physical object in developing an understanding of the screen image. The second is to do with the movement between the elements of the microworld: from computer to object to computer, and the change in their significance and function.

Physical objects provided the Pair with a reference point and spheres in particular are noted by Sean for their familiarity and usefulness (line 25). Sean was able to picture the sphere in his mind and work with that image (lines 15-20). However, the mental representation was built on interactions with physical objects while he was thinking about the projection and the motion of the Turtle. A lack of a corresponding object for Turtle C created difficulties for both participants. Paul, who could use the sphere to help him re-create the projection each time he used the computer, found it difficult, if not impossible, to work with Turtle C. He had an Escher print and a grid to look at, but this was not sufficient for him to operate effectively with the Turtle. Sean did not have a "feel for" the space represented by Turtle C. It was "not real" for him and so he was "reticent" to work with the Turtle.

Sean's comments are interesting in understanding the dynamics of the process of coming to terms with the screen images (lines 39-46). Starting with the confusion engendered by the initial contact with the computational software, Sean described how they had come to terms with the fact that they might be dealing with a curved surface. The introduction of the sphere and the description of the projection enabled Sean and Paul to make sense of the screen images. They began to make hypotheses about the sphere and then checked them on the computer. For the sphere, there was a physical object to handle, trace over, and look at. There was a projection process which could be described and represented visually. Paul and Sean had a familiar object to start thinking about the computer images and a relatively straight forward algorithm for translating their perceptions of the sphere into the dynamics of the Turtle.

Turtle C, however, presented several difficulties. As Paul put it, he "can't comprehend" spaces of negative curvature. There was no familiar object on which to base an understanding; there was no algorithm for translating perceptions of objects into 2-dimensional images. Paul and Sean had an Escher print and grid which attempted to "model" the space concerned. They, in common with the other pairs, were *told* certain things about the print and grid which *clashed* with their perception of the print. Notably that "infinity" was represented by a finite euclidean circle. Sean was unable to develop a "feel" for Turtle C in both a physical and metaphorical sense. The sensitivity of the screen turtle C did not help matters in the sense that it produced behaviour which was "really" nonsense as well as that which was "apparently" nonsense. The difference could be discerned by the trained eye. However, how one develops a "trained eye" is precisely the object of the exercise.

Sean's remarks in lines 39-46 are interesting, however, because they describe his process of using the sphere to understand the screen images. In lines 33-38, he described how they had almost come to an unaided understanding of the fact that they were working with a projected image during the first sessions. Working with an object in subsequent sessions confirmed his intuition (line 40). The movement, for him, was from machine to sphere and, once he had understood and could *see* the surface he was working on, the process was reversed (lines 43-46). However, there was no such process for Turtle C, since there was no "object" to work with. This was interesting since it suggested that the general methodology of creating confusion with the screen images, which was then resolved through the use of physical objects and their projection, provided a satisfactory basis for developing understanding. On the other hand, starting with screen images which could not be related to an object did not lead to a clear understanding of the screen images.

7.4 Reflections on Cycle 2

This section will reflect on technical, cognitive, and pedagogic issues raised by the cycle and also identify those issues carried over to the next cycle.

7.4.1 Cognitive Aspects of the Cycle

An important theme in this cycle was the role of physical surfaces in developing the participants' understanding of the screen image. §7.3.3(a) described the way in which the participants tried to make sense of the confusing screen images in terms of their own experience. Once they had established that the Turtle was moving on some sort of surface, they employed their visual intuition to provide a situation which could be used to interpret the screen images. For one participant, this was a gravitational well produced by the motion across a valley floor, for the other it was a football constructed from polygons with curved edges. However, they both needed some sort of context in which to locate and interpret the Turtle's motion.

In the second episode, §7.3.3(b), the relationship between the physical surfaces and the screen was highlighted. Tim's success in predicting that the screen image did not match the path he had expected from his observation of the sphere had a number of interesting aspects. First, there was the mathematical connection between the surfaces and the screen established by the projection of the sphere, which Tim seemed to have understood. Secondly, there was the experience of him tracing and seeing closed circular paths on the sphere and expecting the Turtle to produce similar paths. Finally, there was the connection, established through the Turtle metaphor, which allowed the participants to talk about it moving across a surface *and* moving on the screen. This was related to a key perceptual link between the surface and the screen: the idea of a path.

The final episode, §7.3.3(c), described two important operational aspects of the cognitive link between the physical surface and the screen from the participants' point of view. First, the extracts show how important it was for the participants to have an "object to think with" and look at while they were trying to interpret the screen images.

The contrast between their understanding of Conformal model A, for which they had a physical surface, and their comparative lack of understanding of Conformal model B, indicates the significance of the physical objects. The second important aspect of the episode was the way in which the sphere was used by the participants. Initially, they used the sphere to understand the projection and the Conformal model and then they used the sphere to make predictions which they checked on the computer. This set up an iterative relationship between the sphere and Conformal model A, in which the participants moved their attention back and forth between the object and the screen as they investigated aspects of the geometry.

Taking these three episodes together, what emerged was the importance of the objects in aiding the participants, since it enabled links to be made between the surface and the screen in a variety of ways: mathematical, perceptual and linguistic. Second, the way in which the participants used the object varied according to their understanding of the screen images, and thirdly, the idea of the "Turtle's path" seemed to provide an important support in developing their understanding.

7.4.2 Technical and Pedagogical Issues for the Next Cycle

Three issues were carried forward to the next cycle as a result of this cycle. The first was the need to review and improve the numerical algorithm for solving the equations of Turtle motion. The second was the need to obtain or create a hyperboloid surface as the basis for introducing Conformal model B, as the previous section indicates. Finally, there was the need to develop suitable activities for Phase 3 of the microworld. Although the participants had worked with the computer, the main focus of this cycle had been in Phases 1 and 2, concentrating on the process of establishing links between the objects, their projected images, and the computer-based versions. There was a need to take this one step further and develop activities which were completely computer-based and assumed that the participants had a reasonable grasp of the previous two phases. The central concern, however, was what constituted a "reasonable grasp" in the context of the microworld, both from the participants' point of view and the aims of the study.

Chapter 8

The Third Developmental Cycle: Plotting A Path

8.0 Introduction

This chapter describes the third developmental cycle of the microworld's design. §8.1 deals with the technical and pedagogical issues identified at the end of the previous cycles. In particular, it gives an account of the changes to the software made during the cycle which improved the speed and accuracy of the algorithm for solving the equations which govern the Turtle's motion. The section also describes the introduction of a physical surface for introducing Conformal model B and the impact which this had on the pedagogic strategy. In the §8.2, new activities for Phase 3 are outlined. §8.3 reviews the technical, pedagogic and cognitive issues which arose from the participants' use of the microworld and the chapter concludes with a reflection on the cycle in §8.4.

8.1 Developments from the Previous Cycle

In this section, the technical and pedagogic changes made to the microworld during Cycle 3 will be described. §8.1.1 recounts the two major areas of development in the technical element of the microworld: the software and the construction of a surface which was to be used for introducing Conformal model B. The changes to the pedagogic objectives and the style which resulted from the introduction of the hyperboloid surface are described in §8.1.2.

8.1.1 Technical Developments: Improving the Software and Surfaces

The changes made to the software were quite extensive and included improvements to its speed and accuracy, together with the introduction of several new features which were intended to aid the participants in developing their understanding of the non-euclidean geometries. These new features included a fuller implementation of the **path** procedure which had been tried during Cycle 2, and the introduction of "dashed" Turtle tracks which emphasised the variation in step length for each of the models. The button pad for Turtle selection was redesigned to include the **path** feature and other buttons were introduced for the selection of the boundary circle for Turtle C and adjusting step size. The "Turtle pairs" buttons were removed, since the work which had originally been intended for the them was not implemented.[1]

A second area of development in the technical element of the microworld was the construction of a hyperboloid surface to introduce Conformal model B. The relationship between the projection of a two-sheet hyperboloid and the Conformal model had been found at the end of the previous cycle, as §7.3.1(b) indicates. It was decided to incorporate the surface into the microworld's pedagogic structure and §8.1.1(f) describes how a suitable surface was made.

8.1.1(a) Algorithm for Solving the Equations of Turtle Motion

A recurrent theme throughout the development of the software was the need to balance its speed of execution against the accuracy of the Turtle's movement on the screen. As §6.1.1(b) and §7.1.1(a) indicate, a variety of things were tried in an attempt to improve the balance of the two factors, such as the use of complex numbers (§7.1.1(a)) and extensive experimentation with values of **:step** in Euler's algorithm (§6.1.1). While they made some difference, the performance of the software was not really satisfactory and led to unintentional difficulties for the participants. It became a matter of some urgency in this cycle, therefore, to try and improve the software's performance.

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[1] It had originally been intended to include some work with pairs of Turtles as a way of introducing the idea that curvature could be measured by the local separation of Turtles with the same initial headings. As the Turtles move on what they consider to be straight lines, their separation changed and by measuring that separation, it was possible to decide whether they were moving on flat or curved surfaces. Apart for some work on parallels during Cycle 2 by Pair B (Tim and Steve) they were not used and even in their activity, Tim and Steve had not found them helpful. It was decided , therefore, to drop the idea of Turtle pairs and concentrate only on single Turtles.

The first area of investigation was that of the algorithm for solving the equations of motion for the Turtle. Euler's method had been used since it had the virtue of being simple to implement for the pair of second-order differential equations which governed the Turtle's motion. Unfortunately, its simplicity had a price in terms of its accuracy. As §3.5.6 indicates, the method relied on a single parameter, called in the software **:step**, to control both its speed and accuracy. A "fast" Turtle required that the algorithm was used with a relatively large value of **:step**, but this produced the sort of inaccuracies described above. An "accurate" Turtle, on the other hand, was relatively slow since the algorithm had to be used with a smaller value of **:step** and that necessitated more calculations, for a given screen distance, than was the case with the "fast" Turtle. The numerical errors for the algorithm were proportional to the square of the **:step** value (Matthews 1992 p.432) and so the larger the value of **:step** the greater the inaccuracy.

The Runge-Kutta technique was tried as an alternative method for solving the equations, which gave an acceptable balance between the speed and accuracy of the software. The reason for not using this technique initially was the complicated nature of the system of differential equations which had to be solved and the fact that Euler's method seemed, at first, to be both simple and effective. However, as the introductory comments indicate, during the first two cycles Euler's method proved less and less satisfactory and, eventually, an alternative to it had to be found. Based on the idea of solving a first-order differential equation by taking a weighted average of its values over a number of points, the Runge-Kutta method is closely related to Simpson's rule. Given a differential equation $\frac{dx}{dt} = f(x, t)$, with an initial value of $x(t_0) = x_0$, it gives the following iterative formula for making estimates of x(t) value:

$$x_{n+1} = x_n + \frac{h}{6}(k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4}) \quad \text{where}$$

$$k_{n,1} = f(x_n, t_n)$$

$$k_{n,2} = f(x_n + \frac{1}{2}hk_{n,1},\ t_n + \frac{1}{2}h)$$

$$k_{n,3} = f(x_n + \frac{1}{2}hk_{n,2},\ t_n + \frac{1}{2}h)$$

$$k_{n,4} = f(x_n + hk_{n,3},\ t_n + h)$$

and h is the step-size for each iteration. The method works by using the differential equation to calculate the slope of the solution ($k_{n,i}$) at a number of points in the interval $[t_n, t_n + h]$ and taking their average. The resulting estimate has a numerical error in the order of h^5, which is three orders of magnitude better than the Euler method of h^2. Its disadvantage is that it requires more calculations per iteration than Euler's method.

In order to use the Runge-Kutta method, the complex second-order differential equation which governed the Turtle's motion,

$$\frac{d^2z}{dt^2} = \frac{2k\bar{z}}{(1 + k|z|^2)}\left(\frac{dz}{dt}\right)^2 \quad \text{.........(1)}$$

had to be re-written as a pair of first-order equations by making the following substitution in equation (1):

$$U = \left(\frac{dz}{dt}\right) \quad \Rightarrow \quad \frac{dU}{dt} = \frac{2k\bar{z}}{(1 + k|z|^2)}U^2 \ .$$

The Runge-Kutta method was then applied at each iteration to both the differential equations with initial conditions z_0 and dz_0 obtained from the Turtle's position and heading respectively . Writing the new first-order equation as :

$$\left.\frac{dU}{dt}\right|_n = \frac{2k\bar{z}_n}{(1 + k|z_n|^2)}U_n^2 = L_n(z_n, U_n)$$

where z_n, and U_{na} were the values of z and U after n iterations, Runge-Kutta gives the following:

$k_{n,1} = U_n$ $\qquad$ $l_{n,1} = L_n(z_n,U_n)$

$k_{n,2} = U_n + 0.5*l_{n,1}$ $\qquad$ $l_{n,2} = L_n((z_n+k_{n,1})(U_n+l_{n,1}))$

$k_{n,3} = U_n + 0.5*l_{n,2}$ $\qquad$ $l_{n,3} = L_n((z_n+k_{n,2})(U_n+l_{n,2}))$

$k_{n,4}= U_n+ 0.5*l_{n,3}$ $\qquad$ $l_{n,4} = L_n((z_n+k_{n,3})(U_n+l_{n,3}))$

$$z_{n+1} = z_n + \frac{h}{6}(k_{n,1} + 2k_{n,2} + 2k_{n,3} + k_{n,4})$$

$$U_{n+1} = U_n + \frac{h}{6}(l_{n,1} + 2l_{n,2} + 2l_{n,3} + l_{n,4})$$

In Object Logo, the variables **:z**, **:dz**, **:k** and **:step** were assigned to the Turtle object referring to z_n, U_n, k and h in the equations and so could be called by any procedure belonging to the Turtle. The system of equations was then placed in a procedure called **direct,** which calculated the Turtle's new position and heading z_{n+1} and heading U_{n+1} from the previous values. It is shown below.

```
ask :t [to direct]
local [k1 k2 k3 k4 l1 l2 l3 l4]
make "k1 :step * :dz
make "l1 :step * (d2z :z :dz)
make "k2 :step * (:dz + :l1/2)
make "l2 :step * (d2z (:z + :k1/2) (:dz + :l1/2))
make "k3 :step * (:dz + :l2/2)
make "l3 :step * (d2z (:z + :k2/2) (:dz + :l2/2))
make "k4 :step * (:dz + :l3/2)
make "l4 :step * (d2z (:z + :k3/2) (:dz + :l3/2))
make "z :z + (:k1 + 2 * :k2 + 2 * :k3 + :k4) / 6
make "dz :dz + (:l1 + 2 * :l2 + 2 * :l3 + :l4) / 6
make "dz :dz * (1 + :k * :z * (cong :z))/(abs :dz)
end
```

direct called another procedure, referred to as **d2z**, which returned the value of $L_n(z_n,U_n)$ with **:z** as the current value of z_n, **:dz** as the current value of U_n and **:k** as the parameter which selected either of the Conformal models (**:k** = 1 or **:k** = -1) or the euclidean model (**:k** = 0). **d2z** implemented equation (1) above and could be used with various values of **:z** and **:dz** needed for the algorithm:

```
ask :t [to d2z :z :dz]
op (2 * :k * (cong :z) * (:dz) ^ 2)/(1 + :k * (abs :z)^2 )
end
```

The system was relatively easy to implement since it involved replacing the old version of **direct** with the new, together with **d2z**. The new system proved to be very effective as regards both speed and accuracy under a number of tests which will be described in a later section.

8.1.1(b) Initial Conditions for the Algorithm to Solve the Equations

A second area of concern was that of the initial values for each iteration of the above procedure. As noted in the previous section, the Runge-Kutta method works by applying the above equations to given initial values which, in this case, were the Turtle's position and heading. In the previous cycle, it had been found that by making the complex number, **:dz**, which represented the Turtle's heading, into a number with unit modulus, it was possible to get an accurate plot near the centre of the screen. However, as the Turtle moved towards the edge of the screen, its behaviour became erratic and less predictable.

On the one hand, this suggested that making the heading have a specific value at each iteration did improve the accuracy, but, on the other hand, the value of the correction was position dependent; making **:dz** have unit modulus worked only for some parts of the screen. The position sensitivity of the "correction factor" suggested that rather than the heading, **:dz**, being given a unit modulus at every position, it was given a value which depended on its position in the Conformal models. Examining the Turtle's behaviour in each model suggested the following: In Conformal model A, the Turtle was taking "bigger screen steps" for each "Turtle step" as it moved towards the edge of the screen. In Conformal model B, the Turtle took progressively smaller and smaller screen steps as it neared the boundary of the model. Both models have unit values for the modulus near the centre of each model.

The factor arrived at was $\dfrac{1 + k|z|^2}{\left|\dfrac{dz}{dt}\right|}$ where k was the parameter which determined the model used. If k = 1, then the heading was scaled-up by a factor $(1 + k\,|z|^2)$ as z increases. If k = -1, then the heading was scaled-down by the same factor as the modulus of z approached 1. The term was divided by the modulus of **:dz** to ensure that it had a unit value before scaling. Again, this proved to be effective, as the tests showed.

8.1.1(c) "Dashed" Turtle Tracks

While thinking about the issues related to initial conditions of each iteration, the researcher was led to reconsider one of the fundamental perceptual differences of the Conformal models: their loss of euclidean congruence. This meant that distance measures varied with the Turtle's position in the Conformal model and so one "Turtle step" produced different screen steps, according both to the Turtle's position and the model it was moving in.

Using the analogy made by Gray (1989), referred to in §1.2.2, the Turtle was thought to be moving in a *Hot-Plate* universe when using Conformal model A and a *Cool-Plate* Universe for Conformal model B. Suppose that the Turtle was using a metal measuring rod to mark out the length of its steps. In the *Hot-Plate* Universe, which gets hotter as the Turtle moves radially out from the centre, its rod would expand and it would take longer strides as it moved outwards. The *Cool-Plate* Universe, on the other hand, gets colder as the Turtle moves away radially from the centre of the model. Hence its measuring rod would contract and it would apparently take shorter and shorter steps as it moved nearer the edge of the model.

These analogies were useful in thinking about the initial conditions in the previous section and it was decided to try and implement them in the software, to give a clearer sense of what was happening. To do this, it was decided to make the Turtle track "dashed" so that the variation in distance measure with screen position could be seen. The Turtle was, therefore, made to lift its pen after every five steps so that a comparison could be made. Samples of this effect are shown in Figures 8.1 and 8.2 for Conformal model A and B respectively.

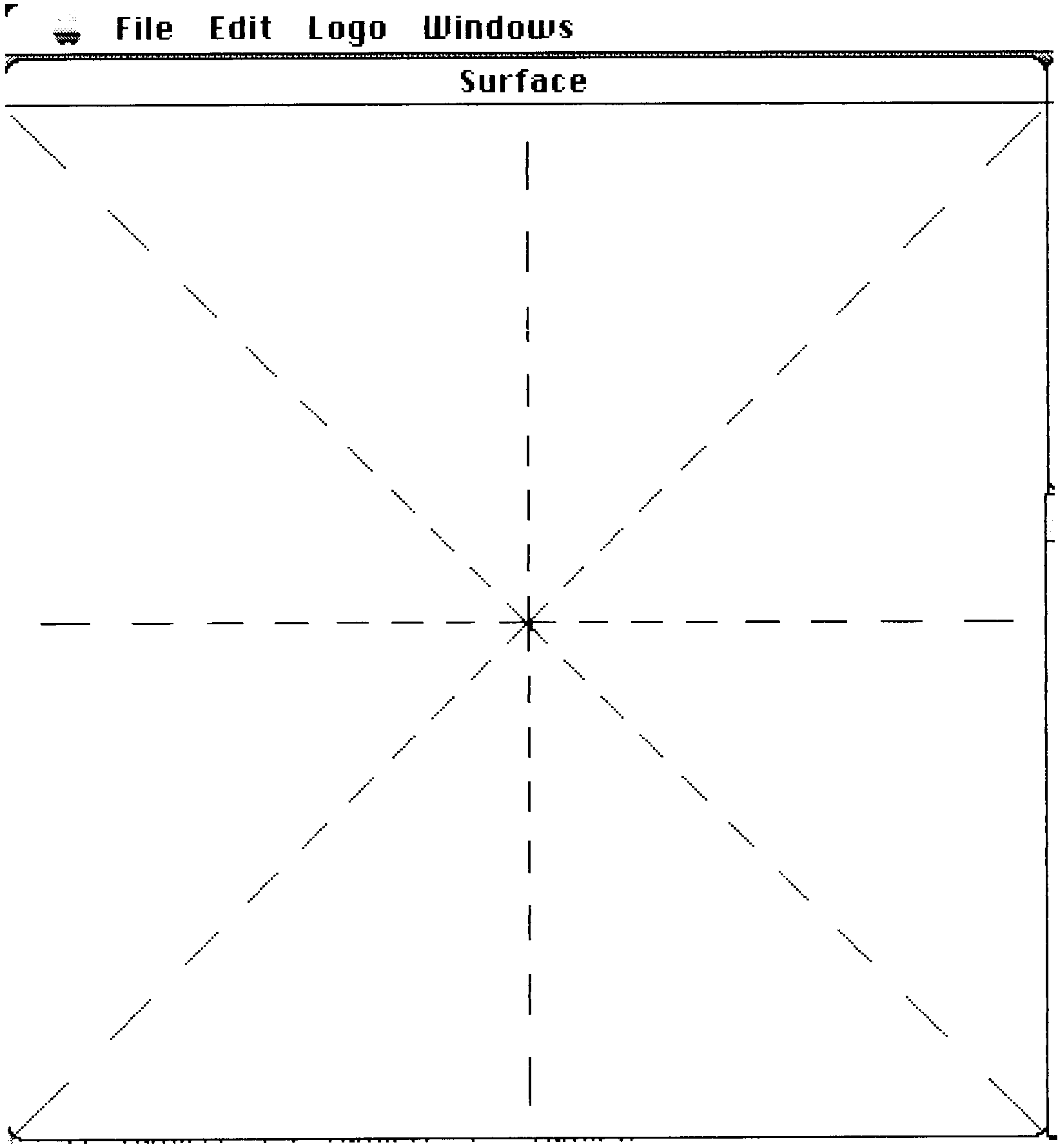


Figure 8.1 Hot-Plate Universe. The Turtle's measuring rod expands towards the edge of the screen as the "temperature" of the screen increases in a radial direction. This is Conformal model A

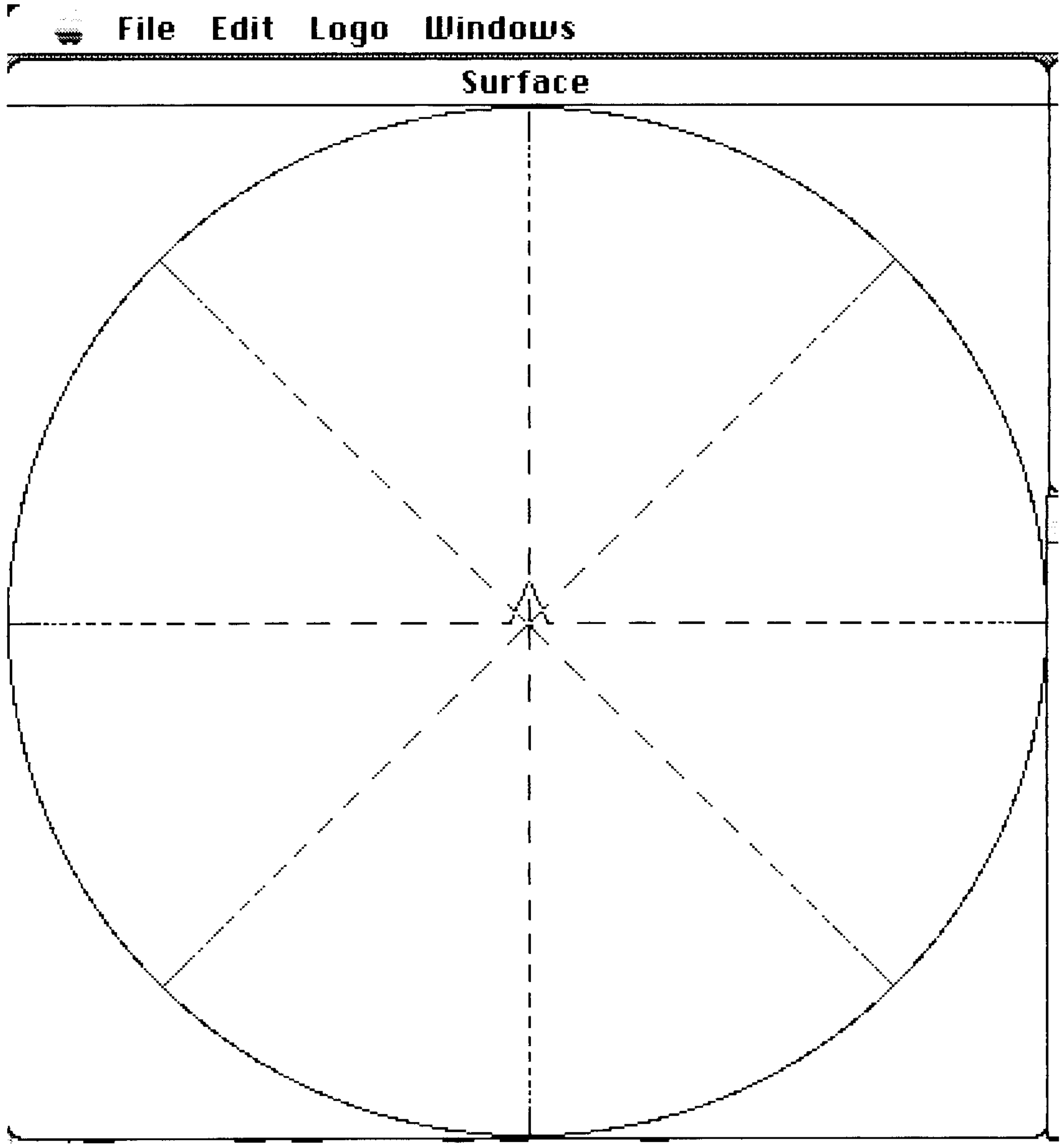


Figure 8.2 Cool-Plate Universe. The Turtle's metal measuring rod contracts as the "screen temperature" decreases radially. This is Conformal model B.

It was decided to try this with the participants to see what their reaction might be. In particular, it was of interest to know whether the "dashing effect" aided their understanding of the models and, if it did, how precisely it helped them.

8.1.1(d) Path Procedure

Recalling §7.3.1(a), the **path** procedure was introduced in an attempt to overcome the difficulties associated with the Turtle's inaccurate and, in the case of Conformal model C, erratic motion near the edge of the screen. The procedure drew the large-scale behaviour of the Turtle without the need for it to actually move the whole way along the path. The result was that the difficulties associated with the large-scale motion could be overcome, whilst retaining the accuracy which the unit speed condition gave near the centre of the screen. However, the participants could only use the **path** command once they had experimented with the software and knew when to apply it to the Turtle's motion. To use it effectively required some knowledge of how the Turtle moved and it could not be used, unaided, in exploring the geometry of the Turtle.

However, the modifications to the software described in the previous two parts resolved the issues of speed and accuracy to such an extent that the **path** procedure was no longer needed. The idea was not dropped for the following reason. While experimenting with the **path** procedure, it was found by the researcher to be useful both for examining the large-scale movement of the Turtle and for finding the direction in which the Turtle should move to close a triangle or illustrate parallel lines, for example. It seemed to provide a useful tool for investigating the geometry associated with each Turtle and so it was decided to introduce it as a tool to the participants. In particular, the **path** command was thought to be useful for Phase 3 of the microworld when the participants were to be encouraged to investigate the Turtles for themselves. It was decided, therefore, to make the procedure into a command which the participants would have available to them any time and so a **path** button would be placed on the screen "button-pad".

When the command was called by clicking on the "path" button, the decision tree shown in Figure 8.3 was invoked by the software. The Conformal models were based on straight euclidean lines or arcs of circles, but the type of path that was drawn depended on the Turtle's position in the model. If it was at the origin, then a straight line had to be drawn, using the procedure **check.line1**. If the Turtle was not at the origin, then a circular arc was to be drawn, according to the Conformal model which was currently in use: **e.region** for Conformal model A and **h.region** for Conformal model B. The coordinates of the centre of the circles, from which the circular arcs for the two models were taken, were found using the equations shown in §3.3, derived from the Turtle's position and heading. They were drawn on the screen using the *QuickDraw* graphics facilities of the Apple Macintosh computer.

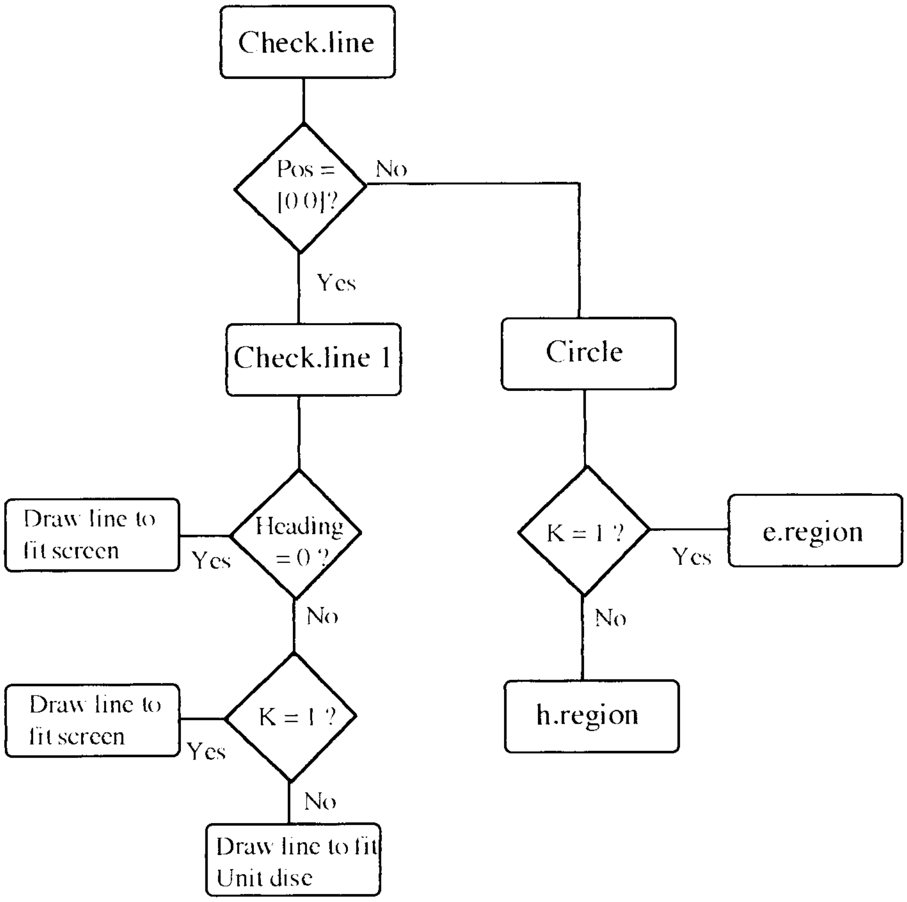


Figure 8.3 Decision Tree for the Path command.
(i) **check.line** decides whether to draw a euclidean straight line.
(ii) **check.line1** decides whether the Turtle is at the origin and draws euclidean straight lines accordingly.
(iii) Circular arcs are drawn using the QuickDraw facilities, according to the Conformal model which is currently in use:
(a) **e.region** for Conformal model A
(b) **h.region** for Conformal model B

The **check.line** procedures were necessary to ensure that straight lines fitted the appropriate region of the screen. In the case of Conformal model A, the straight lines had to fit the screen, but in Conformal model B, the straight lines had to fit within the boundary of the model. Figures 8.1 and 8.2 illustrate the two cases for the straight lines.

Two cases were not accounted for. The first was that of **:k** (the parameter which selected the appropriate geometry) being zero and which gave the euclidean model. It was assumed that any participant who was working with euclidean geometry would have a clear idea of the Turtle's likely path and so it was not implemented. It should also have been "trapped" as a possible error and this was left to a later date. The second and potentially more difficult case, not dealt with in the logic, was that in which the Turtle was not at the origin but had a heading which pointed in the direction of the origin. If the Turtle was pointing directly at the origin, this gave a straight line through the origin. The difficulty was that if the Turtle was pointing a few degrees either side of the origin, it would lie on the arc of a circle, but the procedures **e.region** and **h.region** produced coordinates for the circle's centre which were outside of the range accepted by the *QuickDraw* facilities. An error was flagged by the software which could be ignored but should have been trapped and, occasionally, the error produced a software crash. Again this was an area for further development, although it was thought that the **path** command could be used without serious problems, provided the participants were aware of the potential pitfalls.

8.1.1(e) Button Pad

Having introduced a number of changes to the software, it was necessary to modify the screen "button pad" to include the new features. The new pad layout is shown in Figure 8.4

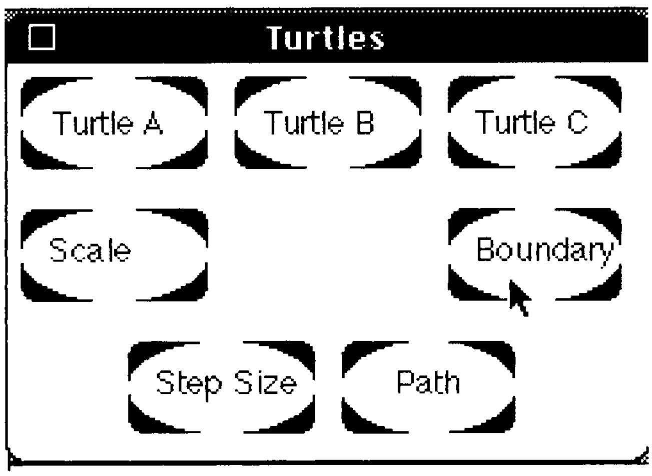


Figure 8.4 .The New Button Pad. The additions to the pad were:
(i) Path which implemented the procedures in Figure 8.3
(ii)Step Size which allowed the participant to modify the value of the **:step** variable in the Runge-Kutta algorithm to give more accurate plot.
(iii) Boundary which worked only with Turtle C placed the boundary of the model on the screen.
(iv) Scale and Turtles A,B and C had the same function as the earlier version.

The additions to the pad included the Path, Step Size and Boundary buttons. The Step Size button allowed the participants to modify the value of the **:step** variable in the Runge-Kutta algorithm to give more accurate plots, if necessary. The Boundary button was designed to work only with Turtle C and selecting it with the mouse placed the boundary of the model on the screen. The other buttons; Scale and Turtles A,B and C had the same function as the earlier version. Scale allowed the participants to modify the scale of Turtle A so that they could "zoom-in" or "zoom-out" as necessary. The Turtles buttons allowed the participants to select either A (for Conformal model A), B (for euclidean model) or C (for Conformal model B).

8.1.1(f) Testing the Software

Three sets of tests were performed on the software to check that the various errors and inaccuracies discovered during the previous two cycles had been corrected. The first check was with Conformal model A, to ensure that the software did in fact produce circles as the image of complete revolutions of the sphere. Recalling §7.3.3(b), one of the participants (Tim) noted that the screen image of Turtle A should have been producing complete circles when, in fact, the Turtle was not. It was important that the software produced accurate images to support the development of the participants' understanding of the projections and geometry. To check that this error had been corrected, a number of complete revolutions were drawn at different distances from the centre of the screen. The first test was for discrepancies in each revolution. Turtle A was made to complete a revolution and then its heading and x-coordinate were reset to 90 and 0 respectively, and then the Turtle was made to perform another revolution. Figure 8.5 illustrates the discrepancies which occurred after each revolution at different screen positions of 40,70, 85 steps from the centre, **:step** = 0.01 and **scale** at 200.

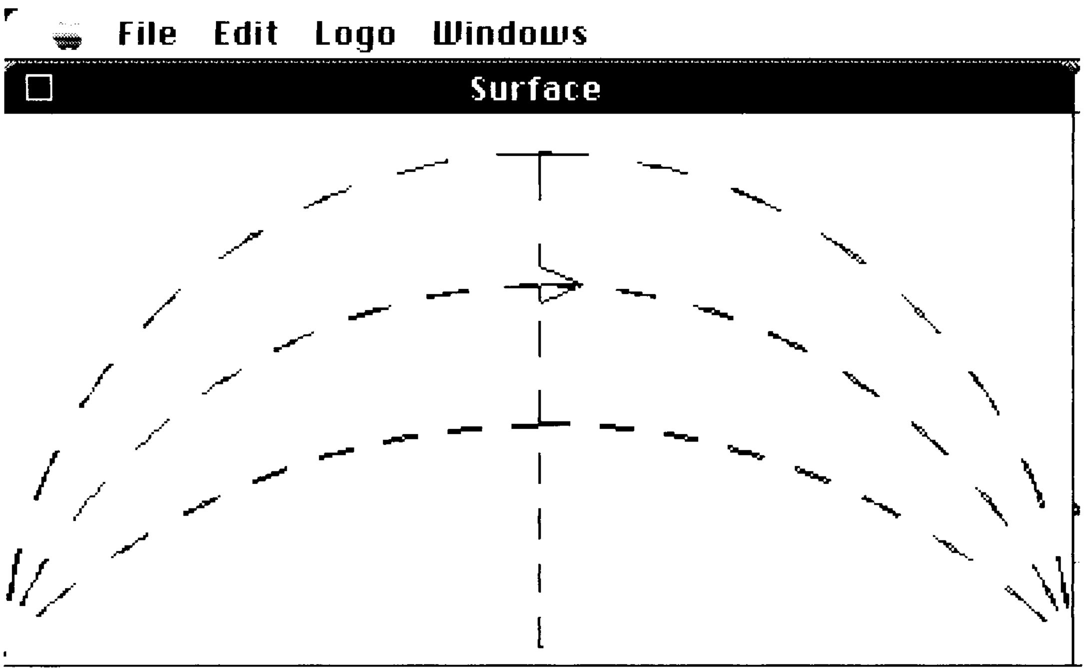


Figure 8.5 Turtle A. with :**step** = 0.01 and **scale** at 200. Complete revolutions were tried at different starting positions. Each path is 314 steps long.

The second test was to let Turtle A complete three successive revolutions at different screen positions and this is shown in Figure 8.6

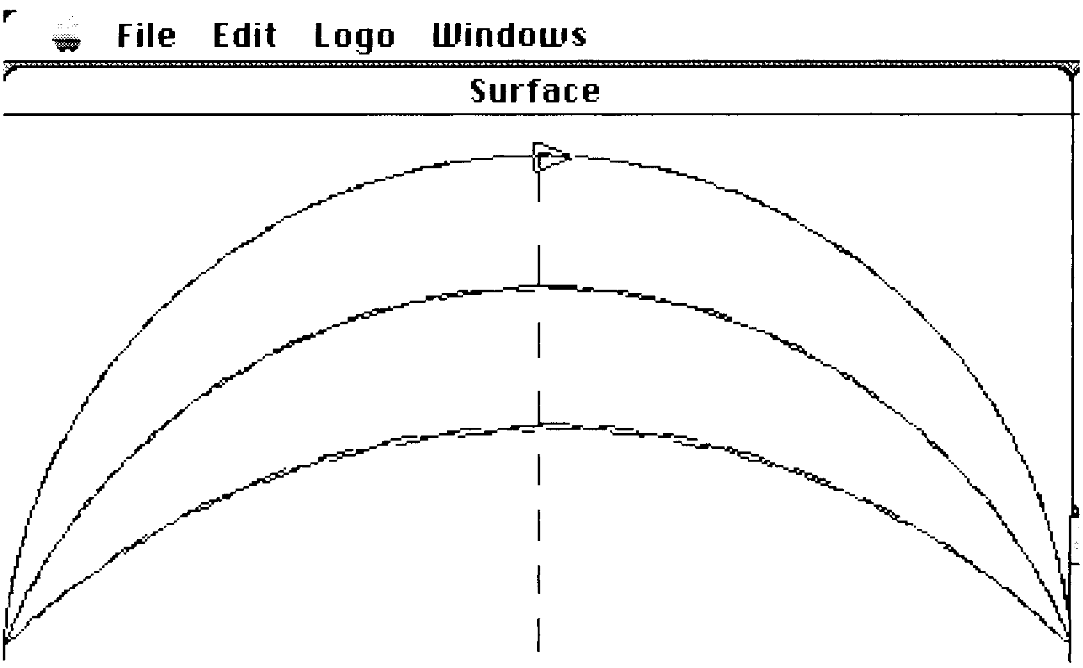


Figure 8.6 Three successive revolutions of Turtle A at different screen position. Each path is 942 = 3 * 314 long.

Besides the improvements in "path closure", the tests also revealed an improvement in accuracy. For any starting position on the screen, it was found that Turtle A always had to travel 314 steps in order to "close the circle" in one screen revolution. This corresponded to it moving on a sphere of radius 50 units, since any path other than that through the point of projection on such a sphere would have that length. The Turtle's movement on the screen also gave a clearer sense of the projection process. First, the length of the "dashed" trail left by the Turtle changed as the Turtle moved away from the centre of the screen, as mentioned in §8.1.1(c). Second, the speed of the Turtle varied with its position, so that it got faster as the Turtle moved away from the centre of the screen and slowed down as it moved closer to the centre. Both of these phenomena were to be expected as a result of the projection process and both were now reproduced by the software.

Turtle C, which had been erratic and inaccurate near the "boundary" of the screen during the previous two cycles, also showed improvements.

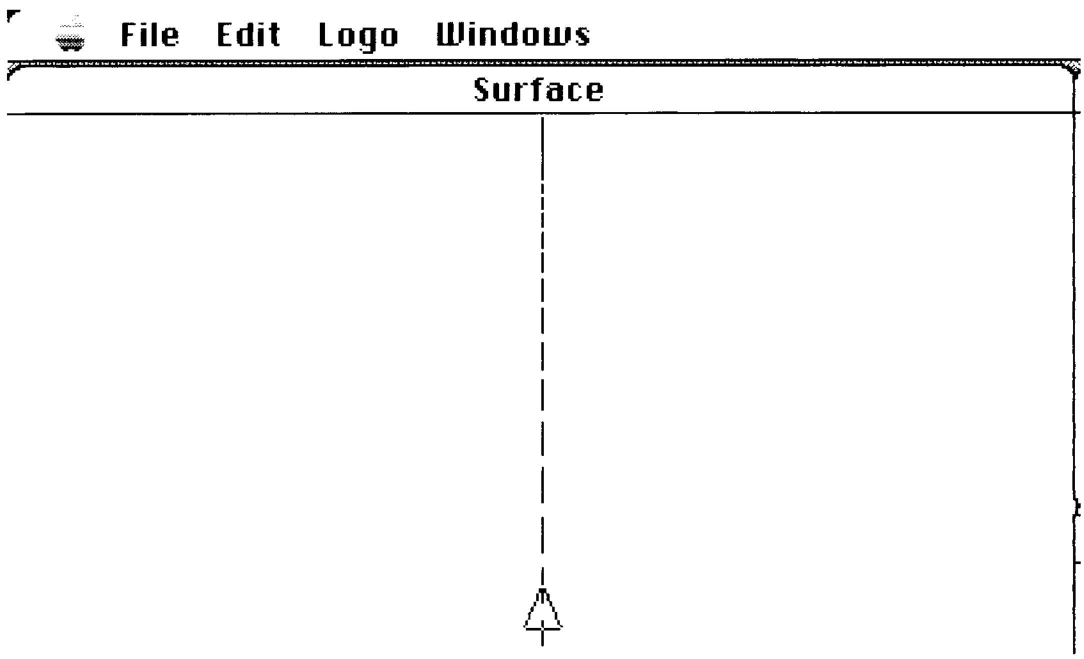


Figure 8.7 Turtle C after **Fd 1000** and **BK 1000**. The error is approximately 4 screen units.

Figure 8.7 illustrates the first test which concentrated on its behaviour near the edge of the model. Turtle C was made to go forward 1000 from the centre and then return. In the past, the return journey had produced an "overshoot" at the origin, but the Turtle now returned to the centre as expected. Although there was an error of about 4 screen units after a round trip of 2000, this was judged to be acceptable.

The Turtle also produced stable behaviour near the boundary. In the previous cycle, Turtle C had tended to rotate if it was positioned near the screen edge of the model, but now it approached the boundary in a steady manner and returned to its original position. As with Turtle A, Turtle C produced a screen motion that corresponded to that expected from the projection process. As §8.1.1(c) shows, its dashed "trail" reduced in size as it approached the screen boundary of the model and it slowed up as it approached the boundary, "hovering" at the boundary but never crossing it. On returning from the boundary, Turtle C speeded up as it moved across the centre of the screen as was expected from the dynamics of the hyperboloid projection.

After the tests, the software was judged to be both accurate and stable. It gave the dynamic behaviour expected from the projections and overcame the difficulties identified in the previous cycles. What inaccuracies that did exist could be compensated for, if necessary, by adjustments to the step size of the algorithm.

8.1.1(f) Object for Conformal Model B.

As §7.3.1(b) indicates, the positive branch of a two-sheet hyperboloid surface, which could be used to introduce Conformal model B, was found towards the end of the second cycle. Although it was not used much during Cycle 2 because of its late arrival, it was decided to make the hyperboloid sheet a main focus of the third developmental cycle.

Initially it was hoped that a solid model of the positive branch of the two-sheet hyperboloid could be made, but this proved not to be possible. A second strategy was to make a paper model of the surface by observing that the hyperboloid sheet was similar to a cone with a slant angle of 45 and a rounded tip. This was achieved, after some experimentation, by constructing a cone and then making a series of triangular cuts at the tip of the cone. These "flaps" were folded inwards to produce a rounded tip and glued together. Although not mathematically accurate, the resulting surface was sufficient for practical purposes since it suggested the asymptotic straightening of the surface and had a profile at the "tip" which matched the hyperboloid approximately. The surface and its construction are illustrated in Figure 8.8.

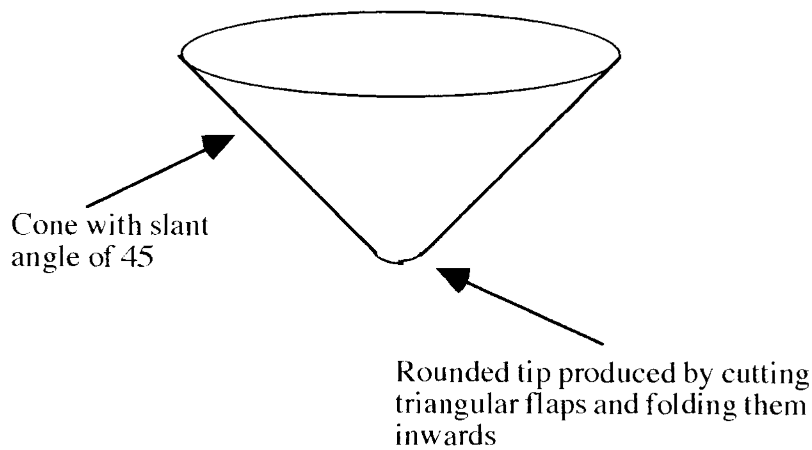


Figure 8.8 Hyperboloid Surface produced from a Cone.

The cone was used in the next set of activities and its impact both on the pedagogical structure and cognitive outcomes of this cycle will be described in the §8.1.2 and §8.4.3.

8.1.2 Pedagogical Developments: Using the Hyperboloid

The pedagogical changes centred on the introduction of the cone representation of the hyperboloid sheet, described in the previous section. This had an impact on the way in which participants were introduced to the Conformal models and the type of activities planned for Phase 3 of the microworld. This section gives an account of the changes to the pedagogical objectives as a result of the use of the conical representation of the hyperboloid and its impact on Phase 3.

A major outcome of the second cycle was the need to have some kind of object to introduce Conformal model B. For example, §7.3.3(c) described Sean's success in using the sphere to understand Conformal model A and the difficulties he experienced in understanding Conformal model B without some kind of physical surface to look at, touch and "think with". These considerations suggested that an objective for this cycle was to incorporate the hyperboloid surface into the pedagogic strategy so that the participants could have "an object to think with".

A second aspect of Cycle 2 that was useful concerned the needed for some activities to be developed for Phase 3. During the previous two cycles, there had not been a real need for such activities since the participants' work had focused on developing an understanding of the projections and the Escher grid. The intention of the cognitive element of the diachronic view was for the participants to develop *fluency* with the software. This meant that they worked purely with the software to investigate aspects of the non-euclidean geometries which interested them. It also implied that once they understood how the projection and the Escher grid worked, they could use the software unaided.

This had three implications for defining the pedagogic objectives of this cycle. First, the introduction of the hyperboloid surface into the pedagogic process had an effect, not only on how the participants understood the models, but also on what type of activities they might engage in during Phase 3. Hence it was important that the participants be introduced to the surface as soon as was possible within the third cycle. Second, the activities for Phase 3 were intended to provide starting points for the participants which they could then develop to suit their own interests. This meant that the activities had to not only stimulate the participants' interest, but also provide sufficient structure to support the participants if the activities did not "spark" anything in them. A third implication was that the cycle should concentrate on the link between Phase 2 and Phase 3, with the emphasis being on moving the participants to working entirely with the computer.

In light of these considerations, Table 8.1 shows the revised pedagogic structure of the microworld, and includes the new objectives for the cycle.

	Phase 1 Physical Objects	Phase 2 Plane	Phase 3 Computer
Technical	Solids with non-zero curvature and their geometry. e.g. sphere.	Flat projections of the solids on paper to produce Conformal models. e.g. stereographic projection.	Dynamic and interactive versions of the Conformal models on computer.
Pedagogic	*Induction* into non-euclidean geometry using objects such as spheres and to challenge euclidean intuitions.	*Scaffolding* to aid the progression from objects to projections of surfaces.	*Fading*. Activities which develop independent use of the computer-based models by participants.
Objectives	•Introduce the sphere. • Introduce the Hyperboloid surface.	•Reiterate the idea of a Conformal model A being the result of a projection of the sphere. •Introduce the idea the Conformal model B was the result of projecting the hyperboloid. • Act as a didact.	•Relate the Logo screen images to the Conformal models obtained in the second phase. • Investigate the properties of the Conformal models using the software through open-ended activities. • Act as counsellor, guide and expert.

Table 8.1 The Pedagogic Objectives for the Third Cycle.

As the Table suggests, the twin foci of Cycle 3 were the introduction of the idea that Conformal model B was the result of projecting a hyperboloid surface and the development of the participants' confidence in exploring the geometries of the Conformal models. Phase 1 was mentioned in Table 8.1 only in so far as it was connected with introducing the participants to the hyperboloid surface. The pedagogic style adopted for Phase 2 was that of the didact, since it was still intended to "instruct" the participants about the projections and then to leave them to appropriate the ideas presented. In Phase 3, however, the pedagogic style was intended to be guided-discovery. It was intended that the researcher would organise the participants' learning through carefully-structured activities at the computer and support them informally through a combination of observation, listening, discussion, and guided reflection.

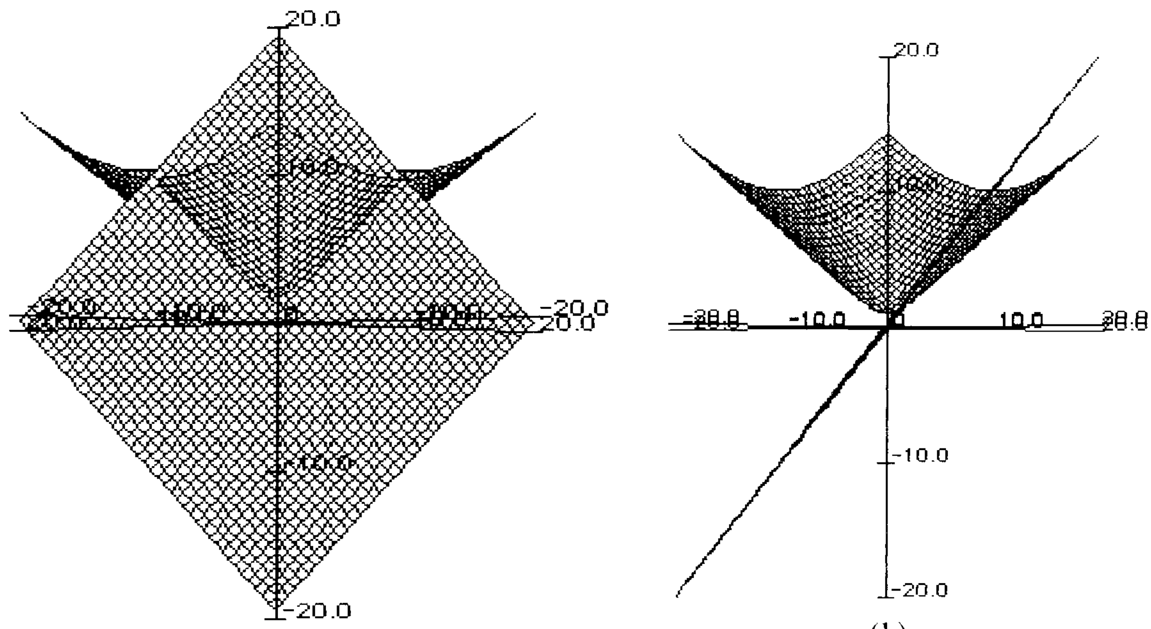
8.2 Activities for Cycle 3

In light of the revised pedagogic objectives, a number of activities were devised to facilitate the change from Phase 2 to Phase 3 and to aid participants in their investigations of Turtle C during Phase 3. §8.2.1 describes the revised introduction of Turtle C using the conical representation of the hyperboloid sheet. §8.2.2 outlines the Phase 3 activities devised in light of the new pedagogical objectives.

8.2.1 Introducing Turtle C

Bringing the hyperboloid-like surface into the pedagogical structure meant that a consistent three-stage method could be used to introduce both of the Conformal models as being the consequence of projecting curved surfaces. The first stage of the process was to introduce the spherical and hyperboloid surfaces as physical objects. Next, the idea that "straight lines" were generated by plane sections of the surfaces which pass through the origin was introduced. These "straight lines" were then projected onto the flat plane for each surface, so that the Conformal model's euclidean representations of the surface's straight lines could be identified as lines and circular arcs.

The participants were introduced to the hyperboloid along with the sphere. They were then told about "straight lines" being produced by plane sections through the origin and the assertions were supported by the images in Figure 8.9. Two views of this intersection between the hyperboloid and a plane were used to illustrate the point. Figure 8.9 (a) represents the plane and hyperboloid viewed so that the intersecting plane is parallel to the viewing position and the curve of intersection is a hyperbola. Figure 8.9 (b) views this situation from the side so the intersecting plane is edge-on to the viewing position. The pictures were produced with the Computer Algebra System MAPLE V.



(a) (b)

Figure 8.9 The Intersection of a Hyperboloid $Z^2 - X^2 - Y^2 = 1$ with a Plane $Z = X + Y$ through the Origin.
(a) The curve of intersection is a hyperbola.
(b) The intersection viewed with the intersecting plane edge-on to the viewing position.

8.2.2 Phase 3 Activities: Introducing New Features of the Software.

The participants were introduced to three new features in the software. These were the "dashing" of the Turtle tracks, the Path button and the Boundary button.

As §8.1.1(c) shows, the "dashed" Turtle Tracks were designed to draw attention to the variation of distance measure with position. This variation was an important perceptual characteristic of the Conformal models. It was intended to draw the participants' attention to the feature as quickly as possible and then to observe what role it played in their understanding of the models.

In §8.1.1(d) , the Path command was described, which enabled the participants to draw the large-scale behaviour of the Turtle without the Turtle having to trace out the full path. The intention in introducing the Path button was to give the participants a "tool" which could be used in different ways according to their needs and it was hoped that they would use it during Phase 3 activities as an aid to their investigations.The

Boundary button was introduced to place the circular boundary of Conformal model B at any time, simply by pointing and clicking with the mouse on the button pad. The button was positioned below that for Turtle C on the button pad so that the participants would associate it with that Turtle.The three features were introduced to the participants, first by demonstrating, and then by allowing them to perform activities which made use of the facilities.

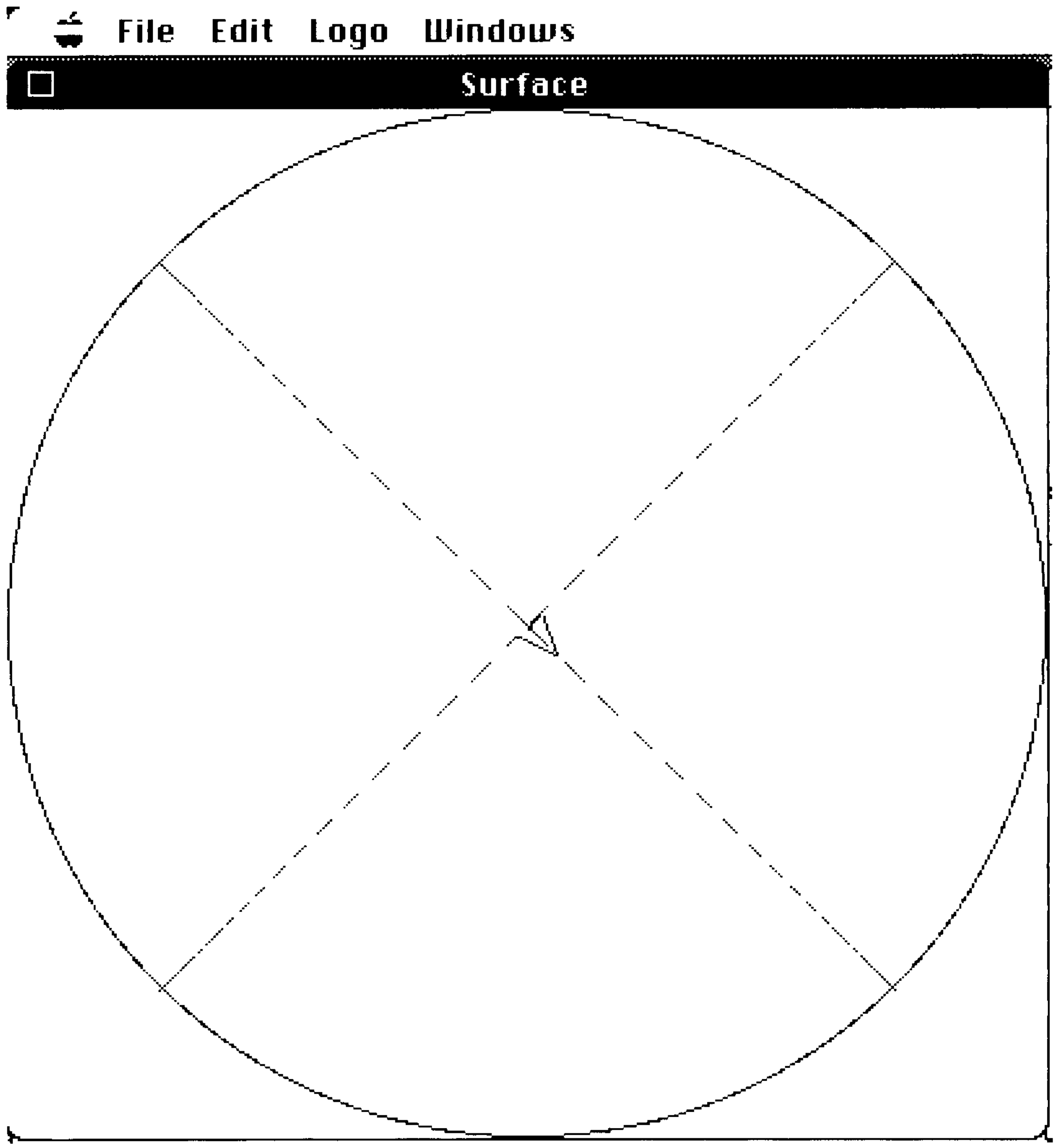


Figure 8.10 Phase 3, Activity 1. The activity challenged the participants to produce this screen with Turtle C. The intention was to familiarise the participants with the new facilities and explore the model.

By way of introduction for the "dashed" Turtle track and the Path button, two Phase 3 activities were given to the participants after they had been introduced to the ideas in the previous section. Both of the activities concentrated on Conformal model

B, since this was found in the previous cycle to be the least understood of the two Conformal models. The first activity challenged the participants to produce the screen shown in Figure 8.10 with Turtle C. It was a relatively simple task, in that the participants were asked to place the Turtle tracks on the screen as shown and use the Boundary button to show the edge of the model. The intention was to familiarise the participants with the new facilities and to explore the model.

The second activity was more demanding and had two parts.

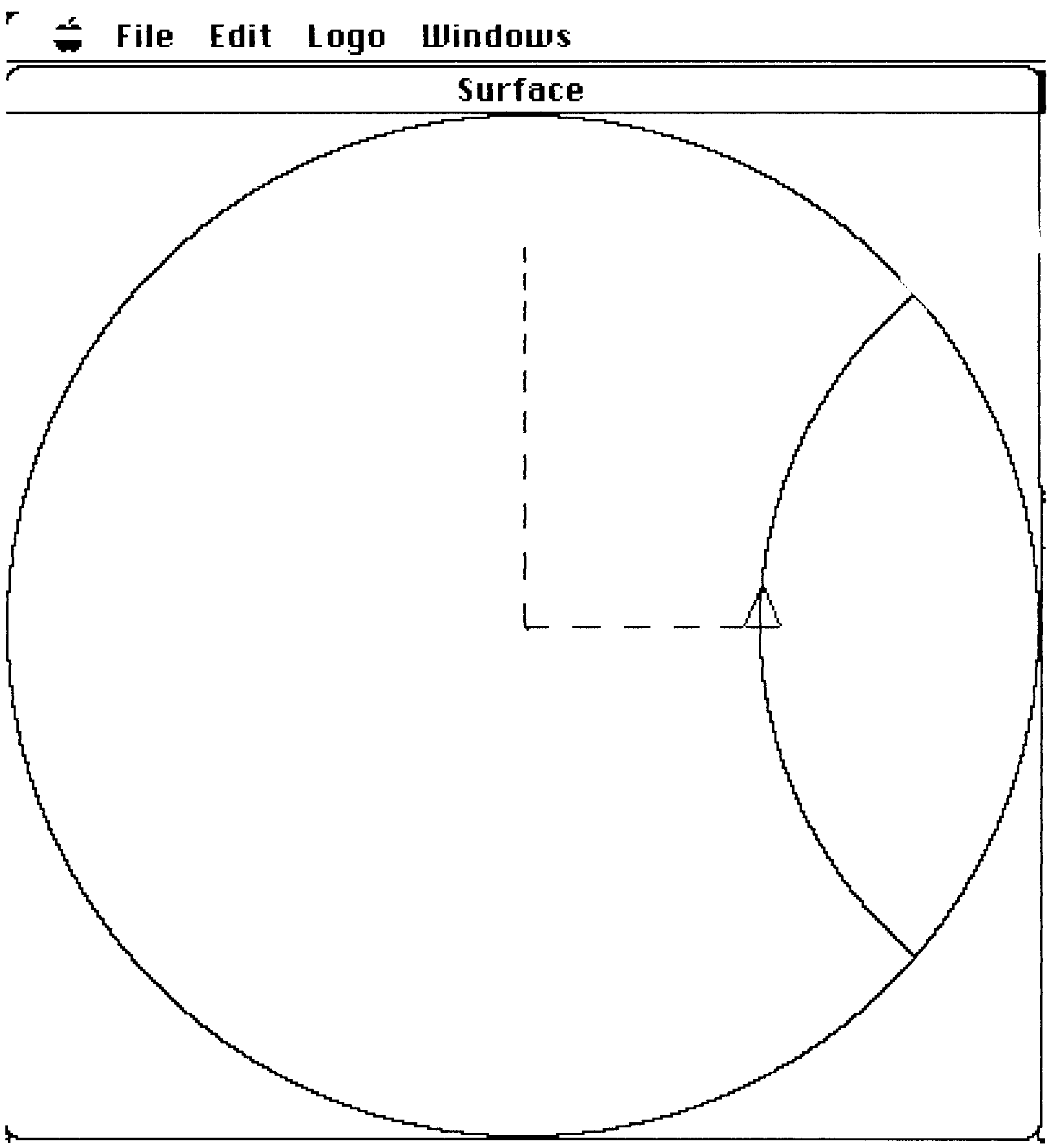


Figure 8.11 Phase 3 Activity Two. The participants were asked to reproduce the screen shown above which consisted of two lines at right-angles to one another. The second part of the activity consisted in the participants being challenged to close the triangle using Path button.

Initially, the participants were asked to reproduce the screen shown in Figure 8.11, which consisted of two lines at right-angles to one another and the Turtle positioned at 50 steps from the origin with a heading of zero. The Path button had been pressed to show the path of the Turtle if it proceeded indefinitely on the given heading. The next part of the activity consisted in the participants being challenged to close the triangle using whatever means they could.

As the activity was set in Phase 3 of the microworld, its chief aim was to encourage and facilitate the participants' fluency with the software and their independent exploration of the Conformal models with the computer. The activity had three objectives. Its first and, perhaps, chief objective, was to provide a task which had enough structure to support the participants' initial work with Turtle C, but could be extended by them if they wished in a direction of their own choosing. The second objective was to introduce them to some aspects of Conformal model B's geometry, which, in this case, was the angle sum of triangles. The third objective was to encourage the participants to use the software facilities provided. The first part of the activity, therefore, was principally concerned with improving the participants' facility with the software and encouraging them to "look at" the geometry of Turtle C in Conformal model B. The second part was concerned with developing their understanding of the geometry and orienting them to explore Conformal model B for themselves.

8.3 Reviewing the Cycle

As with the previous cycles, the outcomes of Cycle 3 will be described using the categories of the synchronic view of the microworld; technical, pedagogic, and cognitive. Since the software had undergone some major changes, it was an objective of this cycle to assess how it had performed and §8.3.1 deals with this aspect. The changes to the pedagogical structure are discussed in §8.3.2 and, in particular, how the participants responded to the introduction of the conical representation of the hyperboloid. Finally, §8.3.3 describes the cognitive outcomes of the cycle.

8.3.1 Reviewing the Technical Element of the Microworld

This section reports the outcomes of the participants' use of the software. It contains their comments about the features of the software introduced during this cycle. These were obtained by direct questioning after the pairs had finished their work with the computer. The participants were asked to comment specifically on the Path button, the "dashing" of Turtle tracks, and the cone representation of the hyperboloid. They also commented, in passing, on the stability of the software and these comments are reported first.

8.3.1(a)Stability and Accuracy of the Software

A major issue in the development of the software was the relationship between its speed, accuracy, and the stability of its behaviour. Many of the changes that were made to the software in this cycle were aimed at improving the accuracy and reliability of the Turtle's behaviour. It was important, therefore, to gain the participants' reactions to the new version of the software and make a comparison with their previous experience. Pair D were emphatic that the new software was much improved. These comments were obtained at the end of their session. S is Steve, T is Tim and R is the researcher.

Extract A

R: As far as you can remember, how do you feel your understanding of Turtle C and how it behaves has changed? Do you think it has? Do you feel more comfortable than you did before?

S: We have an idea now.

T: Yes, I think we were slightly inaccurate somewhere and it (Turtle C) wasn't quite behaving, but it wasn't shooting off in completely the wrong direction like it was last time.

S: Last time it was here, there and everywhere!

R: Yes.

S: With this we've an idea of what we were actually looking for ...the inaccuracies were what you thought you might be doing rather than something that the computer was doing.

T: Yes.

Clearly the accuracy and reliability of the software was crucial to the participants' developing an understanding of the geometry and they felt confident enough in the software to attribute inaccuracies to either the limitations of the hardware or their own errors. Although Pair D's comments were obtained as a result of being asked, their confidence in the software and that of Pair E was implied by the progress that both pairs made with it.

8.3..1 (b) The "Cone" Representation of the Hyperboloid.

Providing some sort of physical representation of the hyperboloid surface as a way of introducing Conformal model B to the participants was an important outcome of the previous cycle, as §7.3.3(c) indicates. A "conical" representation for the hyperboloid, described in §8.1.1(f), was produced for the activities in this cycle. The comments which follow indicate, in general terms, how this surface was received by the participants and later sections will show how it was used by them. The first extract is from Pair D and contains a suggestion for developing it.

Extract B

R: How useful was it to have something like that?(*Pointing to the cone*)

T: That helped (*emphatically*)... I would have actually liked to draw the lines on and then just hold it (*holds the cone in front of his face with apex near to his eye line*)

R: I see, so you would... You'd imagine you were looking at it through here (*pointing to plane between T 's face and the cone.*).

T: You could just imagine looking through glass on to it ...you'd draw the lines on it (the cone) fd 50 turn 90 and looked at it end on.

Tim's account of his use of the cone bears close resemblance to Michael's remarkable recognition, in the first cycle §5.4.3(a), of the fact that the Escher print, and hence the Conformal model, was a projection of the hyperboloid. Tim's positioning of his eye close to the apex of the cone, and his reference both to viewing the cone through glass and using the inverse projection from the plane onto the cone to map lines, suggests that he had interpreted the situation in a similar way to Michael. Pair E's comments also show how useful the conical representation was. S is Sean, P is Paul and R is the researcher.

Extract C

R: First of all, general comments on the cone.

S: That (*pointing to the cone*) is wonderful!!

R: Is it?!

S: The big problem I had last time was that I could not visualise the Turtle (C) at all.

P: I'm sure that you get a better picture (*holding cone over the paper*). The problem that I've got is visualising what that is doing on there (*pointing to the computer screen*).

R: And having that, those two things (*cone and Figure 8.9*) are helpful.

P: Yeah.

S: Yeah....starting off with that (*pointing to Figure 8.9*), that would be useful enough seeing the image of what that's doing, rather...we just had the flat Eschereque drawing.

P: Yeah.

S: But having a three-d model...it's great because you can look. I took the inside even though the Turtle's walking on the outside, where I was ignoring the fact that it was there.

P: I was thinking about what you (Sean) said about doing it backwards ...getting a circle and seeing where that projects to on this (*cone*) and see if there's a recognisable path that we could tell it to take.

S: If that was made of perspex and transparent, or even acetate, you could put it over and see what that path is on here (*the cone*) ....no you couldn't, thinking about it because you'd need the angle as well.

The conical representation was useful for Sean to visualise what Turtle C was doing. His reference to using the cone "on the inside" meant that he traced out the Turtle's path on the inside surface of the cone, although he knew that the Turtle "walked on the outside". This suggests two aspects to his use of the cone in visualisation. First, it enabled him to get a clearer sense of the Turtle's action, more than with the diagrams of the hyperboloid and its projection. But, second, he used the cone in an apparently incorrect manner as far as the projection was concerned. He used the inside of the cone, but was still able to internalise the projection process so that he could make a link between the cone and the screen. Paul, on the other hand, found making the projective link harder to do and was interested in the inverse projection from the plane to the conical representation, which started with something familiar, such as the flat plane, and moved to the less familiar cone.

It was also interesting to note that one person in both of the pairs wished to work with the visual aspects of the projection. Tim in Pair D wanted the flat plane to be clear so that it was easier for him to see lines on the cone and their projection in the plane at the same time. Similarly for Paul, who found Turtle C difficult to visualise, there was a wish to "see" the lines and their projected images, whether one started with the cone and projected it onto the plane or used the inverse projection from the plane to the cone. However, both pairs agreed that the conical representation was valuable.

8.3..2 Reviewing the Pedagogic Element of the Microworld

Recalling the pedagogic objective outlined in Table 8.1, a central concern of this cycle was to use the conical representation of the hyperboloid to introduce Conformal model B for the Turtle C. The significance of the cone was that it enabled a consistent three-stage approach to be developed for introducing both of the Conformal models. This began with the participants working with a physical surface (either a sphere or the hyperboloid) to examine its geometrical properties such as the meaning of "straight line". In both cases, these "straight lines" could be shown to be produced by plane sections of the surface which passed through the origin giving "great circles", in the case of the sphere, and hyperbolas in the case of the Hyperboloid. In the final stage of the process, these "straight lines" were then projected onto the flat plane to produce the Conformal models. During these stages, the researcher adopted a didactic approach in describing the process, explaining the meaning of the term "straight" and checking the participants' understanding through question and answer.

For both pairs, the next part of the session involved them working on Phase 3 activities with Turtle C. The intention was for them to develop fluency with the models mediated by their computer activity. The activities described in §8.2 were prepared for this section as starting points, with the intention that the participants use the activities to generate their own investigations. In the event, the activities were used by Pair E only, and in this section their respective responses to the pedagogic strategy will be described.

8.3.2 (a) Pair D

Tim and Steve's work with Turtle C was generated by Tim's attempt to resolve his difficulties with the plane-section account of straight lines, given in the introduction to the material for the session. He spent some time using the diagrams shown in Figure 8.9 and the conical representation of the hyperboloid, trying to understand how the lines were generated. Steve, who had some experience of non-euclidean geometry, tried to explain to Tim how the projection worked. Tim, however, was an independent thinker who liked to follow through at his own pace and, in that sense, their work was dominated by his attempts to understand both the plane sections and the projection process. In the following extract they have drawn the screen shown in Figure 8.12 and are discussing it using the cone.

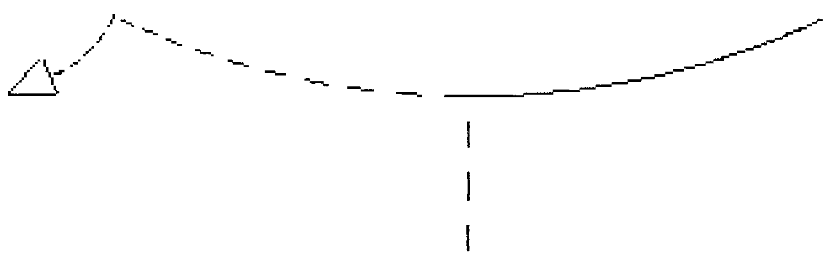


Figure 8.12 Tim and Steve's Exploration of Turtle C. This was the screen which they were discussing as Tim was trying to make sense of the idea that straight lines are produced by plane sections of the hyperboloid. The Turtle had been moved 30 steps out, from the centre, turned rt 90 , move fd 100 and bk 200. Finally, they had made the Turtle do rt 90 and fd 50.

T is Tim, S is Steve and R is the researcher.

Extract D

T: I don't understand yet where the 90 is (*makes a chopping motion down towards the cone surface indicating a plane section*). We've come out from here (*pointing to the apex of the cone*) turned 90, yes (*moves his finger along the surface out from the apex*) it's from that point there isn't it? (*pointing to the point on the cone*).

S: Yeah.

T: Turn 90 and then it's (*chops against the cone indicating a plane section*) cut.

S: Yeah.

T: Round here, turn 90 then it's cut (*makes vertical chopping motion*).

S: It cut into a different plane.

T: Yeah.

Pause

S: It's just seeing whichever one it's cut into. (Pause) If you try and bring it back (*pointing to current position of the Turtle on the screen*) over this way, we'll get a series of arcs.

S enters **lt 90 fd 50 lt 90 fd 50** which are shown in Figure 8.13.

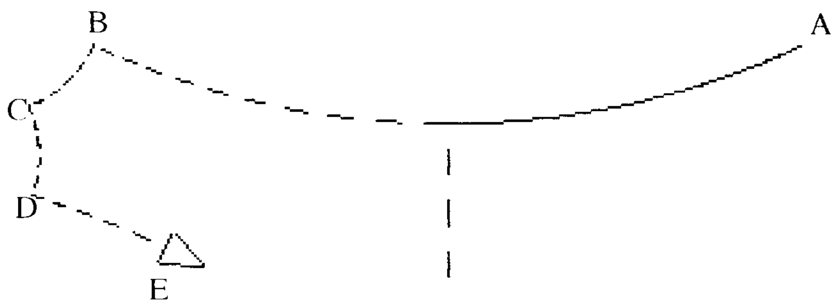


Figure 8.13 Steve's extension of the shape using **lt 90 fd 50 lt 90 fd 50**

T: What have you done? Two 90's?

S: Yeah.

Pause

S: (*Steve picks up the cone with its apex pointing downwards and indicates a point on the side of the cone above the apex*) And we're getting back down towards here now (*Points to Turtle*) because those dashes are getting bigger

T: Erm.

S: So we come out (*Steve moves his finger from the apex, up the side of the cone and traces the curved path on the cone which corresponds to section marked AB in Figure 7.13 ending at B*).

T: Yes. (*points to where Steve's finger is*)

S: Then through there (*traces path on the cone corresponding to BC*).

T: Umm. The same distance all the time.

S: Yes ......No

T: This was a 100 (*indicating AB*) .

S: These three were all 50 (*indicating BC, CD and DE*).

T: Come out a 100 (*traces path up from the vertex*), 50 (*traces arc at right angles to previous path*), 90 degrees.. 50 (*traces path at right angles*), 90 degrees..50 (*traces path at right angles and moves finger off to the right across the cone.*).......Because of the cone shape (*makes cone shape with hand and moves it down the surface to the apex*) you'd have to put the angle of that cone in somewhere.

S: Yeah.

Two things are interesting here. First, Tim's repeated tracing of the path on the cone was an attempt to visualise the sectioning process and to relate it to the screen image, which he clearly found difficult. He seemed unable to relate the motion of the Turtle to the lines produced on the cone by a plane section through the origin (Lines 1 - 5). His repetition of the angle and the accompanying chopping motion while looking at the computer screen indicate that he cannot coordinate the three aspects: physical surface, screen and plane section. This was resolved partially through a dialogue with Steve and through his own thoughts about the image, so that gradually he was able to bring together the Turtle motion and the sectioning of the cone.

Second, Steve took on what might be described as the "teacher's role" in guiding Tim through what had happened. Steve helped him to confirm his understanding of the relationship between the path on the cone, which is produced a plane section, and the screen image. He did this by adding more lines to the screen image as shown in Figure 7.13 and then traced the new path on the cone (lines 19 - 34). In this sense, Steve was using the elements of the microworld to scaffold Tim's understanding by tracing out a path and then connecting it in a "step-by step" manner with the Turtle's motion on the screen. He also used the dashing of the tracks to support his explanation. In line 24, he describes how the Turtle's motion, translated onto the cone, would have placed it near the apex and confirms it by referring to the dashes "getting bigger". He had made the connection between the screen and the cone and was able to use information given by the dashed lines to locate his position on the cone.

The significance of this extract was that the process by which Tim tried to understand the sectioning and projection, with Steve supporting him, led them to generate their own activities. Steve's prior knowledge of the geometry and, presumably, of the hyperboloid surface, gave him the role of peer-tutor. Tim directed the activity of the pair in the sense that he set the goals for the pair's subsequent work. They did not, therefore, try either of the activities which had been prepared, but, in keeping with the overall intentions of Phase 3 of the microworld, they worked on the computer developing Tim's fluency through the use of their own investigation.

The investigation which Tim set them was to create regular closed polygons around the apex of the cone. They decided on trying to draw a square, which they managed to do after some experimentation with lengths and angles. The details of this will be discussed in §8.4.3(a). The significance of their choice lay not so much in *what* they chose to do, but rather that *they* chose to do it as the basis for learning about Conformal model B.

8.3. 2(b) Pair E

Paul and Sean, after being introduced to Turtle C, began to work on making the Turtle "walk round the boundary of the screen". This investigation was instigated by Sean as he looked at the circular rim of the cone and related it to the asymptotic behaviour of the Turtle. This involved them in trying to solve two problems. First, there was their need to get the Turtle to the boundary so that it could walk round it. However, when they could not do this, they focused on how to make the Turtle walk in a circle. Using a short Logo routine for drawing circles, **repeat 360 [fd 1 rt 1]**, they began to consider the need for a "correction factor" in the Turtle's walk, since the surface of the cone did not naturally produce circles, as the sphere had done. Finally, they considered the curvature of the circular arcs to see how much of a correction they would need to add to produce a circle.

At this point, the researcher intervened to give them Activity Two, described in §8.2. The reason for this was that the problem that Sean had set them was based on a misunderstanding of how the cone related to the boundary of the model. He had connected the circular cross-section of the cone, its edge, with the circular boundary, and he felt that if the Turtle could be made to go far enough on the screen, then it would "walk round the boundary". It was interesting that in trying to solve the problem, they investigated several aspects of the geometry. However, it was also clear that, due to the nature of the problem and time constraints, they were not going to obtain anything useful and they had to be moved down a more productive avenue.

This posed a pedagogical dilemma in that the intended style for this Phase 3 of computer work was that of guided discovery, after a didactical introduction. The ideal situation was either for the participants to set their own problems, or to work with activities that structured their investigations in certain directions. Sean had found a problem which he wanted to solve, and which, ideally, he should have been left to work on. However, this could not be accommodated by the researcher's agenda, mainly because of time constraints and so he moved the pair in a different direction. The episode brought out once again the tension between guidance and discovery which had been found in the previous two cycles. However, the researcher must have learned something throughout the developmental cycles, because Sean and Paul did not feel they were being directed, as the following extract indicates.

Extract E

S: It's interesting that we weren't given anything to do this time. Before, you set us tasks, and this time without tasks to do we tried to remember what we were doing before ..we still didn't solve how would you walk around the boundary..we still haven't done that .

R: Can you think what might make it possible and why it may not be possible? When you say walk round the boundary, you mean actually walk round the boundary on the screen here?

S: Yeah, make our image walk round the boundary.

R: What you were doing ..you were taking it out quite a long way and then telling it to go forward.

P: Yeah, but 'cos the boundary is at infinity..you can't actually make the Turtle go to it.

S: No, the Turtle can't actually go to it, but the equivalent for what the Turtle is actually doing walking round inside here (*picks up the cone and points to its inside*) we haven't made it walk round here (*indicating the edge of the cone*).

Paul, by this time, had understood that it was intrinsically impossible to do what Sean wanted to do. First, the boundary was "at infinity" and so the Turtle could never get there; it was not part of the model. Second, the Turtle's screen behaviour indicated that it could not "walk in circles" and, perhaps, Sean was beginning to see that. However, they did not feel that the researcher's pedagogic and research agenda had interfered with their work.

8.3.3 Reviewing the Cognitive Element of the Microworld

The cognitive outcomes of the cycle were again investigated using three questions:

- What was understood by the participants?
- How was it understood?
- What assumptions and knowledge did the participants bring with them to the activities?

This section will discuss three episodes which had significance for understanding how the participants understood the geometry of Turtle C (Conformal model B).

8.3.3 (a) Pair D's Self-Generated Activity

In this extract, Pair D (Tim and Steve) were investigating the regular polygons that Tim was interested in, mentioned in §8.3.2(a), and which he had found using the cone. Typically, he had the cone with its apex facing him and then traced out a "hyperbolic" square so that it was symmetrical about the apex of the cone. Both he and Steve then tried to reproduce the shape on the screen, using trial and error together with the Path button. As the extract begins, the Pair had created the screen shown in Figure 8.14, with the first shape that they produced with an external angle of 147° in the centre, together with their attempts to produce other symmetrical squares around it. The lines produced by the Path button have been removed to show the shape. The full screen will be discussed later in the section.

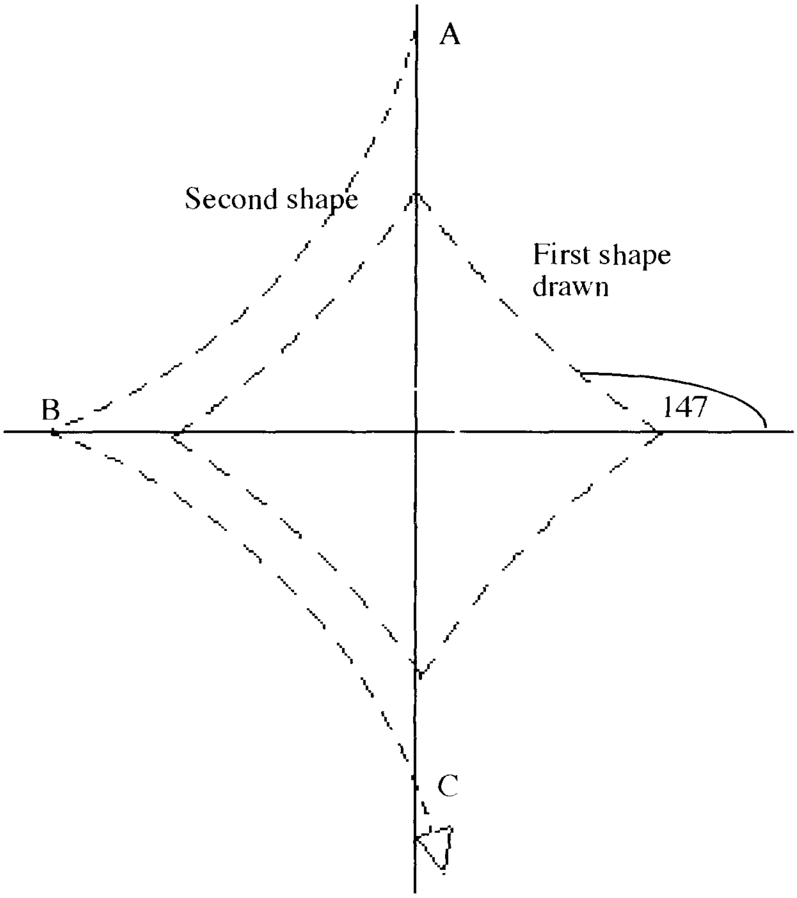


Figure 8.14 Pair D's Shapes. The lines produced by Path button have been removed to aid clarity.

Extract F

S: We've overshot (*referring to the Turtle's position at C*).

T: I'm struggling to understand what goes on there (*indicates B*).

S: Yeah.

T: It seems to have lost its symmetry..We've come in at one angle and we've had to change it to come back out at the same angle. The first one (in the centre), it worked with you know a bit of inaccuracy.

S: Yes. There's not much out.

T: We've not had to do any of this .

S: The further out you go, you lose symmetry......I don't know why.

T: We came in at ..er, was it 165..or did we change it?

S: This first one was 165 (*points to Listener*).

T: Then we changed it (*points to next line in the Listener*).

Some hesitation

S: So in theory, that should have been 29 (*indicates B*) but it wasn't any where near 29.

Pause. Both T and S look at the screen.

S: This is 29 (*points to C*), this is 30.

Pause

T: I can't figure it out.

Pause

S: You need to work out the length of that (*indicates path joining B to C*) using pythagoras, but you can't.

Pause. Tim picks up the cone.

T: Could it be that (*points the apex of the cone towards himself and moved his hand across the apex*) there's this curve on the end..it's straight (*runs his finger along cone away from the apex*), could it be that it's more off the curve (*moving his finger back over the apex*), could it be that one of the patterns that we've got is behaving differently because we're on a different section.

S: Yeah, it could be. It could be that this inner one (*pointing to the inner quadrilateral on the screen*) is on more of a curve than that one (*points to the shape they have just drawn and then points to the apex of the cone*) up here.

T: Yes.

Tim puts down the cone and picks up Figure 8.9.

T: We've come in (*points to lower image and moves his finger from the apex of the diagram up its side*) and you could predict one that very close (*points to the inner shape on the screen*), it just got a different behaviour because it's that bit farther out.

S: Yes. We couldn't work on the idea that because that's (*points to outer shape*) twice the length of that (*points to inner shape*) then that (*points to the outer shape*) would be half the angle. It doesn't work.

T: Yes. Because we're just thinking it's the same right up to the very point (*forms his hands into a cone shape and pushes them upwards to emphasis the pointed nature of the apex*).

S: Urmm.

The investigation was based on the assumption that by doubling the length of the square, one halves its internal angles. Their conjecture was incorrect, but the way in which they interpreted their results was illuminating. First, there was their use of the conical representation to set up the investigation, with Tim wanting to draw a hyperbolic "square" around the apex of the cone. He and Steve then worked on the screen using the Path button until they came to the situation in Figure 8.14, where their calculations have not worked. They first double-checked the numbers that should have been used, with both Tim and Steve noting the failure of symmetry (line 10 and 15) in relation to position. Tim tries to understand what has happened by looking at the cone (lines 30 - 34) and, as a result, modifies his interpretation of the cone. Their failure to get the screen shape to behave as they expected made Tim pay greater attention to the cone's structure (lines 39 - 41) which, as Steve pointed out, varied in curvature. A little later Steve makes the point in a "de-brief" session.

Extract G

T: We seemed to have a rule then, that if you came out, if you take the internal angle off, then that was the angle.

S: The original idea that we worked on was that we were doubling this length and we were halving this internal angle. This one there (*pointing to final screen*). But it didn't quite work. It wasn't far away but it wasn't right.

T: The length on the first two seemed to.....were 75 on each of these, call it a square, we came out 100 instead of 50 and the length went to 150. But that seemed to disappear when we came in (*closer to the centre of the screen*)......once we found the first angle, we used the rule that we learnt on the first square.

R: But it wasn't really working. Have you any thoughts about why it's not working? That if you double the distance, you halve the internal angle?

S: It's got something to do with the curvature of the cone. The curvature of the cone is always changing. It's alright, you've got the representation (*pointing to the cone*) and that's straight, but in actual fact it's..

R: More variable?

S: Yes, it's a hyperbolic function. So it's close to that asymptote without ever reaching it which makes it look straight but in actual fact it isn't.

Steve, who had some experience of non-euclidean geometry, identifies the general type of function which should have produced the real surface: hyperbolic. Tim, on the other hand, treated the conical representation as a cone, as he comments.

T: I think I was thinking of it in terms of just a cone.

S: Whereas when you get there the angles are changing more rapidly than they are here.

R: I see.

S: And I think that has got something to do with it. But why hit on a point somewhere round there (*pointing to middle of cone in Figure 8.9*).

T: Yeah we just seem to....

S: It worked, but we don't know why!

A little later on in the same extract, Tim returns to the idea of a rule.

T: But we were still looking for a rule that we could apply once and it will work going anywhere round.

R: And do you think that that probably isn't correct or do you feel that there is a rule there?

S: There is one there but it would need to take account of the changing curvature

T: I think there's something there.

This episode indicates how prior assumptions can affect the approach that the pair takes. On the one hand, there is Tim's belief that there was "a rule somewhere" which was supported by their initial success in drawing the square. On the other hand, there was Tim's assumption that the hyperboloid was a cone with a linear connection between the angle and distance. The first assumption was, perhaps, a general belief that the computer behaved according to some sort of rule and it was their job to discover it. The second type of assumption was perhaps built on this more general belief of "a rule somewhere" in that Tim, at least, tried to put together the facts which they had established into a coherent strategy. They were encouraged by their initial success and developed the rule that if one doubles the distance from the centre of the screen, then one halves the internal angle of the polygon. However, it was not successful, and they tried to find the "correct numbers". Steve, who had some experience of hyperbolic geometry, suggested that the rule could not be linear since the hyperboloid was not a linear surface. Tim, on the other hand, was thinking of the hyperboloid as a cone and this perhaps supported his view that the connection between angle and distance was correct.

An important issue which this raised was the potential ambiguity of the conical representation of the hyperboloid surface. Steve was aware of this and referred to the cone as a "representation". Clearly, the cone played an important and vital role in aiding both pairs to visualise both Turtle C and the Conformal model. However, as this extract indicated, representations are not the "real thing". The conical representation was a very good qualitative substitute for the hyperboloid, in the sense of having most of the features which one recognises visually (rounded apex; almost 45° for most of the surface), but it also had limitations. These limitations only came to light when the cone was being subjected to the type of close scrutiny which Tim gave it in order to find a quantitative relationship between angle and distance.

The final issue of interest in this episode was Tim and Steve's use of the Path button while investigating the shape. The Path button was introduced to provide a quick way of finding the large-scale behaviour of a Turtle without having to make the Turtle "step" it out. Figure 8.15 shows the paths that Pair D placed on the screen while constructing the shapes.

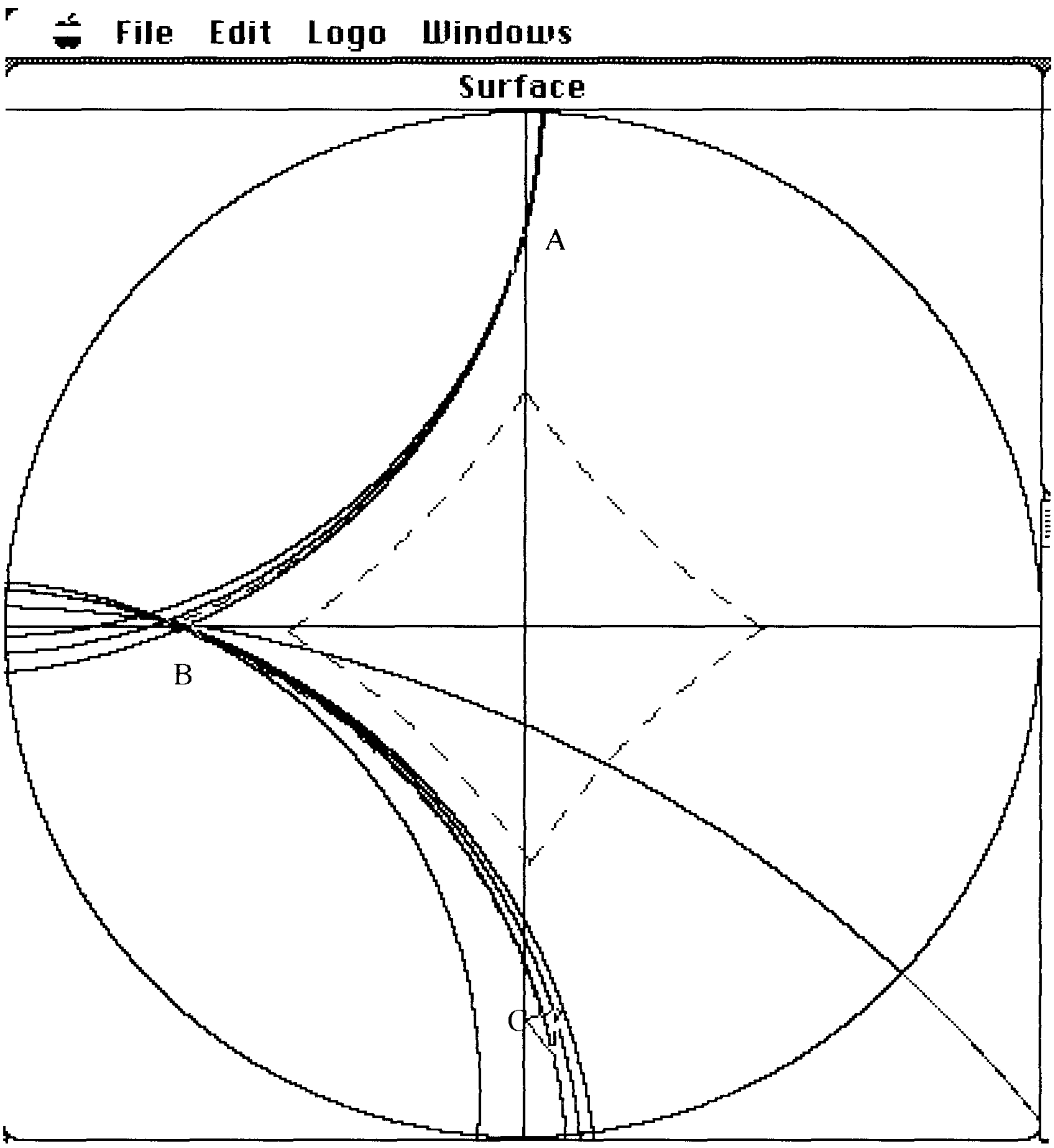


Figure 8.15 Pair D's Use of Path Button.

They used the button as a sort of "ranging" device, modifying the angle according to the point where the path crossed the horizontal axis, to find a suitable path for the Turtle. Here is short extract from their code at A.

fd 100	*Moves Turtle to point A*
lt 165	*Sets the heading*
path	*Uses* Path
lt 1	*Modifies heading*
path	*Tries* Path *again*
rt .5	*Modifies heading by* 0.5
path	*Tries* Path
fd 150	*Moves forward on the chosen path*

Later, in a de-brief session, Pair D commented on the usefulness of Path as a tool.

Extract H

R: What about this Path?

S: That's very ..that's a big help. It's time saving.

R: It can help you to decide what the angle ought to be.

S: Yes, rather than take the Turtle through and bring it back and try again.

T: Some of these (paths), we've tried four or five times to get them.

S: Half a degree or a degree (can make a big difference). The only thing we've found with one or two is that the line actually drawn didn't quite follow.....didn't actually sit on the projected line.

There are three interesting points in these comments. The first is the way in which Pair D made use of Path. They used it to experiment with the selection of directions to close shapes, as Figure 8.15 indicates. This meant that they could experiment with very small changes to the Turtle's direction and, because of the nature of the model, these changes could have a big effect. Observation of the possible Turtle track obtained from the Path procedure enabled them to check what was going to happen and hence make modifications without having to let the Turtle go forward and then return it when it was on an incorrect heading. Secondly, the "point and click" nature of selecting the button made it easy to use and Pair D's comments tended to confirm its tool-like operation. Finally, they noticed some slight inaccuracies between the path that a Turtle might actually take and the path traced out by the Path procedure. Although the discrepancies were not major, they could be noticed. The reasons for this were to do partly with the accuracy of the Turtle's motion and partly to do with the fact that the paths were produced in a different way to the Turtle's motion.

As §8.1.1(d) shows, the paths were obtained by overlaying the screen with circular regions (produced by the *QuickDraw* graphics of the Apple Macintosh), whose centre and radius were determined from the Turtle's position and heading using the equations in §3.4. The *QuickDraw* graphics created the circle from a square region, determined by the diameter of the circle, by "framing" the region on the inside with a line whose thickness could be chosen from the software. This meant that according to the orientation of the region it usually matched the Turtle's track, but occasionally it was to one side or other of the Turtle's path. However, it was always consistently wrong in the sense that the Turtle track and the path were parallel on the screen! This was an area that needed to be investigated, although initial attempts to solve the problem indicated that it was very difficult to predict where the discrepancies would occur.

8.3.3(b) Pythagoras and Angle-sum of a Triangle

This extract describes the way in which Pair E (Paul and Sean) came to understand two geometrical facts about Turtle C (Conformal Model B). They found that neither the angle-sum of triangle nor Pythagoras's Theorem holds for this Turtle. The extract begins with them being given the diagram for Activity 2 (see §8.2.2) and they are asked to reproduce it before "closing the triangle".

They begin by drawing the triangle but decide that, since the diagram they have been given shows them what a left turn of 90° from the horizontal looks like, they choose left 135° and make use of Path to close the triangle. This does not work so they turn the Turtle left by another 5° and use Path. This closes the triangle and they discuss the result.

Extract I

S: I'm assuming it's a triangle.

P: Yes.

S: How many degrees has it got in it?

P: Probably not 90. Probably not 180.

S: We haven't got 180, but it's walking a straight-line path.

P: Yeah, you've probably got to turn.

S: No, you don't have to turn. It's actually drawing a triangle on the surface.

P: Right.

S: Because the surface is curved, that (*indicating the hypotenuse suggested by the* Path line). Now we've turned through 135° to get that., right.

P: 140.

Sean notes that the triangle probably does not have an internal angle-sum of 180. His explanation shows an understanding of the difference between what the Turtle was instructed to do (go forward) and the screen image of a curved track. The resulting track is because the Turtle is walking on a curved surface. They go on to discuss the angle-sum in detail

S: 140 yeah. So what's that leave us with?

P: Which makes that 50.

S:40, it's the internal angle.

P: You're right, 40...so if you were to be up there (*pointing to the top of the vertical line which is 100 steps long*) and turn at 50 or make that angle (internal angle) 50°, would that be your path?

S: Let's have a look at fd 150?

Moving the Turtle along the "hypotenuse" of the triangle, they guess at 150 steps, but this is too much and they come back 30 steps. Figure 8.16 shows their results

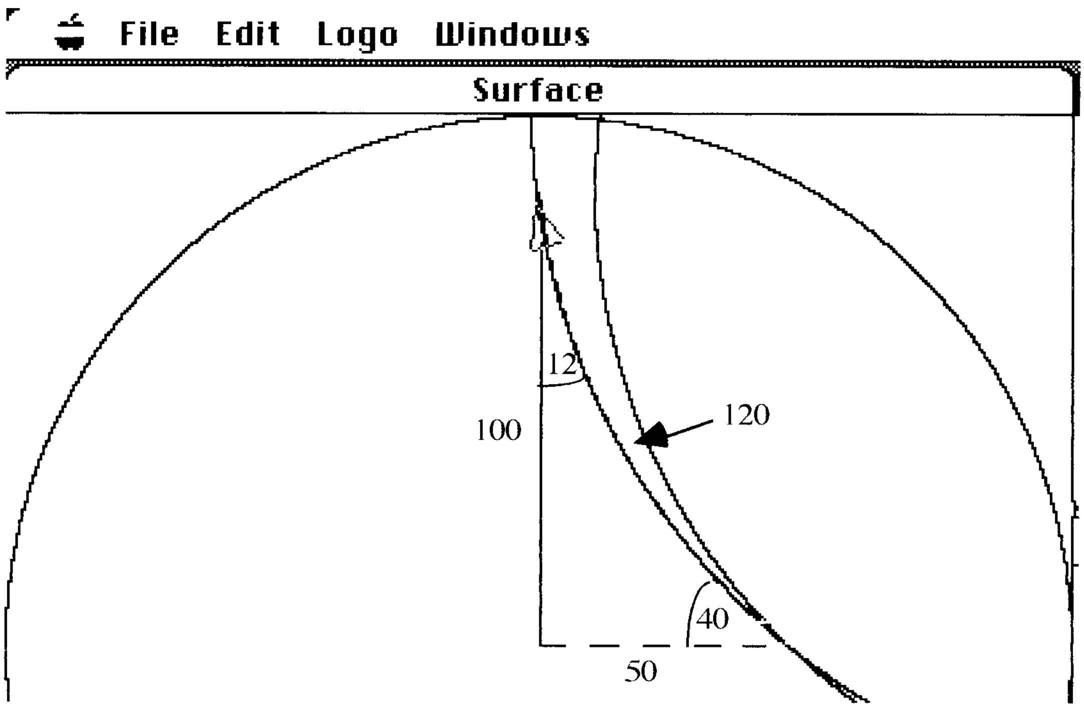


Figure 8.16 Paul and Sean's Triangle

They decide to find the value of the third angle in the triangle by positioning the Turtle at the end of the vertical line, which is 100 steps long, and rotating it. Using a combination of turning the Turtle and Path, they find the value of the external angle to be 168° and this gives an internal angle of 12°. Sean calculates, using Pythagoras, that the hypotenuse for a triangle with sides of 100 and 50 should be 112 and comments.

Extract J

S: The projection defies Pythagoras. No! Hang on, walking on the surface is defying it, isn't it! 'cos we walk straight lines on the surface we just see them as curves on the projection.

(*P picks up the cone and moves finger from the base up and away from him on one side. He then lets his finger run across and off the surface*)

S: It's the angles that throw it out.

P: Yeah, 'cos where in fact you are going from (*running finger from apex of the cone up the side and turns head on one side to look at the position of his finger on the surface of the cone*), you're going from that point there (*indicates apex*) to that point there (*indicates a point on side of cone towards the computer screen*)

S: It's the curve ....it's because of this steps business, because we have to walk further to get the same distance.

P: Yeah.

S: The Turtle walking sort of like that (*draws top row in Figure 8.17*).

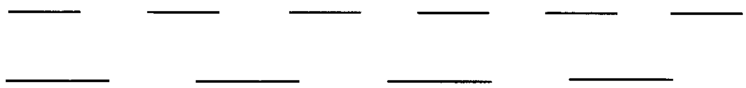


Figure 8.17

and for our straight line it should be doing that (*draws the bottom row of Figure 8.17*). It's having to walk more to keep up. (Pause). What if we make it an equilateral 'cos then we are walking on the same level (*picks up cone*). If we make that a 100 out from here (*indicating point on the cone which is 100 straight out from its apex*) to walk in a straight line from one to the other (*move finger around the inside of the cone until he reaches a point on the cone which is at the same vertical height from the apex*). No, we're not walking on a level aren't we?...No, it wouldn't be..forget that you'd be walking round the curve like that (*Sean indicates the difference between moving directly out from the apex and moving around the cone*) and we'd end up with a curve line between (*Paul traces out two lines from the apex and one around the inside of the cone. Sean does the same*), but you'd hit the same problem (as with the previous triangle). What you could work out is relative step size. You could work out how much more you'd have to walk up here (*Sean moves finger across and up the inside of the cone, indicating the distance to be walked across the surface*) to make it the same distance.

Three points were interesting here. First, Paul and Sean found that Turtle C did not obey euclidean geometry, since it did not produce an angle-sum of 180° nor did it satisfy Pythagoras's Theorem. This had been found by using the various facilities of the software, such as Path, and manoeuvring the Turtle to measure the angles of the triangle. However, Sean was clear that the Turtle was moving in straight lines on the surface of the cone and so was tracing out a triangle, perhaps not a euclidean triangle but nonetheless a polygon with three "straight" sides (lines 1 -4).

A second point was that Pair E were trying to account for the Turtle's behaviour in terms of its motion on a curved surface. Sean, and probably Paul, had noted the difference between the "Turtle's eye view" of moving forward in a straight line and their perception both of the cone and the curved screen motion of the Turtle. The Turtle metaphor had been transferred to thinking about moving on the cone in a local and intrinsic way and this may be related to their use of the cone (lines 15 - 20). When they used the cone, it was mainly to "think through" what they were doing and they both made use of their fingers to trace out what they thought might be likely paths on the cone's surface (Lines 22 - 35). This was particularly noticeable during the last part of the extract, in which Sean was thinking about what the path for an equilateral triangle might have. He traced out the path which he thought would close the triangle, but almost immediately rejected it because it was not curved in the correct manner. Sean then made the suggestion that the path of the Turtle required to close the triangle must be longer, because of the curvature of the cone.

The third point concerned Sean's attempt to explain the failure of Pythagoras which was based on the dashed lines shown in Figure 8.16. As he put it in line 22, the Turtle was "having to walk more to keep up". Comparing the path that a euclidean Turtle took with the one which the non-euclidean followed between the same two points, the latter would have to walk further than the former. The reason for this was that the non-euclidean Turtle was moving on a curved surface. His use of the dashes to illustrate his point in Figure 8.17 was significant because the dashes seemed to help him come to an understanding of the underlying surface and provided him with a means to develop that understanding. This is a point which he developed in the next episode.

8.3.2 (c) Step Size and Geometry

This extract concerns the way in which Pair E developed an understanding of the effect which the curved surface has on the size of step that they take. It begins with them discussing an isosceles triangle that they have just found. It has two sides of 100, perpendicular to one another, and an internal base angle of 15°. The hypotenuse of the triangle is 165 and they compare this with a euclidean triangle, which should have an hypotenuse of 141.42 = ($\sqrt{2}$x100). They calculate the ratio of 165 to 141.42 as 1.17. They draw an equilateral triangle with 60° angle between two lines, each 100 steps, coming from the origin. They estimate its third side to be 100 x 1.17 using the previous figures. But it fails, and Sean asks whether it makes sense.

Extract K

S: We are not in the same region. We've done different things. Over here (*traces finger across the screen in an upward arc*) we start with smaller steps, get bigger and then come back to smaller .....

P: Have not you got ...?

S: Yeah, it's travelling through a different distance on the way, but the steps change every time you move out, don't they? So it gets even more complicated than that.... 'cos the size of the step is related to the ...hang on ...it's not related to how far out you are.....(*picks up cone and indicates from the apex out*). Went out from there a 100.

P: Yeah.

S: We got it to walk that line (*indicates round the inside of the cone*) but this curve takes it ....(*indicates that the curve moves his finger upwards*)..hang on, it should be equal steps at that point. They should be equal.

P: 'Cos you're at the same level.

S: In fact they are equal....roughly (*points to the screen image of the hypotenuse and notes that the dashes are roughly equal*), but when we do this one at 90 (*traces out two lines from the apex of the cone and up its side with 90° between them*) it does not go through the same points (*compares the path around the inside of the cone between the 90 degree pair and 60 degree pair*).

Figure 8.18 below illustrates Sean's point about the relationship between the paths and the angles. Triangle OAB was isosceles and triangle OBC was the triangle which they were hoping to make equilateral. Comparing the dashes on the hypotenuse of the triangles, Sean and Paul noted that they were "roughly" the same length. However, comparing the hypotenuse of the two triangles on the cone, Sean points out that they cannot be at the same level, since they pass across different parts of the cone's surface, as the "curve takes them".

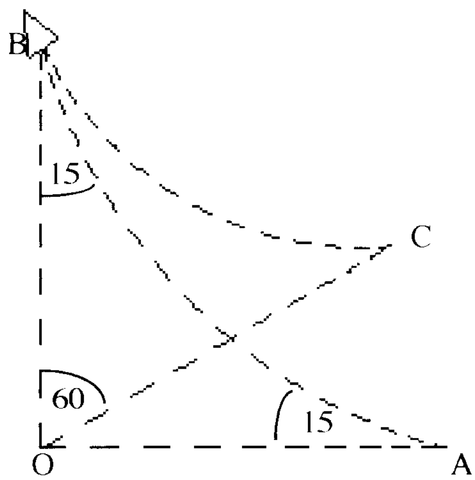


Figure 8.18 Comparison of Isosceles with another Triangle. Screen shot

They try to summarise their work at the request of the researcher.

Extract L

R: So what do you think that you have found out about this now...about Turtle C and the way this works ?

S: It's still so much easier to visualise than the first time we did this, just because we've got a surface to play with. I'm finding it so much easier to work out things like relative steps; things like this wouldn't have occurred to me without having seen that first (*indicates the cone*), because we just had the flat Escher-esque diagram.

R: Umm.

S: And I couldn't relate to that at all. Not the way we related A to the sphere, which meant that we then clicked on relative steps. I've done the same again with C just because of that (*points to the cone*).

R: When you say relative steps, you mean the difference between the steps the Turtle would actually take and the projected step on the screen?

S: Well actually..no..yeah it is that.

P: Are you thinking about relative steps .. at one point relative to a point further out from the centre?

S: That comes as well....on a flat surface we're just used to thinking about the flat all the time and we know all the geometry and the various things that apply to that, but when we try on this (the computer) it doesn't work. The reason why it doesn't work is that the surface we see is bigger than the flat surface, because of this dip, which means the turtle has to walk more..it has to walk a further distance and that's what we've got to make it do ..to .. to finish off the path. There's the other one Paul mentioned. (The) further out you go, that step size changes all the time; the more that you are out, the more you've got to run to make up.

There are two points of interest in this extract. The first was Sean's attempt to grapple with the problem of how step size varied with distance. He tried to relate it to the effect that the curve of the surface had on the Turtle's motion by comparing the two triangles shown in Figure 8.18. Although they both started at the apex of the cone and were produced by moving out the same distance (100 steps), the fact that the two sets of lines were at different angles (one at 60 and the other at 90) meant that the curves which closed the two triangles moved over different parts of the cone's surface and had different lengths. Sean and Paul were, perhaps, still thinking of this in euclidean terms and expecting that there was a circular arc which passed through them both. However, the Turtle moved on hyperbolic paths as its "straight lines" in this surface and so the portions of hyperbolas containing the hypotenuse of each triangle might coincide at B in Figure 8.18, but they would be "taken by the curve" in different directions to reach A and C.

Sean appeared to use the term "relative step size" in at least two ways. He first agreed with the researcher's formulation as it meaning: "the difference between the steps the Turtle would actually take and the projected step on the screen"(line 44-45). He also agreed with Paul's understanding of the "further out you go, that step size changes all the time" (line 55-56). He adds his own gloss that the "more that you are out, the more you've got to run to make up" (line 56). The first sense of relative step size drew attention to the difference between the local and intrinsic sense of the Turtle's regular step, to use the metaphor, which produces a straight line on the curved surface and its projected step with variation in length. Paul's sense of the term, however, was related to the position-sensitive nature of the Conformal model, in which distance measure varied with position. In fact, both are legitimate interpretations of the term "relative", since the first captures the sense in which distance varies as a result of the projection process, while the second was based on the observation of the screen dashes.

Sean's remark about the Turtle on the surface having to run to make up suggests, and his observation that the hyperboloid was "bigger" in some way than the flat plane implies that he was using an interpretation of curvature which could reconcile both aspects (line 51 - 54). His image seemed to be that of the Turtle moving on a hyperbolic path between points on the cone and having to compensate for the "dip" produced by the curvature, which made the surface bigger than euclidean space. This is reflected in the change of length of the dashed Turtle tracks on the screen, which vary according to motion relative to the screen centre and are also "longer" than a comparable euclidean distance measurement between any two screen points.

The second point of interest was the use which Sean and Paul made of the cone in their reasoning. As they were constructing the triangles they worked entirely with the computer screen, making use of Path to find a suitable angle with which to close the Triangles. However, when they needed to think about some aspect of the geometry, they used the cone and traced paths out on the its surface (line 5 -14). Having resolved their problem, they then returned to the screen (lines 15 - 19). The surface was an important aid to their thinking as they became more proficient at using the software. In fact, the development of their facility with the software changed the way in which they used the cone. It was not just something useful to visualise the unusual behaviour of the Turtle, it seemed to become intrinsic to their thinking by providing a means to guide their investigation and interpret its outcomes.

Finally, there was the use of the dashed Turtle tracks to reason about its position on the cone. At the start of the extract, Sean commented about the region that their triangles were in, on the basis of the variation in the length of the steps (line 1 - 4) and concluded that the length of the dashes was related to position on the cone. This was significant because, in the final part of the extract, Sean relates the use of the conical representation to thinking about step size and the underlying curvature of the surface. Here all aspects of the technical element of the microworld were brought together to develop a model of how the Turtle on the screen was related to motion on the cone. Sean was able to look at the cone, imagine the Turtle's behaviour on it and, at the same time, relate it to the screen through the dashes of the tracks. He was then able to make specific comments about the curvature of the cone in which he reasoned from the screen to the cone.

These two elements of physical surface and screen image seemed to mutually condition one another in the following way. The features of the screen, such as the Turtle's speed variation as it moved and its dashed tracks provided a sense that it was moving on a non-flat surface. The cone and its use to interpret the screen images provided a non-flat surface to think about and with. The link between the two was provided perceptually by the visual and tactile experience of the cone and the features of the screen. Sean appeared to be able to "see" the curvature of the cone through the dashed tracks and the Turtle motion and, at the same time, guide his choice for the Turtle's path by using the cone. Linguistically, he spoke of the "Turtle walking on the surface" of the cone and this use of the Turtle metaphor seemed to provide a way of linking the object and the screen.

8.4 Reflecting on Cycle 3

This final section is concerned with drawing together pedagogical and cognitive aspects of the final developmental cycle. The first section deals with the pedagogical element of the microworld and how effective it was in introducing the Conformal models. The second section, §8.4.2, deals with the cognitive aspects of the final cycle. In particular, it discusses the contribution that the software made to their understanding of the geometry.

8.4.1 Reflecting on the Pedagogical Strategy

The main focus of this cycle's activities was the third phase of the microworld, which was concerned with the use of the computer-based models. As §8.2 pointed out, a three-stage introduction to the activities of this phase was facilitated by means of the conical representation of the hyperboloid and the images of plane sections of the hyperboloid. An important question was therefore: how effective had it been?

It was clear from both pairs that the structured introduction of Turtle C gave them a reasonable starting point and enabled them to make sense of its behaviour. Both pairs were able to use the cone and the diagrams of plane sections to relate motion on the curved surface to the screen motion of the Turtle. The two extracts, §8.4.1 and §8.4.2, show, however, that the direction which the pairs took varied as they tried to make sense of the didactic introduction. Pair D's investigation was generated by the needs of one of the pair, Tim, which produced interesting results. Pair E, although again dominated by the interests of one of the pair, went in a direction which was unhelpful within the time-frame of the activity.

The strategy of structured introduction by a "teacher" followed by open-ended enquiry based on activities was effective, provided the activities were carefully monitored. The researcher was able to act as a counsellor / guide when he felt that the participants were going in a productive direction both for themselves and for the agenda of the research. If they did not, then, he, the researcher, needed to intervene. The participants' acceptance of the intervention was, perhaps, affected by their familiarity with the situation and a tacit acknowledgement that this was for the researcher in some way (which of course it was!). Hence, in assessing the effectiveness of the strategy as a pedagogic process, account must be taken of the fact that this was done within the context of research.

The strategy seemed effective in the sense that the participants understood the models and could use them either to investigate what they had been told or explore new avenues. The type of pedagogic intervention was also important. The flexibility of approach suggested by the four roles outlined in Chapter 7 enabled varying degrees of formal and informal support to be given. This allowed the researcher to be involved with the participants where necessary, without coming into conflict with the research agenda. On the few occasions where conflict did occur, the researcher was able to guide the participants in directions which suited the research agenda.

8.4.2 The Cognitive Development of the Participants in Cycle 3

The stability and accuracy of the software, with its new features of Path and dashing, and the use of the conical representation of the hyperboloid, had a considerable impact on the cycle. As the previous section indicates, the pedagogical strategy of the microworld could be given a coherent framework which covered both Conformal models with the introduction of a physical surface for hyperbolic geometry. These reflections will consider the impact of the software and the cognitive developments associated with the participants' use of the technical element of the microworld.

As the introduction to §8.3.3 indicates, two issues were explored in analysing the participants' activities with the microworld: what they understood and how it was understood, both in relation to their use of the technical element but also how this related to their prior experience of geometry. However, a pre-requisite of any sort of development happening was a stable and accurate computerised version of the Conformal models for the participants to work with. This really only came about during this cycle and what characterised the activities of the participants was a greater air of confidence with the software. They were able to work in a detailed way with Conformal models and obtain support from the features of the screen such as the Path button, the dashed Turtle tracks and, to a lesser extent, the Boundary button. These were all important because of the non-intuitive nature of Conformal model B which formed the basis for this cycle's activities.

From a cognitive point of view, the aspects of the cycle which related to specific geometric facts that emerged were Pair E's discovery of the failure of angle-sum and Pythagoras. However, Pair D's investigation into the forming of "hyperbolic" squares yielded an understanding both of the geometry of the Conformal model and the structure of the surface which produced it. In both cases the central axis around which their understandings developed was the relationship between the physical surfaces and the screen images of the Conformal models. In the case of Pair D, the link developed by them between the screen and the cone formed the basis of their investigation. This led them, in turn, to re-assess their understanding of the conical representation's structure. For Pair E, the conical representation was both a stimulus for investigation and a qualitative means of interpreting the screen. However, both Pairs' experiences suggested that the relationship between the screen and the physical surfaces was dialectical in the sense that each made distinct contributions to the development of the participants' understanding, but each was mutually conditioned by the other in that understanding.

This can be illustrated in two ways. The first relates to the way that the participants moved between the physical surface and the screen. Initially, they used the cone to help interpret the screen and this consisted of a rapid movement of attention from one to the other. Later, as the participants investigated the screen in more detail, they worked on the computer in a precise and quantitative manner and referred back to the cone for qualitative information and to think through what they were doing. This process suggested that the link between the surface and the screen was such that the participants needed both qualitative and quantitative aspects to make sense of their experience.

Second, the features of the screen, particularly the dashed Turtle tracks, provided some insight into the curved surface which produced the screen images. This can best be summarised by some comments from both Pairs as they talked about their experience.

Extract M

R: Did the dashed lines help?

S: They did in a way, because you could tell by the length of the dash how far the projection was away from the origin. The shorter the dashes the further we were. Whereas before, when we used it last year, we'd no real idea of distance.

Although not stated explicitly here, Steve had made the connection between variation in distance measure and position, which was crucial to the functioning of the models. Information about the model was available "at a glance" by the dashes, because they could be seen immediately the Turtle moved and comparisons could be made implicitly between its effect in different parts of the screen. Pair E understood this dashing in two ways. First, as giving information about the underlying surface, and second, as providing a sense of perspective. S is Sean, P is Paul.

Extract N

S: It shows you that the surface is changing n some way. The surface is changing as you move along it, so the step size is definitely easier to see. Be interested to see what it's like on B (*Turtle B*); is it B?..spherical, A is the spherical. (*Sean tries Turtle A. This produced a dashed line, which got longer as the Turtle moved radially outwards*). It's the opposite way around. You can tell that something's happening on the surface there, straight away.....with the benefit of hindsight, because we know what we are talking about. It would be interesting to know if someone who did not know could use it!

P: You do get an impression of some kind of perspective coming closer or going further away as you're taking the steps .

Sean was "seeing" the surface through the dashes in the sense they provided a qualitative clue that the surface was changing. Although he noted that this was with hindsight, it suggests how the screen might have conditioned his interpretation of the physical surface. Paul, on the other hand, makes a remark about perspective which related the dashes to a more familiar experience.

This latter point brings out the final aspect of cognitive development: the role of the participants' experiences of geometry and their use of imagery and assumptions in developing an understanding of the geometry of the Conformal models. Paul's reference to perspective indicates how the dashing of the tracks enabled him to relate the screen behaviour of the Turtle to a familiar idea, and this presumably enabled him to make sense of what was happening. The impact of assumptions on how the participants developed their understanding can be seen in Pair D's work. In §8.3.3(a), Tim was convinced that there was a relationship between distance from the centre of the screen and internal angle, and this guided their activities. When it became clear that this was not the case, Tim remained convinced that there was some sort of rule, although he could see physically why it was incorrect and identified the incorrect assumptions that he had about the conical representation. From a different point of view, Steve's mathematical knowledge was also important in developing an argument which showed why their hypothesis could not be correct. Both used assumptions and prior knowledge to guide their actions and interpret its outcomes.

Chapter 9

The Conclusion: Journey's End

9.0 Introduction

The aim of this study was to explore the possibilities for constructing a computer-based context for teaching and learning non-euclidean geometries using their euclidean models. The exploration was also intended to provide windows on the processes of teaching and learning, and the design of a computer-based context. This Chapter will assess the extent to which the study met these aims.

Central to any assessment process is the choice of criteria by which to judge the material being presented. A first step in judging whether the aims of the study have been met is, therefore, to establish how the microworld and its development should be assessed. An immediate problem is the lack of an agreed theoretical base on which to develop a set of criteria for assessing microworlds. Several examples of microworlds have been mentioned in Chapter 3, but, as §2.1 indicates, even the definition of the term "microworld" is debatable. Indeed, this study can be seen as a contribution to the debate both about the meaning of the term "microworld" and the development of a set of principles for describing and assessing microworld construction.

In light of this, a suitable place to start might be to consider what one is trying to achieve with a microworld and the relationship between the "end product" and the process of constructing it. As §2.1.1 shows, microworlds are pedagogical devices in the sense that they are formed from specific knowledge domains with the intention of inducting learners into those domains. However, the intention in using microworlds is to foster an active exploration and construction of the knowledge domain by the learner. Turtle Geometry, implemented in Logo, was designed as "a natural learning environment for an experimental approach to mathematical ideas and process" (Feurzig 1969 cited in Hoyles and Sutherland 1989 p.6). Turtle Geometry was "natural" in the context of Piagetian psychology through the syntonic nature of the Turtle metaphor and it was also intended to be active and experimental. This, perhaps, provides an insight into what a microworld should be aiming for: a *natural* introduction to a knowledge domain which actively engages the learner directly with the structures of the domain, but in a way suited to the learner and not an expert. Papert's reference in *Mindstorms* (1980) to his childhood experiences with cogs and gears acting as transitional objects in developing and consolidating his mathematical understanding provides a sense of this "naturalness". The elements of the microworld must enable the learner to connect with and become embedded in the knowledge domain at a pace and in a way suited to them. A criterion for judging a microworld might be, therefore, the extent to which the learner can *connect* with the knowledge domain in a way which provides cognitive continuity. Alternatively, in the words of Sean from Pair E, does the microworld enable the learner to "develop a feel" for the knowledge domain? In §9.1, this issue will be explored in relation to the microworld developed in this study.

Related to the question of how a microworld should "feel" to the learner, there is the issue of how to design the microworld so that this can be achieved. The exploratory nature of this study was designed to enable the examination of this process of microworld construction, so that critical factors might be identified in relation to teaching and learning non-euclidean geometry. These will be discussed in §9.2 and an attempt will be made to address the more general relationship between design and cognitive development.

In §9.3 the limitations of the study will be discussed and §9.4 will outline a number of issues arising from the study which might merit further consideration. Finally, the implications of the study for teaching and learning will be discussed in §9.5.

9.1 "Developing a Feel" for Non-Euclidean Geometry

An interesting and unexpected aspect to the cognitive development of the participants, was the way in which they appeared to construct their understanding of the geometry at the same time as they gained fluency with the software. It was not the case, as originally thought, that there was a linear development from work with physical objects to fluency with the software. Rather, there was a continual reference by the participants to the physical surface as they worked with the computer.

The development of understanding seemed to be based on the interplay between the technical, pedagogic, and cognitive aspects of the microworld. It is this interplay between the surfaces, computer screen, and the activities of the microworld, which will be considered, and the role of each component in the process will be examined. In particular, this section will discuss the role of visualisation, together with the technical and the pedagogic elements of the microworld in the process of the participant's cognitive development. It will conclude with an attempt to interpret the overall process in terms of the study's theoretical framework.

9.1.1 The Role of Visualisation

Visualisation, both internal and external, played an important role in the participants' development of understanding. The internal form of visualisation took two forms. First, there was the process in which participants mobilised their own visual experience and imagery to interpret images, both on the screen and on paper. This was best illustrated by Michael's experience in Cycle 1, as he was able to work out that the Escher print was the result of projecting a hyperboloid surface onto the flat plane. He did this by gradually drawing together features of the print and, using his own experience of projecting surfaces onto the plane, developed a scenario which provided an explanation of the print. Michael's quite remarkable understanding of the Escher print as a projected image was built on his capacity to visualise the relationship between a curved and a flat surface. The second form of this internal visualisation occurred through the participants' use of their own visual intuition to provide a context in which they could interpret the screen images. For one participant, this was the idea of motion across a valley floor, which he used to interpret Turtle A's behaviour. Another participant interpreted the patterns produced by Turtle A in terms of the type of curved polygons used to construct a football. They both needed some sort of context in which to locate and interpret the Turtle's motion.

A second form of visualisation occurred as participants internalised the external relationship between the physical objects and the computer screen. Two episodes in the second cycle illustrate this. For Sean in Pair E, having the sphere and the computer model enabled him to develop a representation of the connection, which he could use to interpret what was happening on the computer screen. This "internal" representation, based on using the sphere, appeared to give him the means to visualise for himself what the Turtle behaviour on the screen corresponded to on the sphere. Tim, in Pair D, used the sphere to argue that the screen Turtle was behaving incorrectly and was apparently able to visualise what the projection of a line on the sphere *should* look like on the screen. In both cases, the sphere seemed to play an important role in enabling the participants to interpret the Turtle's behaviour and its relationship to the geometry of the surface.

9.1.2 The Role of Objects in the Process of Understanding.

Having physical surfaces to work with played an important part in the participants' developing an understanding of the screen images. As §9.1.1 indicates, having the sphere to look at and think about helped the participants to internalise the connection between it and its Conformal model. Similarly, in the third Cycle, having a conical representation of the hyperboloid enabled the participants to develop an understanding of Turtle C (Conformal model B). However, the way in which the participants used the physical surfaces differed from that implied by the model of the microworld. Initially, the sequence of object-use in the technical element of the microworld was represented as a linear progression in the microworld's model. It began with some work on the computer as part of the induction phase. This moved on to use of curved surfaces, images of their projections, and finished with the computer screen again. Figure 9.1 illustrates this progression.

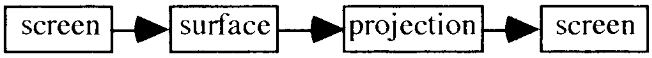


Figure 9.1 Expected Linear Progression of Surface-use

Corresponding to this, the "cognitive" strand was intended to start with confusion and end with confident use of the software by the participants, who no longer would need the support of the surface and projections. As the participants grew progressively more confident with the software, therefore, the scaffolding provided by the pedagogic element would "fade".

In practice, however, the progression from physical objects to screen was not linear but cyclical. The participants used the surfaces in different ways as they developed an understanding of the projections and the screen representations. Initially they worked with the surface, images of projections, and the computer screen. As they grew more confident, they no longer used the projected images, working exclusively with the screen image and the surfaces such as the sphere and the conical representation of the hyperboloid.

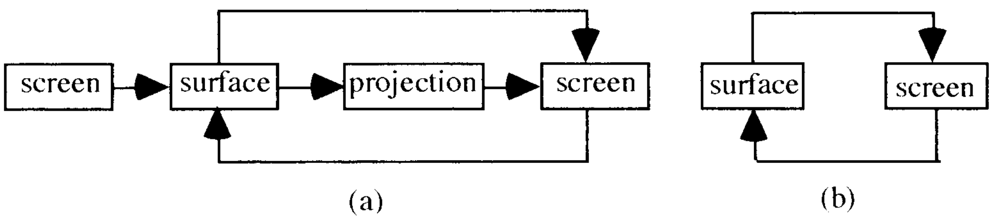


Figure 9.2 Actual Use of Surfaces and Screen.
(a) shows the initial movement from screen to surface to projection to screen was as expected.
(b) actual use of surfaces by the participants.

Figure 9.2 illustrates the actual use made of the surfaces. Figure 9.2(a) shows expected development: screen —> surface —> projection —> screen. However, as the participants started to use the surfaces such as the sphere and conical hyperboloid to understand the Conformal models, they moved between the computer and the physical objects, shown in Figure 9.2(b).

Two things were interesting. First, there was no fading of the scaffolding, in the sense of the participants no longer making use of the physical objects or images of projection, as the pedagogic structure implied. Rather, the participants used both the surfaces and computer images to explore the geometry of the respective Turtles and check their new intuitions. Second, what did "fade" was the need by the participants to refer to the images of projection. These images were no longer required to support the process of understanding how the screen models were obtained from projection and appeared to be internalised by the participants as part of the background which connected the foreground of physical objects and screen images.

From a cognitive point of view, the diachronic view of the microworld also anticipated a linear development, corresponding to the linear progression described in Figure 9.1. However, the use of the surfaces in the technical element of the microworld suggested that this cognitive development was also cyclical. Figure 9.3(a) represents the linear structure of Breakdown —> Construction —> Fluency first envisaged by the microworld's model, and Figure 9.3(b) represents the iterative structure suggested by the three cycles and implied by the participants' use of surfaces.

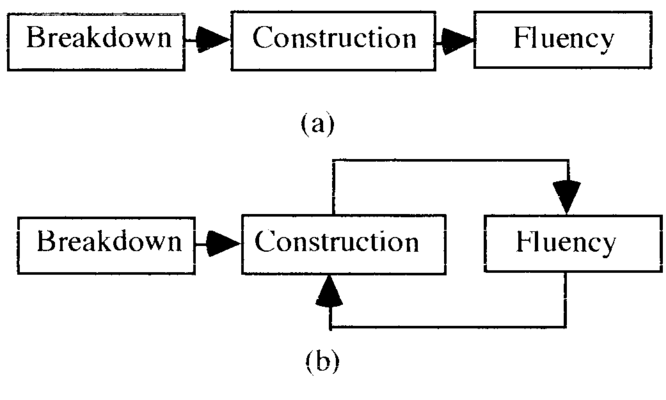


Figure 9.3 New Structure for the Cognitive Element of the Diachronic View.
(a) This represents the linear structure of Breakdown --> Construction --> Fluency first envisaged.
(b) This represents the iterative structure suggested by the cognitive outcomes of the three cycles.

The process outlined was structured around the relationship between the physical surfaces used and the computerised Conformal models. It was iterative and could be divided into three stages. The first stage was concerned with the participants' introduction to the projective connection between physical surfaces and the computerised Conformal models. They were shown images of the projection process which then were related to the behaviour of the screen Turtle. The second stage involved the participants trying to understand for themselves how the projection worked using the components of the technical element. Finally, they moved on to explore the screen images for themselves using both the physical objects and the computer.

The main activity of the participants occurred in stages two and three. Stage two of the process may be characterised as a *learning* phase. It began with the participants moving their attention rapidly between the physical objects and the screen, learning features of the computer-based Conformal model. The participants checked features of the screen against features of the physical objects, as they were engaged in exploring the way in which the computer-based Conformal model represented the physical surfaces. Gradually, the projective relationship seemed to move into the background as they became progressively more confident about the connection between the surface and the computer. This learning phase was followed by an *exploration phase*, Stage 3. In this stage, the physical surfaces were used to "think through" ideas which were then tried with the computer and its results interpreted by referring back to the surface. What characterised this stage was a clearer division between the time spent with the physical objects and with the computer. Participants began by thinking, talking and looking at the physical surface before moving to the work at the computer with the Turtle. If there was a problem, or they did not understand what the computer had produced, they would go back to the physical surface and review what they thought the Turtle had done. In the exploration phase, they seemed to have developed an understanding about the basic structure of the representational relationship and moved on to explore the computer-based models for themselves. However, their reference back to the physical surface, either to guide their investigation or make sense of the screen image, suggested that the distinction between exploration and learning was not absolute.

9.1.3 The Role of the Software

An important achievement of the study was the production of a working computer-based version of the Conformal models that was stable, accurate and reasonably fast. The process of development has been described through Chapters 6, 7 and 8. Without the software there would not have been a study! The choice of an object-oriented paradigm (OOP) for implementing the microworld was very productive. This was due, in the main, to two things. First, the mathematical connections between Turtle and differential geometry, noted in Chapter 3, were explicit in the calculations which the software made to move the Turtle. Second, the Logo version of the OOP enabled the connection between the Turtle and specific non-euclidean geometries to be made explicit. The Turtle object created by the software could exist in three separate forms of geometry, which each had their own specific properties. In this sense, OOP was ideally suited to the task of implementing the Conformal models, since it explicitly provided programming structures to enable the construction of objects in their own "world".

Besides noting the general success of the software, this section will focus on two aspects of it which are thought to be relevant to the participants' development in understanding: Turtle Geometry and the "feature" of the software.

An interesting feature of Turtle Geometry relevant here is the relationship between the local and intrinsic nature of the commands to the Turtle, and the global and extrinsic information often needed to draw shapes. For example, consider the following sequence of "ordinary" commands to draw a square: **repeat 4 [fd 50 rt 90]**. The use of the **repeat** control structure is needed to stop the Turtle after the required number of sides have been drawn. This information, needed to close the square, comes from visual and geometric experience that is both euclidean and extrinsic to the Turtle commands and which must be added to the sequence to obtain the required effect. In the participants' first contacts with the computer-based models, part of the confusion experienced by them was the lack of this visual support provided by their euclidean intuitions. Turtle Geometric commands were issued, but the participants were unable to relate the Turtle's behaviour to their knowledge of euclidean geometry. It may be that the participants' need for the physical surfaces as they are working with the Conformal models was an attempt to provide the geometric support required to make sense of the Turtle. They moved the Turtle, using the local and intrinsic commands, but were unable to draw on the global and extrinsic support usually obtained implicitly from looking at the flat euclidean screen. They then found this by "having an object to think with". The software could be said to have created the need for an object, since ordinarily, Turtle Geometry supported the learners' euclidean intuition in their understanding of the Turtle's behaviour.

The second set of issues related to the software was the impact of the two features added in the third Cycle: the Path button and dashing of the Turtle tracks. The Path button was introduced to overcome some problems with accuracy that arose during Cycle 2. Although the software had been improved considerably in Cycle 3, so that the original issue of inaccuracy which had given rise to Path no longer applied, it was thought to be a useful addition and was implemented as a button. The participants' use of this Path feature confirmed it as a useful "tool" for exploring the Conformal models and, in particular, for finding headings for the Turtle. The dashed Turtle tracks were derived from the idea of the Conformal models as the Hot and Cool Plate Universe[1]. The tracks showed the variation in the length of the Turtle's "step" as it moved in each model and was a key feature of the Conformal models which marked them out from the euclidean case.

In both cases, the features of the software seemed to provide the participants with information on the global behaviour of the Turtle from local variation. The Path button showed the global effect of changes in the heading of the Turtle, by definition a local and intrinsic measure. The dashed tracks gave information about the Turtle's position and the distances it had travelled in each model, by means of a local contrast in the length of the Turtle's step. Both of these features played a significant part in enabling

[1] See §1.2.2 and §1.2.3

the participants to navigate the Turtle and support the relationship between the physical surfaces and the computer at a perceptual level.

As the comments in §8.4.2 indicate, the dashed Turtle tracks also provided the participants with important perceptual information about its position within the Conformal models and about the surface which had produced the model. One participant spoke about the sense of perspective which the dashes gave so that it was possible for him to decide from the screen how the Turtle might be moving on the surface. Another felt that the dashes gave an indication that the Turtle was on a surface which was changing in some way. The dashes seemed, therefore, to refer the participants back to the physical surface that they were using and to support links between the screen and the physical surfaces. They also mobilised the participants' visual intuition and knowledge, which played an important part throughout the cycles in enabling the participants to make sense of the puzzling images that they were presented with. The comments by one of the participants about perspective suggested that the visual effect of the dashing enabled him to connect the screen with visual experiences that were familiar and understood. In a similar manner, the Path button gave information about the model and supported the link with the physical surface through the identification of the screen path with the path that was seen and traced out on the physical object. Using the button enabled the participants to see the global structure of the Conformal models and emphasised the relationship of the structure to the heading of the Turtle, rather than its position.

In both cases, the usefulness of the features seemed to reside in the fact that they highlighted aspects of the knowledge domain which both mobilised visual intuition and previous experiences for the participants and supported the object-screen link. The Path button encapsulated both epistemological and perceptual aspects of non-euclidean geometry into a specific function which was "ready-at-hand". The dashed Turtle tracks integrated aspects of the model's structure with the dynamic behaviour of the Turtle to produce screens in which the Turtle moved faster as its dashed steps got longer and slowed down as its steps got shorter. The participants used the Path button in their investigations and derived information from the dashed tracks which enabled them to interpret the Turtle's behaviour. Together, these visual and dynamic aspects seemed to enable the participants to make sense both of the screen and its link with physical surfaces.

9.1.4 The Role of Pedagogy

The three-stage pedagogical strategy which was developed in Cycle 3 came mainly as a result of introducing the conical representation of the hyperboloid. The first stage was to let the participants use the computer-based models without any support. The intention was to establish what sense they could make of the screen images and to challenge their euclidean intuitions about Turtle geometry. The second step was to introduce the participants to the sphere and the conical representation of the hyperboloid and to discuss with them what they knew of spherical or hyperbolic geometry. This led to an exploration of what terms such as "straight line" might mean in the context of non-euclidean geometry and the introduction of the language-game associated with it, as outlined in §4.4 - §4.5. Straight lines on either the sphere or the hyperboloid were described as being the result of taking plane sections of the sphere or hyperboloid which pass through the origin of coordinates. The final stage in the pedagogic strategy was to introduce the participants to the projections of such lines onto a flat plane and to connect these projected images with the computer-generated Conformal models. The participants were then given activities that helped them to explore the surfaces and computer-based models, or investigate either for themselves.

The creation of meaning through this pedagogical strategy consisted in using a combination of visual images, together with the participants' ability to touch physical surfaces, to introduce particular ways of speaking. In transferring the referent of the term "straight line" from a euclidean to non-euclidean context, the pedagogical strategy was establishing the connection between the two by both linguistic and extra-linguistic means. Some participants found this transfer difficult to understand and required some time with the various technical components, such as the physical surfaces and images of sectioning, before they were ready to accept the connection.

As a consequence of this difficulty in understanding, two sorts of activities were used with the participants. The first type of activity consisted of those tasks created by the researcher to support the process of meaning construction, and to foster independent investigation of the computer-based models by the participants. The second type of activity was generated by the participants, as they tried to make sense of the situation posed by the microworld and the links between the components of the microworld's technical element.

The activities reflected different pedagogic styles. The counter-intuitive nature of the Conformal models necessitated some form of didactic introduction, as the participants were unable to make sense of the images generated by the computer. They had to be guided, formally, through the connection between the physical surfaces and the projection which produced the Conformal models. The first type of activity was related to the didactic introduction of the Conformal models during Phase 2 of the microworld. The researcher led the process, either by formal teaching episodes or by providing structured activities. In the second type of activity, generated by the participants, the researcher played an informal support role. These approaches tend to support Hoyles and Sutherland's (1989) analysis of intervention strategies, which balance participants' autonomy in learning against the pedagogic agenda of the microworld.

9.1.5 "Developing a Feel": How is it Possible?

Having identified the role that the various aspects of the microworld played in aiding the participants to develop an understanding of the Conformal models, the question remains as to their significance. What is the connection between these factors and "developing a feel" for non-euclidean geometry? To address this question, it is necessary to review some of the theoretical ideas which underpinned the microworld and its construction.

The position adopted in Chapter 2 was that cognitive development is rooted in our fundamental experience of thrownness. We are embedded in a social process which conditions our cognitive growth, but to enable that cognitive growth it is necessary to separate oneself from the social process. Such *distancing* of ourselves from others enables us to gain a sense of "me" and "them". This sense of being *simultaneously* embedded in the social process and separate from it, is supported and maintained by the role of language, understood as a mediating sign system. The process of understanding the Conformal models was interpreted as learning a language-game in which the computer-based models were *embedded*, with the intention of the participants gaining fluency with it.

The counter -intuitive nature of the models implied that they had to be explicitly taught as a way of inducting the participants into their use for non-euclidean geometry. This gave rise to the pedagogical strategy built around the projective connection between the physical surfaces, such as the sphere and hyperboloid, and the computer-based models. From the participants' point of view, this gave rise to a process of learning and exploration of the language-game that they were inducted into. A significant feature of the learning-exploration process, described in §9.1.2, was the way in which the participants spent progressively longer periods working with the software as they grew more confident with it. They referred to the physical surfaces when something did not make sense or they wanted to develop a new line of enquiry. Two aspects of this are interesting. First, as the episodes in Cycle 3 indicate, the participants worked increasingly with software as an entity in its own right and learned to use it according to their interests. They developed an understanding of the Turtle and its various geometries, using both the Path button and the dashed Turtle tracks, in ways which showed their developing confidence and competence. As §9.1.3 indicates, these two features of the software provided a tool to explore the Turtle's screen behaviour and perceptual information about its position in the models, respectively. Once the learning-exploration process had enabled the participants to understand and use the modelling relationship between the physical surfaces and the computer-based Conformal models, they tended to operate with the software.

A second dimension to the participants' use of the software was also apparent during the learning-exploration process. This centred on the way in which the features of the software provided a continual perceptual reference to that which lay *beyond* it, namely the physical surfaces. The dashed Turtle tracks served not only to indicate position within the models, but also to show that the computer images *were* models. They enabled the participants to relate their perceptions of the screen to familiar notions of perspective and curvature. Similarly, the Path button had a well-defined meaning within the context of the software, but also referred the participants to the path of things moving on the physical surfaces beyond the computer. These two features emphasised the inverse of the modelling relationship, from computer to surface, in a way which continually reminded the participants of the connection while allowing them to work, *at the same time*, within the discourse of the software. One may conjecture that it was this simultaneous understanding of the computer-based models as *objects* to be used and *symbols* of something which lay beyond the object, fostered by the features of the software, which reflected and reinforced the cognitive development of the participants.

One may conjecture, therefore, that the features of the software performed a similar role in supporting the participants' *embedding* within the protocols of the computer-based models, while *simultaneously* pointing the participants *beyond* the models to the physical surfaces. In doing so, the Path button and the dashed Turtle tracks were serving a *symbolic* role which supported the dialectic of embedding and separation that lies at the heart of this account of cognitive development. Hence, the participants were able to develop their fluency with the software at the same time as they reconstructed their understanding because the symbolic role of the software's features both facilitated and supported the process.

"Developing a feel" for the subject domain may be interpreted, therefore, as inducting the participants into the domain through embedding them in its practices. This embedding gives the participants a sense of direct contact with the structures of the domain. At the same time, the participants are enabled to distance themselves from the practices of the subject domain through the activities and the tools available to them. The symbolic structure of these practices mediates the dialectical relationship between embedding and separation. This relationship was mirrored in the learning-exploration process which the participants engaged in as they made sense of the microworld. The dynamic interplay between the pedagogic and technical elements of the microworld enabled the participants to both engage with and stand apart from the microworld as they grew in confidence and competence with it.

9.2 Microworld Design and "Developing a Feel": Critical Issues

What then is the relationship between the construction of the microworld and developing a feel for non-euclidean geometry? This section will consider two aspects to the question. First, it will describe what actually happened to the microworld by comparing its initial structure with that of the final model. Having summarised the development of the microworld by considering the development of its model, the second aspect to be considered is an attempt to map out the connections between the cognitive and pedagogical issues, identified through the synchronic views, and the processes of microworld design and development. Two areas will be considered. The first relates to the structure and function of the microworld's model in relation to identifying areas of cognitive "significance". The second issue concerns the exploratory nature of the study and how that affected the process of development.

9.2.1 What Happened: The Microworld's Development

The development of the microworld over the period of the three cycles can be summarised by using the diachronic view of the microworld to compare its initial structure with that obtained after three cycles of development. Figure 9.4 shows the initial structure.

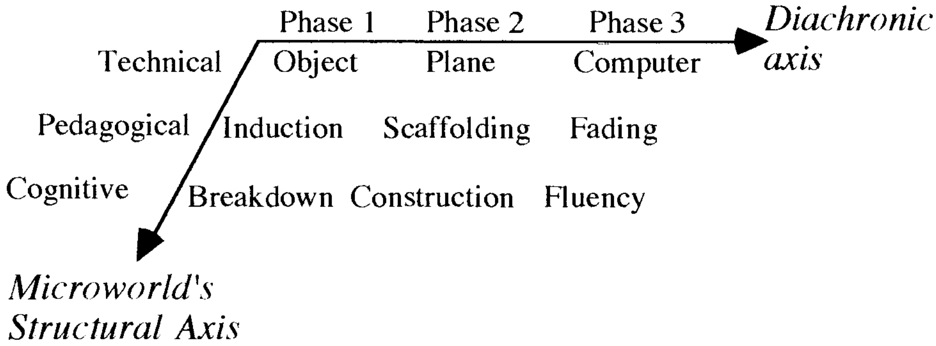


Figure 9.4 The Original Diachronic View of the Microworld. It shows the Diachronic axis moving horizontally from left to right and the elements of the microworld on the other axis, together with their initial structures.

In the original model, the phases of the diachronic view were structured by the technical element of the microworld, with Phase 1 being dominated by the use of physical objects, Phase 2 by projections of the surfaces and Phase 3 by computers. The associated aspects of the pedagogical and cognitive elements are also shown. This model forms the base-line against which to describe the development of the microworld and will be compared with the "finishing point" of the development, shown in Figure 9.5. This brings the changes to the pedagogical and cognitive elements described in §9.1 together into one structure.

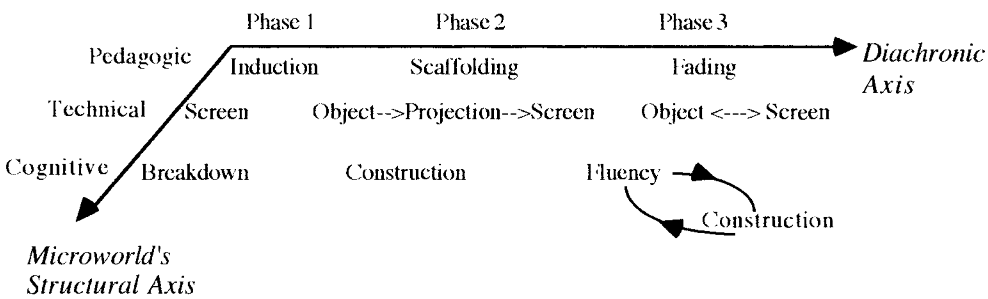


Figure 9.5. The Final Diachronic View.

The final model differs in three respects from the initial one. First, the diachronic view is structured by the pedagogical structure of induction, scaffolding and fading which will be discussed in the next section. Second, the technical element was re-structured to show the various combinations of objects used during both the pedagogic and cognitive elements. Finally, the structure of the cognitive element was changed to reflect the iterative nature of the process, whereby the participants developed fluency with the software as they constructed their understanding of the geometry.

Comparing these two models, it is possible to gain an insight into the process of development which occurred over the three cycles, as the pedagogical element of the microworld "adjusted" to the participants' responses. Two points are significant. First, comparing the initial and final structures of the model shows how the determination of the diachronic structure of the microworld changed from the technical to the pedagogical element. Initially, the microworld was structured by the surfaces used, but this changed to the pedagogical phases of induction, scaffolding and fading. This served to emphasise the fact that the microworld was primarily pedagogic and, consequently, its structure was determined by pedagogical considerations.

Second, comparing the structure of each element within the model indicates two things. In the first place, the pedagogical structure remained a linear progression but both the technical and cognitive elements had non-linear diachronic developments. This reveals an assumption implicit in the initial model; that all three areas, technical, cognitive, and pedagogic, would develop in linear manner. Although this was perhaps the simplest assumption initially, it turned out to be incorrect. In the second place, the different diachronic structures of the pedagogical and cognitive elements of the microworld emphasised the difference between teaching and learning. From a pedagogical point of view, the intention was for the participants to gain fluency so that they no longer needed to be "scaffolded" and this constituted the end point of the pedagogical process. Cognitively, however, the process was more complex, with a diachronic structure that reflected the dynamic nature of coming to understand the Conformal models built around the link between physical surfaces and the computer.

9.2.2 The Structure and Function of the Microworld's Model

Central to this study was the development of a model for the microworld. Both its structure and function played an important part in describing, organising and analysing the activities of the participants. Both the structure and function of the microworld's model will now be considered, to assess their contribution to establishing the cognitive aspects of the microworld.

The structure of the model served two purposes. First, it enabled the articulation both of the microworld's technical, pedagogic and cognitive elements, and the relationship between the elements. This provided a framework with which to describe and analyse the microworld's development. It also enabled the distinction to be made between the diachronic structure of the microworld and the diachronic development of the microworld. This was used to distinguish the microworld's division into three phases from the development of that phase structure over three developmental cycles.

The model functioned as a means of *identifying* the cognitively-significant aspects of the microworld relevant to a particular phase and *supported* their development. Two examples of this process can be seen in the identification of the need for physical objects and the features of the software. In the first case, the participants' need for physical objects as an integral part of the process of learning and exploring the computer-based model emerged from their attempts to try and understand the puzzling screen images. Second, the role of the dashed Turtle tracks in helping the participants make sense of the models was partly the result of reflecting on the Conformal models themselves and the partly the result of the participants' use of them.

Using the phases and cycles, therefore, allowed a systematic investigation of each phase to be carried out to identify or support important cognitive developments. The microworld model could be used to *coarse-tune* for significant features by locating the search within a specific framework with known parameters. It could also be used to *fine-tune* the structures of the microworld to the responses of the participants, thereby aiding their understanding of the underlying knowledge domain.

9.2.3 "An Exploratory Study......"

As the aims of the study indicate, a central objective was to explore the possibilities for constructing a computer-based context for learning non-euclidean geometry. The exploratory nature of the study meant that the activities generated by the microworld had two functions within the framework of the model. First, the activities used in the microworld confirmed and supported those aspects of the participants' cognitive development which had been identified and understood from previous cycles. Second, the activities were also used to search for those factors which were significant for the development of the participants' understanding of the computer-based Conformal models. The process of investigation had three characteristics: iterative, dynamic, and empirically-based. The first two characteristics are summarised in Figure 9.6.

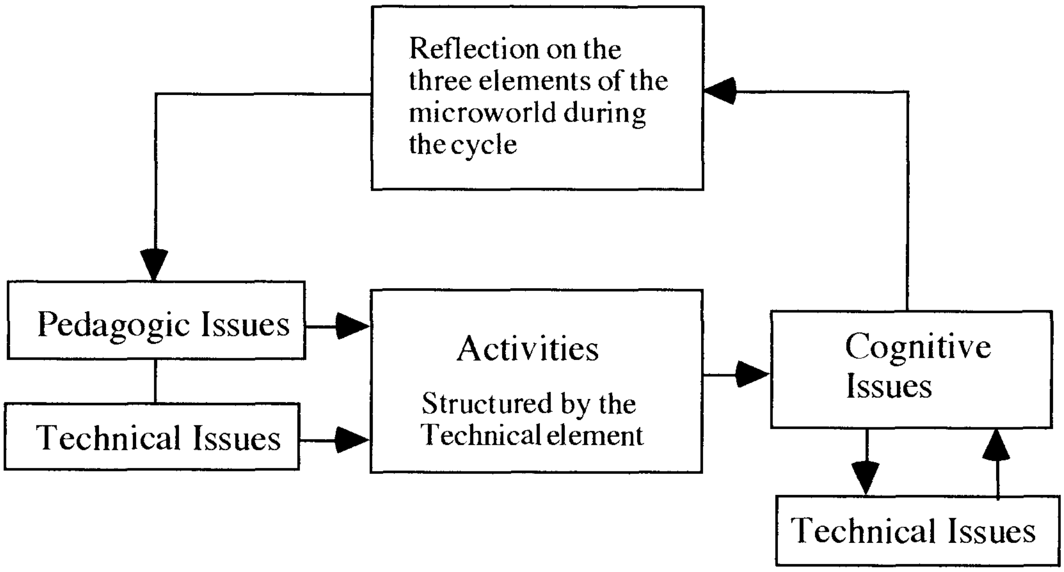


Figure 9.6 The Structure of the Global Window. Each cycle begins with a set of pedagogical and technical issues which have either been carried over from a previous cycle or, in the case of the first cycle were part of the initial attempts to build the microworld. The activities were tried with the participants, and the development in their understanding as a result of their work with the microworld's technical components was reflected on. This generated a new set of pedagogical and technical issues which, together with new ideas, formed the basis for designing the next set of activities.

The *iterative* nature of the design process lay in the fact that the study was divided into three developmental cycles. Each cycle had the same structure, which began with a set of pedagogical and technical issues that had either been carried over from a previous cycle or, in the case of the first cycle, were part of the initial attempts to build the microworld. Using the same structure for the model enabled the outcomes of the various cycles to be used in refining the next cycle. The *dynamic* nature of the process lay in the interaction of the three elements, technical, pedagogic, and cognitive, which made up the microworld. After each cycle, new activities and approaches were generated by reflecting on the cognitive responses of the participants and these played an important role in determining the direction of the next cycle. Finally, the fact that the developmental process was *empirically-based* enabled the technical and pedagogical issues raised by the outcomes of each cycle to be clearly focused on what was happening in the microworld. It was possible to formulate an understanding of the difficulties which the participants were experiencing and how the elements of the microworld either helped or hindered the development of the participants' understanding.

9.2.3 Towards a Principled Design for Microworlds

What may be concluded from this in relation to this study? How did the design of this microworld facilitate the development of a "feel" for non-euclidean geometry? An answer to this question may be found in the metaphor which runs through the chapters devoted to the development of the microworld: exploration.

This study has been concerned with developing a computer-based context for teaching and learning non-euclidean geometry using the euclidean Conformal models. It had two aspects. One was to provide a window on what were the important pedagogical and cognitive issues with regard to the non-euclidean geometry. Another was to provide a window on the way in which these issues could be identified and developed. What bound these two aspects together was the exploratory nature of the process for both learner and designer. The twin principles that might be formulated to draw together this study, therefore, are those of design and learning as explorations.

❑ Design as Exploration

The study has been engaged in a search to identify what was pedagogically, technically and cognitively significant for the domain of non-euclidean geometry. Searching for these significant aspects took place within a framework defined by the microworld's model. It delineated the area for the search and, through the Phases and

Cycles structure, enabled a systematic coarse-tune, a sweep, of the search area. The diachronic and synchronic views gave this coarse-tuning process a definite structure which enabled the identification of those cognitively and pedagogically significant aspects of the search. These could then be developed through the "fine-tuning" process implicit in the iterative cycles of development.

In terms of microworld design, this suggests that the process is an iterative, dynamic and empirically-based activity. It sets out to explore the knowledge domain of the microworld for entry points and modes of cognitive development suited to that domain. This involves a systematic course-tuning of the domain for cognitively and pedagogically significant aspects and a fine-tuning of those aspects identified. All this presupposes a framework which is capable of articulating such a detailed search by providing categories to describe and analyse the "tuning" process.

❑ Learning as Exploration

What distinguished aspects of the microworld as cognitively, technically, or pedagogically significant were those which transformed learning into exploration. This transformation was apparent in the change of the participants' behaviour, described in §9.1. The three-stage pedagogic strategy located the participants within the domain of the Conformal models through its use of physical surfaces, images of projection, and computer-based models. The participants began by ensuring that they understood this "location" process and then began to explore the possibilities offered by the software. As §9.1 indicates, the dynamic interplay between the participants and the technical element of the microworld enabled two things to happen simultaneously. One was that the participants were able to embed themselves in the practices of the Conformal models. The other was that they developed a reflective distance and began to explore the models for themselves. The two aspects were, perhaps, mediated by the features of the software which simultaneously gave the participants a tool to work with and an object to think with.

"Cognitive significance" is a central aspect of what has been described earlier as "developing a feel" for non-euclidean geometry. Two things may be said of it. First it refers to that which is significant "to me" and which orientates "me" towards the knowledge domain on which a microworld is built. Second it expresses a quality and structuring of a relationship rather than a specific content. It connects "me" with the knowledge domain in a direct and, as Wilensky (1991) puts it, a concrete way, which simultaneously embeds me in the knowledge domain and allows me to objectivate it. Learning, then, becomes a personal exploration of the new knowledge domain, built on a structure which supports both the quality of directness and the distancing implied by the term "knowing".

9.3 Limitations of the Study

The major limitations of the study were those of scale and time. The design process which entailed development of the software, the creation of activities, and the detailed analysis of the participants' responses took up a considerable amount of time. It was decided to stop after three cycles for two reasons. First, the software seemed to be both accurate and fast enough for practical purposes. This was an important achievement and it gave the participants in the third cycle a reliable set of computer-generated Conformal models to work with. Secondly, the pedagogic strategy in the third cycle seemed to be effective both as a way of introducing hyperbolic geometry and of overcoming the strains in the didactic contract identified in the first two cycles.

However, as it stands, the conclusion of the study must be tempered by the scale of the developmental process, both in terms of the number of pairs used to trial activities and the time spent by the pairs on the full range of activities. The same pairs were used in Cycles 2 and 3, which had the advantage that they were familiar with the context and the aims of the study. A disadvantage of this was that the pairs used may not have provided either a wide enough range of cognitive responses or explored aspects of the software which were potentially significant. Neither Pair D or E followed the full range of activities from start to finish in one continuous experience with the microworld. There was a gap of a year between the second and third cycle. It was decided to concentrate on Turtle C (Conformal model B) in the third cycle, partly because the pairs seemed to understand the Conformal model for the Sphere after the second cycle and partly because of constraints on their time. This meant that the time that the pairs spent on the Third Cycle was also a limiting factor. Although they seemed to produce interesting results in Cycle 3, it would have been useful to have spent longer on the activities.

From a technical point of view there were two other limitations. The first concerns the use of the conical representation of the hyperboloid in the Third Cycle. Clearly, having the physical surface made a considerable difference to the development of the participants' understanding. However, the fact that the conical representation was precisely that, a representation, led to some confusion. This occurred in two ways. The participants used the circular cross-section to guide their investigations, as in Pair E, or they used the linear aspect of the conical surface to make incorrect inferences about hyperbolic geometry (Pair D). An accurate physical surface is clearly needed. The second technical issue concerns the software. This was not properly error-trapped and although this did not cause any problems during the development of the microworld, there was the potential for difficulties. This required some extra work which was not carried out due to time constraints.

Pedagogically, the activities of the Cycle 3 were concerned with Turtle C, to the exclusion of Turtle A. As was mentioned earlier, this was partly due to the fact that the pairs seemed to understand Turtle A and partly due to time constraints. However, the introduction of the dashed Turtle tracks, which formed a significant contribution to helping the participants understand Turtle C, were not explored in Cycle 3 with Turtle A.

9.4 Implications for Further Research

Connected with this are a number of questions for further considerations. The first is to extend the number of people with which to try out the microworld "from scratch". The participants during the second and third cycles had developed an understanding of what the intentions of the study were. It would be interesting to see how others with no knowledge of the microworld's aims might have faired. Connected with this is the question of whether individuals actually do develop the capacity to use the software unaided or without reference to physical surfaces. Is it only a matter of time before the cyclical process of learning-exploration using the computer and physical surfaces associated with developing fluency, described in §9.1.2, gives way to work entirely on the computer? On the other hand, is it the case that the objects used are *intrinsic* to the understanding developed by the participants and will always play some role in their use of the computer-based models?

From a technical point of view, the software presents a number of possibilities. The OOP and the mathematics developed for the software enabled Turtles to be constructed that acted according to Conformal flat metrics. In General Relativity, there is a standard form of representing known solutions to the field equations by using two-dimensional conformal metrics called Penrose Diagrams (Hawking and Ellis 1973). The Conformal models represented by the software are also of this type. This suggests the possibility that the mathematical techniques used to find the equations for Turtle Geometry can be extended to the case of Penrose Diagrams.

From the design point of view, an issue for further consideration is the style of interface available. OOP provides the possibility for control of the Turtle both semantically and graphically. Although some "direct" manipulation features were developed in this version of the software, such as being able to position the Turtle using the mouse, they were not investigated. The notion of "directness", identified as important for developing a "feel" for the knowledge domain, could be explored by attending both to what features participants might need and to the way in which participants use such direct manipulation features.

9.5 Implications for Teaching and Learning

In this final section, the implications of this study for teaching and learning will be considered. Two areas will be discussed. The first will examine the contribution which the study makes to the teaching and learning of the Conformal models for non-euclidean geometry. The second will outline the implications for microworld design.

9.5.1 Teaching and Learning Non-euclidean Geometry

The study has produced both a possible approach to teaching and learning non-euclidean geometry using the computer-based Conformal models and given some insight into the process of learning to use the models. The Conformal models are counter-intuitive and, as the experience of the participants shows, the models and their properties must be explicitly introduced. The didactic introduction, which this implies and which is outlined in §9.1.4, relies on developing intuitions about physical surfaces as a prelude to showing how the curved surfaces are projected to form the Conformal models. The study also shows the importance of the physical surfaces for learning and using the Conformal models, both as something "to think with" and to support intuition.

The software developed for this study introduces a number of features which are directly related to the models and which facilitate the learner coming to understand the meaning of the models. First, setting the study of the Conformal models in the context of Turtle Geometry provides a context in which to explore the models in a dynamic and interactive way. Second, the dashed Turtle tracks and the Path button provides the learner with tools, specifically related to the models, to investigate the specific structure of the models. This, set within the pedagogical framework mentioned above, provides a new approach to teaching and learning the Conformal models.

9.5.2 Microworld Design

The study provides some insight into designing microworlds. The twin principles described above, based on the idea of design and learning as exploration, provide an approach to designing microworlds which has two aspects.

First, they suggest that the design process involves bringing together knowledge of a particular domain on which the microworld is to be based and approaches to teaching. These provide a starting point for examining how learners may be inducted into the knowledge domain. However, the process of identifying what is important for the learner is an exploratory one and there needs to be a framework in which this exploration can take place. The framework must direct the process of development and provide the means to describe and analyse the outcomes in a systematic way. The model developed for the study seems to be an example of such a framework.

Secondly, what the design process is searching for are ways of inducting the learner into the knowledge domain and developing the means to support their exploration of it. Central to this process is the identification of those aspects of the knowledge domain which enable the learner to connect directly with it, at the same time as supporting their objectivation of it.

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Appendices

A.1 Curvature and Ideas of Space

Space and its nature are issues which occur in philosophy and psychology as well as mathematics. Jammer (1954) describes the early history of philosophical questions about space, tracing its origins in the speculations of the Pythagoreans and early Greek Atomists. For the Pythagoreans, space was intimately related to the "void", being the necessary condition for there to be individuals which could be counted. Their preoccupation with number implied a need to be able to distinguish things: space filled the "gaps". Atomists such as Democritus filled the void with indivisible and indestructible units from which all things were made. Both concepts have influenced thinkers to the present day, centring on the question of whether space is a "something" or a "nothing".

From the production of *The Elements* by Euclid around 300 BC. until the nineteenth century, euclidean geometry was considered to be both *the* expression of mathematics and *the* description of physical space. Kant (1929), for example, regarded space as a "synthetic *apriori*" concept; prior to experience, but which can give us knowledge about the world. Space, as a category, structures our experience and provides the necessity which we experience when we reason geometrically. Kant thought that space was euclidean in nature and this ensured universality of our perception and gave necessity to geometric judgements about the world. This marked a shift from thinking of space as a "thing" to space as a structuring principle of experience. Kant was influenced by the Newtonian world-view (outlined below). He sought to provide philosophical backing for the notion of absolute space underpinning Newtonian mechanics while at the same time insisting that it had no reality, but was "the form of outer intuition" (Smart 1973 p.119).

To describe the role that curvature has played in changing the view of Euclid as the description of physical space, it is necessary to describe the change in conceptions of space and time that occurred at the turn of this century. To set this in context, it is important to describe the view of space and time held prior to 1900 and this is most clearly expressed by Newton.

A.1.1 Space and Time of Newton

Newton defined his notions of space and time under the *Scholium* to the definitions of Part 1 in *Philosophie Naturalis Principia Mathematica*. Space and time were both absolute and completely separate.

> Absolute, true and mathematical time, of itself, and from its own nature, flows equably without relation to anything external.........
>
> Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of absolute space (Newton 1713 Third edition)

Being unaffected by the masses that move within them, space and time formed the context in which laws of motion were formulated and descriptions were expressed. Time and space, being of different ontological categories, were measured in different ways, using different instruments (clocks and rods) and were not affected by the process of measurement. One entity can be used to index the other so that particle position, for example, could be given as a function of time.

An important consequence of this was the idea of simultaneity: being able to specify the position of particles at the same instance of time. It implied that given some initial "snapshot" of the universe at a time T_0, the entire future development can be predicted by means of the laws of motion; an idea that was central to the notions of determinism and science in the nineteenth century. Figure A.1 illustrates this.

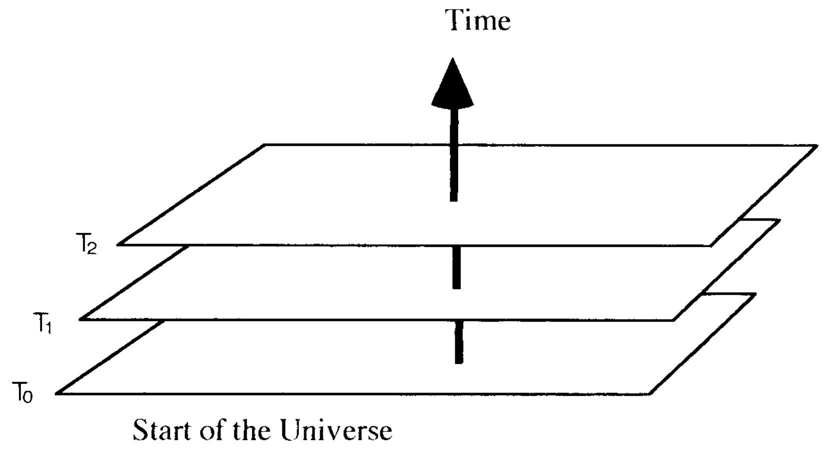

Figure A.1 Newtonian view of the relationship between space and time. Space and time are considered to be distinct entities and could be represented as slices of space moving along a time axis. The initial slice at the "Start" of the Universe contains all the information about the universe. Its subsequent evolution can be described as a function of time.

Given the initial conditions and the laws of physics as known at the turn of the century, some physicists thought that they could completely predict the future. The diagram above shows the first time slice, T_0, at the start of the universe, containing the position of all particles at the same time. These were then "evolved" forword to give the position of every particle at time T_1. The flat two-dimensional slices actually stand for three-dimensional euclidean space.

1.2 Spacetime of Special Relativity

Until the end of the nineteenth century, the Newtonian view of space and time was both dominant and successful. An implication of Newton's conception is that infinite speeds are possible. If space and time are separate, any interval of space may be divided by any temporal interval, thereby suggesting that there was no limit to the speed of particles. In 1887, Michelson and Morley set out to establish the value of the velocity of light in an attempt to settle the question of whether light needed a medium "the ether", to propagate. They found, to within 5 km/s, that the velocity of light is the same whether the light is travelling in the same direction as the earth or in the opposite direction. Further, they found the velocity is the same, irrespective of the frame in which it is measured. Light not only had a constant speed, it was an absolute speed in the sense that its value was independent of the way in which the velocity is measured.

In 1905, Einstein published a paper which brought together the result about the velocity of light and accounted for how different coordinate systems could describe nature so that the laws of physics were preserved. Special Relativity develops an understanding of space and time which takes account of this by proposing that the laws of nature are invariant under a class of coordinate transformation called the Lorentz Transforms.

Suppose that one has a clock and a rod to measure time and distance in one's own coordinate system (s , t), where t is proper time and s is arc length. One also knows that the fastest anything can travel is light, in any coordinate system. Arrange the units so that the velocity of light is 1, it follows that there is a line in the space such that s = t and it is at 45 degrees to the coordinate axes. (t axis horizontal and the s axis vertical). Figure 1.2 illustrates this.

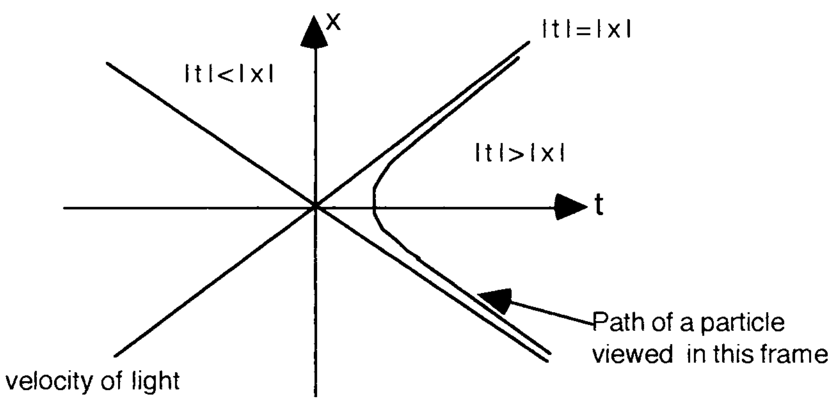

Figure A..2 The diagram shows the path of a particle with proper time t and arc length s as seen from a coordinate system with time t and space coordinate x. The lines in which | t | = | x | represent the speed of light. These line divides the graph into two sets of regions in | t | > | x | and | t | < | x | respectively. The particle follows a hyperbolic path | x | > | t | since it cannot travel faster than the speed of light.

The slope of the line gives the speed of any particle as shown on the diagram. Any particle with coordinate (t , s) must move with a speed less than 1 and so is restricted to the region for which $|t| < |x|$. This implies that any particle moves on a curve $x^2 - t^2 > 0$, which is a hyperbola, as shown in the diagram. The interesting point here is that the path of a particle moving with uniform speed is not a euclidean straight line, but a hyperbola, implying that the geometry is non-euclidean.

Two things emerged from this which were of interest from the point of view of the study. The first was the idea that *space and time are inextricably linked through the constancy of the velocity of light*. Time coordinates are treated in exactly the same way as space coordinates. Unlike Newtonian mechanics, there was no way of measuring space and time independently of one another. Mathematically, the space-time of Special Relativity is four-dimensional rather than the three-plus-one dimensions of Newtonian mechanics. The second thing to emerge was that for purely physical reasons, *the geometry of space and time is non-euclidean*. In fact, it is hyperbolic when viewed from a euclidean point of view.

A.1.3 Spacetime in General Relativity

General Relativity is based on two principles: equivalence and general covariance. The first is concerned with the notion of space and time and so will be the focus of this section. The second is a mathematical condition, which determines the way in which the theory is to be expressed. The principle of equivalence asserts that gravitational and inertial mass are the same. It has its origins in Galileo's observation that freely-falling masses undergo the same acceleration irrespective of their composition. It implies that if one's coordinate system is freely falling in a gravitational field, then one does not experience a gravitational "force", provided the system is small enough. Such coordinate systems are referred to as local inertial frames and they have the geometry of Special Relativity.

Gravity enters the picture when the geometry of spacetime departs from that of Special Relativity. Space-time is flat in Special Relativity; it has zero curvature. General Relativity identifies gravity with the non-zero curvature of the space-time . As

Wheeler *et al.* (1973) put it : "Space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve" (p.5).

Einstein selected the simplest possible equation to express equivalence of mass-energy and curvature: $G = k\ T$, where G represents the average curvature in all directions, T represents the mass-energy and k is a constant. Solving for the above equation to find a Riemannian metric, enables a model to be constructed. Once the metric is found, the local and global geometry of the space-time is given and, unlike the Newtonian model, the metric is not extrinsic to the physics but determines the behaviour of particles. The metric and curvature defined from it gives the geometric and dynamic properties of a spacetime as suggested by the principle of equivalence.

Starting with Newton, in which space and time were the framework within which dynamics was described using a fixed euclidean metric, we have reached the position with Einstein in which spacetime itself is determined by many different metrics. Each metric is equivalent to the distribution of mass-energy and corresponds to different gravitational fields. Central to the theory is that curvature of spacetime represents gravity and so plays a significant role in describing the geometry of physical space.

A.2. Calculations for the Derivation of the Geodesic Equations.

$$\mathbf{V} = v_1 \mathbf{E}_1 + v_2 \mathbf{E}_2 \Rightarrow \dot{\mathbf{V}} = \dot{v}_1 \mathbf{E}_1 + v_1 \dot{\mathbf{E}}_1 + \dot{v}_2 \mathbf{E}_2 + v_2 \dot{\mathbf{E}}_2.$$

The components of $\dot{\mathbf{V}}$, relative to the basis $\{\mathbf{E}_1, \mathbf{E}_2\}$, are given by

$$\dot{\mathbf{V}} \bullet \mathbf{E}_1 = V_1 \text{ and } \dot{\mathbf{V}} \bullet \mathbf{E}_2 = V_2$$

Hence

$$V_1 = \dot{v}_1 (\mathbf{E}_1 \bullet \mathbf{E}_1) + v_1 (\dot{\mathbf{E}}_1 \bullet \mathbf{E}_1) + \dot{v}_2 (\mathbf{E}_2 \bullet \mathbf{E}_1) + v_2 (\dot{\mathbf{E}}_2 \bullet \mathbf{E}_1) \quad ..(1)$$

$$V_2 = \dot{v}_1 (\mathbf{E}_1 \bullet \mathbf{E}_2) + v_1 (\dot{\mathbf{E}}_1 \bullet \mathbf{E}_2) + \dot{v}_2 (\mathbf{E}_2 \bullet \mathbf{E}_2) + v_2 (\dot{\mathbf{E}}_2 \bullet \mathbf{E}_2) \quad ...(2)$$

$(\mathbf{E}_1 \bullet \mathbf{E}_2) = 0$ since the vectors are orthogonal., but the terms containing the derivatives of the vectors are not. Introducing two sets of notation:

$$(\mathbf{E}_i \bullet \mathbf{E}_j) = g_{ij} \text{ and } \Gamma_{ij,k} = \frac{\partial \mathbf{E}_i}{\partial x_j} \bullet \mathbf{E}_k$$

In this notation, the components of the basis vector's derivative are:

$$(\dot{\mathbf{E}}_i \bullet \mathbf{E}_k) = \frac{\partial E_i}{\partial x_1}\frac{dx_1}{ds} \bullet \mathbf{E}_k + \frac{\partial E_i}{\partial x_2}\frac{dx_2}{ds} \bullet \mathbf{E}_k$$

$$(\dot{\mathbf{E}}_i \bullet \mathbf{E}_k) = \Gamma_{i1,k}\frac{dx_1}{ds} + \Gamma_{i2,k}\frac{dx_2}{ds}$$

where s is the arc length for the curve defining **V.**

Hence, for equations (1) and (2)

$$(\dot{\mathbf{E}}_1 \bullet \mathbf{E}_1) = \Gamma_{11,1}\frac{dx_1}{ds} + \Gamma_{12,1}\frac{dx_2}{ds}, \qquad (\dot{\mathbf{E}}_2 \bullet \mathbf{E}_1) = \Gamma_{21,1}\frac{dx_1}{ds} + \Gamma_{22,1}\frac{dx_2}{ds}$$

$$(\dot{\mathbf{E}}_1 \bullet \mathbf{E}_2) = \Gamma_{11,2}\frac{dx_1}{ds} + \Gamma_{12,2}\frac{dx_2}{ds}, \qquad (\dot{\mathbf{E}}_2 \bullet \mathbf{E}_2) = \Gamma_{21,2}\frac{dx_1}{ds} + \Gamma_{22,2}\frac{dx_2}{ds}$$

$$v_1 = \frac{dx_1}{ds}, v_2 = \frac{dx_2}{ds}, \dot{v}_1 = \frac{d^2x_1}{ds^2}, \dot{v}_2 = \frac{d^2x_2}{ds^2}$$

Hence, $V_1 = g_{11}\dot{v}_1 + v_1\left(\Gamma_{11,1}\frac{dx_1}{ds} + \Gamma_{12,1}\frac{dx_2}{ds}\right) + v_2\left(\Gamma_{21,1}\frac{dx_1}{ds} + \Gamma_{22,1}\frac{dx_2}{ds}\right)$

which can be written as : $V_1 = g_{11}\frac{d^2x_1}{ds^2} + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij,1}\frac{dx_i}{ds}\frac{dx_j}{ds}$ ....(3)

Similarly, $V_2 = g_{22}\dot{v}_2 + v_1\left(\Gamma_{11,2}\frac{dx_1}{ds} + \Gamma_{12,2}\frac{dx_2}{ds}\right) + v_2\left(\Gamma_{21,2}\frac{dx_1}{ds} + \Gamma_{22,2}\frac{dx_2}{ds}\right)$

which can be written as : $V_2 = g_{22}\frac{d^2x_2}{ds^2} + \sum_{i=1}^{2}\sum_{j=1}^{2}\Gamma_{ij,2}\frac{dx_i}{ds}\frac{dx_j}{ds}$ ....(4)

A3.Projection of Curved Surfaces

In three-dimensions, A is the projection point with coordinate (0, 0, a), P is the point being projected on a unit sphere with coordinate (X, Y, Z). P′ is the image of P on the X-Y plane with coordinate (x, y). AP is a straight line so that in vector form

$$\mathbf{OP'} = \mathbf{OA} + \lambda\,\mathbf{AP} \Rightarrow \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} + \lambda \begin{pmatrix} X \\ Y \\ Z-a \end{pmatrix}$$

This gives: $x = \lambda X, \qquad y = \lambda Y, \qquad \lambda = \dfrac{a}{a-Z}$ ....(1)

A 3 .1 Projection of the Sphere

For the unit sphere $X^2 + Y^2 + Z^2 = 1$ with the projection point is (0, 0, 1), any point P on the sphere with coordinate (X, Y, Z) will be projected to a point P′ on the x-y plane with coordinate (x, y). From (1):

$$X^2 + Y^2 + Z^2 = 1 \Rightarrow (1-Z)^2 x^2 + (1-Z)^2 y^2 + Z^2 = 1$$

$$\Rightarrow x^2 + y^2 = \frac{1-Z^2}{(1-Z)^2} = \frac{1+Z}{1-Z}$$

$$\Rightarrow Z = \frac{x^2+y^2-1}{x^2+y^2+1}$$

Hence, $X = \dfrac{2x}{x^2+y^2+1}, \qquad Y = \dfrac{2y}{x^2+y^2+1}$ ....(2)

Geodesics on the sphere are obtained by plane sections through the origin To show this, consider the following basis of unit orthogonal vectors $\{\mathbf{E}_1, \mathbf{E}_2\}$ for R^3 such that

$$\mathbf{E}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \text{ and } \mathbf{E}_2 = \begin{pmatrix} \mathrm{Cos}\theta \\ \mathrm{Sin}\theta \\ 0 \end{pmatrix} \text{ for some } \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right).$$

They define a plane through the origin and can be used to give the position vector of a point $\mathbf{r}(t) = \mathrm{Cos}(t)\,\mathbf{E}_1 + \mathrm{Sin}(t)\,\mathbf{E}_2$, on a circle which lies in the plane they define. Now $|\mathbf{r}(t)| = 1$ and so the circle also lies on the surface of the unit sphere. It remains to show that this is a geodesic of the sphere. For this to be the case, $\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}} = 0$ and this follows from the metric $dS^2 = dX^2 + dY^2 + dZ^2$ on the sphere.

A.3.2 Projection of the Two-Sheet Hyperboloid

Using the basis $\{\mathbf{E}_1, \mathbf{E}_2\}$ for R^3 as in the spherical case, it possible to show that the geodesics of the solid are obtained by plane sections . Let $\mathbf{r}(t) = \text{Cosh}(t)\,\mathbf{E}_1 + \text{Sinh}(t)\,\mathbf{E}_2$ be the equation of an hyperbola in the plane defined by the basis vectors. This curve lies on the positive branch of the hyperboloid with $|\mathbf{r}(t)| = 1$ (using the hyperboloid metric for the modulus) and hence its derivative with respect to t is tangent to the surface. Now $\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}} = 0$ using the metric $dS^2 = dZ^2 - dX^2 - dY^2$, imposed on the hyperboloid and so $\mathbf{r}(t)$ is a geodesic of the surface.

A.4 Derivation of the Equations of Governing Turtle Motion

The aim of this section is to find the equations of motion which will govern the Turtle's behaviour. Since the Turtle moves in a manner which is determined by the intrinsic geometry of each model provided by the metric, the first step in finding the equations of motion is to use equations (3) and (4) in §A.2 with the induced metric.

Computing the value of $$\Gamma_{ij,k} = \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial x_j} + \frac{\partial g_{jk}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_k}\right\} \quad \text{............(1)}$$

with the given the metric components:

$$g_{11} = g_{22} = \frac{4}{(1 + k(x^2 + y^2))^2} \text{ and } g_{12} = g_{21} = 0,$$

by substituting the derivatives of the metric components into (1), gives the following:

$$\Gamma_{11,1} = \frac{-8kx}{(1 + k(x^2 + y^2))^3} \qquad \Gamma_{11,2} = \frac{8ky}{(1 + k(x^2 + y^2))^3}$$

$$\Gamma_{12,1} = \frac{-8ky}{(1 + k(x^2 + y^2))^3} \qquad \Gamma_{12,2} = \frac{-8kx}{(1 + k(x^2 + y^2))^3}$$

$$\Gamma_{21,1} = \frac{-8ky}{(1 + k(x^2 + y^2))^3} \qquad \Gamma_{21,1} = \frac{-8kx}{(1 + k(x^2 + y^2))^3}$$

$$\Gamma_{22,1} = \frac{8kx}{(1 + k(x^2 + y^2))^3} \qquad \Gamma_{22,2} = \frac{-8ky}{(1 + k(x^2 + y^2))^3}$$

Substituting the metric components into the equations of the geodesics in equations §A. 2 (3) and (4) gives a pair of second order differential equations which describe how the coordinates $x = x_1$ and $y = x_2$ vary with arc length, s. For geodesics,

$$g_{11}\frac{d^2x_1}{ds^2} + \frac{dx_1}{ds}\left(\Gamma_{11,1}\frac{dx_1}{ds} + \Gamma_{12,1}\frac{dx_2}{ds}\right) + \frac{dx_2}{dt}\left(\Gamma_{21,1}\frac{dx_1}{ds} + \Gamma_{22,1}\frac{dx_2}{ds}\right) = 0$$

$$g_{22}\frac{d^2x_2}{ds^2} + \frac{dx_1}{ds}\left(\Gamma_{11,2}\frac{dx_1}{ds} + \Gamma_{12,2}\frac{dx_2}{ds}\right) + \frac{dx_2}{ds}\left(\Gamma_{21,2}\frac{dx_1}{ds} + \Gamma_{22,2}\frac{dx_2}{ds}\right) = 0$$

which, on substitution and simplification, gives :

$$\frac{d^2x_1}{ds^2} = \frac{2k}{1+k(x_1^2+x_2^2)}\left[x_1\left(\frac{dx_1}{ds}\right)^2 - x_1\left(\frac{dx_2}{ds}\right)^2 + 2x_2\frac{dx_1}{ds}\frac{dx_2}{ds}\right]$$

...(2)

$$\frac{d^2x_2}{ds^2} = \frac{2k}{1+k(x_1^2+x_2^2)}\left[x_2\left(\frac{dx_2}{ds}\right)^2 - x_2\left(\frac{dx_1}{ds}\right)^2 + 2x_1\frac{dx_1}{ds}\frac{dx_2}{ds}\right]$$

(These were checked using MAPLE V and Thorpe 1979)

A. 5 Listing of the Object Logo Code

Setting Up the Listener Window

```
ask first listeners [setwpos [400 210] setwsize [240 230]]
make "surface kindof turtlewindow

ask :surface [to exist]
MAKE "initlist SENTENCE :INITLIST [WTITLE Surface WPOS [0 40] WSIZE [400
400] growp false procid 16]
havemake "surwin [0 40 400 440]
USUAL.EXIST
ht
setbackcolor 409
setpencolor 30
havemake "sc 200
end

ask :surface [TO WCLICK :X :Y :MODS]
local [newmouse winpos]
MAKE "newmouse list :X :Y
make "winpos globaltolocal mousepos
IF PTINRECTP :NEWMOUSE :surwin [ask :t1 [ pu setpos (se (first :winpos) (last
:winpos) + 40) make "z (complex xcor/:scale ycor /:scale) make "dz (complex (first
set.up.initial.heading) (last set.up.initial.heading))*(1 + :k * :z * (cong :z)) make
"oldcoord (se :z :dz)  pd]]
usual.wclick :x :y :mods
end

make "s1 oneof :surface
```

Creating the Turtle

```
make "t kindof turtle

ask :t [to exist]
usual.exist
have  "k
have "z
have "dz
havemake "scale 200
havemake "step 0.01
have "oldcoord
end

make "t1 (oneof :t)


ask :t1 [st]
```

Creating The Button Pad

```
make "buttons kindof turtlewindow

ask :buttons [to exist]
MAKE "initlist SENTENCE :INITLIST [WTITLE Turtles WPOS [400 40] WSIZE
[240 150] growp false procid 16]
USUAL.EXIST
HT
```

```
setbackcolor 341
SETPENMODE "reverse
SETWFONT "Helvetica
setwfontsize 12

HAVEMAKE "Turtle.A.but [5 5 75 40]
setpencolor 33
paintroundRECT :Turtle.A.BUT 12 12
setpencolor 205
paintoval :Turtle.A.BUT
MOVETO 17 27
setpencolor 69
PRINT "|Turtle A|

HAVEMAKE "Turtle.B.but [85 5 155 40]
setpencolor 33
paintroundRECT :Turtle.B.BUT 12 12
setpencolor 205
paintoval :Turtle.B.BUT
MOVETO 96 27
setpencolor 69
PR "|Turtle B|

HAVEMAKE "Turtle.C.but [165 5 230 40]
setpencolor 33
paintroundRECT :Turtle.C.BUT 12 12
setpencolor 205
paintoval :Turtle.C.BUT
MOVETO 175 27
setpencolor 69
PR "|Turtle C|

HAVEMAKE "scale.but [5 55 75 90]
setpencolor 33
paintroundRECT :scale.but 12 12
setpencolor 205
paintoval :scale.but
MOVETO 15 75
setpencolor 69
PRINT "Scale

HAVEMAKE "Boundary.but [165 55 230 90]
setpencolor 33
paintroundRECT :Boundary.but 12 12
setpencolor 205
paintoval :Boundary.but
MOVETO 175 75
setpencolor 69
PR "Boundary


HAVEMAKE "step.but [45 105 115 140]
setpencolor 33
paintroundRECT :step.but 12 12
setpencolor 205
paintoval :step.but
MOVETO 55 125
setpencolor 69
PR "|Step Size|

HAVEMAKE "path.but [125 105 190 140]
```

```
setpencolor 33
paintroundRECT :path.but 12 12
setpencolor 205
paintoval :path.but
MOVETO 145 125
setpencolor 69
PR "Path
ENd

make "b1 oneof :buttons


ask :buttons [TO WCLICK :X :Y :MODS]
MAKE "newmouse LIST :X :Y
IF PTINRECTP :NEWMOUSE :Turtle.A.BUT [ SA]
IF PTINRECTP :NEWMOUSE :Turtle.B.BUT [SB]
IF PTINRECTP :NEWMOUSE :Turtle.C.But [SC]
if PTINRECTP :NEWMOUSE :path.But [paths]
IF PTINRECTP :NEWMOUSE :boundary.BUT [boundary]
IF PTINRECTP :NEWMOUSE :step.But [step.size]
IF PTINRECTP :NEWMOUSE :Scale.But [Res]
usual.wclick :x :y :mods
end


ask :buttons [to sa]
invertroundrect :Turtle.a.but 20 15
ask :t1 [cs st make "k 1 ]
while [buttonp] []
ask first listeners [pr [Turtle A is now active]]
invertroundrect :Turtle.a.but 20 15
setdefaultturtle :t1
end


ask :buttons [to sb]
ask :t1 [cs st make "k 0 ]
invertroundrect :Turtle.B.but 20 10
while [buttonp][]
ask first listeners [pr [Turtle B is now active]]
invertroundrect :Turtle.B.but 20 10
setdefaultturtle :t1
end

ask :buttons [to sc]
ask :t1 [cs st make "k -1 ]
invertroundrect :Turtle.C.but 20 10
while [buttonp][]
ask first listeners [pr [Turtle C is now active]]
invertroundrect :Turtle.C.but 20 10
setdefaultturtle :t1
end

ask :buttons [to boundary]
ask :s1 [frameoval [0 0 400 400] ]
invertroundrect :boundary.but 20 10
while [buttonp][]
invertroundrect :boundary.but 20 10
setdefaultturtle :t1
end
```

```
ask :buttons [to paths]
path
invertroundrect :path.but 20 10
while [buttonp][]
invertroundrect :path.but 20 10
setdefaultturtle :t1
end


ask :buttons [to res]
local [p]
invertroundrect :scale.but 20 10
make "p tscale
while [buttonp][]
LOCAL [list1 list2]
MAKE "list1 DIALOGWORD "|Enter the scale. Default values are scale = 200 | :p
[scaling]
make "list2 dialog (word first :list1 "= last :list1 "| OK?|) []
if :list2 = "Yes [ invertroundrect :scale.but 20 10 INVOKE FIRST :LIST1 LAST
:LIST1 ]
if :list2 = "No [ invertroundrect :scale.but 20 10 res]
if :list2 = "Cancel [ invertroundrect :scale.but 20 10 stop ]
end

ask :buttons [to step.size]
local [p]
invertroundrect :step.but 20 10
make "p tstep
while [buttonp][]
LOCAL [list1 list2]
MAKE "list1 DIALOGWORD "|Enter the step size. Default value is scale = 0.01 | :p
[stepsize]
make "list2 dialog (word first :list1 "= last :list1 "| OK?|) []
if :list2 = "Yes [ invertroundrect :step.but 20 10 INVOKE FIRST :LIST1 LAST :LIST1
]
if :list2 = "No [ invertroundrect :step.but 20 10 res]
if :list2 = "Cancel [ invertroundrect :step.but 20 10 stop ]
end

to tstep
ask :t1 [op :step]
end

to stepsize :a
ask :t1 [make "step :a]
end

to scaling :s
ask :t1 [make "scale :s]
end
```

Shadowing the FORWARD Command

```
ask :t [to fd :s]
initial.state
local [c]
make "c 0
repeat :s [direct dash :c make "c :c + 1 setheading towards (se (realpart :z) * :scale
(imagpart :z) * :scale) USUAL.FD (len) * :scale make "oldcoord (se :z :dz)]
end
```

```
ask :t [to len]
op (abs (:z - (first :oldcoord)))
end

ask :t [to initial.state]
ifelse pos = [0 0] [make "z complex 0 0] [make "z first :oldcoord]
make "dz (complex (first set.up.initial.heading) (last set.up.initial.heading))*(1 + :k * :z
* (cong :z))
make "oldcoord (se :z :dz)
setpencolor 30
end

ask :t [to direct]
local [k1 k2 k3 k4 l1 l2 l3 l4]
make "k1 :step * :dz
make "l1 :step * (d2z :z :dz)
make "k2 :step * (:dz + :l1/2)
make "l2 :step * (d2z (:z + :k1/2) (:dz + :l1/2))
make "k3 :step * (:dz + :l2/2)
make "l3 :step * (d2z (:z + :k2/2) (:dz + :l2/2))
make "k4 :step * (:dz + :l3/2)
make "l4 :step * (d2z (:z + :k3/2) (:dz + :l3/2))
make "z :z + (:k1 + 2 * :k2 + 2 * :k3 + :k4) / 6
make "dz :dz + (:l1 + 2 * :l2 + 2 * :l3 + :l4) / 6
make "dz :dz * (1 + :k * :z * (cong :z))/(abs :dz)
end


ask :t [to d2z :z :dz]
op (2 * :k * (cong :z) * (:dz) ^ 2)/(1 + :k * (abs :z)^2 )
end


ask :t [to cong :z]
op  complex realpart :z minus imagpart :z
end


ask :t [to initialise]
make "z (complex xcor/:scale ycor /:scale)
make "dz (complex (first set.up.initial.heading) (last set.up.initial.heading))*(1 + :k * :z
* (cong :z))
make "oldcoord (se :z :dz)
end


ask :t [to set.up.initial.heading ]
if (and heading >0 heading < 90) [op (se ((sin heading) )  ((cos heading)))]
if (and heading >90 heading < 180) [op (se (cos heading - 90 )  ((-1)*(sin heading -
90)))]
if (and heading >180 heading < 270) [op (se ((-1)*(sin heading - 180) )  ((-1)*(cos
heading - 180)))]
if (and heading >270 heading < 360) [op (se ((-1)*(cos heading - 270))  ((sin heading -
270)))]
if heading = 0 [op [0 1]]
if heading = 90 [op [1 0]]
if heading = 180 [op [0 -1]]
if  heading =270 [op [-1 0]]
end
```

```
to tcoord
ask :t1 [op (se :x1 :x2)]
end
```

Shadowing the BACK Command

```
ask :t [to bk :s]
rt 180
fd :s
rt 180
end
```

Creating Dashed Turtle Tracks

```
ask :t [to dash :s]
ifelse (remainder :s 10) < 5 [pd][pu]
end
```

*Creating the **Path** Command*

```
ask :surface [to set.up.coords]
op (se (ask :t1[ realpart :z]) (ask :t1 [imagpart :z]) (-1/set.up.heading))
end


ask :surface [to set.up.heading]
if (or heading = 0  heading = 180) [op tan 89.95]
if (or heading = 90 heading = 270 ) [op 0.00000001]
if (and heading >0 heading < 90) [op (tan (90-heading))]
if (and heading > 90 heading <270) [op (tan -(abs heading - 90))]
if (and heading >270 heading < 360) [op (tan (450 - heading))]
end

to tscale
ask :t1 [op :scale]
end


ask :surface [to set.up.hcircle :list]
local [p q m s t rad s1 sc]
make "sc tscale
make "p first :list
make "q first bf :list
make "m last :list
make "s1  2 * :p + 2 * :m *:q
make "s (1 / :s1) * (:p^2 + :q^2 + (2 * :p * :q * :m) - (2 * :q^2) + 1)
make "t :m * :s - :m * :p + :q
make "rad (SQRT :s^2 + :t^2 - 1)
op (se (:s * :sc) (:t * :sc)  (:rad * :sc))
end

ask :surface [to set.up.ecircle :list]
local [p q m s t rad s1 sc]
make "sc tscale
make "p first :list
make "q first bf :list
make "m last :list
make "s1  2 * :p + 2 * :m *:q
```

```
make "s (1 / :s1) * (:p^2 + :q^2 + (2 * :p * :q * :m) - (2 * :q^2) - 1)
make "t :m * :s - :m * :p + :q
make "rad (SQRT :s^2 + :t^2 + 1)
op (se (:s * :sc) (:t * :sc)  (:rad * :sc))
end

ask :surface [to circle]
local [ c ]
make "c find.curvature
if (equalp :c  -1) [hregion  set.up.hcircle set.up.coords ]
if (equalp :c 1) [eregion set.up.ecircle set.up.coords]
end

ask :surface [to hregion :list]

local [a b]
setpencolor 409
startrgn
frameoval [0 0 400 400]
make "a getrgn
startrgn
frameoval (se ((first :list) + 199 - (last :list)) (199 - (first bf :list) - (last :list)) ((first :list)
+ 201 + (last :list)) (200 - (first bf :list)  + (last :list)))
make "b getrgn
setpencolor 137
framergn :a
framergn sectrgn :a :b
setpencolor 30
end

ask :surface [to eregion :list]
frameoval (se ((first :list) + 199 - (last :list)) (199 - (first bf :list) - (last :list)) ((first :list)
+ 201 + (last :list)) (200 - (first bf :list)  + (last :list)))
setpencolor 30
end

ask :surface [to check.line]
local [p c]
make "c find.curvature
if (or (equalp heading 0) (equalp pos [0 0])) [lineto  200 0 lineto  200 400]
if (and (heading > 0) (equalp pos [0 0]) (equalp :c -1)) [lineto (200 * sin heading) (200
* cos heading)]
end

to find.curvature
ask :t1 [op :k]
end

to path
ask :s1 [check.line]
end


ask :surface [to check.line]
local [p c h]
ask :t1 [initialise]
make "h (ask :t1 [set.up.initial.heading])
make "c find.curvature
make "p (ask :t1 [first :oldcoord ])
ifelse (equalp pos [0 0]) [check.line1 :h :c][circle]
end
```

```
ask :surface [to check.line1 :h :c ]
ifelse (equalp heading 0)  [lineto 200 0 lineto 200 400] [ifelse (equalp :c -1) [pu setpos
[0 0 ] pd line1 200 :h 1][pu setpos [0 0 ] pd line1 200 :h 2 ]]
end


ask :surface [to line1 :l :h :s]
lineto (:l*(1+ :s*(first :h))) (:l *(1-:s*(last :h)))
lineto (:l*(1-:s*(first :h)))  (:l*(1+:s*(last :h)))
end
```